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# A Least-squares Minimization Approach to Depth Determination from Magnetic Data

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*Abstract* — We have developed a least-squares minimization approach to depth determination from magnetic data. By defining the anomaly value $T(0)$ at the origin and the anomaly value $T(N)$ at any other distance $(N)$ on the profile, the problem of depth determination from magnetic data has been transformed into finding a solution to a nonlinear equation of the form $f(z) = 0$. Formulas have been derived for a sphere, horizontal cylinder, dike, and for a geologic contact. Procedures are also formulated to estimate the effective magnetization intensity and the effective magnetization inclination. A scheme for analyzing the magnetic data has been formulated for determining the model parameters of the causative sources. The method is applied to synthetic data with and without random errors. Finally, the method is applied to two field examples from Canada and Arizona. In all cases examined, the estimated depths are found to be in good agreement with actual values.

**Key words:** Magnetic interpretation, simple structures, least-squares method.

## *1. Introduction*

The aim of magnetic surveys is to locate rocks or minerals with unusual magnetic properties, which reveal themselves as anomalies in the intensity of the earth's magnetic field. Magnetic anomalies are often used in a qualitative way to assist regional geologic interpretations. However, sometimes an individual magnetic anomaly is found which stands out so clearly that it can be separated from the regional background and the neighboring interferences (Li and Oldenburg, 1998) and from remnant magnetization effects (Roest and Pilkington, 1993) and which is so simple in appearance that it can be considered as caused by a single magnetized body. In this case, quantitative methods of interpretation can be used to determine the parameters of the magnetized body by assuming a model with simple geometry. The model is considered realistic if the form and magnitude of the calculated magnetic effects are closed to the observed residual magnetic anomalies.

Estimation of the depth of a buried structure from the residual magnetic anomaly has drawn considerable attention. An excellent review was given by Hinze (1990). It

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is evident that magnetic data contain measurement errors which are compounded when source depths are estimated.

Most of the geologic structures in mineral and petroleum exploration can be classified into four categories: spheres, cylinders, dikes, and geologic contacts. These four simple geometric forms are convenient approximations to common geologic structures often encountered in the interpretation of magnetic data. Several graphical methods have been developed for interpreting residual magnetic anomalies due to simple models (GAY, 1963, 1965; PAUL, 1964; STANLEY, 1977; ATCHUTA RAO and RAM BABU, 1980; PRAKASA RAO *et al.*, 1986; PRAKASA RAO and SUBRAHMANYAN, 1988). However, the drawback with these techniques is that they are highly subjective and therefore, can lead to major errors. Also, none of the above articles discuss in sufficient detail the precision and validity of these models when dealing with noise inherent in measurements.

On the other hand, a very convenient method for finding best-fitting model parameters without involving the interpreter's subjective judgment is the application of least-squares methods because all observations can be taken into consideration. MCGRATH and HOOD (1970) described an essentially trial-and-error method in which each of the model parameters is varied by an arbitrary amount until a set of best-fit parameters is found. RAO *et al.* (1973) described a method of differential correction to improve initially assumed model parameters. Similar works include those of MOO (1965), MCGRATH and HOOD (1973), LINES and TREITEL (1984), and SILVA (1989). However, most of these approaches to nonlinear least-squares inverse problems rely on good initial estimates of the model parameters.

We address the problem confronted in nonlinear least-squares inversion of magnetic data when no good initial guess of the model parameters is available. To make the problem computationally tractable, a simple mathematical model is presented to determine the depth of a buried structure from residual magnetic anomaly. Using previously published formulas for the magnetic anomalies produced by these models, the depth determination problem has been parameterized and transformed into a problem of finding a solution for a nonlinear equation of the form $f(z) = 0$. The solution to this nonlinear equation is obtained by minimizing a function in the least-squares sense. A procedure is developed for semi-automated interpretation of magnetic anomalies attributable to simple geometrical causative sources. The method has been applied to synthetic data with and without random errors, and is tested on two field examples from Canada and Arizona.

