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Results in Mathematics

On Legendre Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds

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Abstract. The present paper is devoted to study the curvature and torsion of Frenet Legendre curves in 3-dimensional normal almost paracontact metric manifolds. Moreover, in this class of manifolds, properties of non-Frenet Legendre curves (with null tangents or null normals or null binormals) are obtained. Many examples of Legendre curves are constructed.

Some of the present results are analogous to those obtained by the author in [10] for Frenet Legendre curves in 3-dimensional normal almost contact manifolds.

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1. Preliminaries

Let M be a $(2n+1)$-dimensional differentiable manifold. Let φ be a $(1,1)$-tensor field, ξ a vector field and η a 1-form on M. Then (φ, ξ, η) is called an almost paracontac structure on M if

(i) $\eta(\xi) = 1, \quad \varphi^2 = \mathrm{Id} - \eta \otimes \xi$,

(ii) the tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D} = ker\, \eta$, that is, the eigendistributions $\mathcal{D}^+$, $\mathcal{D}^-$ corresponding to the eigenvalues 1, -1 of φ, respectively, have equal dimension n.

M is said to be almost paracontact manifold if it is endowed with an almost paracontact structure (cf. [3, 6, 9, 11]).

Let M be an almost paracontact manifold. M will be called an almost paracontact metric manifold if it is additionally endowed with a pseudo-Riemannian

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metric g of signature $(n+1, n)$ such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y).$$

For such a manifold, we additionally have $\eta(X) = g(X, \xi)$, $\varphi\xi = 0$, $\eta \circ \varphi = 0$. Moreover, we can define a skew-symmetric 2-form Φ by $\Phi(X, Y) = g(X, \varphi Y)$, which is called the fundamental form corresponding to the structure. Note that $\eta \wedge \Phi$ is, up to a constant factor, the Riemannian volume element of M.

On an almost paracontact manifold, one defines the $(2, 1)$-tensor filed $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) - 2\, d\eta(X, Y)\xi, \tag{1.1}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y].$$

If $N^{(1)}$ vanishes identically, then the almost paracontact manifold (structure) is said to be normal (cf. [9] and recent papers [3, 11]). The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$ by

$$J\left(X, \lambda\frac{d}{dt}\right) = \left(\varphi X + \lambda\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable (paracomplex).

Similarly as in the class of almost contact metric manifolds [1], a normal almost paracontact metric manifold will be called para-Sasakian if $\Phi = d\eta$ [6] and quasi-para-Sasakian if $d\Phi = 0$. Obviously, the class of para-Sasakian manifolds is contained in the class of quasi-para-Sasakian manifolds. The converse does not hold in general.

2. Normal almost paracontact metric manifolds

In the sequel, we need the following characterization of normality.

Proposition 2.1. *A $(2n + 1)$-dimensional almost paracontact metric manifold is normal if and only if*

$$\varphi(\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)Y + (\nabla_X\eta)(Y)\xi = 0, \tag{2.1}$$

∇ being the Levi–Civita connection.

Proof. Rewrite the normality condition (1.1) with the help of the Levi–Civita connection as

$$\begin{aligned}\varphi(\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)Y + (\nabla_X\eta)(Y)\xi&\\ - \big(\varphi(\nabla_Y\varphi)X - (\nabla_{\varphi Y}\varphi)X + (\nabla_Y\eta)(X)\xi\big) &= 0.\end{aligned} \tag{2.2}$$

Let us define an auxiliary $(0, 3)$-tensor field A by

$$A(X, Y, Z) = g\big(\varphi(\nabla_X\varphi)Y - (\nabla_{\varphi X}\varphi)Y + (\nabla_X\eta)(Y)\xi, Z\big).$$

Obviously, if $A = 0$, then the structure is normal.

