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The Double Semion State in Infinite Volume

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Abstract. We describe in a simple setting how to extract a braided tensor category from a collection of superselection sectors of a two-dimensional quantum spin system, corresponding to abelian anyons. We extract from this category its fusion ring as well as its F and R-symbols. We then construct the double semion state in infinite volume and extract the braided tensor category describing its semion, anti-semion, and bound state excitations. We verify that this category is equivalent to the representation category of the twisted quantum double $\mathcal{D}^{\phi}(\mathbb{Z}_2)$.

1. Introduction

Gapped ground states of two-dimensional quantum lattice systems may support anyonic excitations. Paradigmatic examples include Kitaev's quantum double models [15] and, more generally, the string-net models of Levin and Wen [16]. It is widely believed that the types of anyons supported by a given ground state are organised in a unitary braided fusion category, whose simple objects are the anyon types and whose structure is described by F and R-matrices.

Recent years have seen a lot of progress towards justifying this belief, by adapting the DHR analysis of superselection sectors in algebraic quantum field theory [5,6,9–11] to the setting of microscopic lattice systems [3,19,21]. This body of work manages to associate with a large class of gapped ground states a strict braided C^*-tensor category whose objects are localised and transportable endomorphisms of the observable algebra, and shows that this category is a robust invariant for the gapped phase [1] to which the ground state belongs.

One can extract from any braided tensor category its fusion ring and F-symbols, which encode the tensor structure of the category, as well as R-symbols, which encode the braided structure of the category. This yields in particular a microscopic definition of fusion rules, F-symbols, and R-symbols,

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which are commonly employed in the physics literature to describe anyon theories. We note that a microscopic definition of F and R-symbols has been given before in [14]. Extracting the F and R-symbols from the DHR-type analysis has the advantage that it is clear that these data yield a robust invariant of gapped phases [3, 21].

In Sect. 2 of the paper, we review the construction of a braided tensor category from a given ground state under some simplifying assumptions. We assume in particular that the anyons we consider are abelian. By, moreover, assuming that the set of anyon types is finite, and that each anyon has an anti-particle, we show that corresponding fusion ring is group-like and abelian and is equipped with F and R-symbols that satisfy the pentagon and hexagon equations.

Section 3 is devoted to the construction and analysis of the double semion state [16] in infinite volume. This is the simplest state that supports abelian anyons whose braided tensor category has non-trivial F-symbols. We show that the double semion ground state satisfies the assumptions of Sect. 2 with anyon content consisting of the vacuum, the semion, the anti-semion, and a semion–anti-semion bound state. We extract the fusion ring, F-symbols, and R-symbols from the corresponding braided tensor category. Finally, we use a result from [12] to conclude that the braided tensor category describing these anyons is equivalent to the category $\mathrm{Rep}_f(\mathcal{D}^\phi(\mathbb{Z}_2))$ of finite- dimensional representations of the twisted quantum double algebra of $\mathbb{Z}_2$. Since the double semion model is the string-net model defined by input fusion category $\mathcal{F} = \mathrm{Vec}_{\mathbb{Z}_2}^\phi$ with ϕ a non-trivial 3-cocycle of $\mathbb{Z}_2$, this partially verifies the conjecture that the resulting anyon theory is described by the Drinfeld centre $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2}^\phi) = \mathrm{Rep}_f(\mathcal{D}^\phi(\mathbb{Z}_2))$, see [17].

In appendix, we show various properties of the double semion state that are used in Sect. 3. In particular, we introduce a new technique based on results from [13] to show that the double semion state we construct is pure. This technique is applicable to the construction and proof of purity of ground states and excited states of a large class of lattice spin models including all Levin–Wen string-net models.

2. Braided Tensor Category for Abelian Anyons

2.1. *Setup.*

2.1.1. Algebra of Observables and State. Consider a countable set $\Gamma \subset \mathbb{R}^2$ of sites in the plane. To each site $x \in \Gamma$, we associate an algebra $\mathcal{A}_x \simeq \mathrm{End}(\mathbb{C}^d)$ for some fixed $d \geq 2$. For any finite $X \subset \Gamma$, we set $\mathcal{A}_X = \bigotimes_{x \in X} \mathcal{A}_x$. If $X \subset Y$ are finite subsets of Γ, then there is a natural norm-preserving inclusion $\mathcal{A}_X \hookrightarrow \mathcal{A}_Y$ by tensoring with the identity.

For any infinite subset $Y \subset \Gamma$, we then have a local net of algebras $\mathcal{A}_X$ for finite $X \subset Y$, whose direct limit is $\mathcal{A}_{Y,\mathrm{loc}}$, the algebra of local observables supported in Y. Its norm completion is $\mathcal{A}_Y := \overline{\mathcal{A}_{Y,\mathrm{loc}}}^{\|\cdot\|}$, the algebra of quasi-local observables supported in Y, and we get inclusions $\mathcal{A}_X \hookrightarrow \mathcal{A}_Y$ also for infinite $X \subset Y$. We write $\mathcal{A}_{\mathrm{loc}} = \mathcal{A}_{\Gamma,\mathrm{loc}}$ and $\mathcal{A} = \mathcal{A}_\Gamma$ for the algebra of all local and all quasi-local observables, respectively. The *support* of an observable $O \in \mathcal{A}$ is the smallest set $X \subset \Gamma$ such that $O \in \mathcal{A}_X$.

Similarly, the support of an automorphism w of $\mathcal{A}$ is the smallest set Y such that $w|_{\mathcal{A}_{Y^c}} = \mathrm{id}_{\mathcal{A}_{Y^c}}$. Any subset $Z \subset \mathbb{R}^2$ of the plane determines a subset $\overline{Z} = Z \cap \Gamma$ of Γ, and we denote $\mathcal{A}_Z := \mathcal{A}_{\overline{Z}}$.

A major role is played by *cones*. The cone with apex at $a \in \mathbb{R}^2$, axis $\hat{v} \in \mathbb{R}^2$ of unit length, and opening angle $\theta \in (0, 2\pi)$ is

$$\Lambda_{a,\hat{v},\theta} := \{x \in \mathbb{R}^2 \mid (x-a)\cdot\hat{v} > \|x-a\| \cos(\theta/2)\}. \tag{1}$$

We will assume that $\overline{\Lambda}$ is infinite for any cone Λ. In particular, this holds if Γ is a lattice.

We will consider a pure state ω on $\mathcal{A}$ with GNS representation $(\pi_1, \mathcal{H}, \Omega)$. For any $X \subset \Gamma$, we put $\mathcal{R}(X) := (\pi_1(\mathcal{A}_X))''$, the von Neumann algebra associated with X. For $Z \subset \mathbb{R}^2$, we also write $\mathcal{R}(Z) = \mathcal{R}(\overline{Z})$. We note that if Λ is a cone, then $\mathcal{R}(\Lambda)$ is an infinite factor (Theorem 5.2 of [19]).

2.1.2. Superselection Sectors.

Definition 2.1. An irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is said to satisfy the superselection criterion with respect to π_1 if for any cone Λ there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$U\pi(O)U^* = \pi_1(O) \quad \text{for all } O \in \mathcal{A}_{\Lambda^c} \tag{2}$$

If two representations π, π' are unitarily equivalent, then we write $\pi \simeq \pi'$.

We assume that we have a finite set of irreducible representations $\mathcal{O} = \{\pi_a \mid a \in I\}$ of $\mathcal{A}$ indexed by a labelling set I. We assume moreover that $\pi_a \simeq \pi_b$ if and only if $a = b$, so all sectors in $\mathcal{O}$ are truly distinct. Moreover, $1 \in I$ so that $\pi_1 \in \mathcal{O}$. We call π_1 the *vacuum sector*.

We will now make some additional Assumptions (1–4) on these sectors which will in particular imply that they satisfy the superselection criterion with respect to π_1. These assumptions are not generically satisfied by gapped ground states, but Assumptions (1–3) and the first part of Assumption 4 are verified for the Toric code model in [19] and for all abelian quantum double models in [8]. The second part of Assumption 4 can be shown for these models using similar methods. Below, we will verify these assumptions for the double semion model. More generally, the authors expect that these assumptions hold for all abelian string-net models [16].

Assumption 1. For any cone Λ and any $a \in I$, there is an automorphism $w_{a,\Lambda}$ supported on Λ such that

$$\pi_a \simeq \pi_1 \circ w_{a,\Lambda}. \tag{3}$$

In particular, we take $w_{1,\Lambda} = \mathrm{id}$ for all cones Λ.

The following assumption says that the anyons we study are abelian.

Assumption 2. For any $a, b \in I$ there is a unique $c \in I$ such that for any two cones Λ_1, Λ_2 we have $\pi_c \simeq \pi_1 \circ w_{a,\Lambda_1} \circ w_{b,\Lambda_2}$. We write $c = a \times b$.

The following assumption says that each anyon has an antiparticle.

Assumption 3. For each $a \in I$, there is an $a^* \in I$ such that $a \times a^* = 1$.

The final assumption is of a technical nature it plays an important role in constructing a tensor category in Sect. 2.2.

Assumption 4. Assumption 1 implies that if $\Lambda_1, \Lambda_2 \subset \Lambda$ are cones, then $\pi_1 \circ w_{a,\Lambda_1} \simeq \pi_1 \circ w_{a,\Lambda_2}$. We assume that any unitary $V \in \mathcal{B}(\mathcal{H})$ implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.

Similarly, it follows from Assumption 2 that $\pi_1 \circ w_{a,\Lambda} \circ w_{b,\Lambda} \simeq \pi_1 \circ w_{a \times b,\Lambda}$. We assume that any unitary $V \in \mathcal{B}(\mathcal{H})$ implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.

This assumption is implied by Haag duality (cf. the discussion at the end of Sect. 2.2.4) and can also often be proven directly if the automorphisms $w_{a,\Lambda}$ are known explicitly, for example, for the abelian quantum double models [8, 19] and for the double semion model treated below.

Assumptions 1–3 have a few elementary but important consequences.

Lemma 2.2. *The representations π_a, $a \in I$ satisfy the superselection criterion w.r.t. π_1.*

Proof. Fix a cone Λ. By Assumption 1, there is an automorphism $w_{a,\Lambda}$ supported in Λ such that $\pi_1 \circ w_{a,\Lambda} \simeq \pi_a$. i.e. there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$U\pi_a(O)U^* = \pi_1(w_{a,\Lambda}(O)) \tag{4}$$

for all $O \in \mathcal{A}$. Since $w_{a,\Lambda}$ is supported in Λ, we have $w(O) = O$ for $O \in \mathcal{A}_{\Lambda^c}$ and therefore

$$U\pi_a(O)U^* = \pi_1(O) \quad \text{for all } O \in \mathcal{A}_{\Lambda^c}. \tag{5}$$

□

Lemma 2.3. *The binary operation $\times : I \times I \to I$ makes I into an abelian group with unit 1 and inverse $a^{-1} = a^*$.*

Proof. We first show that $\times$ is abelian. Take $a, b \in I$. Assumption 2 says that for any two cones Λ_1, Λ_2 there are automorphisms w_{a,Λ_1}, w_{b,Λ_2} such that $\pi_{a\times b} \simeq \pi_1 \circ w_{a,\Lambda_1} \circ w_{b,\Lambda_2}$. Exchanging the roles of a and b and of Λ_1 and Λ_2, we have $\pi_{b\times a} \simeq \pi_1 \circ w_{b,\Lambda_2} \circ w_{a,\Lambda_1}$. If we now take Λ_1 and Λ_2 to be disjoint, then certainly $w_{a,\Lambda_1} \circ w_{b,\Lambda_2} = w_{b,\Lambda_2} \circ w_{a,\Lambda_1}$ and therefore $\pi_{a\times b} \simeq \pi_{b\times a}$. But we assumed that two representations in $\mathcal{O}$ are unitarily equivalent only if they are the same, so $a \times b = b \times a$.

We now show that 1 is the identity for the product $\times$. Fix cones Λ_1 and Λ_2. By Assumptions 1 and 2, there are automorphisms $w_{1,\Lambda_1} = \mathrm{id}$ and w_{a,Λ_2} such that $\pi_{1\times a} \simeq \pi_1 \circ \mathrm{id} \circ w_{a,\Lambda_2} = \pi_1 \circ w_{a,\Lambda_2} \simeq \pi_a$, hence $1 \times a = a$. We already know that $\times$ is abelian, so also $a \times 1 = a$.

Finally, Assumption 3 states that a^* is the inverse of a. □

We will often write $ab = a \times b$ for the product of elements $a, b \in I$.

2.2. *Braided Tensor Category.* It is well understood how to associate a braided tensor category with a pure state on a quantum spin system [3,19,21]. In this section, we recap this construction and use Assumptions 1–4 to identify a braided fusion subcategory with abelian fusion rules.

2.2.1. Tensor Category of Localised and Transportable Endomorphisms. Fix a unit vector $\hat{f} \in \mathbb{R}^2$, representing a 'forbidden direction'. We say a cone $\Lambda_{a,\hat{v},\theta}$ with axis $\hat{v}$ and opening angle θ is *forbidden* if it contains the forbidden direction $\hat{f}$, i.e. if $\hat{v} \cdot \hat{f} > \cos(\theta/2)$. A cone that is not forbidden is said to be *allowed.* We define an *allowed algebra* by

$$\mathcal{B} := \overline{\bigcup_{\text{allowed } \Lambda} \mathcal{R}(\Lambda)}^{\|\cdot\|} \subset \mathcal{B}(\mathcal{H}). \tag{6}$$

This is a C^*-algebra that contains $\pi_1(\mathcal{A})$ since any local observable is supported in some allowed cone.

Definition 2.4. We say an endomorphism $\bar{\rho}$ of $\mathcal{B}$ is localised on Λ if $\bar{\rho}(\pi_1(O)) = \pi_1(O)$ for all $O \in \mathcal{A}_{\Lambda^c}$, and that $\bar{\rho}$ is localised if it is localised on some allowed cone Λ. We say that an endomorphism $\bar{\rho}$ of $\mathcal{B}$ that is localised on an allowed cone Λ is transportable if for any allowed cone Λ' there is an endomorphism $\bar{\rho}'$ of $\mathcal{B}$, localised on Λ', and a unitary $U \in \mathcal{B}$ such that $U\,\bar{\rho}(\pi_1(O)) = \bar{\rho}'(\pi_1(O))\,U$ for all $O \in \mathcal{A}$ and such that $U \in \mathcal{R}(\widetilde{\Lambda})$ for any allowed cone $\widetilde{\Lambda}$ that contains the cones Λ and Λ'. We denote by Δ the set of all localised and transportable endomorphisms.

The localised and transportable endomorphisms of $\mathcal{B}$ are the objects of a $\mathbb{C}$-linear category with morphisms

$$(\bar{\rho}, \bar{\sigma}) := \{R \in \mathcal{B} \,:\, R\bar{\rho}(\pi_1(O)) = \bar{\sigma}(\pi_1(O))R \ \text{ for all } \ O \in \mathcal{A}\} \tag{7}$$

for any $\bar{\rho}, \bar{\sigma} \in \Delta$. Morphisms are referred to as intertwiners.

Direct sums of objects can be constructed as in Lemma 6.1 of [19]. Indeed, for any cone Λ, Corollary 5.3 of [19] shows that there are isometries $V_1, V_2 \in \mathcal{R}(\Lambda)$ such that $V_i^* V_j = \delta_{i,j}\,\mathbb{1}$ and $V_1 V_1^* + V_2 V_2^* = \mathbb{1}$. If $\overline{\rho}, \overline{\sigma}$ are localised on Λ, then $\mathrm{Ad}[V_1] \circ \overline{\rho} + \mathrm{Ad}[V_2] \circ \overline{\sigma}$ is a direct sum of $\overline{\rho}$ and $\overline{\sigma}$ which is still localised on Λ. Moreover, if $\overline{\rho}$ and $\overline{\sigma}$ are transportable, then for any allowed cone Λ' there are endomorphisms $\overline{\rho}', \overline{\sigma}'$ localised on Λ' and unitary morphisms $U_1 \in (\overline{\rho}, \overline{\rho}')$ and $U_2 \in (\overline{\sigma}, \overline{\sigma}')$ such that $U_1, U_2 \in \mathcal{R}(\widetilde{\Lambda})$ for any allowed cone $\widetilde{\Lambda} \supseteq \Lambda \cup \Lambda'$. Then take isometries $V_1', V_2' \in \mathcal{R}(\Lambda')$ as above and consider the direct sum $\mathrm{Ad}[V_1'] \circ \overline{\rho}' + \mathrm{Ad}[V_2'] \circ \overline{\sigma}'$ of $\overline{\rho}'$ and $\overline{\sigma}'$, which is localised on Λ'. The unitary $W = V_1'\,U_1\,V_1^* + V_2'\,U_2\,V_2^* \in \mathcal{R}(\widetilde{\Lambda})$ then intertwines the two direct sums, showing that direct sums of localised and transportable endomorphisms are again localised and transportable.

The category of localised and transportable endomorphisms of $\mathcal{B}$ can be equipped with a monoidal structure. For any $\bar{\rho}, \bar{\sigma} \in \Delta$, we define their tensor product by

$$\bar{\rho} \otimes \bar{\sigma} := \bar{\rho} \circ \bar{\sigma}, \tag{8}$$

and for any intertwiners $R \in (\bar{\rho}, \bar{\rho}')$ and $S \in (\bar{\sigma}, \bar{\sigma}')$ we define the tensor product by

$$R \otimes S := R\bar{\rho}(S) \in (\bar{\rho} \otimes \bar{\sigma}, \bar{\rho}' \otimes \bar{\sigma}'). \tag{9}$$

2.2.2. Subcategory of Abelian Anyons. We use Assumptions 1–4 to obtain a subcategory of Δ whose simple objects correspond to the anyon types $a \in I$.

The following lemma shows that the automorphisms $w_{a,\Lambda}$ of Assumption 1 can be extended to localised and transportable endomorphisms of the allowed algebra $\mathcal{B}$, i.e. they yield objects in the category Δ.