## *Theory*

Following GAY (1963), PRAKASA RAO *et al.* (1986), and PRAKASA RAO and SUBRAHMANYAN (1988), the magnetic anomaly produced by most geologic structures with the centre located at $X_i = 0$ can be represented by the following function

$$T(X_i, Z, \theta) = K\frac{\left(aZ^{2r} + bX_i^2\right)(\sin\theta)^m(\cos\theta)^n + cX_iZ^p(\sin\theta)^n(\cos\theta)^m}{(X_i^2 + Z^2)^q},$$
$$i = 1, 2, 3, \ldots, L \ . \tag{1}$$

The geometries are shown in Figure 1. In equation (1), $Z$ is the depth, $K$ is the amplitude coefficient (effective magnetization intensity), $\theta$ is the index parameter (effective magnetization inclination), $X_i$ is the horizontal coordinate position, and $q$ is the shape factor. Values for $a$, $b$, $c$, $m$, $n$, $r$, $p$, and $q$, are given in Table 1.

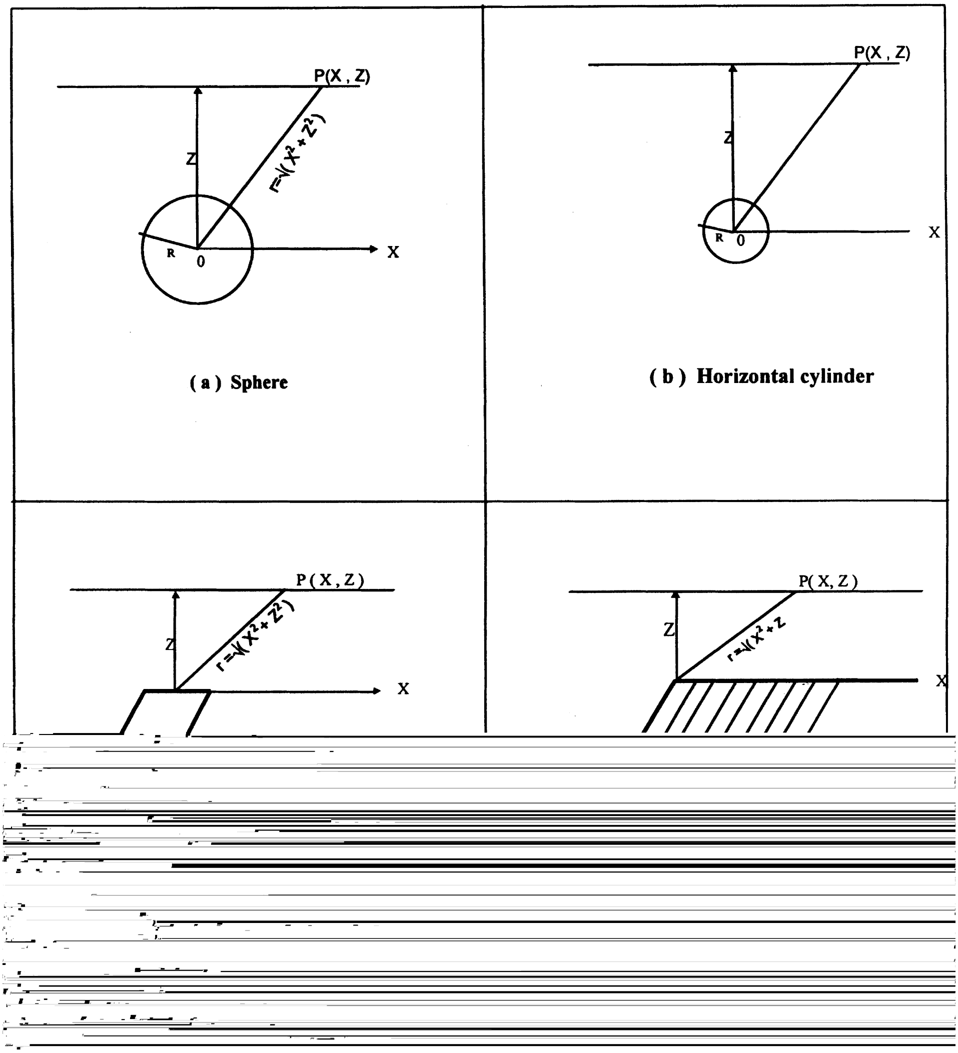


Figure 1
Diagrams for various simple geometrical structures.

