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Representation and Analysis of the Earthquake Size Distribution: A Historical Review and Some New Approaches

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Abstract—The size distribution of earthquakes has been investigated since the early 20th century. In 1932 WADATI assumed a power-law distribution $n(E) = kE^{-w}$ for earthquake energy E and estimated the w value to be $1.7 \sim 2.1$. Since the introduction of the magnitude-frequency relation by GUTENBERG and RICHTER in 1944 in the form of $\log n(M) = a - bM$, the spatial or temporal variation (or stability) of b value has been a frequently discussed subject in seismicity studies. The $\log n(M)$ versus M plots for some data sets exhibit considerable deviation from a straight line. Many modifications of the G-R relation have been proposed to represent such character. The modified equations include the truncated G-R equation, two-range G-R equation, equations with various additional terms to the original G-R equation. The gamma distribution of seismic moments is equivalent to one of these equations.

In this paper we examine which equation is the most suitable to magnitude data from Japan and the world using AIC. In some cases, the original G-R equation is the most suitable, however in some cases other equations fit far better. The AIC is also a powerful tool to test the significance of the difference in parameter values between two sets of magnitude data under the assumption that the magnitudes are distributed according to a specified equation. Even if there is no significant difference in b value between two data sets (the G-R relation is assumed), we may find a significant difference between the same data sets under the assumption of another relation. To represent a character of the size distribution, there are indexes other than parameters in the magnitude-frequency distribution. The η value is one of such numbers. Although it is certain that these indexes vary among different data sets and are usable to represent a certain feature of seismicity, the usefulness of these indexes in some practical problems such as foreshock discrimination has not yet been established.

Key words: Earthquake statistics, size distribution, b value, η value, AIC.

1. *Historical Review*

1.1. *Early Studies in Japan*

It has been recognized since the early years of seismology that smaller earthquakes are considerably more frequent than larger ones. OMORI (1902) illustrated a table of the frequency distribution of maximum amplitudes recorded by a

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seismometer in Tokyo. If his data were plotted in a double-logarithmic diagram, they fit a straight line indicating a power-law distribution. However, the power-law relation for the distribution of amplitudes

$$n(A) = kA^{-m} \tag{1}$$

was first reported in a paper by ISHIMOTO and IIDA (1939) as a result of the observation with a newly designed seismograph.

ENYA (1908a) applied a lognormal distribution to the frequency of maximum velocities recorded at Tokyo station. ENYA (1908b) also tried to represent the distribution of earthquakes with respect to the radius R of the felt region. Before the introduction of the earthquake magnitude, R (or area S) of the felt region was only a measure of earthquake size routinely reported in the Bulletins of the Central Meteorological Observatory, Tokyo since 1885. However, the formula proposed by ENYA (1908b) for the distribution of R was too complicated to be of practical use. OMORI (1908) also discussed the frequency distribution of earthquakes with respect to the felt area S, although no specific distribution functions were suggested.

1.2. Power-law Distribution of Earthquake Energies

WADATI (1932) published a paper titled "*On the Frequency Distribution of Earthquakes*." This paper received slight attention because of its vague title and the language used. In his paper he assumed that the earthquake energy E has a distribution in the form

$$n(E) = kE^{-w} \tag{2}$$

(k and w are constants) and tried to estimate the w value from the observed frequency distribution of $S-P$ times recorded at the Tokyo station. The distribution of $S-P$ is controlled by the spatial distribution of earthquakes around the station, the attenuation of seismic waves, and the frequency distribution of earthquakes with respect to energy. He obtained $w = 1.7$ and $w = 2.1$ under the assumption that the hypocenters were distributed uniformly on a horizontal line and a horizontal surface, respectively. Only geometrical spreading was assumed for attenuation, however he noted that the w value might become smaller if the effect of absorption was included. It should be noted that Wadati's estimates of w are close to 5/3, now generally accepted for the index of the power-law distribution of seismic energies or moments. $w = 5/3$ corresponds to $b = 1$ in Equation (4), since $w = b/1.5 + 1$, where 1.5 is the coefficient of a well-known formula connecting the magnitude M and the seismic energy (or moment) E

$$\log E = 1.5M + \text{constant}. \tag{3}$$

In the present paper, $\log X$ denotes $\log_{10} X$, while $\ln X$ denotes $\log_e X$. We use the same notation for both moment and energy, because they are proportional.

1.3. Exponential Distribution of Earthquake Magnitudes

In the first paper on the instrumental magnitude scale, RICHTER (1935) noted that the number of shocks falls off very rapidly for the higher magnitudes. GUTENBERG and RICHTER (1941) suggested an exponential distribution for earthquake magnitude. The famous equation

$$\log n(M) = a - bM \tag{4}$$

was used by GUTENBERG and RICHTER (1944, 1949). The coefficient b usually takes a value around 1.0. Since then, this formula has been used by many investigators. The b value has been considered as an important parameter which characterizes the seismicity of a region.

Throughout this paper, $n(M)\,dM$ represents the frequency of earthquakes having magnitude between $M - dM/2$ and $M + dM/2$, and $N(M)$ represents the number of earthquakes with magnitude M and larger. If we use the earthquakes with magnitude M_z and larger, the density function for the G-R formula (4) is written in the form

$$f(X) = B\exp(-BX) \quad (X \geq 0) \tag{5}$$

where $X = M - M_z$ and $B = b \ln 10$.

1.4. Temporal and Spatial Variability and Stability of the b Value

Under the assumption that the magnitudes are distributed in accordance with the G-R formula, the b value is only the parameter which characterizes the distribution. If we have a complete magnitude data for earthquakes with magnitude M_z and larger, $M_1, M_2, \ldots, M_N$, the b value is usually calculated from the equation (UTSU, 1965)

$$B = E[X]^{-1} \quad \text{i.e.,} \quad b = (\log e)N \bigg/ \sum_{i=1}^{N} (M_i - M_z) \tag{6}$$

where $E[\cdot]$ denotes the expectancy and N is the total number of earthquakes. This is the maximum likelihood estimate (MLE) of b (AKI, 1965).

Even if the above assumption is not valid, we can determine the b value from Equation (6). Under the above assumption the significance of the difference in b values between two earthquake groups can be tested by using the F distribution or more easily by using AIC (see Section 3).

The spatial or temporal variation of b value has been one of the frequently discussed topics in seismicity studies, since GUTENBERG and RICHTER (1949) estimated the b values for earthquakes occurring in various regions of the world. Numerous papers were published dealing with the b values or m values in Equation (1) ($m = b + 1$, ASADA *et al.*, 1951). Some tried to relate the spatial variation to tectonics, degree of fracturing, material properties, degree of stress concentration, etc. Some tried to relate the temporal variation to changes in stress level, pore-fluid pressure, fracture growth condition, etc. which might be connected with the occurrence of large earthquakes. Recent papers include OGATA *et al.* (1991), FROHLICH and DAVIS (1993), OGATA and KATSURA (1993), KÁRNÍK and KLÍMA (1993), OKAL and KIRBY (1995), ÖNCEL *et al.* (1996), WIEMER and BENOIT (1996), WIEMER and MCNUTT (1997), MOLCHAN *et al.* (1997), MORI and ABERCROMBIE (1997), WIEMER and WYSS (1997), WYSS *et al.* (1997) for spatial variations, and SMITH (1986), IMOTO (1987), JIN and AKI (1989), OGATA and ABE (1991), IMOTO (1991), HENDERSON *et al.* (1992, 1994), TRIFU and SHUMILA (1996) for temporal variations. It is also known that some volcanic earthquakes have quite unique magnitude distribution (e.g., OKADA *et al.*, 1981; MAIN, 1987).

Uncertainties of the published b values are often quite large. The b values are affected by various factors; properties of the magnitude scale used, magnitude range of adopted data, method of determination, data completeness, etc. Care must be taken to accept the geographic variations (in this connection, see UTSU, 1971; FROHLICH and DAVIS, 1993; KAGAN, 1997).

Some authors are of the opinion that the b value for tectonic earthquakes in general does not differ significantly from a universal value. Some of the reported variations in b value must be real (e.g., earthquakes with normal faulting have larger b values, FROHLICH and DAVIS, 1993), however there may be many cases in which occurrence or non-occurrence of relatively few numbers of large events by chance causes an apparent variation in b value. The spatial stability of the magnitude distribution has been suggested or emphasized by SUZUKI (1959), RIZNICHENKO (1959), ALLEN *et al.* (1965), BLOOM and ERDMAN (1980), KAGAN (1991, 1997), among others. The temporal stability is also mentioned in some papers.

If there is no significant difference in b value between different earthquake groups, this does not always mean that the earthquakes have the same size distribution. We can calculate the b value for any earthquake group by the use of Equation (6), whether the magnitude distribution fits the G-R relation or not. We often find two earthquake groups for which the b values are nearly equal but the patterns of the magnitude distribution are quite different. If we assume a distribution function other than the G-R relation, we may find the significant difference between the two groups. We will discuss this problem in Section 3.

1.5. Modifications of the G-R Relation

The $\log n(M)$ versus M plots for some magnitude data exhibit considerable deviation from a straight line expected from the G-R relation. The deviation is either the convex type (Fig. 1, curve $\eta < 2$) or the concave type (curve $\eta > 2$). Many modified equations have been proposed to represent such data. The problem of selecting the equation best representing a given set of data will be discussed in Section 2.

1.5.1. Convex Type Equations. Most of the modified equations have been designed for the convex type distribution. They have the form

$$\log n(M) = a - bM - \phi(M) \quad (b > 0) \tag{7}$$

where $\phi(M)$ is an increasing function of M. Most of the functions proposed hitherto belong to one of the following four groups.

The first group uses a polynomial $\phi(M) = k(M - c)^n$ ($n = 1, 2, \ldots$).

For $n = 1$, only the case in which $\phi(M)$ is truncated at $M = c$ ($\phi(M) = 0$ for $M < c$) is meaningful. This case corresponds to a two-range G-R relation

$$\log n(M) = a_1 - b_1 M \quad (M \leq c), \tag{8a}$$

$$\log n(M) = a_2 - b_2 M \quad (M \geq c). \tag{8b}$$

A condition $a_1 - b_1 c = a_2 - b_2 c$ is required for the continuity of $n(M)$ at $M = c$. Such two-range expression was used by GUTENBERG (1956), PACHEKO and SYKES (1992), OKAL and ROMANOWICZ (1994), SORNETTE *et al.* (1996), among others. TRIEP and SYKES (1997) used the two-range $\log N(M)$ versus M relation, which is different from Equations (8a,b) as $N(M)$ represents the cumulative number. This relation looks somewhat strange because it has a discontinuity of the gradient of the cumulative curve.

