



Bernoulli Variables, Classical Exclusion Processes and Free Probability

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In memory of Krzysztof GAWEDZKI (1947–2022) (https://en.wikipedia.org/wiki/Krzysztof_Gawedzki). We imagine that Krzysztof would have appreciated this manuscript, which intertwines problems in physics and mathematics. We hope that it fits with Krzysztof's rigorous, precise exploration of mathematical physics.

Abstract. We present a new description of the known large deviation function of the classical symmetric simple exclusion process by exploiting its connection with the quantum symmetric simple exclusion processes and using tools from free probability. This may seem paradoxical as free probability usually deals with non-commutative probability while the simple exclusion process belongs to the realm of classical probability. On the way, we give a new formula for the free energy—alias the logarithm of the Laplace transform of the probability distribution—of correlated Bernoulli variables in terms of the set of their cumulants with non-coinciding indices. This latter result is obtained either by developing a combinatorial approach for cumulants of products of random variables or by borrowing techniques from Feynman graphs.

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1. Introduction and Summary

The symmetric simple exclusion process (SSEP) [9,10,20] is an iconic model of out-of-equilibrium classical statistical physics [16,25]. It describes particles on a line, hopping to the left and right but with the exclusion rule that two particles can never be at the same place. The SSEP played an important role in the emergence of the so-called macroscopic fluctuation theory [5,6], which is a general, phenomenological framework, suited for dealing with diffusive out-of-equilibrium classical systems. A quantum version of the classical SSEP [2], named Q-SSEP, has recently been defined. Q-SSEP extends the SSEP in the sense that it keeps track of possible quantum mechanical interferences but in such a way that the classical SSEP is recovered when looking at the average behavior of quantum mechanical observables. It might play a role in a possible quantum extension of the classical macroscopic fluctuation theory [4].

Interestingly, free probability techniques play an important role in the study of the Q-SSEP, either in constructing its invariant measure [7] or in deciphering its dynamics [14]. Since the classical SSEP is embedded in the quantum SSEP, free probability also plays a role in understanding the known

characteristics of SSEP, in particular its large deviation rate function. This may sound surprising as free probability has been introduced in the mathematical literature [8, 21, 24, 27] in order to deal with non-commutative probability while SSEP belongs to the realm of classical probability. The purpose of this manuscript is to explain this hidden role.

On the way, we solve the following problem, apparently simple but for which we did not find an answer in the literature and which reveals nice connections with combinatorial structures. Let $b_i, i = 1, \dots, N$, be a collection of N Bernoulli variables, $b_i = 0$ or 1 and let $K_n(b_{i_1}, \dots, b_{i_n})$ be their cumulants. We call non-coincident these cumulants when the indices i_1, \dots, i_n are pairwise distinct (i.e., there are no $p \neq q$ such that $i_p = i_q$). Since $b_i^2 = b_i$ for all i , all other cumulants, and hence the joint distribution, are determined from the non-coincident cumulants. Let $Z[h] := \mathbb{E} [e^{\sum_i h_i b_i}]$, be the Laplace transform of the joint distribution of the b_i 's. To make contact with physics terminology, we shall call it the partition function. It is clearly fully determined by the non-coincident cumulants, since

$$Z[h] = \mathbb{E} \left[\prod_i (1 + b_i e_i) \right], \tag{1}$$

with $e_i := e^{h_i} - 1$.

The question is then: How to compactly write $W[h] := \log Z[h]$, the generating function of the cumulants, including coincident indices, in terms of the non-coincident cumulants?

Of course, the answer to this question is easy when these variables are independent, since then the generating function factorizes, $Z_{\text{free}}[h] = \prod_i [1 + g_i e_i]$, with $g_i := \mathbb{E}[b_i]$ the mean of b_i , and

$$W_{\text{free}}[h] = \sum_i \log [1 + g_i e_i]. \tag{2}$$

It informs on cumulants at coincident points, say at order two $\mathbb{E}[b_i^2]^c = K_2(b_i, b_i) = g_i(1 - g_i)$ or three $\mathbb{E}[b_i^3]^c = K_3(b_i, b_i, b_i) = g_i - 3g_i^2 + 2g_i^3$, and similarly at higher orders. To later make contact with large deviation rate functions, let $S[n]$ be the Legendre transform of $W[h]$, that is: $S_{\text{free}}[n] = \max_{\{h_i\}} [\sum_i h_i n_i - W_{\text{free}}[h]]$, then

$$S_{\text{free}}[n] = \sum_i \left[n_i \log \left(\frac{n_i}{g_i} \right) + (1 - n_i) \log \left(\frac{1 - n_i}{1 - g_i} \right) \right]. \tag{3}$$

A simple formula such as (2) does not hold in the correlated case. Nevertheless, as explained in Sect. 2, $W[h]$ admits a representation as a sum over bipartite graphs whose weights are determined by the non-coincident cumulants.

$$W[h] = \sum_H \frac{\mu(H^\bullet)}{|\text{Aut } H|} \sum_{\mathcal{L} \in \text{Lab}(H)} w(\mathcal{L}), \tag{4}$$

where the sum is over all connected bipartite graphs H with an arbitrary number of black but at most N white vertices, and \mathcal{L} denotes a labeling of

the white vertices by distinct integer indices in $[1, N]$. To such a labeling \mathcal{L} is associated a weight $w(\mathcal{L})$ described below, see equations (35,36).

Things simplify in the large N scaling limit if we assume that the non-coincident cumulants scale in a specific way at large N . Namely, let us assume that

$$K_n(b_{i_1}, \dots, b_{i_n}) = \frac{1}{N^{n-1}} \psi_n \left(\frac{i_1}{N}, \dots, \frac{i_n}{N} \right) \left(1 + O \left(\frac{1}{N} \right) \right), \quad (5)$$

for $\psi_n(x_1, \dots, x_n)$ a collection of continuous functions and $h_i = h(\frac{i}{N})$ for $h(s)$ a continuous function, then only trees contribute to the graph expansion of the cumulant generating functions and $W[h] \sim NF[h]$ at large N . In accordance with physics terminology, we shall call $F[h]$ the free energy (per unit of volume). As explained in Sects. 2 and 3, the latter can be determined by solving an extremization problem :

$$F[h] = \max_{g(\cdot); q(\cdot)} \left[\int_0^1 dx \log(1 + g(x)e(x)) - q(x)g(x) + F_0[q] \right], \quad (6)$$

with $e(x) := e^{h(x)} - 1$ and $F_0[q]$ the generating function of non-coincident cumulants,

$$F_0[q] := \sum_{n \geq 1} \frac{1}{n!} \int_0^1 \psi_n(x_1, \dots, x_n) \prod_{k=1}^n q(x_k) dx_k. \quad (7)$$

The extremization problem (6) has to be solved over all functions g and q on $[0, 1]$, without specified boundary conditions. Comparing with the free formula (2), we observe that $F[h]$ is given by a mean field like formula—the first term $\int dx \log(1 + g(x)e(x))$ —with effective local density $g(x)$ self-consistently determined from the non-coincident cumulants—by coupling it to an external field $q(x)$ whose Boltzmann distribution is fixed by F_0 .

We shall apply this result to give a new presentation of the known large deviation rate function in the classical SSEP. Recall that SSEP is a stochastic model suited for describing transport and density fluctuations in many particle systems out of equilibrium. Its rate function, denoted $I_{\text{ssep}}[n]$, governs the rare large density fluctuations in the sense that the probability that the SSEP density profile $\mathbf{n}(x)$ approaches a given profile $n(x)$ away from the mean, most probable profile is exponentially small :

$$\mathbb{P}[\mathbf{n}(\cdot) \approx n(\cdot)] \asymp_{N \rightarrow \infty} e^{-N I_{\text{ssep}}[n]}, \quad (8)$$

with N the number of sites. A more precise definition and description shall be given in Sect. 4.

The derivation of the new formula we shall give uses three ingredients : (i) the relation between SSEP and Q-SSEP [2]; (ii) the connections between the invariant measure of the quantum SSEP and free probability [7]; and (iii) the solution of the problem stated above.

Combining these first two ingredients leads to a representation of the generating function for the non-coincident cumulants of the density in the

classical SSEP in terms of appropriate free cumulants. Namely, let $F_0^{\text{ssep}}[a]$ be the generating function of SSEP non-coincident cumulants, then

$$F_0^{\text{ssep}}[a] = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} R_n(\mathbb{I}_{[a]}),$$

where R_n are the free cumulants of the function $\mathbb{I}_{[a]}(x) = \int_x^1 dy a(y)$ viewed as a random variable on the interval $[0, 1]$ equipped with the Lebesgue measure as probability measure.

Knowing the generating function of the non-coincident cumulants, we can then use the solution (6) of the problem stated above to write the large deviation rate function as the solution of the following extremization problem :

$$I_{\text{ssep}}[n] = \max_{g(\cdot), q(\cdot)} \left(\int_0^1 dx \left[n(x) \log \left(\frac{n(x)}{g(x)} \right) + (1 - n(x)) \log \left(\frac{1 - n(x)}{1 - g(x)} \right) + q(x)g(x) \right] - F_0^{\text{ssep}}[q] \right). \quad (9)$$

Comparing with the free formula (3), this formula has a mean field like self consistent flavor, as does the formula (6). It also shows similarities with the formula known in the SSEP literature [9, 10, 20] and we check in Sect. 4.5 that they of course coincide. Its derivation is, however, different, as it makes a detour through Q-SSEP and it reveals the hidden ingredients from free probability in the classical SSEP large deviation rate function.

Since the SSEP large deviation rate function has initially been derived using a matrix product ansatz for the SSEP stationary measure [9, 10, 13, 20], one may wonder if there is any connection between matrix product ansatz, or more generally tensor network techniques, and free probability. In view of the impact of tensor techniques in studies of quantum many-body systems, such connection, if it exists, would provide further evidence for the possible universal role of free probability tools in such systems [14, 23].

The rest of the paper is organized as follows. In Sect. 2, we show how to deal with cumulants of Bernoulli variables, using combinatorial techniques, and we derive the variational problem associated with the large N limit. Another approach to these results, using more standard Feynman diagram tools, is presented in Sect. 3. Finally, in Sect. 4, we make the connection with the Q-SSEP.

2. Bernoulli Partition Functions and Combinatorics

The purpose of this section is to give some combinatorial properties of cumulants, which will then be used to study the asymptotics of the free energy of a family of Bernoulli variables.

2.1. Partition Lattices and Möbius Functions

2.1.1. The Lattice of Partitions of a Finite Set. The set-partitions of $\{1, \dots, n\}$ (or, more generally, of a finite set S) form a lattice for the inverse refinement order, such that $\pi \leq \gamma$ if π is finer than γ . We denote by \mathcal{P}_n (or $\mathcal{P}(S)$)

this lattice. It has a maximal element 1_n (the partition with one part) and a minimal element 0_n (the partition with n parts). Every interval $[\pi_1, \pi_2]$ in this lattice is isomorphic, as a partially ordered set, to a product

$$[\pi_1, \pi_2] \sim \prod_p [0_{k_p}, 1_{k_p}] \quad (10)$$

where the terms in the product are indexed by the parts p of π_2 and k_p is the number of parts of π_1 which are subsets of p .

2.1.2. Lattice of Partitions of a Graph. Let G be a finite, simple and loopless¹ graph (all graphs considered below will satisfy these conditions) with set of vertices V and edges E and let \mathcal{P}_G be the set of partitions of V into *connected* parts. Then, $\mathcal{P}_G \subset \mathcal{P}(V)$ with equality if and only if G is a complete graph. We endow this set with the inverse refinement order $<_G$.

For every partition π of V there exists a maximal partition $\pi^* \in \mathcal{P}_G$ such that $\pi^* \leq \pi$. The parts of this partition are the connected components of the parts of π . It follows that the partially ordered set \mathcal{P}_G is a lattice with

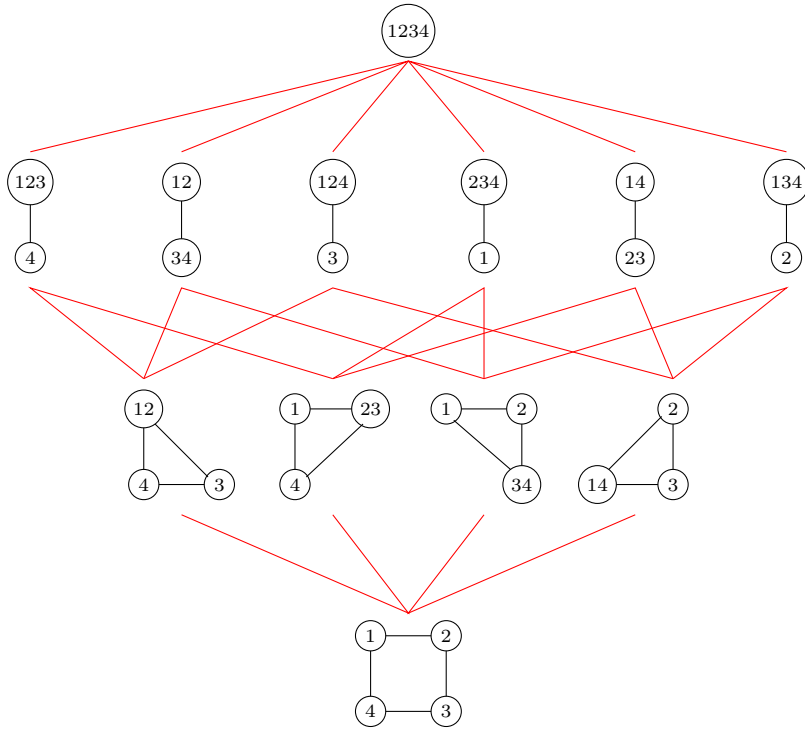
$$\pi_1 \vee_G \pi_2 = \pi_1 \vee \pi_2 \quad \pi_1 \wedge_G \pi_2 = (\pi_1 \wedge \pi_2)^*$$

Again there is a smallest element, 0_G and a maximal element 1_G , whose parts are the connected components of G , moreover every interval $[\pi_1, \pi_2]$ is isomorphic to a lattice of the form $\mathcal{P}_{G'}$ for some graph G' .

