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Renormalization of Higher Currents of the Sine-Gordon Model in pAQFT

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Abstract. In this paper, we show that the higher currents of the sine-Gordon model are super-renormalizable by power counting in the framework of pAQFT. First we obtain closed recursive formulas for the higher currents in the classical theory and introduce a suitable notion of degree for their components. We then move to the pAQFT setting, and by means of some technical results, we compute explicit formulas for the unrenormalized interacting currents. Finally, we perform what we call the piecewise renormalization of the interacting higher currents, showing that the renormalization process involves a number of steps which is bounded by the degree of the classical conserved currents.

1. Introduction

This paper is the first step of a bigger project, which aims at understanding to what extent features of the conservation laws of classical systems are preserved when considering the corresponding quantum counterpart. More specifically, the focus of this paper is on the two-dimensional massless sine-Gordon model, both from the point of view of classical field theory and of perturbative algebraic quantum field theory (pAQFT for short).

During its long history, the sine-Gordon model has been keeping exhibiting a remarkable richness of properties. As a classical relativistic nonlinear scalar field theory, it was found to be an example of integrable system. This encompasses features like: existence of an infinite number of solutions to the sine-Gordon equation (see [1]), related by the Bäcklund transformations, and existence of an infinite number of conserved higher currents, which moreover form a commutative algebra with respect to the Peierl's bracket (see [2]). Particularly relevant to our purpose is their interpretation in terms of Noether's theorem, proposed in [3] and [4].

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As a quantum physical system, the sine-Gordon model admits a non-trivial scattering theory. In recent years, it has revealed remarkable features also in the context of pAQFT. In particular, as shown in [5] and [6], the scattering matrix of the two-dimensional massless sine-Gordon model in Minkowski signature was explicitly constructed and its summability proved, building partially on older results in Euclidean signature (e.g. [7]). These results represent the starting point of this work, which aims at investigating the renormalization properties of the higher currents in the framework of pAQFT.

We adopt the Epstein–Glaser point of view on renormalization and prove, as main result, that the components of the higher currents are super-renormalizable by power counting in pAQFT (see [8]).

We remark that, compared to other approaches, in our setting we do not need Fock techniques and we are hence not concerned with Fock space representations issues.

We also point out that our argument follows from well-known results on scaling-degree-preserving extensions of distributions [9–12] and on a notion of degree that we introduce based on the concrete expressions of the higher currents, which gives a bound on the number of counterterms necessary in the renormalization process. Unlike other renormalization techniques though, we do not compute the explicit counterterms.

However, we believe that the notations and technical results that we introduce along the way might represent a good foundation for further investigations of the summability and convergence properties of the renormalized interacting higher currents. For a discussion of the renormalizability, summability, conservation and other properties of the first of the higher conserved currents of the sine-Gordon model, namely its stress–energy tensor, where the counterterms are explicitly computed, we refer to [13] and [14].

The paper is organized as follows. Section 2 is devoted to the classical theory of the sine-Gordon model. We generalize the work done in [4] to the sine-Gordon with coupling constant (so that the setting considered in [4] is recovered as a special case for the value of the coupling constant $a = 1$), and moreover, we obtain explicit expressions for the components of the higher conserved currents. Along the way, we also introduce a notion of degree that will reveal to be crucial in the discussion of the renormalization of the currents in pAQFT.

In Sect. 3, we prove some technical results, on the star products and on the time-ordered products of fields with specific properties, that allow us to find closed and explicit expressions for the unrenormalized time-ordered products and the retarded components of the currents.

Finally, in Sect. 4 we show the renormalizability of the components of the conserved currents. We do this in three steps: first we further expand the unrenormalized expressions of the retarded components to their very elementary parts, then we piecewise renormalize the elementary parts separately and in the end we show that reassembling the piecewise renormalized parts all together gives a well-defined renormalized version of the retarded components of the currents.

2. Conserved Currents in the Classical Theory

In this section, we explain how the higher conserved currents for the classical sine-Gordon model can be obtained. Conceptually, we follow the same passages as in [4]. However, we extend all the definitions (given there only for the standard sine-Gordon model) to the general case of the sine-Gordon model with coupling constant (in the following referred to as the general sine-Gordon model or simply as the sine-Gordon model). Hence, also our results are more general and, moreover, we derive explicit recursive formulas for the quantities involved.

Let us start introducing some basic notions. The sine-Gordon model is a massless relativistic nonlinear scalar field theory. The rôle of spacetime is played by the two-dimensional Minkowski space $\mathbb{M}_2$. The configuration bundle of the theory is the trivial bundle $\mathbb{M}_2 \times \mathbb{R} \longrightarrow \mathbb{M}_2$. Configurations are sections of this bundle, namely functions $\varphi \in C^\infty(\mathbb{M}_2)$. Adopting Cartesian coordinates $(x^0 =: t, x^1 =: \vec{x})$ on $\mathbb{M}_2$, with Minkowski metric $\eta = \mathrm{diag}(-1, 1)$, the Lagrangian of the sine-Gordon model is written as:

$$L(\varphi)\,\mathrm{d}t \wedge \mathrm{d}\vec{x} = (L_0 + L_{\mathrm{int}})\,\mathrm{d}t \wedge \mathrm{d}\vec{x} = \Big(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \cos(a\varphi)\Big)\,\mathrm{d}t \wedge \mathrm{d}\vec{x}, \qquad (1)$$

where the parameter $a > 0$ is called coupling constant. The corresponding Euler–Lagrange equation, also called sine-Gordon equation, is

$$-\Box\varphi + a\sin(a\varphi) = \partial_t^2\varphi - \partial_{\vec{x}}^2\varphi + a\sin(a\varphi) = 0. \qquad (2)$$

In the sequel, we will always work in another system of coordinates, which turns out to be particularly useful in the description of the conservation laws of the sine-Gordon model. Light-cone coordinates (τ, ξ) are defined by:

$$\tau = \frac{1}{2}(\vec{x} + t), \qquad \xi = \frac{1}{2}(\vec{x} - t). \qquad (3)$$

The sine-Gordon Lagrangian in light-cone coordinates becomes

$$L(\varphi)\,\mathrm{d}\tau \wedge \mathrm{d}\xi = \Big(\frac{1}{2}\varphi_\xi\varphi_\tau + \cos(a\varphi)\Big)\,\mathrm{d}\tau \wedge \mathrm{d}\xi, \qquad (4)$$

and the sine-Gordon equation is

$$\varphi_{\xi\tau} - a\sin(a\varphi) = 0, \qquad (5)$$

where we also adopt the convention that subscripts $_\tau$ and $_\xi$ indicate partial derivation w.r.t. the corresponding coordinate.

Remark 2.0.1. The so-called standard sine-Gordon model, which is more often treated in the literature, for example, also in [4], is the special case for $a = 1$.

2.1. Extended Bäcklund Transformations

Extended Bäcklund transformations are defined in [4] for the standard sine-Gordon model. We extend the definition to the general sine-Gordon model by introducing also a dependence on the coupling constant $a > 0$.

Definition 2.1.1. We say that the configuration $\varphi' \in C^\infty(\mathbb{M}_2)$ is obtained from a given configuration $\varphi \in C^\infty(\mathbb{M}_2)$ by an extended Bäcklund transformation $\hat{B}_\alpha$ of parameter $\alpha \in \mathbb{R}$, in notation $\varphi' = \hat{B}_\alpha\varphi$, if φ' satisfies the following parametric PDE:

$$\frac{1}{2}(\varphi' + \varphi)_\xi = \frac{1}{\alpha}\sin\left[\frac{1}{2}a(\varphi' - \varphi)\right] \tag{6}$$

Assuming that φ' admits a power series expansion in the parameter α, we write:

$$\varphi' = \sum_{\nu=0}^{\infty} A_\nu[a,\varphi]\alpha^\nu, \tag{7}$$

where the coefficients A_ν depend on both the coupling constant a and the initial configuration φ. We can now substitute the power series expansion (7) in Eq. (6), using also the power series expansion of sine, and compare order by order in α. Omitting the dependence on a and φ, we obtain the following expressions for the first coefficients A_ν:

$$A_0 = \varphi, \qquad A_1 = \frac{2}{a}\varphi_\xi, \qquad A_2 = \frac{2}{a^2}\varphi_{\xi\xi}. \tag{8}$$

Carrying out a detailed study of Eq. (6), it is possible to obtain an explicit recursive formula for the higher coefficients A_ν.

Proposition 2.1.1. *For $\nu \geq 2$, the following recursive formula holds:*

$$A_{\nu+1} = \sum_{\beta=0}^{[\frac{\nu}{2}]-1} (-1)^\beta \left(\frac{1}{2}a\right)^{2(\beta+1)} \sum_{\substack{n_0,\dots,n_{\nu-2-2\beta}\geq 0 \\ n_0+\cdots+n_{\nu-2-2\beta}=2\beta+3 \\ \sum_{i=1}^{\nu-2-2\beta} i\cdot n_i=\nu-2-2\beta}} \frac{A_1^{n_0}\cdots A_{\nu-1-2\beta}^{n_{\nu-2-2\beta}}}{n_0!\cdots n_{\nu-2-2\beta}!} + \frac{1}{a}A_{\nu,\xi}, \tag{9}$$

where $[\frac{\nu}{2}]$ denotes the integer part of $\frac{\nu}{2}$.

Proof. The proof is given in Appendix A. □

The first coefficients obtained using formula (9) have the form:

$$\begin{aligned} A_3 &= \frac{2}{a^3}\varphi_{\xi\xi\xi} + \frac{1}{3a}\varphi_\xi^3, \\ A_4 &= \frac{2}{a^4}\varphi_{4\xi} + \frac{2}{a^2}\varphi_\xi^2\varphi_{\xi\xi}. \end{aligned} \tag{10}$$

Remark 2.1.1. We point out two direct consequences of Proposition 2.1.1:

(i) From formula (9) it follows that the coefficients A_ν are all polynomials in the derivatives of the configuration φ with respect only to the light-cone coordinate ξ.

(ii) As a consistency check, the expressions for the coefficients A_ν presented in [4] are recovered from our expressions setting $a = 1$.

We introduce here a notion that turns out to be crucial for the subsequent discussion of the renormalization of the higher currents in pAQFT.

Definition 2.1.2. Consider a configuration $\varphi \in C^\infty(\mathbb{M}_2)$. We assign a degree to its k-th derivative with respect to the light-cone coordinate ξ, by:

$$\deg(\varphi_{k\xi}) = k, \qquad \forall k \in \mathbb{N}.$$

We extend this definition to monomials in the derivatives of φ by additivity:

$$\deg(\varphi_{k_1\xi}\varphi_{k_2\xi}\dots\varphi_{k_N\xi}) = k_1 + k_2 + \dots + k_N.$$

We say that a polynomial in the derivatives of φ is homogeneous of degree d if all its monomial terms have degree d.

Proposition 2.1.2. *For every $\nu \geq 0$, the coefficient A_ν is homogeneous of degree equal to ν.*

Proof. The claim is trivial for $A_0 = \varphi$, $A_1 = \frac{2}{a}\varphi_\xi$ and $A_2 = \frac{2}{a^2}\varphi_{\xi\xi}$. For $\nu \geq 3$, we proceed by induction. For $\nu = 3$, using formulas (10), we have

$$\deg(A_3) = \deg\Big(\frac{2}{a^3}\varphi_{\xi\xi\xi} + \frac{1}{3a}\varphi_\xi^3\Big) = 3.$$

Now suppose the claim is true for $\nu \leq N$. The coefficient A_{N+1} is given by formula (9). The first term is $\frac{1}{a}A_{N,\xi}$ which, due to the additional derivative w.r.t. ξ, has degree $N+1$. The other terms are given by products

$$A_1^{n_0}\cdots A_{N-1-2\beta}^{n_{N-2-2\beta}}, \tag{11}$$

with the conditions on the indexes $n_0, \dots, n_{N-2-2\beta}$:

$$\begin{aligned} n_0 + \dots + n_{N-2-2\beta} &= 2\beta + 3 \\ 1\cdot n_1 + \dots + (N-2-2\beta)\cdot n_{N-2-2\beta} &= N-2-2\beta \end{aligned} \tag{12}$$

From these conditions and additivity of the degree, it follows that

$$\begin{aligned} \deg\big(A_1^{n_0}\cdots A_{N-1-2\beta}^{n_{N-2-2\beta}}\big) &= 1\cdot n_0 + \dots + (N-1-2\beta)\cdot n_{N-2-2\beta} \\ &= n_0 + n_1 + \dots + n_{N-2-2\beta} \\ &\quad + n_1 + \dots + (N-2-2\beta)\cdot n_{N-2-2\beta} \\ &= 2\beta + 3 + N - 2 - 2\beta \\ &= N+1. \end{aligned} \tag{13}$$

□

2.2. The Higher Conserved Currents

First, we restrict on-shell; namely, we assume that φ is a solution of the sine-Gordon equation (5). In [4], it is shown that the one-parameter family of 1-forms $\tilde{s}^{(\alpha)} = -\tilde{s}_1^{(\alpha)}\mathrm{d}\tau + \tilde{s}_2^{(\alpha)}\mathrm{d}\xi$, with components

$$\begin{aligned} \tilde{s}_1^{(\alpha)} &= \cos\Big[\frac{1}{2}(\varphi + \tilde{B}_{-\alpha}\varphi)\Big] + \cos\Big[\frac{1}{2}(\varphi + \tilde{B}_\alpha\varphi)\Big] \\ \tilde{s}_2^{(\alpha)} &= \frac{1}{\alpha^2}\Big\{2 - \cos\Big[\frac{1}{2}(\varphi - \tilde{B}_{-\alpha}\varphi)\Big] - \cos\Big[\frac{1}{2}(\varphi - \tilde{B}_\alpha\varphi)\Big]\Big\}, \end{aligned} \tag{14}$$

where $\tilde{B}_\alpha$ are the extended Bäcklund transformations for the standard sine-Gordon model (see Remark 2.0.1), forms a family of on-shell conserved currents for the standard sine-Gordon model. It turns out that the correct adaptation of these formulas to the more general case of the sine-Gordon model (with parameter $a > 0$) yields the following one-parameter (in $\alpha \in \mathbb{R}$) family of on-shell conserved currents $s^{(\alpha)} = -s_1^{(\alpha)}\mathrm{d}\tau + s_2^{(\alpha)}\mathrm{d}\xi$, with components

$$s_1^{(\alpha)} = \cos\left[\frac{1}{2}a(\varphi + \hat{B}_{-\alpha}\varphi)\right] + \cos\left[\frac{1}{2}a(\varphi + \hat{B}_{\alpha}\varphi)\right] \tag{15a}$$

$$s_2^{(\alpha)} = \frac{1}{\alpha^2}\left\{2 - \cos\left[\frac{1}{2}a(\varphi - \hat{B}_{-\alpha}\varphi)\right] - \cos\left[\frac{1}{2}a(\varphi - \hat{B}_{\alpha}\varphi)\right]\right\}. \tag{15b}$$

We see immediately that the previous formulas for the components of the conserved currents $\tilde{s}_1^{(\alpha)}$, $\tilde{s}_2^{(\alpha)}$ are obtained from formulas (15a) and (15b) simply by setting $a = 1$. Using our definition of extended Bäcklund transformations, in particular Eq. (6), and using also the further relation

$$\frac{1}{2}(\hat{B}_{\alpha}\varphi - \varphi)_{\tau} = \alpha \sin\left[\frac{1}{2}a(\hat{B}_{\alpha}\varphi + \varphi)\right], \tag{16}$$

which holds only on-shell (cfr. [4]), it is easy to check that the currents $s^{(\alpha)}$ satisfy indeed a null-divergence, or continuity, equation, namely an on-shell conservation law:

$$\mathrm{div}(s^{(\alpha)}) := \partial_{\xi}s_1^{(\alpha)} + \partial_{\tau}s_2^{(\alpha)} = 0, \quad \forall\alpha \in \mathbb{R}. \tag{17}$$