Table 1

*Definition of a, b, c, m, n, p, r, and q values shown in equation (1). F.H.D. and S.H.D. are the first and the second horizontal derivatives of the magnetic anomaly, respectively*

| Model | Magnetization | $a$ | $b$ | $c$ | $m$ | $n$ | $p$ | $r$ | $q$ |
|---|---|---|---|---|---|---|---|---|---|
| Sphere | Vertical | 2 | −1 | −3 | 1 | 0 | 1 | 1 | 2.5 |
| Sphere | Horizontal | −1 | 2 | −3 | 0 | 1 | 1 | 1 | 2.5 |
| Horizontal cylinder Dike (F.H.D.) | Total, vertical, horizontal | 1 | −1 | 2 | 0 | 1 | 1 | 1 | 2 |
| Geologic contact (S.H.D.) Dike | Total, vertical, horizontal | 1 | 0 | −1 | 0 | 1 | 0 | 0.5 | 1 |

At the origin ($X_i = 0$), equation (1) gives the following relationship:

$$K = \frac{T(0)Z^{2q-2r}}{a(\sin\theta)^m(\cos\theta)^n}\ , \tag{2}$$

where $T(0)$ is the anomaly value at the origin (Fig. 2).

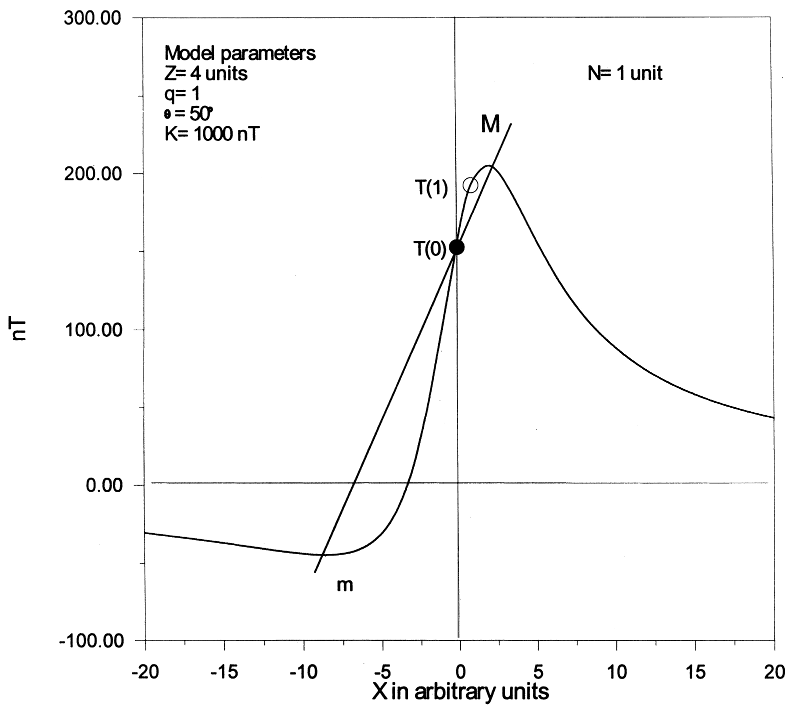


Figure 2

A typical magnetic anomaly profile over a thin dike. The anomaly value at the origin $T(0)$ and the anomaly value $T(N)$ where $N$ is taken to be 1 arbitrary unit in this case, the position of the maximum value ($M$), and the minimum value ($m$) are illustrated.

Using equation (2), equation (1) can be rewritten as

$$T(X_i, Z, \theta) = \frac{T(0)Z^{2q-2r}}{a}\left[\frac{(aZ^{2r} + bX_i^2) + cX_iZ^p(\tan\theta)^{n-m}}{(X_i^2 + Z^2)^q}\right] \ . \tag{3}$$

For all shapes (function of $q$), equation (3) gives the following relationship at $X_i = N$

$$(\tan\theta)^{n-m} = \frac{aT(N)(N^2 + Z^2)^q - T(0)Z^{2q-2r}(aZ^{2r} + bN^2)}{cNZ^pT(0)Z^{2q-2r}} \ , \tag{4}$$

where $T(N)$ is the anomaly value at some arbitrary specified position $X_i = N$ (Fig. 2).

