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A Stochastic Two-node Stress Transfer Model Reproducing Omori's Law

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Abstract—We present an alternative to the epidemic type aftershock sequence (ETAS) model of Ogata (1988). The continuous time two-node network stress release/transfer Markov model is able to reproduce the (modified) Omori law for aftershock frequencies. One node (denoted by A) is loaded by external tectonic forces at a constant rate, with 'events' (main shocks) occurring at random instances with risk given by a function of the 'stress level' at the node. Each event is a random (negative) jump of the stress level, and adds (or removes) a random amount of stress to the second node (B), which experiences 'events' in a similar way, but with another risk function (of the stress level at that node only). When that risk function satisfies certain simple conditions (it may, in particular, be exponential), the frequency of jumps (aftershocks) at node B, in the absence of any new events at node A, follows Omori's law ($\propto (c+t)^{-1}$) for aftershock sequences. When node B is allowed tectonic input, which may be negative, i.e., aseismic slip, the frequency of events takes on a decay form that parallels the constitutive law derived by Dieterich (1994), which fits very well to the modified Omori law. We illustrate the model by fitting it to aftershock data from California post-1973, and from the Valparaiso earthquake of March 3 1985.

Key words: Aftershocks, modified Omori formula, constitutive law, stress release, Markov model.

1. Introduction

Omori (1894a,b) studied the frequency of felt aftershocks following the 1891 Nobi, Central Japan, earthquakes. He showed that the frequency of aftershocks per unit time interval, $n(t)$ at time t is well represented by the *Omori formula* (or hyperbolic law)

$$n(t) = K(t+c)^{-1}, \tag{1}$$

where K and c are constants. Utsu (1957) generalized this to the *modified Omori formula*

$$n(t) = K(t+c)^{-p}, \tag{2}$$

where the constant p usually falls in the range 0.9–1.8 (Utsu *et al.*, 1995). This has been shown by a number of studies (see Utsu *et al.*, 1995, for a review) to be a good

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representation of the temporal variation of aftershock activity. The constant c is small, and strongly influenced by incomplete detection of small aftershocks in the early stage of the sequence.

The parameter of interest is of course the index p. Although no systematic dependence of p has been demonstrated, it differs from sequence to sequence, perhaps as a consequence of the tectonic conditions such as structural heterogeneity, stress and temperature (KISSLINGER, 1996). Depth is probably the most important factor, p increasing sharply (FROHLICH, 1987), although this may be at least partially a consequence of the detection threshold. The value of p is relatively low for intraplate China (ZHAO *et al.*, 1992) but little variation has been observed between inter- and intraplate events in Japan (MATSU'URA, 1993). Three similar events in Japan (1940, 1983, 1993) with similar magnitudes and source regions showed great variability in p. There may be a correlation with the degree of heterogeneity of the fault zone of the main shock.

Stochastic models for temporal shallow seismicity must include the existence of aftershocks. UTSU (1961, 1969) discusses the characterization of individual aftershock sequences by the values p and c, indicating that derived formulae must cater for the variation implicit in (2). This was the motivation for the ETAS model formulated by OGATA (1988), which incorporates (2) together with a constant background activity rate. Our object is to construct an alternative model, which besides producing modified Omori formula decay, also allows for additional behaviors in the underlying rate of main sequence events. Our candidate for the latter is the *Stress Release Model* (SRM) proposed by VERE-JONES (1978). This is a stochastic version of the elastic rebound theory, incorporating a deterministic build-up of stress within a region and its stochastic release through earthquakes. The key variable, or state, is the stress level in a region, which controls the probability of an earthquake occurring. This stress level $X(t)$ can be represented in the form

$$X(t) = X(0) + \rho t - S(t), \tag{3}$$

where $X(0)$ is the initial value, ρ is a constant loading rate from external tectonic forces, and $S(t)$ is the accumulated stress release from earthquakes within the region over the period $(0, t)$, that is, $S(t) = \sum_{t_i<t} S_i$, where t_i and S_i are the origin time and the stress release associated with the i-th earthquake.

The probability intensity of an earthquake occurrence is controlled by a hazard function $\Psi(x)$, with the interpretation that, given $X(t) = x$, the probability of an event occurring in the time interval $(t, t + \Delta)$ is approximately $\Psi(x)\Delta$ for small Δ. Obviously the function Ψ must be nondecreasing. A constant independent of x would result in a random (Poisson) model of occurrences. Using

$$\Psi(x) = \begin{cases} 0 & x \le x_c \\ \infty & x > x_c \end{cases}$$

produces a *time-predictable* model, supposing a fixed crustal strength x_c. An effective compromise (ZHENG and VERE-JONES, 1991, 1994) between these extremes of behavior is the form $\Psi(x) = \exp(\mu + vx)$. It also represents the behavior that might be expected from a region with a locally heterogeneous strength. We can interpret the constant μ (or rather the parameter α that replaces it, see below) as effectively a parameter to be fitted for the unknown initial value of stress, while the constant v is an amalgam of the strength and heterogeneity of the crust in the region.