Using formula (7) to write $\hat{B}_{\pm\alpha}\varphi$ and the power series expansion of cosine, we can expand also $s_1^{(\alpha)}$ and $s_2^{(\alpha)}$ as power series in α. Since formulas (15a) and (15b) are symmetric in α, only even powers will appear. We denote the results of the power series expansions by:

$$s_1^{(\alpha)} = \sum_{N=0}^{\infty} s_1^N \alpha^{2N}, \quad s_2^{(\alpha)} = \sum_{N=0}^{\infty} s_2^N \alpha^{2N}. \tag{18}$$

For every order in α a conserved current is obtained, which we denote by

$$s^N = -s_1^N\mathrm{d}\tau + s_2^N\mathrm{d}\xi. \tag{19}$$

Proposition 2.2.1. *The components s_1^N and s_2^N of the conserved currents have the following form:*

$$s_1^N = \cos(a\varphi)\left[2\sum_{\beta=1}^{N}(-1)^{\beta}\left(\frac{1}{2}a\right)^{2\beta}\sum_{\substack{n_1,\ldots,n_{2N}\geq 0\\ n_1+\cdots+n_{2N}=2\beta\\ \sum_{i=1}^{2N} i\cdot n_i=2N}}\frac{A_1^{n_1}\cdots A_{2N}^{n_{2N}}}{n_1!\cdots n_{2N}!}\right]$$
$$+\sin(a\varphi)\left[2\sum_{\beta=0}^{N-1}(-1)^{\beta+1}\left(\frac{1}{2}a\right)^{2\beta+1}\sum_{\substack{n_1,\ldots,n_{2N}\geq 0\\ n_1+\cdots+n_{2N}=2\beta+1\\ \sum_{i=1}^{2N} i\cdot n_i=2N}}\frac{A_1^{n_1}\cdots A_{2N}^{n_{2N}}}{n_1!\cdots n_{2N}!}\right], \tag{20}$$

where the coefficient of $\sin(a\varphi)$ *is defined only for* $N \geq 1$*, and*

$$s_2^N = 2\sum_{\mu=0}^{N}(-1)^{\mu}\Big(\frac{1}{2}a\Big)^{2(\mu+1)} \sum_{\substack{n_0,\dots,n_{2(N-\mu)}\geq 0\\ n_0+\dots+n_{2(N-\mu)}=2(\mu+1)\\ \sum_{i=1}^{2(N-\mu)} i\cdot n_i=2(N-\mu)}} \frac{A_1^{n_0}\cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}}{n_0!\cdots n_{2(N-\mu)}!}. \tag{21}$$

Proof. The proof is given in Appendix B. □

The expressions of the first components, from formulas (20) and (21), are:

$$\begin{cases} s_1^0 = 2\cos(a\varphi), \\ s_2^0 = \varphi_\xi^2, \end{cases} \qquad \begin{cases} s_1^1 = -\varphi_\xi^2\cos(a\varphi) - \frac{2}{a}\varphi_{\xi\xi}\sin(a\varphi), \\ s_2^1 = \frac{1}{4}\varphi_\xi^4 + \frac{2}{a^2}\varphi_\xi\varphi_{\xi\xi\xi} + \frac{1}{a^2}\varphi_{\xi\xi}^2. \end{cases} \tag{22}$$

Remark 2.2.1. Considering point (i) of Remark 2.1.1 together with formulas (20) and (21), it follows that the coefficients of $\cos(a\varphi)$ and of $\sin(a\varphi)$ in the expression of s_1^N, and the second components s_2^N are all polynomials in the derivatives of the configuration φ with respect to the coordinate ξ.
Again, as a consistency check, we have that we recover the expressions for the components of the conserved currents given in [4] from our expressions setting $a = 1$.

To conclude this section, we study the properties of the degree of the components of the higher conserved currents.

Proposition 2.2.2. *Assign by convention degree equal to* 0 *to* $\cos(a\varphi)$ *and* $\sin(a\varphi)$*. Then, we have that:*

- *The first component* s_1^N *of the conserved current* s^N *is homogeneous of degree equal to* $2N$*.*
- *The second component* s_2^N *of the conserved current* s^N *is homogeneous of degree equal to* $2(N+1)$*.*

Proof. The first claim follows from the observation that the coefficients of $\cos(a\varphi)$ and $\sin(a\varphi)$, in formula (20), are given by sums of products of the form

$$A_1^{n_1}\cdots A_{2N}^{n_{2N}}, \tag{23}$$

with the condition $n_1 + \cdots + 2N\cdot n_{2N} = 2N$. All these products have degree

$$\deg(A_1^{n_1}\dots A_{2N}^{n_{2N}}) = 1\cdot n_1 + \cdots + 2N\cdot n_{2N} = 2N. \tag{24}$$

As for the second claim, from formula (21) we have that s_2^N is given by a finite sum of products of the form

$$A_1^{n_0}\cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}, \tag{25}$$

with the conditions

$$\begin{aligned} n_0 + \cdots + n_{2(N-\mu)} &= 2(\mu+1) \\ 1\cdot n_1 + \dots 2(N-\mu)\cdot n_{2(N-\mu)} &= 2(N-\mu). \end{aligned} \tag{26}$$

The degree of each one of these products is

$$\begin{aligned}\deg\big(A_1^{n_0}\dots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}\big) &= n_0 + \dots + (2(N-\mu)+1)n_{2(N-\mu)} \\ &= n_0 + n_1 + \dots + n_{2(N-\mu)} \\ &\quad + n_1 + \dots + 2(N-\mu)n_{2(N-\mu)} \\ &= 2(\mu+1) + 2(N-\mu) \\ &= 2(N+1).\end{aligned} \tag{27}$$

□

3. Unrenormalized Expressions for the Interacting Higher Currents in pAQFT

Before discussing the technical details, we recall some basic notions regarding the framework of pAQFT. In particular, we restrict to the specific setting of the sine-Gordon model (for more extensive and general treatments, we refer for example to [9,10,15,16]).

Fields, also called observables, are described by a class of smooth functionals $F\colon C^\infty(\mathbb{M}_2) \to \mathbb{C}$, called microcausal functionals and denoted by $\mathcal{F}_{\mu c}$. More generally, in adherence with the perturbative approach, one considers formal power series in $\hbar$ with coefficients in microcausal functionals, denoted by $\mathcal{F}_{\mu c}[[\hbar]]$.

The space of fields is endowed with a non-commutative product

$$F \star G = \sum_{n=0}^{\infty} \frac{\hbar}{n!} \left\langle \Big(\frac{\delta^n}{\delta\varphi^n} F\Big),\, (W)^n\, \frac{\delta^n}{\delta\varphi^n} G \right\rangle, \qquad F, G \in \mathcal{F}_{\mu c}[[\hbar]], \tag{28}$$

where $\frac{\delta^n}{\delta\varphi^n}F$, $\frac{\delta^n}{\delta\varphi^n}G$ denote the n-th functional derivatives of the fields F and G and the bidistribution W denotes a choice of Wightman two-point function. This product defines on $\mathcal{F}_{\mu c}[[\hbar]]$ the structure of a Poisson $*$-algebra, where the Poisson bracket is given by the commutator with respect to the star product and the involution * by complex conjugation. The resulting Poisson $*$-algebra $(\mathcal{F}_{\mu c}[[\hbar]], \star, [\cdot,\cdot]_\star, ^*)$ is called the algebra of free fields.

While the algebra of free fields represents the model algebra for observables, the physical concept of evolution is encoded by the notion of interacting fields. Roughly speaking, considering interacting fields represents the quantum equivalent of the classical restriction to on-shell fields, namely observables evaluated only on configurations that are solutions of the Euler–Lagrange equations of the theory. This is precisely the case for the higher currents of the sine-Gordon model, which are conserved only when evaluated on a configuration that is a solution of the sine-Gordon equation.

Physical interactions are modelled by a subclass of functionals called local functionals, denoted by $\mathcal{F}_{\mathrm{loc}} \subset \mathcal{F}_{\mu c}$, characterized by the property that their functional derivatives of every order are distributions with compact support contained in the diagonal of the appropriate number of copies of spacetime. In order to construct interacting fields, first the time-ordered product is defined as the following commutative product of local fields $F_1, F_2 \in \mathcal{F}_{\mathrm{loc}}[[\hbar]]$:

$$T_2\big(F_1 \otimes F_2\big) = F_1 \star_F F_2 = \sum_{n=0}^{\infty} \frac{\hbar}{n!} \left\langle \Big(\frac{\delta^n}{\delta\varphi^n} F_1\Big), (\Delta^F)^n \frac{\delta^n}{\delta\varphi^n} F_2 \right\rangle, \tag{29}$$

where Δ^F is the unique Feynman propagator (we remark that in general, on curved spacetimes, the Feynman propagator is not unique). The time-ordered product of order l is then:

$$T_l\big(F_1 \otimes \cdots \otimes F_l\big) := F_1 \star_F \cdots \star_F F_l, \qquad F_1, \ldots, F_l \in \mathcal{F}_{\mathrm{loc}}[[\hbar]]. \tag{30}$$

In the formula for the time-ordered product, the Feynman propagator is intended as a symmetric bidistribution defined on $\mathbb{M}_2^2$. Its wavefront set is such that the product of Feynman propagators can be defined using Hörmander's sufficient criterion [11] only outside of the diagonal. This implies that the time-ordered products $T_l\big(F_1(x_1) \otimes \cdots \otimes F_l(x_l)\big)$, seen as observable-valued distributions, can be defined by Hörmander's sufficient criterion only on a subset of $\mathbb{M}_2^l$ denoted by

$$\check{\mathbb{M}}_2^l := \left\{(x_1, \ldots, x_l) \in \mathbb{M}_2^l | x_i \neq x_j \quad \forall 1 \leq i < j \leq l\right\}. \tag{31}$$

The renormalization problem in pAQFT is the problem of extending these products to distributions well-defined on the whole $\mathbb{M}_2^l$. This can be done by combining a study of the properties of their wavefront set with the notion of Steinmann scaling degree [12]. The extension process is not always unique. The ambiguities are represented by the possibility to add derivatives of Dirac deltas up to a certain order. This renormalization framework goes under the name of Epstein–Glaser renormalization. We also remark that the pAQFT setting is defined not only on Minkowski spacetime, but applies more generally to globally hyperbolic spacetimes.

The main ingredient of the interacting picture in pAQFT is the scattering matrix S, defined as the generating function of the time-ordered products:

$$S(F) := T\big(e_{\otimes}^{iF/\hbar}\big) := \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{i}{\hbar}\Big)^n T_n\big(F^{\otimes n}\big), \qquad F \in \mathcal{F}_{\mathrm{loc}}[[\hbar]], \tag{32}$$

where $T_0\big(F\big) = 1$ and $T_1\big(F\big) = F$. Interacting fields $(F)_{\mathrm{int}}$ can then be constructed by means of the Bogoliubov formula[1]:

$$(F)_{\mathrm{int}} := -i\hbar \frac{\mathrm{d}}{\mathrm{d}\lambda} \bigg(S(L_{\mathrm{int}})^{\star -1} \star S(L_{\mathrm{int}} + \lambda F) \bigg)\bigg|_{\lambda=0}, \tag{33}$$

where L_{int} is the interaction Lagrangian of the system under consideration. In particular for the sine-Gordon model, we have $L_{\mathrm{int}} = \cos(a\varphi)$.

[1] The idea of using Bogoliubov formula to compute the interacting components of the higher currents for the sine-Gordon model was presented by the author at the 46th LQP Workshop, Erlangen, 24–25 June 2022. Other approaches are possible. In particular, for considerations on the stress–energy tensor in a different framework, see [13] and [14].

The result has to be intended as a formal power series in $\hbar$ (and in the coupling constant contained in L_{int}) and is denoted by:

$$(F)_{\text{int}} = \sum_{n=0}^{\infty} \frac{1}{n!} R_n\big(L_{\text{int}}^{\otimes n}, F\big). \tag{34}$$

The coefficients of the series are called retarded products and are given by

$$R_t\big(L_{\text{int}}^{\otimes t}, F\big) = \Big(\frac{i}{\hbar}\Big)^t \sum_{l=0}^{t} \binom{t}{l} (-1)^{t-l} \bar{T}_{t-l}\big(L_{\text{int}}^{\otimes(t-l)}\big) \star T_{l+1}\big(L_{\text{int}}^{\otimes l} \otimes F\big), \tag{35}$$

where $\bar{T}_n$ are the anti-chronological products, defined as the coefficients of the inverse (in the sense of formal power series) of the scattering matrix

$$S(F)^{\star -1} = \sum_{n=0}^{\infty} \frac{1}{n!} \Big(\frac{-i}{\hbar}\Big)^n \bar{T}_n\big(F^{\otimes n}\big). \tag{36}$$

3.1. Time-Ordered Products for Components s_2^N

We start the discussion with the components s_2^N because, as pointed out in Remark 2.2.1, these are polynomials in the derivatives of the configuration φ. Thanks to this fact, we can further manipulate the unrenormalized expression of the $(l+1)$-th time-ordered products occurring in formula (35).

Proposition 3.1.1. *The unrenormalized* $(l+1)$*-th time-ordered product for the components* s_2^N *can be written as a finite sum of terms:*

$$T_{l+1}\Big(L_{int}^{\otimes l} \otimes s_2^N\Big) = \sum_{j=0}^{2(N+1)} \hbar^j \sum_{\substack{j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}} \frac{1}{j_1! \dots j_l!} (\Delta^F)^j T_l\Big(\frac{\delta^{j_1}}{\delta\varphi^{j_1}} L_{int} \otimes \dots \otimes \frac{\delta^{j_l}}{\delta\varphi^{j_l}} L_{int}\Big) \frac{\delta^j}{\delta\varphi^j} s_2^N. \tag{37}$$

Proof. The result can be obtained using the formula for the time-ordered product from [5] and splitting the exponential in the following way:

$$\begin{aligned} T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes s_2^N\Big) &= \mu \circ e^{\hbar \sum_{1 \leq i < j \leq l+1} D_F^{ij}} \Big(L_{\text{int}}^{\otimes l} \otimes s_2^N\Big) \\ &= \mu \circ e^{\hbar \sum_{1 \leq i < j \leq l} D_F^{ij}} \circ e^{\hbar \sum_{i=1}^{l} D_F^{i\, l+1}} \Big(L_{\text{int}}^{\otimes l} \otimes s_2^N\Big), \end{aligned} \tag{38}$$

where

$$D_F^{ij} := \Big\langle \Delta^F, \frac{\delta}{\delta\varphi_i} \otimes \frac{\delta}{\delta\varphi_j} \Big\rangle \tag{39}$$

and the index i in $\frac{\delta}{\delta\varphi_i}$ means that the functional derivative is applied to the i-th term of the tensor product.