The truncated G-R equation

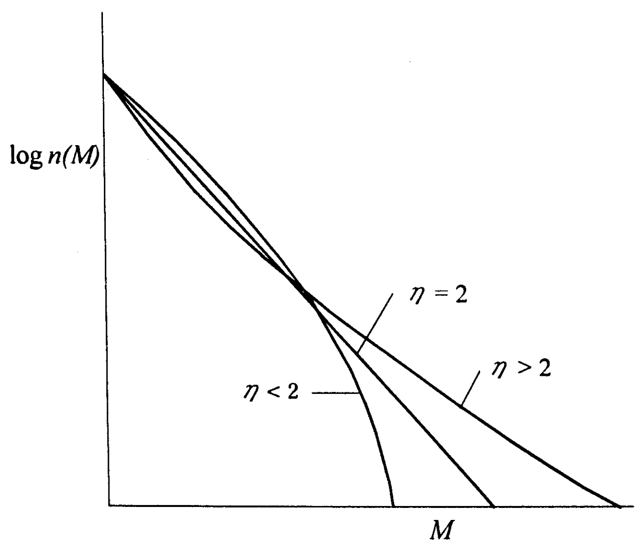


Figure 1
Schematic diagram of the magnitude distribution of earthquakes. Two types of the deviation from the G-R relation ($\eta = 2$), convex type ($\eta < 2$) and concave type ($\eta > 2$) are shown. For the definition of η, see Section 4.2.

$$\log n(M) = a - bM \quad (M \leq c) \tag{9a}$$

$$n(M) = 0 \quad (M > c) \tag{9b}$$

may be a variation of Equations (8a,b), i.e., $b_2 \to \infty$. The truncated G-R equation was used by RIZNICHENKO (1964, 1966), PAGE (1968), CORNELL and VANMARCHE (1969), KAGAN (1969), OKADA (1970), COSENTINO and LUZIO (1976), COSENTINO *et al.* (1977), UTSU (1978), BERRILL and DAVIS (1980), among others. An equation used by LATOUSSAKIS and DRAKOPOULOS (1987), $N(M) = -A_1 + A_2 \exp(-A_3 M)$ is the same as the truncated G-R equation, because Equation (9a) can be transformed to $N(M) = 10^A\{\exp(-BM) - \exp(-Bc)\}$ where $A = a - \log B$ and $B = b \ln 10$.

If $n = 2$, $n(M)$ becomes a normal distribution. This distribution was used by NIAZI (1964), NEUNHÖFER (1969), OLSSON (1986), and SPEIDEL and MATTSON (1993). This corresponds to a lognormal distribution of energy E, which was used by LOMNITZ (1964) and KAGAN (1969). Since no complete data are available below a certain magnitude level M_z (or energy level E_z), the distribution must be truncated at this level. This left-hand truncation makes the density function for $n(M)$ fairly complex and the maximum likelihood estimation of the parameters is not easy.

PURCARU (1975) considered the equation for the cumulative frequency

$$\log N(M) = a - bM - k(c - M)^3. \tag{10}$$

This does not belong to any of the four groups treated here.

The second group uses an exponential function $\phi(M) = k \exp(hM)$ $(k > 0, h > 0)$. SAITO *et al.* (1973) obtained Equation (11) for the frequency distribution of the size E of events generated by a branching model (equivalent to a site percolation model) proposed by OTSUKA (1972).

$$n(E) = kE^{-3/2} \exp(-aE) \tag{11}$$

where k and a are constants. This equation was derived also by VERE-JONES (1976, 1977) and MARUYAMA (1978) through different procedures. If the size is proportional to the energy and the energy is related to the magnitude M by Equation (3), this yields the equation

$$\log n(M) = a - 0.75M - k10^{1.5M}. \tag{12}$$

Since Equation (12) is constrained too tightly, a relaxed form

$$\log n(M) = a - bM - k10^{2bM} \tag{13}$$

has been considered. We shall call (13) the SAITO *et al.* equation.

KAGAN and KNOPOFF (1984) and KAGAN (1991, 1993, 1997) used an equation for seismic moment distribution, which has the form of a gamma distribution

$$n(E) = \kappa E^{-1-\beta} \exp(-E/E_m) \tag{14}$$

where κ, β, and E_m are constants. This is equivalent to a generalized form of Equation (12)

$$\log n(M) = a - bM - k10^{1.5M} \tag{15}$$

where $b = 1.5\beta$. We shall call Equation (15) the generalized SAITO *et al.* equation ($h = 1.5$), since this is a special case of a more general equation

$$\log n(M) = a - bM - k10^{hM}. \tag{16}$$

An equation was obtained by MAIN and BURTON (1984a, 1986) based on the entropy maximization principle. This equation which can be written as

$$n(M) = a \exp(-\lambda_1 M - \lambda_2 E) \tag{17}$$

is the same as the generalized SAITO *et al.* equation ($h = 1.5$), if moment E is converted to magnitude M using Equation (3).

An equation in the form

$$N(M) = \exp\{A - c \exp(BM)\} \tag{18}$$

was proposed by LOMNITZ-ADLER and LOMNITZ (1978, 1979). This can be transformed to

$$\log n(M) = a + bM - c10^{bM} \quad (b > 0). \tag{19}$$

Since the sign of the second term bM is plus, $n(M)$ decreases as M decreases for small M. Due to this unique character, the Lomnitz-Adler and Lomnitz equation fits better than most other equations to incomplete data sets in which small earthquakes are missing. Incomplete data sets should not be used unless special consideration is given such as described by OGATA and KATSURA (1993).

The third group uses a logarithmic function $\phi(M) = -k \log(c - M)$ for $M < c$. $n(M) = 0$ for $M > c$. UTSU (1971, 1978) proposed the equation

$$\log n(M) = a - bM + \log(c - M) \quad (M < c). \tag{20}$$

This is equivalent to the power-law distribution of energy with a logarithmic taper

$$n(E) = \kappa E^{-1-\beta} \log(E_m/E) \quad (E \le E_m) \tag{21}$$

where κ, β, and E_m are constants. The equation of MAKJANIĆ (1972, 1980) can be written as $N(M) = N\{(c - M)/(c - M_z)\}^{k+1}$. This is equivalent to

$$\log n(M) = a + k \log(c - M) \quad (M < c). \tag{22}$$

The equation proposed by PURCARU (1975)

$$\log N(M) = a - bM + k \log(c - M) \quad (M < c) \tag{23}$$

is not the generalization of Equation (20) or (22), since this represents the cumulative frequency.

The fourth group uses a function $\phi(M) = \log[1 - \exp\{-h(c - M)\}]$. $n(M) = 0$ for $M > c$. The equation used by ANDERSON and LUCO (1983) and MAIN and BURTON (1984b), $n(M) = A\{\exp(-BM) - \exp(-Bc)\}$ for $M < c$, can be written as

$$\log n(M) = a - bM + \log\{1 - 10^{-b(c-M)}\} \quad (M < c), \tag{24}$$

i.e., $h = b \ln 10$ in this case. This is a special case ($M_z = m$) of a more general distribution used by CAPUTO (1976)

$$n(M) = \lambda_1 \exp(-BM) \quad (M_z < M \leq m) \tag{25a}$$

$$n(M) = \lambda_2\{\exp(-BM) - \exp(-Bc)\} \quad (m \leq M < c). \tag{25b}$$

Expressions in Caputo's paper are complicated but they are equivalent to (25a,b). Here, we call (24) the Caputo equation.

SEINO *et al.* (1989) used the general form of this group

$$\log n(M) = a - bM - \log[1 - \exp\{-h(c - M)\}] \quad (M < c). \tag{26}$$

It is interesting that this equation degenerates into the G-R relation (4), when $c \to \infty$, into the Utsu Equation (20) when $h \to 0$, and into the truncated G-R Equation (7) when $h \to \infty$. When $h = b \ln 10$, Equation (26) coincides with the Caputo Equation (24).

1.5.2. Concave Type Equations. The equations for the concave type distributions are few. Here we consider an equation

$$n(M) = n_1(M) + n_2(M) \tag{27a}$$

where

$$\log n_1(M) = a_1 - b_1 M \tag{27b}$$

$$\log n_2(M) = a_2 - b_2 M \quad (b_1 > b_2 > 0). \tag{27c}$$

It is readily seen that $n_1(M) = n_2(M)$ at $M = m$, where $m = (a_1 - a_2)/(b_1 - b_2)$. $n_1(M)$ and $n_2(M)$ predominates in (27a) for $M \ll m$ and $M \gg m$, respectively. We call (27a,b,c) the combined G-R equation. If this equation is applied to a data set of convex type, the MLEs of b_1 and b_2 become equal, indicating that the equation degenerates into a single G-R relation.

The two-range G-R relation (8a,b) is concave if $b_1 > b_2$. If the second range is truncated at $M = d(> c)$, i.e., $n(M) = 0$ for $M > d$ in Equation (8b), b_2 may take a negative value. It is possible that Equations (27b) and (27c) are both truncated at different magnitude levels (WARD, 1996).

1.5.3. Other Equations. Many other equations have been proposed in seismological papers, but the MLEs of the parameter values for most of these equations are not easy (though not impossible) to compute, because of the complicated form of the likelihood function. These include the truncated normal distribution (mentioned already), the truncated lognormal distribution (SACUIU and ZORILESCU, 1970; RANALLI, 1975), and various functions (SHLIEN and TOKSÖZ, 1970; MERZ and CORNELL, 1973, 1981; GUARNIERI-BOTTI *et al.*, 1981; CORNELL and WINTERSTEIN, 1998; RUNDLE, 1993, etc.).

2. Parameter Estimation for the Frequency-magnitude Relations: Selection of the Most Suitable One

2.1. General

We assume that we have complete magnitude data for N earthquakes with magnitude M_z and larger, $M_1, M_2, \ldots, M_N$. These are considered as random samples from a population whose magnitude distribution is represented by a density function $f(X)$ $(X = M - M_z)$. The MLEs of the parameters θ_i $(i = 1, 2, \ldots, v)$ in the density function are the values for θ_i which maximize the log-likelihood function

$$\ln L = \sum_{i=1}^{N} \ln f(X_i) \quad (X_i = M_i - M_z). \tag{28}$$

The density function for Equations (4), (8), (9), (12), (13), (15), (16), (19), (20), (22), (24), (26), and (27) are shown below. The MLEs can be computed either by solving simultaneous equations $\partial \ln L/\partial \theta_i = 0$ $(i = 1, 2, \ldots, v)$, or by maximizing $\ln L$ by using some nonlinear optimization procedure.