Statistical analysis is made possible by treating the data in historical earthquake catalogs as a point process in time-stress space with conditional intensity function

$$\lambda(t) = \Psi(X(t)) = \exp[\mu + v(X(0) + \rho t - S(t))] = \exp[\alpha + v(\rho t - S(t))]. \quad (4)$$

Estimates of the parameters can then be found by maximizing the log-likelihood function (see, for example, DALEY and VERE-JONES, 1988). Stochastic process properties have been examined by VERE-JONES (1988), ZHENG (1991) and BOROVKOV and VERE-JONES (2000).

Obviously, stress transfer and interaction cannot be considered in the simple stress release model, motivating a modification of the stress release model as follows. The modeled seismic area is subdivided into n regions. The evolution of stress $X_i(t)$ in the ith region can then be rewritten, generalizing (3), as

$$X_i(t) = X_i(0) + \rho_i t - \sum_{j=1}^{n} \theta_{ij} S^{(j)}(t), \quad (5)$$

where $S^{(j)}(t)$ is the accumulated stress release in region j over the period $(0, t)$, and the coefficient θ_{ij} measures the fixed proportion of stress drop, initiated in region j, which is transferred to region i. Clearly $\theta_{ii} = 1$ for all i. Here, θ_{ij} may be positive or negative, resulting in damping or excitation respectively. This is called a *linked stress release model* (LSRM) (LIU *et al.*, 1998 ; LU *et al.*, 1999). Note that the construction treats space only in the form of regions, which will be of differing size and shape. Hence there is no requirement for symmetry in the transfers, as these are not between points, and hence no distance can be calculated. Instead we fit the transfers statistically.

We shall assume each region to have an exponential risk function $\Psi_i(x_i) = \exp(\mu_i + v_i x_i)$, with differing parameters indicating different tectonic properties by region. In other words, the strength (earthquake triggering condition) and tectonic loading rate can differ in each seismic region. Thus, in a similar manner to (4), we obtain a point process conditional intensity function

$$\lambda_i(t) = \Psi_i(X_i(t)) = \exp\left[\alpha_i + v_i\left(\rho_i t - \sum_{j=1}^{n} \theta_{ij} S^{(j)}(t)\right)\right], \quad (6)$$

for each region i.

In order to fit the model to data we need to estimate the value of stress released during an earthquake. This requires a relationship between the observed quantity, magnitude, and our notional variable, stress. KANAMORI and ANDERSON (1975) show that the magnitude M is proportional to the logarithm of the seismic energy E released during an earthquake according to the relation $M = \frac{2}{3}\log_{10} E + const.$ For simplicity, we assume that the stress drop during an earthquake is

$$S \propto E^{1/2} \tag{7}$$

(Benioff strain), giving the formula

$$S = 10^{0.75(M-M_0)}, \tag{8}$$

where M_0 is the normalized magnitude. The use of the exponent $1/2$ in (7) has been investigated by ZHENG and VERE-JONES (1991), who found that this provided the best fit statistically, although the results were not particularly sensitive to an exponent in the range $(1/3, 1)$. A similar phenomenon for the accelerating moment release model was observed by JAUMÉ and SYKES (1999), who felt that a Benioff strain variable was favored by the use of a restricted magnitude range.

The central tenet of this paper is, using a two region version of the above formulation, to represent main shocks by one region (henceforth *node*), denoted A, and aftershocks by the second node (B). In this case the "regions" will in fact be spatially congruent, with A comprising the largest fault or faults, and B the secondary faults on which corresponding aftershocks will occur. In terms of the linked stress release model the essential distinction is in the nature of the stress loading. For node A, it is tectonic in nature, while for node B it is basically due to the stress release from node A. This later can be thought of as the "secondary redistribution" of MENDOZA and HARTZELL (1988); DIETERICH (1994) similarly supposed aftershocks to represent adjustments on secondary faults to stresses induced by main shock slips. In fact, we will later assume that there can be some tectonic loading at node B as well, in order to produce behavior approximating the modified Omori law. Hence, instead of node B representing a single process, it may be better thought of as the superposition of a large number (cf., KISSLINGER, 1996) of independent statistically identical processes. This enables us to make the simplification that the jumps down in the stress level of node B can be independent of the stress level, presuming that larger events simply load a greater number of secondary faults, in line with the greater extent of the original rupture. KISSLINGER (1996) further notes that there is no theoretical basis for determining the magnitude of an aftershock from that of the main shock, hence we can assign a distribution of stress releases to node B independent of the joint process history. The self-similar/fractal nature of faulting (KNOPOFF, 1996) implies that a large event on the main fault (node A) loads some number of the many smaller faults in the neighborhood of the original slip, which collectively comprise node B. One can then assume that each minor fault has a number of even smaller faults to which it disperses stress, and so on, until the magnitude cutoff in the Omori law is reached. In sequences which include a secondary sequence triggered by a strong event, such as Whittier Narrows and Superstition Hills (KISSLINGER and JONES, 1991) or Landers/Big Bear (JONES, 1994), a third node might be included to represent the intermediate size events.