From Remark 2.2.1 and Proposition 2.2.2 it follows that, as a local field, s_2^N admits nonzero functional derivatives of order at most $2(N+1)$. Hence the second exponential series is in fact a finite sum:

$$e^{\hbar \sum_{i=1}^{l} D_F^{i\,l+1}} \left(L_{\text{int}}^{\otimes l} \otimes s_2^N\right) = \sum_{j=0}^{2(N+1)} \frac{\hbar^j}{j!} \left(\sum_{i=1}^{l} D_F^{i\,l+1}\right)^j \left(L_{\text{int}}^{\otimes l} \otimes s_2^N\right) \tag{40}$$

Expanding the operators

$$\left(\sum_{i=1}^{l} D_F^{i\,l+1}\right)^j = \langle (\Delta^F)^j, \left(\frac{\delta}{\delta\varphi_1} + \cdots + \frac{\delta}{\delta\varphi_l}\right)^j \otimes \frac{\delta^j}{\delta\varphi_{l+1}^j}\rangle, \tag{41}$$

using the multinomial formula

$$\left(\frac{\delta}{\delta\varphi_1} + \cdots + \frac{\delta}{\delta\varphi_l}\right)^j = \sum_{\substack{j_1,\ldots,j_l \geq 0 \\ j_1+\cdots+j_l=j}} \frac{j!}{j_1!\ldots j_l!} \prod_{t=1}^{l} \frac{\delta^{j_t}}{\delta\varphi_t^{j_t}}, \tag{42}$$

and carrying out the computations, we finally arrived at the desired result. □

3.2. Retarded Components s_2^N

We can use Proposition 3.1.1 to make more explicit the expression of the unrenormalized retarded product of order t for components s_2^N:

$$R_t\left(L_{\text{int}}^{\otimes t} \otimes s_2^N\right) = \left(\frac{i}{\hbar}\right)^t \sum_{l=0}^{t} \binom{t}{l} (-1)^{t-l} \bar{T}_{t-l}\left(L_{\text{int}}^{\otimes (t-l)}\right) \star T_{l+1}\left(L_{\text{int}}^{\otimes l} \otimes s_2^N\right). \tag{43}$$

First, we study the $\star$-products $\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes (t-l)}\right) \star T_{l+1}\left(L_{\text{int}}^{\otimes l} \otimes s_2^N\right)$ which, according to formula (37), are in turn composed by terms of the form

$$\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes (t-l)}\right) \star \left(T_l\left(\frac{\delta^{j_1}}{\delta\varphi^{j_1}} L_{\text{int}} \otimes \cdots \otimes \frac{\delta^{j_l}}{\delta\varphi^{j_l}} L_{\text{int}}\right) \frac{\delta^j}{\delta\varphi^j} s_2^N\right). \tag{44}$$

We can prove the following slightly more general technical result that applies in particular to formula (44).

Proposition 3.2.1. *Consider the star product of fields $A, B, C \in \mathcal{F}_{\mu c}[[\hbar]]$*

$$A \star (B\,C),$$

where C is such that $\exists\, c \in \mathbb{N}$ for which $\frac{\delta^i}{\delta\varphi^i} C = 0$ whenever $i > c$, while A and B can possibly admit nonzero functional derivatives of arbitrary order. Then, the product can be written in the form:

$$A \star (B\,C) = \sum_{k=0}^{c} \frac{\hbar^k}{k!} \left(\sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left(\frac{\delta^k}{\delta\varphi^k} \frac{\delta^n}{\delta\varphi^n} A\right) (W)^n \frac{\delta^n}{\delta\varphi^n} B\right) (W)^k \frac{\delta^k}{\delta\varphi^k} C. \tag{45}$$

Proof. The claim is obtained by explicit calculation, applying the Leibniz rule and exchanging the order of the summations as follows:

$$\begin{aligned} A\star(B\,C) &= \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\left(\frac{\delta^n}{\delta\varphi^n}A\right)(W)^n\frac{\delta^n}{\delta\varphi^n}(B\,C)\\ &= \sum_{n=0}^{\infty}\frac{\hbar^n}{n!}\left(\frac{\delta^n}{\delta\varphi^n}A\right)(W)^n\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\delta^{n-k}}{\delta\varphi^{n-k}}B\,\frac{\delta^k}{\delta\varphi^k}C\right)\\ &= \sum_{k=0}^{c}\left(\sum_{n=k}^{\infty}\frac{\hbar^n}{n!}\binom{n}{k}\left(\frac{\delta^n}{\delta\varphi^n}A\right)(W)^n\frac{\delta^{n-k}}{\delta\varphi^{n-k}}B\right)\frac{\delta^k}{\delta\varphi^k}C. \end{aligned} \tag{46}$$

Rescaling the index of the second summation $n\to n-k$, we conclude. □

Substituting formula (37) in the product $\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes(t-l)}\right)\star T_{l+1}\left(L_{\text{int}}^{\otimes l}\otimes s_2^N\right)$, we obtain, as noted above, a sum of products of the form

$$\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes(t-l)}\right)\star\left(T_l\left(\frac{\delta^{j_1}}{\delta\varphi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\varphi^{j_l}}L_{\text{int}}\right)\frac{\delta^j}{\delta\varphi^j}s_2^N\right). \tag{47}$$

Applying Proposition 3.2.1 to each one of these terms, setting $A=\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes(t-l)}\right)$, $B=T_l\left(\frac{\delta^{j_1}}{\delta\varphi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\varphi^{j_l}}L_{\text{int}}\right)$ and $C=\frac{\delta^j}{\delta\varphi^j}s_2^N$, we finally get:

$$\begin{aligned} &\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes(t-l)}\right)\star T_{l+1}\left(L_{\text{int}}^{\otimes l}\otimes s_2^N\right)\\ &\quad=\sum_{j=0}^{2(N+1)}\hbar^j\sum_{\substack{j_1,\dots,j_l\geq 0\\ j_1+\cdots+j_l=j}}\frac{1}{j_1!\dots j_l!}\sum_{i=0}^{2(N+1)-j}\frac{\hbar^i}{i!}\\ &\qquad\times\left(\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}\frac{\delta^i}{\delta\varphi^i}\frac{\delta^k}{\delta\varphi^k}\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes(t-l)}\right)(W)^k\frac{\delta^k}{\delta\varphi^k}T_l\left(\frac{\delta^{j_1}}{\delta\varphi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\varphi^{j_l}}L_{\text{int}}\right)\right)\\ &\qquad\times(\Delta^F)^j(W)^i\frac{\delta^i}{\delta\varphi^i}\frac{\delta^j}{\delta\varphi^j}s_2^N. \end{aligned} \tag{48}$$

We can now plug this equation in formula (43) and finally arrive at the explicit expression for the unrenormalized retarded components s_2^N:

$$\begin{aligned} &R_t\left(L_{\text{int}}^{\otimes t}\otimes s_2^N\right)\\ &\quad=\left(\frac{i}{\hbar}\right)^t\sum_{l=0}^{t}\binom{t}{l}(-1)^{t-l}\sum_{j=0}^{2(N+1)}\sum_{i=0}^{2(N+1)-j}\frac{\hbar^j\hbar^i}{i!}\sum_{\substack{j_1,\dots,j_l\geq 0\\ j_1+\cdots+j_l=j}}\frac{1}{j_1!\dots j_l!}\\ &\qquad\times\left(\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}\frac{\delta^i}{\delta\varphi^i}\frac{\delta^k}{\delta\varphi^k}\bar{T}_{t-l}\left(L_{\text{int}}^{\otimes(t-l)}\right)(W)^k\frac{\delta^k}{\delta\varphi^k}T_l\left(\frac{\delta^{j_1}}{\delta\varphi^{j_1}}L_{\text{int}}\otimes\cdots\otimes\frac{\delta^{j_l}}{\delta\varphi^{j_l}}L_{\text{int}}\right)\right)\\ &\qquad\times(\Delta^F)^j(W)^i\frac{\delta^i}{\delta\varphi^i}\frac{\delta^j}{\delta\varphi^j}s_2^N. \end{aligned} \tag{49}$$

3.3. Time-Ordered Products for Components s_1^N

We know from formula (21) that components s_1^N are given by the sum of a homogeneous part of degree $2N$ multiplied by $\cos(a\varphi)$ and another homogeneous part of degree $2N$ multiplied by $\sin(a\varphi)$. We rename the two homogeneous parts q_1^N and r_1^N, respectively, and write

$$s_1^N = \cos(a\varphi) q_1^N + \sin(a\varphi) r_1^N. \tag{50}$$

By linearity of the time-ordered products, we have

$$T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes s_1^N\Big) = T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\cos(a\varphi) q_1^N)\Big) + T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\sin(a\varphi) r_1^N)\Big). \tag{51}$$

The two terms on the right hand side can be treated exactly in the same way, so we discuss only the second one. In this case it is not any longer true that the field $\sin(a\varphi) r_1^N$ admits nonzero functional derivatives only up to a finite order, so we cannot directly apply Proposition 3.1.1. Nevertheless, we can prove a similar technical result.

Proposition 3.3.1. *Consider a time-ordered product of the form*

$$T_{l+1}\Big(A^{\otimes l} \otimes (B\, C)\Big), \qquad A, B, C \in \mathcal{F}_{\mu c}[[\hbar]],$$

where C is such that $\exists\, c \in \mathbb{N}$ for which $\frac{\delta^i}{\delta\varphi^i} C = 0$ whenever $i > c$, while A and B can possibly admit nonzero functional derivatives of arbitrary order. Then, the following equation holds:

$$\begin{aligned} &T_{l+1}\big(A^{\otimes l} \otimes (B\, C)\big) \\ &\quad = \sum_{i=0}^{c} \hbar^i (\Delta^F)^i \sum_{\substack{i_1,\dots,i_l \geq 0 \\ i_1+\dots+i_l = i}} \frac{1}{i_1! \dots i_l!} T_{l+1}\Big(\frac{\delta^{i_1}}{\delta\varphi^{i_1}} A \otimes \dots \otimes \frac{\delta^{i_l}}{\delta\varphi^{i_l}} A \otimes B\Big) \frac{\delta^i}{\delta\varphi^i} C. \end{aligned} \tag{52}$$

Proof. We start from the formula for the time-ordered products as in the proof of Proposition 3.1.1:

$$T_{l+1}\big(A^{\otimes l} \otimes (B\, C)\big) = \mu \circ e^{\hbar \sum_{1 \leq i<j \leq l} D_F^{ij}} \circ e^{\hbar \sum_{i=1}^{l} D_F^{i\, l+1}} \big(A^{\otimes l} \otimes (B\, C)\big). \tag{53}$$

The second exponential acts on the fields as:

$$\begin{aligned} &e^{\hbar \sum_{i=1}^{l} D_F^{i\, l+1}} \Big(A^{\otimes l} \otimes (B\, C)\Big) \\ &\quad = \sum_{j=0}^{\infty} \frac{\hbar^j}{j!} (\Delta^F)^j \left(\frac{\delta}{\delta\varphi_1} + \dots + \frac{\delta}{\delta\varphi_l}\right)^j \otimes \frac{\delta^j}{\delta\varphi_{l+1}^j} \Big(A^{\otimes l} \otimes (B\, C)\Big). \end{aligned} \tag{54}$$

Then, we proceed studying separately the derivatives of the product $\frac{\delta^j}{\delta\varphi_{l+1}^j}(B\, C)$. Applying the Leibniz rule, we have:

$$\sum_{j=0}^{\infty} \frac{\delta^j}{\delta\varphi_{l+1}^j} (B\, C) = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \binom{j}{k} \frac{\delta^{j-k}}{\delta\varphi_{l+1}^{j-k}} B \, \frac{\delta^k}{\delta\varphi_{l+1}^k} C. \tag{55}$$

We can rewrite the sums using indices k and $i = j - k$. Recalling also that by hypothesis, the field C admits nonzero derivatives only up to order c, we get:

$$\sum_{j=0}^{\infty} \frac{\delta^j}{\delta\varphi_{l+1}^j}(B\,C) = \sum_{k=0}^{c}\left(\sum_{i=0}^{\infty}\binom{i+k}{k}\frac{\delta^i}{\delta\varphi_{l+1}^i}B\right)\frac{\delta^k}{\delta\varphi_{l+1}^k}C. \tag{56}$$

Substituting in Eq. (54) and using the new indices i and k for the sums, we obtain:

$$\begin{aligned} &e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(A^{\otimes l}\otimes(B\,C)\right)\\ &\quad= \sum_{k=0}^{c}\sum_{i=0}^{\infty}\frac{(\hbar\Delta^F)^{i+k}}{(i+k)!}\binom{i+k}{k}\left(\frac{\delta}{\delta\varphi_1}+\cdots+\frac{\delta}{\delta\varphi_l}\right)^{i+k}(A^{\otimes l})\otimes\left(\frac{\delta^i}{\delta\varphi_{l+1}^i}B\frac{\delta^k}{\delta\varphi_{l+1}^k}C\right)\\ &\quad= \sum_{k=0}^{c}\frac{(\hbar\Delta^F)^k}{k!}\sum_{i=0}^{\infty}\frac{(\hbar\Delta^F)^i}{i!}\left(\frac{\delta}{\delta\varphi_1}+\cdots+\frac{\delta}{\delta\varphi_l}\right)^{i+k}(A^{\otimes l})\otimes\left(\frac{\delta^i}{\delta\varphi_{l+1}^i}B\frac{\delta^k}{\delta\varphi_{l+1}^k}C\right). \end{aligned}$$

We can expand the operators $\left(\frac{\delta}{\delta\varphi_1}+\cdots+\frac{\delta}{\delta\varphi_l}\right)^k$ using the multinomial formula. Substituting it in the formula above, we obtain:

$$\begin{aligned} &e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(A^{\otimes l}\otimes(B\,C)\right)\\ &\quad= \sum_{\substack{k=0\\ k_1,\ldots,k_l\geq 0\\ k_1+\cdots+k_l=k}}^{c}\frac{(\hbar\Delta^F)^k}{k_1!\cdots k_l!}\sum_{i=0}^{\infty}\frac{(\hbar\Delta)^i}{i!}\left(\frac{\delta}{\delta\varphi_1}+\cdots+\frac{\delta}{\delta\varphi_l}\right)^{i}\left(\frac{\delta^{k_1}}{\delta\varphi^{k_1}}A\otimes\cdots\otimes\frac{\delta^{k_l}}{\delta\varphi^{k_l}}A\right)\\ &\qquad\otimes\left(\frac{\delta^i}{\delta\varphi_{l+1}^i}B\frac{\delta^k}{\delta\varphi_{l+1}^k}C\right). \end{aligned} \tag{57}$$

It is now clear that the sum over the index i in the last formula corresponds to the exponential notation for the time-ordered product. Hence, we can write:

$$\begin{aligned} &e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(A^{\otimes l}\otimes(B\,C)\right)\\ &\quad= \sum_{\substack{k=0\\ k_1,\ldots,k_l\geq 0\\ k_1+\cdots+k_l=j}}^{c}\frac{(\hbar\Delta^F)^k}{k_1!\cdots k_l!}e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(\frac{\delta^{k_1}}{\delta\varphi^{k_1}}A\otimes\cdots\otimes\frac{\delta^{k_l}}{\delta\varphi^{k_l}}A\otimes B\right)\frac{\delta^k}{\delta\phi_{l+1}^k}C. \end{aligned}$$