Lemma 2.5 (Proposition 4.6 [19]). *For each allowed cone Λ and each $a \in I$, the automorphism $w_{a,\Lambda}$ of Assumption 1 has a unique extension to an endomorphism $\overline{w}_{a,\Lambda}$ of $\mathcal{B}$ that is weakly continuous on $\mathcal{R}(\Lambda')$ for any allowed cone Λ'. If $\Lambda' \supset \Lambda$, then $\overline{w}_{a,\Lambda}(\mathcal{R}(\Lambda')) = \mathcal{R}(\Lambda')$. Moreover, $\overline{w}_{a,\Lambda}$ is localised on Λ, and is transportable. In particular, $\overline{w}_{a,\Lambda} \in \Delta$.*

Proof. Let $\Lambda' \supset \Lambda$ be any allowed cone that contains Λ. From Assumption 1, we have for any forbidden cone Λ'' with $\Lambda' \cap \Lambda'' = \emptyset$ that $\pi_1 \circ w_{a,\Lambda} \simeq \pi_1 \circ w_{a,\Lambda''}$. Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary implementing this equivalence. We have for any $O \in \mathcal{A}_{\Lambda'}$ that

$$\pi_1\big(w_{a,\Lambda}(O)\big) = V\pi_1\big(w_{a,\Lambda''}(O)\big)V^* = V\pi_1(O)V^* \tag{10}$$

where we used that $w_{a,\Lambda''}$ is supported on Λ'', which is disjoint form Λ'. We define the action of $\overline{w}_{a,\Lambda}$ on $\mathcal{R}(\Lambda')$ by $\mathrm{Ad}(V)$, which is weakly continuous, and is uniquely determined by the action of $w_{a,\Lambda}$ on $\mathcal{A}_{\Lambda'}$. Clearly this action on $\mathcal{R}(\Lambda')$ does not depend on the choice of Λ''. It follows that the extensions to $\mathcal{R}(\Lambda')$ for different Λ' are consistent with each other. Together with the weak continuity, this shows that the extension $\overline{w}_{a,\Lambda}$ is well defined on all of $\mathcal{B}$. Moreover, we have $w_{a,\Lambda}(\mathcal{A}_{\Lambda'}) = \mathcal{A}_{\Lambda'}$ and weak continuity then implies $\overline{w}_{a,\Lambda}(\mathcal{R}(\Lambda')) = (w_{a,\Lambda}(\mathcal{A}_{\Lambda'}))'' = \mathcal{R}(\Lambda')$ as required. This implies the inclusion $\overline{w}_{a,\Lambda}(\mathcal{B}) \subseteq \mathcal{B}$, so $\overline{w}_{a,\Lambda}$ is indeed an endomorphism of $\mathcal{B}$.

The endomorphism $\overline{w}_{a,\Lambda}$ is localised on Λ by construction. To see that it is transportable, let Λ' be any allowed cone and let $\overline{w}_{a,\Lambda'}$ be the extension of the automorphism $w_{a,\Lambda'}$ to the allowed algebra $\mathcal{B}$. By Assumption 4, there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $U \in \mathcal{R}(\Lambda'')$ for any allowed cone $\Lambda'' \supset \Lambda \cup \Lambda'$ and

$$U\,\pi_1\big(w_{a,\Lambda}(O)\big) = \pi_1(w_{a,\Lambda'}(O))\,U \tag{11}$$

for all $O \in \mathcal{A}$. It follows from this and the construction of $\overline{w}_{a,\Lambda}$ and $\overline{w}_{a,\Lambda'}$ that $U \in (\overline{w}_{a,\Lambda}, \overline{w}_{a,\Lambda'})$, showing that $\overline{w}_{a,\Lambda}$ is transportable. $\square$

Lemma 2.6. *The endomorphisms $\overline{w}_{a,\Lambda}$ are simple objects of Δ, and two such objects $\overline{w}_{a,\Lambda}$ and $\overline{w}_{b,\Lambda'}$ are isomorphic if and only if $a = b$. That is,*

$$(\overline{w}_{a,\Lambda}, \overline{w}_{b,\Lambda'}) \simeq \begin{cases} \mathbb{C}\,\mathbb{1} & \text{if } a = b \\ \{0\} & \text{otherwise.} \end{cases} \tag{12}$$

Proof. Suppose $R \in (\overline{w}_{a,\Lambda}, \overline{w}_{b,\Lambda'})$, i.e. the operator $R \in \mathcal{B}$ satisfies

$$R\,\overline{w}_{a,\Lambda}(\pi_1(O)) = \overline{w}_{b,\Lambda'}(\pi_1(O))R \tag{13}$$

for all $O \in \mathcal{A}$. By construction of $\overline{w}_{a,\Lambda}$, this implies

$$R\,(\pi_1 \circ w_{a,\Lambda})(O) = (\pi_1 \circ w_{b,\Lambda'})(O)\,R \tag{14}$$

for all $O \in \mathcal{A}$. By Assumption 1, we have $\pi_1 \circ w_{a,\Lambda} \simeq \pi_a$ and $\pi_1 \circ w_{b,\Lambda} \simeq \pi_b$, so there are unitaries $U_a, U_b \in \mathcal{B}(\mathcal{H})$ such that

$$RU_a\,\pi_a(O)\,U_a^* = U_b\,\pi_b(O)\,U_b^* R \tag{15}$$

for all $O \in \mathcal{A}$. We see that $U_b^* R U_a$ intertwines the irreducible representations π_a and π_b. By assumption, we have $\pi_a \simeq \pi_b$ if and only if $a = b$. So if $a = b$ then we must have $U_b^* R U_a \in \mathbb{C}\,\mathbb{1}$ which holds if and only if $R \in \mathbb{C}I$, and if $a \neq b$ then we must have $U_b^* R U_a = 0$ hence $R = 0$. □

Recall from Assumption 2 and Lemma 2.3 that the set of anyon types I is equipped with an abelian product $\times$.

Lemma 2.7. *If Λ and Λ' are allowed cones and $a, b \in I$, then $\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'} \simeq \overline{w}_{a\times b,\Lambda''}$ for any allowed cone Λ''. If $\widetilde{\Lambda} \supseteq \Lambda \cup \Lambda' \cup \Lambda''$, then the intertwiners that realise this isomorphism are elements of $\mathcal{R}(\widetilde{\Lambda})$.*

Proof. By Assumption 4, we have $\pi_1 \circ w_{a,\Lambda} \simeq \pi_1 \circ w_{a,\Lambda''}$ implemented by unitaries $V \in \mathcal{R}(\widetilde{\Lambda})$. It follows from this and Lemma 2.6 that $(\overline{w}_{a,\Lambda}, \overline{w}_{a,\Lambda''})$ is spanned by a unitary $U_a \in \mathcal{R}(\widetilde{\Lambda})$. Similarly, $(\overline{w}_{b,\Lambda'}, \overline{w}_{b,\Lambda''})$ is spanned by a unitary $U_b \in \mathcal{R}(\widetilde{\Lambda})$.

Again by Assumption 4, we have $\pi_1 \circ w_{a,\Lambda''} \circ w_{b,\Lambda''} \simeq \pi_1 \circ w_{a\times b,\Lambda''}$ with unitary intertwiners all belonging to $\mathcal{R}(\Lambda'')$. It follows that $\overline{w}_{a,\Lambda''} \otimes \overline{w}_{b,\Lambda''}$ is isomorphic to $\overline{w}_{a\times b,\Lambda''}$ and is therefore simple. Moreover, $(\overline{w}_{a,\Lambda''} \otimes \overline{w}_{b,\Lambda''}, \overline{w}_{a\times b,\Lambda''})$ is spanned by a unitary $V_{a\times b} \in \mathcal{R}(\Lambda'') \subset \mathcal{R}(\widetilde{\Lambda})$.

We now find that $V_{a\times b}(U_a \otimes U_b) \in \mathcal{R}(\widetilde{\Lambda})$ is a unitary intertwiner in $(\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'}, \overline{w}_{a\times b,\Lambda''})$. Since $\overline{w}_{a\times b,\Lambda''}$ is simple, so is $\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'}$, and it follows that $(\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'}, \overline{w}_{a\times b,\Lambda''})$ is actually spanned by $V_{a\times b}(U_a \otimes U_b)$. This proves the claim. □

We now identify a full subcategory of Δ whose isomorphism classes correspond to sums of anyons types $a \in I$.

For any allowed cone Λ, we fix isometries $V_1, V_2 \in \mathcal{R}(\Lambda)$ such that $V_i^* V_j = \delta_{i,j}\,\mathbb{1}$ and $V_1 V_1^* + V_2 V_2^* = \mathbb{1}$ (existence follows from Corollary 5.3 of [19]) and define for any $\bar{\rho}, \bar{\sigma} \in \Delta$ the concrete direct sum $\bar{\rho} \oplus_\Lambda \bar{\sigma} := \mathrm{Ad}[V_1] \circ \bar{\rho} + \mathrm{Ad}[V_2] \circ \bar{\sigma}$.

Definition 2.8. We let Δ_Λ^I be the full tensor subcategory of Δ generated by the simple objects $\{\overline{w}_{a,\Lambda}\}_{a\in I}$ using the tensor product $\otimes$ and the direct sum $\oplus_\Lambda$.

By construction, all objects of Δ_Λ^I are localised on Λ. Moreover, the category Δ_Λ^I is semisimple as show in the following lemma.

Lemma 2.9. *Every object of Δ^I_Λ is isomorphic to an object of the form*

$$\overline{\rho}_1 \oplus_\Lambda \overline{\rho}_2 \oplus_\Lambda \cdots \oplus_\Lambda \overline{\rho}_n \tag{16}$$

where the $\overline{\rho}_i$ are tensor products of the simple objects $\overline{w}_{a,\Lambda}$ and the isomorphism is given by a unitary in $\mathcal{R}(\Lambda)$.

(Note that the direct sum $\oplus_\Lambda$ is not associative, so the expression (16) is to be interpreted as being defined by some choice of bracketing. Different bracketings are isomorphic through a unitary in $\mathcal{R}(\Lambda)$).

Proof. For the duration of this proof, we write $\oplus = \oplus_\Lambda$. Denote by $V_1, V_2 \in \mathcal{R}(\Lambda)$ the isometries used to construct the direct sum $\oplus_\Lambda$, and for any $\overline{\rho} \in \Delta$ we write $\oplus_{\overline{\rho}}$ for the direct sum constructed using the isometries $\overline{\rho}(V_1), \overline{\rho}(V_2)$.

Suppose the claim is true for two objects $\overline{\rho}$ and $\overline{\sigma}$ of Δ^I_Λ, i.e. $\overline{\rho} \simeq \overline{\rho}_1 \oplus \cdots \oplus \overline{\rho}_n$ and $\overline{\sigma} \simeq \overline{\sigma}_1 \oplus \cdots \oplus \overline{\sigma}_m$ with isomorphisms implemented by unitaries in $\mathcal{R}(\Lambda)$, and where the $\overline{\rho}_i$ and $\overline{\sigma}_j$ are finite tensor products of the simple objects $\overline{w}_{a,\Lambda}$. The claim then holds trivially for $\overline{\rho} \oplus \overline{\sigma}$. Let us now show that the claim holds for the tensor product $\overline{\rho} \otimes \overline{\sigma}$. We have

$$\begin{aligned}\overline{\rho} \otimes \overline{\sigma} &\simeq (\overline{\rho}_1 \oplus \cdots \oplus \overline{\rho}_n) \otimes (\overline{\sigma}_1 \oplus \cdots \oplus \overline{\sigma}_m) \\ &= \bigoplus_{\kappa=1}^{n} (\overline{\rho}_\kappa \otimes \overline{\sigma}_1) \oplus_{\overline{\rho}_\kappa} \cdots \oplus_{\overline{\rho}_\kappa} (\overline{\rho}_\kappa \otimes \overline{\sigma}_m)\end{aligned}$$

where the isomorphism is implemented by a unitary in $\mathcal{R}(\Lambda)$. Noting that the direct sums $\oplus_{\overline{\rho}_\kappa}$ are isomorphic to $\oplus$ through a unitary in $\mathcal{R}(\Lambda)$ we obtain the required equivalence of $\overline{\rho} \otimes \overline{\sigma}$ to an object of the form (16). We have shown that if $\overline{\rho}$ and $\overline{\sigma}$ both satisfy the claim, then so do $\overline{\rho} \oplus \overline{\sigma}$ and $\overline{\rho} \otimes \overline{\sigma}$. Since the claim holds trivially for the simple objects $\{\overline{w}_{a,\Lambda}\}_{a \in I}$ and the category Δ^I_Λ is by definition generated by these simple objects using $\otimes$ and $\oplus$, we conclude that every object of Δ^I_Λ is isomorphic to an object of the form (16) through a unitary in $\mathcal{R}(\Lambda)$, as we wanted to show. □

Lemma 2.10. *Let Λ be an allowed cone and $\Lambda_1, \Lambda_2 \subset \Lambda$ two allowed subcones. For any two objects $\overline{\rho} \in \Delta^I_{\Lambda_1}$ and $\overline{\sigma} \in \Delta^I_{\Lambda_2}$, we have $(\overline{\rho}, \overline{\sigma}) \subset \mathcal{R}(\Lambda)$.*

Proof. Let $\overline{\rho}_1, \overline{\rho}_2 \in \Delta^I_{\Lambda_1}$ and $\overline{\sigma}_1, \overline{\sigma}_2 \in \Delta^I_{\Lambda_2}$ and suppose $(\overline{\rho}_k, \overline{\sigma}_l) \subset \mathcal{R}(\Lambda)$ for $k, l \in \{1, 2\}$. We will show that $(\overline{\rho}_1 \oplus_{\Lambda_1} \overline{\rho}_2, \overline{\sigma}_1 \oplus_{\Lambda_2} \overline{\sigma}_2) \subset \mathcal{R}(\Lambda)$. According to Lemma 2.9, the result then follows by induction on the number of summands in (16), since it holds for the simple objects $\{\overline{w}_{a,\Lambda_1}\}_{a \in I}$ and $\{\overline{w}_{a,\Lambda_2}\}_{a \in I}$ by Lemma 2.6 and for finite tensor products of these by repeated application of Lemma 2.7.

For $i \in \{1, 2\}$, let $V_1^{(i)}, V_2^{(i)} \in \mathcal{R}(\Lambda_i)$ be the isometries used to define $\oplus_{\Lambda_i}$, and let $p_k^{(i)} = V_k^{(i)} (V_k^{(i)})^*$ for $k = 1, 2$. Then $(V_k^{(i)})^* p_l^{(i)} = \delta_{k,l} (V_k^{(i)})^*$ and $p_k^{(i)} V_l^{(i)} = \delta_{k,l} V_k^{(i)}$. Suppose $R \in (\overline{\rho}_1 \oplus_{\Lambda_1} \overline{\rho}_2, \overline{\sigma}_1 \oplus_{\Lambda_2} \overline{\sigma}_2)$, then

$$\begin{aligned}&R\big(\mathrm{Ad}[V_1^{(1)}] \circ \overline{\rho}_1(O) + \mathrm{Ad}[V_2^{(1)}] \circ \overline{\rho}_2(O)\big) \\ &\quad = \big(\mathrm{Ad}[V_1^{(2)}] \circ \overline{\sigma}_1(O) + \mathrm{Ad}[V_2^{(2)}] \circ \overline{\sigma}_2(O)\big) R\end{aligned} \tag{17}$$

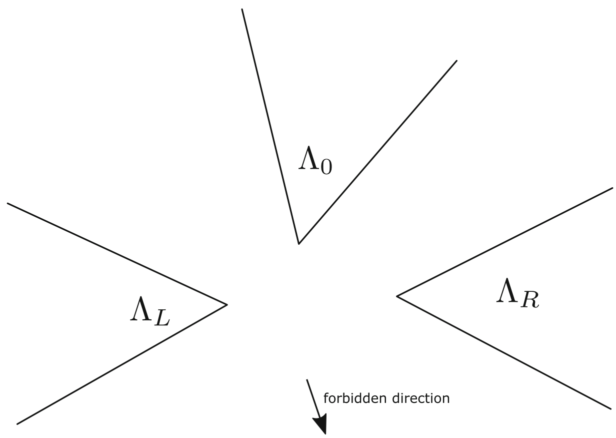

FIGURE 1. Cones Λ_0, Λ_L and Λ_R used in the definition of the braiding intertwiners $\epsilon(\rho, \sigma)$

for all $O \in \mathcal{A}$. Multiplying from the left with $(V_k^{(2)})^*$ and from the right with $V_l^{(1)}$ yields

$$(V_k^{(2)})^* R\, V_l^{(1)}\, \overline{\rho}_l(O) = \overline{\sigma}_k(O)\, (V_k^{(2)})^* R\, V_l^{(1)} \tag{18}$$

for all $O \in \mathcal{A}$, so $(V_k^{(2)})^* R\, V_l^{(1)} \in (\overline{\rho}_l, \overline{\sigma}_k)$. So then by hypothesis, $(V_k^{(2)})^* R\, V_l^{(1)} \in \mathcal{R}(\Lambda)$ for all $k, l = 1, 2$, and also $p_k^{(2)} R\, p_l^{(1)} \in \mathcal{R}(\Lambda)$. Therefore, $R = \sum_{k,l} p_k^{(2)} R\, p_l^{(1)} \in \mathcal{R}(\Lambda)$ as required. $\square$

2.2.3. Braided Structure. Fix an allowed cone Λ_0 and consider two endomorphisms $\bar{\rho}, \bar{\sigma} \in \Delta^I_{\Lambda_0}$. Pick allowed cones Λ_L and Λ_R as in Fig. 1. i.e. the disjoint allowed cones Λ_R, Λ_0 and Λ_L are arranged in a counterclockwise order from the forbidden direction, and there are allowed cones $\widetilde{\Lambda}_L \supset \Lambda_L \cup \Lambda_0$ and $\widetilde{\Lambda}_R \supset \Lambda_R \cup \Lambda_0$ such that $\widetilde{\Lambda}_L \cap \Lambda_R = \widetilde{\Lambda}_R \cap \Lambda_L = \emptyset$. We say Λ_L is to the left of Λ_0, and Λ_R is to the right of Λ_0. By transportability, there are endomorphisms $\bar{\rho}_L \in \Delta^I_{\Lambda_L}$ and $\bar{\sigma}_R \in \Delta^I_{\Lambda_R}$, and unitary intertwiners $U \in (\bar{\rho}, \bar{\rho}_L)$ and $V \in (\bar{\sigma}, \bar{\sigma}_R)$ such that $U \in \mathcal{R}(\widetilde{\Lambda}_L)$ and $V \in \mathcal{R}(\widetilde{\Lambda}_R)$.

Definition 2.11. The braiding intertwiner $\epsilon(\bar{\rho}, \bar{\sigma}) \in (\bar{\rho} \otimes \bar{\sigma}, \bar{\sigma} \otimes \bar{\rho})$ is given by

$$\epsilon(\bar{\rho}, \bar{\sigma}) := (V^* \otimes U^*)(U \otimes V) = V^* \bar{\rho}(V). \tag{19}$$

To get the last equality, we use $\bar{\sigma}_R(U) = U$ which holds because $\bar{\sigma}_R$ is localised on Λ_R while $U \in \mathcal{R}(\widetilde{\Lambda}_L)$. Using $\bar{\rho}_L \otimes \bar{\sigma}_R = \bar{\sigma}_R \otimes \bar{\rho}_L$, one easily verifies that $\epsilon(\bar{\rho}, \bar{\sigma})$ is indeed an intertwiner from $\bar{\rho} \otimes \bar{\sigma}$ to $\bar{\sigma} \otimes \bar{\rho}$.