Substituting equation (4) into equation (3), we obtain the following nonlinear equation in $Z$

$$T(X_i,Z) = \frac{NT(0)Z^{2q-2r}(aZ^{2r} + bX_i^2) + aX_iT(N)(N^2 + Z^2)^q - X_iT(0)Z^{2q-2r}(aZ^{2r} + bN^2)}{aN(X_i^2 + Z^2)^q} \ . \tag{5}$$

The unknown depth $Z$ in equation (5) can be obtained by minimizing

$$\psi(Z) = \sum_i^N [L(X_i) - T(X_i, Z)]^2 \ , \tag{6}$$

where $L(X_i)$ denotes the observed magnetic anomaly at $X_i$.

Minimization of $\psi(Z)$ in the least-squares sense, i.e., $(d/dZ)\psi(Z) = 0$ leads to the following equation:

$$f(Z) = \sum_i^N [L(X_i) - T(X_i, Z)]T^*(X_i, Z) = 0 \ , \tag{7}$$

where

$$T^*(X_i, Z) = \left(\frac{d}{dz}\right)T(X_i, Z) \ .$$

Equation (7) can be solved for $Z$ using the standard methods for solving nonlinear equations such as Newton's method, steepest descent method, simple iteration method, and Muller's iterative method (PRESS *et al.*, 1986). Here, it is solved by a simple iteration method. The iteration form of equation (7) is given as

$$Z_f = f(Z_j) \ , \tag{8}$$

where $Z_j$ is the initial depth and $Z_f$ is the revised depth. $Z_f$ will be used as the $Z_j$ for the next iteration. The iteration stops when $|Z_f - Z_j| \le e$, where $e$ is a small predetermined real number close to zero.

The source body depth is determined by solving one nonlinear equation in $Z$. Any initial guess for $Z$ works well because there is only one minimum. Experience with minimization techniques for two or more unknowns shows that they produce good results for synthetic data with or without random noise. In the case of field data, good results may only be obtained when using very good initial estimates for the model parameters. The optimization problem for the depth parameter is highly nonlinear. Increasing the number of parameters to be solved simultaneously also increases the dimensionality and complexity of the error surface, thereby greatly increasing the probability of the optimization stalling in a local minimum. Thus common sense dictates that the nonlinear optimization should be restricted to as few

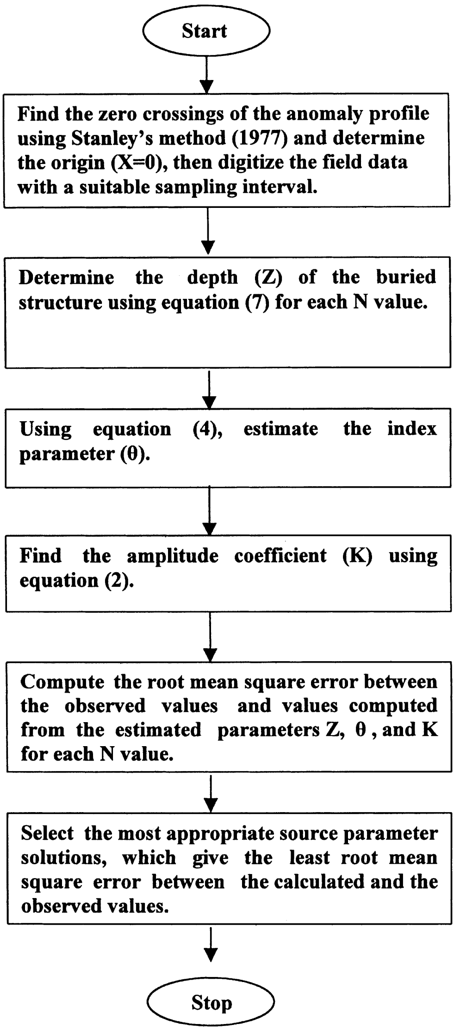


Figure 3
Generalized scheme for semi-automated depth, index parameter, and amplitude coefficient estimation.

parameters as is consistent with obtaining useful results. This is why we propose a solution for only one unknown, $Z$.