We will show that the simple scheme outlined above not only displays aftershock sequences, but also that such sequences follow patterns specified by the modified Omori formula. We shall next derive, analytically, decay formulae for the aftershock frequency, and show how the approximation can closely fit the modified Omori law (2). Following that we shall fit the entire point process model to a catalog of aftershocks from California, in order to demonstrate that the model captures most of the dynamics producing aftershocks. Using the aftershock sequence from the Valparaiso earthquake of March 3, 1985, we will then show how a very close estimate of the decay parameter p can be extracted from the fitted SRM parameters.

2. Derivation of the Decay Formula

Let $\{X_1(t), X_2(t)\}$ be a bivariate process of the type (5), $X_1(t)$ representing the stress at node A and $X_2(t)$ standing for the stress level at node B. In accordance with our construction, we set $\theta_{12} = 0$ (no stress transfer from B to A), with $\theta_{ii} = 1$, as usual. Since there is no stress transfer to A from node B, the process $X_1(t)$ is just a simple SRM whose behavior is quite well studied. In particular, in the case of the exponential hazard function Ψ_1, we know the conditions for stability of the process in the long run and can compute some important characteristics of the process, such as the stationary distribution (BOROVKOV and VERE-JONES, 2000). Thus we can concentrate on the behavior of node B only.

The node B also evolves generally as an isolated SRM process, but, from time to time, the value of $X_2(t)$ increases by a random jump due to a stress discharge at node A, $\Delta X_2(t) = \xi > 0$ when $\Delta X_1(t) \neq 0$. Interpreting a large jump of that sort as a major seismic event, we want now to analyze how X_2 behaves during a (relatively) short period after it. More precisely, we want to analyze the "typical behavior" of the hazard function $\Psi_2(X_2(t))$ in the absence of any further jumps at node A, when the starting level of stress at node B is rather high. If the stress drops at node B are relatively small and the hazard function value initially is high, the node will be losing stress in a rather long series of drops. The "typical value" of the hazard function t time units after a sharp increase of the risk due to a large stress transfer from node A would then give us the frequency at which jumps occur at node B at that time, i.e., the frequency law for the aftershocks.

To simplify notation, we introduce a new process, $Z(t)$, which describes the behavior of such an isolated node with a fixed initial value z, and put $\rho = \rho_2, \Psi = \Psi_2$. In this context it is convenient to use generators (see, for example, KURTZ, 1981, Chapter 3) to derive properties of the process. The generator, $\mathscr{A}$, of the process $Z(t)$ is specified by its action on a function h as

$$\begin{aligned}\mathscr{A}h(x) &= \lim_{\Delta\to 0}\Delta^{-1}\Big[\mathbf{E}(h(Z(t+\Delta))|Z(t)=x)-h(x)\Big]\\ &= \lim_{\Delta\to 0}\Delta^{-1}\Big[h(x+\rho\Delta)(1-\Psi(x)\Delta)+\mathbf{E}h(x+\rho\Delta-\xi)\Psi(x)\Delta\\ &\quad -h(x)+o(\Delta)\Big] = \rho h'(x)+\Psi(x)(\mathbf{E}h(x-\xi)-h(x)),\end{aligned} \tag{9}$$

where $\mathbf{E}(\cdot)$ denotes the expected value, h is a bounded continuously differentiable function and ξ a random variable which has the jump distribution J at node B (see e.g., Borovkov and Vere-Jones, 2000).

What could be meant by the "typical behavior" of the hazard function at node B after a large jump at A? Recall that, for two processes $\{X(t)\}$ and $\{Y(t)\}$, for any bounded continuous "test" function $\varphi(x)$, one has $\mathbf{E}\varphi(X(t))=\mathbf{E}\varphi(Y(t))$ if and only if $X(t)$ has the same distribution as $Y(t)$. So if the "typical value" of $X(t)$ is given by a deterministic function $m(t)$, and the variation of $X(t)$ (in a certain range of t) is not high, then one may expect that

$$\mathbf{E}\varphi(X(t)) \approx \varphi(m(t)).$$

In fact, this last relation can be taken as a definition of the "typical behavior" of $X(t)$. It will of course depend on the choice of the test function $\varphi(x)$. A standard choice is to take $\varphi(x)\equiv x$ which takes the typical behavior to be the mean function $m(t)=\mathbf{E}X(t)$. Choosing $\varphi(x)\equiv x^2$ leads to $m(t)=\sqrt{\mathbf{E}X^2(t)}$, and so on. In the general case, if $\varphi(x)$ has an inverse φ^{-1}, we set

$$m(t)=\varphi^{-1}(\mathbf{E}\varphi(X(t))). \tag{10}$$

This approach is, in a sense, similar to the method of moments in statistics where one equates theoretical moments to the sample ones to get estimates for parameters. The particular choice of φ is a rather subjective issue, depending often on computational convenience. The choice $\varphi(x)=x$ yielding the mean function does not appear to be any better than any other φ. One usually uses means simply because they are easier to compute, since the value of interest is a sum of random variables. On the other hand, when the variation of the process value is relatively small for any given t from a time interval, any reasonable choice of the test function φ would lead to about the same answer for this time interval.