Lemma 2.12. *The braiding $\epsilon(\bar{\rho}, \bar{\sigma})$ is independent of the choice of cones Λ_L, Λ_R, the choice of objects $\bar{\rho}_L \in \Delta^I_{\Lambda_L}$ and $\bar{\sigma}_R \in \Delta^I_{\Lambda_R}$ and the choice of intertwiners $U \in (\bar{\rho}, \bar{\rho}_L)$ and $V \in (\bar{\sigma}, \bar{\sigma}_R)$.*

Proof. Choose different cones Λ'_L and Λ'_R to the left and to the right of Λ_0. Then there are allowed cones $\Lambda''_L \supseteq \Lambda_L \cup \Lambda'_L$ and $\Lambda''_R \supseteq \Lambda_R \cup \Lambda'_R$ that are also to the left and to the right of Λ_0, respectively. Choose objects $\bar{\rho}'_L \in \Delta^I_{\Lambda'_L}$ and $\bar{\sigma}'_R \in \Delta^I_{\Lambda'_R}$, as well as morphisms $U' \in (\bar{\rho}, \bar{\rho}'_L)$ and $V' \in (\bar{\sigma}, \bar{\sigma}'_R)$. Then $V'V^* \in (\bar{\sigma}_R, \bar{\sigma}'_R) \subset \mathcal{R}(\Lambda''_R)$ by Lemma 2.10. The new choice $\bar{\rho}'_L, \bar{\sigma}'_R$ leads to a braiding intertwiner

$$\begin{aligned}\epsilon'(\bar{\rho}, \bar{\sigma}) &= (V'^* \otimes U'^*)(U' \otimes V') = V'^* \bar{\rho}(V') \\ &= V^*(V'V^*)^* \bar{\rho}\big((V'V^*)V\big) = V^* \bar{\rho}(V) \\ &= \epsilon(\bar{\rho}, \bar{\sigma})\end{aligned}$$

where we used $\bar{\rho}(V'V^*) = V'V^*$ since $V'V^* \in \mathcal{R}(\Lambda''_R)$ and $\bar{\rho}$ is supported in Λ_0, which is disjoint from Λ''_R. □

Lemma 2.13. *The braiding intertwiners satisfy the braid equations*

$$\begin{aligned}\epsilon(\rho \otimes \sigma, \tau) &= \big(\epsilon(\rho, \tau) \otimes 1_\sigma\big)\big(1_\rho \otimes \epsilon(\sigma, \tau)\big) \\ \epsilon(\rho, \sigma \otimes \tau) &= \big(1_\sigma \otimes \epsilon(\rho, \tau)\big)\big(\epsilon(\rho, \sigma) \otimes 1_\tau\big)\end{aligned}$$

where $1_\rho = \mathbb{1} \in (\rho, \rho)$.

Proof. Let us prove the first equation, the second is shown in the same way. Choose ρ_L, σ_L supported in Λ_L and morphisms $U_\rho \in (\rho, \rho_L)$, $U_\sigma \in (\sigma, \sigma_L)$. Choose τ_R supported in Λ_R and a morphism $V_\tau \in (\tau, \tau_R)$. Then

$$\begin{aligned}\epsilon(\rho \otimes \sigma, \tau) &= \big(V^*_\tau \otimes (U_\rho \otimes U_\sigma)^*\big)\big((U_\rho \otimes U_\sigma) \otimes V_\tau\big) \\ &= V^*_\tau \overline{\rho \otimes \sigma}(V_\tau) = V^*_\tau \overline{\rho}\big(\overline{\sigma}(V_\tau)\big) \\ &= V^*_\tau \overline{\rho}(V_\tau) \overline{\rho}(V^*_\tau) \overline{\rho}\big(\overline{\sigma}(V_\tau)\big) \\ &= \big(\epsilon(\rho, \tau) \otimes 1_\sigma\big)\big(\overline{\rho}(V^*_\tau \overline{\sigma}(V_\tau))\big) \\ &= \big(\epsilon(\rho, \tau) \otimes 1_\sigma\big)\big(1_\rho \otimes \epsilon(\sigma, \tau)\big).\end{aligned}$$

□

2.2.4. Relation to Previous Work. The state ω is said to satisfy Haag duality for cones if $\mathcal{R}(\Lambda) = \mathcal{R}(\Lambda^c)'$ for all cones Λ. Haag duality for cones has been verified for abelian quantum double models in [8,20]. We believe that the double semion state introduced below can also be shown to satisfy Haag duality for cones using similar methods.

Under the assumption that the pure state ω satisfies (approximate) Haag duality for cones, it is shown in [21] that the category of localised and transportable endomorphisms Δ is a braided C*-tensor category with isomorphism classes of simple objects in one-to-one correspondence with (equivalence classes of) irreducible representations of the observable algebra that satisfy the superselection criterion (Definition 2.1).

The braided tensor categories Δ^I_Λ constructed above are full subcategories of Δ. If ω satisfies Haag duality for cones, then Δ is equipped with a braiding given by Definition 4.11 of [21]. The restriction of this braiding to Δ^I_Λ agrees with the braiding defined in 2.11. It follows that the category Δ^I_Λ

completely describes the closed system of abelian anyons corresponding to the irreducible representations $\{\pi_a\}_{a\in I}$ in a way that is consistent with the theory of [21]. These abelian anyons form a subset of all anyon types present in the model (anyon types correspond to irreducible representations of the observables algebra that satisfy the superselection criterion). In contrast, the theory of [21] captures all anyon types, in particular also all non-abelian anyons.

The reason that (approximate) Haag duality allows a description of all anyon types is twofold. Most importantly, Haag duality allows one to construct for any representation π that satisfies the superselection criterion with respect to π_1, and for any cone Λ, a transportable endomorphism $\overline{\rho}_{\pi,\Lambda}$ localised on Λ such that $\pi \simeq \overline{\rho}_{\pi,\Lambda} \circ \pi_1$ (Definition 2.13 and Lemma 2.14 of [21]). This ensures that Δ contains objects corresponding to any superselection sector. In our setting, we obtain localised and transportable endomorphisms for the superselection sectors $\{\pi_a\}_{a\in I}$ using Assumptions 1 and 4, see Lemma 2.5.

Haag duality also allows a braiding to be defined for the entire tensor category Δ. Note that for the braiding of definition 2.11 to be well defined, we must show that it is independent of the choice of intertwiners U and V used in the definition. This is done in Lemma 2.12 using the fact that if $\bar{\rho} \in \Delta^I_{\Lambda_1}$ and $\bar{\sigma} \in \Delta^I_{\Lambda_2}$, then all morphisms in $(\bar{\rho}, \bar{\sigma})$ are elements of $\mathcal{R}(\Lambda)$ for any allowed cone $\Lambda \supseteq \Lambda_1 \cup \Lambda_2$. In our setting, this follows from Assumption 4 (Lemma 2.10). With Haag duality for cones, this locality property of morphisms follows immediately. Indeed, suppose $\overline{\rho}$ and $\overline{\sigma}$ are both localised on an allowed cone Λ, and suppose $R \in (\overline{\rho}, \overline{\sigma})$. Then for any $O \in \mathcal{A}_{\Lambda^c}$ we have

$$\overline{\rho}(\pi_1(O)) = \overline{\sigma}(\pi_1(O)) = \pi_1(O), \tag{20}$$

and hence

$$R\pi_1(O) = R\overline{\rho}(\pi_1(O)) = \overline{\sigma}(\pi_1(O))R = \pi_1(O)R. \tag{21}$$

We see that $R \in \pi_1(\mathcal{A}_{\Lambda^c})' = \mathcal{R}(\Lambda^c)' = \mathcal{R}(\Lambda)$ by Haag duality.

Using the same argument, one can use Haag duality for cones to prove Assumption 4, as mentioned above.

2.3. *Fusion Ring, F-Symbols, and R-Symbols.*

2.3.1. Fusion Ring of Δ^I_Λ.

To any semisimple tensor category, one may associate its fusion ring, see Section 4.5 of [7] for details. For the category Δ^I_Λ, the construction goes as follows. First, note that the isomorphism classes of simple objects of Δ^I_Λ are labelled by the elements of I, they are precisely the classes $[\overline{w}_{a,\Lambda}]$ for $a \in I$. Since Δ^I_Λ is semisimple, any object $\bar{\rho}$ of Δ^I_Λ is isomorphic to a direct sum of simple objects $\overline{w}_{a,\Lambda}$. The number of times that $\overline{w}_{a,\Lambda}$ appears in such a direct sum decomposition is independent on the particular choice of direct sum decomposition and is called the multiplicity of a in $\bar{\rho}$, and denoted by $[\bar{\rho} : a]$. The fusion ring of Δ^I_Λ is the free abelian group generated by the isomorphism classes of simple objects $\{[\overline{w}_{a,\Lambda}]\}_{a\in I}$, which we can identify with the elements of I. Addition in this group corresponds to the direct sum in the category. The isomorphism class of a general object $\bar{\rho} \in \Delta_\Lambda^I$ corresponds to an element of fusion ring given by

$$[\bar{\rho}] := \sum_{a \in I} [\bar{\rho} : a]\, a. \tag{22}$$

The multiplication of the fusion ring is given by the tensor product of the category. It is sufficient to define the multiplication on the generators by

$$a \times b := [\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda}] = [\overline{w}_{a \times b,\Lambda}] = ab, \tag{23}$$

which corresponds precisely to the multiplication on I introduced in Assumption 2 and Lemma 2.3. We see that the fusion ring of Δ_Λ^I is given by $\mathbb{Z}[I]$, the ring of polynomials over the abelian group I with integer coefficients.

Two tensor categories that are monoidally equivalent have isomorphic fusion rings. The converse certainly does not hold; there are inequivalent tensor categories that have isomorphic fusion rings (for example, all categories Vec_G^ω for 3-cocycle $\omega : G^{\times 3} \to \mathbb{C}^\times$ have fusion ring $\mathbb{Z}[G]$). It turns out that the fusion ring together with certain cohomological data is sufficient information to characterise a tensor category up to monoidal equivalence, see, for example, Proposition 1.1 of [23]. Below, we will describe these cohomological data for the category Δ_Λ^I in terms of 'F-symbols', a nomenclature common in the physics literature. Since we are interested in *braided* tensor categories, the question arises to what extent the fusion ring together with the F-symbols characterises a braided tensor category. The braiding induces more structure on the fusion ring in the form of 'R-symbols', which will be described for our Δ_Λ^I below. To the best of the authors' knowledge, it is not known if a fusion ring together with F and R-symbols is enough information to characterise a braided tensor category completely. In the special case where the fusion ring takes the form $\mathbb{Z}[G]$ for a finite abelian group G however, this does turn out to be the case, see Proposition 7.5.2 of [12].

2.3.2. Fusion and F-Symbols. Fix an allowed cone Λ_0 and write $\overline{w}_a := \overline{w}_{a,\Lambda_0}$. Pick unitary intertwiners $\Omega(a,b) \in (\overline{w}_a \otimes \overline{w}_b, \overline{w}_{a\times b}) \subset \mathcal{R}(\Lambda_0)$ called fusion operators. Note that the $\Omega(a,b)$ are unique up to phase. The unitaries

$$\Omega(ab,c)(\Omega(a,b) \otimes 1_c) = \Omega(ab,c)\Omega(a,b)$$
$$\Omega(a,bc)(1_a \otimes \Omega(b,c)) = \Omega(a,bc)\overline{w}_a(\Omega(b,c))$$

are both intertwiners from $\overline{w}_a \otimes \overline{w}_b \otimes \overline{w}_c$ to $\overline{w}_{abc}$. Since $(\overline{w}_a \otimes \overline{w}_b \otimes \overline{w}_c, \overline{w}_{abc})$ is one-dimensional, there are phases $F(a,b,c) \in U(1)$ such that

$$\Omega(ab,c)\Omega(a,b) = F(a,b,c) \times \Omega(a,bc)\overline{w}_a(\Omega(b,c)). \tag{24}$$

These $F(a,b,c)$ are the F-symbols. Figure 2 gives a graphical representation of Eq. (24).

The F-symbols satisfy a pentagon equation, which in our setting of abelian anyons takes the form of a cocycle relation.

Proposition 2.14. *The F-symbols satisfy*

$$(\mathrm{d}F)(a,b,c,d) := \frac{F(a,b,c)F(a,bc,d)F(b,c,d)}{F(ab,c,d)F(a,b,cd)} = 1. \tag{25}$$

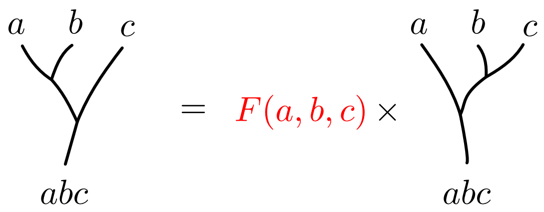

Figure 2. Graphical representation of Eq. (24), defining the F-symbols $F(a, b, c)$. Each node represents a fusion operator. The diagrams represent two different compositions of fusion operators both yielding intertwiners from $w_a \otimes w_b \otimes w_c$ to w_{abc}

Proof. A graphical proof is shown in Fig. 3. In equations, we have

$$\begin{aligned}
&\Omega(abc, d)\Omega(ab, c)\Omega(a, b)\\
&\quad = F(ab, c, d) \times \Omega(ab, cd)\overline{w}_{ab}\big(\Omega(c, d)\big)\Omega(a, b)\\
&\quad = F(ab, c, d) \times \Omega(ab, cd)\Omega(a, b)\overline{w}_a\big(\overline{w}_b\big(\Omega(c, d)\big)\big)\\
&\quad = F(ab, c, d)F(a, b, cd) \times \Omega(a, bcd)\overline{w}_a(\Omega(b, cd))\overline{w}_a(\overline{w}_b(\Omega(c, d)))\\
&\quad = F(ab, c, d)F(a, b, cd) \times \Omega(a, bcd)\overline{w}_a\big(\Omega(b, cd)\overline{w}_b(\Omega(c, d))\big)
\end{aligned}$$

but also

$$\begin{aligned}
&\Omega(abc, d)\Omega(ab, c)\Omega(a, b)\\
&\quad = F(a, b, c) \times \Omega(abc, d)\Omega(a, bc)\overline{w}_a(\Omega(b, c))\\
&\quad = F(a, b, c)F(a, bc, d) \times \Omega(a, bcd)\overline{w}_a(\Omega(bc, d))\overline{w}_a(\Omega(b, c))\\
&\quad = F(a, b, c)F(a, bc, d) \times \Omega(a, bcd)\overline{w}_a\big(\Omega(bc, d)\Omega(b, c)\big)\\
&\quad = F(a, b, c)F(a, bc, d)F(b, c, d) \times \Omega(a, bcd)\overline{w}_a\big(\Omega(b, cd)\overline{w}_b(\Omega(c, d))\big).
\end{aligned}$$

And the desired equality follows. □

2.3.3. Braiding and R-Symbols. We simply set $\epsilon(a, b) := \epsilon(\overline{w}_a, \overline{w}_b)$ for any $a, b \in I$. The unitaries $\Omega(b, a)\epsilon(a, b)$ and $\Omega(a, b)$ are both intertwiners from $\overline{w}_a \otimes \overline{w}_b$ to $\overline{w}_{ab}$. Since $(\overline{w}_a \otimes \overline{w}_b, \overline{w}_{ab})$ is one-dimensional, there exist phases $R(a, b) \in U(1)$ such that

$$\Omega(b, a)\epsilon(a, b) = R(a, b) \times \Omega(a, b). \tag{26}$$

The phases $R(a, b)$ are the R-symbols. Figure 4 gives a graphical representation of Eq. (26).

2.3.4. Yang–Baxter Equation. The braidings $\epsilon(a, b)$ and fusions $\Omega(a, b)$ satisfy the Yang–Baxter equations, see Fig. 5.

Proposition 2.15. *We have*

$$\overline{w}_c\big(\Omega(a, b)\big)\epsilon(a, c)\overline{w}_a\big(\epsilon(b, c)\big) = \epsilon(ab, c)\Omega(a, b) \tag{27}$$

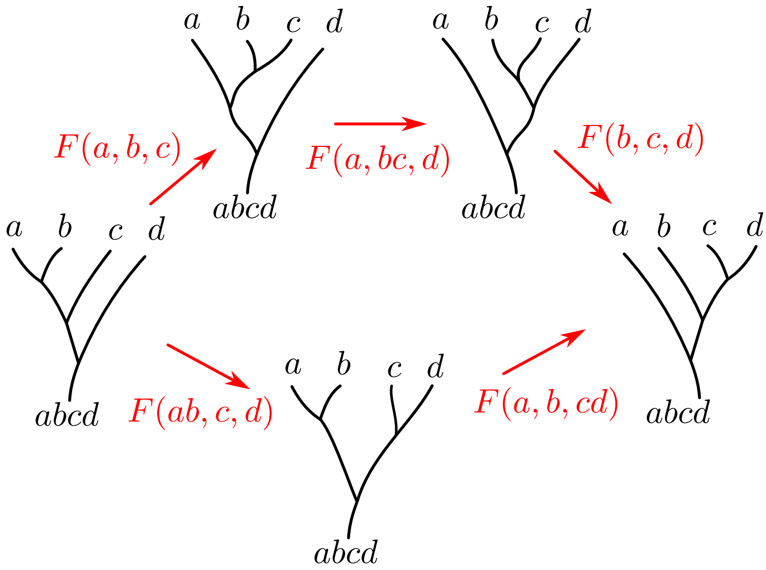

FIGURE 3. A graphical proof of the Pentagon equation

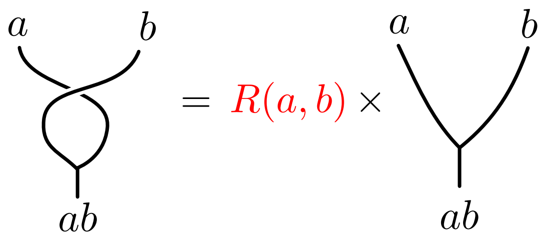

FIGURE 4. Graphical representation of Eq. (26), defining the R-symbols $R(a, b)$. The point where the a-line passes under the b-line represents the braiding intertwiner $\epsilon(a, b)$

and

$$\Omega(b, c)\overline{w}_b\big(\epsilon(a, c)\big)\epsilon(a, b) = \epsilon(a, bc)\overline{w}_a\big(\Omega(b, c)\big). \tag{28}$$

Proof. We prove Eq. (27); the proof of Eq. (28) is similar. Choose a cone Λ_R, an endomorphism $\overline{w}_c^R \in \Delta^I_{\Lambda_R}$, and a unitary $V \in (\overline{w}_c, \overline{w}_c^R)$ as in Definition 2.11. Then

