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Anisotropic Ising Model in $d + s$ Dimensions

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Abstract. In this note, we consider the asymmetric nearest neighbor ferromagnetic Ising model on the $(d+s)$-dimensional unit cubic lattice $\mathbb{Z}^{d+s}$, at inverse temperature $\beta = 1$ and with coupling constants $J_s > 0$ and $J_d > 0$ for edges of $\mathbb{Z}^s$ and $\mathbb{Z}^d$, respectively. We obtain a lower bound for the critical curve in the phase diagram of (J_s, J_d). In particular, as J_d approaches its critical value from below, our result is directly related to the so-called dimensional crossover phenomenon.

Mathematics Subject Classification. 82B20, 82B26.

1. Introduction

In this paper, we consider an asymmetric nearest neighbor ferromagnetic Ising model on the $(d + s)$-dimensional unit cubic lattice $\mathbb{Z}^{d+s}$, with coupling constants $J_d > 0$ and $J_s > 0$ in the hyperplanes $\mathbb{Z}^d$ and $\mathbb{Z}^s$, respectively.

Anisotropic lattice spin systems have been the subject of great interest within the physics community since the sixties. The study of such models has been tackled both numerically, mainly via Monte Carlo simulations (see, e.g., [1–4] and references therein), and theoretically, via mean-field, Bethe approximation, truncated high-temperature expansion of the susceptibility, etc. (see, e.g., [4–11]). A strong motivation to study anisotropic systems is to investigate finite-size effects in realistic materials modeled by quasi-two-dimensional (thin films) and quasi-one-dimensional spin systems. Furthermore, exploring these systems could provide valuable insights into isotropic models, notably the three-dimensional Ising model (see, e.g., [12] and references therein).

Rigorous results on the asymmetric Ising model on $\mathbb{Z}^{d+s}$ have been obtained mainly in the case $\mathbb{Z}^{1+s}$ with strong coupling in one dimension and

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small coupling in the remaining directions. In particular, in a well-known article (see [13]), Fisher derived an asymptotic bound on the critical temperature of the $\mathbb{Z}^{1+s}$ anisotropic Ising model in the limit $\frac{J_s}{J_1} \to 0$. It has also been shown rigorously (see [14]) that the free energy of the Ising model on $\mathbb{Z}^{1+s}$ is analytic for any inverse temperature β if J_s is small enough (depending on J_1 and the inverse temperature). These rigorous results rely heavily on the fact that when $d = 1$ the one-dimensional system is in the gas phase at all temperatures so that the standard high-temperature expansion can be used with effectiveness. On the other hand, to obtain the same kind of results of references [13,14] in the case $\mathbb{Z}^{d+s}$, $d \geq 2$ is expected to be trickier since the $d-$dimensional system exhibits a phase transition and the usual high-temperature expansion turns to be much more difficult to control.

In this paper, we somehow extend the results obtained in [13] and [14] for the case $\mathbb{Z}^{1+s}$ to the case $\mathbb{Z}^{d+s}$ with $d \geq 1$. Namely, we show that for all J_d above the critical reduced temperature J_d^c, the susceptibility of the $(d+s)$-dimensional system is finite when the coupling J_s is sufficiently small (inversely proportional to the susceptibility $\chi(J_d)$ of the d-dimensional system). A similar result was obtained by two of us in [15] for the Bernoulli anisotropic bond percolation model on $\mathbb{Z}^{d+s}$, in which edges in the $\mathbb{Z}^d$ hyperplane are open with probability $p < p_c(d)$ and edges parallel in the $\mathbb{Z}^s$ hyperplane are open with probability q. In [15], probabilistic arguments were applied and in particular, a crucial use of the van den Berg–Kesten (BK) inequality has been made. In the present paper, to get the analogous result for the anisotropic Ising model in $\mathbb{Z}^{d+s}$ (for which BK inequalities are not available), we use an alternative (w.r.t. the high temperature) expansion of the Ising partition function, namely the so-called random current representation. This powerful technique, introduced in the eighties by Aizenman [16], was widely used by several authors in the following decades, and recently, it has employed as a crucial tool in several remarkable papers, e.g., [17–19]. We remind that the high-temperature phase of any d dimensional Ising model satisfies strong spatial mixing, which is equivalent to the Dobrushin–Shlosman "Complete Analyticity-CA" condition. Therefore also in $d+s$ dimensions under a sufficiently weak s-dimensional perturbation, the CA-condition holds implying analyticity of the free energy (see [20] for a recent proof of CA).