Applying the remaining operators $\mu\circ e^{\hbar\sum_{1\leq i<j\leq l} D_F^{ij}}$, we finally arrive at:

$$\begin{aligned} \mu\circ e^{\hbar\sum_{1\leq i<j\leq l} D_F^{ij}}\circ e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(A^{\otimes l}\otimes(B\,C)\right) &= \sum_{\substack{k=0\\ k_1,\ldots,k_l\geq 0\\ k_1+\cdots+k_l=k}}^{c}\frac{(\hbar\Delta^F)^k}{k_1!\cdots k_l!}\\ &\times\mu\circ e^{\hbar\sum_{1\leq i<j\leq l} D_F^{ij}}\circ e^{\hbar\sum_{i=1}^{l} D_F^{i\,l+1}}\left(\frac{\delta^{k_1}}{\delta\varphi^{k_1}}A\otimes\cdots\otimes\frac{\delta^{k_l}}{\delta\varphi^{k_l}}A\otimes B\right)\frac{\delta^k}{\delta\varphi_{l+1}^k}C \end{aligned}$$

$$= \sum_{\substack{k=0 \\ k_1,\dots,k_l \geq 0 \\ k_1+\dots+k_l=k}}^{c} \frac{(\hbar\Delta^F)^k}{k_1! \cdots k_l!} T_{l+1}\left(\frac{\delta^{k_1}}{\delta\varphi^{k_1}} A \otimes \cdots \otimes \frac{\delta^{k_l}}{\delta\varphi^{k_l}} A \otimes B\right) \frac{\delta^k}{\delta\varphi^k} C.$$

□

In our case we have $A = L_{\text{int}}$, $B = \sin(a\varphi)$ and $C = r_1^N$. From Remark 2.2.1 and Proposition 2.2.1, we know that functional derivatives of r_1^N of order greater than $2N$ are all zero. We can hence apply Proposition 3.3.1 and obtain:

$$\begin{aligned} &T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\sin(a\varphi) \cdot r_1^N)\Big) \\ &= \sum_{i=0}^{2N} \hbar^i (\Delta^F)^i \sum_{\substack{i_1,\dots,i_l \geq 0 \\ i_1+\dots+i_l=i}} \frac{1}{i_1! \dots i_l!} T_{l+1}\Big(\frac{\delta^{i_1}}{\delta\varphi^{i_1}} L_{\text{int}} \otimes \cdots \otimes \frac{\delta^{i_l}}{\delta\varphi^{i_l}} L_{\text{int}} \otimes \sin(a\varphi)\Big) \\ &\quad \times \frac{\delta^i}{\delta\varphi^i} r_1^N. \end{aligned} \tag{58}$$

A completely analogous expression occurs for term $T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\cos(a\varphi) q_1^N)\Big)$ in equation (51), with $\cos(a\varphi)$ in place of $\sin(a\varphi)$ and q_1^N in place of r_1^N.

3.4. Retarded Components s_1^N

Using formula (50) and linearity of the retarded products, we have

$$R_t\Big(L_{\text{int}}^{\otimes t} \otimes s_1^N\Big) = R_t\Big(L_{\text{int}}^{\otimes t} \otimes (\cos(a\varphi) q_1^N)\Big) + R_t\Big(L_{\text{int}}^{\otimes t} \otimes (\sin(a\varphi) r_1^N)\Big). \tag{59}$$

The two terms on the right hand side are completely analogous, so we consider only the second one. By formula (35), we can expand the retarded product as

$$\begin{aligned} &R_t\Big(L_{\text{int}}^{\otimes t} \otimes (\sin(a\varphi) r_1^N)\Big) \\ &= \left(\frac{i}{\hbar}\right)^t \sum_{l=0}^{t} \binom{t}{l} (-1)^{t-l} \bar{T}_{t-l}\Big(L_{\text{int}}^{\otimes(t-l)}\Big) \star T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\sin(a\varphi) r_1^N)\Big). \end{aligned} \tag{60}$$

First we consider the star products $\bar{T}_{t-l}\Big(L_{\text{int}}^{\otimes(t-l)}\Big) \star T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\sin(a\varphi) r_1^N)\Big)$. We substitute equation (58), renaming the index $i \to j$, and then apply Proposition 3.2.1 to every term to obtain:

$$\begin{aligned} &\bar{T}_{t-l}\Big(L_{\text{int}}^{\otimes(t-l)}\Big) \star T_{l+1}\Big(L_{\text{int}}^{\otimes l} \otimes (\sin(a\varphi) r_1^N)\Big) \\ &= \sum_{j=0}^{2N} \hbar^j \sum_{\substack{j_1,\dots,j_l \geq 0 \\ j_1+\dots+j_l=j}} \frac{1}{j_1! \dots j_l!} \sum_{i=0}^{2N-j} \frac{\hbar^i}{i!} \Big(\sum_{k=0}^{\infty} \frac{\hbar^k}{k!} \frac{\delta^i}{\delta\varphi^i} \frac{\delta^k}{\delta\varphi^k} \bar{T}_{t-l}\Big(L_{\text{int}}^{\otimes(t-l)}\Big) \\ &\quad \times (W)^k \frac{\delta^k}{\delta\varphi^k} T_{l+1}\Big(\frac{\delta^{j_1}}{\delta\varphi^{j_1}} L_{\text{int}} \otimes \cdots \otimes \frac{\delta^{j_l}}{\delta\varphi^{j_l}} L_{\text{int}} \otimes \sin(a\varphi)\Big)\Big) \\ &\quad \times (\Delta^F)^j (W)^i \frac{\delta^i}{\delta\varphi^i} \frac{\delta^j}{\delta\varphi^j} r_1^N. \end{aligned} \tag{61}$$

Then, we plug this expression in Eq. (60) and arrive at:

$$\begin{aligned} &R_t\Big(L_{\text{int}}^{\otimes t}\otimes(\sin(a\varphi)\cdot r_1^N)\Big)\\ &\quad=\Big(\frac{i}{\hbar}\Big)^t\sum_{l=0}^{t}\binom{t}{l}(-1)^{t-l}\sum_{j=0}^{2N}\hbar^j\sum_{\substack{j_1,\dots,j_l\geq 0\\ j_1+\dots+j_l=j}}\frac{1}{j_1!\dots j_l!}\sum_{i=0}^{2N-j}\frac{\hbar^i}{i!}\\ &\qquad\times\Bigg(\sum_{k=0}^{\infty}\frac{\hbar^k}{k!}\frac{\delta^i}{\delta\varphi^i}\frac{\delta^k}{\delta\varphi^k}\bar{T}_{t-l}\Big(L_{\text{int}}^{\otimes(t-l)}\Big)(W)^k\\ &\qquad\times\frac{\delta^k}{\delta\varphi^k}T_{l+1}\Big(\frac{\delta^{j_1}}{\delta\varphi^{j_1}}L_{\text{int}}\otimes\dots\otimes\frac{\delta^{j_l}}{\delta\varphi^{j_l}}L_{\text{int}}\otimes\sin(a\varphi)\Big)\Bigg)(\Delta^F)^j(W)^i\frac{\delta^i}{\delta\varphi^i}\frac{\delta^j}{\delta\varphi^j}r_1^N. \end{aligned} \tag{62}$$

A completely analogous expression occurs for the term $R_t\Big(L_{\text{int}}^{\otimes t}\otimes(\cos(a\varphi)q_1^N)\Big)$ in Eq. (59), with $\cos(a\varphi)$ in place of $\sin(a\varphi)$ and q_1^N in place of r_1^N.

4. Renormalization of the Interacting Currents

In the following sections, we perform the renormalization of the retarded components $R_t\Big(L_{\text{int}}^{\otimes t}\otimes s_2^N\Big)$ and $R_t\Big(L_{\text{int}}^{\otimes t}\otimes(\sin(a\varphi)\cdot r_1^N)\Big)$ (recall that the other term in Eq. (59) for the retarded component s_1^N is completely analogous). Specifically, we show that it is possible to extend their unrenormalized expressions, as distributions defined on the subset $\check{\mathbb{M}}_2^{t+1}\subseteq\mathbb{M}_2^{t+1}$ (see formula (31)), to distributions defined on the whole space $\mathbb{M}_2^{t+1}$.

More importantly, in Lemma 4.1.1 we prove that the scaling degree of the distributions appearing in the unrenormalized retarded components of the currents is uniformly bounded by the corresponding degree of the component, according to Definition 2.1.2 and Proposition 2.2.2. This directly implies the main result of this paper, namely, the super-renormalizability of the retarded components of the higher currents of the sine-Gordon model, see Theorem 4.3.1.

We adopt an approach which we call piecewise renormalization. It consists of three steps: expansion of the expression to be renormalized in its elementary parts, renormalization of each one of the elementary parts separately and finally showing that reassembling the renormalized elementary parts all together gives a well-defined result.

Before starting with our program, we recall (see [5] and [6] for more details) that vertex operators V_a, $a>0$, act on configurations $\varphi\in C^\infty(\mathbb{M}_2)$, returning a distribution (a smooth function in fact), in the following way:

$$V_a(x)\colon\varphi\mapsto V_a(x)[\varphi]:=e^{ia\varphi(x)}. \tag{63}$$

The critical property of vertex operators is that functional derivatives of vertex operators have the form:

$$\frac{\delta^k}{\delta\varphi(y_1)\dots\delta\varphi(y_k)}V_a(x)=(ia)^k\delta(y_1-x)\dots\delta(y_k-x)V_a(x), \tag{64}$$

thus they are essentially again vertex operators, modulo constant coefficients. We use vertex operators to write the interaction Lagrangian of the sine-Gordon model as $L_{\text{int}} = \cos(a\varphi) = \frac{1}{2}(V_a + V_{-a})$ and to write $\sin(a\varphi) = \frac{1}{2i}(V_a - V_{-a})$.

We also introduce some notation. By slight abuse, we denote with $x := (\tau, \xi)$ the set of light-cone coordinates on $\mathbb{M}_2$, and consequently with $(x_1, \dots, x_n) := (\tau_1, \xi_1, \dots, \tau_n, \xi_n)$ the set of light-cone coordinates on $\mathbb{M}_2^n$. Substituting the vertex operators, using formula (64) and omitting the numerical coefficients, we can write the generic term of Eq. (49) in the form:

$$\begin{aligned}&\frac{\delta^i}{\delta\varphi^i}\frac{\delta^k}{\delta\varphi^k}\bar{T}_{t-l}\Big(V_{a_{l+1}}(x_{l+1}) \otimes \cdots \otimes V_{a_t}(x_t)\Big)\big(W(x_{\{l+1\leq\cdot\leq t\}} - x_{\{1\leq\cdot\leq l\}})\big)^k \\ &\quad\times\frac{\delta^k}{\delta\varphi^k}T_l\Big(V_{a_1}(x_1) \otimes \cdots \otimes V_{a_l}(x_l)\Big)\big(\partial^{\cdot}_{\xi_{t+1}} W(x_{\{l+1\leq\cdot\leq t\}} - x_{t+1})\big)^i \\ &\quad\times\big(\partial^{\cdot}_{\xi_{t+1}}\Delta^F(x_{\{1\leq\cdot\leq l\}} - x_{t+1})\big)^j\frac{\delta^i}{\delta\varphi^i}\frac{\delta^j}{\delta\varphi^j}s_2^N(x_{t+1}),\end{aligned} \tag{65}$$

where, moreover, we have that:

- $a_1, \dots, a_t \in \{+a, -a\}$, $a \in \mathbb{R}_+$;
- $\big(W(x_{\{l+1\leq\cdot\leq t\}} - x_{\{1\leq\cdot\leq l\}})\big)^k$ denotes products of k Wightman two-point functions and for each one of them the first argument belongs to the set $x_{\{l+1\leq\cdot\leq t\}} := \{x_{l+1}, \dots, x_t\}$, while the second argument belongs to the set $x_{\{1\leq\cdot\leq l\}} := \{x_1, \dots, x_l\}$, in all the possible combinations;
- $\big(\partial^{\cdot}_{\xi_{t+1}} W(x_{\{l+1\leq\cdot\leq t\}} - x_{t+1})\big)^i$ denotes products of i Wightman two-point functions where for each one of them the first argument belongs to the set $x_{\{l+1\leq\cdot\leq t\}}$, the second argument is x_{t+1} and moreover we take a derivative of unspecified order $\partial^{\cdot}_{\xi_{t+1}}$ with respect to the coordinate ξ_{t+1} of $x_{t+1} = (\tau_{t+1}, \xi_{t+1})$ (the order of the derivative depends on the results of the functional derivatives $\frac{\delta^i}{\delta\varphi^i}$ of $s_2^N(x_{t+1})$);
- $\big(\partial^{\cdot}_{\xi_{t+1}}\Delta^F(x_{\{1\leq\cdot\leq l\}} - x_{t+1})\big)^j$ denotes products of j Feynman propagators where for each one of them the first argument belongs to the set $x_{\{1\leq\cdot\leq l\}}$, the second argument is x_{t+1} and moreover we take a derivative of unspecified order $\partial^{\cdot}_{\xi_{t+1}}$ with respect to the coordinate ξ_{t+1} of $x_{t+1} = (\tau_{t+1}, \xi_{t+1})$ (the order of the derivative depends on the results of the functional derivatives $\frac{\delta^j}{\delta\varphi^j}$ of $s_2^N(x_{t+1})$).

For the generic term of Eq. (62), we have the only difference that also the time-ordered products depend on x_{t+1}. In this case we have:

$$\begin{aligned}&\frac{\delta^i}{\delta\varphi^i}\frac{\delta^k}{\delta\varphi^k}\bar{T}_{t-l}\Big(V_{a_{l+2}}(x_{l+2}) \otimes \cdots \otimes V_{a_{t+1}}(x_{t+1})\Big) \\ &\quad\times\big(W(x_{\{l+2\leq\cdot\leq t+1\}} - x_{\{1\leq\cdot\leq l+1\}})\big)^k\big(\partial^{\cdot}_{\xi_{l+1}} W(x_{\{l+2\leq\cdot\leq t+1\}} - x_{l+1})\big)^i \\ &\quad\times\frac{\delta^k}{\delta\varphi^k}T_{l+1}\Big(V_{a_1}(x_1) \otimes \cdots \otimes V_{a_l}(x_l) \otimes V_{a_{l+1}}(x_{l+1})\Big) \\ &\quad\times\big(\partial^{\cdot}_{\xi_{l+1}}\Delta^F(x_{\{1\leq\cdot\leq l\}} - x_{l+1})\big)^j\frac{\delta^i}{\delta\varphi^i}\frac{\delta^j}{\delta\varphi^j}r_1^N(x_{l+1}),\end{aligned} \tag{66}$$

with same notations as above.

We are now ready to discuss the renormalization of the generic terms (65) and (66). In Sect. 4.1, we consider time-ordered products of vertex operators together with derivatives of Feynman propagators and perform what we call their piecewise renormalization. In Sect. 4.2, we consider anti-chronological products of vertex operators and we perform their piecewise renormalization.

As for the derivatives of the components of the currents, they are smooth functions, there is no need for renormalization. The products of Wightman two-point functions and their derivatives, instead, are always well-defined distributions on $\mathbb{M}_2^{t+1}$ according to Hörmander's sufficient criterion.

Finally, in Sect. 4.3 we reassemble the pieces all together and show, by a careful study of the wavefront sets of all the elements involved, that the result is well-defined.