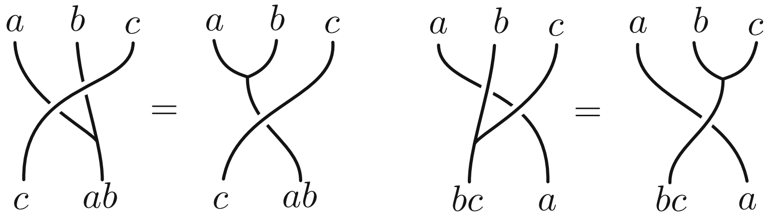

FIGURE 5. Graphical representation of the Yang–Baxter equations

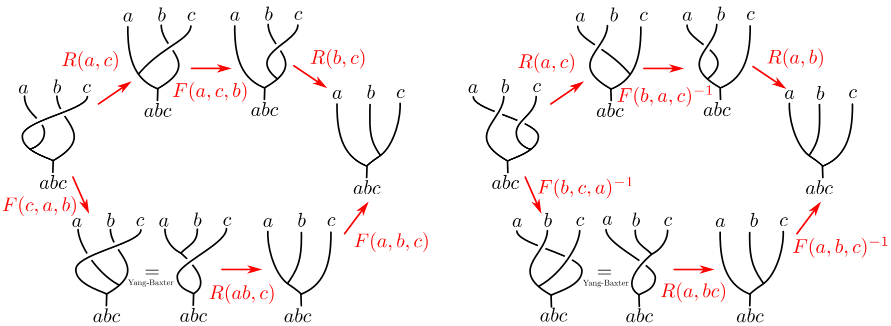

FIGURE 6. Graphical representations of the first and second hexagon equations

$$
\begin{aligned}
\epsilon(ab,c)\Omega(a,b) &= \epsilon(\overline{w}_{ab},\overline{w}_c)\Omega(a,b) \\
&= V^*\,\overline{w}_{ab}\big(V\big)\Omega(a,b) \\
&= V^*\Omega(a,b)\overline{w}_a\big(\overline{w}_b(V)\big) \\
&= V^*\overline{w}_c^R\big(\Omega(a,b)\big)\overline{w}_a(V)\,\overline{w}_a\big(V^*\overline{w}_b(V)\big) \\
&= \overline{w}_c\big(\Omega(a,b)\big)\,V^*\overline{w}_a(V)\,\overline{w}_a\big(V^*\overline{w}_b(V)\big) \\
&= \overline{w}_c\big(\Omega(a,b)\big)\,\epsilon(a,c)\,\overline{w}_a\big(\epsilon(b,c)\big)
\end{aligned}
$$

where we used $\Omega(a,b) \in (\overline{w}_a \otimes \overline{w}_b, \overline{w}_{ab})$ to obtain the third line. We used the fact that $\Omega(a,b) \in \mathcal{R}(\Lambda_0)$ so $\overline{w}_c^R(\Omega(a,b)) = \Omega(a,b)$ to obtain the fourth line. The fifth line follows from $V \in (\overline{w}_c, \overline{w}_c^R)$ and the fact that V is unitary, and the final line follows by the definition of the braidings $\epsilon(a,c)$ and $\epsilon(b,c)$. $\square$

2.3.5. Hexagon Equation. Using the Yang–Baxter equation, we obtain the Hexagon equation, see Fig. 6.

Proposition 2.16. *The F and R-symbols satisfy the hexagon equations*

$$\frac{F(a,b,c)F(c,a,b)}{F(a,c,b)} = \frac{R(a,c)R(b,c)}{R(ab,c)} \tag{29}$$

and

$$\frac{F(a,b,c)F(b,c,a)}{F(b,a,c)} = \frac{R(a,bc)}{R(a,b)R(a,c)}. \tag{30}$$

Proof. The left diagram in Fig. 6 suggests the following two equalities of morphisms:

$$\begin{aligned}
&\Omega(ca,b)\Omega(c,a)\epsilon(a,c)\overline{w}_a(\epsilon(b,c)) \\
&\quad = R(a,c) \times \Omega(ca,b)\Omega(a,c)\overline{w}_a(\epsilon(b,c)) \\
&\quad = R(a,c)F(a,c,b) \times \Omega(a,cb)\overline{w}_a(\Omega(c,b))\overline{w}_a(\epsilon(b,c)) \\
&\quad = R(a,c)F(a,c,b) \times \Omega(a,cb)\overline{w}_a\big(\Omega(c,b)\epsilon(b,c)\big) \\
&\quad = R(a,c)F(a,c,b)R(b,c) \times \Omega(a,cb)\overline{w}_a(\Omega(b,c))
\end{aligned}$$

and

$$\begin{aligned}
&\Omega(ca,b)\Omega(c,a)\epsilon(a,c)\overline{w}_a(\epsilon(b,c)) \\
&\quad = F(c,a,b) \times \Omega(c,ab)\overline{w}_c(\Omega(a,b))\epsilon(a,c)\overline{w}_a(\epsilon(b,c)) \\
&\quad = F(c,a,b) \times \Omega(c,ab)\epsilon(ab,c)\Omega(a,b) \\
&\quad = F(c,a,b)R(ab,c) \times \Omega(ab,c)\Omega(a,b) \\
&\quad = F(c,a,b)R(ab,c)F(a,b,c) \times \Omega(a,bc)\overline{w}_a(\Omega(b,c))
\end{aligned}$$

where we used the Yang–Baxter equation to obtain the second line. The coefficients of the right-hand sides must be equal, yielding the first hexagon equation.

The second hexagon equation is obtained in exactly the same way, following the right diagram in Fig. 6. □

2.3.6. Dependence of F and R-Symbols on the Choice of $\boldsymbol{\Lambda_0}$ and the Phases of $\boldsymbol{\Omega(a,b)}$. Suppose we chose different phases for the intertwiners $\Omega(a,b)$, i.e. we consider

$$\Omega'(a,b) = \chi(a,b)\Omega(a,b) \tag{31}$$

for phases $\chi(a,b)$. This yields new F-symbols by

$$\Omega'(ab,c)\Omega'(a,b) = F'(a,b,c) \times \Omega'(a,bc)\overline{w}_a(\Omega'(b,c)) \tag{32}$$

which are related to the original F-symbols by

$$F'(a,b,c) = (\mathrm{d}\chi)(a,b,c)F(a,b,c) = \frac{\chi(b,c)\chi(a,bc)}{\chi(ab,c)\chi(a,b)}F(a,b,c). \tag{33}$$

i.e. F' is related to F by the coboundary $\mathrm{d}\chi$. It follows that only the cohomology class $[F] \in H^3(I,U(1))$ is well defined.

The R-symbols are also affected by the different choice of phases. Indeed, the new R-symbols defined by

$$\Omega'(b,a)\epsilon(a,b) = R'(a,b) \times \Omega'(a,b) \tag{34}$$

are related to the old by

$$R'(a,b) = \frac{\chi(b,a)}{\chi(a,b)}R(a,b). \tag{35}$$

It follows that the self-statistics $R(a,a)$ and the double braidings $R(a,b)R(b,a)$ are invariants.

Next, we investigate the dependence of the F and R-symbols on the choice of allowed cone Λ_0. We will find no additional ambiguity beyond the one just discussed.

Let Λ_0' be another allowed cone. Then there is an allowed cone $\widetilde{\Lambda}_0$ containing $\Lambda_0 \cup \Lambda_0'$. Denote $\overline{w}_a' = \overline{w}_{a,\Lambda_0'}$. Then there are unitaries $W_a \in \mathcal{R}(\widetilde{\Lambda}_0)$ such that

$$W_a \in (\overline{w}_a, \overline{w}_a'). \tag{36}$$

These unitaries are unique up to phase.

This leads to new fusion intertwiners $\Omega'(a,b) \in (\overline{w}_a' \otimes \overline{w}_b', \overline{w}_{ab}')$ given by

$$\Omega'(a,b) = W_{ab}\Omega(a,b)(W_a \otimes W_b)^*. \tag{37}$$

The new F-symbols are determined by

$$\Omega'(ab,c)\Omega'(a,b) = F'(a,b,c) \times \Omega'(a,bc)\bar{w}_a'(\Omega'(b,c)). \tag{38}$$

Using Eq. 37, we compute

$$\Omega'(ab,c)\Omega'(a,b) = W_{abc}\Omega(ab,c)\Omega(a,b)\bar{w}_a\big(\bar{w}_b(W_c^*)W_b^*\big)W_a^* \tag{39}$$

and

$$\Omega'(a,bc)\bar{w}_a'(\Omega'(b,c)) = W_{abc}\Omega(a,bc)\bar{w}_a(\Omega(b,c))\bar{w}_a\big(\bar{w}_b(W_c^*)W_b^*\big)W_a^*. \tag{40}$$

It follows that $F'(a,b,c) = F(a,b,c)$ for all $a,b,c \in I$.

Recall that the braiding intertwiners $\epsilon(a,b)$ are defined in terms of endomorphisms $\overline{w}_a$ localised on Λ_0, $\overline{w}_a^L$ localised on Λ_L and $\overline{w}_a^R$ localised on Λ_R as follows. Pick intertwiners (unique up to phase) $U_a \in (\overline{w}_a, \overline{w}_a^L)$ and $V_a \in (\overline{w}_a, \overline{w}_a^R)$, then

$$\epsilon(a,b) = (V_b^* \otimes U_a^*)(U_a \otimes V_b) = V_b^*\overline{w}_a(V_b). \tag{41}$$

It is shown in Lemma 2.12 that this braiding intertwiner is independent of the choice of cones Λ_L, Λ_R. Moreover, $\epsilon(a,b)$ is independent of the choice of phase for the intertwiners U_a and V_a.

In order to make the comparison with the braiding on $\Delta^I_{\widetilde{\Lambda}_0}$, let us choose the left and right cones Λ_L and Λ_R such that they are to the left and right of both Λ_0 and $\widetilde{\Lambda}_0$.

With the new endomorphisms $\overline{w}_a'$ related to the old $\overline{w}_a$ by Eq. (36), we get new intertwiners $U_a' = U_a W_a^* \in (\overline{w}_a', \overline{w}_a^L)$ and $V_a' = V_a W_a^* \in (\overline{w}_a', \overline{w}_a^R)$ and therefore new braiding intertwiners

$$\epsilon'(a,b) = ((V_b')^* \otimes (U_a')^*)(U_a' \otimes V_b') = (V_b')^*\bar{w}_a'\big(V_b'\big). \tag{42}$$

A short computation relates this to the braiding $\epsilon(a,b)$ as

$$\epsilon'(a,b) = W_b V_b^* W_a V_b \epsilon(a,b)\overline{w}_a(W_b^*)W_a^*. \tag{43}$$

The new R-symbol is determined by

$$\Omega'(b,a)\epsilon'(a,b) = R'(a,b) \times \Omega'(a,b). \tag{44}$$

Using Eqs. (37) and (43), the left-hand side becomes

$$\Omega'(b,a)\epsilon'(a,b) = W_{ba}\Omega(b,a)\epsilon(a,b)\overline{w}_a(W_b^*)W_a^* \tag{45}$$

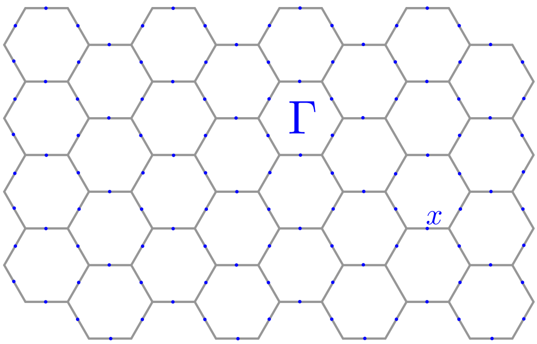

FIGURE 7. Degrees of freedom of the double semion state live on the edges of a hexagonal lattice

and

$$\Omega'(a,b) = W_{ab}\Omega(a,b)\overline{w}_a(W_b^*)W_a^* \tag{46}$$

so, noting that $ab = ba$, we find $R'(a,b) = R(a,b)$.

We conclude that Eqs. (33) and (35) are the only ambiguities in the F and R-symbols.

We call two sets of F and R-symbols (F,R) and (F',R') on the fusion ring $\mathbb{Z}(I)$ gauge equivalent if they are related by (33) and (35) for some phases $\chi(a,b)$.

3. The Double Semion State

We construct an infinite volume version of the ground state of the double semion model, first introduced in [16]. We identify superselection sectors corresponding to semion, anti-semion, and bound state anyons and find that the braided fusion category describing these anyons corresponds to the representation category of a twisted quantum double algebra $\mathcal{D}^\phi(\mathbb{Z}_2)$.

3.1. *Construction of the Double Semion State.* Let $\Gamma^V \subset \mathbb{R}^2$ be the vertices of the hexagonal lattice. We take $\Gamma = \Gamma^E$ to be the (midpoints of the) edges of the hexagonal lattice (Fig. 7) and to each edge $e \in \Gamma$ we associate a degree of freedom $\mathcal{A}_e \simeq \mathrm{End}(\mathbb{C}^2)$. We fix Pauli matrices $\sigma_e^X, \sigma_e^Y, \sigma_e^Z$ in each $\mathcal{A}_e$. We denote by $\Gamma^F = (\Gamma^V)^*$ be the set of faces of the hexagonal lattice.

We say an edge $e \in \Gamma$ belongs to a hexagon $p \in \Gamma^F$ and write $e \in p$ if e is one of the six boundary edges of p. We write ∂p for the set of six edges that belong to p. For any subset $\Pi \subset \Gamma^F$, we write $\Pi^E = \cup_{p\in\Pi}\partial p$ for all edges that belong to some hexagon in Π and by $\partial\Pi = \Pi^E \cap (\Pi^c)^E$ the collection of edges that belong to exactly one hexagon in Π. That is, $\partial\Pi$ is the boundary of Π.

We interpret $\sigma_e^Z = -1$ as the edge e being occupied by a string, while $\sigma_e^Z = 1$ means that the edge is unoccupied.

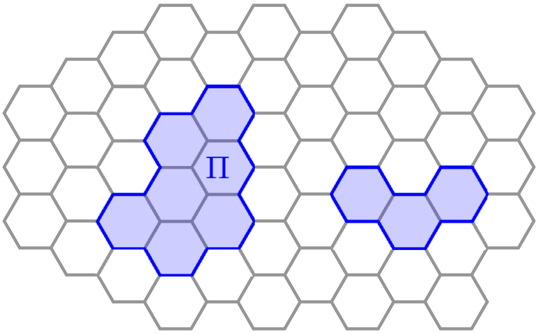

FIGURE 8. Π is the set of edges of the hexagons shaded blue. Acting with A_Π on the state ω_0 yields a string configuration with two connected components (color figure online)

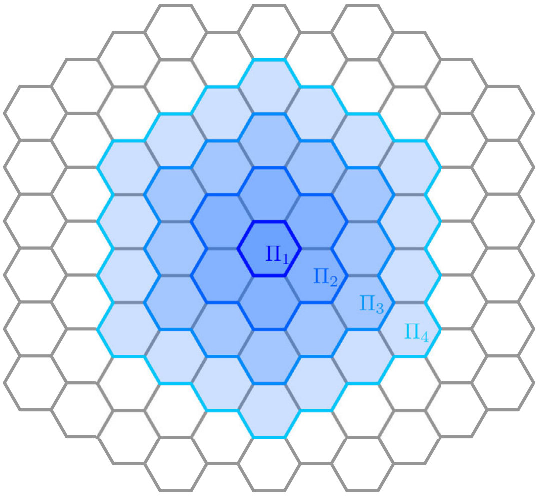

FIGURE 9. Increasing sequence of balls Π_n depicted in the primal hexagonal lattice

For any hexagon $p \in \Gamma^F$, let

$$A_p = \prod_{e \in \partial p} \sigma_e^X \tag{47}$$

and for any finite set Π of hexagons, let

$$A_\Pi = \prod_{p \in \Pi} A_p = \prod_{e \in \partial \Pi} \sigma_e^X . \tag{48}$$

Note that A_Π produces a string around the region Π when it acts on ω_0, see Fig. 8.

Let us fix a hexagon $p_0 \in \Pi^F$ as an origin and define $\Pi_n = \{p \in \Pi^F : \mathrm{dist}(p, p_0) \leq n\}$, where $\mathrm{dist}(\cdot, \cdot)$ is the graph distance for Π^F, see Fig. 9.

Let ω_0 be the pure product state without any strings, i.e. $\omega_0(\sigma_e^Z) = 1$ for all $e \in \Gamma$. Let $(\pi_0, \mathcal{H}_0, \Omega_0)$ be the GNS triple for ω_0, and let

$$\Omega_n := \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\Pi \subset \Pi_n} (-1)^{\sharp \Pi} A_\Pi \Omega_0 \tag{49}$$

where $\sharp\Pi$ is the number of connected components of Π. i.e. Ω_n is a superposition of closed string configurations supported in Π_n^E, with phases determined by the parity of the number of components of the string configuration.

The vectors Ω_n determine a sequence of pure states ω_n on $\mathcal{A}$. The following theorem is proved in Appendix A.

Theorem 3.1. *The sequence ω_n converges in the weak-* topology to a pure state ω.*

We call this pure state ω the *double semion state* and denote its GNS triple by $(\pi_1, \mathcal{H}, \Omega)$.

3.2. *String Operators.* An oriented edge is a pair (v_0, v_1) of neighbouring vertices of Γ^V. We say v_0 (v_1) is the initial (final) vertex of (v_0, v_1). A *path* P is a collection of oriented edges such that there is a sequence of vertices $(\cdots, v_{i-1}, v_i, v_{i+1} \cdots)$ such that each oriented edge in P is of the form $e = (v_i, v_{i+1})$ for some i. Such a sequence is called a vertex sequence for P. The set of vertices appearing in any vertex sequence for P is uniquely determined by P and denoted by P^V. We call P^V the vertex set of P; it is the set of vertices that are the initial or final vertex of some edge in P. We, moreover, require paths to be self-avoiding in the sense that any vertex sequence for P consists of distinct vertices, except for possibly the initial vertex, which may be equal to the final vertex of the sequence, if these exist. In the latter case, we say the path P is closed. Similarly, if a vertex sequence for P is bi-infinite, we also say P is closed. If P has finite vertex sequence with all its vertices being distinct, then the vertex sequence is uniquely determined by P and we denote by $\partial_0 P$ the first vertex of the vertex sequence and by $\partial_1 P$ the final vertex of the vertex sequence. We say an edge e belongs to P if P contains an oriented edge corresponding to e. With slight abuse of notation, we write $e \in P$ if e belongs to P.

