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Results in Mathematics

On Genuine Bernstein–Durrmeyer Operators

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Dedicated to Academician Dimitrie D. Stancu on the occasion of his 80th birthday

Abstract. We continue the studies on the so–called genuine Bernstein–Durrmeyer operators U_n by establishing a recurrence formula for the moments and by investigating the semigroup $T(t)$ approximated by U_n. Moreover, for sufficiently smooth functions the degree of this convergence is estimated. We also determine the eigenstructure of U_n, compute the moments of $T(t)$ and establish asymptotic formulas.

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1. Introduction

The present note continues and supplements previous research on the so-called genuine Bernstein–Durrmeyer operators which – according to our present knowledge – were first considered by W. Chen [2] and T. N. T. Goodman and A. Sharma [4] around 1987. They constitute an approximation process for functions $f \in C[0,1]$ which produces algebraic polynomials and is related to the well-known Bernstein operators B_n as follows. The latter are given by $(f \in C[0,1], x \in [0,1], n \in \mathbb{N})$

$$\begin{aligned} B_n(f;x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \\ &:= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \,. \end{aligned}$$

Furthermore, in 1972 A. Lupaş [8] investigated a Beta-type operator $\overline{\mathbb{B}}_n$ defined by

$$\overline{\mathbb{B}}_n(f;x) := \begin{cases} f(x)\,, & \text{for } x \in \{0,1\} \\ \frac{1}{B(nx,n-nx)} \cdot \int_0^1 t^{nx-1}(1-t)^{n-1-nx} f(t)dt\,, & \text{for } x \in (0,1) \end{cases}$$

with $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt,\ \ a,b > -1$. Both B_n and $\overline{\mathbb{B}}_n$ are positive linear operators reproducing linear functions and thus interpolating every function $f \in C[0,1]$ at 0 and 1. Hence these properties are shared by their composition $U_n := B_n \circ \overline{\mathbb{B}}_n$, the genuine Bernstein–Durrmeyer operators, given explicity by

$$U_n(f;x) = f(0)\cdot p_{n,0}(x) + f(1)\cdot p_{n,n}(x) + \sum_{k=1}^{n-1} p_{n,k}(x)\cdot \int_0^1 p_{n-2,k-1}(t)f(t)dt\,.$$

Here $p_{-1,k} \equiv 0$ and $p_{0,k} \equiv 1$.

Among the many articles written on the U_n, we mention here only the ones by P. Parvanov and B. Popov [11], by T. Sauer [15], by S. Waldron [16], and the book of R. Păltănea [12]. The reader's attention is drawn to the fact that the U_n are sometimes also called "the modified Bernstein–Durrmeyer operators", "modified Bernstein–Schoenberg operators", "Bernstein-type operators", "limit case of Bernstein's operators with Jacobi weights", etc. We prefer to use Sauer's genuine naming.

In the present note Section 2 is devoted to the computation of the moments of U_n and the images of the monomials under U_n. In Section 3 we investigate the semigroup approximated by U_n and include a statement for the degree of approximation of sufficiently smooth functions. Section 4 deals with the eigenstructure of U_n, and Section 5 describes the moments of the semigroup approximated by the Bernstein operators (and thus those of the semigroup associated to the U_n). The final Section 6 provides information concerning the asymptotic behavior of U_{2n} and some consequences of it.

2. The moments of U_n

First we describe a method to compute recursively the moments of U_n and the images of the monomials under U_n. Then we establish a recurrence formula for the moments.

Let $H_{n,x,k}(t) := U_n(e_1 - xe_0)^k(t)$, with $e_i(t) := t^i$.