4.1. Piecewise Renormalization of Time-Ordered Products and Derivatives of Feynman Propagators

According to our plan, we first expand the expressions to be renormalized in their most elementary parts. In [5] it is shown that the unrenormalized time-ordered products of vertex operators can be written in exponential form as:

$$T_l\Big(V_{a_1}(x_1)\otimes\cdots\otimes V_{a_l}(x_l)\Big)=e^{i\big(a_1\varphi(x_1)+\cdots+a_l\varphi(x_l)\big)}\prod_{1\le i<j\le l}e^{-a_ia_j\hbar\Delta^F(x_i-x_j)}. \tag{67}$$

Omitting the exponentials of configurations, which do not need renormalization, we can expand the exponentials of Feynman propagators as a formal power series in $\hbar$. The coefficient of the power $\hbar^p$ is given by:

$$\begin{aligned}\sum_{\substack{\{p_{i,j}\ge 0,\,1\le i<j\le l\\ \text{s.t.}\ \sum_{i,j}p_{i,j}=p\}}}&\frac{(-1)^p(a_1a_2)^{p_{1,2}}\cdots(a_{l-1}a_l)^{p_{l-1,l}}}{p_{1,2}!\cdots p_{l-1,l}!}\\ &\times(\Delta^F)^{p_{1,2}}(x_1-x_2)\cdots(\Delta^F)^{p_{l-1,l}}(x_{l-1}-x_l),\end{aligned} \tag{68}$$

4.1.1. Discussion for Components s_2^N. We now concentrate specifically on the time-ordered products of vertex operators and derivative of Feynman propagators appearing in formula (65). We write the products of derivatives of Feynman propagators as:

$$\begin{aligned}&\partial^{i_{1,1}}_{\xi_{t+1}}\Delta^F(x_1-x_{t+1})\cdots\partial^{i_{1,n_1}}_{\xi_{t+1}}\Delta^F(x_1-x_{t+1})\\ &\quad\times\partial^{i_{2,1}}_{\xi_{t+1}}\Delta^F(x_2-x_{t+1})\cdots\partial^{i_{2,n_2}}_{\xi_{t+1}}\Delta^F(x_2-x_{t+1})\\ &\quad\vdots\\ &\quad\times\partial^{i_{l,1}}_{\xi_{t+1}}\Delta^F(x_l-x_{t+1})\cdots\partial^{i_{l,n_l}}_{\xi_{t+1}}\Delta^F(x_l-x_{t+1}),\end{aligned} \tag{69}$$

where $i_{r\,s}\ge 1$, if $n_r\ge 1$, and otherwise, if $n_r=0$, then there is no product of Feynman propagators with argument (x_r-x_{t+1}). Using formulas (69) and (68) in expression (65), we obtain that the coefficient of $\hbar^p$, modulo multiplicative constants, is:

$$\sum_{\substack{\{p_{i,j}\geq 0,\, 1\leq i<j\leq l \\ \text{s.t. } \sum_{i,j} p_{i,j}=p\}}} \frac{(-1)^p (a_1 a_2)^{p_{1,2}} \cdots (a_{l-1}a_l)^{p_{l-1,l}}}{p_{1,2}! \cdots p_{l-1,l}!}$$
$$\times (\Delta^F)^{p_{1,2}}(x_1 - x_2) \cdots (\Delta^F)^{p_{l-1,l}}(x_{l-1} - x_l)$$
$$\times \partial^{i_{1,1}}_{\xi_{t+1}} \Delta^F(x_1 - x_{t+1}) \cdots \partial^{i_{1,n_1}}_{\xi_{t+1}} \Delta^F(x_1 - x_{t+1})$$
$$\vdots$$
$$\times \partial^{i_{l,1}}_{\xi_{t+1}} \Delta^F(x_l - x_{t+1}) \cdots \partial^{i_{l,n_l}}_{\xi_{t+1}} \Delta^F(x_l - x_{t+1}). \tag{70}$$

We consider each one of the elementary parts separately, as distributions defined on $\mathbb{M}_2 \setminus \{0\}$, and denote them by:

$$\left.\begin{aligned} D_{1,2} &:= (\Delta^F)^{p_{1,2}}, \\ &\vdots \\ D_{l-1,l} &:= (\Delta^F)^{p_{l-1,l}}, \\ D_{1,t+1} &:= (\partial^{i_{1,1}}_{\xi_{t+1}} \Delta^F) \cdots (\partial^{i_{1,n_1}}_{\xi_{t+1}} \Delta^F), \\ &\vdots \\ D_{l,t+1} &:= (\partial^{i_{l,1}}_{\xi_{t+1}} \Delta^F) \cdots (\partial^{i_{l,n_l}}_{\xi_{t+1}} \Delta^F), \end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2 \setminus \{0\}). \tag{71}$$

In order to proceed with the second step of the piecewise renormalization process of formula (70), we take into account the Steinmann scaling degree of the Feynman propagator. On the two-dimensional Minkowski space $\mathbb{M}_2$, the Feynman propagator Δ^F scales homogeneously with scaling degree $\mathrm{sd}(\Delta^F) = 0$, and every derivative potentially increases by one the scaling degree (see [10, 12]).

Lemma 4.1.1. *For the products of derivatives of Feynman propagators appearing in formula (71), the following estimate on the scaling degrees holds:*

$$\left.\begin{aligned} sd\Big((\partial^{i_{1,1}}_{\xi_{t+1}} \Delta^F) \cdots (\partial^{i_{1,n_1}}_{\xi_{t+1}} \Delta^F)\Big) &\leq \textstyle\sum_{s=1}^{n_1} i_{1,s}, \\ &\vdots \\ sd\Big((\partial^{i_{l,1}}_{\xi_{t+1}} \Delta^F) \cdots (\partial^{i_{l,n_l}}_{\xi_{t+1}} \Delta^F)\Big) &\leq \textstyle\sum_{s=1}^{n_l} i_{l,s}, \end{aligned}\right\} \leq deg(s_2^N) = 2(N+1), \tag{72}$$

for every $N \in \mathbb{N}$.

Proof. The result follows from Remark 2.2.1 and Proposition 2.2.2 on the structure of the components s_2^N and from the general fact that, given two distributions $u, v \in \mathscr{D}'$ whose distributional product is well-defined, it holds

$$\mathrm{sd}(uv) \leq \mathrm{sd}(u) + \mathrm{sd}(v). \tag{73}$$

□

Remark 4.1.1. Formula (73) also implies that any power of the Feynman propagator has scaling degree equal to 0 on $\mathbb{M}_2 \setminus \{0\}$.

Knowing an estimate on the scaling degree of each one of the elements in formula (71), we can apply well-known results [9,10,17] to extend them to distributions defined on the whole $\mathbb{M}_2$ in such a way to also preserve the scaling degree. We denote these extensions by:

$$\left.\begin{aligned} &[D_{1,2}] := [(\Delta^F)^{p_{1,2}}],\\ &\quad\vdots\\ &[D_{l-1,l}] := [(\Delta^F)^{p_{l-1,l}}],\\ &[D_{1,t+1}] := [(\partial^{i_{1,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{1,n_1}}_{\xi_{t+1}}\Delta^F)],\\ &\quad\vdots\\ &[D_{l,t+1}] := [(\partial^{i_{l,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{l,n_l}}_{\xi_{t+1}}\Delta^F)], \end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2). \tag{74}$$

Remark 4.1.2. We point out two important observations:

(i) For powers of Feynman propagators, since their scaling degree is equal to 0, the extensions are direct and unique. For products of derivatives of Feynman propagators, when the scaling degree is $\mathrm{sd} \geq 2$, the extension is unique up to adding a finite number of derivatives of the Dirac delta, namely derivatives up to order $\mathrm{sd} - 2$ (see also beginning of Sect. 3).

(ii) Considering the wavefront set of the Feynman propagator and the fact that the extensions are realized by possibly adding Dirac deltas, we have that the wavefront set of each element of (74) is contained in the set

$$\begin{aligned}\Gamma_0 := &\left\{(w,k) \in T^*\mathbb{M}_2 \,|\, |w|^2 = 0,\, w \neq 0,\, k = \frac{\eta_\flat(w)}{\lambda},\, \lambda > 0\right\} \cup \\ &\cup \{(0,k) \in T^*\mathbb{M}_2 | k \neq 0\}, \end{aligned} \tag{75}$$

where $|w|^2 = \eta(w,w)$, and $\eta_\flat \colon T\mathbb{M}_2 \to T^*\mathbb{M}_2$ is the isomorphism induced by the Minkowski metric.

The piecewise renormalization process has to maintain the translation invariance of the unrenormalized expressions. Starting from the extended elementary parts (74), we obtain translation-invariant distributions defined on $\mathbb{M}_2^2$ performing the pull-back of every element via appropriate maps.

Lemma 4.1.2. *Consider the maps*

$$\begin{aligned} s_{i,j} \colon \quad \mathbb{M}_2^2 \quad &\to \quad \mathbb{M}_2 \\ (x_i, x_j) &\mapsto w_{i,j} = x_i - x_j, \end{aligned} \tag{76}$$

where $i,j \in \{1,\dots,l\}$ and $i<j$, or $i=1,\dots,l$ and $j=t+1$. Then, the following are well-defined translation-invariant distributions:

$$\left.\begin{aligned}
s^*_{1,2}\big([D_{1,2}]\big) &= [(\Delta^F)^{p_{1,2}}](x_1-x_2),\\
&\vdots\\
s^*_{l-1,l}\big([D_{l-1,l}]\big) &= [(\Delta^F)^{p_{l-1,l}}](x_{l-1}-x_l),\\
s^*_{1,t+1}\big([D_{1,t+1}]\big) &= [(\partial^{i_{1,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{1,n_1}}_{\xi_{t+1}}\Delta^F)](x_1-x_{t+1}),\\
&\vdots\\
s^*_{l,t+1}\big([D_{l,t+1}]\big) &= [(\partial^{i_{l,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{l,n_l}}_{\xi_{t+1}}\Delta^F)](x_l-x_{t+1}),
\end{aligned}\right\} \in \mathscr{D}'(\mathbb{M}_2^2). \quad (77)$$

Proof. Pull-back along the maps $s_{i,j}$ is well-defined in general for any distribution, hence a fortiori for our distributions (74). Indeed, the transpose of the tangent maps $(s'_{i,j})^t$ have the form:

$$\begin{aligned}
(s'_{i,j})^t\colon \qquad & T^*\mathbb{M}_2 \quad \rightarrow \quad T^*\mathbb{M}_2^2\\
& (w_{i,j}=x_i-x_j,k) \mapsto (x_i,k;x_j,-k).
\end{aligned} \quad (78)$$

Hence, the condition ensuring the well-posedness of the pull-back (see [11])

$$(s'_{i,j})^t\big(\Gamma_0\big) \cap \big\{(x_i,0;x_j,0) \subset T^*\mathbb{M}_2^2\big\} = \emptyset \quad (79)$$

is always satisfied, for any i,j as above. □

Remark 4.1.3. From the properties of the wavefront set under the operation of pull-back and formula (75), we obtain that the wavefront set of the distributions (77) are contained, respectively, in the sets

$$\begin{aligned}
\Gamma_{i,j} = (s'_{i,j})^t\big(\Gamma_0\big) = \Big\{&(x_i,k;x_j,-k) \in T^*\mathbb{M}_2^2|\, |x_i-x_j|^2=0,\, x_i \neq x_j,\\
&k = \frac{\eta_\flat(x_i-x_j)}{\lambda},\, \lambda>0\Big\} \cup \big\{(x,k;x,-k) \in T^*\mathbb{M}_2^2 | k \neq 0\big\},
\end{aligned} \quad (80)$$

with $i,j \in \{1,\dots,l\}$ and $i<j$, or $i=1,\dots,l$ and $j=t+1$.

We have thus completed the piecewise renormalization of the elementary parts. Reassembling them together, we arrive at the following piecewise renormalized expression for the coefficient of $\hbar^p$:

$$\begin{aligned}
\sum_{\substack{\{p_{i,j}\geq 0,\, 1\leq i<j\leq l\\ \text{s.t. } \sum_{i,j} p_{i,j}=p\}}} & \frac{(-1)^p(a_1a_2)^{p_{1,2}}\cdots(a_{l-1}a_l)^{p_{l-1,l}}}{p_{1,2}!\cdots p_{l-1,l}!}\\
& \times[(\Delta^F)^{p_{1,2}}](x_1-x_2)\cdots[(\Delta^F)^{p_{l-1,l}}](x_{l-1}-x_l)\\
& \times[(\partial^{i_{1,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{1,n_1}}_{\xi_{t+1}}\Delta^F)](x_1-x_{t+1})\\
& \quad \vdots\\
& \times[(\partial^{i_{l,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{l,n_l}}_{\xi_{t+1}}\Delta^F)](x_l-x_{t+1}).
\end{aligned} \quad (81)$$

We can repeat the piecewise renormalization in the same way for the coefficients of every power of $\hbar^p$. Summing together all the orders, we finally obtain the piecewise renormalized expression of the time-ordered products of vertex operators and derivatives of Feynman propagators appearing in formula (65)

$$\begin{aligned}&\big[T_l\Big(V_{a_1}(x_1)\otimes\cdots\otimes V_{a_l}(x_l)\Big)\big]\cdot[(\partial^{i_{1\,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{1\,n_1}}_{\xi_{t+1}}\Delta^F)](x_1-x_{t+1})\cdots\\&\quad\cdots[(\partial^{i_{l\,1}}_{\xi_{t+1}}\Delta^F)\cdots(\partial^{i_{l\,n_l}}_{\xi_{t+1}}\Delta^F)](x_l-x_{t+1}).\end{aligned}\tag{82}$$

We show that the distributional products in this formula are actually well-defined according to Hörmander's sufficient criterion in Theorem 4.3.1.

Remark 4.1.4. We stress that, from point (i) of Remark 4.1.2, the extensions regarding the Feynman propagators and consequently the time-ordered products of vertex operators are unique and direct. Instead, the extensions regarding the products of derivatives of Feynman propagators are not unique. Nevertheless, the number of derivatives of Dirac deltas that can be added in the extension process is finite and bounded by $\deg(s_2^N)-2=2N$.

4.1.2. Discussion for Components s_1^N. The piecewise renormalization of the time-ordered products of vertex operators and derivatives of Feynman propagators in the case of the retarded components s_1^N proceeds in a completely analogous way. We only point out some slight differences.

The first one is that the time-ordered products appearing in formula (66) depend also on the same argument x_{l+1} as the term r_1^N. This translates in the fact that the coefficient of the power $\hbar^p$ takes the form

$$\begin{aligned}\sum_{\substack{\{p_{i,j}\geq 0,\,1\leq i<j\leq l+1\\ \text{s.t.}\,\sum_{i,j}p_{i,j}=p\}}}&\frac{(-1)^p(a_1a_2)^{p_{1,2}}\cdots(a_la_{l+1})^{p_{l,l+1}}}{p_{1,2}!\cdots p_{l,l+1}!}\\&\times(\Delta^F)^{p_{1,2}}(x_1-x_2)\cdots(\Delta^F)^{p_{l,l+1}}(x_l-x_{l+1})\\&\times\partial^{i_{1,1}}_{\xi_{l+1}}\Delta^F(x_1-x_{l+1})\cdots\partial^{i_{1,n_1}}_{\xi_{l+1}}\Delta^F(x_1-x_{l+1})\\&\quad\vdots\\&\times\partial^{i_{l,1}}_{\xi_{l+1}}\Delta^F(x_l-x_{l+1})\cdots\partial^{i_{l,n_l}}_{\xi_{l+1}}\Delta^F(x_l-x_{l+1}).\end{aligned}\tag{83}$$

The second difference is that, from Proposition 2.2.2, $\deg(r_1^N)=2N$. So the same argument as in Lemma 4.1.1 tells us that the scaling degrees of the products of derivatives of Feynman propagators in (83) are bounded by $2N$.