Let P be a path. An edge e is said to be a *leg* of a path P at $v \in P^V$ if e is the unique edge with endpoint v such that e does not belong to P. If e is a leg of P, then e either lies to the left or to the right of P w.r.t. the orientation of P and the standard orientation of the plane. A leg that lies to the left is called an L-leg of P, and a leg of P that lies to the right is called an R-leg of P. Let P^V be the vertex set of P. If $v \in P^V$ is the endpoint of an L$-$leg of P, then we say v is an L$-$vertex of P. Similarly, if $v \in P^V$ is the endpoint of an R-leg of P, then we say v is an R-vertex of P, see Fig. 10.

Following [16], we define three types of non-trivial string operators.

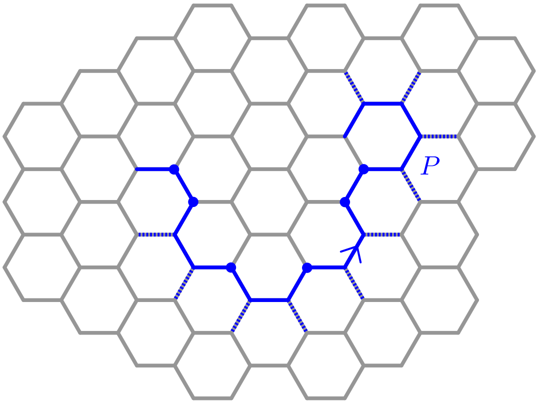

FIGURE 10. An oriented path P in solid blue with its L-vertices fattened and its R-legs marked with dotted lines (color figure online)

The *semion string* is given by

$$W_S[P] := \left(\prod_{e\in P}\sigma_e^X\right)\left(\prod_{\text{R-legs } e} \mathrm{i}^{\frac{1-\sigma_e^Z}{2}}\right)\left(\prod_{\text{L-vertices } v}(-1)^{s_v}\right) \tag{50}$$

where $s_v = \frac{1}{4}(1-\sigma_e^Z)(1+\sigma_{e'}^Z)$ and e, e' are the edges of P that go in and out of the vertex v, respectively.

The *anti-semion* string is given by

$$W_{\bar{S}}[P] := \left(\prod_{e\in P}\sigma_e^X\right)\left(\prod_{\text{R-legs } e} (-\mathrm{i})^{\frac{1-\sigma_e^Z}{2}}\right)\left(\prod_{\text{L-vertices } v}(-1)^{s_v}\right) \tag{51}$$

and the *bound-state string* is given by

$$W_B[P] := \left(\prod_{\text{R-legs } e}\sigma_e^Z\right). \tag{52}$$

We further define string operators for the vacuum sector $W_1[P] = \mathbb{1}$. We set $I = \{1, S, \bar{S}, B\}$, and we denote by $w_a[P]$ the automorphism defined by conjugation with the (possibly formal) unitary $W_a[P]$. We say $W_a[P]$ or $w_a[P]$ is a closed string operator whenever P is a closed path.

One easily checks that $W_{\bar{S}}[P] = W_S[P]W_B[P] = W_B[P]W_S[P]$, and $W_B[P]^2 = \mathbb{1}$. We will see later that this implies that the anyons $1, S, \bar{S}$ and B satisfy fusion rules $S\times B = \bar{S}$, $\bar{S}\times B = S$ and $B\times B = 1$. The fusion rules $S\times S = 1$ and $\bar{S}\times\bar{S} = 1$ do not follow so simply. For example, $W_S[P]^2 \neq \mathbb{1}$, see Eq. (54). This failure of the string operators to form a strict representation of the fusion rules is the origin of the non-trivial F-symbols of the double semion model, see Sect. 3.3.

These string operators have the following important property:

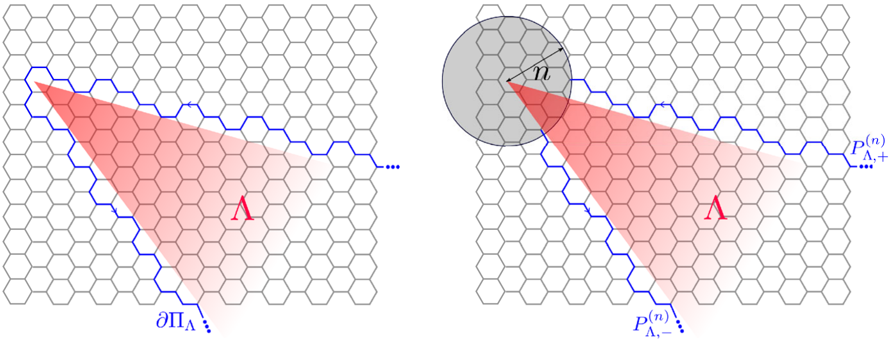

FIGURE 11. A cone Λ with the oriented path $\partial\Pi_\Lambda$ going around it in a counterclockwise direction. The paths $P^{(n)}_{\Lambda,+}, P^{(n)}_{\Lambda,-}$ straddle the left and right side of Λ, respectively

Proposition 3.2. *Closed string operators leave the ground state invariant. i.e. if P is a closed string, then*

$$\omega \circ w_a[P] = \omega \tag{53}$$

for all $a \in I$.

The proof is in Appendix B.

3.2.1. Definition of $w_{a,\Lambda}$. For any set $Z \subset \mathbb{R}^2$, let Π_Z be the set of hexagons (regarded as open subsets of $\mathbb{R}^2$) that have some overlap with the set Z. For a cone Λ, we interpret the boundary $\partial\Pi_\Lambda$ as an infinite closed path oriented counterclockwise around Λ. Assuming the opening angle of Λ is less than π, the edges of $\partial\Pi_\Lambda$ whose centre lies a distance further than $n > 2$ from the apex of Λ form two half-infinite oriented paths $P^{(n)}_{\Lambda,+}$ and $P^{(n)}_{\Lambda,-}$, as shown in Fig. 11.

Let Λ be a cone, and let $\Lambda^{(L)}$ and $\Lambda^{(R)}$ be its left- and right half-cones, see Fig. 12. Take $n > 2$ sufficiently large such that $w_{a,\Lambda} := w_a[P^{(n)}_{\Lambda^{(R)},+}]$ and $v_{a,\Lambda} := w_a[P^{(n)}_{\Lambda^{(L)},-}]$ are supported in Λ for all $a \in I$. Denote $P_\Lambda := P^{(n)}_{\Lambda^{(R)},+}$ and $\overline{P}_\Lambda := P^{(n)}_{\Lambda^{(L)},-}$.

3.2.2. Fusion Rules. In this section, we show the fusion rules for the string operators $w_{a,\Lambda}$. In particular, we will show that $w_{a,\Lambda} \circ w_{b,\Lambda}$ is unitarily equivalent to $w_{ab,\Lambda}$.

Let us begin with semion-semion fusion. For any oriented path P, the automorphism $w_S[P] \circ w_S[P]$ is given by conjugation with the formal unitary

$$W_S[P]^2 = \Omega_{S,S}[P] := \left(\prod_{\text{R-legs } e} \sigma^Z_e \right) \left(\prod_{\substack{\text{L-vertices} \\ v=(e,e')}} \sigma^Z_e \sigma^Z_{e'} \right). \tag{54}$$

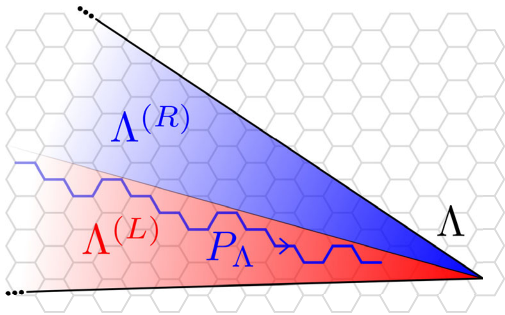

FIGURE 12. A cone Λ divided into its left and right cones $\Lambda^{(L)}$ and $\Lambda^{(R)}$. The path P_Λ is the largest part of $\partial\Pi_{\Lambda^{(R)}}$ such that $w_a[P_\Lambda]$ is supported in Λ

Let e_i and e_f be the initial and final edges of the path P (if they exist) and let $u[P]$ be the automorphism given by conjugation with

$$U[P] := \Omega_{S,S}[P] \times \sigma^Z_{e_i}\sigma^Z_{e_f}. \tag{55}$$

Lemma 3.3. *We have $\omega \circ u[P] = \omega$ for any path P.*

Proof. We first take P finite and show that $u[P]$ leaves any ω_n invariant.

Recall that ω_n is a vector state in the GNS representation of ω_0, given by

$$\Omega_n = \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\Pi \subset \Pi_n} (-1)^{\sharp\Pi} A_\Pi \Omega_0. \tag{56}$$

We have

$$U[P]A_\Pi\Omega_0 = A_\Pi\Omega_0. \tag{57}$$

Indeed, A_Π is a product of σ^X_e for all $e \in \partial\Pi$, a finite closed path. Now, any such closed path supports an even number of factors σ^Z_e of the unitary $U[P]$. Indeed, if $\partial\Pi$ travels along P, then the two edges of P along an R-leg carry no σ^Z, while the two edges along an L-vertex both have a σ^Z. The closed path $\partial\Pi$ must enter/leave the path P an even number of times. If it enters through an R-leg, it picks up a σ^Z from the R-leg. If it enters through an L-vertex, then it picks up exactly one of the σ^Z's of the two edges of P next to the L-vertex. Finally, if $\partial\Pi$ enters P through an endpoint of P, then the factors σ^Z_i, σ^Z_f at the initial/final edges ensure that a factor σ^Z is picked up. In all, we see that $U[P]A_\Pi = A_\Pi U[P]$, because the computation involves an even number of commutations of a σ^X with a σ^Z. Obviously $U[P]\Omega_0 = \Omega_0$ so $U[P]A_\Pi\Omega_0 = A_\Pi\Omega_0$ and $U[P]\Omega_n = \Omega_n$. It follows that $\omega_n \circ u[P] = \omega_n$ for any n and hence $\omega \circ u[P] = \omega$ for any finite P.

If P is infinite, then for any strictly local observable O we can find a finite P' such that $u[P](O) = u[P'](O)$ so $(\omega \circ u[P])(O) = (\omega \circ u[P'])(O) = \omega(O)$. Since the strictly local observables are dense in $\mathcal{A}$, this proves the claim. $\square$

Lemma 3.4. *If $u[P]$ is supported in a cone Λ, then $\pi_1 \circ u[P] \simeq \pi_1$, and the unitary implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. The unitary equivalence $\pi_1 \circ u[P] \simeq \pi_1$ follows immediately from Lemma 3.3. Let U be the unitary implementing this equivalence, i.e.

$$UOU^* = u[P](O) \tag{58}$$

for all $O \in \mathcal{A}$, and $U\Omega = \Omega$. (We identify $\mathcal{A}$ with its image under the faithful representation π_1.)

If P is finite, then actually $U \in \mathcal{A}_\Lambda \subset \mathcal{R}(\Lambda)$. If P is infinite, let P_n be the path consisting of edges of P whose midpoints lie in Π_n. Then $U[P_n] \in \mathcal{A}_\Lambda$ has $U[P_n]\Omega = \Omega$ for all n, and for any strictly local observables O, O' we have

$$\begin{aligned} \langle O\Omega, UO'\Omega\rangle &= \langle O\Omega, u[P](O')U\Omega\rangle = \lim_{n\uparrow\infty} \langle O\Omega, u[P_n](O')\Omega\rangle \\ &= \lim_{n\uparrow\infty} \langle O\Omega, U[P_n]O'\Omega\rangle. \end{aligned} \tag{59}$$

Since the vectors $O\Omega$, $O'\Omega$ for O, O' strictly local observables are dense in $\mathcal{H}$, this shows that the sequence $U[P_n]$ converges weakly to U. Since $U[P_n] \in \mathcal{A}_\Lambda$ for all n, it follows that $U \in \mathcal{R}(\Lambda)$. $\square$

Lemma 3.5. *For any cone Λ, we have that $\pi_1 \circ w_{S,\Lambda} \circ w_{S,\Lambda} \simeq \pi_1$, and the unitary U_Λ implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. By definition, $w_{S,\Lambda} = w_S[P_\Lambda]$ so $w_{S,\Lambda} \circ w_{S,\Lambda} = \mathrm{Ad}(\sigma_f^Z) \circ u[P_\Lambda]$ where e_f is the final edge of the half-infinite path P_Λ (cf. Eqs. (54) and (55)). From Lemma 3.4, we find that there exists a unitary $U_\Lambda \in \mathcal{R}(\Lambda)$ such that $u[P_\Lambda] = \mathrm{Ad}(U_\Lambda)$, hence

$$\pi_1 \circ w_{S,\Lambda} \circ w_{S,\Lambda} = \pi_1 \circ \mathrm{Ad}(\sigma_{e_f}^Z U_\Lambda), \tag{60}$$

proving the claim. $\square$

We can now easily show

Proposition 3.6. *For each cone Λ, there are unitaries $\Omega(a,b) \in \mathcal{R}(\Lambda)$ such that*

$$\mathrm{Ad}(\Omega(a,b)) \circ w_{a,\Lambda} \circ w_{b,\Lambda} = w_{a\times b,\Lambda} \tag{61}$$

for all $a, b \in I = \{1, S, \bar{S}, B\}$ and where $\times$ is an abelian product on I given by

$\times$	1	S	$\bar{S}$	B
1	1	S	$\bar{S}$	B
S	S	1	B	$\bar{S}$
$\bar{S}$	$\bar{S}$	B	1	S
B	B	$\bar{S}$	S	1

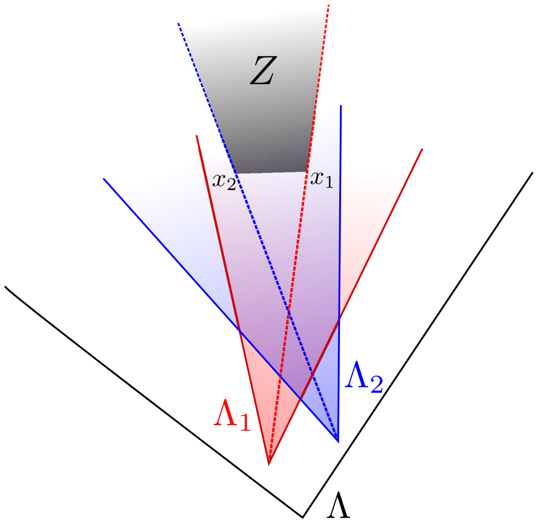

FIGURE 13. Axis of Λ_1 points to the right of the axis of Λ_2 relative to the cone Λ. The region Z has $w_a[\partial\Pi_Z]$ supported in Λ. Moreover, the path $\partial\Pi_Z$ differs from the union of the paths P_{Λ_1}, $\overline{P}_{\Lambda_2}$ by a finite number of edges

Proof. Lemma 3.5 shows that the claim holds for $S \times S = 1$. The rest of the claim follows from this case and

$$w_{\bar{S},\Lambda} = w_{S,\Lambda} \circ w_{B,\Lambda} = w_{B,\Lambda} \circ w_{S,\Lambda}, \quad w_{B,\Lambda} \circ w_{B,\Lambda} = \mathrm{id}. \tag{62}$$

□

3.2.3. Transportability.

Lemma 3.7. *If Λ_1 and Λ_2 are cones with axes $\hat{w}_1$ and $\hat{w}_2$, both contained in a cone Λ and such that $\hat{w}_1$ points to the right of $\hat{w}_2$ relative to Λ (see Fig. 13). Then $\pi_1 \circ w_{a,\Lambda_1} \simeq \pi_1 \circ v_{a,\Lambda_2}^{-1}$ and the unitary implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. Fix points x_1, x_2 on the central axes of the cones Λ_1, Λ_2 such that the region Z bounded by the half-infinite parts of these axes starting at x_1, x_2, and the line between x_1 and x_2 is convex and has $w_{a,\partial\Pi_Z}$ supported in Λ, see Fig. 13.

By construction, $w_{a,\partial\Pi_Z}$ differs from $w_{a,\Lambda_1} \circ v_{a,\Lambda_2}$ by the action of a local unitary W supported on Λ. Since $\partial\Pi_Z$ is a closed path, it follows from Proposition 3.2 that there exists a unitary $V \in \mathcal{B}(\mathcal{H})$ such that $\pi_1 \circ w_{a,\partial\Pi_Z} = \mathrm{Ad}(V) \circ \pi_1$ and $V\Omega = \Omega$; hence,

$$\pi_1 \circ w_{a,\Lambda_1} = \mathrm{Ad}(VW) \circ \pi_1 \circ v_{a,\Lambda_2}^{-1}. \tag{63}$$

This shows the required unitary equivalence. It remains to show that $V \in \mathcal{R}(\Lambda)$.