We are going to show that if the structure is normal, then $A = 0$. First, by (2.2), we see that

$$A(X, Y, Z) = A(Y, X, Z)\,. \tag{2.3}$$

Next a simply calculation shows that

$$\begin{aligned} A(X, Y, Z) + A(X, Z, Y) = & -g\big((\nabla_X\varphi)Y, \varphi Z\big) \\ & -g\big(\varphi Y, (\nabla_X\varphi)Z\big) \\ & +\eta(Y)g(\nabla_X\xi, Z) + \eta(Z)g(\nabla_X\xi, Y)\,. \end{aligned} \tag{2.4}$$

By the compatibility condition $g(\varphi Y, \varphi Z) = -g(Y, Z) + \eta(Y)\eta(Z)$ we have

$$g\big((\nabla_X\varphi)Y, \varphi Z\big) + g\big(\varphi Y, (\nabla_X\varphi)Z\big) = \eta(Y)g(\nabla_X\xi, Z) + \eta(Z)g(\nabla_X\xi, Y)\,,$$

which applied to (2.4) gives $A(X, Y, Z) = -A(X, Z, Y)$. But this together with (2.3) implies the following

$$\begin{aligned} A(X, Y, Z) &= -A(X, Z, Y) = -A(Z, X, Y) = A(Z, Y, X) \\ &= A(Y, Z, X) = -A(Y, X, Z) = -A(X, Y, Z)\,. \end{aligned}$$

Hence it follows that $A = 0$, completing the proof. □

In the sequel, we are mainly interested in dimension 3-dimensional manifolds.

Proposition 2.2. *For a 3-dimensional almost paracontact metric manifold M it holds*

$$(\nabla_X\varphi)Y = g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi\,. \tag{2.5}$$

Proof. The 3-form $\eta \wedge \Phi$ is equal to the volume element of M up to a constant factor. Therefore $\nabla_X(\eta \wedge \Phi) = 0$, and hence

$$\begin{aligned} &(\nabla_X\eta)(Y)\Phi(Z, W) + \eta(Y)(\nabla_X\Phi)(Z, W) \\ &+ (\nabla_X\eta)(Z)\Phi(W, Y) + \eta(Z)(\nabla_X\Phi)(W, Y) \\ &+ (\nabla_X\eta)(W)\Phi(Y, Z) + \eta(W)(\nabla_X\Phi)(Y, Z) = 0\,. \end{aligned}$$

Substituting $W = \xi$ in the above equality, we derive

$$\begin{aligned} (\nabla_X\Phi)(Z, Y) = & -\eta(Z)(\nabla_X\Phi)(Y, \xi) + \eta(Y)(\nabla_X\Phi)(Z, \xi) \\ &= g\big(Z, g(\varphi\nabla_X\xi, Y)\xi - \eta(Y)\varphi\nabla_X\xi\big)\,, \end{aligned}$$

which gives (2.5). □

Proposition 2.3. *For a 3-dimensional almost paracontact metric manifold M the following three conditions are mutually equivalent*

(a) *M is normal,*

(b) *there exist functions α, β on M such that*

$$(\nabla_X\varphi)Y = \beta\big(g(X, Y)\xi - \eta(Y)X\big) + \alpha\big(g(\varphi X, Y)\xi - \eta(Y)\varphi X\big)\,, \tag{2.6}$$

(c) *there exist functions α, β on M such that*

$$\nabla_X\xi = \alpha\big(X - \eta(X)\xi\big) + \beta\varphi X\,. \tag{2.7}$$

Proof. Using (2.5), we note that for our manifold the normality condition (2.1) is equivalent to

$$\nabla_{\varphi X}\xi = \varphi\nabla_X\xi\,. \tag{2.8}$$

It is clear that (2.7) implies (2.8). To prove that (2.8) implies (2.7), we choose an adapted frame (E_0, E_1, E_2), i.e. a frame such that

$$E_0 = \xi\,,\quad \varphi E_1 = E_2\,,\quad \varphi E_2 = E_1\,,\quad g(E_1, E_1) = -1\,,\quad g(E_0, E_0) = g(E_2, E_2) = 1\,.$$

Taking (2.8) into consideration, we have

$$\nabla_{E_0}\xi = 0\,,\quad \nabla_{E_1}\xi = \alpha E_1 + \beta E_2\,,\quad \nabla_{E_2}\xi = \beta E_1 + \alpha E_2$$

for certain α, β. Hence, we can deduce (2.7).