Theorem 2.1. (i) *$H_{n,x,0}(t) = 1$, $H_{n,x,1}(t) = t - x$, and for $k \geq 2$,*

$$t(1-t)H''_{n,x,k} + k(k-1)H_{n,x,k} = k(k-1)\big[(1-2x)H_{n,x,k-1} + x(1-x)H_{n,x,k-2}\big]\,. \tag{2.1}$$

(ii) *Given $n \geq 1, x \in [0,1], k \geq 2$, we have*

$$H_{n,x,k}(t) = a_k t^k + \cdots + a_1 t + a_0 \tag{2.2}$$

where

$$a_k = \frac{(n-1)(n-2)\dots(n-k+1)}{(n+1)(n+2)\dots(n+k-1)} \tag{2.3}$$

and, if the right-hand side member of (2.1) *is represented as* $c_{k-1}t^{k-1}+\dots+c_1t+c_0$*, then*

$$a_{k-j} = \frac{1}{j(2k-j-1)}\big[c_{k-j} - (k-j+1)(k-j)a_{k-j+1}\big]\,, \quad j=1,\dots,k\,. \tag{2.4}$$

Proof. Applying Lemma 4.2 of [11] to the function $(e_1 - xe_0)^k$ we get

$$\begin{aligned} t(1-t)\frac{d^2}{dt^2}U_n(e_1-xe_0)^k(t) &= U_n\big[(e_1-e_2)k(k-1)(e_1-xe_0)^{k-2}\big](t) \\ &= k(k-1)U_n\big[(1-2x)(e_1-xe_0)^{k-1} \\ &\quad + x(1-x)(e_1-xe_0)^{k-2} - (e_1-xe_0)^k\big](t)\,. \end{aligned}$$

This yields (2.1). Now recall the recurrence formula

$$U_ne_{p+1}(t) = \frac{(n-p)t+2p}{n+p}U_ne_p(t) - \frac{p(p-1)(1-t)}{(n+p)(n+p-1)}U_ne_{p-1}(t)\,, \tag{2.5}$$

established in [7]; using it, we derive (2.3) Finally, (2.4) is a consequence of (2.1). □

Consider the moments $T_{n,k}(x) := U_n(e_1-xe_0)^k(x)$; clearly $T_{n,k}(x) = H_{n,x,k}(x)$. By using Theorem 2.1 it is possible to compute recursively $H_{n,x,0}, H_{n,x,1}, H_{n,x,2},\dots$ and then $T_{n,0}, T_{n,1}, T_{n,2},\dots$. In particular,

$$\begin{aligned} &T_{n,0}(x) = 1\,, \quad T_{n,1}(x) = 0\,, \quad T_{n,2}(x) = \frac{2x(1-x)}{n+1}\,, \\ &T_{n,3}(x) = \frac{6x(1-x)(1-2x)}{(n+1)(n+2)}\,, \quad T_{n,4}(x) = \frac{12(n-7)x^2(1-x)^2+24x(1-x)}{(n+1)(n+2)(n+3)}\,. \end{aligned} \tag{2.6}$$

These moments have been determined also in [7].

Now suppose that the function $H_{n,x,k}(t)$ has been determined for some given n and k; let us represent it as

$$H_{n,x,k}(t) = \sum_{i=0}^{k} b_i x^i\,, \quad \text{for suitable } b_i\,.$$

Then $\sum_{i=0}^k b_ix^i = \sum_{i=0}^k (-1)^i\binom{k}{i}U_ne_{k-i}(t)x^i$. It follows that

$$U_ne_j(t) = (-1)^{k-j}\binom{k}{j}^{-1} b_{k-j}\,, \quad j=0,\dots,k\,. \tag{2.7}$$

In conclusion, after determining the function $H_{n,x,k}(t)$ we get the moment of order k of U_n and all the images $U_ne_j, j=0,\dots,k$.

Theorem 2.2 (Recurrence formula for the moments).

$$(n+k)T_{n,k+1}(x) = x(1-x)T'_{n,k}(x)+k(1-2x)T_{n,k}(x)+2kx(1-x)T_{n,k-1}(x)\,. \quad (2.8)$$

Proof. Let $T^{bern}_{n,k}(x) := (1-x)^n(-x)^k + x^n(1-x)^k$ and

$$T^{durr}_{n,k}(x) := (n-1)\sum_{i=1}^{n-1} p_{ni}(x)\int_0^1 p_{n-2,i-1}(t)(t-x)^k dt\,.$$

Using the method of [3], Prop. II. 3, we find that $T^{durr}_{n,k}$ satisfies (2.8). Then, by a direct computation, we see that $T^{bern}_{n,k}$ satisfies the same equation (2.8). It follows that $T_{n,k} = T^{bern}_{n,k} + T^{durr}_{n,k}$ satisfies (2.8). □