Let $Z_n = Z \cap B_n$ where B_n is the disk of radius n centred at the origin of $\mathbb{R}^2$. Then $\partial\Pi_{Z_n}$ are closed paths and the automorphisms $w_{\partial\Pi_{Z_n}} = \mathrm{Ad}(W_a[\partial\Pi_{Z_n}])$ leave the ground state invariant, and are supported in Λ. In particular, there exist phases ϕ_n such that $V_n := \phi_n W_a[\partial\Pi_{Z_n}]$ satisfies $V_n\Omega = \Omega$. For any strictly local observables $O, O' \in \pi_1(\mathcal{A})$ we have

$$\begin{aligned}\langle O\Omega, VO'\Omega\rangle &= \langle O\Omega, w_{a,\partial\Pi_Z}(O')V\Omega\rangle = \lim_{n\uparrow\infty}\langle O\Omega, w_{a,\partial\Pi_{Z_n}}(O')V_n\Omega\rangle \\ &= \lim_{n\uparrow\infty}\langle O\Omega, V_nO'\Omega\rangle.\end{aligned} \tag{64}$$

Since the vectors $O\Omega, O'\Omega$ are dense in $\mathcal{H}$, this shows that V_n converges weakly to V. Since each V_n is in $\mathcal{A}_\Lambda$, we conclude that $V \in \mathcal{R}(\Lambda)$. □

Proposition 3.8. *If Λ_1 and Λ_2 are cones both contained in a cone Λ, then $\pi_1 \circ w_{a,\Lambda_1} \simeq \pi_1 \circ w_{a,\Lambda_2}$ and the unitary implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. Let $\hat{w}_1, \hat{w}_2$ be the axes of the cones Λ_1, Λ_2 and take a cone $\Lambda_3 \subset \Lambda$ such that its axis $\hat{w}_3$ points to the right of both $\hat{w}_1$ and $\hat{w}_2$ relative to Λ. Then, Lemma 3.7 implies that there are unitaries $V_1, V_2 \in \mathcal{R}(\Lambda)$ such that

$$\pi_1 \circ w_{a,\Lambda_1} = \mathrm{Ad}(V_1) \circ \pi_1 \circ v_{a,\Lambda_3}^{-1}, \quad \pi_1 \circ w_{a,\Lambda_2} = \mathrm{Ad}(V_2) \circ \pi_1 \circ v_{a,\Lambda_3}^{-1}, \tag{65}$$

hence

$$\pi_1 \circ w_{a,\Lambda_1} = \mathrm{Ad}(V_2^*V_1) \circ \pi_1 \circ w_{a,\Lambda_2}. \tag{66}$$

Since $V_2^*V_1 \in \mathcal{R}(\Lambda)$, this proves the claim. □

3.2.4. Distinct Sectors. Fix a cone Λ_0 with axis $(0,1)$ and let $\pi_a := \pi_1 \circ w_{a,\Lambda_0}$ for $a \in I$.

Proposition 3.9. *For all $a, b \in I$, we have $\pi_a \simeq \pi_b$ if and only if $a = b$.*

Proof. For any n large enough such that the endpoint of P_{Λ_0} is contained in Π_{n-2}, consider the S-matrix

$$S_{ab} := \frac{1}{2}(\omega \circ w_{a,\Lambda_0})(W_b[\partial\Pi_n]). \tag{67}$$

An easy calculation shows that these quantities are independent of n and given by

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{68}$$

It follows that for any $a \neq b$ there is a c such that $(\omega \circ w_{a,\Lambda_0})(W_c[\partial\Pi_n]) = -(\omega \circ w_{b,\Lambda_0})(W_c[\partial\Pi_n])$ for all n sufficiently large. Corollary 2.6.11 of [2] then implies that π_a and π_b are disjoint. □

Table 1. Fusion intertwiners $\Omega(a,b)$ for the double semion state

$\Omega(a,b)$	1	S	$\bar{S}$	B
1	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
S	$\mathbb{1}$	$U\sigma^Z_{e_f}$	$U\sigma^Z_{e_f}$	$\mathbb{1}$
$\bar{S}$	$\mathbb{1}$	$U\sigma^Z_{e_f}$	$U\sigma^Z_{e_f}$	$\mathbb{1}$
B	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$

3.2.5. Verification of Assumptions. The four faithful irreducible representations $\pi_1, \pi_S, \pi_{\bar{S}}, \pi_B$ defined by $\pi_a = \pi_1 \circ w_{a,\Lambda_0}$ for $a \in \{1, S, \bar{S}, B\} = I$ are pairwise disjoint by Proposition 3.9.

For any cone Λ and any $a \in I$, we defined an automorphism $w_{a,\Lambda}$ supported in Λ. This collection of automorphisms satisfies Assumption 1 by Proposition 3.8. Assumptions 2 and 3 are verified by Proposition 3.6. Finally, Assumption 4 holds by Propositions 3.6 and 3.8.

3.3. *Computation of F-Symbols.* Having fixed the cone Λ_0 with axis $(0,1)$, we use for all $a \in I$ the shorthand notations $w_a := w_{a,\Lambda_0}$ and $\overline{w}_a := \overline{w}_{a,\Lambda_0}$, where the latter are the extensions of w_a to the allowed algebra $\mathcal{B}$ constructed in Lemma 2.5.

Let e_f be the final edge of the path P_{Λ_0} and let $U \in \mathcal{R}(\Lambda_0)$ be the unitary such that $\pi_1 \circ u[P_{\Lambda_0}] = \mathrm{Ad}(U) \circ \pi_1$ and $U\Omega = \Omega$ provided by Lemma 3.4. The proof of Lemma 3.5 shows that

$$\mathrm{Ad}(\Omega(S,S)) \circ (\overline{w}_S \circ \overline{w}_S) = \overline{w}_1 \tag{69}$$

with $\Omega(S,S) = \sigma^Z_{e_f} U$.

Using

$$w_{\bar{S}} = w_B \circ w_S = w_S \circ w_B, \quad w_B \circ w_B = \mathrm{id}, \tag{70}$$

we find that

$$\mathrm{Ad}(\Omega(a,b)) \circ (w_a \circ w_b) = w_{a\times b} \tag{71}$$

for all $a, b \in I$ with fusion intertwiners $\Omega(a,b)$ given in Table 1. It follows that $\Omega(a,b) \in (\overline{w}_a \otimes \overline{w}_b, \overline{w}_{a\times b})$ for all $a, b \in I$.

In order to compute the F-symbols, we first show

Lemma 3.10.

$$\overline{w}_S(U\sigma^Z_{e_f}) = -U\sigma^Z_{e_f}, \quad \overline{w}_B(U\sigma^Z_{e_f}) = U\sigma^Z_{e_f}, \quad \overline{w}_{\bar{S}}(U\sigma^Z_{e_f}) = -U\sigma^Z_{e_f}. \tag{72}$$

Proof. Since e_f is the final edge of the path P_{Λ_0}, we have $w_S(\sigma^Z_{e_f}) = w_{\bar{S}}(\sigma^Z_{e_f}) = -\sigma^Z_{e_f}$ and $w_B(\sigma^Z_{e_f}) = \sigma^Z_{e_f}$. It remains to show that $\overline{w}_S(U) = \overline{w}_{\bar{S}}(U) = \overline{w}_B(U) = U$.

Since U is the weak limit of the sequence $U_n = U[P_n]$ where P_n is the path consisting of edges of P_{Λ_0} whose midpoints lie in Π_n (cf. the proof of Lemma 3.4), it is sufficient to show $w_S(U_n) = w_{\bar{S}}(U_n) = w_B(U_n) = U_n$. This follows similar to the argument in the proof of Lemma 3.4. Since U_n is a product of σ^Z's, we have that $w_B(U_n) = U_n$, and

$$w_S(U_n) = w_{\bar{S}}(U_n) = \left(\prod_{e \in P_n} \sigma_e^X\right) U_n \left(\prod_{e \in P_n} \sigma_e^X\right). \tag{73}$$

By design, the unitary U_n has an even number of σ^Z's on the path P_n. Indeed, there are two factors of σ^Z for every L-vertex, zero for every R-leg, and another two for the endpoints. We conclude that $w_S(U_n) = W_{\bar{S}}(U_n) = U_n$ for all n.

□

We can now start computing the F-symbols. If in Eq. (24) we take $a = 1$, then

$$\Omega(b, c)\Omega(1, b) = F(1, b, c)\Omega(1, bc)\Omega(B, C). \tag{74}$$

Since $\Omega(1, b) = \Omega(1, bc) = \mathbb{1}$, we find that $F(1, b, c) = 1$ for all b, c.

Similarly we find $F(a, 1, c) = F(a, b, 1) = 1$ for all a, b, c.

Let us now consider F-symbols that involve the bound state B, for example,

$$\Omega(Bb, c)\Omega(B, b) = F(B, b, c)\Omega(B, bc)\overline{w}_B(\Omega(b, c)). \tag{75}$$

Since $\Omega(B, b) = \Omega(B, bc) = \mathbb{1}$, this reduces to

$$\Omega(Bb, c) = F(B, b, c)\overline{w}_B(\Omega(b, c)). \tag{76}$$

If $b = B$ or $c = B$, then then $\Omega(Bb, b) = \Omega(b, c) = \mathbb{1}$ so $F(B, b, c) = 1$. If $b, c \in \{S, \bar{S}\}$, then $\Omega(Bb, c) = \Omega(b, c) = U\sigma^Z_{e_f}$, so using Lemma 3.10 we find again $F(B, b, c) = 1$ for all b, c.

Similar considerations show that $F(B, b, c) = F(a, B, c) = F(a, b, B) = 1$ for all a, b, c.

Finally, we consider the case where $a, b, c \in \{S, \bar{S}\}$. Then since $ab, bc \in \{1, B\}$ we have $\Omega(ab, c) = \Omega(a, bc) = \mathbb{1}$ so

$$U\sigma^Z_{e_f} = F(a, b, c)\overline{w}_a(U\sigma^Z_{e_f}). \tag{77}$$

Using Lemma 3.10, we conclude that $F(a, b, c) = -1$ for $a, b, c \in \{S, \bar{S}\}$.

3.4. *Computation of R-Symbols.* Choose cones Λ_L with axis $(-1, 0)$ and Λ_R with axis $(1, 0)$, both disjoint from Λ_0 as in Fig. 14. Let $\widetilde{\Lambda}_L \supseteq \Lambda_0 \cup \Lambda_L$ and $\widetilde{\Lambda}_R \supseteq \Lambda_0 \cup \Lambda_R$ be allowed cones such that $\widetilde{\Lambda}_L \cap \Lambda_R = \widetilde{\Lambda}_R \cap \Lambda_L = \emptyset$.

To compute the braiding intertwiners $\epsilon(a, b) = \epsilon(\overline{w}_a, \overline{w}_b)$, set $\overline{v}_a^L = \overline{v}_a [\Lambda_L]^{-1}$, $\overline{v}_a^R = \overline{v}_a[\Lambda_R]^{-1}$ as well as $\overline{w}_a^L = \overline{w}_{a,\Lambda_L}$ and $\overline{w}_a^R = \overline{w}_{a,\Lambda_R}$ for all $a = 1, S, \bar{S}, B$. (Recall that the automorphisms $v_{a,\Lambda}$ are defined in Sect. 3.2.1.)

It follows from Lemma 3.7 that there are unitaries $U_a \in (\overline{w}_a, \overline{v}_a^L)$ and $V_b \in (\overline{w}_b, \overline{v}_b^R)$ such that $U_a \in \mathcal{R}(\widetilde{\Lambda}_L)$ and $V_b \in \mathcal{R}(\widetilde{\Lambda}_R)$. By the same lemma, there are unitaries $U'_a \in (\overline{v}_a^L, \overline{w}_a^L)$ and $V'_b \in (\overline{v}_b^R, \overline{w}_b^R)$ such that $U'_a \in \mathcal{R}(\Lambda_L)$ and $V'_b \in \mathcal{R}(\Lambda_R)$. We therefore have unitary morphisms $U'_a U_a \in (\overline{w}_a, \overline{w}_a^L)$ and

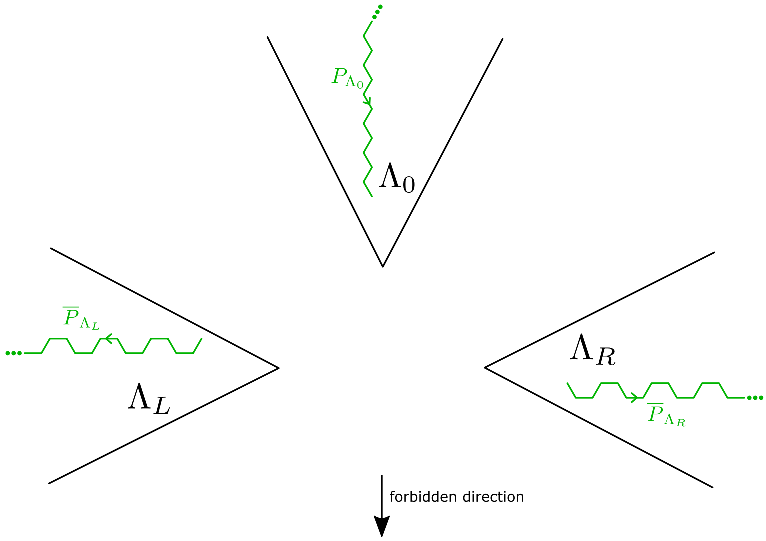

FIGURE 14. Cones Λ_0, Λ_L and Λ_R used to define the braiding intertwiners

$V_b'V_b \in (\overline{w}_b, \overline{w}_b^R)$ with $U_a'U_a \in \mathcal{R}(\widetilde{\Lambda}_L)$ and $V_b'V_b \in \mathcal{R}(\widetilde{\Lambda}_R)$. By definition 2.11 and the fact that $\overline{w}_a(V_b') = V_b'$, we have

$$\epsilon(a,b) = V_b^* \, \overline{w}_a\big(V_b\big). \tag{78}$$

In order to compute $\overline{w}_a(V_b)$, let us realise V_b as the weak limit of a sequence of strictly local unitaries.

Let K be the cone whose legs coincide with the central axes of Λ_0 and Λ_R, see Fig. 15. Then the path $\partial\Pi_K$ contains P_{Λ_0} and $\overline{P}_{\Lambda_R}$ and the path $Q = \partial\Pi_K \backslash (P_{\Lambda_0} \cup P_{\Lambda_R})$ is finite. For each n, let $K_n = K \cap B_n$ where $B_n \subset \mathbb{R}^2$ is the disk of radius n centred at the origin of $\mathbb{R}^2$. Consider the sequence of paths $P_n = \partial\Pi_{K_n} \setminus Q$ and set $V_b^{(n)} := W_b[P_n]$.

Lemma 3.11. *There are phases ϕ_n such that the sequence $\phi_n V_b^{(n)}$ converges weakly to V_b.*

Proof. Consider first the sequence of finite closed paths $\partial\Pi_{K_n} = P_n \cup Q$ and corresponding string operators $W_b[\partial\Pi_{K_n}]$. By Proposition 3.2, the unitaries $W_b[\partial\Pi_{K_n}]$ leave the ground state invariant up to a phase, so there are phases ϕ_n such that $\widetilde{W}^{(n)} := \phi_n W_b[\partial\Pi_{K_n}]$ satisfy $\widetilde{W}^{(n)}\Omega = \Omega$.

Since $\partial\Pi_K$ is a closed path, the automorphism $w_b[\partial\Pi_K]$ leaves the grounds state invariant by Proposition 3.2. It follows that there is a unique unitary $\widetilde{W} \in \mathcal{B}(\mathcal{H})$ such that $\widetilde{W}\Omega = \Omega$ and $w_b[\partial\Pi_K] = \mathrm{Ad}(\widetilde{W})$ (as automorphisms on $\pi_1(\mathcal{A})$).

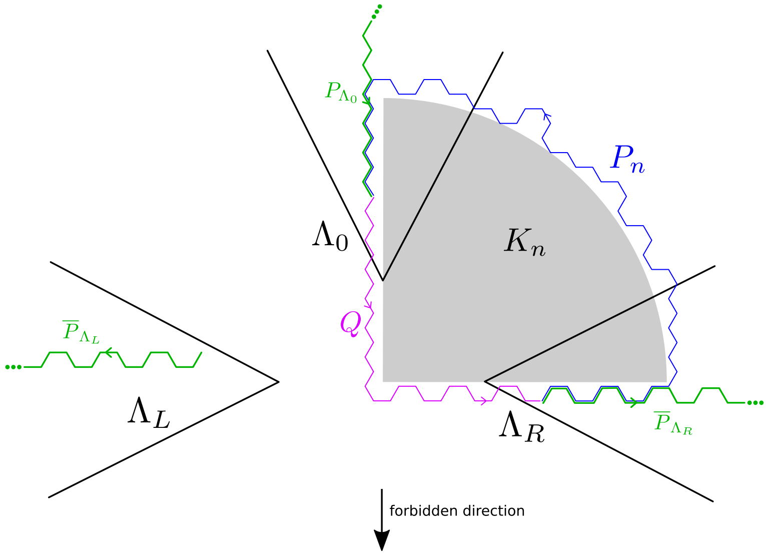

FIGURE 15. Sets K_n and the paths P_n and Q used in the construction of the sequence of unitaries $V_b^{(n)}$ that converge weakly to the intertwiner V_b

Now, for any strictly local observable O we have $w_b[\partial\Pi_K](O) = w_b[\partial\Pi_{K_n}](O)$ for all n large enough. Thus, for any strictly local operators O, O' we have

$$\begin{aligned}
\langle O\Omega, \widetilde{W}\, O'\Omega\rangle &= \langle O\Omega, w_b[\partial\Pi_K](O')\, \widetilde{W}\Omega\rangle \\
&= \lim_{n\uparrow\infty} \langle O\Omega, w_b[\partial\Pi_{K_n}](O')\, \Omega\rangle \\
&= \lim_{n\uparrow\infty} \langle O\Omega, \widetilde{W}^{(n)} O'\Omega\rangle,
\end{aligned}$$

showing that the sequence $\widetilde{W}^{(n)}$ converges weakly to $\widetilde{W}$.

Now note that the paths $\partial\Pi_{K_n}$ and $P_n = \partial\Pi_{K_n}\setminus Q$ differ by the same path $Q = \partial\Pi_{K_n}\setminus P_n$ for all n. It follows that the corresponding string operators $W_b[\partial\Pi_{K_n}]$ and $V_b^{(n)} = W_b[P_n]$ satisfy $V_b^{(n)}(W_b[\widetilde{P}_n])^* = W$ for a unitary W that is independent of n and is supported on the path Q and edges adjacent to Q. (In fact, W is equal to $W_b[Q]^*$ up local operators supported near the endpoints of the path Q.) Therefore, $V_b^{(n)} = WW_b[\partial\Pi_{K_n}]$ and $\phi_n V_b^{(n)} = W\widetilde{W}^{(n)}$.

Since $\widetilde{W}^{(n)}$ converges weakly to $\widetilde{W}$, it follows that the sequence $\phi_n V_b^{(n)} = W\widetilde{W}^{(n)}$ converges weakly to $W\widetilde{W}$. By construction, $\mathrm{Ad}(W\widetilde{W}) = w_b \circ (w_b^R)^{-1} = \mathrm{Ad}(V_b)$ so $W\widetilde{W} = \mu V_b$ for some phase μ. We then find that the sequence $\mu^*\phi_n V_n^{(n)}$ converges weakly to V_b. This proves Lemma. □

We can now compute the braiding intertwiners.

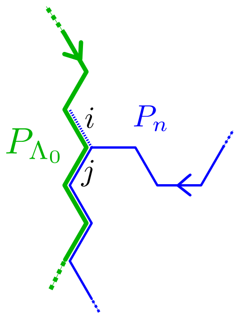

FIGURE 16. Edges e and e' playing a role in the computation of $\epsilon(S,S)$

Obviously $\overline{w}_1 = \mathrm{id}$ and $V_1 = I$ so $\epsilon(1,a) = \epsilon(a,1) = \mathbb{1}$ for all $a \in I$. It is also easy to see that

$$w_B(V_a^{(n)}) = V_a^{(n)} \tag{79}$$

for any $a \in \{1, S, \bar{S}, B\}$, so $\epsilon(B,a) = \mathbb{1}$ for all a, while

$$w_S(V_B^{(n)}) = w_{\bar{S}}(V_B^{(n)}) = -V_B^{(n)}, \tag{80}$$

because the path P_{Λ_0} contains a single R-leg of the path P_n. So $\epsilon(S,B) = \epsilon(\bar{S},B) = -\,\mathbb{1}$.