Assuming the exponential hazard function Ψ, we will derive first an approximate form of the mean function $\mathbf{E}\Psi(Z(t))$ and then an explicit closed form for the "typical behavior function" for the risk $\Psi(Z(t))$ corresponding to the test function choice $\varphi(x)\equiv 1/x$. We will see that the expressions will be very close to each other.

Setting, for a bounded smooth (in an interval D such that $\mathbf{P}(\Psi(Z(t))\in D)=1$; in the case of the exponential hazard function we can put $D=(0,\infty)$) function φ,

$$f(t)=\mathbf{E}\varphi(\Psi(Z(t)))\equiv\mathbf{E}h(Z(t)) \quad \text{for} \quad h(x)=\varphi(\Psi(x)),$$

we observe from (9) that

$$\frac{d}{dt}f(t) = \mathbf{E}\mathscr{A}h(Z(t)) = \rho\mathbf{E}h'(Z(t)) + \mathbf{E}\Psi(Z(t))[\mathbf{E}(h(Z(t)-\xi)|Z(t)) - h(Z(t))],$$

ξ being a random variable, independent of $Z(t)$, following the jump distribution J. Choosing $\varphi(x) \equiv x^j$, $x > 0$, $j = \ldots, -1, 0, 1, \ldots$, leads to $h_j(x) = \Psi^j(x) \equiv \exp(j(vx+\mu))$ and hence, for the functions

$$f_j(t) = \mathbf{E}\Psi^j(Z(t)), \quad j = \ldots, -1, 0, 1, 2, \ldots,$$

we get

$$f_j'(t) = jv\rho f_j(t) + (q(jv) - 1)f_{j+1}(t), \quad j \neq 0, \tag{11}$$

where $q(\lambda) = \mathbf{E}e^{-\lambda\xi}$ is the Laplace transform of the jump size ξ (when $j = 0$, we simply get the identity $0 = 0$).

The "upper half" (case $j > 0$) of (11) is an infinite hierarchy of differential equations. There is apparently no simple way of solving the hierarchy. In statistical mechanics, to "close" such infinite systems, one makes a plausible simplifying assumption (such as the ergodic hypothesis used to derive the Boltzmann equation). Similarly, we can assume here that, when the risk is rather high (just after the inflow of stress at time $t = 0$ from an event at node A, the value of the hazard function is large), and our node B is releasing stress in a series of relatively small jumps, the coefficient of variation of the risk is small:

$$\frac{\mathbf{E}\Psi^2(Z(t))}{[\mathbf{E}\Psi(Z(t))]^2} \approx 1, \quad \text{or} \quad f_2(t) \approx f_1^2(t)$$

on some time interval. Then, instead of (11) for $j = 1$, we have the following equation for only one unknown function $y(t)$ to be used as an approximation to $f_1(t)$:

$$y'(t) = sy(t) - ay^2(t), \quad s = v\rho, \ a = 1 - q(v) > 0.$$

The solution to this equation is easily seen to be given by

$$y(t) = \frac{s}{a(1 - Ce^{-st})},$$

C being the integration constant. Now clearly $C = 1 - s/a\psi$, where $\psi = \Psi(Z(0))$ is the initial risk at node B just after the event at A. This leads to the following expression for our approximation $y(t)$ to the mean function:

$$y(t) = \frac{\psi}{e^{-st} + (1 - e^{-st})a\psi/s}. \tag{12}$$

In the case where there is no tectonic input at node B, $s = 0$, and (12) reduces to $\psi(1 + a\psi t)^{-1}$ which is exactly the Omori law (1).

The "lower half" (case $j < 0$) of (11) is, in contrast to the case $j > 0$, a "closed" sequence of differential equations which can be solved recursively. Choosing $j = -1$ in (11), we get, for the mean reciprocal hazard function $f_{-1}(t)$ the equation

$$f'_{-1}(t) = -sf_{-1}(t) + a^*, \quad a^* = q(-v) - 1 > 0$$

(we have to assume now that $q(-v) = \mathbf{E}e^{v\xi} < \infty$, which is usually the case since the distribution J is truncated). From this we immediately conclude that

$$f_{-1}(t) = Ce^{-st} + \frac{a^*}{s} = \frac{1}{\psi}e^{-st} + \frac{a^*}{s}(1 - e^{-st})$$