Once the MLEs are obtained we can compute AIC (Akaike information criterion, AKAIKE, 1974)

$$\mathrm{AIC} = -2 \ln L_m + 2v \tag{29}$$

where L_m is the maximum of L and v is the number of parameters in the density function $f(X)$.

When AIC values are calculated for each of these distributions for a given data set, we can judge which is the most suitable distribution for the data set. The one which provides the smallest AIC is the best, though the difference less than about 2 in AIC is considered insignificant.

2.2. Density Functions

In the following density functions (a) to (n), $B = b \ln 10$ and $C = c - M_z$ unless otherwise noted, where b and c are the parameters in the respective distribution and M_z is the threshold magnitude.

(a) Gutenberg–Richter Equation (4)

$$f(X) = B\exp(-BX) \quad (X \geq 0).$$

(b) Truncated G-R Equation (9)

$$f(X) = B\exp(-BX)/\{1 - \exp(-BC)\} \quad (0 \leq X \leq C).$$

(c) Utsu Equation (20)

$$f(X) = \exp(-BX)(C - X)B^2/\{\exp(-BC) + BC - 1\} \quad (0 \leq X \leq C).$$

(d) Makjanić Equation (22)

$$f(X) = (1 - X/C)^{1/k-1}/(kC) \quad (0 \leq X \leq C).$$

(e) Saito *et al.* Equation ($b = 0.75$) (12): $B = 0.75 \ln 10$ in (f).
(f) Saito *et al.* Equation (13)

$$f(X) = 2BC^{-0.5}\exp\{-BX - C\exp(2BX)\}/G(0.5, C) \quad (X \geq 0)$$

where $C = c\,(\ln 10)\exp(2BM_z)$ and $G(\cdot\,,\cdot)$ denotes the incomplete gamma function.
(g) Generalized Saito *et al.* Equation ($h = 1.5$) (15): $H = 1.5 \ln 10$ in (h).
(h) Generalized Saito *et al.* Equation (16)

$$f(X) = HC^{-B/H}\exp\{-BX - C\exp(HX)\}/G(-B/H, C) \quad (X \geq 0)$$

where $C = c\,(\ln 10)\exp(HM_z)$, $H = h \ln 10$.
(i) Caputo Equation (24)

$$f(X) = B\{\exp(-BX) - \exp(-BC)\}/\{1 - (1 + BC)\exp(-BC)\} \quad (0 \leq X \leq C).$$

(j) Seino *et al.* Equation (26)

$$f(X) = \exp(-BX)[1 - \exp\{-h(C - X)\}]/F \quad (0 \leq X \leq C)$$

where $F = \{1 - \exp(-BC)\}/B + \{\exp(-BC) - \exp(-hC)\}/(B - h)$.
(k) Lomnitz-Adler and Lomnitz Equation (19)

$$f(X) = BC\exp(BX)\exp[-C\{\exp(BX) - 1\}] \quad (X \geq 0)$$

where $C = c(\ln 10)\exp(BM_z)$.

(l) Combined G-R Equation (27)

$$f(X) = \lambda B_1\exp(-B_1X) + (1 - \lambda)B_2\exp(-B_2X) \quad (1 > \lambda > 0,\ X \geq 0).$$

(m) Two-range G-R Equation (8)

$$f(X) = \lambda B_1\exp(-B_1X) \quad (0 \leq X \leq X_c)$$

$$f(X) = \mu B_2\exp(-B_2X) \quad (X \geq X_c)$$

where $\lambda = \{1 - (1 - B_1/B_2)\}^{-1}\exp(B_1X_c)$ and $\mu = \lambda(B_1/B_2)\exp(B_2X_c)/\exp(B_1X_c)$.

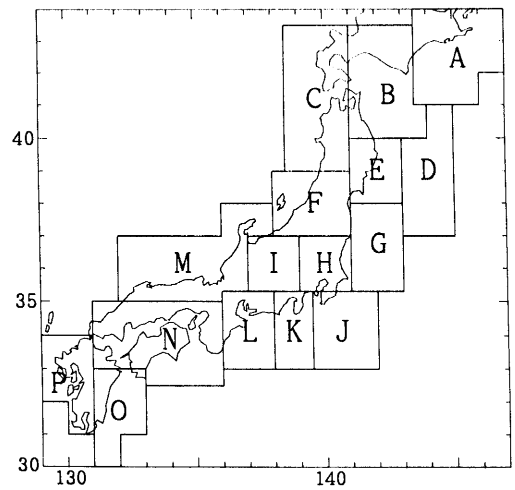


Figure 2
Index map of 16 regions of Japan.

To obtain MLEs of X_c, B_1, and B_2, we assume a certain value for X_c and calculate the B_1 and B_2 values which maximize

$$\ln L = N_1 \ln(\lambda B_1) - B_1 \sum_{X_i \leq X_c} X_i + N_2 \ln(\mu B_2) - B_2 \sum_{X_i > X_c} X_i \tag{30}$$

where N_1 is the number of events with $X \leq X_c$ and N_2 is the number of events with $X > X_c$. We search a value for X_c which maximizes $\ln L$ by changing X_c in a systematic manner. Although the number of the parameters ν is 3, the penalty term of AIC must be larger than 2ν $(=6)$ by α, because the parameter X_c represents the changing point of the slope (for this problem, see OGATA, 1992). The increment of the penalty α is about 6 at most.

(n) Truncated two-range G-R Equation

$$f(X) = \lambda B_1 \exp(-B_1 X) \quad (0 \leq X \leq X_c)$$

$$f(X) = \mu B_2 \exp(-B_2 X) \quad (X_c \leq X \leq C)$$

where $\lambda = [1 - \exp(-B_1 X_c) + [1 - \exp\{(B_2(C - X_c)\}](B_1/B_2)\exp(-B_1 X_c)]^{-1}$ and $\mu = \lambda(B_1/B_2)\exp(B_2 X_c)/\exp(B_1 X_c)$. The same comment on the penalty for AIC as (m) is needed in this case.

2.3. Results from Selected Data Sets

To demonstrate the applicability of our approach, the following data sets (2.3.1–2.3.4) are analyzed.

2.3.1. Shallow Earthquakes in 16 Regions of Japan (1926–1997, $0 \le h \le 100\,km$). We deal with the 16 regions of Japan (A to P shown in Fig. 2). The boundaries of the regions were determined with seismotectonic considerations. They do not cross the aftershock zones of major earthquakes. Magnitudes are M_J given by the Japan Meteorological Agency (JMA). The threshold magnitudes M_z differ among regions as shown in Table 3 (Section 3.1). The largest earthquake in this period was the Tokachi-oki earthquake of 1952 in region A ($M_J = 8.2$, $M_w = 8.0$), or the Sanriku earthquake of 1933 in region D ($M_J = 8.1$, $M_w = 8.4$). There is a small systematic difference between M_J and M_w (or M_s) (UTSU, 1982), so care is needed in the comparison of the results, with the results from the worldwide data using M_w or M_s.

The MLEs of the parameters in the 14 density functions (a) to (n) presented in Section 2.2 and the corresponding AIC values are computed. Here the results for region B are shown in Figure 3. The AIC values for each distribution for each region are shown in Figure 4, where δAIC represents the difference from the smallest AIC for each region. From this figure we notice the following points.

(1) The G-R relation (a) is most suitable for 8 regions.

(2) For the remaining 8 regions except region C, one of the convex equations provides the smallest AIC. For 6 regions, δAIC for the G-R relation is larger than about 2 or more.

(3) For 11 regions δAIC for the truncated G-R relation (b) is less than about 1. No other distributions perform so well. The mean of δAIC for (b) is 1.94, which is the smallest among the 14 equations tested. The second smallest mean δAIC, 2.57, is attained by the Utsu equation (c). Since mean values of δAIC for (d), (f), (g), (i), (k) fall in the range 2.61–2.72, it can be said that no appreciable difference in the performance seems to exist among these 6 two-parameter equations.

(4) One-parameter equations (a) and (e) and three-or-more-parameter equations (h), (j), (l), (m), and (n) have larger mean δAIC.

(5) In computing the parameters, we often encounter the case in which some parameter value increases infinitely or converges to zero during the iteration. This means that the equation degenerates into the G-R relation or another equation with fewer parameters.

2.3.2. Shallow Earthquakes in the World (1904–1980, $M_s \ge 7.0$). Shallow earthquakes in the world (depth less than about 65 km). The magnitudes are M_s given by ABE (1981) with corrections by ABE (1984), ABE and NOGUCHI (1983a,b). M_s for great earthquakes ($M_w \ge 8.5$) is replaced by M_w taken mostly from KANAMORI (1977). The largest one is the M_w 9.5 Chilean earthquake of 1960. Other corrections (e.g., PACHECO and SYKES, 1992) are not considered.

The whole data (W) are divided into two groups (H and L). Group H includes the quakes in the high-latitude zone (south of 38°S and north of 38°N) and group L those in the low-latitude zone (38°S to 38°N). This is to reconfirm the result by MOGI (1979) who demonstrated that the magnitude distribution is quite different between high- and low-latitude zones. Mogi selected the zone boundary at 40°S and 40°N, but here we use 38°S and 38°N to include the 1960 Chilean sequence in the high latitude group. This sequence is located south of 40°S in the catalog used by Mogi, however they are located north of 40°S in the catalog used here.

The AIC values for the 14 equations (a) to (n) for groups W, H, and L are shown in Table 1. The smallest AIC values and the AIC values not different from the smallest value by 0.5 are shown in bold italic letters.

In Figure 5 (left) the magnitude distributions are shown for group W, H, and L. The curves for the G-R equation and equation of the smallest AIC are drawn in the figure.