Once $Z$ is known, the effective magnetization inclination $\theta$ can be determined from equation (4). Finally, knowing $\theta$, the effective magnetization intensity, $K$, can be determined from equation (2).

Then we measure the goodness of fit between the observed and the computed magnetic data for each $N$ value. The simplest way to compare two magnetic profiles is to compute the root-mean-square (rms) of the differences between the observed and the fitted anomalies. The model parameters which give the least root-mean-square error are the best.

To this point we have assumed knowledge of the axes of the magnetic profile so that $T(0)$ can be found. $T(0)$ is determined using methods described by STANLEY (1977). As illustrated in Figure 2, the line $M - m$ intersects the anomaly profile at $X_i = 0$. The base line of the anomaly profile lies a distance $M - T(0)$ above the minimum.

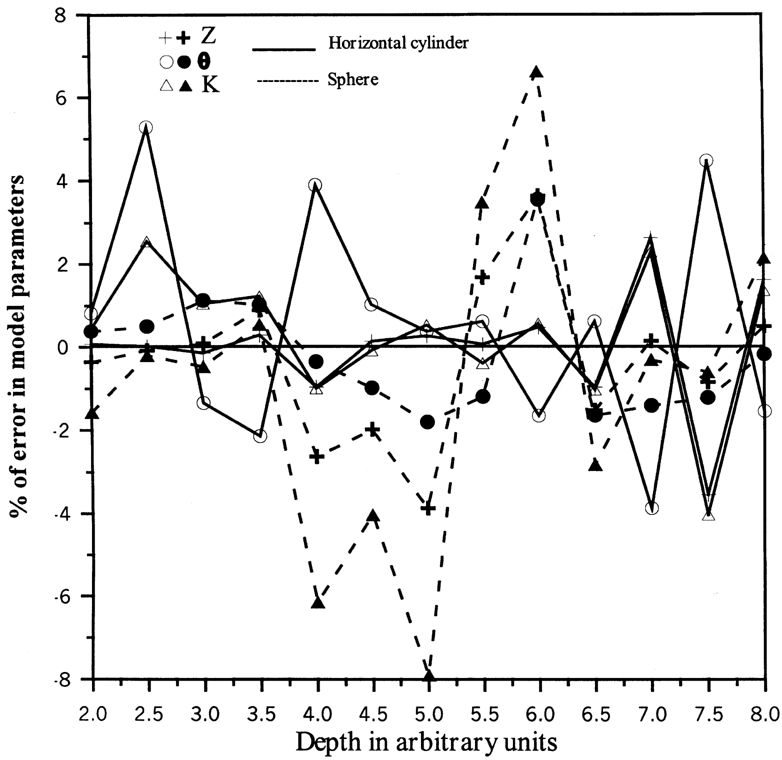


Figure 4
Error response in model parameters estimates. Abscissa: model depth. Ordinate: percent error in model parameters.

A semi-automated interpretation scheme based on the above equations for analyzing field data is illustrated in Figure 3.

## *Synthetic Examples*

### *Effect of Random Noise*

Synthetic examples of spheres and horizontal cylinders buried at different depths (profile length = 40 units, $K = 100$ units, $\theta = 60$ degrees, sampling interval = 1 unit) were interpreted using the present method [equations (7), (4), and (2)] to determine, respectively, depth, index parameter, and amplitude coefficient. In each case, the starting depth was 5 units. In all cases examined, the exact values of $Z$, $\theta$, and $K$ were obtained. However, in studying the error response of the least-squares method, synthetic examples contaminated with 5% random errors were considered. Following the interpretation scheme, values of the most appropriate model parameters $(Z, \theta, K)$ were computed and percentage of error in model parameters was plotted against the model depth for comparison (Fig. 4).

We verified numerically that the depth obtained is within 3% for horizontal cylinders and 4% for spheres. The index parameter obtained is within 5.5% whereas the amplitude coefficient is within 8% (Fig. 3). Good results are obtained by using the present algorithm, particularly for depth estimation, which is a primary concern in magnetic prospecting and other geophysical work.