Applying (2.5), one claims that (2.7) implies (2.6). Conversely, assume that (2.6) is fulfilled. Then from (2.6) with $Y = \xi$ we derive (2.7). □

Corollary 2.4. *The functions* α, β *realizing* (2.6) *as well as* (2.7) *are given by*

$$2\alpha = \operatorname{Trace}\{X \to \nabla_X\xi\}\,,\quad 2\beta = \operatorname{Trace}\{X \to \ \varphi\nabla_X\xi\}\,. \tag{2.9}$$

Proposition 2.5. *For a* 3*-dimensional almost paracontact metric manifold* M*, the following three conditions are mutually equivalent*

(a) M *is quasi-para-Sasakian,*

(b) *there exists a function* β *on* M *such that*

$$(\nabla_X\varphi)Y = \beta\big(g(X,Y)\xi - \eta(Y)X\big)\,, \tag{2.10}$$

(c) *there exists a function* β *on* M *such that*

$$\nabla_X\xi = \beta\varphi X\,. \tag{2.11}$$

Especially, such a manifold is para-Sasakian if and only if the condition (2.11) *or, equivalently,* (2.10) *is fulfilled with* $\beta = -1$.

Proof. By virtue of Proposition 2.3, to prove the first assertion, we should only show that, for a 3-dimensional normal almost paracontact metric manifold, $d\Phi = 0$ if and only if $\alpha = 0$. To do it, apply (2.6):

$$\begin{aligned}
3d\Phi(X,Y,Z) &= (\nabla_X\Phi)(Y,Z) + (\nabla_Y\Phi)(Z,X) + (\nabla_Z\Phi)(X,Y)\\
&= g\big(Y, (\nabla_X\varphi)Z\big) + g\big(Z, (\nabla_Y\varphi)X\big) + g\big(X, (\nabla_Z\varphi)Y\big)\\
&= 2\alpha\big(g(X,\varphi Y)\eta(Z) + g(Y,\varphi Z)\eta(X) + g(Z,\varphi X)\eta(Y)\big)\\
&= 2\alpha\big(\Phi(X,Y)\eta(Z) + \Phi(Y,Z)\eta(X) + \Phi(Z,X)\eta(Y)\big)\\
&= 6\alpha(\Phi\wedge\eta)(X,Y,Z)\,.
\end{aligned}$$

Since $\Phi\wedge\eta$ is non-zero at any point of our manifold, we claim that $d\Phi = 0$ if and only if $\alpha = 0$.

To have the second assertion completed, we should guarantee the condition $d\eta = \Phi$. Note that, by (2.11),

$$\begin{aligned}
2d\eta(X,Y) &= (\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X)\\
&= -2\beta g(X,\varphi Y) = -2\beta\Phi(X,Y)\,.
\end{aligned}$$

Hence, $d\eta = \Phi$ if and only if $\beta = -1$. □

3. Frenet Legendre curves

Let (M, g) be an 3-dimensional pseudo-Riemannian manifold. Let $\gamma\colon I \to M$, I being an interval, be a curve in M such that $g(\dot\gamma,\dot\gamma) = \varepsilon_1$, $\varepsilon_1 = \pm 1$, and let $\nabla_{\dot\gamma}$ denote the covariant differentiation along γ. We say that γ is a Frenet curve if one of the following three cases hold

(a) γ is of osculating order 1, i.e., $\nabla_{\dot\gamma}\dot\gamma = 0$ (geodesics);

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $E_1(= \dot\gamma)$, E_2, $(g(E_2, E_2) = \varepsilon_2 = \pm 1)$ and a positive function κ (the curvature) along γ such that

$$\nabla_{\dot\gamma}E_1 = \kappa\varepsilon_2 E_2\,, \quad \nabla_{\dot\gamma}E_2 = -\kappa\varepsilon_1 E_1\,;$$

(c) γ is of osculating order 3, i.e., there exist three orthonormal vector fields $E_1(= \dot\gamma)$, E_2, E_3, $(g(E_2, E_2) = \varepsilon_2 = \pm 1$, $g(E_3, E_3) = \varepsilon_3 = \pm 1)$ and two positive functions κ (the curvature) and τ (the torsion) along γ such that

$$\nabla_{\dot\gamma}E_1 = \kappa\varepsilon_2 E_2\,, \quad \nabla_{\dot\gamma}E_2 = -\kappa\varepsilon_1 E_1 + \tau\varepsilon_3 E_3\,, \quad \nabla_{\dot\gamma}E_3 = -\tau\varepsilon_2 E_2\,.$$

In contact geometry, a curve is called Legendre if it is an integral curve of the contact distribution, see e.g. [1].