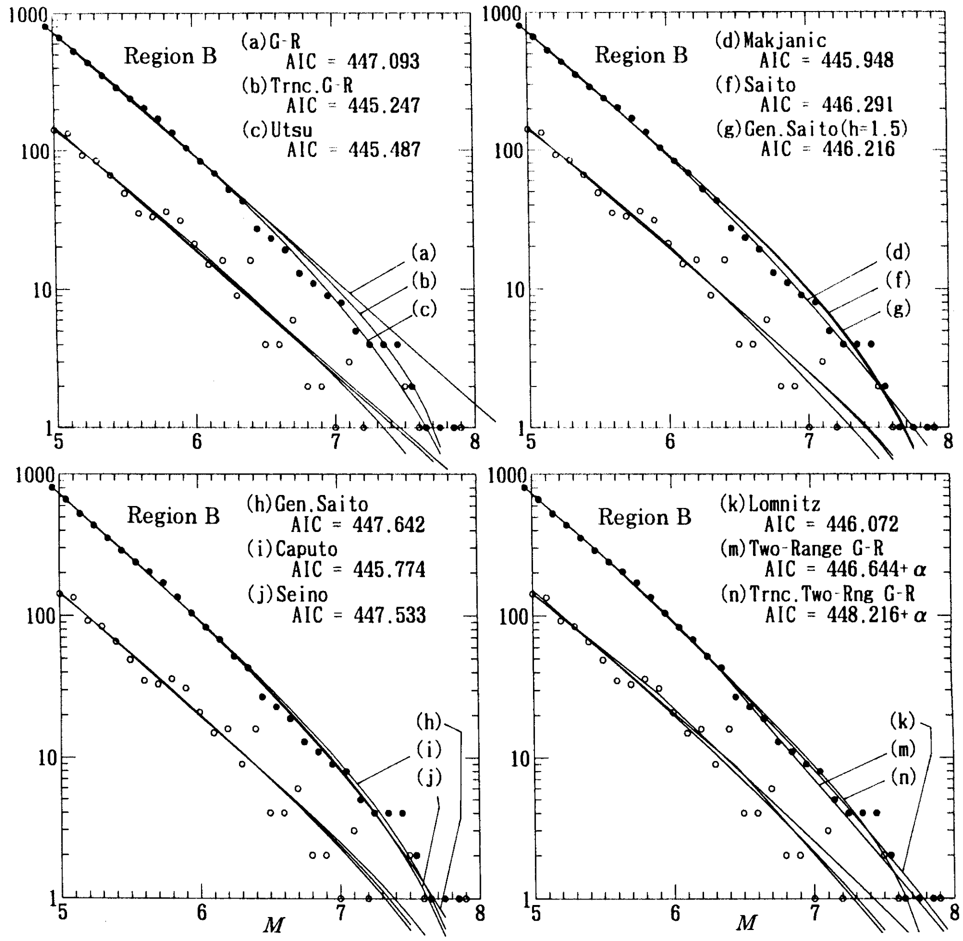


Figure 3
Magnitude-frequency diagrams for shallow earthquakes in region B, 1926–1997. Solid and open circles represent $N(M)$ and $n(M)\,dM$ ($dM = 0.1$), respectively. Three different curves fitted to the data are drawn in each diagram.

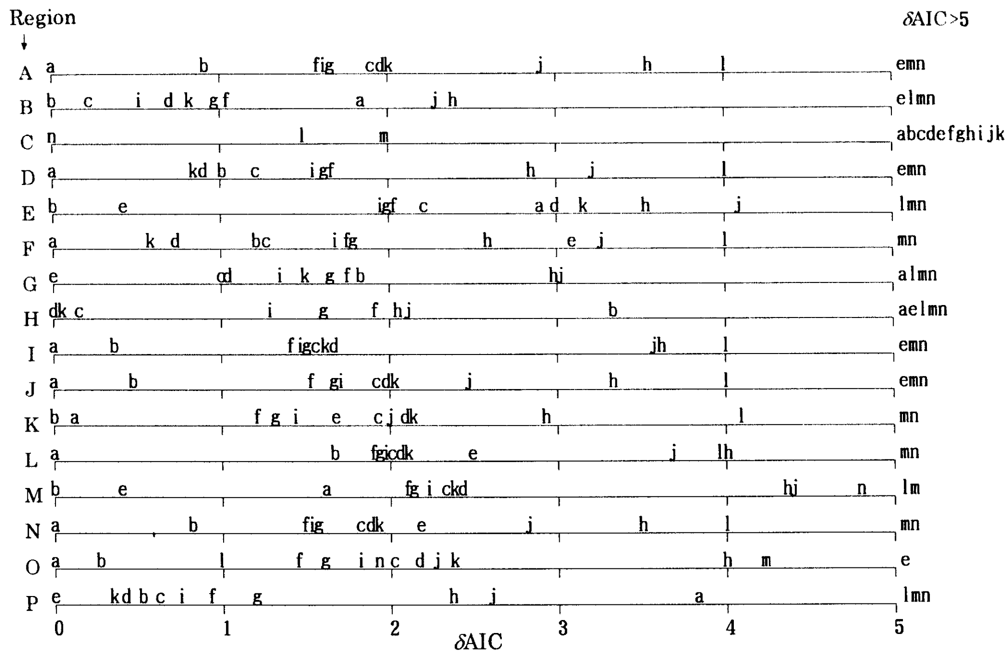


Figure 4
δAIC for 14 distributions (a) to (n) (see Section 2.2) for 16 regions A to P (see Fig. 2). δAIC represents the difference in AIC from the smallest AIC for each region. If two or more AIC values are nearly equal, the letters are slightly displaced so as not to overlap. The letters on the right (δAIC > 5) are arranged in alphabetical order.

Although the two b values, $b = 1.040$ for H, $b = 1.102$ for L, are not significantly different, the shape of the distribution is very different. This point will be discussed in Section 3.

2.3.3. Shallow and Deep Earthquakes in the World (1977–1997, $M_w \geq 5.50$). All magnitudes used are M_w calculated to two decimal places from the seismic moment (M_0 (dyn-cm) in the Harvard University CMT catalog using Equation (3) with the constant equal to 16.1. The largest one is the 1977 Sumba Is. earthquake which has a moment of 3.59×10^{28} dyn-cm ($M_w = 8.31$). The data are divided into high and low-latitude groups in the same manner as before.

Table 2 lists the AIC values. For W and L groups the G-R equation provides a very poor fit. The AIC values for the G-R equation differ by more than 24 from the smallest AIC values provided by the Utsu and Caputo equations. The AIC values for the other equations are larger by more than about 2. For H group the truncated G-R and Utsu equations give the smallest AIC.

Figure 5 (right) shows the magnitude distributions for groups W, H, and L. The curves for the G-R and Utsu equations are drawn.

Table 1

AIC values for shallow earthquakes of $M_s \geq 7.0$ occurring during 1904–1980 in the whole world (W) and high- and low-latitude zones (H and L) for 14 equations (a) to (n) given in Section 2.2. For values with asterisk, increments of penalty of about 6 (at most) should be added to compare with other AIC values. N is the number of earthquakes

Zone (N)	(a) G-R	(b) T. G-R	(c) Utsu	(d) Mak	(e) Sai-1	(f) Sai-2	(g) Sai-3
W (764)	134.40	133.65	132.01	***130.36***	179.63	134.52	134.06
H (232)	***60.90***	61.82	62.91	62.90	77.42	62.62	62.73
L (532)	74.95	***57.72***	***57.35***	58.71	58.74	58.05	57.85

Zone	(h) Sai-4	(i) Capu	(j) Seino	(k) Lom	(1) C. G-R	(m) R. G-R	(n) TR. G-R
W	132.15	133.66	134.48	***130.13***	138.40	129.59*	130.61*
H	64.90	62.80	63.82	63.06	64.64	64.18*	63.98*
L	59.83	***57.45***	59.35	60.45	78.95	60.51*	60.81*

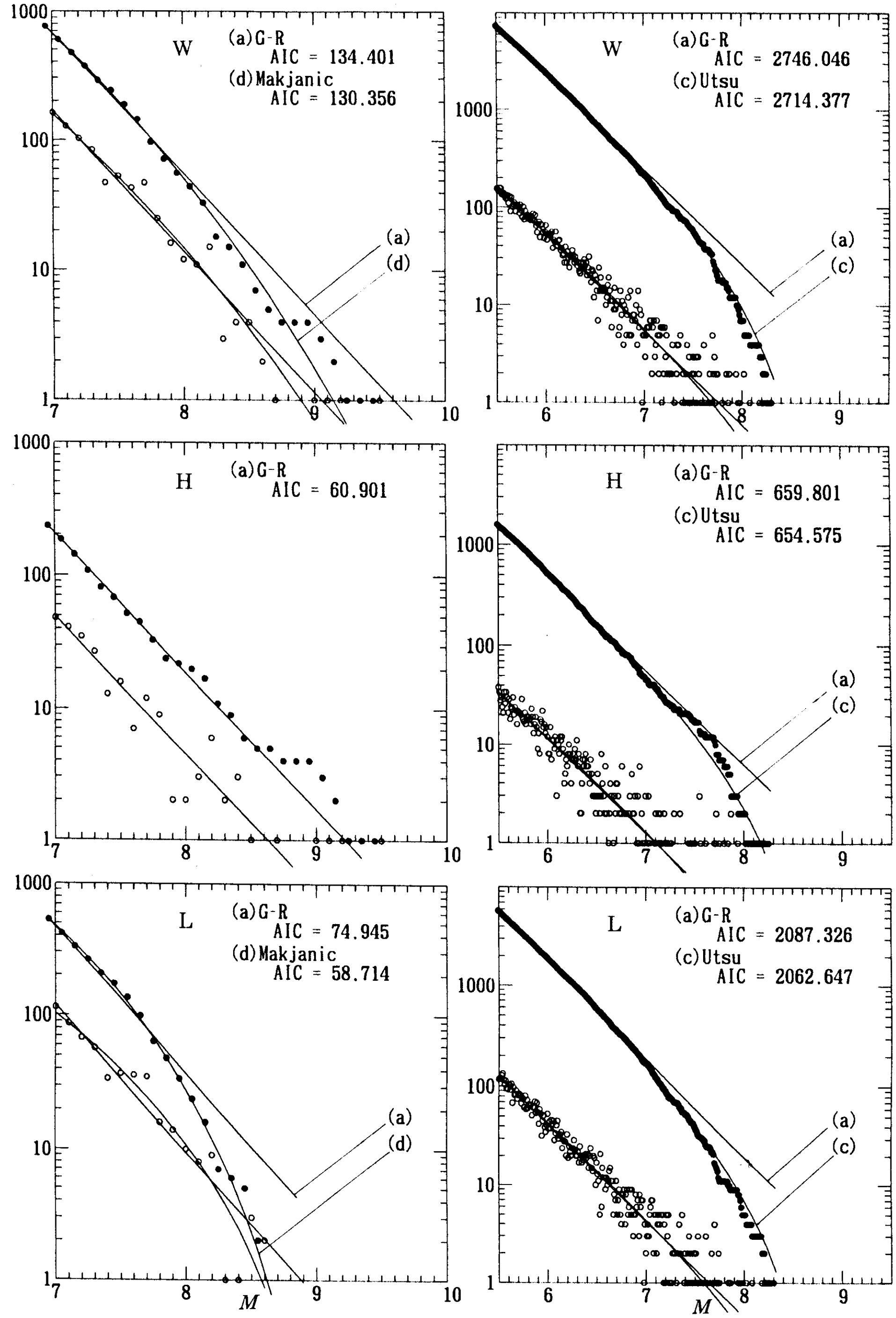
2.3.4. Aftershocks of the 1995 Hyogoken–Nanbu (Kobe) Earthquake ($M_J \geq$ 2.5). This devastating earthquake (January 16, 1995, $M_J = 7.2$) produced relatively weak but remarkably regular aftershock activity. The data are taken from the preliminary catalog of JMA for the first 1,000 days. 4157 shocks of focal depths less than 40 km (most shocks are less than 20 km) and of $M_J \geq 2.0$ occurred in the quadrangular region defined by the four points (34.55°N, 134.65°E), (35.0°N, 135.3°E), (34.75°N, 135.55°E), and (34.3°N, 134.9°E). These shocks are regarded as