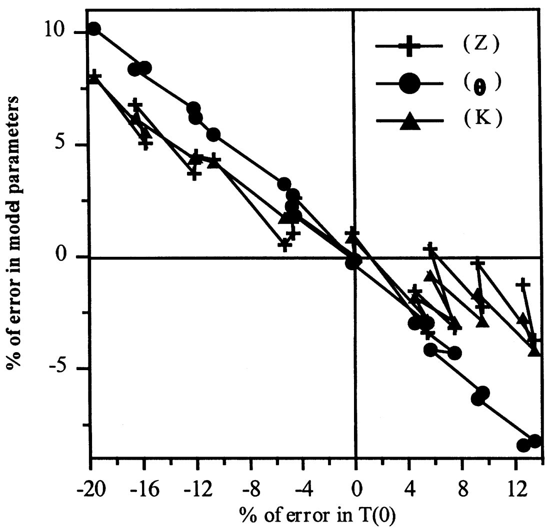


Figure 5
Error response in model parameters estimates. Abscissa: percent error in $T(0)$. Ordinate: percent error in model parameters.

*Effect of Using a Wrong Origin*

This procedure begins with selecting the origin using STANLEY's method (1977) and may lead to errors in the solution for model parameters when interpreting real data. We introduced errors of $\pm 0.1$, $\pm 0.2$, $\pm 0.3, \ldots, \pm 1$ units to the horizontal coordinate $(X_i)$ in equation (1) of a thin dike model (profile length $=$ 40 units, $Z = 10$ units, $\theta = 60$ degrees, and $K = 100$ units). The magnetic anomaly is also corrupted with 5% random noise. In each case we estimated $T(0)$ and $T(N)$ from the noisy magnetic anomaly profile thus obtained. In all cases examined, the initial estimate for depth was 5 units. Following the same interpretation method, the results are shown in Figure 5.

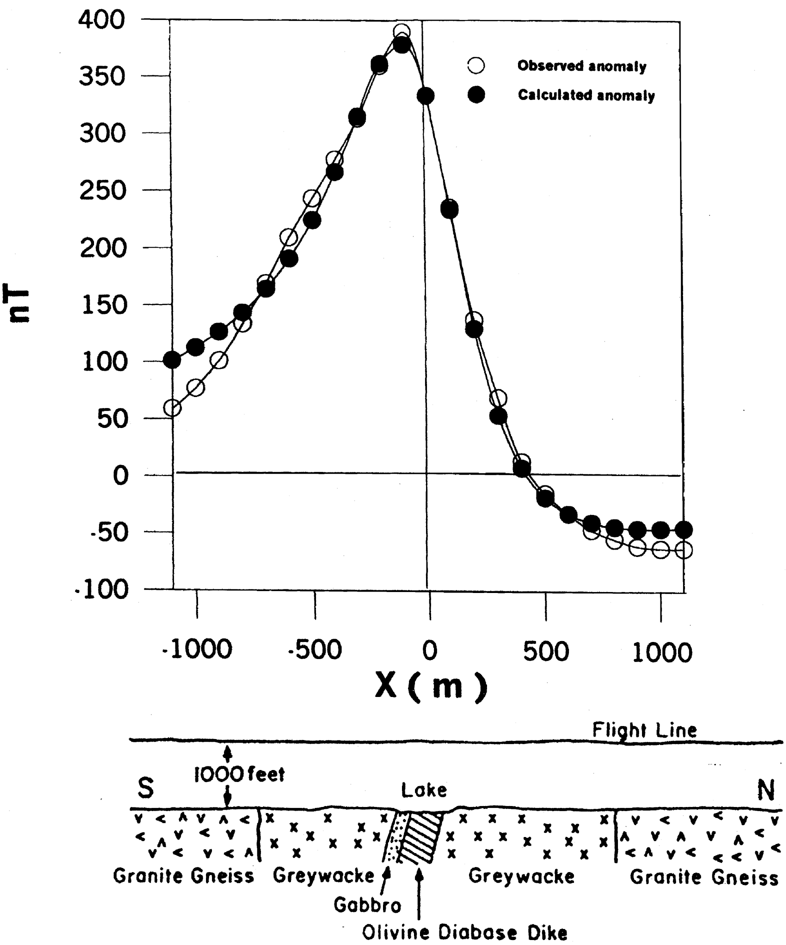


Figure 6
Total magnetic anomaly (above) over an outcropping diabase dike (below), Pishabo Lake, Ontario, Canada (MCGRATH and HOOD, 1970). The base line and zero crossing shown are determined using STANLEY's method (1977).