Theorem 2.2 can be also used in order to determine the moments (2.6). We have

Corollary 2.3. *Let* $X := x(1-x)$*. Then, for* $j \geq 1$,

$$T_{n,2j}(x) = \frac{P_{j-1,j}(n,X)}{(n+1)(n+2)\dots(n+2j-1)}\,, \quad (2.9)$$

$$T_{n,2j+1}(x) = \frac{Q_{j-1,j}(n,X)}{(n+1)(n+2)\dots(n+2j)}(1-2x)\,, \quad (2.10)$$

where $P_{j-1,j}$ *and* $Q_{j-1,j}$ *are polynomials in* n *and* X *of degree* $j-1$ *with respect to* n *and* j *with respect to* X.

3. The semigroup approximated by U_n

Let B_n be the classical Bernstein operators. Consider the differential operators $A := \frac{x(1-x)}{2}\frac{d^2}{dx^2}$ and $B := 2A = x(1-x)\frac{d^2}{dx^2}$ with common domain

$$D(A) = D(B) := \left\{u \in C[0,1] \cap C^2(0,1) : \lim_{x\to 0} x(1-x)u''(x) = \lim_{x\to 1} x(1-x)u''(x) = 0\right\}.$$

Theorem 3.1 (see [1]). (i) *For all* $f \in C^2[0,1]$,

$$\lim_{n\to\infty} n(B_n f - f) = Af\,. \quad (3.1)$$

(ii) $(A, D(A))$ *is the infinitesimal generator of a positive contractive semigroup* $(T(t))_{t\geq 0}$ *on* $C[0,1]$*, and*

$$T(t)f = \lim_{n\to\infty} B_n^{[nt]} f\,, \quad f \in C[0,1]\,. \quad (3.2)$$

For the operators U_n, the corresponding result is

Theorem 3.2 (see also [14, p. 63]). (i) *For all* $f \in C^2[0,1]$,

$$\lim_{n\to\infty} n(U_n f - f) = Bf\,. \quad (3.3)$$

(ii) $(B, D(B))$ *is the infinitesimal generator of a positive contractive semigroup* $(S(t))_{t\geq 0}$ *on* $C[0,1]$, *and*

$$S(t)f = \lim_{n\to\infty} U_n^{[nt]} f\,, \quad f \in C[0,1]\,. \tag{3.4}$$

Moreover, $S(t) = T(2t), t \geq 0$.

Proof. The Voronovskaja-type formula (3.3) can be proved – as (3.1) – by using the properties of the moments; see also [5]. The proof of (3.4) is similar to that of (3.2); one applies (3.3) and Trotter's approximation Theorem (see [1]). Since the infinitesimal generators satisfy $B = 2A$, we have $S(t) = T(2t), t \geq 0$. □

Proposition 3.3 (see also [14]). *For all* $t \geq 0$ *and* $m \geq 1$,

$$S(t)U_m = U_m S(t)\,. \tag{3.5}$$

Proof. Let $f \in C[0,1]$. According to [5], we have $U_nU_m = U_mU_n,\ m,n \geq 1$, so that

$$\begin{aligned} U_m\big(S(t)f\big) &= U_m(\lim_{n\to\infty} U_n^{[nt]} f) = \lim_{n\to\infty} U_m(U_n^{[nt]} f) \\ &= \lim_{n\to\infty} U_n^{[nt]}(U_m f) = S(t)(U_m f)\,. \end{aligned}$$

□

The rate of approximation of $T(t)$ by iterates of Bernstein operators was investigated in [6]; here the following estimate can be found:

$$\|B_n^m f - T(t)f\| \leq \frac{1}{24}\left(3\left|\frac{m}{n} - t\right| + \frac{13}{2n-1}\right)\|f^{(2)}\| + \frac{5}{32(2n-1)}\|f^{(4)}\|\,, \tag{3.6}$$

for all $m, n \geq 1, t \geq 0$, and $f \in C^4[0,1]$. Estimates involving functions in $C^3[0,1]$ can be found in [10].