As mentioned earlier, our results may be of interest in the study of realistic quasi-two-dimensional magnets which can be modeled by a two-dimensional sub-critical Ising bilayer. When the interaction between the two two-dimensional layers is weak and the transverse interaction is sub-critical, the bilayer system is expected to still exhibit sub-critical behavior. However, as the coupling between the layers strengthens, the overall system may exhibit spontaneous magnetization. Our result implies rigorously that as long as the inter-layer interaction between layers is below the inverse two-dimensional susceptibility (with a constant factor $1/(2\,\mathrm{s})$), the global system remains sub-critical. The understanding of how several layers of 2D slightly sub-critical systems with small interactions between them can start to behave as a $(2+s)$-dimensional system, is the so-called dimensional crossover phenomenon (see, e.g., Sec. VI in [6]). This phenomenon is characterized by a critical exponent, which is believed to depend on the original dimension of the layers, but not on the target dimension. Moreover, numerical simulations and formal calculations (see, e.g., [6,7]), suggest that this critical exponent equals the exponent of the susceptibility of the original dimension. Our results imply rigorously an inequality between the two exponents.

2. The Model and Results

Let $\mathbb{Z}^{d+s} = \mathbb{Z}^d \times \mathbb{Z}^s$ be the $(d+s)$-dimensional unit cubic lattice. We will denote by $\mathbb{E}^{d+s}$ the set of nearest neighbor pairs of $\mathbb{Z}^{d+s}$ so that $\mathbb{G}^{d+s}$ is the graph with vertex set $\mathbb{Z}^{d+s}$ and edge set $\mathbb{E}^{d+s}$. Given two vertices $x, y \in \mathbb{Z}^{d+s}$, we denote by $|x-y|$ the usual graph distance between x and y (i.e., the edge length of the shortest path between x and y). We will suppose that $\mathbb{Z}^{d+s}$ is equipped with the usual operation of sum. We represent hereafter a site $x \in \mathbb{Z}^{d+s}$ as $x = (u, t)$, where $u \in \mathbb{Z}^d$ and $t \in \mathbb{Z}^s$.

Given an integer N, we denote by $\Lambda_N \subset \mathbb{Z}^{d+s}$ the hypercube with side length $2N + 1$, centered at the origin, so that $\Lambda_N \to \infty$ means that $N \to \infty$. We denote by E_N the set of edges of $\mathbb{E}^{d+s}$ with both endpoints in Λ_N, so that $\mathbb{G}^{d+s}|_{\Lambda_N} = (\Lambda_N, E_N)$ is the restriction of $\mathbb{G}^{d+s}$ to Λ_N. Note that $\Lambda_N = \bar{\Lambda}_N \times \hat{\Lambda}_N$ where $\bar{\Lambda}_N$ denotes the d-dimensional hypercube in $\mathbb{Z}^d$ of size $2N+1$ centered at the origin and $\hat{\Lambda}_N$ denotes the s-dimensional hypercube in $\mathbb{Z}^s$ of size $2N + 1$ centered at the origin. Given $w \in \hat{\Lambda}_N$, we set $\Lambda_N^w = \{(u, t) \in \mathbb{Z}^{d+s} : t = w\}$. Namely Λ_N^w is the subset of Λ_N formed by sites of Λ_N with w as the second set of coordinates. Similarly, E_N^w will denote the set of edges with both endpoints in Λ_N^w. Observe that Λ_N^w is a d-dimensional box of side length $2N + 1$ centered at $(0, w)$.

To each vertex $x \in \Lambda_N$, we associate a random variable σ_x taking values in the set $\{+1, -1\}$. A spin configuration in Λ_N is a function $\boldsymbol{\sigma} : \Lambda_N \to \{+1, -1\} : x \mapsto \sigma_x$. The energy of a configuration $\boldsymbol{\sigma}$ is given by the (free boundary condition) Hamiltonian

$$H_{\Lambda_N}(\boldsymbol{\sigma}) = - \sum_{\{x,y\} \in E_N} J_{\{x,y\}} \sigma_x \sigma_y,$$

where

$$J_{\{x,y\}} = \begin{cases} J_s & \text{if } \; x = (u,t) \; \text{ and } \; y = (u,t') \; \text{ with } \; |t - t'| = 1, \\ J_d & \text{if } \; x = (u,t) \; \text{ and } \; y = (u',t) \; \text{ with } \; |u - u'| = 1, \end{cases}$$

with $J_s > 0$ and $J_d > 0$. In what follows, an edge $\{x, y\} \in E_N$ is called *vertical* if $x = (u, t)$, $y = (u, t')$ with $|t - t'| = 1$, and called *planar* if $x = (u, t)$, $y = (u', t)$ with $|u - u'| = 1$. So $J_{\{x,y\}} = J_s$ if $\{x, y\}$ is vertical and $J_{\{x,y\}} = J_d$ if $\{x, y\}$ is planar.