Every partition $\pi \in \mathcal{P}_G$ defines a graph G_π whose vertices are the parts of π and two vertices are connected by an edge if and only if the union of the corresponding parts is connected in G . In terms of the graphs G_π , the covering relations for the order on \mathcal{P}_G can be described as the contraction of an edge: $\pi_1 <_G \pi_2$ is a covering relation if and only if G_{π_2} can be obtained from G_{π_1} by contracting some edge (and possibly removing spurious edges to keep the graph simple).

¹There is a small terminology mismatch between communities here. In graph theory loopless means that there is no edge with its two ends at the same vertex. For Feynman graphs in physics, the term loop is used either for what is called a cycle in graph theory or a cycle class in homology, and this is the convention used in Sect. 3 and the Appendix. This should cause no confusion. Feynman graphs are neither simple—they may have multiple edges—nor loopless in general. However, they are for the situations covered in Sect. 3.

For example, here is \mathcal{P}_G when G is a cycle of size 4. Each partition is denoted by its associated graph G_π .



2.1.3. Möbius Functions. Recall that, for a partially ordered set, its *zeta function* is the function

$$\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if not} \end{cases}$$

The *Möbius function* $\mu(x, y)$, defined for $x \leq y$, satisfies, for all $x \leq z$:

$$\sum_{y; x \leq y \leq z} \mu(x, y)\zeta(y, z) = \delta_{xz}$$

The Möbius functions for the lattices \mathcal{P}_n and, more generally, \mathcal{P}_G play an important role in the following. The Möbius function on \mathcal{P}_n is multiplicative namely if $[\pi_1, \pi_2]$ is as in (10) then

$$\mu(\pi_1, \pi_2) = \prod_p \mu(0_{k_p}, 1_{k_p})$$

and

$$\mu(0_k, 1_k) = (-1)^{k-1}(k-1)!$$

In order to compute the Möbius function on \mathcal{P}_G , we will need some facts about chromatic polynomials.

2.1.4. Chromatic Polynomials. A *proper coloring* of a finite graph G is a coloring of its vertices such that, for any edge, the adjacent vertices have different colors. The chromatic polynomial of G , denoted χ_G , is the unique polynomial such that, for any integer $k \geq 1$, the number of proper colorings of G with at most k colors is equal to $\chi_G(k)$. If ω_r denotes the number of proper colorings of G which use exactly r colors, then one has

$$\chi_G(k) = \sum_r \omega_r \binom{k}{r}. \tag{11}$$

Since $\omega_r = 0$ for $r > |V|$ this shows that χ_G is indeed a polynomial.

For example, the complete graph with n vertices has

$$\chi_{K_n}(z) = (z)_n := z(z-1)(z-2)\dots(z-n+1)$$

while, if T is a tree with n vertices, then

$$\chi_T(z) = z(z-1)^{n-1}.$$

We note the following properties of the chromatic polynomial: if G is the union of two disjoint graphs G_1, G_2 , then

$$\chi_G(z) = \chi_{G_1}(z)\chi_{G_2}(z) \tag{12}$$

whereas, if G is the join of G_1, G_2 , namely $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \{v\}$ with no edge joining $V_1 \setminus \{v\}$ to $V_2 \setminus \{v\}$, then

$$\chi_G(z) = \frac{1}{z}\chi_{G_1}(z)\chi_{G_2}(z). \tag{13}$$

The Möbius function of \mathcal{P}_G has been computed by Rota [26], one has

$$\mu(0_G, 1_G) = [z]\chi_G(z), \tag{14}$$

the coefficient of z in the polynomial $\chi_G(z)$, moreover, if $\pi_1 \leq \pi_2$ in \mathcal{P}_G , then $[\pi_1, \pi_2] \sim \mathcal{P}_{G'}$ for some graph G' and

$$\mu(\pi_1, \pi_2) = \mu(0_{G'}, 1_{G'})$$

Note that, by (11), one has

$$[z]\chi_G(z) = \sum_r \frac{(-1)^{r-1}}{r} \omega_r \tag{15}$$

In the following we will use the notation $\mu(G) = \mu(0_G, 1_G)$ when the context is clear.

The proof of (14) is based on inclusion–exclusion. The number of all colorings of G using at most k colors is $k^{|V|}$, moreover any such coloring determines a partition $\pi \in \mathcal{P}_G$ into connected unicolor components, so that the graph G_π is properly colored. It follows that

$$k^{|V|} = \sum_{\pi \in \mathcal{P}_G} \chi_{G_\pi}(k)$$

and formula (14) is obtained by Möbius inversion, see [26] for details.

2.2. Moments and Cumulants

Let \mathcal{A} be a complex algebra with unit and $\varphi : \mathcal{A} \rightarrow \mathbf{C}$ a linear form such that $\varphi(1) = 1$. For most applications below \mathcal{A} will be an algebra of complex random variables defined over some probability space, in particular it will be commutative, but it is not more difficult to consider here the general case of an arbitrary algebra over the complex numbers.

The *cumulants* are a sequence of n -multilinear forms $K_n, n = 1, 2, \dots$ on \mathcal{A} , implicitly defined by

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in \mathcal{P}_n} K_\pi(a_1, \dots, a_n) \tag{16}$$

with

$$K_\pi(a_1, \dots, a_n) = \prod_{p \in \pi} K_{|p|}(a_{i_1}, \dots, a_{i_{|p|}}) \tag{17}$$

the product being over the parts of π with $p = \{i_1, \dots, i_{|p|}\}$ and $i_1 < i_2 < \dots < i_{|p|}$. This formula can be inverted to express the cumulants in terms of the “moments,” i.e., φ evaluated on products. For example,

$$\begin{aligned} \varphi(a_1) &= K_1(a_1) \\ \varphi(a_1 a_2) &= K_2(a_1, a_2) + K_1(a_1)K_1(a_2) \end{aligned}$$

gives

$$K_2(a_1, a_2) = \varphi(a_1 a_2) - \varphi(a_1)\varphi(a_2)$$

while

$$\begin{aligned} \varphi(a_1 a_2 a_3) &= K_3(a_1, a_2, a_3) + K_1(a_1)K_2(a_2, a_3) + K_2(a_1, a_3)K_1(a_2) \\ &\quad + K_2(a_1, a_2)K_1(a_3) + K_1(a_1)K_1(a_2)K_1(a_3) \end{aligned}$$

gives

$$\begin{aligned} K_3(a_1, a_2, a_3) &= \varphi(a_1 a_2 a_3) - \varphi(a_1 a_2)\varphi(a_3) - \varphi(a_1 a_3)\varphi(a_2) \\ &\quad - \varphi(a_1)\varphi(a_2 a_3) + 2\varphi(a_1)\varphi(a_2)\varphi(a_3) \end{aligned}$$

In the general case there is an expression using the Möbius function on \mathcal{P}_n :

$$K_n(a_1, \dots, a_n) = \sum_{\pi \in \mathcal{P}_n} \varphi_\pi(a_1, \dots, a_n)\mu(\pi, 1_n)$$

In the case where \mathcal{A} is commutative the cumulants are symmetric multilinear forms and their generating function is

$$\log \varphi[e^{\sum_{i=1}^N \lambda_i a_i}] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{I: i_1 + \dots + i_N = n} \lambda_1^{i_1} \dots \lambda_N^{i_N} K_n(a_I) \tag{18}$$

where one sums over all sequences $a_I = (a_{j_1}, \dots, a_{j_n})$ with i_k occurrences of a_k . Each such sequence determines a partition of $\{1, \dots, n\}$ into parts corresponding to the value of the indices. One can thus rewrite (18) as

$$\log \varphi[e^{\sum_{i=1}^N \lambda_i a_i}] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\Gamma \in \mathcal{L}P_n} \lambda^\Gamma K_n(a_\Gamma) \tag{19}$$

where the sum is over labeled partitions Γ of $\{1, \dots, n\}$ into at most N parts, where each part γ of Γ has a label $\nu(\gamma)$ in $\{1, 2, \dots, N\}$ (the parts having distinct labels) and $\lambda^\Gamma = \prod_{\gamma \in \Gamma} \lambda_{\nu(\gamma)}^{|\gamma|}$.

2.2.1. Cumulants with Products as Entries. Let $\Gamma := \gamma_1 \cup \dots \cup \gamma_k$ be a partition of $\{1, \dots, n\}$ into intervals, i.e., each γ_l is of the form $\{j_l + 1, j_l + 2, \dots, j_{l+1}\}$ with $0 = j_1 < j_2 < \dots < j_{k+1} = n$.

Let us define

$$K_n^\Gamma(a_1, \dots, a_n) = K_k(A_1, \dots, A_k)$$

where $A_l = a_{j_l+1} a_{j_l+2} \dots a_{j_{l+1}}$, the product of the a_i with indices $i \in \gamma_l$ and, more generally,

$$K_\pi^\Gamma(a_1, \dots, a_n) = \prod_{p \in \pi} K_{|p|}^{\Gamma|_p}(a_{i_1}, \dots, a_{i_{|p|}}) \tag{20}$$

Here, $\Gamma|_p$ is the partition of $p \in \pi$ induced by Γ . Observe that one has also

$$K_\pi^\Gamma(a_1, \dots, a_n) = K_\pi^{\Gamma \wedge \pi}(a_1, \dots, a_n). \tag{21}$$

One has

$$\begin{aligned} K_\pi^{0n}(a_1, \dots, a_n) &= K_\pi(a_1, \dots, a_n) \\ K_\pi^{1n}(a_1, \dots, a_n) &= \varphi_\pi(a_1 \dots a_n) \end{aligned}$$

so that the K_π^Γ , for $0_n \leq \Gamma \leq 1_n$, interpolate between cumulants and moments. The following formula, attributed to Leonov and Shiryaev [19], expresses the K^Γ in terms of ordinary cumulants.

Theorem 1.

$$K_\xi^\Gamma(a_1, \dots, a_n) = \sum_{\pi: \pi \vee \Gamma = \xi} K_\pi(a_1, \dots, a_n), \quad \text{for } \xi \geq \Gamma. \tag{22}$$

In particular

$$K_n^\Gamma(a_1, \dots, a_n) = \sum_{\pi: \pi \vee \Gamma = 1_n} K_\pi(a_1, \dots, a_n). \tag{23}$$

Proof. This follows easily by comparing the two formulas:

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \sum_{\pi} K_\pi(a_1, \dots, a_n) = \sum_{\xi \geq \Gamma} \left(\sum_{\pi: \pi \vee \Gamma = \xi} K_\pi(a_1, \dots, a_n) \right) \\ \varphi(a_1 \dots a_n) &= \varphi(A_1 \dots A_n) = \sum_{\xi \geq \Gamma} K_\xi^\Gamma(a_1, \dots, a_n) \end{aligned}$$

□

When $\Gamma = 0_n$ the formula (23) is trivially true and when $\Gamma = 1_n$ it is the moments-cumulants formula (16). Also if $\xi \not\leq \Gamma$ one can use (21) to get

$$K_\xi^\Gamma(a_1, \dots, a_n) = K_\xi^{\Gamma \wedge \xi}(a_1, \dots, a_n) = \sum_{\pi: \pi \vee (\Gamma \wedge \xi) = \xi} K_\pi(a_1, \dots, a_n) \quad (24)$$

In the general case, the formula (23) can be inverted. For this, we introduce a graph $G_{\pi, \Gamma}$, whose vertices are the parts of π , and there is an edge between p and q if there exists a part γ of Γ such that $p \cap \gamma \neq \emptyset$ and $q \cap \gamma \neq \emptyset$. One has then

Theorem 2.

$$K_n(a_1, \dots, a_n) = \sum_{\pi: \pi \vee \Gamma = 1_n} K_\pi^\Gamma(a_1, \dots, a_n) \mu(G_{\pi, \Gamma}) \quad (25)$$

Proof. This formula can be verified by plugging it into the right-hand side of (22) and checking that it reduces to $K_n^\Gamma(a_1, \dots, a_n) = K_n^\Gamma(a_1, \dots, a_n)$ after using the properties of the Möbius function.