By using the approach of [6], D. Kacsó [7] obtained

$$\|U_n^m f - S(t)f\| \leq \frac{1}{4}\left(\left|\frac{m}{n} - t\right| + \frac{10}{2n-1}\right)\|f^{(2)}\| + \frac{5}{8(2n-1)}\|f^{(4)}\|\,. \tag{3.7}$$

The next theorem contains an estimate comparable with (3.7); the proof is based on Proposition 3.3.

Theorem 3.4. *For all* $m, n \geq 1, t \geq 0,$ *and* $f \in C^4[0,1]$,

$$\|U_n^m f - S(t)f\| \leq \frac{1}{4}\left(\left|\frac{m}{n} - t\right| + \frac{6m}{n^2}\right)\|f^{(2)}\| + \frac{5m}{16n^2}\|f^{(4)}\|\,. \tag{3.8}$$

Proof. Due to (3.5) we have

$$U_n^m - S\left(\frac{m}{n}\right) = \left(U_n^{m-1} + U_n^{m-2}S\left(\frac{1}{n}\right) + \cdots + U_nS\left(\frac{m-2}{n}\right) + S\left(\frac{m-1}{n}\right)\right)\left(U_n - S\left(\frac{1}{n}\right)\right). \tag{3.9}$$

But $\|U_n\| = \|S(t)\| = 1$, so that

$$\left\|U_n^m g - S\left(\frac{m}{n}\right) g\right\| \le m \left\|U_n g - S\left(\frac{1}{n}\right) g\right\|, \quad g \in C[0,1]\,. \tag{3.10}$$

Let $f \in C^4[0,1]$. As in [6] we get

$$\|S(t)f - f - tBf\| \le \frac{t^2}{2}\|B^2 f\| \le \frac{t^2}{32}\left(8\|f^{(2)}\| + 8\|f^{(3)}\| + \|f^{(4)}\|\right).$$

Setting $t = 1/n$ leads to

$$\left\|S\left(\frac{1}{n}\right) f - f - \frac{1}{n}Bf\right\| \le \frac{1}{32n^2}\left(8\|f^{(2)}\| + 8\|f^{(3)}\| + \|f^{(4)}\|\right). \tag{3.11}$$

By using Lemma 5.1 of [11] we obtain

$$\max_{x\in[0,1]}\left|U_n f(x) - f(x) - \frac{1}{n}Bf(x)\right| \le \frac{1}{8n^2}\max_{x\in[0,1]}\left|x(1-x)f^{(4)}(x) + 2(1-2x)f^{(3)}(x) - 2f^{(2)}(x)\right|,$$

so that

$$\left\|U_n f - f - \frac{1}{n}Bf\right\| \le \frac{1}{32n^2}\left(8\|f^{(2)}\| + 8\|f^{(3)}\| + \|f^{(4)}\|\right). \tag{3.12}$$

From (3.11) and (3.12) we infer

$$\left\|U_n f - S\left(\frac{1}{n}\right) f\right\| \le \frac{1}{16n^2}\left(8\|f^{(2)}\| + 8\|f^{(3)}\| + \|f^{(4)}\|\right). \tag{3.13}$$

Now (3.10) and (3.13) yield

$$\left\|U_n^m f - S\left(\frac{m}{n}\right) f\right\| \le \frac{m}{16n^2}\left(8\|f^{(2)}\| + 8\|f^{(3)}\| + \|f^{(4)}\|\right). \tag{3.14}$$

Let us remark that $\|S(\frac{m}{n})f - S(t)f\| \le |\frac{m}{n} - t| \cdot \|Bf\| \le |\frac{m}{n} - t| \cdot \frac{1}{4}\|f^{(2)}\|$; combined with (3.14) this gives

$$\begin{aligned}\|U_n^m f - S(t)f\| &\le \left\|U_n^m f - S\left(\frac{m}{n}\right) f\right\| + \left\|S\left(\frac{m}{n}\right) f - S(t)f\right\| \\ &\le \frac{m}{16n^2}\left(8\|f^{(2)}\| + 8\|f^{(3)}\| + \|f^{(4)}\|\right) + \frac{1}{4}\left|\frac{m}{n} - t\right| \cdot \|f^{(2)}\|\,.\end{aligned}$$