(using the initial condition $f_{-1}(0) = 1/\Psi(Z(0)) = 1/\psi$). Applying our definition (10) yields the "typical behavior function"

$$\frac{1}{f_{-1}(t)} = \frac{\psi}{e^{-st} + (1 - e^{-st})a^*\psi/s}, \tag{13}$$

which is basically identical to (12), with the only difference that, instead of a we have now a^*. But this difference is, in fact, very small. Indeed, our assumption that the stress change due to a single jump of size ξ at node B is small relative to the total stress accumulated there means that the hazard value $\Psi(x) = \exp(\mu + vx)$ changes insignificantly when the value of x decreases by ξ. This implies that the values our random variable $v\xi$ takes are usually rather small, so that one can expect that

$$q(\pm v) = \mathbf{E}e^{\mp v\xi} \approx 1 \mp v\mathbf{E}\xi.$$

Therefore

$$a = 1 - q(v) \approx v\mathbf{E}\xi \approx q(-v) - 1 = a^*.$$

Having found $f_{-1}(t)$, we can then easily use (11) to compute $f_{-2}(t)$ and so on. The resulting expressions for the moments $f_{-j}(t)$, $j = -1, -2, \ldots,$ will be linear combinations of exponential functions of t (case $\rho \neq 0$) or polynomials of the respective order j (when $\rho = 0$). We can then, in particular, find the variance of the reciprocal risk process which would indicate how large the deviations from the Omori-type law could be. To illustrate this statement, we will find now the variance of the reciprocal risk process in the case when $\rho = 0$ (corresponding to the classical form of the Omori law). Clearly, in this case we have $f_{-1}(t) = 1/\psi + a^*t$ from (13), and hence (11) for $j = -2$ and the initial condition $f_{-2}(0) = \psi^{-2}$ immediately yield

$$f_{-2}(t) = (q(-2v) - 1)\left(\frac{1}{\psi} + \frac{a^*}{2}t\right)t + \frac{1}{\psi^2}.$$

Therefore

$$\mathrm{Var}\left[\frac{1}{\Psi(Z(t))}\right] = f_{-2}(t) - f_{-1}^2 = b\left[\frac{1}{\psi} + \frac{a^*}{2}t\right]t$$

with $b = q(-2v) - 1 - 2a^* = \mathbf{E}(e^{-v\xi} - 1)^2$. So the variance is small for small values of t, and it is small even for moderate t's when the initial risk ψ is large enough and a^* is small (which, as we have already said, is the case when $v\xi$ is a "small" random variable). This indicates that, in the respective range of t, our "typical behavior" functions (12) and (13) will be rather close to the observed frequencies of the events on node B which we interpret as aftershocks, and so the (modified) Omori law will pretty well describe the behavior of the node.

DIETERICH (1994) constructed a general analytic physical model of earthquake nucleation and time to instability which results in a formula for the seismicity rate of aftershocks. The approach involves calculating the change in an assumed steady background seismicity rate r due to some variation in the shear stressing rate. Aftershocks arise as the effect of a step in shear stress (due to the main shock) superimposed on the background shear stressing rate, and followed by a constant shear stressing rate. In the case where the latter is nonzero, the derived seismicity rate is

$$R = \frac{r\dot{\tau}/\dot{\tau}_r}{\left[\frac{\dot{\tau}}{\dot{\tau}_r}\exp\left(\frac{-\Delta\tau}{A\sigma}\right) - 1\right]\exp\left(\frac{-t}{t_a}\right) + 1}, \tag{14}$$

where A is a fault constitutive parameter, r the reference seismicity rate, and t_a the characteristic time for the seismicity to return to steady state, defined as the time at which the activity returns to the background rate. Parameters σ, τ and τ_r are respectively the normal, shear and reference (before the drop) shear stress, and $\Delta\tau < 0$ is the stress change initiating the process. A most interesting observation is that, by setting

$$s = 1/t_a, \qquad \psi = r\exp\left(\frac{\Delta\tau}{A\sigma}\right), \qquad a^* = \frac{\dot{\tau}_r}{r\dot{\tau}t_a},$$

(12)–(13) are equivalent to the formula (14) for the seismicity rate following a stress step derived by DIETERICH (1994). Hence, provided $s > 0$, (13) gives the Omori law for $t < t_a$, after which seismicity merges to the background rate given by $\rho > 0$. Note that the two node model described above is fitted, as we shall see later, through the observed event data, rather than to physical quatities as in the formula (14). The stochastic nature provides a mechanism for the noise of individual aftershocks around the mean rate. The model also allows for repeated, possibly overlapping, main-shock aftershock sequences, as does the ETAS model. The model can be used to forecast activity in a probabilistic sense (VERE-JONES, 1995) and, furthermore, these forecasts can be updated in the light of activity or the absence thereof.