By analogy, a curve γ in an almost paracontact metric manifold M will be called Legendre if it is an integral curve of the paracontact distribution $\mathcal{D}$, that is, $\eta(\dot\gamma) = 0$.

Theorem 3.1. *Let M be a 3-dimensional normal almost paracontact metric manifold. If a Frenet Legendre curve $\gamma\colon I \to M$ is not a geodesic, then its curvature and torsion are given by*

$$\kappa = \sqrt{|\alpha^2 - \varepsilon_1\delta^2|}\,, \tag{3.1}$$

$$\tau = \left|\beta + \frac{\alpha\dot\delta - \dot\alpha\delta}{\alpha^2 - \varepsilon_1\delta^2}\right|, \tag{3.2}$$

where δ is the function defined by $\delta = g(\nabla_{\dot\gamma}\dot\gamma, \varphi\dot\gamma)$, and for simplicity we write α, β instead of the composed functions $\alpha\circ\gamma, \beta\circ\gamma$ with α, β being the same as in (2.9).

Proof. Let γ be a Frenet Legendre curve in a 3-dimensional normal almost paracontact metric manifold. Note that $\dot\gamma, \varphi\dot\gamma, \xi$ are orthonormal vector fields along γ. Differentiating $g(\dot\gamma, \xi) = 0$ along γ and using (2.7), we see that $g(\nabla_{\dot\gamma}\dot\gamma, \xi) = -\varepsilon_1\alpha$. Moreover, let $\delta = g(\nabla_{\dot\gamma}\dot\gamma, \varphi\dot\gamma)$. Then we have

$$\nabla_{\dot\gamma}E_1 = \nabla_{\dot\gamma}\dot\gamma = -\varepsilon_1\alpha\xi - \varepsilon_1\delta\varphi\dot\gamma\,. \tag{3.3}$$

Therefore,

$$\varepsilon_2\kappa^2 = g(\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma) = \alpha^2 - \varepsilon_1\delta^2 \tag{3.4}$$

so that the curvature is given by (3.1), $\varepsilon_2 = \operatorname{sgn}(\alpha^2 - \varepsilon_1\delta^2) = \pm 1$ and

$$E_2 = \frac{1}{\varepsilon_2\kappa}\nabla_{\dot\gamma}E_1 = -\frac{\varepsilon_1\varepsilon_2\alpha}{\kappa}\xi - \frac{\varepsilon_1\varepsilon_2\delta}{\kappa}\varphi\dot\gamma\,.$$

Differentiating E_2 along γ and next applying (2.7), (2.6) and (3.3), we obtain

$$\begin{aligned}\nabla_{\dot\gamma}E_2 = &-\varepsilon_1\varepsilon_2\dot\gamma\left(\frac{\alpha}{\kappa}\right)\xi-\varepsilon_1\varepsilon_2\frac{\alpha}{\kappa}\nabla_{\dot\gamma}\xi \\ &-\varepsilon_1\varepsilon_2\dot\gamma\left(\frac{\delta}{\kappa}\right)\varphi\dot\gamma-\varepsilon_1\varepsilon_2\frac{\delta}{\kappa}\big((\nabla_{\dot\gamma}\varphi)\dot\gamma+\varphi\nabla_{\dot\gamma}\dot\gamma\big) \\ = &-\varepsilon_1\kappa\dot\gamma-\varepsilon_1\varepsilon_2 a\xi-\varepsilon_1\varepsilon_2 b\varphi\dot\gamma\,,\end{aligned} \tag{3.5}$$

where

$$a=\frac{\varepsilon_1\delta\beta}{\kappa}+\frac{\dot\alpha\kappa-\alpha\dot\kappa}{\kappa^2}\,,\quad b=\frac{\alpha\beta}{\kappa}+\frac{\dot\delta\kappa-\delta\dot\kappa}{\kappa^2}\,.$$

With the help of (3.4), the functions a, b can be transformed into

$$a=\frac{\varepsilon_1\delta c}{\kappa}\,,\quad b=\frac{\alpha c}{\kappa}\quad\text{with}\quad c=\beta+\varepsilon_2\frac{\alpha\dot\delta-\dot\alpha\delta}{\kappa^2}\,.$$