Let us now compute $\epsilon(S,S)$. Note that the path P_n enters the path P_{Λ_0} at an L-vertex of P_{Λ_0}. Let (e,e') be the edges of P_{Λ_0} before and after this L-vertex, see Fig. 16. We find

$$\begin{aligned}&(V_S^{(n)})^* \, w_S(V_S^{(n)})\\ &= \left(\sigma_{e'}^X \left(\mathrm{i}^{\frac{1-\sigma_e^Z}{2}}\right)\right) \left(\sigma_e^X \sigma_{e'}^X (-1)^{s_I}\right) \left(\left(\mathrm{i}^{\frac{1-\sigma_e^Z}{2}}\right)^* \sigma_{e'}^X\right) \left((-1)^{s_I} \sigma_{e'}^X \sigma_e^X\right) = \mathrm{i}\,\mathbb{1},\end{aligned}$$

which implies $\epsilon(S,S) = \mathrm{i}\,\mathbb{1}$.

We now use the braid equations (Lemma 2.13)

$$\begin{aligned}\epsilon(\rho, \sigma\otimes\tau) &= (\mathbb{1}_\sigma \otimes \epsilon(\rho,\tau))(\epsilon(\rho,\sigma)\otimes \mathbb{1}_\tau)\\ \epsilon(\rho\otimes\sigma, \tau) &= (\epsilon(\rho,\tau)\otimes \mathbb{1}_\sigma)(\mathbb{1}_\rho)\otimes \epsilon(\sigma,\tau))\end{aligned}$$

(where I_ρ denotes the identity intertwiner from ρ to itself) to compute

$$\begin{aligned}\epsilon(\bar{S},S) &= \epsilon(S\times B, S) = \epsilon(S,S)\overline{w}_S(\epsilon(B,S)) = \mathrm{i}\,\mathbb{1}\\ \epsilon(S,\bar{S}) &= \epsilon(S, S\times B) = \overline{w}_S(\epsilon(S,B))\epsilon(S,S) = -\mathrm{i}\,\mathbb{1}\\ \epsilon(\bar{S},\bar{S}) &= \epsilon(S\times B, \bar{S}) = \epsilon(S,\bar{S})\overline{w}_S(\epsilon(B,S)) = -\mathrm{i}\,\mathbb{1}\,.\end{aligned}$$

Thus, we have computed all braiding intertwiners,see Table 2 for a summary.

The R-symbols are defined (26) by

$$\Omega(b,a)\epsilon(a,b) = R(a,b)\times\Omega(a,b). \tag{81}$$

Since $\Omega(a,b) = \Omega(b,a)$ for all a,b, we find that the R-symbols are shown in Table 3.

Table 2. Braiding intertwiners $\epsilon(a,b)$ for the double semion state

$\epsilon(a,b)$	1	S	$\bar{S}$	B
1	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
S	$\mathbb{1}$	$\mathrm{i}\,\mathbb{1}$	$-\mathrm{i}\,\mathbb{1}$	$-\,\mathbb{1}$
$\bar{S}$	$\mathbb{1}$	$\mathrm{i}\,\mathbb{1}$	$-\mathrm{i}\,\mathbb{1}$	$-\,\mathbb{1}$
B	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$

Table 3. R-symbols $R(a,b)$ for the double semion state

$R(a,b)$	1	S	$\bar{S}$	B
1	1	1	1	1
S	1	i	$-\mathrm{i}$	-1
$\bar{S}$	1	i	$-\mathrm{i}$	-1
B	1	1	1	1

One can verify that the F and R-symbols indeed satisfy the pentagon and hexagon equations.

3.5. *Anyons are Described by* $\mathrm{Rep}_f\mathcal{D}^{\phi}(\mathbb{Z}_2)$. In Sect. 3.2.5, we verified Assumptions 1–4 for the superselection sectors π_a with $a \in I = \{1, S, \bar{S}, B\}$ of the double semion state. Let us denote by $\Delta^I_{\Lambda_0}$ the corresponding braided tensor category constructed in Sect. 2.2.

In this section, we show that $\Delta^I_{\Lambda_0}$ is braided monoidal equivalent to the representation category $\mathrm{Rep}_f\mathcal{D}^{\phi}(\mathbb{Z}_2)$, where ϕ is a non-trivial 3-cocycle on $\mathbb{Z}_2$. We will do this by showing that both categories have equivalent fusion rings, F-symbols, and R-symbols, and appealing to Proposition 7.5.2 of [12].

This identification is relevant, because it partially verifies a conjecture about string-net models [16]. A given string-net model is defined by an input fusion category $\mathcal{F}$, and the topological order of the model is conjectured [17] to correspond to the Drinfeld centre $\mathcal{Z}(\mathcal{F})$ of $\mathcal{F}$. In the case where $\mathcal{F} = \mathrm{Vec}^{\phi}_G$ for a finite group G and a 3-cocycle ϕ of G, we have $\mathcal{Z}(\mathcal{F}) = \mathrm{Rep}_f\mathcal{D}^{\phi}(G)$ (cf. [18]), where $\mathcal{D}^{\phi}(G)$ is the twisted quantum double algebra first described in [4].

3.5.1. The Braided Fusion Ring of $\Delta^I_{\Lambda_0}$. We have extracted from the braided tensor category Δ^I_{Λ} its fusion ring, generated by the elements of $I = \{1, S, \bar{S}, B\}$ with abelian fusion rules given by the group structure on I described in Proposition 3.6. In other words, the fusion ring of $\Delta^I_{\Lambda_0}$ is isomorphic to $\mathbb{Z}(I)$. In Sect. 3.3, we obtained the F-symbols, and in Sect. 3.4 we obtained the R-symbols, Cf. Table 3 derived from the braided tensor category $\Delta^I_{\Lambda_0}$. It follows from Proposition 7.5.2 of [12] that these data completely determine the category $\Delta^I_{\Lambda_0}$ up to braided monoidal equivalence.

3.5.2. Description of $\mathbf{Rep}_f\,\mathcal{D}^\phi(\mathbb{Z}_2)$. We describe the quasi Hopf algebra $\mathcal{D}^\phi(\mathbb{Z}_2)$ first introduced in [4]. We follow the presentation in [22].

Let $\phi : (\mathbb{Z}_2)^3 \to U(1)$ be the normalised representative of the non-trivial class in $H^3(\mathbb{Z}_2, U(1))$:

$$\phi(-,-,-) = -1, \quad \text{all other components equal to 1.} \tag{82}$$

Let

$$c_x(f,g) := (\iota_x\phi)(f,g) = \frac{\phi(x,f,g)\phi(f,g,x)}{\phi(f,x,g)}. \tag{83}$$

for all $x, f, g \in \mathbb{Z}_2$. For each $x \in \mathbb{Z}_2$ the map $c_x : (\mathbb{Z}_2)^2 \to U(1)$ is a 2-cocycle, it satisfies

$$c_x(f,g)c_x(fg,h) = c_x(f,gh)c_x(g,h). \tag{84}$$

The quasi-quantum double $\mathcal{D}^\phi(\mathbb{Z}_2)$ is an algebra spanned by $\{P_x f\}_{x,f\in\mathbb{Z}_2}$ with multiplication

$$(P_x f)(P_y g) = \delta_{x,y}(P_x fg)\, c_x(f,g). \tag{85}$$

The unit for this multiplication is $\sum_{x\in\mathbb{Z}_2}(P_x 1)$.

The quasi-quantum double is, moreover, equipped with a coproduct $\Delta : \mathcal{D}^\phi(\mathbb{Z}_2) \to \mathcal{D}^\phi(\mathbb{Z}_2) \otimes \mathcal{D}^\phi(\mathbb{Z}_2)$ given by

$$\Delta(P_x f) = \sum_{yz=x} c_f(y,z)(P_y f) \otimes (P_z f). \tag{86}$$

Associativity and quasicoassociativity follow readily from Eq. (84), in particular

$$(\mathrm{id} \otimes \Delta)\Delta(P_x f) = \varphi \cdot (\Delta \otimes \mathrm{id})\Delta(P_x f) \cdot \varphi^{-1} \tag{87}$$

with $\varphi = \sum_{f,g,h\in\mathbb{Z}_2} \phi^{-1}(f,g,h)\,(P_f 1) \otimes (P_g 1) \otimes (P_h 1)$. That Δ is an algebra morphism follows from the identity

$$\frac{c_x(f,g)c_y(f,g)}{c_{xy}(f,g)} \times \frac{c_f(x,y)c_g(x,y)}{c_{fg}(x,y)} = 1. \tag{88}$$

There is a counit $\epsilon : \mathcal{D}^\phi(\mathbb{Z}_2) \to \mathbb{C}$ and an antipode $S : \mathcal{D}^\phi(\mathbb{Z}_2) \to \mathcal{D}^\phi(\mathbb{Z}_2)$ given by

$$\epsilon(P_x f) = \delta_{x,1}, \quad S(P_x f) = (P_{x^{-1}} f^{-1})\, c_{x^{-1}}(f, f^{-1})^{-1} c_f(x, x^{-1})^{-1}. \tag{89}$$

These give $\mathcal{D}^\phi(\mathbb{Z}_2)$ the structure of a quasi Hopf algebra. This quasi-Hopf algebra is, moreover, quasitriangular with universal R-matrix

$$R = \sum_{x,y} (P_x 1) \otimes (P_y x). \tag{90}$$

3.5.3. Category of Representations and Its Fusion Ring. Since $\mathcal{D}^{\phi}(\mathbb{Z}_2)$ is a quasitriangular Hopf algebra, its category of finite-dimensional representations $\mathrm{Rep}_f\mathcal{D}^{\phi}(\mathbb{Z}_2)$ is a braided tensor category. We extract the fusion ring, F-symbols, and R-symbols of this braided tensor category. See [18] for a more in depth analysis of this category of representations.

There are four irreducible representations of $\mathcal{D}^{\phi}(\mathbb{Z}_2)$, labelled by pairs $(x,\chi) \in \mathbb{Z}_2 \times \mathbb{Z}_2^*$. ($\mathbb{Z}_2^*$ consists of the characters of $\mathbb{Z}_2$, namely 1 and sgn.) They are given by

$$\Pi_{(x,\chi)}(P_y f) = \delta_{x,y}\, \varepsilon_x(f)\chi(f) \tag{91}$$

with

$$\varepsilon_x(f) := \exp\left(\frac{\pi \mathrm{i}}{2}[x].[f]\right) \tag{92}$$

where $[x], [f]$ are the additive representation of x and f. i.e. $\varepsilon_-(-) = \mathrm{i}$ and all other components are equal to one. ε_x is a cocycle and

$$c_x(f,g) = (d\varepsilon_x)(f,g) = \frac{\varepsilon_x(fg)}{\varepsilon_x(f)\varepsilon_x(g)}. \tag{93}$$

Since we have a coproduct, we have the following tensor product of representations:

$$(\Pi_1 \otimes \Pi_2)(P_x f) := ((\Pi_1 \otimes \Pi_2) \circ \Delta)(P_x f) = \sum_{yz=x} c_f(y,z)\, \Pi_1(P_y f) \otimes \Pi_2(P_z f). \tag{94}$$

One easily verifies that

$$\Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)} = \Pi_{(xy,\chi\sigma)}. \tag{95}$$

The representation $\Pi_{(1,1)}$ is an identity for this tensor product (with trivial left and right unitors). In particular, the fusion ring of the representation category is $\mathbb{Z}(G)$ with G the abelian group with elements $\{(x,\chi)\}_{(x,\chi)\in\mathbb{Z}_2\times\mathbb{Z}_2^*}$ and group multiplication given by $(x,\chi)\cdot(y,\sigma) = (xy,\chi\sigma)$.

With this tensor product, the representations of $\mathcal{D}^{\phi}(\mathbb{Z}_2)$ form a tensor category with simple objects $\Pi_{(x,\chi)}$ and associators between simple objects

$$\alpha_{(x,\chi),(y,\sigma),(z,\tau)} : (\Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)}) \otimes \Pi_{(z,\tau)} \to \Pi_{(x,\chi)} \otimes (\Pi_{(y,\sigma)} \otimes \Pi_{(z,\tau)}) \tag{96}$$

given by multiplication with $\phi(x,y,z)$. This shows that the F-symbols of the representation category $\mathrm{Rep}_f\mathcal{D}^{\phi}(\mathbb{Z}_2)$ are given by the 3-cocycle α on G given by $\alpha((x,\chi),(y,\sigma),(z,\tau)) = \phi(x,y,z)$.

The braiding $\epsilon_{(x,\chi),(y,\sigma)} : \Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)} \to \Pi_{(y,\sigma)} \otimes \Pi_{(x,\chi)}$ of simple objects of $\mathcal{D}^{\phi}(\mathbb{Z}_2)$ is given by multiplication with

$$(\Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)})(R) = \varepsilon_y(x)\sigma(x), \tag{97}$$

where R is the universal R-matrix given in Eq. (90). These braidings are summarised in Table 4, and it follows from (95) that the R-symbols of $\mathcal{D}^{\phi}(\mathbb{Z}_2)$ are given by the same table.

TABLE 4. Braiding isomorphisms of $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ for simple objects

$\epsilon_{(x,\chi),(y,\sigma)}$	$(1,1)$	$(-1,1)$	$(-1,\mathrm{sgn})$	$(1,\mathrm{sgn})$
$(1,1)$	1	1	1	1
$(-1,1)$	1	i	−i	−1
$(-1,\mathrm{sgn})$	1	i	−i	−1
$(1,\mathrm{sgn})$	1	1	1	1

3.5.4. Braided Monoidal Equivalence of $\boldsymbol{\Delta^I_{\Lambda_0}}$ and $\boldsymbol{\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)}$. By the identification

$$(1,1) \leftrightarrow 1, \quad (-1,1) \leftrightarrow S, \quad (-1,\mathrm{sgn}) \leftrightarrow \bar{S}, \quad (1,\mathrm{sgn}) \leftrightarrow B. \tag{98}$$

we see that the groups G and I and therefore the fusion rings $\mathbb{Z}(G)$ and $\mathbb{Z}(I)$ of $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ and $\Delta^I_{\Lambda_0}$ are isomorphic.

Under this identification, the F-symbols of $\Delta^I_{\Lambda_0}$ computed in Sect. 3.3 match precisely with the F-symbols α of the representation category. Furthermore, comparing Tables 3 and 4 we see that also the R-symbols match precisely. Thus, the braided tensor category $\Delta^I_{\Lambda_0}$ and the representation category $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ have the same fusion rings $\mathbb{Z}(G) \simeq \mathbb{Z}(I)$, the same F-symbols, and the same R-symbols. It follows from Proposition 7.5.2 of [12] that $\Delta^I_{\Lambda_0}$ and $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ are isomorphic as braided monoidal categories.

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Appendix A. Purity of the Double Semion State

In appendix, we prove Theorem 3.1, stating that the double semion state ω constructed in Sect. 3.1 is pure.

A.1. *Restrictions of ω to finite regions.* Denote by Π_n^E the set of edges belonging to some hexagon of Π_n and set $\mathcal{M}_n = \mathcal{A}_{\Pi_n^E}$. We investigate the restrictions $\omega|_n := \omega|_{\mathcal{M}_n}$.

Lemma A.1. *For any $m > n$, we have that $\omega|_n = \omega_m|_n$.*

Proof. It is sufficient to note that ω_m has the same expectation value for any operator in $\mathcal{M}_n$ as ω does. Indeed, for any $A \in \mathcal{M}_n$ we have $\omega(A) = \lim \omega_m(A)$, and the latter sequence becomes constant as soon as Π_m contains all hexagons containing edges in the support of A. i.e. $\omega_m(A) = \omega(A)$ for all $m \geq n+1$. $\square$

Thus, we can restrict our attention to states $\omega_m|_n$. Recall that ω_m is given by the expectation value in the vector state

$$\Omega_m = \sqrt{\frac{1}{2^{|\Pi_m|}}} \sum_{\Pi \subset \Pi_m} (-1)^{\sharp \Pi} A_{\Pi} \Omega_0 = \sqrt{\frac{1}{2^{|\Pi_m|}}} \sum_{\Pi \subset \Pi_m} (-)^{\sharp \partial \Pi} |\partial \Pi\rangle \tag{99}$$

where $\sharp \partial \Pi$ is the number of closed loops in the loop soup $\partial \Pi$, and we chose to write A_Π instead of $\pi_0(A_\Pi)$ because the representation π_0 is faithful, and $|\partial \Pi\rangle$ is the product state with all degrees of freedom spin up, except those on the edges along the path $\partial \Pi$, which are spin down.

Note that every closed path α supported on Π_m^E is of the form $\partial \Pi$ for a unique $\Pi \subset \Pi_m$, so we have written Ω_m as a uniform superposition over all closed-loop soups supported on Π_m^E. Moreover, for closed paths α and β we have $|\alpha\rangle = |\beta\rangle$ if and only if $\alpha = \beta$, and these states are orthogonal otherwise.

We will show that $\omega_m|_n$ is a mixed state which is an equal-weight convex combination of pure states $\eta_n(b)$ where b is a *boundary condition*, namely an assignment of up-or down to each out edge of the region Π_n such that an even number of edges are up, see Fig. 17

The state $\eta_n(b)$ is then given by a uniform superposition of all loop soups that satisfy the boundary condition b, weighed by ± 1 depending on whether a fixed 'closure' of the boundary condition has an even or an odd number of closed loops.

Lemma A.2. *There are $2^{|\Pi_n|}$ such loop soups for each boundary condition b.*

Proof. For the boundary condition with all spins up, this is obvious, because then the loop soups are precisely closed-loop soups in Π_n^E.

To obtain loop soups for an arbitrary boundary condition b, act on any closed-loop soup with A_p on the hexagons between pairs of boundary edges where b forces a loop to end (choose one of two possible pairings). This yields a loop soup that satisfies the boundary condition, and two different closed-loop soups give two different loop soups satisfying the boundary condition. Conversely, every loop soup satisfying the boundary condition arises in this

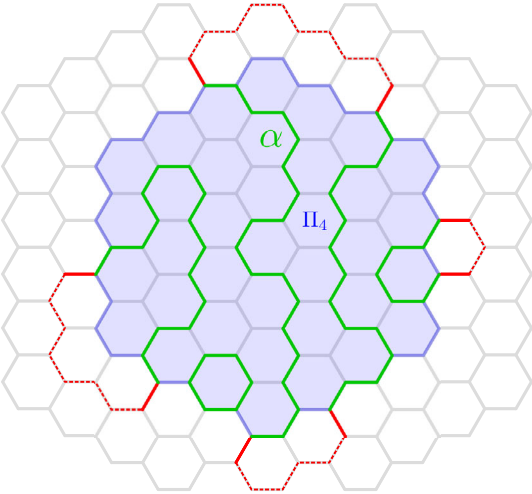

FIGURE 17. A loop soup $\alpha \in \mathcal{P}_4^{(b)}$ with boundary condition b corresponding to the red edges. The dotted red paths indicate one of two ways of pairing neighbouring red edges, resulting in a closed-loop soup (color figure online)

way, because acting on loop soups satisfying b with A_v's on the vertices between pairs of boundary edges where b forces a loop to end yields a closed-loop soup. $\square$

Write $\mathcal{P}_n^{(b)}$ for the loop soups in Π_n^E that satisfy the boundary condition b. For a given boundary condition b, any $\alpha \in \mathcal{P}_n^{(b)}$ can be 'closed up' in precisely two ways by connecting neighbouring marked edges using edges in $\Pi_{n+1}^E \setminus \Pi_n^E$, see Fig. 17. Pick one such 'pairing' of marked boundary edges, and let $\sharp\alpha$ be the number of loops of α closed up with the chosen pairing. Then we have normalised vectors

$$|\eta_n^{(b)}\rangle = \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\alpha \in \mathcal{P}_n^{(b)}} (-1)^{\sharp\alpha} |\alpha\rangle. \tag{100}$$

We have $\langle \eta_n^{(b)}, \eta_n^{(b')} \rangle = \delta_{b,b'}$, i.e. these vectors form an orthonormal set. Denote by $\eta_n^{(b)}$ the pure state on $\mathcal{M}_n$ corresponding to the vector $|\eta_n^{(b)}\rangle$.