The partition function of the system is given by

$$Z_{\Lambda_N}(J_d, J_s) = \int d\mu_{\Lambda_N}(\boldsymbol{\sigma}) e^{-H_{\Lambda_N}(\boldsymbol{\sigma})} = \int d\mu_{\Lambda_N}(\boldsymbol{\sigma}) \prod_{\{x,y\} \in E_N} e^{J_b \sigma_x \sigma_y},$$

where $\int d\mu_{\Lambda_N}(\boldsymbol{\sigma})$ is a short notation for $\prod_{x\in\Lambda_N}\frac{1}{2}\sum_{\sigma_x=\pm1}$ (a product probability measure). Moreover, without loss of generality, we have set the inverse temperature $\beta = 1$.

The two-point correlation function of the $(d+s)$-system is then defined as

$$\langle\sigma_x\sigma_y\rangle_{\Lambda_N} = \frac{\int d\mu_{\Lambda_N}(\boldsymbol{\sigma})\sigma_x\sigma_y e^{-H_{\Lambda_N}(\sigma_\Lambda)}}{Z_{\Lambda_N}(J_d,J_s)} = \frac{\int d\mu_{\Lambda_N}(\boldsymbol{\sigma})\sigma_x\sigma_y\prod_{\{x,y\}\in E_N} e^{J_b\sigma_x\sigma_y}}{Z_{\Lambda_N}(J_d,J_s)}. \tag{1}$$

In general, for any set $U\subset\Lambda_N$, letting $E_U = \{\{x,y\}\in E_N : \{x,y\}\subset U\}$, we set

$$H_U(\boldsymbol{s}) = -\sum_{\{x,y\}\in E_U} J_{\{x,y\}}\sigma_x\sigma_y,$$

$$Z_U(J_d,J_s) = \int d\mu_{\Lambda_N}(\boldsymbol{\sigma})e^{-H_U(\boldsymbol{\sigma})},$$

and, for any $x,y\in U$,

$$\langle\sigma_x\sigma_y\rangle_U = \frac{\int d\mu_{\Lambda_N}(\boldsymbol{\sigma})\sigma_x\sigma_y e^{-H_U(\sigma_\Lambda)}}{Z_U(J_d,J_s)}.$$

According to the above notations, for any $t\in\hat{\Lambda}_N$, we have that

$$\langle\sigma_x\sigma_y\rangle_{\Lambda_N^t} = \frac{\int d\mu_{\Lambda_N}(\boldsymbol{\sigma})\sigma_x\sigma_y e^{-H_{\Lambda_N^t}(\sigma_\Lambda)}}{Z_{\Lambda_N^t}(J_d,J_s)},$$

The finite volume susceptibility function of the system is defined as

$$\chi_{\Lambda_N}(J_d,J_s) := \sup_{x\in\Lambda_N}\left\{\sum_{y\in\Lambda_N}\langle\sigma_x\sigma_y\rangle_{\Lambda_N}\right\},$$

so that

$$\chi_{d+s}(J_d,J_s) = \lim_{N\to\infty}\chi_{\Lambda_N}(J_d,J_s) \tag{2}$$

is the susceptibility of the anisotropic $(d+s)$-dimensional Ising model.

For any $t\in\hat{\Lambda}_N$, let

$$\chi_{\Lambda_N^t}(J_d) = \sup_{x\in\Lambda_N^t}\left\{\sum_{y\in\Lambda_N^t}\langle\sigma_x\sigma_y\rangle_{\Lambda_N^t}\right\}, \tag{3}$$

so that

$$\chi_d(J_d) = \lim_{N\to\infty}\chi_{\Lambda_N^t}(J_d) \tag{4}$$

is the susceptibility of the d-dimensional Ising model with ferromagnetic interaction J_d.

We are now in a position to state our main result.

Theorem 1. *Take any J_d such that*

$$\chi_d(J_d) < +\infty,$$

and let J_s such that

$$\tan h(J_s) < \frac{1}{2s\chi_d(J_d)}.$$

Then,

$$\chi_{d+s}(J_d, J_s) < +\infty.$$

Remark 1. As mentioned in the introduction, Theorem 1 is related to the so-called dimensional crossover phenomenon. One can define, for any $J_d > 0$, the function $J_s^c(J_d) : [0, \infty) \to [0, \infty)$, where

$$J_s^c(J_d) = \sup\{J_s : \chi_{d+s}(J_d, J_s) < +\infty\}. \tag{5}$$

Denoting by J_d^c the critical inverse reduced temperature of the d-dimensional system, we have that by definition, $J_s^c(J_d) = 0$ if $J_d \geq J_d^c$. Theorem 1 determines a region in the (J_d, J_s) plane where no phase transition occurs in the $(d+s)$-system and it also gives an upper bound for the function $J_s^c(J_d)$ as J_d varies in the interval $[0, J_d^c)$. There is a strong interest in understanding the behavior of the function $J_s^c(J_d)$ when $J_d < J_d^c$, and in particular, it is widely believed that there exists a constant $\phi_d > 0$ such that