Introducing the unknown function $\mu_\Gamma(\pi)$ such that

$$K_n(a_1, \dots, a_n) = \sum_{\pi: \pi \vee \Gamma = 1_n} K_\pi^\Gamma(a_1, \dots, a_n) \mu_\Gamma(\pi)$$

and using (24) one has

$$\begin{aligned} K_n(a_1, \dots, a_n) &= \sum_{\pi: \pi \vee \Gamma = 1_n} K_\pi^\Gamma(a_1, \dots, a_n) \mu_\Gamma(\pi) \\ &= \sum_{\pi: \pi \vee \Gamma = 1_n} \sum_{\xi: (\pi \wedge \Gamma) \vee \xi = \pi} K_\xi(a_1, \dots, a_n) \mu_\Gamma(\pi) \\ &= \sum_{\xi: \xi \vee \Gamma = 1_n} K_\xi(a_1, \dots, a_n) \left(\sum_{\pi: (\pi \wedge \Gamma) \vee \xi = \pi} \mu_\Gamma(\pi) \right) \end{aligned}$$

so that μ_Γ has to satisfy

$$\sum_{\pi: (\pi \wedge \Gamma) \vee \xi = \pi} \mu_\Gamma(\pi) = 1, \quad \text{if } \xi = 1_n, \quad = 0 \quad \text{if not.} \quad (26)$$

Define an order relation on the set of partitions π such that $\pi \vee \Gamma = 1_n$ by requiring

$$\pi_1 \leq_\Gamma \pi_2 \quad \text{if and only if} \quad \pi_1 \leq \pi_2 \quad \text{and} \quad \pi_1 \vee (\pi_2 \wedge \Gamma) = \pi_2$$

Indeed, if $\pi_1 \leq_\Gamma \pi_2$ and $\pi_2 \leq_\Gamma \pi_3$, then $\pi_1 \vee (\pi_3 \wedge \Gamma) \geq \pi_1 \vee (\pi_2 \wedge \Gamma) = \pi_2$; therefore, $\pi_1 \vee (\pi_3 \wedge \Gamma) \geq \pi_2$ and $\pi_1 \vee (\pi_3 \wedge \Gamma) \geq \pi_3 \wedge \Gamma$ so that, finally $\pi_1 \vee (\pi_3 \wedge \Gamma) \geq \pi_2 \vee (\pi_3 \wedge \Gamma) = \pi_3$. This relation is transitive as claimed. Taking $\mu_\Gamma(\pi) = \mu([\pi, 1_n])$ where μ is the Möbius function for this order yields (26). It is easy to check that the interval $[\pi, 1_n]$ for this order is isomorphic à $\mathcal{P}_{G_{\pi, \Gamma}}$. One has thus $\mu_\Gamma(\pi) = [z]\chi_{G_{\pi, \Gamma}} = \mu(G_{\pi, \Gamma})$. □

In the commutative case, one can define the cumulants K^Γ for any partition Γ , not just interval partitions, since the cumulants are symmetric and (22), (23), (25) still hold.

2.3. Non-crossing Partitions and Cumulants

In this section, we define non-crossing cumulants which will be used later in the asymptotic analysis of the Q-SSEP. Many information about the combinatorics of the non-crossing cumulants may be found in the book by Nica and Speicher [22].

A partition of $\{1, \dots, n\}$ has a crossing if there exists two parts of the partition and $i < j < k < l$ such that i, k belong to the first part and j, l to the second part. Partitions without crossing are called *non-crossing*. The set of non-crossing partitions of $\{1, \dots, n\}$, denoted $NC(n)$, is a lattice under the inverse refinement order, and each interval $[\pi_1, \pi_2]$ is isomorphic, as a partially ordered set, to a product $\prod_i NC(k_i)$ for some integers k_i . The Möbius function is again multiplicative and one has

$$\mu_{NC(n)}(0_n, 1_n) = (-1)^{n-1} \text{Cat}_{n-1}$$

where $\text{Cat}_n = \frac{1}{n+1} \binom{2n}{n}$ is a Catalan number.

Non-crossing cumulants R_n are defined similarly as the cumulants K_n using an implicit formula:

$$\varphi(a_1 \dots a_n) = \sum_{\pi \in NC(n)} R_\pi(a_1, \dots, a_n) \tag{27}$$

which can be inverted as

$$R_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \dots, a_n) \mu_{NC(n)}(\pi, 1_n)$$

Every partition $\pi \in \mathcal{P}_n$ has a least non-crossing majorant $\hat{\pi}$. Using this one can write

$$\begin{aligned} \varphi(a_1 \dots a_n) &= \sum_{\pi \in \mathcal{P}_n} K_\pi(a_1, \dots, a_n) = \sum_{\xi \in NC(n)} \left(\sum_{\hat{\pi}=\xi} K_\pi(a_1, \dots, a_n) \right) \\ \varphi(a_1 \dots a_n) &= \sum_{\xi \in NC(n)} R_\xi(a_1, \dots, a_n) \end{aligned}$$

from which one can easily deduce that

$$R_\xi(a_1, \dots, a_n) = \sum_{\pi: \hat{\pi}=\xi} K_\pi(a_1, \dots, a_n).$$

In particular, the relation

$$R_n(a_1, \dots, a_n) = \sum_{\pi: \hat{\pi}=1_n} K_\pi(a_1, \dots, a_n) \tag{28}$$

expresses non-crossing cumulants in terms of cumulants. The formula can be reversed using again the Möbius function of a certain lattice \mathcal{P}_G . For this, define the *crossing graph* G_π^c of a partition π as the graph whose vertices are the parts of π and two parts of π are connected if they contain a crossing. Using this graph one has, by Möbius inversion,

Proposition 3.

$$K_n(a_1, \dots, a_n) = \sum_{\pi: \hat{\pi}=\xi} R_\pi(a_1, \dots, a_n)\mu(G_\pi^c). \tag{29}$$

An equivalent formula was first derived in [15] by different means, see also [1].

There is also, for non-crossing cumulants, an analog of the formula (23), due to Krawczyk and Speicher [17].

2.4. Cumulants of Bernoulli Variables

2.4.1. Non-coincident Cumulants. Let $b_i; i = 1, 2, \dots, N$ be a sequence of (commuting) Bernoulli random variables, taking values in $\{0, 1\}$. They satisfy $b_i^r = b_i$ for all $r \geq 1$; therefore, all the information about the joint distribution of the b_i is contained in the $2^N - 1$ “non-coincident moments,” i.e., the quantities $E[b_{i_1}b_{i_2} \dots b_{i_k}]$ where $1 \leq i_1 < i_2 < \dots < i_k \leq N$ (here E denotes the expectation), or in the $2^N - 1$ “non-coincident cumulants” $K_k(b_{i_1}, b_{i_2}, \dots, b_{i_k})$. It is therefore of interest to express an arbitrary cumulant $K_n(b_{j_1}, \dots, b_{j_n})$, for a sequence of indices $1 \leq j_k \leq N$, in terms of these non-coincident cumulants.

Let Γ be the partition of $\{1, \dots, n\}$ such that k and l are in the same part of Γ if and only if $i_k = i_l$. Using the fact that $(b_i)^r = b_i$ for any $r \geq 1$ we see that the Γ -cumulants defined by (20) and the formula (25) express any cumulant as a polynomial in the non-coincident cumulants.

$$K_n(b_{i_1}, \dots, b_{i_n}) = \sum_{\pi: \pi \vee \Gamma = 1_n} K_\pi^\Gamma(b_{i_1}, \dots, b_{i_n})\mu(G_{\pi, \Gamma}) \tag{30}$$

2.4.2. Free Energy. Recall the generating function of the cumulants (19)

$$\log E[e^{\sum_{i=1}^N h_i b_i}] = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\Gamma} h^\Gamma K_n(b_\Gamma) \tag{31}$$

One can use (30) in each term of this sum to obtain a sum over pairs (π, Γ) :

$$\log E[e^{\sum_{i=1}^N h_i b_i}] = \sum_n \frac{1}{n!} \sum_{\Gamma} h^\Gamma \left(\sum_{\pi: \pi \vee \Gamma = 1_n} K_\pi^\Gamma(b_\Gamma)\mu(G_{\pi, \Gamma}) \right). \tag{32}$$

Let us introduce a labeled bipartite graph $\Delta_{\pi, \Gamma}$ with the parts of Γ as set of white vertices and the parts of π as set of black vertices. The white vertices, corresponding to the parts of Γ , are labeled by $1, 2, \dots$ (the indices of the Bernoulli variables) each index appearing at most once. There is an edge between a part of Γ and a part of π if they have a non-empty intersection. The condition $\pi \vee \Gamma = 1_n$ ensures that this graph is connected. Observe that one can associate to every edge of the graph a subset of $\{1, \dots, n\}$ by taking the intersection of the part of π corresponding to its black extremity and the part of Γ corresponding to its white extremity. These sets form a partition of $\{1, \dots, n\}$, indexed by the edges of the graph, and one can reconstruct the partitions π and Γ from this edge-indexed partition by taking the union over

edges adjacent to a white vertex to get the parts of Γ or to a black vertex, to get the parts of π .

The graph $G_{\pi,\Gamma}$ is obtained from the bipartite graph $\Delta_{\pi,\Gamma}$ by keeping the black vertices and putting an edge between two such vertices if they have at least one white neighbor in common.

The factor associated with the pair π, Γ can then be written as

$$\mu(G_{\pi,\Gamma}) \frac{h^\Gamma}{n!} \prod_{\bullet} K(b_\bullet) \tag{33}$$

where the product is over the black vertices \bullet of $\Delta_{\pi,\Gamma}$ and, for each such vertex, the factor

$$K(b_\bullet) = K_k(b_{u_1}, b_{u_2}, \dots, b_{u_k})$$

the indices u_1, u_2, \dots, u_k being those of the white neighbors of \bullet in $\Delta_{\pi,\Gamma}$.

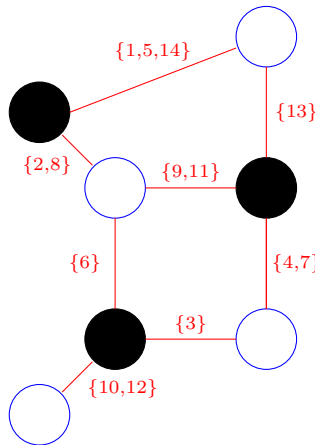
As an example, here is the graph $\Delta_{\pi,\Gamma}$ associated with the partitions

$$\pi = \{1, 2, 5, 8, 14\} \cup \{3, 6, 10, 12\} \cup \{4, 7, 9, 11, 13\}$$

and

$$\Gamma = \{1, 5, 13, 14\} \cup \{2, 6, 8, 9, 11\} \cup \{3, 4, 7\} \cup \{10, 12\}$$

where we show, near each edge, the associated set:



Let H be a bipartite connected graph (with at least two vertices), then the pairs (π, Γ) such that H is the underlying unlabeled graph of $\Delta_{\pi,\Gamma}$ can be obtained by

1. Labeling the white vertices of H with distinct labels in $\{1, 2, \dots, N\}$ (we call $Lab(H)$ the set of such labelings).
2. Choosing a partition of $\{1, \dots, n\}$ indexed by the edges of H .

Denote H^\bullet the graph whose vertices are the black vertices of H and with edges between vertices sharing a white neighbor in H . Using the fact that we are

summing over partitions indexed by edges of H , which are counted by multinomial coefficients, we see that the sum of all contributions (33) corresponding to H is

$$\frac{\mu(H^\bullet)}{|\text{Aut } H|} \sum_{\mathcal{L} \in \text{Lab}(H)} w(\mathcal{L}) \tag{34}$$

where \mathcal{L} runs over all labelings of the white vertices of H by distinct indices and

$$w(\mathcal{L}) = \prod_{\bullet} K(b_\bullet) \prod_{\text{edges of } H} e_i \tag{35}$$

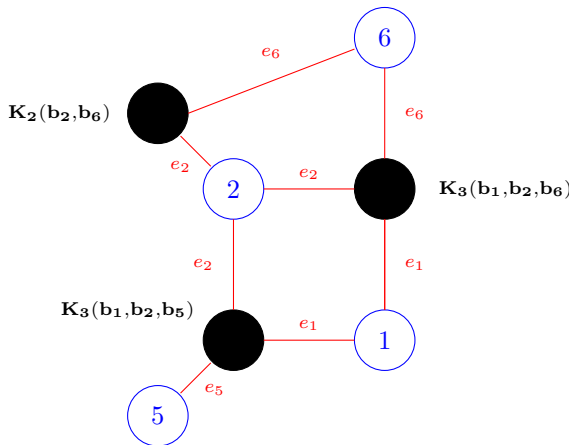
(for any edge e of H one denotes $e_i := e^{h_i} - 1$, where i is the index of the white vertex adjacent to the edge e). The term $\frac{1}{|\text{Aut } H|}$, as usual, is here to avoid overcountings due to symmetries. The automorphism group is that of H considered as a bipartite graph, i.e., automorphisms should send black vertices to black vertices and white vertices to white vertices. We can thus rewrite (32) as

Proposition 4.

$$W[h] = \log E[e^{\sum_{i=1}^N h_i b_i}] = \sum_H \frac{\mu(H^\bullet)}{|\text{Aut } H|} \sum_{\mathcal{L} \in \text{Lab}(H)} w(\mathcal{L}), \tag{36}$$

where the sum is over connected bipartite graphs and the weight $w(\mathcal{L})$ as in Eq. (35).