Proposition A.3. *For $m > n \geq 1$,*

$$\omega_m|_n = \frac{1}{2^{6n-1}} \sum_b \eta_n^{(b)}. \tag{101}$$

Since the $|\eta_n^{(b)}\rangle$ form an orthonormal set, this is a Schmidt decomposition of $\omega_m|_n$.

Here, 2^{6n-1} is the number of boundary conditions b. Indeed, there are $6n$ outer edges where the boundary condition either forces or does not force a string to pass, and the number of edges where a string is forced to end must be even. There are as many even boundary conditions as there are odd boundary conditions. Indeed, flipping a fixed edge gives a bijection.

Proof. By Lemma A.1, it is sufficient to consider $m = n+1$. The state ω_{n+1} on Π^E_{n+1} is a uniform superposition of closed-loop soups in Π^E_{n+1}. Any such loop soup α defines a boundary condition $b(\alpha)$ by the outer edges of Π_n that are occupied by strings of α. We can therefore organise the α according to which boundary condition they induce:

$$|\Omega_{n+1}\rangle = \sqrt{\frac{1}{2^{|\Pi_{n+1}|}}} \sum_b \sum_{\alpha : b(\alpha)=b} (-1)^{\sharp\alpha} |\alpha\rangle. \tag{102}$$

The states $|\alpha\rangle$ are orthonormal product states. If O is supported on Π^E_n, then the matrix elements $\langle\beta, O\alpha\rangle$ only depend on the configuration of α and β on Π^E_n. This information still allows us to deduce the boundary conditions $b(\alpha)$ and $b(\beta)$. Moreover, the matrix element vanishes if $b(\alpha) \neq b(\beta)$, hence

$$\begin{aligned}
\omega_{n+1}(O) = \langle \Omega_{n+1}, O\Omega_{n+1}\rangle &= \frac{1}{2^{|\Pi_{n+1}|}} \sum_{\alpha,\beta} (-1)^{\sharp\alpha+\sharp\beta} \langle\beta, O\alpha\rangle \\
&= \frac{1}{2^{|\Pi_{n+1}|}} \sum_b \sum_{\substack{\alpha : b(\alpha)=b \\ \beta : b(\beta)=b}} (-1)^{\sharp\alpha+\sharp\beta} \langle\beta, O\alpha\rangle \\
&= \frac{2}{2^{|\Pi_{n+1}|}} \sum_b \sum_{\alpha',\beta' \in \mathcal{P}^{(b)}_n} (-1)^{\sharp\alpha'+\sharp\beta'} \langle\beta', O\alpha'\rangle \\
&= \frac{1}{2^{6n-1}} \sum_b \frac{1}{2^{|\Pi_n|}} \sum_{\alpha',\beta' \in \mathcal{P}^{(b)}_n} (-1)^{\sharp\alpha'+\sharp\beta'} \langle\beta', O\alpha'\rangle \\
&= \frac{1}{2^{6n-1}} \sum_b \eta^{(b)}_n(O).
\end{aligned}$$

The factor of 2 appearing in the third line is the number of choices of completing a loop soup α' in Π^E_n with boundary condition b to a closed-loop soup α in Π^E_{n+1}. The phase $(-)^{\sharp\alpha'+\sharp\beta'}$ does not depend on which (common) completion is chosen. Indeed, changing the completion changes both $\sharp\alpha'$ and $\sharp\beta'$ by an odd amount if the number of marked edges is a multiple of 4, and both by an even amount otherwise (Lemma A.4). To get the fourth line, we used that $|\Pi_{n+1}| - |\Pi_n| = 6n - 1$. □

Lemma A.4. *Given $\alpha \in \mathcal{P}^{(b)}_n$, denote by $\sharp_1\alpha$ and $\sharp_2\alpha$ the number of loops in the two possible completions. Then $\sharp_1\alpha - \sharp_2\alpha$ is odd if the number of marked points for b is a multiple of 4, and even otherwise.*

Proof. Assume first that α has no closed loops. Let the number of marked points be $2n$. The following construction is illustrated in Fig. 18. Abstract the region Π_n to a disk with the marked points sitting on the boundary. Then the two completions correspond to two sets of n intervals that 'interlace' along the

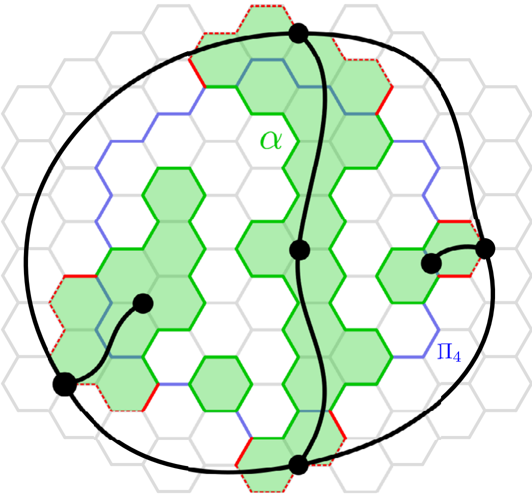

FIGURE 18. Red dotted paths completing α to a closed-loop soup are marked with vertices (black), and so are the closed regions (green) resulting from this completion. The white regions correspond one-to-one to faces of the black graph. Each such white region corresponds to a loop of the alternative completion of α to a closed-loop soup (color figure online)

boundary of the disk. Choose one of them. The loop soup α connects these n intervals into groups. The number of groups g is the number of closed loops in this completion, say $\sharp_1\alpha = g$. Put a vertex on each interval for this completion, and add a vertex in each group. Connect this vertex by edges to the vertices of the intervals in the group. Finally, connect the vertices on the intervals by edges along the boundary of the disk. This gives a connected graph with $V = n + g$ vertices and $E = 2n$ edges. By the Euler formula, this graph has $F = 1 - V + E = 1 + n - g$ internal faces. The number of internal faces corresponds precisely to $\sharp_2\alpha$, and we find

$$\sharp_1\alpha - \sharp_2\alpha = g - (1 + n - g) = 2g - n - 1, \tag{103}$$

which is odd if n is even and vice versa.

Any closed loops of α remain connected components of both completions, so internal loops do not contribute to $\sharp_1\alpha = \sharp_2\alpha$.

□

We further show

Lemma A.5. *For any boundary condition b and any O supported on Π^E_{n-1}, we have $\eta^{(b)}_n(O) = \omega(O)$.*

Proof. From Lemma A.1, it is sufficient to show that $\eta^{(b)}_n(O) = \omega_n(O)$. Note that $\omega_n = \eta^{\emptyset}_n$, where $\emptyset$ stand for the trivial boundary condition.

For any other boundary condition b, let A_b be the product of A_p operators over hexagons between pairs of marked edges of b. Clearly, A_b is supported outside Π^E_{n-1}, so $A^*_b O A_b = O$, and since A_b bijectively maps loop soups satisfying b to closed-loop soups, we find

$$\eta^{(b)}_n(O) = \eta^{(b)}_n(A^*_b O A_b) = \eta^{\emptyset}_n(O) = \omega_n(O) = \omega(O). \tag{104}$$

□

A.2. *Purity of the Limit State.* We will now show that ω is a pure state by making use of the following lemma, which is a special case of Lemma 2.1. of [13].

Lemma A.6 (Lemma 2.1 of [13]). *A state ω on a UHF algebra realised as the inductive limit of a sequence of finite matrix algebras $\{\mathcal{M}_m\}$ is pure if the following holds:*

For each n, there exists $m > n$ such that if ρ is a linear functional on $\mathcal{M}_m$ that satisfies

$$\omega|_{\mathcal{M}_m} \geq \rho \geq 0, \tag{105}$$

then

$$\rho|_{\mathcal{M}_n} = \lambda\omega|_{\mathcal{M}_n} \tag{106}$$

for some $\lambda \in \mathbb{R}$.

In applying this theorem to our setting, we take $\mathcal{M}_n$ to be the algebra supported on Π^E_n.

Fix n and take $m \geq n+1$. Let ρ be a linear functional on $\mathcal{M}_m$ such that Eq. (105) is satisfied. From Proposition A.3 and Lemma A.1, we have

$$\omega|_m = \omega_{m+1}|_m = \frac{1}{2^{6m-1}} \sum_b \eta^{(b)}_m, \tag{107}$$

which is a Schmidt decomposition for $\omega|_m$. The assumption $\omega|_m \geq \rho \geq 0$ implies that ρ is a mixture of pure states in the span of the $\eta^{(b)}_m$'s (Lemma A.7). It then follows from Lemma A.5 and $m > n$ that

$$\rho|_n = \lambda\omega|_n \tag{108}$$

for some $0 \leq \lambda \leq 1$.

We conclude by Lemma A.6 that ω is pure.

We have used the following lemma:

Lemma A.7. *Let ω and ρ be linear functionals on a finite matrix algebra such that $\omega \geq \rho \geq 0$. Suppose*

$$\omega = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \tag{109}$$

is a Schmidt decomposition of ω. Then any Schmidt decomposition

$$\rho = \sum_\beta q_\beta |\phi_\beta\rangle\langle\phi_\beta| \tag{110}$$

satisfies

$$\mathrm{span}\{|\phi_\beta\rangle\}_\beta \subset \mathrm{span}\{|\psi_\alpha\rangle\}_\alpha. \tag{111}$$

i.e. the Schmidt states of ρ span a subspace of the space spanned by the Schmidt states of ω.

Proof. Suppose the conclusion is false, so one of the ϕ_β, say ϕ_1, lies outside of $\mathcal{V} = \mathrm{span}\{\psi_\alpha\}$. Then there is a vector χ orthogonal to $\mathcal{V}$ and such that $c = \langle \phi_1, \chi \rangle \neq 0$.

Consider now the positive operator $P = |\chi\rangle\langle\chi|$. We have $\omega(P) = 0$ and

$$\rho(P) = \sum_\beta q_\beta |\langle \phi_\beta, \chi\rangle|^2 > 0 \tag{112}$$

where all terms are non-negative and at least the term $\beta = 1$ is strictly positive.

It follows that $(\omega - \rho)(P) < 0$, violating the assumption. □

Appendix B. Properties of String Operators

For each vertex v, let

$$A_v := \frac{1}{2}\left(1 + \prod_{e\sim v} \sigma_e^Z\right) \tag{113}$$

where the product runs over the three edges connected to v. For each hexagon p, regarded as a closed loop with counterclockwise orientation, let

$$B_p := \frac{1}{2}\left(1 + W_S[\partial p]\right)\left(\prod_{v\in p} A_v\right). \tag{114}$$

The operators B_p and A_v are orthogonal projections, and they all commute with each other.

Let Π_n be a finite set of hexagons as in Fig. 9. Let Π_n^V be the set of vertices belonging to some hexagon in Π_n. We set

$$H_{\Pi_n} := \sum_{v\in\Pi_n^V} (1 - A_v) + \sum_{p\in\Pi_n} B_p. \tag{115}$$

Let us also introduce terms imposing boundary conditions:

$$H_{\partial\Pi_n} = \sum_{e\in\partial\Pi_n} \frac{1}{2}\left(1 - \sigma_e^Z\right). \tag{116}$$

This is also a sum of orthogonal projections, and they all commute with each other and with the B_p and A_v appearing in H_{Π_n}.

We now consider the commuting projection Hamiltonians

$$H_n := H_{\Pi_n} + H_{\partial\Pi_n}. \tag{117}$$

Let $\widetilde{\Pi}_n$ be the collection of edges that have an endpoint in Π_n^V. Then $\mathcal{H}_n \in \mathcal{A}_{\widetilde{\Pi}_n}$. Moreover, the state ω_n restricts to $\mathcal{A}_{\widetilde{\Pi}_n}$ as a pure state. Let us continue to denote this restriction by ω_n. We have

Lemma B.1. *The state ω_n on $\mathcal{A}_{\tilde{\Pi}_n}$ is the unique ground state of H_n.*

Proof. The state ω_n is defined by the expectation in the vector state

$$\Omega_n = \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\Pi \subset \Pi_n} (-1)^{\sharp(\Pi)} A_\Pi \Omega_0 \tag{118}$$

where Ω_0 has all $\sigma_e^Z = 1$.

The state Ω_n is a superposition of closed string configurations in Π_n. Each such closed string configuration satisfies

$$(1 - A_v) A_\Pi \Omega_0 = 0, \quad \frac{1}{2}(1 - \sigma_e^Z) A_\Pi \Omega_0 = 0 \tag{119}$$

for all $v \in \Pi_n^V$, all $e \in \partial \Pi_n$ and all $\Pi \subset \Pi_n$.

To see that Ω_n is a ground state of H_n it remains to show that it is in the kernel of all B_p for $p \in \Pi_n$. One can check that

$$W_S[\partial p] A_\Pi \Omega_0 = \phi(p, \Pi) A_{p \triangle \Pi} \Omega_0 \tag{120}$$

where $\phi(p, \Pi) = -1$ if $p \triangle \Pi$ has the same parity of connected components as Π, and $\phi(p, \Pi) = 1$ otherwise. i.e.

$$\phi(p, \Pi) = (-1)^{\sharp \Pi + \sharp(p \triangle \Pi) + 1} \tag{121}$$

It follows that $W_S[\partial p] \Omega_n = -\Omega_n$ for any $p \in \Pi_n$, hence $B_p \Omega_n = 0$.

To see that Ω_n is the unique ground state, observe that any ground state must be in the kernel of all the $1 - A_v$ for $v \in \Pi_n^V$ and all the $\frac{1}{2}(1 - \sigma_e^Z)$ for $e \in \partial \Pi_n$. The space of states that are simultaneously in the kernels of all these projections is spanned by the closed string states

$$A_\Pi \Omega_0, \quad \Pi \subset \Pi_n. \tag{122}$$

We must find in this space a state that is in the kernel of all the B_p, equivalently a -1 eigenstate of all the $W_S[\partial p]$ for $p \in \Pi_n$. Consider a general state

$$\Psi = \sum_{\Pi \subset \Pi_n} \psi(\Pi) A_\Pi \Omega_0 \tag{123}$$

where $\psi(\Pi) \in \mathbb{C}$ are arbitrary. Then

$$W_S[\partial p] \Psi = \sum_{\Pi \subset \Pi_n} \psi(\Pi) \phi(p, \Pi) A_{p \triangle \Pi} \Omega_0, \tag{124}$$

so $W_S[\partial p] \Psi = -\Psi$ only if

$$\psi(\Pi)(-1)^{\sharp \Pi} - \psi(p \triangle \Pi)(-1)^{\sharp(p \triangle \Pi)}. \tag{125}$$

If any of the $\psi(\Pi)$ is non-zero (which must be the case, otherwise $\Psi = 0$), then this enforces

$$\psi(\Pi') = (-1)^{\sharp \Pi + \sharp \Pi'} \psi(\Pi') \tag{126}$$

for all $\Pi' \subset \Pi_n$. Indeed, and Π can be related to any Π' by a sequence of symmetric differences with elementary hexagons p. This shows that $\Psi \simeq \Omega_n$, so Ω_n is indeed the unique ground state of H_n on $\mathcal{A}_{\tilde{\Pi}_n}$. $\square$

Lemma B.2. *If P is a closed path entirely contained in Π_n, then $W_a[P]$ commutes with H_n.*

Proof. This is shown for the string operators of any Levin–Wen model using a graphical representation of the string operators in [16]. In our case of the double semion model, we can also show it by brute force. That $W_S[P]$ commutes with the star operators A_v and with the boundary terms in $H_{\partial\Pi_n}$ is obvious. Let us show that $W_S[P]$ commutes with B_p for $p \in \Pi_n$.

To this end, note simply that if Q is the path, possibly consisting of multiple components, made up of edges of P that are also edges or R-legs of p, oriented with the same orientation as P, then

$$W_S[P]W_S[\partial p]W_S[P]^* = W_S[\partial p]U[Q] \tag{127}$$

where the string operators $U[Q]$ are defined in Eq. (55). Since $U[Q](\prod_{v\in p} A_v) = \prod_{v\in p} A_v$, and all A_v's commute with $W_S[P]$ we find

$$\begin{aligned} W_S[P]\left(W_S[\partial p](\prod_{v\in p} A_v)\right)W_S[P]^* &= W_S[\partial p]U[Q]\left(\prod_{v\in p} A_v\right) \\ &= W_S[\partial p]\left(\prod_{v\in p} A_v\right). \end{aligned} \tag{128}$$

The claim for semion string operators $W_S[P]$ follows.

The required result is easy to verify for bound state strings $W_B[P]$, and since $W_{\bar{S}}[P] = W_S[P]W_B[P]$, the claim also holds for the anti-semion string operators. □

Lemma B.3. *If P is a finite closed string, then*

$$\omega \circ w_a[P] = \omega \tag{129}$$

for any $a \in \{1, S, \bar{S}, B\}$.

Proof. By Lemmas B.1 and B.2, we have

$$W_a[P]\Omega_n \sim \Omega_n \tag{130}$$

for n sufficiently large. Hence, $\omega_n \circ w_a[P] = \omega_n$ for n sufficiently large. The double semion state ω is by definition the weak-* limit of the sequence ω_n so

$$\omega \circ w_a[P] = \lim_{n\uparrow\infty} \omega_n \circ w_a[P] = \lim_{n\uparrow\infty} \omega_n = \omega. \tag{131}$$

□

We are now ready to give the

Proposition 3.2. Let O be a strictly local observable. Then we can find a finite closed loop P' such that $w_a[P](O) = w_a[P'](O)$. From Lemma B.3, we then find $(\omega \circ w_a[P])(O) = (\omega \circ w_a[P'])(O) = \omega(O)$. Since the strictly local operators are dense in $\mathcal{A}$, it follows that $\omega \circ w_a[P] = \omega$.

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