$$J_s^c(J_d) \approx |J_d - J_d^c|^{\phi_d}, \text{ as } J_d \uparrow J_d^c,$$

where the symbol $\approx$ stands for log equivalence, that is, $f(x) \approx g(x)$ if $\lim_{\beta\uparrow\beta_c} \frac{\log f(x)}{\log g(x)} = 1$. The constant ϕ_d is the so-called crossover critical exponent. On the other hand, the d-dimensional Ising susceptibility $\chi_d(J_d)$ is known to behave like

$$\chi_d(J_d) \approx |J_d - J_d^c|^{-\gamma_d}, \text{ as } J_d \uparrow J_d^c,$$

where $\gamma_d > 0$ is the susceptibility d-dimensional critical exponent.

As an immediate corollary, Theorem 1 implies that $\phi_d \leq \gamma_d$ for all $d \geq 2$, and a lower bound for $J_s^c(J_d)$ still inversely proportional to $\chi_d(J_d)$ would imply that $\phi_d = \gamma_d$. The conjectured equality of the crossover critical exponent ϕ_d and susceptibility critical exponent γ_d has been discussed in the physics literature by several authors, see, for instance, [3,5,6,9,21]. See also [22] for a recent example of dynamical approach to dimensional crossover.

3. The Random Current Representation

As mentioned in the introduction, in order to prove Theorem 1 we will use the so-called *random current representation* for the Ising model introduced by Aizenman in [16]. We will describe this technique here below following mainly reference [23].

Let $\mathfrak{F}(E_N)$ be the set of all functions $\eta : E_N \to \mathbb{N} : b \mapsto \eta_b$. Then, we can expand the exponential inside the product and rewrite the partition function $Z_{\Lambda_N}(J_d, J_s)$ as

$$Z_{\Lambda_N}(J_d, J_s) = \sum_{\eta \in \mathfrak{F}(E_N)} W(\eta) \int d\mu_{\Lambda_N}(\boldsymbol{\sigma}) \prod_{x \in \Lambda_N} (\sigma_x)^{\sum_{b \ni x} \eta_b},$$

where we have denoted shortly

$$\sum_{b \ni x} \eta_b = \sum_{\substack{b \in E_N \\ x \in b}} \eta_b$$

and we have set

$$W(\eta) = \prod_{b \in E_N} \frac{(J_b)^{\eta_b}}{\eta_b!}.$$

Observe that the integral $I(\eta) = \int d\mu_{\Lambda_N}(\boldsymbol{\sigma}) \prod_{x \in \Lambda_N} (\sigma_x)^{\sum_{b \ni x} \eta_b}$ is zero, unless $\sum_{b \ni x} \eta_b$ is even for all $x \in \Lambda$, in which case $I(\eta) = 1$. Hence,

$$Z_{\Lambda_N}(J_d, J_s) = \sum_{\substack{\eta \in \mathfrak{F}(E_N) \\ \partial\eta = \emptyset}} W(\eta),$$

where we have set

$$\partial\eta = \left\{ x \in \Lambda : \sum_{b \ni x} \eta_b \text{ is odd} \right\}.$$

In general, given any $\eta \in \mathfrak{F}(E_N)$, the vertices in $\partial\eta$ are called *sources* of η, and if $\partial\eta = \emptyset$, then η is called *sourceless*. Proceeding similarly, we have that the random current expansion for the two-point function is

$$\langle \sigma_x \sigma_y \rangle_{\Lambda_N} = \sum_{\substack{\eta \in \mathfrak{F}(E_N) \\ \partial\eta = \{x,y\}}} \frac{W(\eta)}{Z_{\Lambda_N}(J_d, J_s)}.$$

We now rewrite the function $\langle \sigma_x \sigma_y \rangle_{\Lambda_N}$ as a sum of edge-self-avoiding walks from x to y. Given an edge $\{x, y\} \in E_N$, the ordered pair (x, y) will be called a **step** from x to y. For any $x \in \Lambda_N$, we establish an arbitrary order (denoted by $\preceq$) for the set of steps emerging from x (i.e., for those (x, y) such that $|x - y| = 1$). For each site x, and each *step* (x, z), we consider the set $\Gamma_{(x,z)}$ formed by the edges $b = \{x, y\}$ such that $(x, y) \preceq (x, z)$. This set will be referred to as the set of edges *canceled* by (x, z). In particular, since $(x, y) \preceq (x, y)$, a step $\{x, y\}$ cancels itself.