Here, the graph H corresponding to the pair π, Γ above, with a labeling by 1, 2, 5, 6.



The graph H^\bullet is a complete graph with three vertices so that $\mu(H^\bullet) = 2$ and there are no nontrivial automorphisms moreover the weight of the labeling is

$$w(\mathcal{L}) = e_1^2 e_2^3 e_5 e_6^2 K_3(b_1, b_2, b_6) K_2(b_2, b_6) K_3(b_1, b_2, b_5)$$

2.4.3. Another Proof of Formula (36). We sketch another derivation of (36), which does not rely on the theory of cumulants of products. One has

$$e^{\sum_{i=1}^N h_i b_i} = \prod_i (1 + e_i b_i) = 1 + \sum_{I \subset \{1, \dots, N\}; I \neq \emptyset} e_I b_I$$

where e_I is the product $\prod_{i \in I} e_i$. Using the moment-cumulant formula, we get

$$E[e^{\sum_{i=1}^N h_i b_i}] = 1 + \sum_{I \neq \emptyset} e_I \sum_{\pi \in \mathcal{P}(I)} K_\pi(b_I) \tag{37}$$

and taking the logarithm

$$W[h] = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} \left(\sum_{I \neq \emptyset} e_I \sum_{\pi \in \mathcal{P}(I)} K_\pi(b_I) \right)^r$$

One has

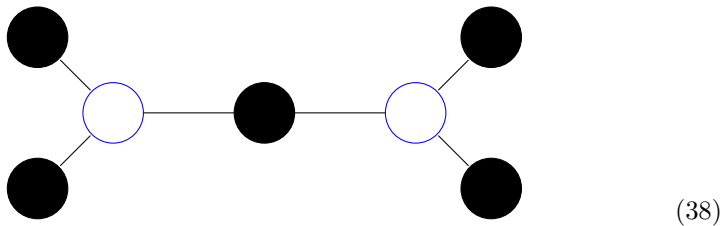
$$\left(\sum_{I \neq \emptyset} e_I \sum_{\pi \in \mathcal{P}(I)} K_\pi(b_I) \right)^r = \sum_{I_1, \dots, I_r, \pi_1, \dots, \pi_r} \prod_{k=1}^r K_{\pi_k}(b_{I_k}) e_{I_k}$$

where we sum over I_1, \dots, I_r , non-empty subsets of $[1, N]$ and π_k partition of I_k .

We now introduce a bipartite graph with white vertices labeled by the $i \in \cup_k I_k$ and black vertices corresponding to the parts of the partitions π_k . There is an edge between a white and a black vertex if the index of the white vertex is in the part corresponding to the black vertex. This bipartite graph induces a graph structure on the black vertices: two vertices share an edge if they have a common white neighbor. For each such graph, we have to sum over all proper colorings of the black vertices using exactly r colors. Using relation (11), we identify the combinatorial term associated with a graph to the z coefficient in the chromatic polynomial of the black graph. By (12) this coefficient is zero if the graph is not connected so that the sum can be taken over connected graphs. We leave details to the reader and give an example: it is easy to see that the monomial

$$e_1^3 e_2^3 K_1(b_1)^2 K_1(b_2)^2 K_2(b_1, b_2)$$

is obtained from only one graph H , the one depicted below.



The graph H^\bullet is the joining of two complete graphs therefore $\mu(H^\bullet) = 4$ while $|\text{Aut}(H)| = 8$; moreover, there are two labelings of the white vertices by

1, 2; therefore, the sum of coefficients of this graph is 1, which should be the coefficient of the monomial.

On the other hand, one can obtain the coefficient of this monomial by expanding the expression $\frac{1}{3}w^3 - \frac{1}{4}w^4 + \frac{1}{5}w^5$ (other powers of w do not contribute) where

$$w = e_1 K_1(b_1) + e_2 K_1(b_2) + e_1 e_2 K_2(b_1, b_2) + e_1 e_2 K_1(b_1) K_1(b_2)$$

Using multinomial coefficients we find

$$\frac{1}{3} \frac{3!}{2!1!} - \frac{1}{4} \frac{4!}{1!1!1!} + \frac{1}{5} \frac{5!}{1!2!2!} = 1,$$

so that the weight of this graph is effectively 1.

2.5. Asymptotic Behavior and Legendre Transform

2.5.1. Reduction to Trees. We suppose now that, as $N \rightarrow \infty$, the non-coincident cumulants have a specific asymptotic behavior: there exists some compact space Σ , some functions $\rho_N : [1, N] \rightarrow \Sigma$ and continuous functions ψ_n on Σ^n such that as $N \rightarrow \infty$

$$K_n(b_{i_1}, \dots, b_{i_n}) \sim N^{1-n} \psi_n(\rho_N(i_1), \dots, \rho_N(i_n)) \tag{39}$$

Moreover the measures $\frac{1}{N} \sum_i \delta_{\rho_N(i)}$ converge to some diffuse measure ds on Σ . We are mainly interested in the case where $\rho_N(i) = i/N$ and $\Sigma = [0, 1]$ but the analysis works in greater generality and can be adapted to deal with other topologies, e.g., lattices in higher dimension.

We will study the asymptotic behavior of the free energy and for this we assume that the h_i converge also to some bounded function $h(s)$ on Σ . We can then estimate the contribution of a graph H in (36). Indeed, the number of labelings is

$$(N)^{\#\text{white vertices}} \sim N^{\#\text{white vertices}}$$

while, by (39), the contribution of the product of cumulants is of the order

$$O(N^{\#\text{black vertices} - \#\text{edges}})$$

It follows that the contribution coming from trees, for which

$$\#\text{white vertices} + \#\text{black vertices} - \#\text{edges} = 1$$

is of the order $O(N)$, while the contribution of other graphs is of lower order in N . For H a tree, the combinatorial factor is $\mu(H^\bullet) = \prod_i (-1)^{k_i-1} (k_i - 1)!$ the product being over white vertices and k_i being the number of black neighbors of the white vertex indexed by i . This follows from the computation of the chromatic polynomial for the complete graph and the formula for the joining of two graphs given by (13). In this case, we can thus rewrite

$$\frac{\mu(H^\bullet)}{|\text{Aut } H|} w(\mathcal{L}) = \frac{1}{|\text{Aut}(H)|} \prod_{\bullet} K_{\bullet}(b_{\bullet}) \prod_{\circ} (-1)^{k_{\circ}-1} (k_{\circ} - 1)! \prod_e e_i$$

where the product \prod_{\circ} is over white vertices and k_{\circ} denotes the number of neighbors of the white vertex \circ .

2.5.2. Gradient of the Free Energy. Let us now compute $e_i \frac{\partial W}{\partial e_i}$ in the large N limit. Since, in the weight $w(\mathcal{L})$, there is a factor e_i for each edge adjacent to a white vertex labeled i , one sees that this derivative is given by a sum over pairs T, e of a tree T and an edge e of T

$$e_i \frac{\partial W}{\partial e_i} \sim \sum_T \frac{\mu(T^\bullet)}{|\text{Aut } T|} \sum_{e \text{ edge of } T} \sum_{\mathcal{L}; e_\circ \sim i} w(\mathcal{L}) \tag{40}$$

where we sum over all labelings \mathcal{L} such that the white vertex of e is labeled by i . Cutting the edge e splits the tree into two rooted trees, T_\bullet and T_\circ , one of them containing the black vertex of e as a root and the other the white vertex, with labelings $\mathcal{L}_\bullet, \mathcal{L}_\circ$. The automorphism subgroup fixing e is the product of the automorphism groups of these two rooted trees (that is, the automorphisms fixing the roots): $\text{Aut}_e(T) \sim \text{Aut}(T_\bullet) \times \text{Aut}(T_\circ)$, while the term

$$\prod_{\bullet} K_\bullet(b_\bullet) \prod_{\circ} (-1)^{k_\circ - 1} (k_\circ - 1)! e_i \prod_{e' \neq e} e_j$$

splits into a product over the two trees. One can sum over all edges in the orbit of e by $\text{Aut}(T)$ (whose size is $|\text{Aut}(T)|/|\text{Aut}_e(T)|$) and get a sum

$$e_i \frac{\partial W}{\partial e_i} \sim \sum_T \sum_{e \in E(T)/\text{Aut}(T)} \frac{1}{|\text{Aut}(T_\bullet)| |\text{Aut}(T_\circ)|} \sum_{\mathcal{L}; e_\circ \sim i} z_i^\bullet(\mathcal{L}_\bullet) z_i^\circ(\mathcal{L}_\circ) \tag{41}$$

In this sum the weights z_i^\bullet, z_i° are computed on labeled bipartite trees with a black (resp. white) root and one has

$$z_i^\bullet(\mathcal{L}_\bullet) = e_i K_{root}(b_i, \dots) \prod_{\bullet \neq root} K_\bullet(b_\bullet) \prod_{\circ} (-1)^{k_\circ - 1} (k_\circ - 1)! \prod_{e'} e_j$$

for a tree with a black root, where $K_{root}(b_i, \dots)$ is the non-coincident cumulant evaluated on b_i and the neighbors of the black root. Similarly,

$$z_i^\circ(\mathcal{L}_\circ) = (-1)^{k_{root}} k_{root}! \prod_{\bullet} K_\bullet(b_\bullet) \prod_{\circ \neq root} (-1)^{k_\circ - 1} (k_\circ - 1)! \prod_{e'} e_j$$

for a tree with a white root. Let us introduce the functions

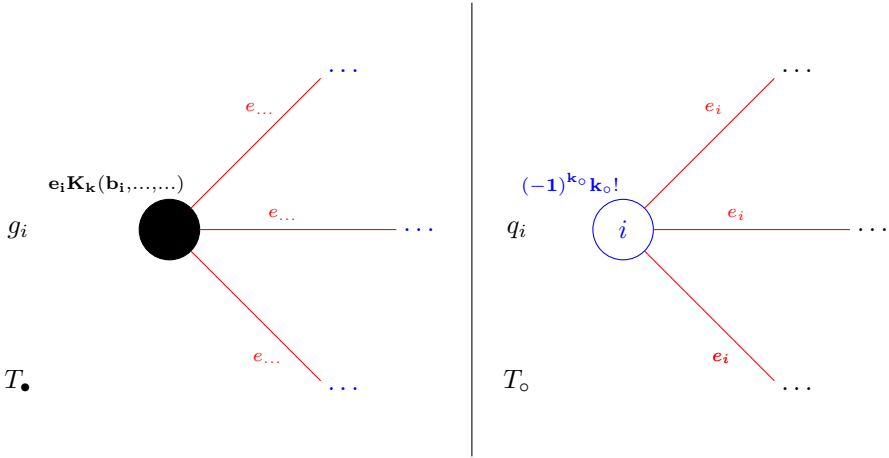
$$q_i = \sum_{T_\circ} \frac{1}{|\text{Aut}(T_\circ)|} \sum_{\mathcal{L}_\circ} z_i^\circ(\mathcal{L}_\circ) \tag{42}$$

$$g_i = \sum_{T_\bullet} \frac{1}{|\text{Aut}(T_\bullet)|} \sum_{\mathcal{L}_\bullet} z_i^\bullet(\mathcal{L}_\bullet) \tag{43}$$

If we compare the expression on the rhs of (41) with the product $g_i q_i$, we see that they coincide up to possibly some repetitions in the labelings in the expansion of qg (since we consider the product of labelings of T_\bullet and T_\circ); however, the number of terms with repetition in the labelings is of smaller order in N ; therefore, as $N \rightarrow \infty$, one has

$$e_i \frac{\partial W}{\partial e_i} \sim g_i q_i.$$

One can depict the trees and the weights involved in the definition of the functions g and q as follows:



2.5.3. A Variational Principle. Reasoning as in (2.5.2) by cutting the tree in either (42) or (43) at its root to form a forest, we find the following relations in the continuous limit between $q(s), g(s), e(s) := e^{h(s)} - 1$:

$$\begin{aligned}
 e(s) \frac{\partial W}{\partial e(s)} &= g(s)q(s) \\
 q(s) &= \frac{e(s)}{1+e(s)g(s)} \\
 g(s) &= \frac{\delta}{\delta q(s)} F_0(q)
 \end{aligned}$$

where $F_0(q)$ is the large N limit of the generating functional of the cumulants:

$$F_0(q) = \sum_n \frac{1}{n!} \int_{\Sigma^n} q(s_1)q(s_2) \dots q(s_n) \psi_n(s_1, \dots, s_n) ds_1 \dots ds_n$$

It follows that

Proposition 5. *In the scaling limit, the free energy is obtained by solving the following variational problem*

$$\lim_{N \rightarrow \infty} \frac{1}{N} W = \max_{g,q} \left[\int [\log(1 + e(s)g(s)) - q(s)g(s)] ds + F_0(q) \right] \quad (44)$$

We shall use this variational formula in the case of SSEP in Sect. 4.

3. Bernoulli Partition Functions and Feynman Graphs

In this section, we employ standard field theory techniques (mainly the semi-classical expansion and Feynman graphs) to encode the combinatorics of Bernoulli cumulants.

3.1. Integral Representation of $\log Z$

We start with another description of $\log Z$ in terms of a graphical expansion. The starting point is a somehow tautological representation of Z as a formal Gaussian integral (see subsection A.2 and the following for some background if needed).