We can repeat the same passages as in the previous section, obtaining a piecewise renormalized version of the coefficient of $\hbar^p$ for every order p:

$$\begin{aligned}\sum_{\substack{\{p_{i,j}\geq 0,\,1\leq i<j\leq l+1\\ \text{s.t.}\,\sum_{i,j}p_{i,j}=p\}}}&\frac{(-1)^p(a_1a_2)^{p_{1,2}}\cdots(a_la_{l+1})^{p_{l,l+1}}}{p_{1,2}!\cdots p_{l,l+1}!}\\&\times[(\Delta^F)^{p_{1,2}}](x_1-x_2)\cdots[(\Delta^F)^{p_{l,l+1}}](x_l-x_{l+1})\end{aligned}$$

$$\times[(\partial_{\xi_{l+1}}^{i_{1,1}}\Delta^F)\cdots(\partial_{\xi_{l+1}}^{i_{1,n_1}}\Delta^F)](x_1-x_{l+1})$$
$$\vdots$$
$$\times[(\partial_{\xi_{l+1}}^{i_{l,1}}\Delta^F)\cdots(\partial_{\xi_{l+1}}^{i_{l,n_l}}\Delta^F)](x_l-x_{l+1}). \tag{84}$$

Summing together all the orders, we finally get the piecewise renormalized expression of the time-ordered products of vertex operators and derivatives of Feynman propagators appearing in formula (66):

$$[T_{l+1}\Big(V_{a_1}(x_1)\otimes\cdots\otimes V_{a_l}(x_l)\otimes V_{a_{l+1}}(x_{l+1})\Big)]$$
$$\times[(\partial_{\xi_{l+1}}^{i_{1,1}}\Delta^F)\cdots(\partial_{\xi_{l+1}}^{i_{1,n_1}}\Delta^F)](x_1-x_{l+1})\cdots[(\partial_{\xi_{l+1}}^{i_{l,1}}\Delta^F)\cdots(\partial_{\xi_{l+1}}^{i_{l,n_l}}\Delta^F)](x_l-x_{l+1}). \tag{85}$$

We show that the distributional products in this formula are actually well-defined according to Hörmander's sufficient criterion in Theorem 4.3.1.

Remark 4.1.5. As previously pointed out in Remark 4.1.4 for the retarded components s_2^N, also in this case the ambiguity in the renormalization process, given by number of derivatives of Dirac deltas that can be added, is finite and bounded by $\deg(s_1^N)-2=2(N-1)$.

4.2. Piecewise Renormalization of Anti-chronological Products of Vertex Operators

We now consider the anti-chronological products appearing in formulas (65) and (66). In both cases we have anti-chronological products of vertex operators with $t-l$ arguments. We indicate them by:

$$\bar{T}_{t-l}\Big(V_{a_{l+1}}(x_{l+1})\otimes\cdots\otimes V_{a_t}(x_t)\Big). \tag{86}$$

The anti-chronological products of vertex operators can be written in the following exponential form:

$$\bar{T}_{t-l}\Big(V_{a_{l+1}}(x_{l+1})\otimes\cdots\otimes V_{a_t}(x_t)\Big) = e^{i\big(a_{l+1}\varphi(x_{l+1})+\cdots+a_t\varphi(x_t)\big)}\prod_{l+1\le i<j\le t} e^{-a_ia_j\hbar\Delta^{AF}(x_i-x_j)}, \tag{87}$$

where Δ^{AF} is the anti-Feynman propagator, defined as the complex conjugate of the Feynman propagator $\Delta^{AF}=\overline{\Delta^F}$. For its scaling degree, it holds:

$$\mathrm{sd}(\Delta^{AF})=\mathrm{sd}(\overline{\Delta^F})=\mathrm{sd}(\Delta^F)=0. \tag{88}$$

We can expand the product of exponentials of anti-Feynman propagators and collect the coefficient of $\hbar^q$:

$$\sum_{\substack{\{q_{i,j}\ge 0,\, l+1\le i<j\le t\\ \text{s.t. } \sum_{i,j} q_{i,j}=q\}}} \frac{(-1)^q(a_{l+1}a_{l+2})^{q_{l+1,l+2}}\cdots(a_{t-1}a_t)^{q_{t-1,t}}}{q_{l+1,l+2}!\cdots q_{t-1,t}!}$$
$$\times(\Delta^{AF})^{q_{l+1,l+2}}(x_{l+1}-x_{l+2})\cdots(\Delta^{AF})^{q_{t-1,t}}(x_{t-1}-x_t). \tag{89}$$

Each one of the elementary parts has scaling degree equal to 0. So they admit direct and unique extensions (cfr. point (i) of Remark 4.1.2). Repeating once more the same passages as in Sect. 4.1.1, we arrive at the piecewise renormalized expression of the coefficient of $\hbar^q$:

$$\sum_{\substack{\{q_{i,j}\geq 0,\, l+1\leq i<j\leq t\\ \text{s.t. } \sum_{i,j} q_{i,j}=q\}}} \frac{(-1)^q (a_{l+1}a_{l+2})^{q_{l+1,l+2}}\cdots(a_{t-1}a_t)^{q_{t-1,t}}}{q_{l+1,l+2}!\cdots q_{t-1,t}!} \times [(\Delta^{AF})^{q_{l+1,l+2}}](x_{l+1}-x_{l+2})\cdots[(\Delta^{AF})^{q_{t-1,t}}](x_{t-1}-x_t). \tag{90}$$

Summing together all the orders, we finally get the piecewise renormalized expression of the anti-chronological products of vertex operators appearing in formulas (65) and (66), which we denote by:

$$\left[\bar{T}_{t-l}\Big(V_{a_{l+1}}(x_{l+1})\otimes\cdots\otimes V_{a_t}(x_t)\Big)\right]. \tag{91}$$

4.3. Well-Posedness of the Piecewise Renormalized Expressions

We complete the renormalization of the retarded components of the higher currents showing that the piecewise renormalized expressions obtained in the previous sections are indeed well-defined. We proceed as follows.

First we prove the well-posedness of the piecewise renormalized time-ordered products of vertex operators and derivatives of Feynman propagators and of the piecewise renormalized anti-chronological products of vertex operators by explicitly computing their wavefront set. Then we estimate the wavefront set of the products of (derivatives of) Wightman two-point functions. Finally, we show that these elements satisfy Hörmander's sufficient criterion and thus their product is well-defined.

Let us start with the piecewise renormalized coefficient of the power $\hbar^p$ of time-ordered products of vertex operators and derivatives of Feynman propagators for components s_1^N, formula (84). We regard it as the result of the pull-back of the tensor product of all its elementary parts

$$\begin{aligned}\mathrm{TOF}^p_{l+1} := \sum_{\substack{\{p_{i,j}\geq 0,\, 1\leq i<j\leq l+1\\ \text{s.t. } \sum_{i,j} p_{i,j}=p\}}} &\frac{(-1)^p (a_1a_2)^{p_{1,2}}\cdots(a_la_{l+1})^{p_{l,l+1}}}{p_{1,2}!\cdots p_{l,l+1}!}\\ &[(\Delta^F)^{p_{1,2}}](w_{1,2})\otimes\cdots\otimes[(\Delta^F)^{p_{l,l+1}}](w_{l,l+1})\\ &\otimes[(\partial^{i_{1,1}}_{\xi_{l+1}}\Delta^F)\cdots(\partial^{i_{1,n_1}}_{\xi_{l+1}}\Delta^F)](\tilde{w}_{1,l+1})\\ &\vdots\\ &\otimes[(\partial^{i_{l,1}}_{\xi_{l+1}}\Delta^F)\cdots(\partial^{i_{l,n_l}}_{\xi_{l+1}}\Delta^F)](\tilde{w}_{l,l+1}),\end{aligned} \tag{92}$$

seen as a distribution in $\mathscr{D}'(\mathbb{M}_2^K)$, $K=\binom{l+1}{2}+l$, via the map

$$\begin{aligned} s:\qquad \mathbb{M}_2^{l+1} &\to \mathbb{M}_2^K\\ (x_1,\ldots,x_{l+1}) &\mapsto (w_{i,j}=x_i-x_j, \tilde{w}_{k,l+1}=x_k-x_{l+1}),\end{aligned} \tag{93}$$

for $1 \leq i < j \leq l+1$ and $1 \leq k \leq l$.

The question of the well-posedness of the coefficient (84) is now rephrased in terms of the well-posedness of the pull-back $s^*(\mathrm{TOF}^p_{l+1})$. In other words, we ask whether the condition

$$(s')^t\big(\mathrm{WF}(\mathrm{TOF}^p_{l+1})\big) \bigcap \big(\mathbb{M}_2 \times \{0\}\big)^{l+1} = \emptyset, \tag{94}$$

where $(s')^t$ is the transpose of the tangent map of s and $\mathrm{WF}(\mathrm{TOF}^p_{l+1})$ is the wavefront set of TOF^p_{l+1}, is satisfied.

A graph notation, introduced in [17] to describe the wavefront set of products of Feynman propagators in the context of algebraic quantum field theory on curved spacetimes, turns out to be the proper tool to answer this question. We recall this notation from [9], adapting it to our spacetime $\mathbb{M}_2$:

- Denote by $\mathcal{G}_n$ the set of non-oriented graphs with vertexes $V = \{1, \ldots, n\}$, and by E^G the set of edges of a given graph $G \in \mathcal{G}_n$. For any edge $e \in E^G$ between vertexes $i < j$, we set source $\sigma(e) = i$ and target $\tau(e) = j$;
- A couple of maps (χ, κ) is an immersion of the graph $G \in \mathcal{G}_n$ into our spacetime $\mathbb{M}_2$ if:
  - $\chi\colon V \to \mathbb{M}_2$ maps vertexes i of G to points $x_i \in \mathbb{M}_2$, with the condition that if the vertexes $i < j$ are connected by an edge, then $|x_i - x_j|^2 = \eta(x_i - x_j, x_i - x_j) = 0$;
  - $\kappa\colon E^G \to T^*\mathbb{M}_2$ with the condition that, if the vertexes $i < j$ are connected by the edge $e \in E^G$, then the covector $\kappa(e) =: k_e$ is

$$\begin{cases} k_e = \lambda_{ij}\eta_\flat(x_i - x_j) \quad \text{for some} \quad \lambda_{ij} > 0, & \text{if} \quad x_i \neq x_j, \\ k_e \in \big(\mathbb{M}_2 \setminus \{0\}\big), \qquad \text{if} \quad x_i = x_j. \end{cases} \tag{95}$$

  The covector k_e is said to be outgoing for the point x_i and incoming for the point x_j.

Using this notation, the wavefront set $\Lambda_{l+1} := (s')^t\big(\mathrm{WF}(\mathrm{TOF}_p)\big)$ of $s^*(\mathrm{TOF}^p_{l+1})$ can be described as:

$$\begin{aligned}\Lambda_{l+1} = \Bigg\{ &(x_1, k_1; \ldots; x_{l+1}, k_{l+1}) \in T^*\mathbb{M}_2^{l+1} \mid \exists\, G \in \mathcal{G}_{l+1} \quad \text{and} \\ &\exists \quad \text{an immersion} \quad (\chi, \kappa) \quad \text{of} \quad G \quad \text{such that} \\ &k_i = \sum_{\substack{e \in E^G \\ \sigma(e) = i}} k_e - \sum_{\substack{f \in E^G \\ \tau(f) = i}} k_f \Bigg\}.\end{aligned} \tag{96}$$

Proposition 4.3.1. *For every order p, the piecewise renormalized coefficient of* $\hbar^p$*, formula (84), is a well-defined distribution on* $\mathbb{M}_2^{l+1}$*. Namely, the condition*

$$\Lambda_{l+1} \cap \big(\mathbb{M}_2 \times \{0\}\big)^{l+1} = \emptyset \tag{97}$$

for the well-posedness of the pull-back $s^*(TOF^p_{l+1})$ *is satisfied. Consequently the piecewise renormalized time-ordered products of vertex operators and derivatives of Feynman propagators, formula (85), are also well-defined.*

Proof. Consider first immersed graphs in $\mathbb{M}_2$ with no loops, i.e. suppose that the immersion map χ is injective. In this case, we obtain the thesis from the following argument. For every immersed vertex x_i, the corresponding covector k_i is given by a sum of covectors which are coparallel to the directions of connection of the vertex to its adjacent vertexes in the immersed graph. The directions of the connections always lie on the boundary of the light-cone. This means that, in order to have all covectors k_i equal to zero, every vertex x_i of the immersed graph has to be connected to its adjacent vertexes in opposite directions. But this can never be the case. In fact, each connected component of every immersed graph has a finite number of vertexes and if we consider for example, in a connected component, the vertex $\bar{x}$ with maximum time coordinate, then this vertex will be connected to its adjacent vertexes only in past-directed directions. Hence the covectors over $\bar{x}$ cannot sum up to zero.

Suppose now that the immersed graph contains loops, namely that the immersion χ maps vertexes $I = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, l+1\}$, $n \leq l+1$, to the same point $x_I \in \mathbb{M}_2$. Let us denote by E_I^G the set of loops, namely E_I^G is the subset of edges $e \in E^G$ such that $\sigma(e) \in I$ and $\tau(e) \in I$. Then the conditions that the covectors $k_{i_1}, \ldots, k_{i_n}$ over the points $x_{i_1} = \cdots = x_{i_n} = x_I$ are all equal to zero can be written as:

$$
\begin{aligned}
k_{i_1} &= \sum_{\substack{e \in E^G \setminus E_I^G \\ \sigma(e)=i_1}} k_e - \sum_{\substack{f \in E^G \setminus E_I^G \\ \tau(f)=i_1}} k_f + \sum_{\substack{e \in E_I^G \\ \sigma(e)=i_1}} k_e - \sum_{\substack{f \in E_I^G \\ \tau(f)=i_1}} k_f = 0 \\
&\vdots \\
k_{i_n} &= \sum_{\substack{e \in E^G \setminus E_I^G \\ \sigma(e)=i_n}} k_e - \sum_{\substack{f \in E^G \setminus E_I^G \\ \tau(f)=i_n}} k_f + \sum_{\substack{e \in E_I^G \\ \sigma(e)=i_n}} k_e - \sum_{\substack{f \in E_I^G \\ \tau(f)=i_n}} k_f = 0.
\end{aligned}
\tag{98}
$$

From these equations, we see that each one of the covectors k_e associated with an edge $e \in E_I^G$ appears twice, with opposite signs. If we sum up the equations above, we are then left with the condition:

$$
k_I = k_{i_1} + \cdots + k_{i_n} = \sum_{\substack{e \in E^G \setminus E_I^G \\ \sigma(e) \in I}} k_e - \sum_{\substack{f \in E^G \setminus E_I^G \\ \tau(f) \in I}} k_f = 0. \tag{99}
$$

This corresponds to the condition that we get if we look at the immersed graph G, without considering the loops. We are then reduced to the situation discussed above and we can apply the same argument to conclude. □

Completing the characterization of the renormalized time-ordered products of vertex operators and derivatives of Feynman propagators, we have the following result.

Proposition 4.3.2. *The wavefront set Λ_{l+1} of the renormalized time-ordered products of vertex operators and derivatives of Feynman propagators satisfies the microlocal condition*

$$
\Lambda_{l+1} \cap \left(\left(\mathbb{M}_2 \times \overline{V}_-\right)^{l+1} \cup \left(\mathbb{M}_2 \times \overline{V}_+\right)^{l+1} \right) = \emptyset, \tag{100}
$$

where $\overline{V}_-$ and $\overline{V}_+$ are the closure of the past and future light cones, respectively.