By Landau's inequality, $\|f^{(3)}\| \le 2\|f^{(2)}\| + \frac{1}{2}\|f^{(4)}\|$, so that $\|U_n^m f - S(t)f\| \le (\frac{3m}{2n^2} + \frac{1}{4}|\frac{m}{n} - t|)\|f^{(2)}\| + \frac{5m}{16n^2}\|f^{(4)}\|$. □

Remark 3.5. Theorem 3.4 can be used as in [6] in order to obtain estimates for $f \in C[0,1]$ in terms of moduli of smoothness.

4. The eigenstructure of U_n

Theorem 4.1. *For each $n \geq 1$ the operator $U_n : \pi_n \to \pi_n$ has the eigenvalues*

$$\omega_{nk} := \frac{(n-1)!n!}{(n-k)!(n+k-1)!}\,, \quad k = 0, 1, \ldots n\,. \tag{4.1}$$

The corresponding eigenpolynomials are $p_0(x) = 1$, $p_1(x) = x$, $p_k(x) = x(1-x)\mathcal{J}_{k-2}^{(1,1)}(x)$, $k = 2, \ldots, n$, where $\mathcal{J}_{k-2}^{(1,1)}(x)$ are the Jacobi polynomials on $[0, 1]$.

Proof. The above result was established in [5], with a proof based on the known eigenstructure of the Durrmeyer operators; see also [13]. Here we sketch a different proof, based on Proposition 3.3 and some properties of U_n established in [11].

The eigenvalues of $B : \pi_n \to \pi_n$ are $-(k-1)k, k = 0, 1, \ldots, n$, with corresponding eigenpolynomials p_k. As a consequence, the eigenvalues of $S(t) : \pi_n \to \pi_n$ are $e^{-(k-1)kt}$, with corresponding eigenpolynomials $p_k, k = 0, 1, \ldots, n$.

By virtue of Proposition 3.3 we have

$$S(t)(U_n p_k) = U_n\big(S(t)p_k\big) = e^{-(k-1)kt} U_n p_k\,.$$

We infer that there exists $\omega_{nk} \in \mathbb{R}$ such that

$$U_n p_k = \omega_{nk} p_k\,, \quad k = 0, 1, \ldots, n\,. \tag{4.2}$$

(A second proof of (4.2) reads as follows. Let $\varphi(x) = x(1-x)$. Since $Bp_k = -(k-1)kp_k$, we get $\varphi p_k'' = -(k-1)kp_k$, and so $U_n(\varphi p_k'') + (k-1)kU_n p_k = 0$. According to Lemma 4.3 in [11], this yields $\varphi(U_n p_k)'' + (k-1)kU_n p_k = 0$, i.e., $U_n p_k$ is a polynomial solution of the equation $x(1-x)y''(x) + (k-1)ky(x) = 0$; this entails (4.2).)

Furthermore, it is easy to see that

$$\omega_{n0} = \omega_{n1} = 1\,, \quad n \geq 1\,.$$

For $n \geq 2$ we have, according to Lemma 4.1 in [11],

$$U_{n-1}p_k - U_n p_k = \frac{\varphi}{n(n-1)}(U_n p_k)''\,.$$

By using again Lemma 4.2 of [11] we get

$$n(n-1)(U_{n-1}p_k - U_n p_k) = U_n(\varphi p_k'') = U_n\big(-(k-1)kp_k\big)\,.$$

This leads to $n(n-1)(\omega_{n-1,k} - \omega_{nk}) = -(k-1)k\omega_{nk}$, i.e.,

$$(n-k)(n+k-1)\omega_{nk} = (n-1)n\omega_{n-1,k}\,, \quad k = 0, \ldots, n-1\,. \tag{4.3}$$

Using (4.3) we obtain

$$\omega_{nk} = \frac{(n-1)!n!}{(n-k)!(n+k-1)!} \cdot \frac{(2k-1)!}{(k-1)!k!} \cdot \omega_{kk}\,, \quad k = 1, \ldots, n-1\,. \tag{4.4}$$