The basic observation is that, J being an arbitrary index set, $(\bar{\lambda}_i)_{i \in J}$ and $(\lambda_I)_{I \overset{\circ}{\subset} J}$ being formal variables (the notation $I \overset{\circ}{\subset} J$ means that I is a finite, non-empty subset of J):

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi} \exp(-z_i \bar{z}_i) (1 + \bar{\lambda}_i z_i) \right) \exp \left(\sum_{I \overset{\circ}{\subset} J} \lambda_I \bar{z}_I \right) = 1 + \sum_{I \overset{\circ}{\subset} J} \bar{\lambda}_I \left(\sum_{\pi \in \mathcal{P}(I)} \lambda_\pi \right),$$

where $\bar{\lambda}_I := \prod_{i \in I} \bar{\lambda}_i$ (and analogously $\bar{z}_I := \prod_{i \in I} \bar{z}_i$), $\mathcal{P}(I)$ is the set of partitions of I and, for $I \neq \emptyset$ and $\pi : I = \sqcup_\alpha I_\alpha \in \mathcal{P}(I)$, $\lambda_\pi := \prod_\alpha \lambda_{I_\alpha}$. This formula is checked by expanding

$$\prod_{i \in J} (1 + \bar{\lambda}_i z_i) = 1 + \sum_{I \overset{\circ}{\subset} J} \bar{\lambda}_I z_I.$$

Concentrating on a given $\bar{\lambda}_I$, formal integration amounts to selecting, in the expansion of $\exp(\sum_{I' \overset{\circ}{\subset} J} \lambda_{I'} \bar{z}_{I'})$, precisely the terms involving the monomial \bar{z}_I , which by inspection come with an overall factor $\sum_{\pi \in \mathcal{P}(I)} \lambda_\pi$.

Though this is a formal integral, if the index set J is finite the result is a polynomial in $(\bar{\lambda}_i)_{i \in J}$ and $(\lambda_I)_{I \overset{\circ}{\subset} J}$, and can be evaluated for “numerical” arguments. As established in the previous section, if b_i are Bernoulli variables then

$$Z = \mathbb{E} \left(\prod_{i \in J} (1 + e_i b_i) \right) = 1 + \sum_{I \overset{\circ}{\subset} J} e_i \sum_{\pi \in \mathcal{P}(I)} K_\pi(b_I),$$

with notations as above. Thus, we may write

$$Z = \int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi} \exp(-z_i \bar{z}_i) (1 + e_i z_i) \right) \exp \left(\sum_{I \overset{\circ}{\subset} J} K(b_I) \bar{z}_I \right).$$

In the sequel we do not distinguish between the formal variables $(\bar{\lambda}_i)_{i \in J}$, $(\lambda_I)_{I \overset{\circ}{\subset} J}$ and their embodied counterparts $(e_i)_{i \in J}$, $(K(b_I))_{I \overset{\circ}{\subset} J}$, which we shall use in the formulæ.

We can turn the crank of Feynman graphs and rules as recalled in Appendix B and subsection B.1. Writing

$$\prod_{i \in J} (1 + e_i z_i) = \exp \sum_{i \in J} \sum_{k \geq 1} \frac{(-1)^{k-1} (k-1)! e_i^k z_i^k}{k!}$$

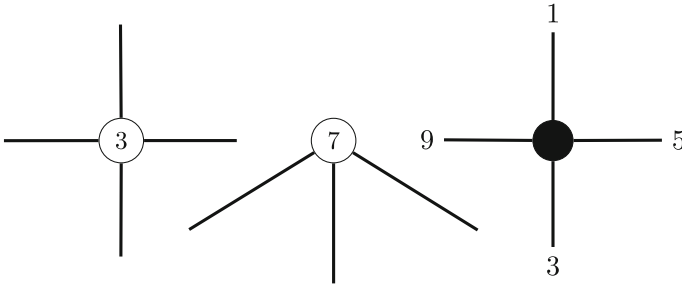
we infer that

$$\log Z = \sum_G w(G)$$

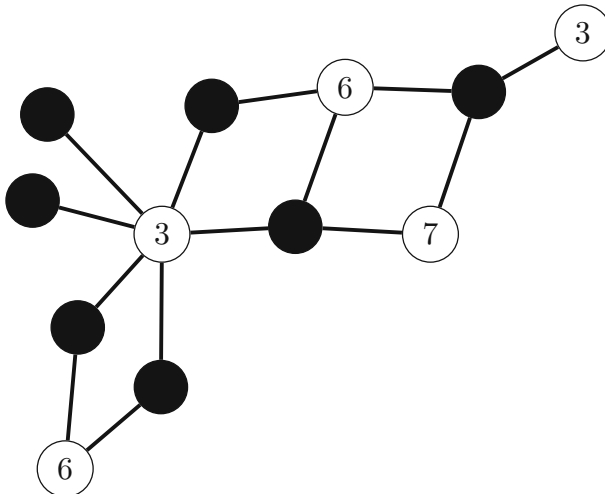
where the sum is over connected bicolored graphs with white and black vertices whose edges carry a type $i \in J$, the type of the edges at a white vertex being all the same (constraint \circ) and at a black vertex all different (constraint \bullet). Each white vertex with k edges of type $i \in J$ contributes a factor $(-1)^{k-1}(k-1)!e_i^k$ to $w(G)$. Each black vertex with edges whose types build a subset $I \subset J$ contributes a factor K_I to $w(G)$. There is an additional factor $1/|\text{Aut } G|$ in $w(G)$. The two constraints (\circ) and (\bullet) allow to “transfer” the types of edges to the white vertices and to consider connected bicolored graphs with white and black vertices, the white vertices carrying a tag $i \in J$ and the tags of white vertices connected to a black vertex being different.

We give some examples, assuming that J is a set of integers:

– Examples: A white vertex of order 4 associated to site $i = 3$, with weight $(-1)^{4-1}(4-1)!e_3^4 = -6e_3^4$, a white vertex of order 3 associated to site 7, with weight $(-1)^{3-1}(3-1)!e_7^3 = 2e_7^3$ and a black vertex connected to sites 1, 3, 9, 5 with weight $K_4(b_1, b_3, b_9, b_5)$:



– Example: A diagram



with weight (reading more or less from left to right, in that simple case symmetries are “local” on the graph)

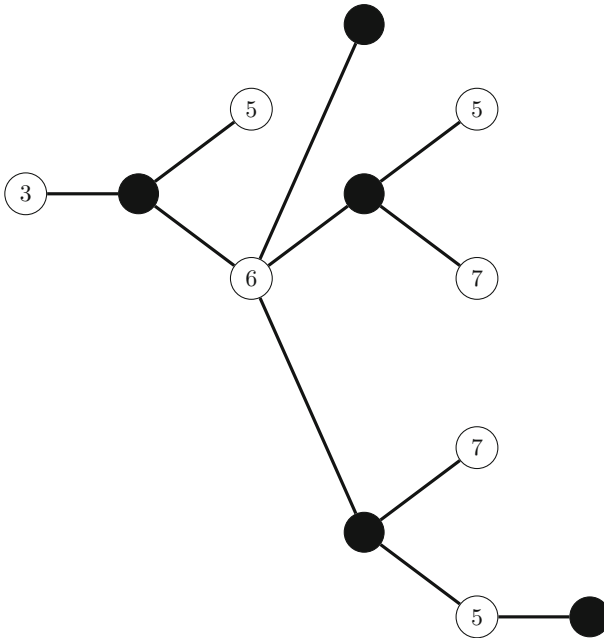
$$\frac{1}{2!} K_3^2 (-1)^{2-1} (2-1)! e_6^2 \frac{1}{2!} K_{3,6}^2 (-1)^{6-1} (6-1)! e_3^6 K_{3,6} K_{3,6,7} (-1)^{3-1} (3-1)! e_6^3 (-1)^{2-1} (2-1)! e_7^2 K_{3,6,7} (-1)^{1-1} (1-1)! e_3,$$

where K_3 , $K_{3,6}$, $K_{3,6,7}$ are a shorthand notation for the cumulants $K_1(b_3)$, $K_2(b_3, b_6)$ and $K_3(b_3, b_6, b_7)$, respectively, a convention already used above.

The weight reduces to $-60 e_3^7 e_6^5 e_7^2 K_3^2 K_{3,6}^3 K_{3,6,7}^2$.

From now on, we could reproduce with very little changes the discussion leading to the continuum limit, to which only tree diagrams contribute.

– Here is an example of a tree:



the computation of whose weight is left to the reader. We pause for a moment to compare with the other (call it *chromatic*) graphical description of $\log Z$ given in Sect. 2.4.

For this, we need to rewrite some previous formulæ, in particular Eqs. (34) and (35). In the chromatic description, instead of summing over unlabeled graphs H and then over labelings by (distinct) elements of $J := \{1, \dots, N\}$, we can as well sum over graphs G whose white vertices are labeled by $\{1, \dots, N\}$. If G is such a graph, define $\mathfrak{w}(G) := \prod_{\bullet} K(b_{\bullet}) \prod_{\text{edges of } G} e_i$, so that, if G is obtained from an unlabeled graph H via the labeling \mathcal{L} , $\mathfrak{w}(G) = w(\mathcal{L})$. A moment thinking shows that

$$\frac{1}{|\text{Aut } H|} \sum_{\mathcal{L} \in \text{Lab}(H)} w(\mathcal{L}) = \sum_G \frac{1}{|\text{Aut } G|} \mathfrak{w}(G)$$

where the sum on the right-hand side is over the distinct graphs G that can be obtained by labeling the white vertices of H , and $\text{Aut } G$ is the group of automorphisms of G respecting the labeling, which may be smaller than $\text{Aut } H$ because several labelings of H may induce the same G , as for instance in the example (38). As all white labels of G are distinct, the automorphism group of G is in fact very easy to describe: saying that two black vertices are equivalent if they are connected to the same set of white vertices, $\text{Aut } G$ is the group of permutations of equivalent black vertices. The definition of the operation \bullet for G can be copied on that of H^\bullet and clearly $G^\bullet = H^\bullet$. Thus, the complete contribution of G to the free energy $\log Z$ in the chromatic description is $\frac{\mu(G^\bullet)}{|\text{Aut } G|} \mathfrak{w}(G)$.

The graphs G we have used for the Feynman graph description have a tag, an element of $J = \{1, \dots, N\}$, assigned to each white vertex, but this is not a labeling of the white vertices in general because several white vertices with the same tag are allowed.² Consequently, their automorphism group is more complicated to describe in general because it may permute white vertices as well. But the definition of the monomial $\mathfrak{w}(G)$ carries over without changes for Feynman graphs. On top of that, a white vertex of order k contributes a multiplicative factor $(-1)^{k-1}(k-1)!$. Defining G° as the collection of white vertices of G with their pending edges (i.e., one just removes the black vertices from the picture) and $\eta(G^\circ) := \prod_{\text{white vertices of } G} (-1)^{k-1}(k-1)!$, we can put things together and write the complete contribution of G to the free energy $\log Z$ in the Feynman graph description as $\frac{\eta(G^\circ)}{|\text{Aut } G|} \mathfrak{w}(G)$.

Working as above with (connected) graphs whose white vertices are tagged, let us denote by \mathcal{C}_C (resp. \mathcal{C}_F) the class of graphs involved in the chromatic (resp. Feynman graph) expansion of $\log Z$. We have just seen that $\mathcal{C}_C \subset \mathcal{C}_F$: the chromatic graphical description which is tailored for the problem at hand and is more economical because there are less graphs to consider. We have

$$\log Z = \sum_{G \in \mathcal{C}_C} \frac{\mu(G^\bullet)}{|\text{Aut } G|} \mathfrak{w}(G) = \sum_{G \in \mathcal{C}_F} \frac{\eta(G^\circ)}{|\text{Aut } G|} \mathfrak{w}(G).$$

We may refine this identity using the obvious observation that for any finite collection (possibly with repetition) of non-empty finite subsets of J , say $(I_a)_{a \in A}$ there is a single graph G in \mathcal{C}_C with black vertices indexed by the I_a , i.e., “black” weight $\prod_{a \in A} K_{I_a}$. In particular, a graph $G \in \mathcal{C}_C$ can be reconstructed from the sole knowledge of $\mathfrak{w}(G)$, leading to the identity

$$\frac{\mu(G^\bullet)}{|\text{Aut } G|} = \sum_{G' \in \mathcal{C}_F, \mathfrak{w}(G') = \mathfrak{w}(G)} \frac{\eta((G')^\circ)}{|\text{Aut } G'|} \quad \text{for } G \in \mathcal{C}_C.$$

This result is perhaps more suggestive if one introduces a partial ordering on \mathcal{C}_F : for $G, G' \in \mathcal{C}_F$ say that $G' \succcurlyeq G$, or that G' covers G if G is obtained from G' by identifying some white vertices carrying the same tag. This is clearly a partial ordering on \mathcal{C}_F . The maximal elements are the trees, and the minimal elements are the elements of \mathcal{C}_C . If $G' \succ G$, G has less white

²However, the tags of white vertices adjacent to a given black vertex are different.

vertices than G' and G has more loops than G' . The graphs G and G' have the same number of black vertices and the same number of edges, and in fact the equality $\mathfrak{w}(G) = \mathfrak{w}(G')$ holds. In particular, given G there are finitely many $G' \succ G$ and the previous identity rewrites

$$\frac{\mu(G^\bullet)}{|\text{Aut } G|} = \sum_{G' \in \mathcal{C}_F, G' \succ G} \frac{\eta((G')^\circ)}{|\text{Aut } G'|} \quad \text{for } G \in \mathcal{C}_C. \tag{45}$$

The above argument gives a rigorous but indirect proof of this identity. A direct proof for trees is easy: if G in \mathcal{C}_C is a tree, there is only G itself in the sum on the right-hand side (G is minimal and maximal for \succ),³ and the $(-1)^{k-1}(k-1)!$ s coming from complete graphs chromatic factors in $\mu(G^\bullet)$ match precisely with Feynman graph contributions for white vertices in $\eta((G)^\circ)$. Note, however, that the ‘‘Feynman trees’’ allow for several white vertices with the same tag. They cover graphs with loops in the chromatic expansion. In both the chromatic and the Feynman graph expansion, loops are suppressed with respect to trees when the number of sites, $|J|$, grows without bounds. This is not in contradiction with the semiclassical expansion: for a fixed number of white vertices, the factor suppressing trees with multiple vertices carrying the same tag is just due to their rarity compared to trees with all white vertices carrying a different tag, and this matches precisely with the scaling in the covering formula. The partial order \succ on \mathcal{C}_F suggests that a recursive approach might help to give a direct proof of (45) in general, but we have not tried to follow this path.