The signature of our metric g is $(1,2)$, and therefore $\varepsilon_3=-\varepsilon_1\varepsilon_2$. Using (3.4), one checks that $a^2-\varepsilon_1 b^2=-\varepsilon_1\varepsilon_2c^2=\varepsilon_3c^2$. On the other hand, having (3.5), we obtain

$$\varepsilon_3\tau E_3=\nabla_{\dot\gamma}E_2+\varepsilon_1\kappa E_1=\varepsilon_3(a\xi+b\varphi\dot\gamma)\,.$$

Consequently, using also (3.1), we derive $\varepsilon_3\tau^2=a^2-\varepsilon_1b^2=\varepsilon_3c^2$. This gives (3.2), completing the proof. □

As an immediate consequence of the above theorem, we obtain

Theorem 3.2. *Let M be a 3-dimensional quasi-para-Sasakian manifold. If a Frenet Legendre curve $\gamma\colon I\to M$ is not a geodesic, then its torsion is strictly related to the structure function β of M by the formula $\tau=|\beta|$.*

4. Non-Frenet Legendre curves

At first, we study the case when a Legendre curve γ has null tangent (is a null curve), that is, $g(\dot\gamma,\dot\gamma)=0$.

Theorem 4.1. *Let M be a 3-dimensional normal almost paracontact metric manifold. If $\gamma\colon I\to M$ is a Legendre curve with null tangent, then $\nabla_{\dot\gamma}\dot\gamma=\vartheta\dot\gamma$, ϑ being a function, and consequently, γ is a geodesic after a reparametrization.*

Proof. Let γ be a Legendre curve with null tangent in a 3-dimensional normal almost paracontact metric manifold. Then we can locally choose vector fields $\{\dot\gamma, V, W\}$ along the curve γ such that (cf. [4,5])

$$\begin{aligned}g(\dot\gamma,\dot\gamma)=g(V,V)=0\,,\\ g(W,W)=g(\dot\gamma,V)=1\,,g(\dot\gamma,W)=g(V,W)=0\,,\end{aligned}\tag{4.1}$$

In view of (4.1), we can write

$$\nabla_{\dot\gamma}\dot\gamma=c\dot\gamma+dW\,,\tag{4.2}$$

for certain functions c, d. Differentiating $g(\xi, \dot{\gamma}) = 0$ along γ and using (2.7) and (4.2), we obtain $dg(W, \xi) = 0$. Let us assume that $d \neq 0$. Then $g(W, \xi) = 0$. Therefore and by (4.1), we must have $\xi = a\dot{\gamma} + bV$ along the curve, for some functions a, b. Hence, using (4.1) again, we obtain

$$1 = g(\xi, \xi) = 2ab\,, \quad 0 = \eta(\dot{\gamma}) = g(\xi, \dot{\gamma}) = b\,,$$

which gives a contradiction. Therefore, $d = 0$ and by (4.2), $\nabla_{\dot{\gamma}}\dot{\gamma} = c\dot{\gamma}$, which completes the proof. □

Next, we investigate curves with null normals (for such curves in Minkowski spaces $\mathbb{E}^3_1$, see among others [2,7,8]). We say $\gamma \colon I \to M$ is a curve with null normal if

$$g(\dot{\gamma}, \dot{\gamma}) = \varepsilon_1 = \pm 1\,, \quad \nabla_{\dot{\gamma}}\dot{\gamma} \neq 0\,, \quad g(\nabla_{\dot{\gamma}}\dot{\gamma}, \nabla_{\dot{\gamma}}\dot{\gamma}) = 0\,. \tag{4.3}$$

Theorem 4.2. *Let M be a 3-dimensional normal almost paracontact metric manifold. $\gamma \colon I \to M$ is a Legendre curve with null normal if and only if $g(\dot{\gamma}, \dot{\gamma}) = \varepsilon_1 = 1$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = -\alpha(\xi \pm \varphi\dot{\gamma})$, where α is a function defined in* (2.9)*, and it is non-zero along γ.*