3. Numerical Fitting

As noted by Dieterich (1994) (14), and hence (12)–(13), give the Omori law provided $s > 0$. However, in our formulation, it is also possible for s (and hence ρ) to be negative, representing a decrease in stress over time in addition to the decrease due to aftershocks. We would expect faster decay in the seismicity rate in such cases, but do the decay functions (12)–(13) still mimic the modified Omori law (2)? A second question is the range of values obtainable for p. One way to investigate this is to numerically find values of K, c, p to minimize the mean-squared error between the two functions for given values of s, ψ, a. In order to improve the stability of the algorithm, and because of the scale and nature of c, we choose to make this a fixed value of 0.1, thus identifying the time units. Figure 1 shows an example of the fit with $a = 0.5, \psi = 25, s = -0.25$, producing estimates of $K = 1.929, p = 1.367$, for the corresponding modified Omori formula. As we can see, the two curves are unlikely to be separated on the basis of observations. Experimentation resulted in values of p ranging from 0.5 to 1.0 ($s > 0$) and 0.8 to 1.7 ($s < 0$), although these are certainly not limits. Figures 2 and 3 show how the fitted p-value depends on the parameters a, ψ and s. We see that s is the primary factor, p decreasing almost linearly with s. While p also increases somewhat with a and ψ, it is in a nonlinear fashion, and soon reaches saturation point. Dieterich (1994) observed that $p > 1$ could be a result of stress decreasing logarithmically with time. This is paralleled here by the fact that values of $p > 1$ are produced only by $s < 0$, as this implies $\rho < 0$, and hence a logarithmically decreasing point process intensity.

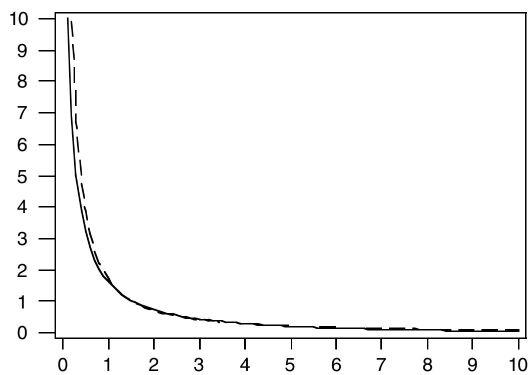

Figure 1
Decay of aftershock frequency for modified Omori law (dashed line) and two-node model (12)–(13) (solid line).

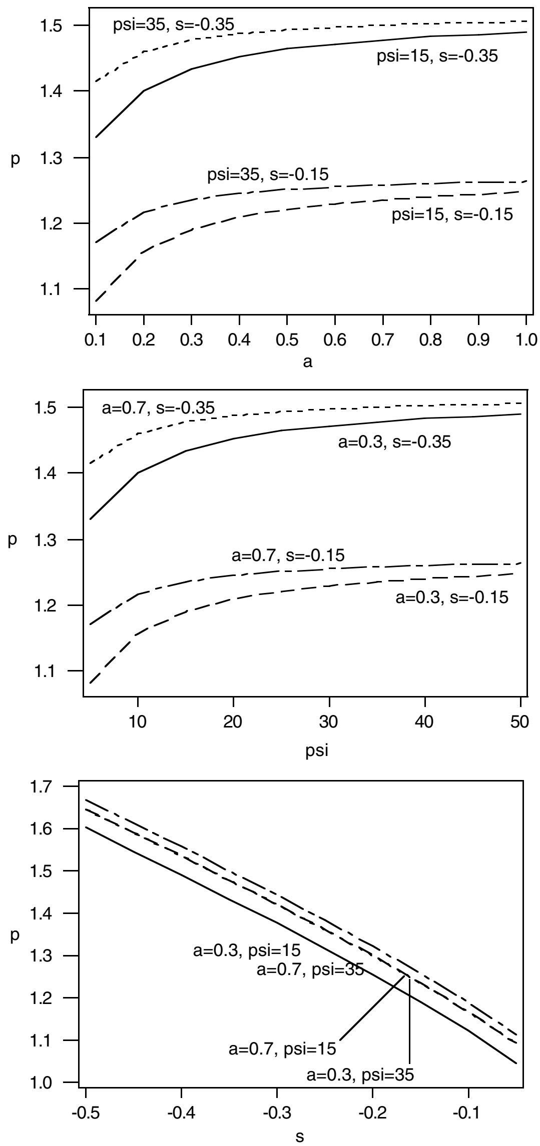

Figure 2
Dependence of p on parameters from (13) for $s < 0$.

4. Examples

The single most important feature of the linked stress release model is that it can be objectively fitted to data by means of maximum likelihood estimation. To do this we numerically maximize the log-likelihood