A final remark: the computation of $|\text{Aut } G|$ in the class \mathcal{C}_F is *NP*-hard, just as is the computation of $\mu(G^\bullet)$ in the class \mathcal{C}_C , whereas the computation of $\eta(G^\circ)$ in \mathcal{C}_F or $|\text{Aut } G|$ in \mathcal{C}_C is trivial.

3.2. Thermodynamic Limits

One small thing that speaks in favor for the redundant description of $\log Z$ by Feynman graphs is that it generalizes plainly to related counting problems.

Suppose for instance that J is finite, that e_i does not depend on i and K_I depends only on the size $|I|$ of $I \subset J$.⁴ We fix a family $(t_k)_{k \geq 1}$ of formal variables. We set $e_i := e$ and $K_I := t_k$ if $|I| = k$. For $I \overset{\circ}{\subset} J$ the number of partitions of I made of m_1 parts of size 1, m_2 parts of size 2, and so on with $\sum_{k \geq 1} km_k = |I|$ is $\frac{(\sum_{k \geq 1} km_k)!}{\prod_{k \geq 1} m_k!(k!)^{m_k}}$, leading to

$$Z = \sum_{\underline{m}} \frac{|J|!}{(|J| - \sum_{k \geq 1} km_k)!} \prod_{k \geq 1} \frac{1}{m_k!} \left(\frac{e^k t_k}{k!} \right)^{m_k}$$

³Thus, though the number of terms on the right-hand side of (45) can be arbitrarily large for a general $G \in \mathcal{C}_C$, the overhead of using the Feynman graph description disappears in the thermodynamic/continuous limit.

⁴Without bothering if this can happen for actual Bernoulli variables expectations, in fact it does at least for the trivial case of independent identically distributed Bernoulli when e_i does not depend on i because K_I is the $|I|$ th power of a single variable expectation.

where the sum is over sequences of integers $\underline{m} := (m_1, m_2, \dots)$. The point is that the combinatorics is precisely recovered in a formal Gaussian integral as

$$Z = \int \frac{d\bar{z} \wedge dz}{2i\pi\hbar} \exp\left(-\frac{z\bar{z}}{\hbar}\right) (1 + ez)^{|J|} \exp \sum_{1 \leq k \leq |J|} \frac{t_k}{k!} \left(\frac{\bar{z}}{\hbar}\right)^k.$$

Notice that \hbar , whether a formal variable or a numerical value, plays a purely spectator role in this formula. The expansion of Z in terms of Feynman graphs would follow straightforwardly.

We use rescaled variables $\Phi_k =: (|J|)^{k-1} t_k$ for $k \geq 1$. Choosing $\hbar := (|J|)^{-1}$ and precisising the variables involved in Z we are led to

$$Z_{|J|}(e, \Phi_\bullet) = \int \frac{d\bar{z} \wedge dz}{2i\pi(|J|)^{-1}} \exp \frac{1}{(|J|)^{-1}} \left(-z\bar{z} + \log(1 + ez) + \sum_{k \geq 1} \frac{\Phi_k}{k!} \bar{z}^k \right).$$

Thus, $(|J|)^{-1}$ plays the role that \hbar plays in the general discussion of A.3 and B and letting $|J|$ grow without bounds with $\Phi_\bullet := (\Phi_k)_{k \geq 1}$ fixed we obtain that in the thermodynamic limit

$$\lim_{|J| \rightarrow \infty} \frac{\log Z_{|J|}(e, \Phi_\bullet)}{|J|} = F^*(e, \Phi_\bullet)$$

where $F^*(e, \Phi_\bullet) = -g^* h^* + \log(1 + eg^*) + \sum_{k \geq 1} \Phi_k \frac{h^{*k}}{k!}$ with (g^*, h^*) solving the equations $h = \frac{e}{1+eg}$ and $g = \sum_{k \geq 1} \Phi_k \frac{h^{k-1}}{(k-1)!}$. The formal power series g (resp. h) is what we denoted by U (resp. \bar{V}) in the general discussion of subsection A.3 and subsection B.1.

As a slight generalization of this extreme case, suppose that J is finite and $J = \cup_{a \in A} J_a$ is a partition of J indexed by some set A . Impose that e_i is the same for all i s in a given J_a , and write it as e_a . Fix a family $(t_k)_{k \geq 1}$ where each t_k is a symmetric function on A^k and for $I \overset{\circ}{\subset} J$ set $K_I = t_k(a_1, \dots, a_k)$ if $I = \{i_1, \dots, i_k\}$ has size k and $i_l \in J_{a_l}$ for $l = 1, \dots, k$. The extreme case is recovered when A is a singleton. The counting of partitions in the extreme case generalizes straightforwardly and leads to the integral representation

$$Z = \int \left(\prod_{a \in A} \frac{d\bar{z}_a \wedge dz_a}{2i\pi\hbar} \exp(-z_a \bar{z}_a / \hbar) (1 + e_a z_a)^{|J_a|} \right) \exp \bar{\mathcal{L}}_{|J|}(\bar{z}_\bullet)$$

where

$$\bar{\mathcal{L}}_N(\bar{z}_\bullet) := \sum_{1 \leq k \leq N} \frac{1}{\hbar^k k!} \sum_{a_1, \dots, a_k \in A} t_k(a_1, \dots, a_k) \bar{z}_{a_1} \cdots \bar{z}_{a_k}.$$

Again \hbar is a spectator role in this representation. We use rescaled variables $\Phi_k =: (|J|)^{k-1} t_k$ for $k \geq 1$. Choosing $\hbar := (|J|)^{-1}$ and precisising the variables involved in Z we are led to

$$Z_{|J|}(e_\bullet, \Phi_\bullet) = \int \prod_{a \in A} \frac{d\bar{z}_a \wedge dz_a}{2i\pi(|J|)^{-1}} \exp \frac{1}{(|J|)^{-1}}$$

$$\times \left(\sum_{a \in A} (-z_a \bar{z}_a + p_a \log(1 + e_a z_a)) + \bar{L}_{|J|}(\bar{z}_\bullet) \right),$$

where $p_a := \frac{|J_a|}{|J|}$ for $a \in A$ and

$$\bar{L}_N(\bar{z}_\bullet) := \sum_{1 \leq k \leq N} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} \Phi_k(a_1, \dots, a_k) \bar{z}_{a_1} \cdots \bar{z}_{a_k}.$$

Thus, there is a semiclassical expansion in powers of $(|J|)^{-1}$, with A, Φ_\bullet and $(p_a)_{a \in A}$ fixed. The first contribution, proportional to $|J|$ is given by the saddle point $F^*(e_\bullet, \Phi_\bullet)$ where $F^*(e_\bullet, \Phi_\bullet) = -\sum_{a \in A} g_a^* h_a^* + \sum_{a \in A} p_a \log(1 + e_a g_a^*) + \bar{L}_\infty(h_\bullet^*)$ with $(g_a^*, h_a^*)_{a \in A}$ solving the equations $h_a = \frac{e_a}{1 + e_a g_a}$ and $g_a = \frac{\partial \bar{L}}{\partial \bar{z}_a}(h_\bullet)$, $a \in A$. Thus, letting $|J|$ grow without bounds with A, Φ_\bullet and $(p_a)_{a \in A}$ fixed (this might require taking $|J|$ along some subsequence) we obtain

$$\lim_{|J| \rightarrow \infty} \frac{\log Z_{|J|}(e_\bullet, \Phi_\bullet)}{|J|} = F^*(e_\bullet, \Phi_\bullet)$$

The full semiclassical expansion is valid only for $(p_a)_{a \in A}$ fixed, but this equation for the dominant contribution holds if the $(p_a)_{a \in A}$ are allowed to depend on $|J|$ with corrections $o((|J|)^{-1})$ and $|J| \rightarrow \infty$ without restrictions.

This formula is a special case of Proposition 5 as we explain now. Take $(f_k)_{k \geq 1}$ a sequence of symmetric integrable functions, $f_k : [0, 1]^k \rightarrow \mathbb{R}$ and consider the functional

$$F(e, f_\bullet, g, q) := - \int_{[0,1]} g(x) q(x) dx + \int_{[0,1]} \log(1 + e(x)g(x)) dx + \sum_{k \geq 1} \int_{0 < x_1 < \dots < x_k} f_k(x_1, \dots, x_k) q(x_1) dx_1 \cdots q(x_k) dx_k,$$

where e, g, q are plain functions. Note that the multiple integral could also be written as $\frac{1}{k!} \int_{[0,1]^k} f_k(x_1, \dots, x_k) q(x_1) dx_1 \cdots q(x_k) dx_k$, because the diagonals (where several x 's coincide) do not contribute. The general formulæ come via the extremization of F with respect to the functions g and q . One way to discretize this problem is to partition $[0, 1]$ with measurable subsets $(M_a)_{a \in A}$ with size $\int_{M_a} dx = p_a > 0$ for $a \in A$ and take e, g, q as simple functions, explicitly

$$e(x) = \sum_a e_a \mathbf{1}_{x \in M_a} \quad g(x) = \sum_a g_a \mathbf{1}_{x \in M_a} \quad q(x) = \sum_a q_a p_a^{-1} \mathbf{1}_{x \in M_a}.$$

Taking Φ_\bullet as the average of f_\bullet over rectangles, explicitly

$$\Phi_k(a_1, \dots, a_k) := \frac{1}{p_{a_1} \cdots p_{a_k}} \int_{M_{a_1} \times \dots \times M_{a_k}} f_k(x_1, \dots, x_k) dx_1 \cdots dx_k,$$

(the diagonals where several a_i s may coincide do count in the discrete object Φ_\bullet) we find that the discretized version of $F(e, f_\bullet, g, q)$ is

$$- \sum_{a \in A} g_a q_a + \sum_{a \in A} p_a \log(1 + e_a g_a) + \sum_{k \geq 1} \frac{1}{k!} \sum_{a_1, \dots, a_k \in A} \Phi_k(a_1, \dots, a_k) q_{a_1} \cdots q_{a_k},$$

which coincides as announced with the thermodynamic limit functional above, to be extremized with respect to $(g_a, q_a)_{a \in A}$. If one feels uneasy with formal integrals in infinite dimensions and a semiclassical expansion with respect to the formal parameter \hbar , one might feel more comfortable with—still formal but—finite-dimensional integrals and a semiclassical expansion with respect to the actual number J of Bernoulli random variables. The general formulæ would then follow by a density argument via the approximation of the $K(b_I)$ s, $I \overset{\circ}{\subset} J$ by step functions.

4. Classical SSEP and Free Probability

4.1. The Classical SSEP

The aim of this section is to recall the definition of the classical SSEP and its large deviation function. Results are taken from the SSEP literature [9, 10, 20] where further details may be found.

The classical SSEP is a time continuous Markov chain describing particles moving along a finite 1D lattice, with sites indexed by $i = 1, \dots, N$ (the sites i and $i + 1$ are adjacent). Let τ_i be the occupation number of the site i : $\tau_i = 0$ (resp. $\tau_i = 1$) if the site i is unoccupied (resp. occupied). Each configuration is specified by the data of these occupancies $\{\tau_i; i = 1, \dots, N\}$. The particles are allowed to jump on their nearby positions, to their left or right with equal probability rate, if the target position is unoccupied. The allowed local moves are therefore $[01] \rightarrow [10]$ or $[10] \rightarrow [01]$, while the local configurations $[00]$ and $[11]$ are frozen. Particles are injected and extracted at the two ends of the interval to drive the system out of equilibrium. The SSEP Markov matrix is defined accordingly to take these moves into account in a natural way. We denote by \mathbb{E}_{ssep} the SSEP invariant measure (which is known to be unique).