Proof. Let γ be a Legendre curve in a 3-dimensional normal almost paracontact metric manifold. In analogy to the proof of Theorem 3.1, we can write

$$\nabla_{\dot{\gamma}}E_1 = \nabla_{\dot{\gamma}}\dot{\gamma} = -\varepsilon_1\alpha\xi - \varepsilon_1\delta\varphi\dot{\gamma}\,,$$

where $\delta = g(\nabla_{\dot{\gamma}}\dot{\gamma}, \varphi\dot{\gamma})$. Hence, it follows that

$$g(\nabla_{\dot{\gamma}}\dot{\gamma}\, \nabla_{\dot{\gamma}}\dot{\gamma}) = \alpha^2 - \varepsilon_1\delta^2\,.$$

In view of (4.3), γ is a curve with null normal if and only if $\varepsilon_1 = 1$ and $\alpha = \pm\delta \neq 0$. This completes the proof. □

As an immediate consequence of Proposition 2.5 and Theorem 4.2, we obtain

Corollary 4.3. *In a 3-dimensional quasi-para-Sasakian manifold, there are no Legendre curves with null normals.*

Finally, we consider curves with null binormal. We say $\gamma \colon I \to M$ is a curve with null binormal if $g(\dot{\gamma}, \dot{\gamma}) = \varepsilon_1 = \pm 1$ and there exist two orthonormal vector fields $E_1(= \dot{\gamma})$, E_2, $(g(E_2, E_2) = \varepsilon_2 = \pm 1)$ and a positive function κ (the curvature) along γ for which

$$\begin{aligned} &\nabla_{\dot{\gamma}}E_1 = \kappa\varepsilon_2E_2\,, \quad \nabla_{\dot{\gamma}}E_2 + \kappa\varepsilon_1E_1 \neq 0\,, \\ &g(\nabla_{\dot{\gamma}}E_2 + \kappa\varepsilon_1E_1, \nabla_{\dot{\gamma}}E_2 + \kappa\varepsilon_1E_1) = 0\,. \end{aligned} \tag{4.4}$$

Theorem 4.4. *In a 3-dimensional normal almost paracontact metric manifold, there are no Legendre curves with null binormals.*

Proof. Let γ be a Legendre curve in M and $g(\dot{\gamma}, \dot{\gamma}) = \varepsilon_1 = \pm 1$. In analogy to the proof of Theorem 3.1, we can get

$$\begin{aligned} \nabla_{\dot{\gamma}}E_1 = \nabla_{\dot{\gamma}}\dot{\gamma} &= -\varepsilon_1\alpha\xi - \varepsilon_1\delta\varphi\dot{\gamma} = \kappa\varepsilon_2E_2\,, \\ \nabla_{\dot{\gamma}}E_2 + \varepsilon_1\kappa E_1 &= -\varepsilon_1\varepsilon_2(a\xi + b\varphi\dot{\gamma})\,, \end{aligned} \tag{4.5}$$

where

$$\delta = g(\nabla_{\dot\gamma}\dot\gamma, \varphi\dot\gamma)\,, \quad \kappa = \sqrt{|\alpha^2 - \varepsilon_1\delta^2|}\,, \quad \varepsilon_2 = \operatorname{sign}(\alpha^2 - \varepsilon_1\delta^2) = \pm 1\,, \tag{4.6}$$

$$a = \frac{\varepsilon_1\delta c}{\kappa}\,, \quad b = \frac{\alpha c}{\kappa}\,, \quad c = \beta + \varepsilon_2\frac{\alpha\dot\delta - \dot\alpha\delta}{\kappa^2}\,. \tag{4.7}$$

Suppose additionally that γ is a curve with null binormal. By (4.5), we find

$$g(\nabla_{\dot\gamma}E_2 + \varepsilon_1\kappa E_1, \nabla_{\dot\gamma}E_2 + \varepsilon_1\kappa E_1) = a^2 - \varepsilon_1 b^2 = 0\,.$$

If $\varepsilon_1 = -1$, then $a = b = 0$. This together with (4.5) contradicts to (4.4).