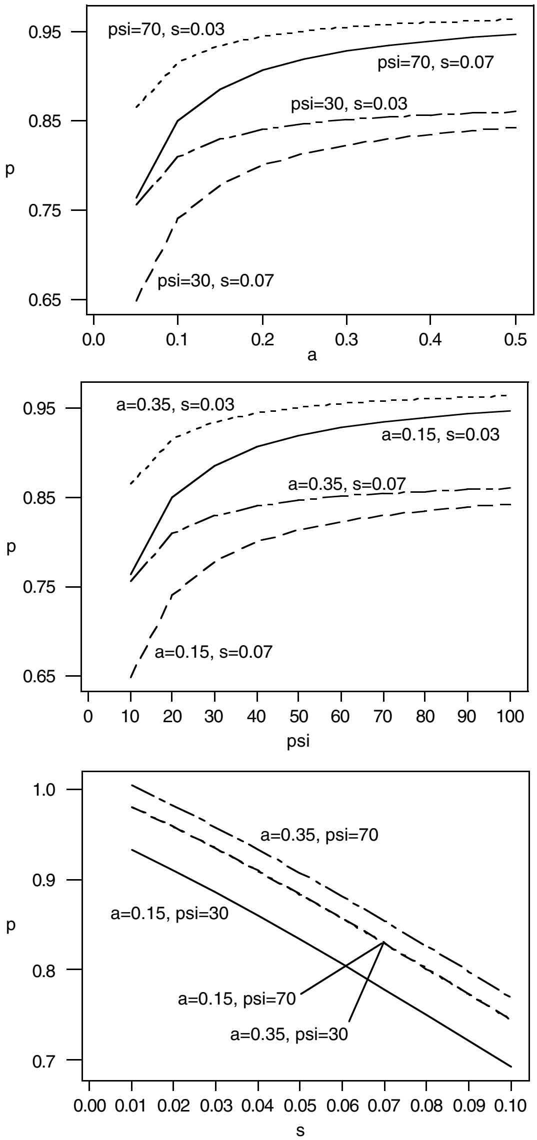

Figure 3
Dependence of p on parameters from (13) for $s > 0$.

$$\log L(T_1, T_2) = \sum_{k:T_1 \le t_k \le T_2} \log \lambda(t_k) - \int_{T_1}^{T_2} \lambda(t)\, dt,$$

(see, for example, DALEY and VERE-JONES, 1988) where events occur at $\{t_k\}$, and $T_1 < t_1 < \cdots < t_n < T_2$ is the observation interval.

This naturally raises the question of how well the model explains actual data. We shall take the PDE catalog from 1973 to present, for a region corresponding roughly to California. Events were identified as either main shocks or aftershocks by the M8 declustering procedure (KOSSOBOKOV, 1997), and trimmed to minimum magnitudes of 4.5 (main shocks) and 4.0 (aftershocks). The difference in the magnitude cutoffs corresponds to the accepted designation of aftershocks as being of smaller magnitude than the main shock and matches the observation of KISSLINGER and JONES (1991) of a smallest observed difference of 0.3–0.6 between the magnitudes of the main shock and largest aftershock. The result was a catalog of 425 main shocks and 1273 aftershocks, as shown in Figure 4. We note that this is of course a superposition of multiple sequences, but repeated sequences, if they even exist in the time frame, are hard to identify. However, the practice of superimposing sequences is an accepted one in the analysis of aftershocks (see, for example, PAPAZACHOS, 1974, or DAVIS and FROHLICH, 1991), and this is the equivalent in the framework we are using. It also accords very well with our object of determining if there are any effects the model is unable to deal with.

Three two-region linked stress release models (6) (cf. BEBBINGTON and HARTE, 2001) were fitted to the data. These were the independent regions (or unlinked) model ($\theta_{ij} = 0, i \neq j$), the full (linked) model ($\theta_{ij} \neq 0, \forall i, j$) and the A/B node model presented above with $\theta_{12} = 0, \theta_{21} \neq 0$. The goodness of fit of the model is measured by the Akaike Information Criterion (AIC), defined as

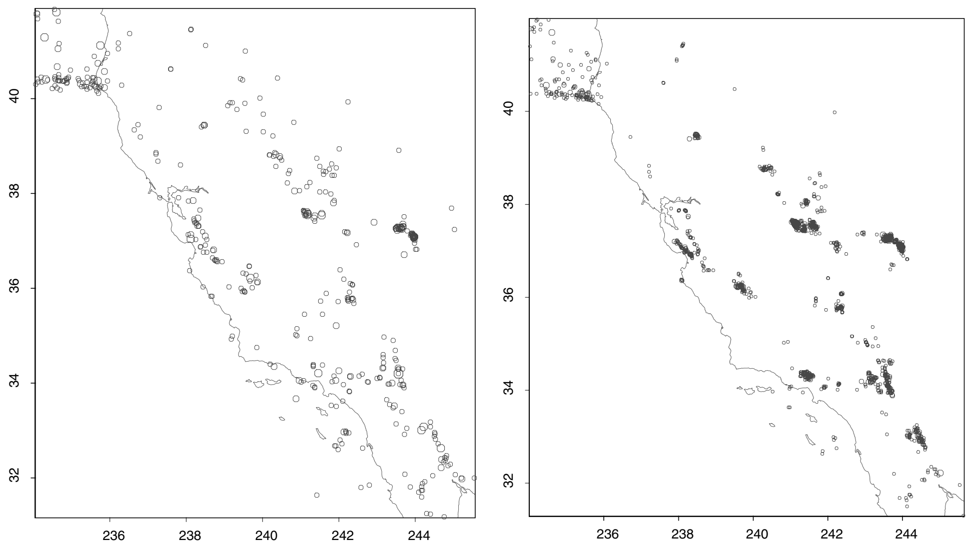

Figure 4
California epicentres; main shocks (left) and aftershocks (right). Symbol size scales with magnitude. Longitude is given in degrees E of Greenwich.