One is interested in the continuum scaling limit $N \rightarrow \infty$, $x = i/N$ fixed, $0 < x < 1$. The occupation configurations $\{\tau_i\}$ then become continuous density profiles $\mathbf{n}(x)$ on the interval $[0, 1]$. In this scaling limit, the densities at the two ends of the interval are fixed by the injection–extraction processes: $\mathbf{n}(0) = n_a$ and $\mathbf{n}(1) = n_b$ with n_a and n_b specified by the injection–extraction rates at the corresponding boundary. We shall use the convention $n_a = 0$, $n_b = 1$ (without loss of generality).

In the scaling limit, the mean density profile interpolates linearly between the two boundary densities: $\lim_{N \rightarrow \infty} \mathbb{E}_{\text{ssep}}[\tau_{i=[xN]}] = x$ (with the convention $n_a = 0$, $n_b = 1$). Fluctuations of the density profiles satisfy a large deviation principle [11, 12]. Namely,

$$\mathbb{P}[\tau_{i=[xN]} \approx n(x)] \underset{N \rightarrow \infty}{\asymp} e^{-N I_{\text{ssep}}[n]} \tag{46}$$

with I_{ssep} the so-called large deviation rate function.

Let F_{ssep} be the generating function of the density cumulants in the scaling limit. It is such that

$$\mathbb{E}_{\text{ssep}}[e^{\sum_i \tau_i h_i}] \underset{N \rightarrow \infty}{\asymp} e^{N F_{\text{ssep}}[h]}, \tag{47}$$

for $h_i = h(x = i/N)$ with $h(x)$ a smooth function over the interval $[0, 1]$. If the occupation configuration $\{\tau_i\}$ approaches the density profiles $\mathbf{n}(x)$, then $\sum_i \tau_i h_i$ approaches $N \int dx h(x) \mathbf{n}(x)$, so that F_{ssep} can alternatively be defined by $\mathbb{E}_{\text{ssep}}[e^{N \int dx h(x) \mathbf{n}(x)}] \underset{\varepsilon \rightarrow 0}{\asymp} e^{N F_{\text{ssep}}[h]}$. Assuming Eq. (47) to be true implies that the p th-order density cumulants scale like N^{1-p} in the scaling limit.

As is well known, the two functions $I_{\text{ssep}}[n]$ and $F_{\text{ssep}}[h]$ are related by Legendre transform :

$$F_{\text{ssep}}[h] = \max_{n(\cdot)} \left[\int_0^1 dx h(x) n(x) - I_{\text{ssep}}[n] \right]. \quad (48)$$

The large deviation function $F_{\text{ssep}}[h]$ has been given in the SSEP literature [9–12, 20] as the solution of an extremization problem (with the convention $n_a = 0, n_b = 1$) :

$$F_{\text{ssep}}[h] = \max_{g(\cdot)} F[h; g], \quad F[h; g] := \int_0^1 dx [\log(1 + g(x)e(x)) - \log(g'(x))], \quad (49)$$

with $e(x) = e^{h(x)} - 1$, as above, and $g(x)$ solution of the nonlinear differential equation,

$$(1 + g(x)e(x)) g''(x) = g'(x)^2 e(x), \quad (50)$$

with boundary conditions $g(0) = 0$ and $g(1) = 1$ (for $n_a = 0, n_b = 1$). Equation(50) is the Euler–Lagrange equation for $F[h; g]$ to be extremal with respect to variations of g .

Expanding $F_{\text{ssep}}[h]$ in power of h yields the first few density cumulants (up to sub-leading terms in $1/N$) :

$$\begin{aligned} \mathbb{E}_{\text{ssep}}[\tau_{i=[xN]} \tau_{i=[yN]}]^c &= -N^{-1} x(1 - y), \\ \mathbb{E}_{\text{ssep}}[\tau_{i=[xN]} \tau_{i=[yN]} \tau_{i=[zN]}]^c &= N^{-2} x(1 - 2y)(1 - z), \end{aligned}$$

for $0 < x < y < z < 1$. More generally, the n th-order SSEP cumulants scale as N^{1-n} in the scaling limit, see Refs. [9, 10, 20]. We let $\psi_n^{\text{ssep}}(x_1, \dots, x_n)$ be the scaled cumulants, at non-coincident points, defined as

$$\psi_n^{\text{ssep}}(x_1, \dots, x_n) := \lim_{N \rightarrow \infty} N^{n-1} \mathbb{E}_{\text{ssep}} [\tau_{[x_1 N]} \cdots \tau_{[x_n N]}]^c, \quad (51)$$

for $x_k \in [0, 1]$, all distincts. The limit is known to exist and to be smooth (actually piecewise polynomial) at non-coincident points.

4.2. The Quantum to Classical SSEP Correspondence

The aim of this section is to recall the definition of the quantum SSEP (Q-SSEP) as well as its relation with the classical SSEP. Results explained below are taken from Refs. [2, 3] where further details may be found.

The quantum SSEP is a model of stochastic quantum dynamics describing fermions hopping along a 1D chain, with sites indexed by $i = 1, \dots, N$ (the sites i and $i + 1$ are adjacent). For an open chain in contact with external reservoirs at their boundaries, the quantum SSEP dynamics results from the

interplay between unitary, but stochastic, bulk flows with dissipative, but deterministic, boundary couplings. The bulk flows induce unitary evolutions of the system density matrix ρ_t onto $e^{-idH_t} \rho_t e^{idH_t}$ with Hamiltonian increments

$$dH_t = \sqrt{J} \sum_{j=1}^N (c_{j+1}^\dagger c_j dW_t^j + c_j^\dagger c_{j+1} d\bar{W}_t^j), \tag{52}$$

for a chain of length L , where c_j and c_j^\dagger are canonical fermionic operators, one pair for each site of the chain, with $\{c_j, c_k^\dagger\} = \delta_{j;k}$, and W_t^j and \bar{W}_t^j are pairs of complex conjugated Brownian motions, one pair for each edge along the chain, with quadratic variations $dW_t^j d\bar{W}_t^k = \delta^{j;k} dt$. The contacts with the external leads are modeled by Lindblad terms [18]. The resulting equation of motion for the system density matrix ρ_t reads

$$d\rho_t = -i[dH_t, \rho_t] - \frac{1}{2}[dH_t, [dH_t, \rho_t]] + \mathcal{L}_{\text{bdry}}(\rho_t)dt, \tag{53}$$

with dH_t as above and $\mathcal{L}_{\text{bdry}}$ the boundary Lindbladian. The two first terms result from expanding the unitary increment $\rho_t \rightarrow e^{-idH_t} \rho_t e^{idH_t}$ to second order as indicated by Itô calculus. The third term codes for the dissipative boundary dynamics representing injection–extraction at the two the boundaries. We do not need here the precise expression for $\mathcal{L}_{\text{bdry}}$ but the latter can be found in the literature [2, 4].

The classical SSEP is embedded in Q-SSEP because the average Q-SSEP dynamics on density matrix diagonal in the particle number basis reduces to the classical SSEP. At each site along the chain, the full $|\bullet\rangle$ and empty $|\emptyset\rangle$ states, with, respectively, one and zero fermion, form a basis of states and diagonalize the particle number operators $\hat{n}_i := c_i^\dagger c_i$ (with eigenvalue 1 or 0). The states $|\mathbf{n}\rangle$ diagonalizing all the particle numbers along the chain are thus indexed by the classical configurations $\mathbf{n} = (\tau_1, \dots, \tau_N)$, with $\tau_i = 0, 1$, the particle number at site i . A density matrix diagonal in this particle number basis specifies a probability measure on classical configurations since it can be written as $\rho_{\text{diag}} = \sum_{\mathbf{n}} Q_{\mathbf{n}} \Pi_{\mathbf{n}}$, with $\Pi_{\mathbf{n}} := |\mathbf{n}\rangle\langle\mathbf{n}|$ the projector on the classical configuration \mathbf{n} and $Q_{\mathbf{n}}$ a probability measure on $\mathbf{n} : \sum_{\mathbf{n}} Q_{\mathbf{n}} = 1, Q_{\mathbf{n}} \geq 0$.

By the Markov property of the Brownian motions, the average dynamics deduced from Eq. (53) defines a semi-group on the average density matrix $\bar{\rho}_t := \mathbb{E}[\rho_t]$, generated by a Lindbladian $\mathcal{L}_{\text{ssep}}$:

$$\partial_t \bar{\rho}_t = \mathcal{L}_{\text{ssep}}(\bar{\rho}_t). \tag{54}$$

The latter is obtained by averaging the Q-SSEP stochastic equation of motion (53). It preserves diagonal density matrices and thus defines a flow—a Markov chain—on probability measures on classical configurations. Locally, $\mathcal{L}_{\text{ssep}}$ acts as follows :

$$\begin{aligned} \mathcal{L}_{\text{ssep}}(|\emptyset\emptyset\rangle\langle\emptyset\emptyset|) &= 0, \\ \mathcal{L}_{\text{ssep}}(|\emptyset\bullet\rangle\langle\emptyset\bullet|) &= J(-|\emptyset\bullet\rangle\langle\emptyset\bullet| + |\bullet\emptyset\rangle\langle\bullet\emptyset|), \\ \mathcal{L}_{\text{ssep}}(|\bullet\emptyset\rangle\langle\bullet\emptyset|) &= J(+|\emptyset\bullet\rangle\langle\emptyset\bullet| - |\bullet\emptyset\rangle\langle\bullet\emptyset|), \end{aligned}$$

$$\mathcal{L}_{\text{ssep}}(|\bullet\bullet\rangle\langle\bullet\bullet|) = 0 .$$

This coincides with the Markov matrix of the classical SSEP. This coincidence also holds for the boundary processes. Thus, the average Q-SSEP dynamics, when reduced to density matrices which are diagonal in the particle number basis, is that of the classical SSEP, as claimed.

As a consequence, the generating function of the steady fluctuations of the classical SSEP occupancies $\tau_j = 0, 1$ can be expressed as a quantum expectation value w.r.t. the steady averaged Q-SSEP density matrix :

$$\mathbb{E}_{\text{ssep}} \left[e^{\sum_j h_j \tau_j} \right] = \mathbb{E}_{\infty} \left[\text{Tr}(\rho e^{\sum_i h_i \hat{n}_i}) \right] = \text{Tr} \left(\bar{\rho}_{\infty} e^{\sum_i h_i \hat{n}_i} \right) , \quad (55)$$

with $\hat{n}_i := c_i^\dagger c_i$ the quantum number operators, $\bar{\rho}_{\infty} := \mathbb{E}_{\infty}[\rho]$ the mean Q-SSEP state, averaged w.r.t. the Q-SSEP steady measure denoted \mathbb{E}_{∞} . In particular, the multi-point correlation functions of the occupation numbers in the classical SSEP coincide with the quantum expectation values of the number operators w.r.t. to steady averaged Q-SSEP density matrix.

To complete this correspondence we need to express the quantum expectation values of the number operators in terms of known data relative to the Q-SSEP steady measure. The latter is constructed by looking at the expectation values of the fermion two-point functions $G_{ij} := \text{Tr}(\rho c_j^\dagger c_i)$. It has been shown [2] that the leading cumulants of this matrix are those for which the matrix indices are organized along a loop, namely of the form $\mathbb{E}_{\infty}[G_{i_1 i_n} \cdots G_{i_3 i_2} G_{i_2 i_1}]^c$. These cumulants scale as N^{1-n} in the scaling limit. We let $\psi_n^\#(x_1, \dots, x_n)$ be the scaled cumulants at non-coincident points:

$$\psi_n^\#(x_1, \dots, x_n) := \lim_{N \rightarrow \infty} N^{n-1} \mathbb{E}_{\infty} \left[G_{[x_1 N][x_n N]} \cdots G_{[x_3 N][x_2 N]} G_{[x_2 N][x_1 N]} \right]^c , \quad (56)$$

for $x_k \in [0, 1]$ all distincts. The limit is known to exist and to be smooth at non-coincident points. Equations characterizing the $\psi_n^\#$'s have been written and analyzed in [3]. They were later shown [7] to be related to free cumulants, as we shall recall below.

Having introduced the main players, we can now state the relation between the non-coincident cumulants in the classical and quantum SSEP, see Ref. [2].

Proposition 6. *For $0 < x_k < 1$ all different, we have*

$$\psi_n^{\text{ssep}}(x_1, \dots, x_n) = (-1)^{n-1} \sum_{\sigma \in S_n / \mathbb{Z}_n} \psi_n^\#(x_{\sigma_1}, \dots, x_{\sigma_n}) . \quad (57)$$

The sum is over all permutations σ modulo cyclic permutations. There are $(n - 1)!$ terms in the sum.

The relation (57) essentially follows from Wick's theorem, see Ref. [2]. Together with the link between the Q-SSEP cumulants and free probability that we shall describe below, Eq. (57) is the starting point of the following new construction of the classical SSEP large deviation function.

4.3. Non-coincident SSEP Cumulants from Free Cumulants

The aim of this section is, on the one hand, to relate the generating function of non-coincident SSEP cumulants to free probability and, on the other hand, to use this relation to derive a simple integral representation of this generating function.