Table 2

AIC values for earthquakes of $M_w \geq 5.50$ occurring during 1977–1997 in the whole world (W) and high- and low-latitude zones (H and L) for 14 equations (a) to (n). For values with asterisk, see Table 1

Zone (N)	(a) G-R	(b) T. G-R	(c) Utsu	(d) Mak	(e) Sai-1	(f) Sai-2	(g) Sai-3
W (7340)	2746.05	2720.71	***2714.38***	2719.66	2754.86	2717.19	2716.16
H (1580)	659.80	***654.24***	***654.58***	655.16	700.08	655.45	655.57
L (5760)	2087.33	2068.69	***2062.65***	2067.46	2053.65	2064.46	2063.15

Zone	(h) Sai-4	(i) Capu	(j) Seino	(k) Lom	(1) C. G-R	(m) R. G-R	(n) TR. G-R
W	2717.13	***2714.48***	2715.08	2720.77	2750.05	2724.60*	2714.66*
H	657.18	655.98	656.72	655.17	663.80	657.19*	656.94*
L	2064.12	***2062.39***	2063.19	2068.57	2091.33	2068.30*	2063.62*


W
(a)G-R
AIC = 134.401
(d)Makjanic
AIC = 130.356
(a)
(d)
W
(a)G-R
AIC = 2746.046
(c)Utsu
AIC = 2714.377
(a)
(c)
H
(a)G-R
AIC = 60.901
H
(a)G-R
AIC = 659.801
(c)Utsu
AIC = 654.575
(a)
(c)
L
(a)G-R
AIC = 74.945
(d)Makjanic
AIC = 58.714
(a)
(d)
L
(a)G-R
AIC = 2087.326
(c)Utsu
AIC = 2062.647
(a)
(c)
M
M

aftershocks in this study. The magnitude distribution of the aftershocks of $M_J \geq 2.5$ is studied, dividing the sequence into four periods I (0.1–1 day), II (1–10 days), III (10–100 days), and IV (100–1000 days).

The G-R relation provides the smallest AIC for periods II and III. For period I, the Lomnitz-Adler and Lomnitz equation has the smallest AIC. Perhaps some shocks of M near 2.5 are missing. For $M_J \geq 3.0$ the truncated G-R equation has the smallest AIC. For period IV, the Makjanić equation provides the smallest AIC, but the AIC values for the Utsu, Caputo, Lomnitz-Adler and Lomnitz, and generalized Saito ($h = 1.5$) equations do not differ by more than 0.5.

The b value of the G-R relation for periods I, II, III, and IV are 1.096 ($N = 154$, $M_J \geq 3.0$), 1.104 ($N = 455$, $M_J \geq 2.5$), 1.230 ($N = 356$), and 1.061 ($N = 258$), respectively. A significant difference in b values is not found among the four periods (using the method described in Section 3). The temporal stability of the magnitude-frequency relation in this aftershock sequence is also supported by the nearly constant p value for different threshold magnitudes. The p value is an index in the modified Omori formula introduced by Utsu (1961). The MLEs of p for aftershocks with $M_J \geq 2.0$, $M_J \geq 2.5$, $M_J \geq 3.0$, $M_J \geq 3.5$, $M_J \geq 4.0$ and $M_J \geq 4.5$ are 1.134, 1.160, 1.116, 1.169, 1.301, and 1.197, respectively (the data between 0.1 day and 1000 days from the mainshock are used). Such magnitude stability in aftershock sequences was first pointed out by UTSU (1962).

3. *Difference in the Size Distribution Between Two Groups of Earthquakes*

3.1. Significance Test of the Difference in b Value Between Two Groups

When we obtain fairly different MLEs of b, b_1 and b_2 ($b_1 > b_2$), for two groups of earthquakes, we have the problem of deciding whether this difference is statistically significant or not. If the earthquakes in both groups are random samples from the same population obeying the G-R relation, b_1/b_2 has the F distribution with $2N_1$ and $2N_2$ degrees of freedom (N_1 and N_2 are the number of earthquakes in each group). A significance test using this property was introduced by UTSU (1966).

A similar test can be performed by using AIC (UTSU, 1992). We use two hypotheses, a null hypothesis that the two groups have the same b value (MLE of b for the combined group is b_0) and an alternative one that the b values are

Figure 5

Magnitude-frequency distribution for world earthquakes. Solid and open circles represent $N(M)$ and $n(M)\,dM$, respectively. Top: the whole world. Center: the high-latitude zone, Bottom: the low-latitude zone. Left: Shallow earthquakes of $M_s \geq 7.0$ ($dM = 0.1$), 1904–1980. M_s for earthquakes of $M_w \geq 8.5$ has been replaced by M_w. For the high-latitude zone, the Makjanić equation degenerates into the G-R equation as $c \to \infty$. Right: Earthquakes of all depths, $M_w \geq 5.50$ ($dM = 0.01$), 1977–1997.

Table 3

b values for 16 regions A to P in Japan during 1926–1997. The significance of the difference between every combination of regions is shown by ◎ (highly significant difference), ○ (significant difference), and × (no significant difference)

N	M_z	b	Region (A to P)	
224	5.5	0.957	A	
806	5.0	0.896	× B	◎: $\Delta AIC \geq 5$
127	5.0	1.072	× × C	○: $5 > \Delta AIC \geq 2$
195	5.7	1.067	× ○ × D	×: $\Delta AIC < 2$
185	5.0	0.761	○ ○ ◎ ◎ E	
183	4.5	0.879	× × × × × F	
290	5.5	0.932	× × × × ○ × G	
820	4.5	0.934	× × × × ○ × × H	
185	4.5	1.014	× × × × ◎ × × × I	
158	5.2	1.043	× × × × ◎ × × × × J	
139	5.0	0.919	× × × × × × × × × × K	
78	5.0	0.913	× × × × × × × × × × × L	
311	4.5	0.797	○ × ◎ ◎ × × × ○ ○ ◎ × × M	
461	4.5	0.819	× × ○ ◎ × × × ○ ○ ○ × × × N	
217	4.5	0.952	× × × × ○ × × × × × × × ○ × O	
96	4.5	0.806	× × ○ ○ × × × × × ○ × × × × × P	

different (MLEs of b are b_1 and b_2). AIC for the former and latter hypothesis is denoted by AIC_0 and AIC_{12}, respectively. It is easy to show that

$$AIC_0 = -2(N_1 + N_2)\ln(b_0 \ln 10) + 2(N_1 + N_2) + 2 \tag{31}$$

$$AIC_{12} = AIC_1 + AIC_2 = -2N_1 \ln(b_1 \ln 10) + 2N_1 - 2N_2 \ln(b_2 \ln 10) + 2N_2 + 4. \tag{32}$$

If AIC_{12} is significantly smaller than AIC_0, we can reject the null hypothesis and believe that the two groups have different b values. Usually the difference in AIC is considered significant if ΔAIC $(= AIC_0 - AIC_{12})$ exceeds about 2. If $\Delta AIC > 5$, the difference is highly significant. If the present case, by using Equation (6), ΔAIC can be written as

$$\Delta AIC = -2(N_1 + N_2)\ln(N_1 + N_2) + 2N_1 \ln(N_1 + N_2 b_1/b_2) + 2N_2 \ln(N_1 b_2/b_1 + N_2) - 2. \tag{33}$$

It is noted here that we can test the two data sets with different magnitude threshold M_z, because the G-R relation we assume is perfectly self-similar.

As an example, Table 3 shows the result of the test of the difference in b value among the regions A to P in Japan (Section 2.3.1). It is seen that most of the double and single circles are related to the small b values in regions E and M.

3.2. *Difference in Size Distribution by Using a Formula Other than the G-R Relation*

It is possible that the MLEs of b for two groups are nearly equal (no significant difference based on the test under the assumption of the G-R relation), but the patterns of the distribution are quite different. The distributions for high- and low-latitude zones (Section 2.3.2, Fig. 5 left) provide a good example. In this case, putting $b_1 = 1.1023$, $N_1 = 532$, $b_2 = 1.0398$, $N_2 = 232$ in Equation (33), we obtain $\Delta\mathrm{AIC} = -1.45$ (this is confirmed from $\Delta\mathrm{AIC} = \mathrm{AIC}_0 - \mathrm{AIC}_1 - \mathrm{AIC}_2$ using the value shown in Table 1). This ΔAIC is smaller than 1 (even smaller than 0). Therefore we conclude that the b values for the two zones do not differ significantly.

Since the whole world's data fit the Makjanić equation best, we use this equation as representative of the population. From Table 1 $\mathrm{AIC}_0 = 130.36$, $\mathrm{AIC}_1 = 62.90$, $\mathrm{AIC}_2 = 58.71$, then $\Delta\mathrm{AIC} = 8.75$. Since this is larger than 5, the difference between H and L groups is highly significant. This indicates that the comparison of the b values of the G-R relation only is not always enough to find a variation in the size distribution. ΔAIC values for other distributions except (1) are also larger than 5.