$$\mathrm{AIC} = -2 \log L + 2k$$

(Akaike, 1977), where L is the maximum likelihood, and k the number of fitted parameters. The best model is that with the smallest AIC. The results are presented in Table 1. Firstly we note the negative fitted values of ρ, indicating that the dominant p-values in the catalog are greater than 1. Kisslinger and Jones (1991) likewise determined that the median p-value in Southern California is greater than 1. The high relative AIC value of the independent model indicates that the linked stress release model does pick up the dependence of the aftershocks. Further, the negative values for v in this case indicates that the independent model considers the aftershocks to be self-exciting (cf. (6)), which is physically implausible. The fact that the remaining two AICs are very close indicates that the proposed model fits the data very well. The difference, as can be seen by the fitted parameters, is due to the main shocks, probably because of the inclusion of main shocks without aftershocks in the catalog. We found that raising the magnitude cutoff for the main shocks to 5.0, corresponding to the average observed difference of 1.0 between a main shock and its largest aftershock (Kisslinger and Jones, 1991), results in the A/B model having a superior AIC. Further increasing the main shock and aftershock cutoffs to 6.0 and 5.0, respectively, makes this difference statistically significant. This difficulty with smaller magnitudes seems to be a characteristic of the stress release model that is currently being investigated further. It does indicate that detailed analysis of a single sequence will be best done for a case where aftershock magnitudes are large. Such a sequence is that following the Valparaiso (Chile) earthquake of March 3, 1985.

The Valparaiso aftershock sequence has 88 events of magnitude 5.0 or greater in the 802 days following the main shock. The decay rate fits very well the modified Omori formula with $p = 1.038$ (Kisslinger, 1988). Fitting a SRM intensity (4) to the sequence of events produces parameter estimates $\alpha = 5.344, v = 0.0264, \rho = -0.000676$. We now need to transform these into estimates of s, ψ, a^*. Firstly, $s = v\rho = -1.76 \times 10^{-5}$. From (4) we see that $\psi = \Psi(Z(0)) = e^{\alpha} = 209.5$. Finally, $a^* = \mathbf{E}(e^{v\xi})$ which, averaging over the events using (8), we obtain as

Table 1

Estimated parameters and fits for the California aftershock data. The top values are for the main shock node, the bottom for the aftershock node

Model	k	α	v	ρ	Θ		AIC
Indep.	6	−2.947	0.00035	0.513	1	0	11193.6
		−1.999	−0.00091	0.575	0	1	
Full	8	−2.898	7.6×10^{-5}	4.410	1	7.87	11040.9
		−1.510	2.2×10^{-4}	−5.665	−9.04	1	
A/B	7	−2.947	0.00035	0.513	1	0	11042.9
		−1.510	0.00022	−5.665	−9.04	1	

$$a^* = \sum_k \exp(0.0264 \times 10^{0.75(M_k - 5.0)})/88 = 1.167.$$

These values produce an estimate of $p = 1.041$ by numerical fitting of (13) to (2), which is satisfactorily close to the original estimate. This confirms that the two node model is able to reproduce the modified Omori formula in practice.

5. Discussion

We have presented an alternative to the ETAS model (Ogata, 1988) for point process data from aftershock sequences. It is equally capable of being fitted to data, with 7 as opposed to 5 parameters, but does allow for elastic rebound effects to trigger main shocks at semi-regular intervals. The model is capable of reproducing the modified Omori law with a range of p values. In the case where the rate of tectonic input $\rho > 0$, it provides the form of the decay law derived by Dieterich (1994), when the stressing rate $\dot{\tau} \neq 0$. When the tectonic input at the second (aftershock-producing) node of the model $\rho = 0$ we obtain the classical Omori formula. If $\rho < 0$, the decay rate fits well the modified Omori formula, typically with $p > 1$. This parallels the observation of Dieterich (1994), that $p > 1$ can be obtained by a decreasing stressing rate and the result of Mikumo and Miyatake (1979) that high p values were obtained for the shortest maximum relaxation time.

The strong dependence of the estimated value of p on the parameter $s = v\rho$ is consistent with the absence of any strong dependence on main shock magnitude. The latter is introduced into the decay formula through the parameter ψ in the model.

Fitting the model to California data indicated that it captured most of the information concerning dependence on main shocks, although the fit was improved by higher magnitude cutoffs. The fitted result of $\rho < 0$ is in line with the determination of Kisslinger and Jones (1991) of a median $p > 1$. Fitting only the aftershocks of the Valparaiso earthquake of 3 March 1985 disclosed an estimated p almost identical to that obtained through direct analysis of the catalog.

Forecasting (in a probabilistic sense, see Vere-Jones, 1995) of aftershocks can be achieved by repeated forward simulation of the fitted model (see, for example, Lu *et al.*, 1999). As the fitting process incorporates the history of the process, such forecasts can be updated as aftershocks (or quiescent periods) occur.

One obvious cause of difficulties with the model is large aftershocks which themselves are accompanied by many secondary aftershocks. This would best be handled by extending the model to a third 'cascading' node, which stands in relation to the second node as that stands to the first. Such sequences are in practice modeled by a combination of the modified Omori formula (Utsu, 1970; Ogata 1983).

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