On the other hand, $e^{-(k-1)k}p_k = S(1)p_k = \lim_{n\to\infty} U_n^n p_k = (\lim_{n\to\infty} \omega_{nk}^n)p_k$, i.e.,

$$e^{-(k-1)k} = \lim_{n\to\infty} \left(\frac{(n-1)!n!}{(n-k)!(n+k-1)!} \cdot \frac{(2k-1)!}{(k-1)!k!} \cdot \omega_{kk} \right)^n .$$

Since $\lim_{n\to\infty} \left(\frac{(n-1)!n!}{(n-k)!(n+k-1)!} \right)^n = e^{-(k-1)k}$, it follows that $\frac{(2k-1)!}{(k-1)!k!} \cdot \omega_{kk} = 1$, and so (4.4) implies (4.1). □

Theorem 4.2. *For each* $p \in \pi$ *we have*

$$\lim_{n\to\infty} n^3 \left(S\left(\frac{1}{n}\right) - U_n \right) p(x) = \frac{x^2(1-x)^2}{6} p^{(4)}(x) + \frac{x(1-x)(1-2x)}{3} p^{(3)}(x) - \frac{x(1-x)}{3} p^{(2)}(x) \quad (4.5)$$

uniformly on $[0,1]$.

Proof. It is a matter of calculus to show that

$$\lim_{n\to\infty} n^3 \left(e^{-\frac{(k-1)k}{n}} - \omega_{nk} \right) = \frac{(k-1)^2 k^2}{6}, \quad k = 0, 1, \ldots . \quad (4.6)$$

It follows that

$$\lim_{n\to\infty} n^3 \left(S\left(\frac{1}{n}\right) - U_n \right) p_k = \frac{(k-1)^2 k^2}{6} p_k, \quad k = 0, 1, \ldots . \quad (4.7)$$

On the other hand, $Bp_k = -(k-1)kp_k$, so that $B^2 p_k = (k-1)^2 k^2 p_k$; together with (4.7), this gives

$$\lim_{n\to\infty} n^3 \left(S\left(\frac{1}{n}\right) - U_n \right) p_k = \frac{1}{6} B^2 p_k, \quad k = 0, 1, \ldots . \quad (4.8)$$

Since $\{p_k : k = 0, 1, \ldots\}$ is a basis of π, we get

$$\lim_{n\to\infty} n^3 \left(S\left(\frac{1}{n}\right) - U_n \right) p = \frac{1}{6} B^2 p, \quad p \in \pi . \quad (4.9)$$

Recalling that $B = x(1-x)\frac{d^2}{dx^2}$, from (4.9) we deduce (4.5). □

5. The moments of $T(t)$

Consider again the semigroup $(T(t))_{t\geq 0}$ generated by A (see Section 3). Given $t \geq 0, x \in [0,1], k \geq 0$, we define

$$G_{t,x,k}(z) = T(t)(e_1 - xe_0)^k(z), \quad z \in [0,1] . \quad (5.1)$$

Theorem 5.1. (i) $G_{t,x,0}(z) = 1, G_{t,x,1}(z) = z - x$, *and for* $k \geq 2$,

$$z(1-z)G''_{t,x,k} + k(k-1)G_{t,x,k} = k(k-1)\left[(1-2x)G_{t,x,k-1} + x(1-x)G_{t,x,k-2}\right] . \quad (5.2)$$

(ii) *Given* $t \geq 0, x \in [0,1], k \geq 2$, *we have*

$$G_{t,x,k}(z) = a_k z^k + \cdots + a_1 z + a_0$$

where $a_k = e^{-k(k-1)t/2}$ *and, if the right-hand side of* (5.2) *is represented as* $c_{k-1}z^{k-1} + \cdots + c_0$, *then*

$$a_{k-j} = \frac{1}{j(2k-j-1)}\big[c_{k-j} - (k-j+1)(k-j)a_{k-j+1}\big]\,, \quad j = 1, \ldots, k\,. \tag{5.3}$$