Proof. For each connected component of each immersed graph, we have a vertex $\bar{x}_+$ with maximum time coordinate and another vertex $\bar{x}_-$ with minimum time coordinate. This means that $\bar{x}_+$ is connected to its adjacent vertexes only by past-directed directions, and hence the covector over $\bar{x}_+$ is past-directed. Conversely the vertex $\bar{x}_-$ is connected to its adjacent vertexes only by future-directed directions, and hence the covector over $\bar{x}_-$ is future-directed.

This situation is not affected by the presence of loops at the vertexes $\bar{x}_+$ or $\bar{x}_-$. In fact, suppose that $\bar{x}_+$ is the image, via the immersion map χ, of the vertexes $I := \{i_1, \dots, i_n\} \subseteq \{1, \dots, l+1\}$, $n \leq l+1$. Then, similarly as in the proof of Proposition 4.3.1, we have that the covectors over the immersed vertexes $x_{i_1}, \dots, x_{i_n}$ can be summed up to give:

$$k_+ = k_{i_1} + \cdots + k_{i_n} = \sum_{\substack{e \in E^G \setminus E^G_I \\ \sigma(e) \in I}} k_e - \sum_{\substack{f \in E^G \setminus E^G_I \\ \tau(f) \in I}} k_f, \tag{101}$$

which is precisely the expression that we get if we look at the immersed graph G, without considering the loops. If we now assume that all covectors belong to $\overline{V}_+$, then also $k_+ \in \overline{V}_+$. This is a contradiction, because from the argument at the beginning we know that for immersed graphs without loops the covector k_+ over $\bar{x}_+$ must belong to $\overline{V}_-$. If we assume, on the contrary, that all covectors belong to $\overline{V}_-$ and repeat the previous reasoning for $\bar{x}_-$, we get a contradiction since we know that for immersed graphs without loops the covector over $\bar{x}_-$ must belong to $\overline{V}_+$. □

Remark 4.3.1. For what concerns the piecewise renormalized time-ordered products of vertex operators and derivatives of Feynman propagators in the case of components s_2^N, formulas (81) and (82), it suffices to repeat the same passages substituting the subscript $l+1$ with $t+1$.

We now consider the piecewise renormalized coefficient of the power $\hbar^q$ of the anti-chronological products of vertex operators, formula (91). We regard it as the pull-back of the tensor product

$$\begin{aligned} \mathrm{ACV}^q_{t-l} := \sum_{\substack{\{q_{i,j} \geq 0,\, l+1 \leq i < j \leq t \\ \text{s.t. } \sum_{i,j} q_{i,j} = q\}}} & \frac{(-1)^q (a_{l+1} a_{l+2})^{q_{l+1,l+2}} \cdots (a_{t-1} a_t)^{q_{t-1,t}}}{q_{l+1,l+2}! \cdots q_{t-1,t}!} \\ & \times [(\Delta^{AF})^{q_{l+1,l+2}}](w_{l+1,l+2}) \otimes \cdots \otimes [(\Delta^{AF})^{q_{t-1,t}}](w_{t-1,t}), \end{aligned} \tag{102}$$

as a distribution defined on $\mathbb{M}_2^{\tilde{K}}$, $\tilde{K} = \binom{t-l}{2}$, via the map

$$\begin{aligned} \tilde{s}: \qquad \mathbb{M}_2^{t-l} \quad & \rightarrow \quad \mathbb{M}_2^{\tilde{K}} \\ (x_{l+1}, \dots, x_t) & \mapsto (w_{i,j} = x_i - x_j), \end{aligned} \tag{103}$$

for $l+1 \leq i < j \leq t$. The condition for the well-posedness of the pull-back $\tilde{s}^*(\mathrm{ACV}^q_{t-l})$ becomes then

$$(\tilde{s}\,')^t \big(\mathrm{WF}(\mathrm{ACV}^q_{t-l})\big) \bigcap \big(\mathbb{M}_2 \times \{0\}\big)^{t-l} = \emptyset. \tag{104}$$

The set $\tilde{\Lambda}_{t-l} := (\tilde{s}\,')^t\big(\mathrm{WF}(\mathrm{ACV}^q_{t-l})\big)$ can be described slightly adapting the graph notation. Recalling that the anti-Feynman propagator is defined as $\Delta^{AF} = \overline{\Delta^F}$, we have the relation:

$$\mathrm{WF}(\Delta^{AF}) = -\mathrm{WF}(\Delta^F) = \big\{(w,-k) \in T^*\mathbb{M}_2 | (w,k) \in \mathrm{WF}(\Delta^F)\big\}\,. \tag{105}$$

This means that in this case, in the definition of immersion $(\tilde{\chi}, \tilde{\kappa})$ of a graph, the prescription is:

$$\begin{cases} \tilde{k}_e = -\lambda_{ij}\eta_\flat(x_i - x_j) \quad \text{for some} \quad \lambda_{ij} > 0, & \text{if} \quad x_i \neq x_j, \\ \tilde{k}_e \in \big(\mathbb{M}_2 \setminus \{0\}\big), & \text{if} \quad x_i = x_j. \end{cases} \tag{106}$$

We have then:

$$\tilde{\Lambda}_{t-l} := \Bigg\{ (\ x_{l+1}, k_{l+1}; \dots; x_t, k_t) \in T^*\mathbb{M}_2^{t-l} \mid \exists\, G \in \mathcal{G}_{t-l} \quad \text{and} \\ \exists \quad \text{an immersion} \quad (\tilde{\chi}, \tilde{\kappa}) \quad \text{of} \quad G \quad \text{such that} \\ k_i = \sum_{\substack{e \in E^G \\ \sigma(e)=i}} \tilde{k}_e - \sum_{\substack{f \in E^G \\ \tau(f)=i}} \tilde{k}_f \Bigg\}. \tag{107}$$

This modification does not affect the validity of the arguments in the proofs of Propositions 4.3.1 and 4.3.2, whose passages can be repeated to obtain the expected results.

Proposition 4.3.3. *For every order q, the piecewise renormalized coefficient of $\hbar^q$, formula (90), is a well-defined distribution on $\mathbb{M}_2^{t-l}$. Namely, the condition*

$$\tilde{\Lambda}_{t-l} \cap \big(\mathbb{M}_2 \times \{0\}\big)^{t-l} = \emptyset. \tag{108}$$

for the well-posedness of the pull-back $\tilde{s}^*\big(ACV^q_{t-l}\big)$ is satisfied. Consequently the piecewise renormalized anti-chronological products of vertex operators, formula (91), are also well-defined.

Proposition 4.3.4. *The wavefront set of the renormalized anti-chronological products of vertex operators $\tilde{\Lambda}_{t-l}$ satisfies the microlocal condition*

$$\tilde{\Lambda}_{t-l} \cap \Big(\big(\mathbb{M}_2 \times \overline{V}_-\big)^{t-l} \cup \big(\mathbb{M}_2 \times \overline{V}_+\big)^{t-l}\Big) = \emptyset. \tag{109}$$

It remains to consider the products of Wightman two-point functions and their derivatives. We recall that these products are always well-defined, hence no renormalization is needed in this case. Without loss of generality, we can consider as working example the product appearing in formula (66)

$$\big(W(x_{\{l+2\leq\cdot\leq t+1\}} - x_{\{1\leq\cdot\leq l+1\}})\big)^k \big(\partial_{\xi_{l+1}} W(x_{\{l+2\leq\cdot\leq t+1\}} - x_{l+1})\big)^i. \tag{110}$$

The analogous product appearing in formula (65) can be treated in the same way. The only difference between formulas (66) and (65) is in the way the dependence of the various elements on the coordinates $(x_1, \dots, x_{t+1})$ is distributed.

Once more, we can estimate the wavefront set of such products by means of the graph notation introduced above. The wavefront set of the Wightman two-point function W is given by [10]:

$$\left\{(x,k) \in T^*\mathbb{M}_2 |\, |x|^2 = 0,\, |k|^2 = 0,\, \lambda k = \eta_\flat(x),\, \lambda \in \mathbb{R} \text{ s.t. } k \in \partial\overline{V}_+ \setminus \{0\}\right\}. \tag{111}$$

Hence, we have to modify the convention (95) in the following way: for vertexes $1 \le i < j \le t+1$ connected by an edge e, we set source $\sigma(e) := j$ and target $\tau(e) := i$, and define an immersion $(\hat{\chi}, \hat{\kappa})$ by

$$\begin{cases} \hat{k}_e = \lambda_{ji}\eta_\flat(x_j - x_i) \quad \text{with } \lambda_{ji} \in \mathbb{R} \quad \text{s.t.} \quad \hat{k}_e \in \partial\overline{V}_+ \setminus \{0\}, \quad \text{if} \quad x_j \neq x_i, \\ \hat{k}_e \in \partial\overline{V}_+ \setminus \{0\}, \qquad \text{if} \quad x_j = x_i. \end{cases} \tag{112}$$

We obtain then the following description for the wavefront set Ω_{t+1} of the product of Wightman two-point functions and their derivatives (110):

$$\Omega_{t+1} = \Bigg\{ (x_1, k_1; \ldots; x_{t+1}, k_{t+1}) \in T^*\mathbb{M}_2^{t+1} \mid \exists\, G \in \mathcal{G}_{t+1} \quad \text{and} \\ \exists \quad \text{an immersion} \quad (\hat{\chi}, \hat{\kappa}) \quad \text{of} \quad G \quad \text{such that} \\ k_i = \sum_{\substack{e \in E^G \\ \sigma(e)=i}} \hat{k}_e - \sum_{\substack{f \in E^G \\ \tau(f)=i}} \hat{k}_f \Bigg\}. \tag{113}$$

Remark 4.3.2. Considering how the coordinates $(x_1, \ldots, x_{t+1})$ are distributed in formula (110), we see that the vertexes $\{x_{l+2}, \ldots, x_{t+1}\}$ only have outgoing edges. Conversely the vertexes $\{x_1, \ldots, x_{l+1}\}$ only have incoming edges. This means that the wavefront set Ω_{t+1} of the product (110) can be estimated by a more explicit expression, namely:

$$\Omega_{t+1} \subseteq \tilde{\Omega}_{t+1} := \Big\{ (x_1, k_1; \ldots; x_{l+1}, k_{l+1}; x_{l+2}, k_{l+2}; \ldots; x_{t+1}, k_{t+1}) \in T^*\mathbb{M}_2^{t+1} \\ \text{s.t.} \quad k_1, \ldots, k_{l+1} \in \overline{V}_- \quad \text{and} \quad k_{l+2}, \ldots, k_{t+1} \in \overline{V}_+ \Big\}. \tag{114}$$

Theorem 4.3.1. *The retarded components $R_t\left(L_{int}^{\otimes t} \otimes s_1^N\right)$ and $R_t\left(L_{int}^{\otimes t} \otimes s_2^N\right)$ of the higher conserved currents of the sine-Gordon model are super-renormalizable by power counting in pAQFT.*

Proof. In the previous part of this section, we have collected almost all the elements to prove our conclusive result. It is sufficient now to prove that the distributional product of renormalized anti-chronological products of vertex operators, renormalized time-ordered products of vertex operators and derivatives of Feynman propagators and Wightman two-point functions and their derivatives is well-defined on $\mathbb{M}_2^{t+1}$ according to Hörmander's criterion.

We know from formula (114), an explicit estimate on the wavefront set of the product of Wightman two-point functions and their derivatives. On the other hand, we can regard the product of renormalized anti-chronological products of vertex operators with renormalized time-ordered products of vertex operators and derivatives of Feynman propagators, defined, respectively, on $\mathbb{M}_2^{t-l}$ and $\mathbb{M}_2^{l+1}$, as a tensor product of distributions. We denote it by:

$$\begin{aligned}\mathrm{ACV}_{t-l}\otimes\mathrm{TOF}_{l+1} := &\Big(\big[\bar{T}_{t-l}\big(V_{a_{l+2}}(x_{l+2})\otimes\cdots\otimes V_{a_{t+1}}(x_{t+1})\big)\big]\Big)\\ &\otimes\Big(\big[T_{l+1}\big(V_{a_1}(x_1)\otimes\cdots\otimes V_{a_l}(x_l)\otimes V_{a_{l+1}}(x_{l+1})\big)\big]\\ &\times[(\partial^{i_{1,1}}_{\xi_{l+1}}\Delta^F)\cdots(\partial^{i_{1,n_1}}_{\xi_{l+1}}\Delta^F)](x_1-x_{l+1})\cdots\\ &\cdots[(\partial^{i_{l,1}}_{\xi_{l+1}}\Delta^F)\cdots(\partial^{i_{l,n_l}}_{\xi_{l+1}}\Delta^F)](x_l-x_{l+1})\Big)\in\mathscr{D}'\big(\mathbb{M}_2^{t+1}\big).\end{aligned}\tag{115}$$

From the properties of the tensor product of distributions [11], we have that the wavefront set of $\mathrm{ACV}_{t-l}\otimes\mathrm{TOF}_{l+1}$ is contained in the set:

$$\begin{aligned}\Lambda_{l+1,t-l} := &\Big(\Lambda_{l+1}\times\tilde{\Lambda}_{t-l}\Big)\cup\Big(\Lambda_{l+1}\times\big(\mathbb{M}_2\times\{0\}\big)^{t-l}\Big)\\ &\cup\Big(\big(\mathbb{M}_2\times\{0\}\big)^{l+1}\times\tilde{\Lambda}_{t-l}\Big)\subseteq T^*\mathbb{M}_2^{t+1},\end{aligned}\tag{116}$$

where Λ_{l+1} and $\tilde{\Lambda}_{t-l}$ are defined by formula (96) and formula (107), respectively. From Proposition 4.3.2 and Proposition 4.3.4, we have that

$$\begin{aligned}\Lambda_{l+1,t-l}\cap\bigg(&\Big(\big(\mathbb{M}_2\times\bar{V}_-\big)^{l+1}\cup\big(\mathbb{M}_2\times\bar{V}_+\big)^{l+1}\Big)\\ &\times\Big(\big(\mathbb{M}_2\times\bar{V}_-\big)^{t-l}\cup\big(\mathbb{M}_2\times\bar{V}_+\big)^{t-l}\Big)\bigg)=\emptyset.\end{aligned}\tag{117}$$

If we now consider the set

$$\begin{aligned}\Lambda_{l+1,t-l}+\Omega_{t+1} := \big\{&(x_1,r_1+s_1;\ldots;x_{t+1},r_{t+1}+s_{t+1})\in T^*\mathbb{M}_2^{t+1}\mid\\ &(x_1,r_1;\ldots;x_{t+1},r_{t+1})\in\Lambda_{l+1,t-l},\\ &\text{and}\,(x_1,s_1;\ldots;x_{t+1},s_{t+1})\in\Omega_{t+1}\big\}\end{aligned}\tag{118}$$

and compare formula (114) and formula (117), we see immediately that

$$\big(\Lambda_{l+1,t-l}+\Omega_{t+1}\big)\bigcap\big(\mathbb{M}_2^{t+1}\times\{0\}\big)=\emptyset.\tag{119}$$

Hence Hörmander's sufficient criterion is satisfied. □

Conclusion. We have shown that the renormalization of the time-ordered products and of the anti-chronological products of interactions does not increase the scaling degree estimates for the piecewise renormalized components of the currents, which makes them super-renormalizable.