We recall that a path in Λ_N is a sequence $p = \{x_0, x_1, ..., x_n\}$ of vertices of Λ_N such that $\{x_{i-1}, x_i\} \in E_N$ for all $i = 1, \cdots, n$. We say that a path $p = \{x_0, x_1, ..., x_n\}$ is *consistent* if, for each $k = 1, ..., n$, we have that $\{x_{k-1}, x_k\} \notin \cup_{i=1}^{k-1} \Gamma_{(x_{i-1}, x_i)}$. That is, each step used in this path is not associated with an edge that was canceled by the previous steps. If $p = \{x_0, x_1, ..., x_n\}$ is a consistent path, we denote by p^* the set of all edges canceled by p, that is,

$p^* = \cup_{i=1}^n \Gamma_{(x_{i-1},x_i)}$. Clearly, by construction, a consistent path is always edge-self-avoiding. We denote by $C_{xy}(\Lambda_N)$ the set of all consistent paths in Λ_N from x to y.

We now define a function Ω, which associates with each current configuration η with $\partial\eta = \{x, y\}$, a consistent path $\omega = \Omega(\eta)$ from x to y, which belongs to $C_{xy}(\Lambda_N)$. As in [23], such a consistent path will be called the *backbone* of η.

Given η with $\partial\eta = \{x, y\}$, let Γ_η be the set of edges $b \in E$ such η_b is odd. Then, Γ_η forms a subgraph of (Λ_N, E_N) (in general not connected) such that every vertex has degree either even or zero, except on x and y, whose degrees are odd. The graph Γ_η necessarily contain a connected component, say $\gamma_\eta^{x,y}$, which contains x and y. Therefore, we can look at this connected component $\gamma_\eta^{x,y}$ (seen as a set of edges in E_N), uniquely determined by η, and associate with it a consistent path $\omega = \Omega(\eta)$. This is the path $\omega = \{z_0 = x, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k = y\}$ for some $k \geq |x - y|$ such that for any $i = 1, 2, \ldots, k$, (z_{i-1}, z_i) is the minimal step according to the order established among the steps emerging from z_{i-1} associated with edges of $\gamma_\eta^{x,y} - \{z_0, z_1\} \cup \{z_1, z_2\} \cup \cdots \cup \{z_{1-2}, z_{i-1}\}$.

Once the function Ω is defined, we now can rewrite $\langle\sigma_x\sigma_y\rangle_{\Lambda_N}$ as

$$\langle\sigma_x\sigma_y\rangle_{\Lambda_N} = \sum_{\omega\in C_{xy}(\Lambda_N)} \sum_{\substack{\eta\in\mathfrak{F}(E_N)\\ \partial\eta=\{x,y\},\Omega(\eta)=\omega}} \frac{W(\eta)}{Z_{\Lambda_N}(J_d, J_s)}.$$

Note that if $\Omega(\eta) = \omega$, then η is odd on the edges of the set ω and is even on the edges of $\omega^* \setminus \omega$. Also, η restricted to $E_N \setminus \omega$, as well as to $E_N \setminus \omega^*$, is such that $\partial\eta = \emptyset$. Therefore, setting shortly $Z_N = Z_{\Lambda_N}(J_d, J_s)$, we have that

$$\begin{aligned}
\sum_{\substack{\eta\in\mathfrak{F}(E_N)\\ \partial\eta=\{x,y\},\Omega(\eta)=\omega}} \frac{W(\eta)}{Z_N} &= \prod_{b\in\omega}\sinh(J_b) \prod_{b\in\omega^*\setminus\omega} \cosh(J_b) \sum_{\substack{\eta\in\mathfrak{F}(E_N\setminus\omega^*)\\ \partial\eta=\emptyset}} \frac{W(\eta)}{Z_N}\\
&= \prod_{b\in\omega}\tan\, h(J_b) \prod_{b\in\omega^*} \cosh(J_b) \sum_{\substack{\eta\in\mathfrak{F}(E_N\setminus\omega^*)\\ \partial\eta=\emptyset}} \frac{W(\eta)}{Z_N}\\
&= \prod_{b\in\omega}\tan\, h(J_b) \sum_{\substack{\eta\in\mathfrak{F}(E_N):\ \partial\eta=\emptyset\\ \eta \text{ even on } \omega^*}} \frac{W(\eta)}{Z_N},
\end{aligned}$$

where the last summation is over all sourceless current configurations η on E_N with the additional restriction that η_b is even on all edges b canceled by ω. Hence, we can rewrite $\langle\sigma_x\sigma_y\rangle_{\Lambda_N}$ as

$$\langle\sigma_x\sigma_y\rangle_{\Lambda_N} = \sum_{\omega\in C_{xy}(\Lambda_N)} \rho_{E_N}(\omega), \tag{6}$$

where

$$\rho_{E_N}(\omega) = \prod_{b\in\omega} \tan\, h(J_b) \sum_{\substack{\eta\in\mathfrak{F}(E_N):\, \partial\eta=\emptyset \\ \eta \text{ even on } \omega^*}} \frac{W(\eta)}{Z_N}. \tag{7}$$