Proof. From the general theory of semigroups we know that $AT(t)f = T(t)Af$ for all $f \in D(A)$. Taking $f := (e_1 - xe_0)^k$ we get (5.2); see also the proof of Theorem 2.1. Concerning the value of a_k, see, e.g., [9] and the references therein. (5.3) is a consequence of (5.2). □

Having determined the functions $G_{t,x,k}$, the moments of $T(t)$ are

$$T(t)(e_1 - xe_0)^k(x) = G_{t,x,k}(x)\,, \quad x \in [0,1],\ k \geq 0\,. \tag{5.4}$$

In particular,

$$\begin{aligned} &T(t)(e_1 - xe_0)^2(x) = (1-e^{-t})x(1-x)\,; \\ &T(t)(e_1 - xe_0)^3(x) = \frac{1}{2}(1-e^{-t})^2(2+e^{-t})x(1-x)(1-2x)\,; \\ &T(t)(e_1 - xe_0)^4(x) = -\frac{1}{5}(1-e^{-t})^2x(1-x)\Big[(e^{-4t} + 2e^{-3t} + 3e^{-2t} - 3)\cdot \\ &\qquad\qquad \cdot\big(1 - 5x(1-x)\big) - e^{-t} - 2\Big]\,. \end{aligned} \tag{5.5}$$

Remark 5.2. Writing $G_{t,x,k}(z) = \sum_{i=0}^{k} b_i x^i$ we get, as in Section 2,

$$T(t)e_j(z) = (-1)^{k-j}\binom{k}{j}^{-1} b_{k-j}\,, \quad j = 0, \ldots, k\,. \tag{5.6}$$

Other results concerning the images of the monomials under $T(t)$ can be found in [9] and the references therein.

6. Asymptotic formulas

Let $M_{n,k}(x) := B_n(e_1 - xe_0)^k(x)$ be the moments of the Bernstein operators. As a consequence of the well-known recurrence formula

$$nM_{n,k+1}(x) = XM'_{n,k}(x) + kXM_{n,k-1}(x)\,, \quad \big(X = x(1-x)\big) \tag{6.1}$$

we get for $j \geq 1$:

$$M_{n,2j}(x) = \frac{C_{j-1,j}(n,X)}{n^{2j-1}}\,; \quad M_{n,2j+1}(x) = \frac{D_{j-1,j}(n,X)}{n^{2j}}(1-2x) \tag{6.2}$$

where $C_{j-1,j}$ and $D_{j-1,j}$ are polynomials in n and X of degree $j-1$ with respect to n and j with respect to X. In particular,

$$\begin{aligned} &M_{n,0}(x) = 1\,, \quad M_{n,1}(x) = 0\,, \quad M_{n,2}(x) = \frac{X}{n}\,, \\ &M_{n,3}(x) = \frac{X}{n^2}(1-2x)\,, \quad M_{n,4}(x) = \frac{3(n-2)X^2 + X}{n^3}\,. \end{aligned} \tag{6.3}$$

Theorem 6.1. *For all* $p \in \pi$,

$$\lim_{n\to\infty} n^2(U_{2n} - B_n)p(x) = \frac{X}{12}(1-2x)p^{(3)}(x) - \frac{X}{4}p^{(2)}(x)\,, \tag{6.4}$$

uniformly on $[0,1]$.

Proof. Let $p \in \pi_s$ and $x \in [0,1]$. Then

$$p = \sum_{i=0}^{s} \frac{p^{(i)}(x)}{i!}(e_1 - xe_0)^i\,. \tag{6.5}$$

It follows that

$$(U_{2n} - B_n)p(x) = \sum_{i=0}^{s} \frac{p^{(i)}(x)}{i!}\big(T_{2n,i}(x) - M_{n,i}(x)\big)\,. \tag{6.6}$$

According to (2.6) and (6.3),

$$\begin{aligned} &\lim_{n\to\infty} n^2\big(T_{2n,2}(x) - M_{n,2}(x)\big) = -\frac{X}{2} \\ &\lim_{n\to\infty} n^2\big(T_{2n,3}(x) - M_{n,3}(x)\big) = \frac{X}{2}(1-2x) \\ &\lim_{n\to\infty} n^2\big(T_{2n,4}(x) - M_{n,4}(x)\big) = 0 \end{aligned}$$

uniformly on $[0,1]$. From Corollary 2.3 and (6.2) we deduce that $\lim_{n\to\infty} n^2(T_{2n,i}(x) - M_{n,i}(x)) = 0, i \geq 5$. □

Corollary 6.2. *For all* $p \in \pi$,

$$\lim_{n\to\infty} n^2\left(T\left(\frac{1}{n}\right) - B_n\right)p(x) = \frac{X}{12}(1-2x)p^{(3)}(x) - \frac{X}{4}p^{(2)}(x)\,, \tag{6.7}$$

uniformly on $[0,1]$.