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A. Proof of Proposition 2.1.1

Proof. First we substitute (7) in (6) and use the power series expansion of sine to get

$$\sum_{\nu=0}^{\infty} A_{\nu,\xi}\alpha^{\nu} = -\varphi_{\xi} + \frac{2}{\alpha}\sum_{\mu=0}^{\infty}\frac{(-1)^{\mu}}{(2\mu+1)!}\left(\frac{1}{2}a\right)^{2\mu+1}\left(\sum_{\nu=0}^{\infty}A_{\nu}\alpha^{\nu}-\varphi\right)^{2\mu+1}. \quad (120)$$

Requiring that the limit for $\alpha \to 0$ of this equation exists gives $A_0 = \varphi$, hence formula above becomes

$$\sum_{\nu=0}^{\infty} A_{\nu,\xi}\alpha^{\nu} = -\varphi_{\xi} + 2\sum_{\mu=0}^{\infty}\frac{(-1)^{\mu}}{(2\mu+1)!}\left(\frac{1}{2}a\right)^{2\mu+1}\alpha^{2\mu}\left(\sum_{\nu=0}^{\infty}A_{\nu+1}\alpha^{\nu}\right)^{2\mu+1}. \quad (121)$$

Now we start comparing the coefficients from the left hand side and the right hand side of Eq. (121) for the first orders:

- At order 0, we have: $A_{0,\xi} = -\varphi_{\xi} + 2\cdot\frac{1}{2}aA_1 \quad\longrightarrow\quad A_1 = \frac{2}{a}\varphi_{\xi}$.
- At order 1 we get: $A_{1,\xi} = 2\cdot\frac{1}{2}aA_2 \quad\longrightarrow\quad A_2 = \frac{2}{a^2}\varphi_{\xi\xi}$.

For orders ≥ 2, we rearrange the summation on the right hand side of equation (121) in the following way:

$$\begin{aligned}\sum_{\mu=0}^{\infty}\frac{(-1)^{\mu}}{(2\mu+1)!}\Big(\frac{1}{2}a\Big)^{2\mu+1}\alpha^{2\mu}\left(\sum_{\nu=0}^{\infty}A_{\nu+1}\alpha^{\nu}\right)^{2\mu+1}\\ =\sum_{\mu,\rho=0}^{\infty}(-1)^{\mu}\Big(\frac{1}{2}a\Big)^{2\mu+1}\left(\sum_{\substack{n_0,\dots,n_\rho\geq 0\\ n_0+\dots+n_\rho=2\mu+1\\ 1\cdot n_1+\dots+\rho\cdot n_\rho=\rho}}\frac{A_1^{n_0}\cdots A_{\rho+1}^{n_\rho}}{n_0!\cdots n_\rho!}\right)\alpha^{\rho+2\mu}.\end{aligned}\tag{122}$$

We rewrite the double summation using indexes $\nu := \rho + 2\mu$ and $\beta := \mu$, so to get the expression:

$$\sum_{\nu=0}^{\infty}\alpha^{\nu}\left(\sum_{\beta=0}^{[\frac{\nu}{2}]}(-1)^{\beta}\Big(\frac{1}{2}a\Big)^{2\beta+1}\sum_{\substack{n_0,\dots,n_{\nu-2\beta}\geq 0\\ n_0+\dots+n_{\nu-2\beta}=2\beta+1\\ \sum_{i=1}^{\nu-2\beta}i\cdot n_i=\nu-2\beta}}\frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}\right),\tag{123}$$

where $[\frac{\nu}{2}]$ is the integer part of $\frac{\nu}{2}$.

We now observe that we can decompose the coefficient of α^{ν} in two parts, one corresponding to $\beta = 0$ and the other for $\beta \geq 1$, respectively:

$$\begin{aligned}\sum_{\beta=0}^{[\frac{\nu}{2}]}(-1)^{\beta}\Big(\frac{a}{2}\Big)^{2\beta+1}\sum_{\substack{n_0,\dots,n_{\nu-2\beta}\geq 0\\ n_0+\dots+n_{\nu-2\beta}=2\beta+1\\ \sum_{i=1}^{\nu-2\beta}i\cdot n_i=\nu-2\beta}}\frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}=\frac{a}{2}\sum_{\substack{n_0,\dots,n_{\nu}\geq 0\\ n_0+\dots+n_{\nu}=1\\ \sum_{i=1}^{\nu}i\cdot n_i=\nu}}\frac{A_1^{n_0}\cdots A_{\nu+1}^{n_{\nu}}}{n_0!\cdots n_{\nu}!}\\ +\sum_{\beta=1}^{[\frac{\nu}{2}]}(-1)^{\beta}\Big(\frac{1}{2}a\Big)^{2\beta+1}\sum_{\substack{n_0,\dots,n_{\nu-2\beta}\geq 0\\ n_0+\dots+n_{\nu-2\beta}=2\beta+1\\ \sum_{i=1}^{\nu-2\beta}i\cdot n_i=\nu-2\beta}}\frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}.\end{aligned}\tag{124}$$

In particular the first term on the right hand side reduces to $\frac{a}{2}A_{\nu+1}$. Comparing the coefficients of the power α^{ν}, for $\nu \geq 2$, from equation (121), we get:

$$A_{\nu,\xi}=aA_{\nu+1}+2\sum_{\beta=1}^{[\frac{\nu}{2}]}(-1)^{\beta}\Big(\frac{a}{2}\Big)^{2\beta+1}\sum_{\substack{n_0,\dots,n_{\nu-2\beta}\geq 0\\ n_0+\dots+n_{\nu-2\beta}=2\beta+1\\ \sum_{i=1}^{\nu-2\beta}i\cdot n_i=\nu-2\beta}}\frac{A_1^{n_0}\cdots A_{\nu-2\beta+1}^{n_{\nu-2\beta}}}{n_0!\cdots n_{\nu-2\beta}!}.\tag{125}$$

Extracting $A_{\nu+1}$ and rescaling the summation over β, we conclude. □

B. Proof of Proposition 2.2.1

Proof. First we introduce some notation. We define:

$$\begin{aligned}
\varphi + \hat{B}_\alpha \varphi &=: \sum_{\nu=0}^{\infty} A_\nu^+ \alpha^\nu, \quad \text{where} \quad \begin{cases} A_0^+ = 2\varphi \\ A_\nu^+ = A_\nu \qquad \forall \nu \geq 1, \end{cases} \\
\varphi - \hat{B}_\alpha \varphi &=: \sum_{\nu=0}^{\infty} A_\nu^- \alpha^\nu, \quad \text{where} \quad \begin{cases} A_0^- = 0 \\ A_\nu^- = -A_\nu \qquad \forall \nu \geq 1. \end{cases}
\end{aligned} \tag{126}$$

We use the power series expansion of cosine and substitute equations above to get the following expressions for the components of the conserved currents:

$$\begin{aligned}
s_1^{(\alpha)} &= \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{(2\mu)!} \left(\frac{1}{2}a\right)^{2\mu} \left[\left(\sum_{\nu=0}^{\infty} A_\nu^+ \alpha^\nu \right)^{2\mu} + \left(\sum_{\nu=0}^{\infty} A_\nu^+ (-\alpha)^\nu \right)^{2\mu} \right], \\
s_2^{(\alpha)} &= -\frac{1}{\alpha^2} \sum_{\mu=1}^{\infty} \frac{(-1)^\mu}{(2\mu)!} \left(\frac{1}{2}a\right)^{2\mu} \left[\left(\sum_{\nu=0}^{\infty} A_\nu^- \alpha^\nu \right)^{2\mu} + \left(\sum_{\nu=0}^{\infty} A_\nu^- (-\alpha)^\nu \right)^{2\mu} \right].
\end{aligned} \tag{127}$$

We remark that both formulas above are symmetric in α, so only even powers will appear. We further manipulate the two components separately. Starting with $s_1^{(\alpha)}$, we expand $\left(\sum_{\nu=0}^{\infty} A_\nu^+ \alpha^\nu\right)^{2\mu}$ and $\left(\sum_{\nu=0}^{\infty} A_\nu^+ (-\alpha)^\nu\right)^{2\mu}$, collect the coefficients of the even powers $\alpha^{2\rho}$ and obtain:

$$s_1^{(\alpha)} = \sum_{\rho=0}^{\infty} \alpha^{2\rho} \left[2 \sum_{\mu=0}^{\infty} (-1)^\mu \left(\frac{1}{2}a\right)^{2\mu} \Bigg(\sum_{\substack{n_0,\ldots,n_{2\rho} \geq 0 \\ n_0 + \cdots + n_{2\rho} = 2\mu \\ \sum_{i=1}^{2\rho} i \cdot n_i = 2\rho}} \frac{(A_0^+)^{n_0} \cdots (A_{2\rho}^+)^{n_{2\rho}}}{n_0! \cdots n_{2\rho}!} \Bigg) \right].$$

We now concentrate on the coefficient of the power $\alpha^{2\rho}$, we call it s_1^ρ:

$$s_1^\rho = 2 \sum_{\mu=0}^{\infty} (-1)^\mu \left(\frac{1}{2}a\right)^{2\mu} \left(\sum_{\substack{n_0,\ldots,n_{2\rho} \geq 0 \\ n_0 + \cdots + n_{2\rho} = 2\mu \\ \sum_{i=1}^{2\rho} i \cdot n_i = 2\rho}} \frac{(A_0^+)^{n_0} \cdots (A_{2\rho}^+)^{n_{2\rho}}}{n_0! \cdots n_{2\rho}!} \right). \tag{128}$$

Specifically, we want to extract the dependence of the powers of A_0^+ on μ. Introducing the index β to account for the possible values of the exponent n_0, we can rewrite formula (128) in the following manner:

$$2 \sum_{\beta=0}^{2\rho} \sum_{\mu \geq \frac{\beta}{2}} (-1)^\mu \left(\frac{1}{2}a\right)^{2\mu} \frac{(A_0^+)^{2\mu-\beta}}{(2\mu-\beta)!} \left(\sum_{\substack{n_1,\ldots,n_{2\rho} \geq 0 \\ n_1 + \cdots + n_{2\rho} = \beta \\ \sum_{i=1}^{2\rho} i \cdot n_i = 2\rho}} \frac{(A_1^+)^{n_1} \ldots (A_{2\rho}^+)^{n_{2\rho}}}{n_1! \ldots n_{2\rho}!} \right). \tag{129}$$

Then we distinguish the cases when β is even or odd. The terms for β even can be collected in the expression:

$$2\sum_{\beta=0}^{\rho}\sum_{\mu=\beta}^{\infty}(-1)^{\mu}\left(\frac{1}{2}a\right)^{2\mu}\frac{(A_0^+)^{2(\mu-\beta)}}{(2(\mu-\beta))!}\left(\sum_{\substack{n_1,\dots,n_{2\rho}\geq 0\\ n_1+\dots+n_{2\rho}=2\beta\\ \sum_{i=1}^{2\rho} i\cdot n_i=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}\right). \tag{130}$$

Rescaling the summation over μ we recognize the power series expansion of $\cos\left(\frac{1}{2}aA_0^+\right)=\cos(a\varphi)$. Hence, for β even we obtain the coefficient:

$$\cos(a\varphi)\left[2\sum_{\beta=0}^{\rho}(-1)^{\beta}\left(\frac{1}{2}a\right)^{2\beta}\sum_{\substack{n_1,\dots,n_{2\rho}\geq 0\\ n_1+\dots+n_{2\rho}=2\beta\\ \sum_{i=1}^{2\rho} i\cdot n_i=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}\right]. \tag{131}$$

On the other hand, assuming $\rho\geq 1$, the terms for β odd are

$$2\sum_{\beta=0}^{\rho-1}\sum_{\mu=\beta+1}^{\infty}(-1)^{\mu}\left(\frac{1}{2}a\right)^{2\mu}\frac{(A_0^+)^{2\mu-2\beta-1}}{(2\mu-2\beta-1)!}\sum_{\substack{n_1,\dots,n_{2\rho}\geq 0\\ n_1+\dots+n_{2\rho}=2\beta+1\\ \sum_{i=1}^{2\rho} i\cdot n_i=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}. \tag{132}$$

Rescaling the summation over μ, we recognize the power series expansion of $\sin\left(\frac{1}{2}aA_0^+\right)=\sin(a\varphi)$. Hence, for β odd we obtain the coefficient:

$$\sin(a\varphi)\left[2\sum_{\beta=0}^{\rho-1}(-1)^{\beta+1}\left(\frac{1}{2}a\right)^{2\beta+1}\sum_{\substack{n_1,\dots,n_{2\rho}\geq 0\\ n_1+\dots+n_{2\rho}=2\beta+1\\ \sum_{i=1}^{2\rho} i\cdot n_i=2\rho}}\frac{(A_1^+)^{n_1}\cdots(A_{2\rho}^+)^{n_{2\rho}}}{n_1!\cdots n_{2\rho}!}\right]. \tag{133}$$

Using the fact that $A_\nu^+=A_\nu$, for $\nu\geq 1$, and changing the name of the upper index ρ to N, we obtain the expected result for s_1^N.

For what concerns $s_2^{(\alpha)}$, we use the fact that $A_0^-=0$ to extract a power $\alpha^{2\mu}$, then we divide by α^2 and finally rewrite the summations rescaling the indexes, thus obtaining:

$$\begin{aligned} s_2^{(\alpha)} &= \sum_{\mu=0}^{\infty}\frac{(-1)^{\mu}}{(2(\mu+1))!}\left(\frac{1}{2}a\right)^{2(\mu+1)}\alpha^{2\mu}\\ &\quad\times\left[\left(\sum_{\nu=0}^{\infty}A_{\nu+1}^-\alpha^{\nu}\right)^{2(\mu+1)}+\left(\sum_{\nu=0}^{\infty}A_{\nu+1}^-(-\alpha)^{\nu}\right)^{2(\mu+1)}\right]. \end{aligned} \tag{134}$$

Expanding $\left(\sum_{\nu=0}^{\infty} A^-_{\nu+1}\alpha^\nu\right)^{2(\mu+1)}$ and $\left(\sum_{\nu=0}^{\infty} A^-_{\nu+1}(-\alpha)^\nu\right)^{2(\mu+1)}$ we see that again only the even powers of α survive and they give:

$$s_2^{(\alpha)} = \sum_{\mu=0}^{\infty} \frac{(-1)^\mu}{(2(\mu+1))!}\left(\frac{1}{2}a\right)^{2(\mu+1)}\alpha^{2\mu} \times \left[2\sum_{\rho=0}^{\infty}\alpha^{2\rho}\left(\sum_{\substack{n_0,\dots,n_{2\rho}\geq 0\\ n_0+\cdots+n_{2\rho}=2(\mu+1)\\ \sum_{i=1}^{2\rho} i\cdot n_i=2\rho}} \frac{(2(\mu+1))!}{n_0!\dots n_{2\rho}!}(A_1^-)^{n_0}\dots(A^-_{2\rho+1})^{n_{2\rho}}\right)\right]. \tag{135}$$

Collecting the powers of α, rewriting the summation using indexes $N := \mu+\rho$ and μ and recalling that $A^-_\nu = -A_\nu$ for $\nu \geq 1$, we finally obtain that the coefficient of α^{2N} is:

$$s_2^N = 2\sum_{\mu=0}^{N}(-1)^\mu\left(\frac{1}{2}a\right)^{2(\mu+1)} \sum_{\substack{n_0,\dots,n_{2(N-\mu)}\geq 0\\ n_0+\cdots+n_{2(N-\mu)}=2(\mu+1)\\ \sum_{i=1}^{2(N-\mu)} i\cdot n_i=2(N-\mu)}} \frac{A_1^{n_0}\cdots A_{2(N-\mu)+1}^{n_{2(N-\mu)}}}{n_0!\cdots n_{2(N-\mu)}!}.$$

□

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