If $\varepsilon_1 = 1$, then $a = \pm b$. This by (4.7) is equivalent to $\alpha = \pm\delta$. Then by (4.6), $\kappa = 0$, which contradicts to (4.4). $\square$

5. Examples illustrating theorems

Example. Let $\mathbb{R}^3$ be the Cartesian space and (x, y, z) be the Cartesian coordinates in it. Define the standard almost paracontact structure (φ, ξ, η) on $\mathbb{R}^3$ by

$$\varphi\partial_1 = \partial_2 - 2x\partial_3\,, \quad \varphi\partial_2 = \partial_1\,, \quad \varphi\partial_3 = 0\,, \quad \xi = \partial_3\,, \quad \eta = 2xdy + dz\,, \tag{5.1}$$

where $\partial_1 = \frac{\partial}{\partial x}$, $\partial_2 = \frac{\partial}{\partial y}$ and $\partial_3 = \frac{\partial}{\partial z}$. By direct calculations one verifies that

$$[\varphi, \varphi](\partial_i, \partial_j) - 2d\eta(\partial_i, \partial_j)\xi = 0\,, \quad 1 \le i < j \le 3\,,$$

so that (1.1) is satisfied and the structure is normal.

Suppose that $M = \mathbb{R}^2 \times \mathbb{R}_+ \subset \mathbb{R}^3$ and consider a normal almost paracontact metric structure on M defined in the following way: (φ, ξ, η) is the structure (5.1) restricted to M and g is the Lorentz metric given by

$$\big[g(\partial_i, \partial_j)\big] = \begin{bmatrix} -2z & 0 & 0 \\ 0 & 4x^2 + 2z & 2x \\ 0 & 2x & 1 \end{bmatrix}.$$

For the Levi–Civita connection, we have

$$\nabla_{\partial_1}\partial_1 = -\frac{x}{z}\,\partial_2 + \left(1 + \frac{2x^2}{z}\right)\partial_3\,, \quad \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \frac{x}{z}\,\partial_2 + \left(1 - \frac{2x^2}{z}\right)\partial_3\,,$$

$$\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = \nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = \frac{1}{2z}\,\partial_1 + \frac{1}{2z}\,\partial_2 - \frac{x}{z}\,\partial_3\,,$$

$$\nabla_{\partial_2}\partial_2 = \frac{2x}{z}\,\partial_1 + \frac{x}{z}\,\partial_2 - \left(1 + \frac{2x^2}{z}\right)\partial_3\,, \quad \nabla_{\partial_3}\partial_3 = 0\,.$$

Using the above and (2.7), we find $\alpha = \beta = (2z)^{-1}$. In view of Proposition 2.5, the structure (φ, ξ, η, g) is not quasi-para-Sasakian.

Below we give a list of curves in M possessing various properties:

$$\gamma(t) = \left(0,\, at,\, -\frac{1}{2a^2}\right), \quad a \ne 0\,, \tag{a}$$

is a Frenet Legendre curve with $\varepsilon_1 = -1$ and $\kappa = \tau = a^2$ (this is a helix since κ and τ are constant);

$$\gamma(t) = \left(c\sqrt{t},\ -2c\sqrt{t},\ 2c^2 t\right), \quad t > 0\,, \quad c = \frac{1}{\sqrt[4]{3}}\,, \tag{b}$$

is a Frenet Legendre curve with $\varepsilon_1 = 1$, $\kappa = \frac{\sqrt{2}}{4t}$ and $\tau = \frac{\sqrt{3}}{4t}$ (this is a generalized helix since $\frac{\kappa}{\tau}$ is constant);

$$\gamma(t) = \left(\frac{t}{\sqrt{2c}},\ d,\ c\right), \quad c > 0\,, \tag{c}$$

is a Frenet Legendre curve with $\varepsilon_1 = -1$, $\kappa = \frac{\sqrt{c^2+t^2}}{2c^2}$ and $\tau = \frac{c^2-t^2}{2c(c^2+t^2)}$;

$$\gamma(t) = (-t, t, t^2) \tag{d}$$

is a Legendre curve with null tangent;

$$\gamma(t) = \left(\sqrt{t}, -a\sqrt{t}, at\right), \quad t > 0\,, \quad a = \sqrt[3]{1-b} + \sqrt[3]{1+b}\,, \quad b = \sqrt{\frac{26}{27}}\,, \tag{e}$$

is a Legendre curve with null normal and $\varepsilon_1 = 1$.