To test the significance of the difference between H and L groups for the data set of 1977–1997 (Section 2.3.3, Fig. 5 right), ΔAIC is calculated for the 14 formulas (a) to (n). We find that ΔAIC is smaller than 2 for all formulas (smaller than 0 in most cases). This indicates that there is no significant difference under the assumption of any of the 14 formulas. The striking difference between H and L groups found for the data set of 1904–1980 is mainly due to the occurrence of several great earthquakes during 1952–1965.

4. *Other Indexes for the Earthquake Size Distribution*

4.1. *Examples of the Indexes*

The character of the size distribution of earthquakes is very often indicated by the b value of the G-R relation, or in some cases by the parameter values in other magnitude-frequency relations. Some investigators suggested the use of indexes other than these parameters. These are the η value (UTSU, 1978), the H value (OUCHI and YOKOTA, 1979), the R index (KAYANO, 1982), the C value (OKUDA *et al.*, 1992) among others.

The η value indicates the degree of departure of the $\log n(M)$ versus M plots from a straight line (the G-R relation) to either concave side or convex side (see Fig. 1). The H, R, and C values provide some measure to indicate the diversity of the earthquake sizes or the deviation from the G-R relation. Here the η value and the R index will be discussed in detail.

4.2. η Value

The η value was introduced by UTSU (1978). It is defined by

$$\eta = E[X^2]/E[X]^2 \tag{34}$$

where $E[\cdot]$ denotes the expectancy and $X = M - M_z$ (M_z is the threshold magnitude). The theoretical value of η for the G-R relation (a) is 2. $\eta < 2$ for concave distributions and $\eta > 2$ for convex distributions. For example, theoretical η for the truncated G-R relation (b) is given by

$$\eta = [2 - BC(BC+2)/\{\exp(BC)-1\}]/[1 - BC/\{\exp(BC)-1\}] \tag{35}$$

and for the Utsu equation (c)

$$\eta = \frac{\{\exp(-BC)(B^2C^2+4BC+6)+2BC-6\}\{\exp(-BC)+BC-1)\}}{\{\exp(-BC)(BC+2)+BC-2\}^2}. \tag{36}$$

These are smaller than 2 for any positive values of B and C. As an example of concave distributions, η for the combined G-R equation (l) is given by

$$\eta = 2\{\lambda B_1^{-2} + (1-\lambda)B_2^{-2}\}/\{\lambda B_1^{-1} + (1-\lambda)B_2^{-1}\}^2 \tag{37}$$

which is larger than 2 if $B_1 \neq B_2$.

It is apparent that earthquake swarms usually have smaller η values compared with mainshock–aftershock sequences. UTSU (1988) suggested that foreshock sequences tend to have smaller η values as compared with the aftershock sequences of the same mainshocks. Examination of more data indicates that this property of foreshock sequences is not so clear, as will be shown in the next section. η values were also used by ZHANG and HUANG (1990) and OKUDA *et al.* (1992).

The formula (6) for computing the b value by UTSU (1965) was obtained by the method of moment, which equates the theoretical and empirical first moments. If we use the nth moment, we obtain another equation for estimating b

$$b_{(n)} = (\log e)\{N(n!)/\Sigma(M_i - M_z)^n\}^{1/n} \tag{38}$$

$b_{(1)}$ is identical to the MLE of b. Since the weight given to the high end range of magnitude increases with increasing n, $b_{(n)} - b_{(1)}$ is positive and increases with n for the convex distribution. We obtain $\eta = 2(b_{(1)}/b_{(2)})^2$ from (38).

4.3. R Index

The R index (relative entropy) used by KAYANO (1982) is defined by

$$R = \left(\sum_{i=1}^{K} p_i \log_2 p_i\right)\Big/\log_2 K, \quad p_i = E_i\Big/\sum_{i=1}^{K} E_i \tag{39}$$

where E_i is the energy of the ith largest earthquake in a sequence. K is an integer larger than 1. If $E_1 = E_2 = \cdots = E_K$, $R = 1$. If $E_1 \gg \Sigma_{i=2}^{K} E_i$, $R \approx 0$. Usually R is large for swarms and small for mainshock–aftershock sequences.

4.4. *b*, *η*, *and R Values for Earthquake Sequences*

Here a result of the study of b, η, and R values for earthquake sequences in Japan is shown. We call the largest shock in a sequence "the mainshock," and the shocks occurring before and after the mainshock, "foreshocks" and "aftershocks," respectively. The method for identifying earthquake sequences is the same as MBC described in OGATA *et al.* (1995).

The earthquakes of $M \geq 3.0$ listed in the JMA catalog for the period from January 1926 through September 1997 are used as the database. There are 432 sequences consisting of 10 or more shocks of $M \geq 3.0$, whose mainshock magnitude is 5.0 or larger. Among these 432 sequences, 48 sequences include 10 or more foreshocks of $M \geq 3.0$, and 326 sequences include 10 or more aftershocks of $M \geq 3.0$. The b, η, and R values have been computed using the largest 10 shocks in each whole sequence (W), in each foreshock sequence (F), and in each aftershock sequence (A). These foreshock sequences are divided into two groups, (F1 and F2) based on the magnitude difference between the mainshock and the largest foreshock. The difference is larger than 0.45 for F1 and smaller than 0.45 for F2.

It is natural that the b and R values are small and the η value is large for W group, because the mainshock usually has a magnitude exceedingly larger than those of the other shocks in the same sequence. There seems to be some difference between F1 and F2 groups, although more data should be collected to draw a reliable conclusion. Many tables similar to Table 4 have been prepared for various combinations of M_m (lowest limit for mainshock magnitude), M_z (lowest limit of foreshock and aftershock magnitudes), δM (magnitude difference for dividing into F1 and F2), and K (the number of the largest earthquakes used in computing b, η, and R values), but no particularly interesting results have been found.

Table 4

Mean b, η, and R values (± standard deviation) for foreshock sequences (F1 and F2), aftershock sequences (A), and the whole sequences (W) which include foreshocks, the mainshock, and aftershocks. $M_m = 5.0$, $M_z = 3.0$, $dM = 0.45$, and $K = 10$

Group	N_G	$\bar{b}$	$\bar{\eta}$	$\bar{R}$
F1	35	0.991 ± 0.354	1.660 ± 0.356	0.646 ± 0.211
F2	13	0.746 ± 0.173	1.833 ± 0.432	0.469 ± 0.234
W	432	0.733 ± 0.413	2.055 ± 0.357	0.255 ± 0.241
A	326	1.024 ± 0.217	1.706 ± 0.498	0.592 ± 0.215

For the purpose of probabilistic earthquake prediction, the earliest K shocks of $M \geq M_z$ forming a sequence must be used. The algorithm MBC is not adequate for this purpose because it uses the information pertaining to the mainshock. This approach requires a rather complex procedure and the final results are not yet achieved.

5. Conclusions

In the history of the study of the size distribution of earthquakes, introduction of the power-law distribution of earthquake energy by WADATI (1932) and the exponential distribution of earthquake magnitude by GUTENBERG and RICHTER (1941, 1944) is of prime importance. These two distributions are equivalent if a linear relationship between the logarithm of energy and the magnitude is accepted. The power-law distribution of amplitude by ISHIMOTO and IIDA (1939) is also equivalent to the above distributions under some natural assumptions.

The $\log n(M)$ versus M plots or the $\log n(E)$ versus $\log E$ plots (E denotes earthquake energy or moment) for some data sets display considerable curvature, especially near the high end of magnitude range. To represent such data, natural modification of the power-law distribution is the truncation at some level E_m (i.e., $n(E) = 0$ for $E > E_m$), or the multiplication of an exponential taper $\exp(-E/E_m)$ or a logarithmic taper $\log(E_m/E)$ (for $E \leq E_m$). In the magnitude domain, these tapers yield the generalized Saito *et al.* equation ($h = 1.5$) and the Utsu equation, respectively.

In addition to the above, various modifications of the G-R relation have been proposed. The question of which is the most suitable relation among these candidates for a given set of data can be answered by using AIC. Application of this method to many data sets indicates that the original G-R relation is the most suitable (has the smallest AIC) for some data sets, although for other data sets one of the modified formulas is found to be the most suitable. In some cases several modified formulas have AIC within 1 unit of the smallest AIC. There is no appreciable difference in performance between these formulas and the most suitable one.

To establish the spatial or temporal variation of b value, the statistical significance of the difference between different groups of earthquakes must be tested. The test can be done easily by the use of AIC. Of course the test is based on the assumption that the magnitude distribution obeys the G-R relation. Under the assumption of distribution functions other than the G-R relation, we can perform a similar significance test. It is possible that two earthquake groups have nearly equal b values, however the magnitude distribution is significantly different if another distribution function is adopted.

Some indexes other than the parameters in the formulas for magnitude-frequency distribution have been proposed. It is certain that these indexes indicate some characteristic feature of the size distribution of earthquakes concerned. However, it is not yet established that these indexes are useful in some practical problems such as detection of precursory change in seismicity, foreshock discrimination, etc.

In this paper, we are concerned only with the statistical problems of representing and analyzing the size distribution of earthquakes. It is beyond the scope of this paper to discuss the mechanism responsible for the size distribution and share the knowledge of the distribution for the studies of the complexity of seismic activity, though numerous papers taking this approach have been published. For reviews of some of these studies, see MAIN (1996), TURCOTTE (1997, Chapters 4 and 16) and KOYAMA (1997, Chapter 8).

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REFERENCES

ABE, K. (1981), *Magnitudes of Large Shallow Earthquakes from 1904 to 1980*, Phys. Earth Planet. Inter. *27*, 72–92.

ABE, K. (1984), *Compliments to "Magnitudes of Large Shallow Earthquakes from 1904 to 1980*," Phys. Earth Planet. Inter. *34*, 17–23.

ABE, K., and NOGUCHI, S. (1983a), *Determination of Magnitude for Large Shallow Earthquakes 1898–1917*, Phys. Earth Planet. Inter. *32*, 45–59.

ABE, K., and NOGUCHI, S. (1983b), *Revision of Magnitudes of Large Earthquakes, 1897–1912*, Phys. Earth Planet. Inter. *33*, 1–11.

AKAIKE, H. (1974), *A New Look at the Statistical Model Identification*, IEEE Trans. Automatic Control *19*, 716–723.