Observing that

$$\sum_{\substack{\eta\in\mathfrak{F}(E_N):\, \partial\eta=\emptyset \\ \eta \text{ even on } \omega^*}} \frac{W(\eta)}{Z_N} \leq \frac{\sum_{\eta\in\mathfrak{F}(E_N):\, \partial\eta=\emptyset} W(\eta)}{Z_N} = 1,$$

we obtain straightforwardly the following upper bound

$$\rho_E(\omega) \leq \prod_{b\in\omega} \tanh(J_b). \tag{8}$$

4. Proof of Theorem 1

To prove Theorem 1, we shall use two properties of the weights $\rho_{E_N}(\omega)$ defined in (7). The interested reader can check their proofs in Section 4.2 of [23].

a) Let $U \subset E_N$ be a set of edges of Λ_N, and let and $\omega \subset U$ be a consistent path. Then,

$$\rho_{E_N}(\omega) \leq \rho_U(\omega). \tag{9}$$

b) If $\omega_1 \circ \omega_2$ is a consistent path, where $\circ$ denotes the usual concatenation of paths, then

$$\rho_{E_N}(\omega_1 \circ \omega_2) = \rho_{E_N}(\omega_1)\rho_{E_N\setminus\omega_1^*}(\omega_2).$$

As shown in Sect. 3, the backbone expansion (6) for the two-point function on Λ_N is given by

$$\langle\sigma_x\sigma_y\rangle_{\Lambda_N} = \sum_{\omega\in C_{xy}(\Lambda_N)} \rho_{E_N}(\omega),$$

where $x = (u_0, t_0)$, $y = (u, t)$ and $C_{xy}(\Lambda_N)$ is the set of all consistent paths ω with extremes $\partial\omega = \{x, y\}$ using edges of E_N.

Let ω be a consistent path connecting x to y. It is possible to break this path into $n + 1$ "planar" pieces ω_i, and n "vertical" steps s_i (i.e., such that $|s_i| = 1$) connecting two d-dimensional hyperplanes. (Note that there are $2s$ possibilities for the choice of s_i.) Namely, we can write

$$\omega = \omega_1 \circ s_1 \circ \omega_2 \circ s_2 \circ \ldots \circ s_n \circ \omega_{n+1}. \tag{10}$$

We are denoting by ω_1 the initial piece of the path ω all contained in $\Lambda_N^{t_0}$. This initial piece ω_1 of the path ω is a "planar" path connecting the site (u_0, t_0) to the site (u_1, t_0), which is the last site of $\Lambda_N^{t_0}$ visited by ω before leaving $\Lambda_N^{t_0}$; this path only uses edges of $E_N^{t_0}$. Then, s_1 is the first vertical step, that is, the edge connecting (u_1, t_0) to (u_1, t_1) (where $t_1 = t_0 + s_1$), which is the first site visited by the path ω after reaching a new hyperplane. Similarly, for each $k = 1, \ldots, n$, we denote by ω_k the consistent piece of ω that connects (u_{k-1}, t_{k-1}) to (u_k, t_{k-1}), using only edges of $E(\Lambda_N^{t_{k-1}})$. Here, (u_{k-1}, t_{k-1}) is