Figure 5 shows that the percentage of error in model parameters increases with increasing the percentage of error in $T(0)$. In all cases examined, the maximum error in $Z$, $\theta$, and $K$ is $\pm 8\%$, $\pm 10\%$, $\pm 8\%$, respectively, which is still less than the maximum error in $T(0)$ (Fig. 5). This demonstrates that the present method will give reliable model parameters even when the origin is determined approximately using STANLEY's method (1977).

## *Field Examples*

To examine the applicability of the present method, the following two field examples are presented.

### *Pishabo Anomaly*

Figure 6 shows a total magnetic anomaly above an olivine diabase dike, Pishabo Lake, Ontario (MCGRATH and HOOD, 1970). The geological cross section of the dike is shown beneath the anomaly. The depth to the outcropping dike (sensor height) is 304 m (Fig. 6). The anomaly profile was digitized at an interval of 100 m. Equations (7), (4), and (2) were used to determine depth, index parameter, and amplitude coefficient, respectively, using all possible cases on $N$ values. The starting depth used in this field example was 50 m. Then we computed the root-mean-square of the differences between the observed and the fitted anomalies. The best fit model parameters are: $Z = 306$ m, $\theta = -38$ degrees, and $K = 1422\, nT * Z$ (Table 2). MCGRATH and HOOD (1970) applied a computer curve-matching method to the same magnetic data employing a least-squares method and obtained a depth of

Table 2

*Numerical results of the Pishabo field example*

| $N$ (m) | Depth, $Z$ (m) | Index parameter, $\theta$ (deg.) | Amplitude coefficient, $K(nT * Z)$ | Root-mean-square error ($nT$) |
|---|---|---|---|---|
| −200 | 305 | −37.9 | 1411.7 | 16.4 |
| −100 | 290 | −42.1 | 1435.9 | 21.9 |
| 100 | 306 | −37.9 | 1422.8 | 16.5 |
| 200 | 314 | −37.5 | 1443.5 | 16.6 |
| 300 | 321 | −36.7 | 1460.1 | 17.6 |
| 400 | 311 | −37.6 | 1434.5 | 16.5 |
| 500 | 310 | −37.7 | 1429.3 | 16.4 |
| **600** | **306** | **−38.3** | **1422.7** | **16.3** |
| 700 | 299 | −39.8 | 1425.4 | 17.5 |
| 800 | 297 | −41.3 | 1447.6 | 20.1 |
| 900 | 298 | −43.0 | 1491.1 | 24.6 |
| 1000 | 302 | −43.9 | 1533.8 | 27.9 |
| 1100 | 307 | −44.5 | 1570.6 | 30.1 |

301 m. Moreover, Figure 6 shows that there is a lack of "thinness" because the thickness of the dike is only slightly less than the depth. However, because many of the characteristics of the thick dike anomalies are similar to those for thin dike (GAY, 1963), our method can be applied not only to magnetic anomalies due to thin dikes but also to those due to thick dikes to obtain reliable depth estimates.

*Pima Copper Mine Anomaly*

Figure 7 presents a vertical anomaly profile from the Pima copper mine, Arizona (GAY, Fig. 10, p. 198), which represents the anomaly due to a thin dike. Drilling information established that the mineralized zone to be 11 m thick, which is much less than the actual depth to the top of the body (64 m). This profile of 750 m length was digitized at an interval of 25 m. The initial guess for $Z$ used in this field example was 50 m. The model parameters determined by our method are: $Z = 68$ m, $\theta = -52$ degrees, and $K = 1611\, nT * Z$ (Table 3). The depth agrees very well with the depth of 64 m obtained from drilling.