AKI, K. (1965), *Maximum Likelihood Estimates of b in the Formula* $\log N = a - bM$ *and its Confidence Limits*, Bull. Earthq. Res. Inst. Univ. Tokyo *43*, 237–239.

ALLEN, C. R., ST. AMAND, P., RICHTER, C. F., and NORDQUIST, J. M. (1965), *Relationship Between Seismicity and Geologic Structure in the Southern California Region*, Bull. Seismol. Soc. Am. *55*, 753–797.

ANDERSON, J. G., and LUCO, J. E. (1983), *Consequences of Slip Constraints on Earthquake Occurrence Relations*, Bull. Seismol. Soc. Am. *76*, 273–290.

ASADA, T., SUZUKI, Z., and TOMODA, Y. (1951), *Notes on the Energy and Frequency of Earthquakes*, Bull. Earthq. Res. Inst. Univ. Tokyo *29*, 289–293.

BERRILL, J. B., and DAVIS, R. D. (1980), *Maximum Entropy and the Magnitude Distribution*, Bull. Seismol. Soc. Am. *70*, 1823–1831.

BLOOM, E. D., and ERDMAN, R. C. (1980), *The Observation of a Universal Shape Regularity in Earthquake Frequency Magnitude Distributions*, Bull. Seismol. Soc. Am. *70*, 349–362.

CAPUTO, M. (1976), *Model and Observed Seismicity Represented in a Two-dimensional Space*, Ann. Geofis. *29*, 277–288.

CORNELL, C. A., and VANMARCHE, E. H. (1969), *The Major Influence on Seismic Risk*, Proc. 4th World Conf. Earthq. Engin. *A1*, 69–83.

CORNELL, C. A., and WINTERSTEIN, S. R. (1988), *Temporal and Magnitude Dependence in Earthquake Recurrence Models*, Bull. Seismol. Soc. Am. *78*, 1522–1537.

COSENTINO, P., and LUZIO, D. (1976), *A Generalization of the Frequency-magnitude Relation in the Hypothesis of a Maximum Regional Magnitude*, Ann. Geofis. *29*, 3–8.

COSENTINO, P., FICARRA, V., and LUZIO, D. (1977), *Truncated Exponential Frequency-magnitude Relationship in Earthquake Statistics*, Bull. Seismol. Soc. Am. *67*, 1615–1623.

ENYA, M. (1908a), *Relation Between the Maximum Velocity of Motion and the Frequency of the Earthquakes Observed in Tokyo*, Rep. Earthq. Invest. Com. *61*, 77–78 (in Japanese).

ENYA, M. (1908b), *Relation Between the Frequency of Large Earthquakes and that of Small Ones*, Rep. Earthq. Invest. Com. *61*, 71–75 (in Japanese).

FROHLICH, C., and DAVIS, S. S. (1993), *Teleseismic b Values; or, Much Ado About 1.0*, J. Geophys. Res. *98*, 631–644.

GUARNIERI-BOTTI, L., PASQUALE, V., and ANGHINOLFI, M. (1980), *A New General Frequency-magnitude Relationship*, Pure appl. geophys. *119*, 196–206.

GUTENBERG, B. (1956), *The Energy of Earthquakes*, Quart. J. Geol. Soc. London *112*, 1–14.

GUTENBERG, B., and RICHTER, C. F. (1941), *Seismicity of the Earth*, Geol. Soc. Am. Spec. Pap. *No. 34*, 1–131.

GUTENBERG, B., and RICHTER, C. F. (1944), *Frequency of Earthquakes in California*, Bull. Seismol. Soc. Am. *34*, 185–188.

GUTENBERG, B., and RICHTER, C. F., *Seismicity of the Earth and Associated Phenomena* (Princeton Univ. Press, 1949) 273 pp.

HENDERSON, J., MAIN, I., MEREDITH, P., and SAMMONDS, P. (1992), *The Evolution of Seismicity of Parkfield: Observation, Experiment and a Fracture-mechanical Interpretation*, J. Struct. Geol. *14*, 905–913.

HENDERSON, J., MAIN, I. G., PEARCE, R. G., and TAKEYA, M. (1994), *Seismicity in North-eastern Brazil: Fractal Clustering and the Evolution of the b Value*, Geophys. J. Int. *116*, 217–226.

IMOTO, M. (1987), *A Bayesian Method for Estimating Earthquake Magnitude Distribution and Changes in the Distribution with Time and Space in New Zealand*, New Zealand J. Geol. Geophys. *30*, 1063–1116.

IMOTO, M. (1991), *Changes in the Magnitude-frequency b Value Prior to Large ($M \geq 6$) Earthquakes in Japan*, Tectonophys. *193*, 311–325.

ISHIMOTO, M., and IIDA, K. (1939), *Observations of Earthquakes Registered with the Microseismograph Constructed Recently (I)*, Bull. Earthq. Res. Inst. Univ. Tokyo *17*, 443–478 (in Japanese).

JIN, A., and AKI, K. (1989), *Spatial and Temporal Correlation Between Coda Q^{-1} and Seismicity and its Physical Mechanism*, J. Geophys. Res. *94*, 14,041–14,059.

KAGAN, Y. Y. (1969), *A Study of the Energy of the Seismoacoustic Pulses Arising during Bursts in a Coal Bed*, Izv. Phys. Solid Earth, 85–91 (English translation).

KAGAN, Y. Y. (1991), *Seismic Moment Distribution*, Geophys. J. Int. *106*, 123–134.

KAGAN, Y. Y. (1993), *Statistics of Characteristic Earthquakes*, Bull. Seismol. Soc. Am. *83*, 4–24.

KAGAN, Y. Y. (1997), *Seismic Moment-frequency Relation for Shallow Earthquakes: Regional Comparison*, J. Geophys. Res. *102*, 2835–2852.

KAGAN, Y. Y., and KNOPOFF, L. (1984), *A Stochastic Model of Earthquake Occurrence*, Proc. 8th Int. Conf. Earthq. Eng. *1*, 295–302.

KANAMORI, H. (1977), *The Energy Release in Great Earthquakes*, J. Geophys. Res. *82*, 2981–2987.

KÁRNÍK, V., and KLÍMA, K. (1993), *Magnitude-frequency Distribution in the European–Mediterranean Earthquake Regions*, Tectonophys. *220*, 309–323.

KAYANO, I. (1982), *A Characteristic of Earthquake Sequences—An Index for Energy Distribution of Major Earthquakes Belonging to Earthquake Sequences*, Bull. Earthq. Res. Inst. Univ. Tokyo *57*, 317–336 (in Japanese).

KOYAMA, J., *The Complex Faulting Process of Earthquakes* (Kluwer Academic Press, 1997) 208 pp.

LATOUSSAKIS, J., and DRAKOPOULOS, J. (1987), *A Modified Formula for Frequency-magnitude Distribution*, Pure appl. geophys. *25*, 753–764.

LOMNITZ, C. (1964), *Estimation Problems in Earthquake Series*, Tectonophys. *2*, 193–203.

LOMNITZ-ADLER, J., and LOMNITZ, C. (1978), *A New Magnitude-frequency Relation*, Tectonophys. *49*, 237–245.

LOMNITZ-ADLER, J., and LOMNITZ, C. (1979), *A Modified Form of the Gutenberg-Richter Magnitude-frequency Relation*, Bull. Seismol. Soc. Am. *69*, 1209–1214.

MAIN, I. G. (1987), *A Characteristic Earthquake Model of the Seismicity Preceding the Eruption of Mount St. Helens on 18 May 1980*, Phys. Earth Planet. Inter. *49*, 283–293.

MAIN, I. G. (1996), *Statistical Physics, Seismogenesis, and Seismic Hazard*, Rev. Geophys. *34*, 433–462.

MAIN, I. G., and BURTON, P. W. (1984a), *Information Theory and the Earthquake Frequency-magnitude Distribution*, Bull. Seismol. Soc. Am. *74*, 140–1426.

MAIN, I. G., and BURTON, P. W. (1984b), *Physical Links Between Crustal Deformation, Seismic Moment and Seismic Hazard for Regions of Varying Seismicity*, Geophys. J. R. Astr. Soc. *79*, 469–488.

MAIN, I. G., and BURTON, P. W. (1986), *Long-term Earthquake Recurrence Constrained by Tectonic Seismic Moment Release Rates*, Bull. Seismol. Soc. Am. *76*, 297–304.

MAKJANIĆ, B. (1972), *A Contribution to the Statistical Analysis of Zagreb Earthquakes in the Period 1869–1968*, Pure appl. geophys. *95*, 80–88.

MAKJANIĆ, B. (1980), *On the Frequency Distribution of Earthquake Magnitude and Intensity*, Bull. Seismol. Soc. Am. *70*, 2253–2260.

MARUYAMA, T. (1978), *Frequency Distribution of the Sizes of Fractures Generated in the Branching Process—Elementary Analysis*, Bull. Earthq. Res. Inst. Univ. Tokyo *53*, 407–421 (in Japanese).

MERZ, H. A., and CORNELL, C. A. (1973), *Seismic Risk Analysis Based on a Quadratic Magnitude-frequency Law*, Bull. Seismol. Soc. Am. *63*, 1999–2006.

MOGI, K. (1979), *Global Variation of Seismic Activity*, Tectonophys. *57*, T43–T50.

MOLCHAN, G., KRONROD, T., and PANZA, G. P. (1997), *Multiscale Seismicity Model for Seismic Risk*, Bull. Seismol. Soc. Am. *87*, 1220–1229.

MORI, J., and ABERCROMBIE, R. E. (1997), *Depth Dependence of Earthquake Frequency-magnitude Distributions in California: Implications for the Rupture Initiation*, J. Geophys. Res. *102*, 15,081–15,090.

NEUNHÖFER, H. (1969), *Non-linear Energy-frequency Curves in Statistics of Earthquakes*, Pure appl. geophys. *72*, 76–83.

NIAZI, M. (1964), *Seismicity of Northern California and Western Nevada*, Bull. Seismol. Soc. Am. *54*, 845–850.

OGATA, Y. (1992), *Detection of Precursory Relative Quiescence Before Great Earthquakes Through a Statistical Model*, J. Geophys. Res. *97*, 19,845–19,871.

OGATA, Y., and ABE, K. (1991), *Some Statistical Features of the Long-term Variation of the Global and Regional Seismic Activity*, Int. Statist. Rev. *59*, 139–161.