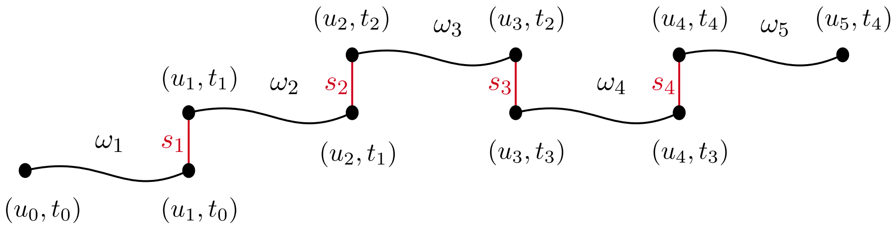

FIGURE 1. A sketch of a possible consistent path ω

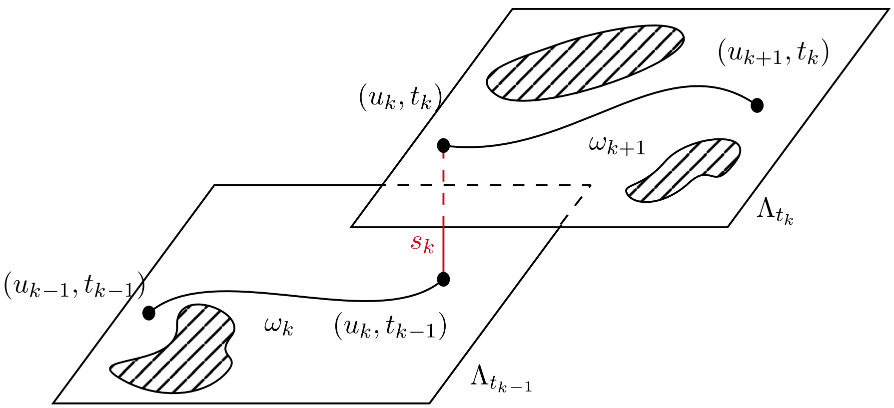

FIGURE 2. A sketch of a transition between hyperplanes

the first site of $\Lambda_N^{t_{k-1}}$ visited after the last vertical step s_{k-1} and (u_k, t_{k-1}) is the last site of this hyperplane visited by ω before it makes another jump, that is, before it reaches another hyperplane. Also, we denote by s_k the vertical jump, that is, the single bond connecting (u_k, t_{k-1}) to (u_k, t_k), the first site visited by the path ω in a hyperplane different from $\Lambda_N^{t_{k-1}}$. Finally, the last piece ω_{n+1} of the path ω connects (u_n, t_n) to $(u_{n+1}, t_n) = (u, t) = y$, using only edges of $E_N^{t_n}$. Note that $t_k = t_0 + \sum_{j=1}^{k} s_j$, for any $k = 1, \ldots, n$. See Fig. 1 for a sketch of this construction.

We stress that since ω is consistent, each one of its pieces ω_i is also consistent. Let us set $F_1 = \emptyset$ and, for $k = 2, \ldots, n+1$, we set $F_k = \omega_1 \circ s_1 \circ \cdots \circ \omega_{k-1} \circ s_{k-1}$ so that F_k^* is the set of edges of Λ_N canceled by the steps preceding ω_k and s_k.

By definition, the piece ω_k of the path ω is in the d-dimensional hypercube $\Lambda_N^{t_{k-1}}$. This hypercube may have already been visited by some piece ω_i of the path ω with $i < k-1$ (e.g., in Fig. 1, ω_4 is in the same hyperplane as ω_2). Since the path ω is consistent, ω_k must avoid edges of the set F_k. Therefore, ω_k is a consistent path which is a subset of $E_N^{t_{k-1}} \backslash F_k$, and we denote by C_k the set of all such paths with these properties (Fig. 2).

Now, for $n = 0, 1, 2, \ldots$, set $\mathcal{U}_n = (u_1, ..., u_{n+1})$ and $\mathcal{S}_n = (s_1, ..., s_n)$, with the convention that $\mathcal{S}_0 = \emptyset$. Then, we can write

$$\langle \sigma_x \sigma_y \rangle_{\Lambda_N} = \sum_{n \geq 0} \sum_{\substack{\mathcal{U}_n, \mathcal{S}_n \\ u_{n+1}=u \\ t_n = t}} \sum_{\omega_1 \in C_1} \cdots \sum_{\omega_{n+1} \in C_{n+1}} \rho_{E_N}(\omega),$$

with ω given by (10). Summing over $y \in \Lambda_N$, we get

$$\sum_{y \in \Lambda_N} \langle \sigma_x \sigma_y \rangle_{\Lambda_N} = \sum_{n \geq 0} \sum_{\mathcal{U}_n, \mathcal{S}_n} \sum_{\omega_1 \in C_1} \cdots \sum_{\omega_{n+1} \in C_{n+1}} \rho_{E_N}(\omega). \tag{11}$$

By Property **b)** given at the beginning of this section, we get

$$\rho_{E_N}(\omega) = \rho_{E_N - F_{n+1}}(\omega_{n+1}) \prod_{k=1}^{n} \rho_{E_N - F_k}(\omega_k) \rho_{E_N - F_k^*}(s_k),$$

where we recall $F_1 := \varnothing$.

Now, using the bound (8), and recalling that s_k is a single vertical edge (and thus with $J_{s_k} = J_s$), it holds that

$$\rho_{E_N - F_k^*}(s_k) \leq \tan\, h(J_s),$$

for any $k = 1, \cdots, n$. Therefore,

$$\rho_{E_N}(\omega) \leq [\tan\, h(J_s)]^n \prod_{k=1}^{n+1} \left[\rho_{E_N - F_k}(\omega_k)\right].$$

Observe that $E_N^{t_{k-1}} - F_k \subset E_N - F_k$. Moreover, since ω is consistent, ω_k only uses edges of $E_N^{t_{k-1}} - F_k$. Hence, we can apply Property **a)** and inequality (9) to obtain

$$\rho_{E_N - F_k}(\omega_k) \leq \rho_{E_N^{t_{k-1}} - F_k}(\omega_k),$$

for any $k = 1 \cdots, n+1$, yielding

$$\rho_{E_N}(\omega) \leq [\tan\, h(J_s)]^n \prod_{k=1}^{n+1} \rho_{E_N^{t_{k-1}} - F_k}(\omega_k). \tag{12}$$