Proof. One has

$$\begin{aligned} \lim_{n\to\infty} n^2\left(T\left(\frac{1}{n}\right) - B_n\right)p(x) = &\lim_{n\to\infty} n^2\left(S\left(\frac{1}{2n}\right) - U_{2n}\right)p(x) \\ &+ \lim_{n\to\infty} n^2(U_{2n} - B_n)p(x)\,. \end{aligned}$$

Now (6.7) is a consequence of (6.4) and (4.5). □

By using (6.5) with $p := (e_1 - xe_0)^k$, and (6.2), it can be proved that for $m, n, j \geq 1$,

$$B_n B_m (e_1 - xe_0)^{2j}(x) = \frac{V_{3j-2;j}(n, m; X)}{n^{2j-1} m^{2j-1}}, \tag{6.8}$$

$$B_n B_m (e_1 - xe_0)^{2j+1}(x) = \frac{W_{3j-1;j}(n, m; X)}{n^{2j} m^{2j}}(1 - 2x), \tag{6.9}$$

where $V_{3j-2;j}(n, m; X)$ is a polynomial in $n, m, X$of degree j with respect to X and global degree $3j - 2$ with respect to $n, m; W_{3j-1;j}$ has a similar meaning. In particular,

$$B_n B_m (e_1 - xe_0)^2(x) = \frac{n + m - 1}{nm} X, \tag{6.10}$$

$$B_n B_m (e_1 - xe_0)^3(x) = \frac{n^2 + m^2 + 3nm - 3n - 3m + 2}{n^2 m^2} X(1 - 2x), \tag{6.11}$$

$$B_n B_{n+1} (e_1 - xe_0)^2(x) = \frac{2}{n + 1} X, \tag{6.12}$$

$$B_n B_{n+1} (e_1 - xe_0)^3(x) = \frac{5n - 1}{n(n + 1)^2} X(1 - 2x), \tag{6.13}$$

$$B_n B_{n+1} (e_1 - xe_0)^4(x) = \frac{12(n^3 - 6n^2 + 4n - 1)}{n^2 (n + 1)^3} X^2 + \frac{15n^2 - 9n + 2}{n^2 (n + 1)^3} X. \tag{6.14}$$

Corollary 6.3. *For all* $p \in \pi$,

$$\lim_{n\to\infty} n^2 (U_n - B_n B_{n+1}) p(x) = \frac{X(1 - 2x)}{6} p^{(3)}(x), \tag{6.15}$$

$$\lim_{n\to\infty} n^2 \left(T\left(\frac{2}{n}\right) - B_n B_{n+1} \right) p(x) = \frac{X(1 - 2x)}{6} p^{(3)}(x), \tag{6.16}$$

uniformly on $[0, 1]$.

Proof. (6.15) is a consequence of (6.5), (2.6), (6.12)–(6.14), (2.9), (2.10), (6.8) and (6.9). Since

$$T\left(\frac{2}{n}\right) - B_n B_{n+1} = \left(S\left(\frac{1}{n}\right) - U_n \right) + (U_n - B_n B_{n+1}),$$

(6.16) follows from (6.15) and (4.5). □

We conclude with the following

Conjecture. Let $f \in C^3[0, 1]$. The following are equivalent:

(a) $f^{(2)}(x) \geq 0, \quad (2x - 1) f^{(3)}(x) \geq 0, \quad x \in [0, 1]$;

(b) $B_n B_{n+1} f \geq U_n f \geq B_n f \geq U_{2n} f \geq f, \quad n \geq 1$.

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