Example. Suppose that $M = \mathbb{R}_+ \times \mathbb{R}^2$ and consider a normal almost paracontact metric structure on M defined in the following way: (φ, ξ, η) is the structure (5.1) restricted to M and g is the Lorentz metric given by

$$\big[g(\partial_i, \partial_j)\big] = \begin{bmatrix} -x^2 & 0 & 0 \\ 0 & 5x^2 & 2x \\ 0 & 2x & 1 \end{bmatrix}.$$

For the Levi–Civita connection, we have

$$\begin{aligned}
&\nabla_{\partial_1}\partial_1 = \frac{1}{x}\,\partial_1\,, \quad \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = \frac{3}{x}\,\partial_2 - 5\,\partial_3\,,\\
&\nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = \frac{1}{x^2}\,\partial_2 - \frac{2}{x}\,\partial_3\,,\\
&\nabla_{\partial_2}\partial_2 = \frac{5}{x}\,\partial_1\,, \quad \nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = \frac{1}{x^2}\,\partial_1\,, \quad \nabla_{\partial_3}\partial_3 = 0\,.
\end{aligned}$$

Using the above and (2.7), we find $\alpha = 0$ and $\beta = x^{-2}$. By Proposition 2.5, the almost paracontact metric structure (φ, ξ, η, g) is quasi-para-Sasakian and not para-Sasakian.

The following curves γ in M are Frenet Legendre:

$$\gamma(t) = \left(\frac{1}{c},\ ct,\ -2t\right), \quad c > 0\,, \tag{a}$$

- a helix with $\varepsilon_1 = 1$, $\kappa = \tau = c^2$;

$$\gamma(t) = \left(\sqrt{t^2+1},\ -t,\ t\sqrt{t^2+1} + \ln\left(t + \sqrt{t^2+1}\right)\right), \quad t > 0\,, \tag{b}$$

- a curve $\varepsilon_1 = 1$, $\kappa = \frac{2}{\sqrt{t^2+1}}$ and $\tau = \frac{1}{t^2+1}$;

$$\gamma(t) = \left(\sqrt{t^2-1}\,, t\,, -t\sqrt{t^2-1} + \ln\left(t + \sqrt{t^2-1}\,\right)\right), \quad t > 1\,, \tag{c}$$

- a curve with $\varepsilon_1 = -1$, $\kappa = \frac{2}{\sqrt{t^2-1}}$ and $\tau = \frac{1}{t^2-1}$.

Example. Suppose that $M = \mathbb{R}^2 \times \mathbb{R}_+$. Define a normal almost paracontact structure (φ, ξ, η) on M by

$$\varphi\partial_1 = \partial_2\,, \quad \varphi\partial_2 = \partial_1\,, \quad \varphi\partial_3 = 0\,, \quad \xi = \partial_3\,, \quad \eta = dz, \tag{5.2}$$

and, compatible with this structure, a Lorentz metric

$$\big[g(\partial_i, \partial_j)\big] = \begin{bmatrix} -2z & 0 & 0 \\ 0 & 2z & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadruple (φ, ξ, η, g) becomes a normal almost paracontact metric structure on M. For the Levi–Civita connection, we find

$$\nabla_{\partial_1}\partial_1 = \partial_3\,, \quad \nabla_{\partial_1}\partial_2 = \nabla_{\partial_2}\partial_1 = 0\,, \quad \nabla_{\partial_1}\partial_3 = \nabla_{\partial_3}\partial_1 = \frac{1}{2z}\,\partial_1\,,$$

$$\nabla_{\partial_2}\partial_2 = -\partial_3\,, \quad \nabla_{\partial_2}\partial_3 = \nabla_{\partial_3}\partial_2 = \frac{1}{2z}\,\partial_2\,, \quad \nabla_{\partial_3}\partial_3 = 0\,.$$

Using the above and (2.7), we get $\alpha = (2z)^{-1}$ and $\beta = 0$. By Proposition 2.5, the structure (φ, ξ, η, g) is not quasi-para-Sasakian.

The curve

$$\gamma(t) = \left(\cosh t, \sinh t, \frac{1}{2}\right)$$

is a Legendre curve with $\varepsilon_1 = 1$ and null normal in M.

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