Plugging (12) in (11), we write

$$\sum_{y \in \Lambda_N} \langle \sigma_x \sigma_y \rangle_{\Lambda_N} \leq \sum_{n \geq 0} [\tan\, h(J_s)]^n \sum_{\mathcal{U}_n, \mathcal{S}_n} S_1 \cdot \ldots \cdot S_{n+1},$$

where, for $k = 1, ..., n+1$, we have set

$$S_k = S_k(u_{k-1}, u_k, t_{k-1}) = \sum_{\omega_k \in C_k} \rho_{\Lambda_N^{t_{k-1}} - F_k}(\omega_k).$$

Then, we can write

$$\sum_{\mathcal{U}_n, \mathcal{S}_n} S_1 \cdot \ldots \cdot S_{n+1} = \sum_{u_1} S_1 \sum_{s_1} \sum_{u_2} S_2 \cdots \sum_{u_{n-1}} S_{n-1} \sum_{s_{n-1}} \sum_{u_n} S_n \sum_{s_n} \sum_{u_{n+1}} S_{n+1}.$$

Observe now that, for any $k = 1, \cdots, n$,

$$
\begin{aligned}
S_k &= \sum_{\omega_k \in C_k} \rho_{\Lambda_N^{t_{k-1}} - F_k}(\omega_k) \\
&= \left\langle \sigma_{(u_{k-1},t_{k-1})} \sigma_{(u_k,t_{k-1})} \right\rangle_{\Lambda_N^{t_{k-1}} - F_k} \\
&\leq \left\langle \sigma_{(u_{k-1},t_{k-1})} \sigma_{(u_k,t_{k-1})} \right\rangle_{\Lambda_N^{t_{k-1}}} .
\end{aligned}
$$

where the last line follows by the GKS inequalities. Therefore, for any fixed $u_1, \ldots, u_n$, and any fixed $s_1, \ldots, s_{n-1}$, we get

$$
\begin{aligned}
\sum_{s_n} \sum_{u_{n+1}} S_{n+1} &\leq \sum_{s_n} \sum_{u_{n+1}} \left\langle \sigma_{(u_n,t_n)} \sigma_{(u_{n+1},t_n)} \right\rangle_{\Lambda_N^{t_n}} \\
&\leq \sum_{s_n} \sup_{u \in \Lambda_N^{t_n}} \sum_{u_{n+1}} \left\langle \sigma_{(u,t_n)} \sigma_{(u_{n+1},t_n)} \right\rangle_{\Lambda_N^{t_n}} \\
&= \sum_{s_n} \chi_{\Lambda_N^{t_n}}(J_d) \\
&= 2s \chi_{\Lambda_N^{t_n}}(J_d) \\
&= 2s \chi_{\Lambda_N^{0}}(J_d).
\end{aligned}
$$

where in the penultimate line we remind that that there are $2s$ possibilities for the choice of the vertical steps s_i. Proceeding iteratively for the sums $\sum_{s_{k-1}} \sum_{u_k} S_k$ (with $k = n, n-1, \ldots, 1$), and with the convention that $\sum_{s_0} = 1$, we obtain

$$
\sum_{\mathcal{U}_n, \mathcal{S}_n} S_1 \cdot \ldots \cdot S_{n+1} \leq (2s)^n [\chi_{\Lambda_N^0}(J_d)]^{n+1},
$$

whence, for any $x \in \Lambda_N$,

$$
\sum_{y \in \Lambda_N} \langle \sigma_x \sigma_y \rangle_{\Lambda_N} \leq \sum_{n \geq 0} (2s \tan\, h(J_s))^n [\chi_{\Lambda_N^0}(J_d)]^{n+1}.
$$

Finally, taking the limit $N \to \infty$ and recalling the definitions of $\chi_{d+s}(J_d, J_s)$ and $\chi_d(J_d)$ given in (2) and (4), respectively, we get

$$
\chi_{d+s}(J_d, J_s) \leq \sum_{n \geq 0} (2s \tan\, h(J_s))^n [\chi_d(J_d)]^{n+1}.
$$

The r.h.s. of the inequality above is finite provided that

$$
\tan\, h(J_s) < \frac{1}{2s \chi_d(J_d)},
$$

and thus, the proof of Theorem 1 is concluded.

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declaration

Conflict of interest The authors have no conflict of interest to declare. All co-authors have seen and agree with the contents of the manuscript, and there is no financial interest to report. We certify that the submission is original work and is not under review at any other publication.

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