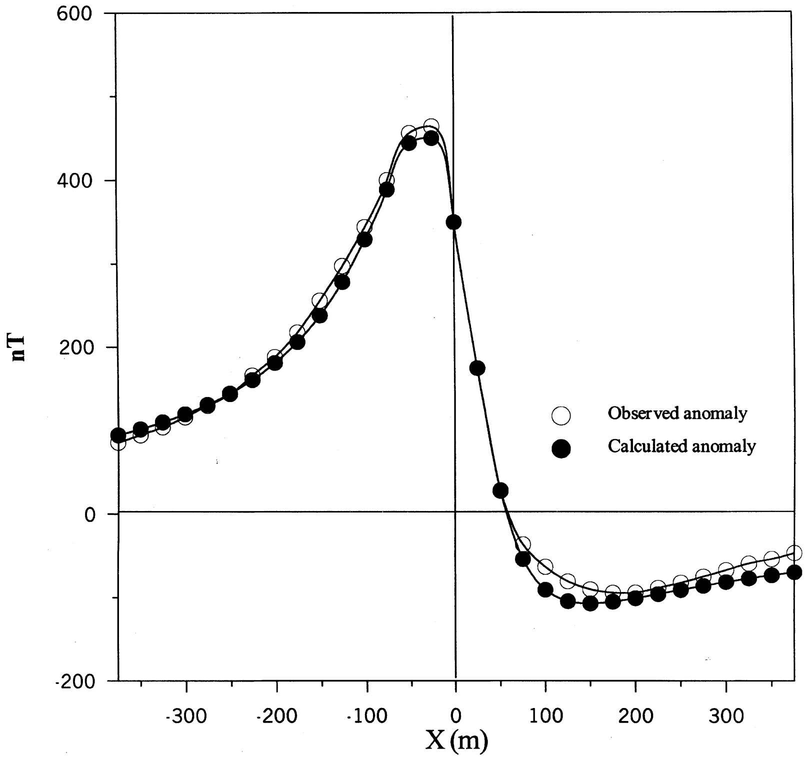


Figure 7
Vertical magnetic anomaly over the Pima copper deposit in Arizona. The base line and the zero crossing shown are determined using STANLEY's method (1977).

Table 3

*Numerical results of the Pima copper mine field example*

| $N$ (m) | Depth, $Z$ (m) | Index parameter, θ (deg.) | Amplitude coefficient, $K(nT * Z)$ | Root mean square error $(nT)$ |
|---|---|---|---|---|
| −50 | 71 | −52.9 | 1726.8 | 16.6 |
| −25 | 64.6 | −53.9 | 1614.7 | 20.4 |
| **25** | **68** | **−51.7** | **1611** | **11.4** |
| 50 | 68.3 | −51.6 | 1616.6 | 13.2 |
| 75 | 71.9 | −50.6 | 1658.6 | 11.8 |
| 100 | 75.2 | −49.3 | 1684.4 | 13.7 |
| 125 | 75.5 | −48.8 | 1674.5 | 14.7 |
| 150 | 74.3 | −48.9 | 1654.9 | 14.1 |
| 175 | 72.9 | −49.5 | 1643.4 | 12.9 |
| 200 | 72.1 | −49.9 | 1641.6 | 12.3 |
| 225 | 72 | −49.5 | 1625.9 | 12.9 |
| 250 | 71.8 | −48.9 | 1602.5 | 14.4 |
| 275 | 71.4 | −48 | 1563.7 | 17.6 |
| 300 | 70.7 | −46.8 | 1514.6 | 22.4 |
| 325 | 70 | −45.2 | 1454.9 | 29.1 |
| 350 | 69 | −44.3 | 1414.9 | 33.1 |
| 375 | 68.3 | −42.3 | 1355.1 | 40.9 |

Finally, these two field examples underscore one of the principle advantages of the least-squares methods: a reliable depth can be obtained in spite of irregularities in the anomaly curves (Figs. 6 and 7), that would more seriously affect methods of depth estimation based on only a few points and distances of the curves.

## *Conclusions*

The problem of determining the depth of a buried structure from the magnetic anomaly has been transformed into the problem of solving a nonlinear equation. The method presented is very simple to execute. The advantages of the present method over previous techniques, which use only a few points, distances, standardized curves, and nomograms are: (1) all observed values can be used, (2) the method is semi-automatic, and (3) the method is not sensitive to errors in the magnetic anomaly. Lastly, the advantage of the proposed method over previous least-squares techniques is that any initial estimate for the depth parameter works well.

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