OGATA, Y., IMOTO, M., and KATSURA, K. (1991), *3-D Spatial Variation of b Values of Magnitude-frequency Distribution Beneath the Kanto District, Japan*, Geophys. J. Int. *104*, 135–146.

OGATA, Y., and KATSURA, K. (1993), *Analysis of Temporal and Spatial Heterogeneity of Magnitude Frequency Distribution Inferred from Earthquake Catalogues*, Geophys. J. Int. *113*, 727–738.

OGATA, Y., UTSU, T., and KATSURA, K. (1995), *Statistical Features of Foreshocks in Comparison with Other Earthquake Clusters*, Geophys. J. Int. *121*, 233–254.

OKADA, H., WATANABE, H., YAMASHITA, H., and YOKOYAMA, I. (1981), *Seismological Significance of the 1977–1978 Eruptions and Magma Intrusion Process of Usu Volcano, Hokkaido*, J. Volc. Geotherm. Res. *9*, 311–334.

OKADA, M. (1970), *Magnitude-frequency Relationship of Earthquakes and Estimation of Supremum—A Statistical Study*, Kenkyujiho (J. Meteorol. Res. JMA) *22*, 8–19 (in Japanese).

OKAL, E. A., and ROMANOWICZ, B. A. (1994), *On the Variation of b Value with Earthquake Size*, Phys. Earth Planet. Inter. *87*, 55–76.

OKAL, E. A., and KIRBY, S. H. (1995), *Frequency-moment Distribution of Deep Earthquakes; Implications for the Seismogenic Zone at the Bottom of Slabs*, Phys. Earth Planet. Inter. *92*, 169–187.

OKUDA, S., OUCHI, T., and TERASHIMA, T. (1992), *Deviation of Magnitude Frequency Distribution of Earthquakes from the Gutenberg–Richter Law: Detection of Precursory Anomalies Prior to Large Earthquakes*, Phys. Earth Planet. Inter. *73*, 229–238.

OLSSON, R. (1986), *Are Earthquake Magnitudes Exponentially or Normally Distributed?*, Tectonophys. *82*, 379–382.

OMORI, F. (1902), *Macroseismic Measurements in Tokyo, II and III*, Pub. Earthq. Invest. Com. *11*, 1–95.

OMORI, F. (1908), *List of Stronger Japan Earthquakes, 1902–1907*, Bull. Earthq. Inv. Com. *2*, 58–88.

ÖNCEL, A. O., MAIN, I., ALPTEKIN, Ö., and COWIE, P. (1996), *Spatial Variations of the Fractal Properties of Seismicity in the Anatolian Fault Zones*, Tectonophys. *275*, 189–202.

OTSUKA, M. (1972), *A Chain-Reaction-Type Source Model as a Tool to Interpret the Magnitude-frequency Relation of Earthquakes*, J. Phys. Earth *20*, 35–45.

OUCHI, T., and YOKOTA, T. (1979), *A New Scale of Magnitude-frequency Distribution of Earthquakes*, Zisin (J. Seism. Soc. Japan) Ser. 2, *32*, 415–421 (in Japanese).

PACHECO, J. F., and SYKES, L. R. (1992), *Seismic Moment Catalog of Large Shallow Earthquakes, 1900–1989*, Bull. Seismol. Soc. Am. *82*, 1306–1349.

PAGE, R. (1968), *Aftershocks and Microaftershocks of the Great Alaska Earthquake of 1964*, Bull. Seismol. Soc. Am. *58*, 1131–1168.

PURCARU, G. (1975), *A New Magnitude-frequency Relation for Earthquakes and a Classification of Relation Types*, Geophys. J. R. Astr. Soc. *42*, 61–79.

RANALLI, G. (1975), *A Test of the Lognormal Distribution of Earthquake Magnitude*, Veroff. Zentralinst. Phys. Erde, DAW *31*, 163–180.

RICHTER, C. F. (1935), *An Instrumental Magnitude Scale*, Bull. Seismol. Soc. Am. *25*, 1–32.

RIZNICHENKO, YU. V. (1959), *On Quantitative Determination and Mapping of Seismicity*, Ann. Geofis. *12*, 227–237.

RIZNICHENKO, YU. V. (1964), *The Investigation of Seismic Activity by the Method of Earthquake Summation*, Izv. Phys. Solid Earth 589–593 (English translation).

RIZNICHENKO, YU. V. (1966), *Problems of Earthquake Physics*, Izv. Phys. Solid Earth 73–86 (English translation).

RUNDLE, J. B. (1993), *Magnitude-frequency Relations for Earthquakes Using a Statistical Mechanical Approach*, J. Geophys. Res. *98*, 21,943–21,949.

SACUIU, I., and ZORILESCU, D. (1970), *Statistical Analysis of Seismic Data on Earthquakes in the Area of Vrancea Focus*, Bull. Seismol. Soc. Am. *60*, 1089–1099.

SAITO, M., KIKUCHI, M., and KUDO, K. (1973), *Analytical Solution of Go-game Model of Earthquakes*, Zisin (J. Seism. Soc. Japan), Ser. 2, *26*, 19–25 (in Japanese).

SEINO, M., FUKUI, K., and CHUREI, M. (1989), *Magnitude vs. Frequency Distributions of Earthquakes with Upper Bound Magnitude*, Zisin (J. Seism. Soc. Japan), Ser. 2, *42*, 73–80 (in Japanese).

SHLIEN, S., and TOKSÖZ, M. N. (1970), *Frequency-magnitude Statistics of Earthquake Occurrences*, Earthq. Notes *41* (1), *5–18*.

SMITH, W. D. (1986), *Evidence for Precursory Changes in the Frequency-magnitude b Value*, Geophys. J. Roy. Astr. Soc. *86*, 815–838.

SORNETTE, D., KNOPOFF, L., KAGAN, Y. Y., and VANNESTE, C. (1996), *Rank-ordering Statistics of Extreme Events: Application to the Distribution of Large Earthquakes*, J. Geophys. Res. *101*, 13,883–13,893.

SPEIDEL, D. H., and MATTSON, P. H. (1993), *The Polymodal Frequency-magnitude Relationship of Earthquakes*, Bull. Seismol. Soc. Am. *83*, 1893–1901.

SUZUKI, Z. (1959), *A Statistical Study on the Occurrence of Small Earthquakes*, *IV*, Sci. Rep. Tohoku Univ., Ser. 5, *11*, 10–54.

TRIEP, E. G., and SYKES, L. R. (1997), *Frequency of Occurrence of Moderate to Great Earthquakes in Intracontinental Regions: Implication for Changes in Stress, Earthquake Prediction, and Hazard Assessments*, J. Geophys. Res. *102*, 9923–9948.

TRIFU, C.-I., and SHUMILA, V. I. (1996), *A Method for Multidimensional Analysis of Earthquake Frequency-magnitude Distribution with an Application to the Vrancea Region of Romania*, Tectonophys. *261*, 9–22.

TURCOTTE, D. L., *Fractals and Chaos in Geology and Geophysics*, 2nd Ed. (Cambridge University Press 1997) 398 pp.

UTSU, T. (1961), *A Statistical Study on the Occurrence of Aftershocks*, Geophys. Mag. *30*, 521–605.

UTSU, T. (1962), *On the Nature of Three Alaskan Aftershock Sequences of 1957 and 1958*, Bull. Seismol. Soc. Am. *52*, 279–297.

UTSU, T. (1965), *A Method for Determining the Value of b in a Formula* $\log n = a - bM$ *Showing the Magnitude-frequency Relation for Earthquakes*, Geophys. Bull. Hokkaido Univ. *13*, 99–103 (in Japanese).

UTSU, T. (1966), *A Statistical Significance Test of the Difference in b Value Between Two Earthquake Groups*, J. Phys. Earth *14*, 37–40.

UTSU, T. (1971), *Aftershocks and Earthquake Statistics (III)*, J. Fac. Sci. Hokkaido Univ. Ser. VII *3*, 379–441.

UTSU, T. (1978), *Estimation of Parameter Values in Formulas for Magnitude-frequency Relation of Earthquake Occurrence*, Zisin (J. Seism. Soc. Japan), Ser. 2, *31*, 367–382 (in Japanese).

UTSU, T. (1982), *Relationships Between Earthquake Magnitude Scales*, Bull. Earthq. Res. Inst. Univ. Tokyo *57*, 465–497 (in Japanese).

UTSU, T. (1988), *η Value for Magnitude Distribution and Earthquake Prediction*, Rep. Coord. Comm. Earthq. Pred. *39*, 280–386 (in Japanese).

UTSU, T. (1992), *Introduction to Seismicity*, Surijishingaku (Mathematical Seismology) (VII), Inst. Statis. Math. 139–157 (in Japanese).

VERE-JONES, D. (1976), *A Branching Model for Crack Propagation*, Pure appl. geophys. *114*, 711–725.

VERE-JONES, D. (1977), *Statistical Theories of Crack Propagation*, J. Math. Geol. *9*, 455–481.

WADATI, K. (1932), *On the Frequency Distribution of Earthquakes*, Kishoshushi (J. Meteorol. Soc. Japan), Ser. 2, *10*, 559–568 (in Japanese).

WARD, S. N. (1996), *A Synthetic Seismicity Model for Southern California: Cycles, Probabilities, and Hazard*, J. Geophys. Res. *101*, 22,393–22,418.

WIEMER, S., and BENOIT, J. (1996), *Mapping the b Value Anomaly at 100 km Depth in the Alaska and New Zealand Subduction Zones*, Geophys. Res. Lett. *23*, 1557–1560.

WIEMER, S., and MCNUTT, S. (1997), *Variations in the Frequency-magnitude Distribution with Depth in Two Volcanic Areas: Mount St. Helens, Washington, and Mt. Spurr, Alaska*, Geophys. Res. Lett. *24*, 189–192.

WIEMER, S., and WYSS, M. (1997), *Mapping the Frequency-magnitude Distribution in Asperities: An Improved Technique to Calculate Recurrence Times?*, J. Geophys. Res. *102*, 15,115–15,128.

WYSS, M., SHIMAZAKI, K., and WIEMER, S. (1997), *Mapping Active Magma Chamber by b Values Beneath the Off-Ito Volcano, Japan*, J. Geophys. Res. *102*, 20,413–20,422.

ZHANG, Y.-X., and HUANG, D.-Y. (1990), *On the η Value as a Parameter for Earthquake Prediction*, Earthq. Res. China *4*, 63–70.

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