In order to avoid confusion between the large deviation generating and rate functions as given in the previous SSEP literature [9–12, 20]—namely $F_{\text{ssep}}[h]$ and $I_{\text{ssep}}[n]$ defined above in Eqs. (47, 46)—and the ones that we shall determine using free cumulant techniques, we shall denote the latter with script letters—namely $\mathfrak{F}_{\text{ssep}}[h]$ and $\mathfrak{I}_{\text{ssep}}[n]$. We shall prove in Sect. 4.5 that they (of course) coincide.

The Q-SSEP steady measure, and hence the functions $\psi_n^\#$, have been shown to be related to free cumulants [7].

Proposition 7. *Let the interval $[0, 1]$ equipped with the Lebesgue measure, denoted μ_L , be viewed as a probability space. Let $\mathbb{I}_x = 1_{[0,x]}$ be the indicator function of the interval $[0, x]$ with $0 < x < 1$. We have $\mu_L(\mathbb{I}_{x_1} \cdots \mathbb{I}_{x_n}) = \min(x_1, \dots, x_n) =: x_1 \wedge \cdots \wedge x_n$.*

The loop-expectation values $\psi_n^\#$ are identified as the free cumulants R_n of those random variables with respect to the measure μ_L . Namely,

$$\psi_n^\#(x_1, \dots, x_n) = R_n(\mathbb{I}_{x_1}, \dots, \mathbb{I}_{x_n}). \tag{58}$$

The proof of this fact relied on a combinatorial analysis, see Ref. [7] for a set of characterizing equations for the steady cumulants $\psi_n^\#$ [3]. See Ref. [14] for an alternative analytic proof based on analyzing the time evolution equations of the Q-SSEP cumulants.

As an illustration, we write the first few terms for low values of n , using the defining relation between moments and free cumulants [8, 21, 24, 27]. For $n = 2, 3$, we have :

$$\begin{aligned} \psi_2^\#(x, y) &= x \wedge y - xy, \\ \psi_3^\#(x, y, z) &= (x \wedge y \wedge z) - (x \wedge y)z - (x \wedge z)y - (y \wedge z)x + 2xyz, \end{aligned}$$

If $0 < x < y < z < 1$, we get

$$\begin{aligned} \psi_2^\#(x, y) &= x(1 - y), \\ \psi_3^\#(x, y, z) &= x(1 - 2y)(1 - z). \end{aligned}$$

For $n \leq 3$ there is no difference between free and standard cumulants. The difference starts at $n \geq 4$. For $n = 4$, we have, for any order between the points x_j :

$$\begin{aligned} \psi_4^\#(x_1, x_2, x_3, x_4) &= (x_1 \wedge x_2 \wedge x_3 \wedge x_4) - ((x_1 \wedge x_2 \wedge x_3)x_4 + \text{[4 terms in total]}) \\ &\quad + ((x_1 \wedge x_2)x_3x_4 + \text{[6 terms in total]}) - 3x_1x_2x_3x_4 \\ &\quad - ((x_1 \wedge x_2) - x_1x_2)((x_3 \wedge x_4) - x_3x_4) \\ &\quad - ((x_1 \wedge x_4) - x_1x_4)((x_2 \wedge x_3) - x_2x_3). \end{aligned}$$

If we choose to order them on the segment $[0, 1]$, i.e., $0 < x_1 < x_2 < x_3 < x_4 < 1$, we get:

$$\begin{aligned}\psi_4^\#(x_1, x_2, x_3, x_4) &= x_1(1 - 3x_2 - 2x_3 + 5x_2x_3)(1 - x_4), \\ \psi_4^\#(x_1, x_3, x_4, x_2) &= x_1(1 - 3x_2 - 2x_3 + 5x_2x_3)(1 - x_4), \\ \psi_4^\#(x_1, x_3, x_2, x_4) &= x_1(1 - 4x_2 - x_3 + 5x_2x_3)(1 - x_4).\end{aligned}$$

We observe that they depend on the ordering of the points on the line (for $n \geq 4$). Higher-order cumulants can similarly be computed recursively. Of course, the computation becomes more and more involved and we need to package it.

Let us introduce the generating function of the classical SSEP non-coincident cumulants—recall the latter are linked to the Q-SSEP expectation values via Eq. (57). It is defined by

$$\mathfrak{F}_0[a; v] := \sum_{n \geq 1} \frac{v^n}{n!} \psi_n^{\text{ssep}}[a], \quad \psi_n^{\text{ssep}}[a] := \int_0^1 \psi_n^{\text{ssep}}(x_1, \dots, x_n) \prod_{k=1}^n a(x_k) dx_k, \quad (59)$$

with ψ_n^{ssep} the scaled non-coincident cumulants. Here, v is simply a counting parameter that we introduced for later convenience. To avoid confusion and to allow for future comparison with the formula obtained in the SSEP literature, we have used a specific notation for the SSEP large deviation function computed using free cumulant technique. We set $\mathfrak{F}_0[a] := \mathfrak{F}_0[a; v = 1]$. Of course we have $\mathfrak{F}_0[a; v] = \mathfrak{F}_0[av]$.

Lemma 8. *The generating function \mathfrak{F}_0 of the classical SSEP non-coincident cumulants can be expressed in terms of free cumulants R_n w.r.t. the Lebesgue measure as*

$$\mathfrak{F}_0[a] = - \sum_{k \geq 1} \frac{(-1)^k}{k} R_k(\mathbb{I}_{[a]}), \quad (60)$$

with $\mathbb{I}_{[a]} := \int dx a(x) \mathbb{I}_x$. Equivalently, $\psi_n^{\text{ssep}}[a] = (-1)^{n-1} (n-1)! R_n(\mathbb{I}_{[a]})$.

Proof. This is a direct consequence of the classical-to-quantum SSEP correspondence (57) and the multi-linearity of the free cumulants which imply

$$\psi_n^{\text{ssep}}[a] = (-1)^{n-1} (n-1)! R_n(\mathbb{I}_{[a]}). \quad (61)$$

□

The function $\mathbb{I}_{[a]}$ is a classical variable on $[0, 1]$, equipped with the Lebesgue measure, and $R_n(\mathbb{I}_{[a]})$ its n th free cumulants. For a single, hence commuting, variable, there is a simple relation between the generating function of the free cumulants and that of the moments [8, 21, 24, 27]. This relation goes through the resolvent. We shall now explain how this yields to an efficient way to compute the classical SSEP non-coincident cumulants and how it can be used it to derive a simple integral representation of $\mathfrak{F}_0[a]$.

Let $R_{[a]}(z)$ be the generating function of the free cumulants $R_n(-\mathbb{I}_{[a]})$:

$$R_{[a]}(z) := \frac{1}{z} + \sum_{p \geq 1} R_p(-\mathbb{I}_{[a]})z^{p-1}.$$

Let $G_{[a]}(z) := \mu_L(\frac{1}{z+\mathbb{I}_{[a]}})$ be the generating function of the moments $\mu_L(-\mathbb{I}_{[a]}^p)$:

$$G_{[a]}(z) = \sum_{p \geq 0} z^{-p-1} \mu_L(-\mathbb{I}_{[a]}^p).$$

From Refs. [8, 21, 24, 27], these two generating functions are inverse functions, i.e., $R_{[a]}(G_{[a]}(z)) = z$, which reads

$$1 + \sum_{p \geq 1} R_p(-\mathbb{I}_{[a]})G_{[a]}(z)^p = zG_{[a]}(z). \tag{62}$$

Let $b(x) := -\int_x^1 dy a(y)$, so that $b'(x) = a(x)$ with $b(1) = 0$. As a function on $[0, 1]$, we have $\mathbb{I}_{[a]} = -b$, since $\mathbb{I}_{[a]}(x) = \int_0^1 dy a(y)\mathbb{I}_{x < y} = \int_x^1 dy a(y)$. Thus, $\varphi(-\mathbb{I}_{[a]}^p) = \int_0^1 dx b(x)^p =: \bar{b}^p$, and

$$G_{[a]}(z) = \int_0^1 \frac{dx}{z + \mathbb{I}_{[a]}(x)} = \int_0^1 \frac{dx}{z - b(x)}. \tag{63}$$

We can turn the logic around and view x as a function of b : $x(b)$ is then interpreted as the cumulative probability for b , i.e., $dx(b)$ is the probability density for the variable b . The fact that $x \in [0, 1]$ is then natural.

Formulas (62,63) yield a simple recursive way to compute the free cumulants R_p in terms of the moments \bar{b}^p and hence the generating function of the classical SSEP cumulants. We have :

$$\begin{aligned} R_1(-\mathbb{I}_{[a]}) &= \bar{b}, \\ R_2(-\mathbb{I}_{[a]}) &= \bar{b}^2 - \bar{b}^2, \\ R_3(-\mathbb{I}_{[a]}) &= \bar{b}^3 - 3\bar{b}^2\bar{b} + 2\bar{b}^3, \\ R_4(-\mathbb{I}_{[a]}) &= \bar{b}^4 - 4\bar{b}^3\bar{b} + 10\bar{b}^2\bar{b}^2 - 2\bar{b}^2^2 + 5\bar{b}^4, \text{ etc.} \end{aligned} \tag{64}$$

Higher free cumulants can be computed recursively. Using Eq. (61) this yields the classical SSEP non-coincident cumulants. As a consequence :

Lemma 9. *Let $b(x) := -\int_x^1 dy a(y)$. We have the following integral representation of the generating function of non-coincident SSEP cumulants :*

$$\mathfrak{F}_0[a] = \int_0^1 dx \log(z - b(x)) - z + 1, \quad \text{with} \quad \int_0^1 \frac{dy}{z - b(y)} = 1. \tag{65}$$

Proof. Recall that $\mathfrak{F}_0[a; v] = \mathfrak{F}_0[av]$. By definition, Eq. (59), we have

$$v\partial_v \mathfrak{F}_0[a; v] = -\sum_{k \geq 1} v^k R_k(-\mathbb{I}_{[a]}) = 1 - vR_{[a]}(v),$$

with $R_{[a]}(z) := \frac{1}{z} + \sum_{p \geq 1} \kappa_p(-\mathbb{I}_{[a]})z^{p-1}$ as above. Thus, the relation $R_{[a]}(G_{[a]}(z)) = z$ becomes

$$1 - v\partial_v \mathfrak{F}_0[a; v] = vz, \quad \text{with } v = \int_0^1 \frac{dy}{z - b(y)}. \tag{66}$$

We simply have to check that the function (65) is indeed solution of the above equation, with the appropriate boundary condition. Assuming $\mathfrak{F}_0[a]$ given by the r.h.s. of (65), we have

$$\mathfrak{F}_0[a; v] = \int_0^1 dx \log[v(z - b(x))] - vz + 1,$$

with $\int_0^1 \frac{dy}{z - b(y)} = v$. Computing its derivative with respect to v , we get (using that $z = z[b, v]$ is actually a function of v and b)

$$v\partial_v \mathfrak{F}_0[a; v] = 1 - vz + v \left(\frac{\partial z}{\partial v} \right) \left(\int_0^1 \frac{dx}{z - b(x)} - v \right).$$

The last term vanish by definition of $z = z[b, v]$ and we get $v\partial_v \mathfrak{F}_0[a; v] = 1 - vz$, as required. It is easy to check that the function (65) has the appropriate behavior at small v . \square

This representation can also be used to recursively compute $\mathfrak{F}_0[a; v]$. Equation (66) can alternatively be written as

$$v\partial_v \mathfrak{F}_0[a; v] = 1 - vz, \quad \text{with } v = G_{[a]}(z). \tag{67}$$

The last relation, $v = G_{[a]}(z)$, determines z as a function of v , recursively :

$$\begin{aligned} v(z(v)) = & 1 + \bar{b}v + (\bar{b}^2 - \bar{b}^2)v^2 + (\bar{b}^3 - 3\bar{b}^2\bar{b} + 2\bar{b}^3)v^3 \\ & + (\bar{b}^4 - 4\bar{b}^3\bar{b} + 10\bar{b}^2\bar{b}^2 - 2\bar{b}^2^2 + 5\bar{b}^4)v^4 + \dots, \end{aligned}$$

This is of course the generating function of the free cumulants of b , see Eq. (64). This is a very efficient way to compute the multi-point SSEP cumulants at non-coincident points.

4.4. The Classical SSEP Large Deviation Function from Free Probability

Once the generating function $\mathfrak{F}_0[a]$ of non-coincident SSEP cumulants has been identified, the SSEP large deviation generating function $\mathfrak{F}_{\text{ssep}}[h]$ can be computed using the variational principle established in the previous Sects. 2 & 3:

$$\mathfrak{F}_{\text{ssep}}[h] = \max_{g(\cdot); q(\cdot)} \left[\int_0^1 dx \left[\log(1 + g(x)(e^{h(x)} - 1)) - q(x)g(x) \right] + \mathfrak{F}_0[q] \right], \tag{68}$$

with $\mathfrak{F}_0[q]$ defined in Eq. (65). Since the rate function $\mathfrak{I}_{\text{ssep}}$ is the Legendre transform of the large deviation generating function, we have

Corollary 10.

$$\mathfrak{J}_{\text{ssep}}[n] = \max_{g(\cdot), q(\cdot)} \left(\int_0^1 dx \left[n(x) \log \left(\frac{n(x)}{g(x)} \right) + (1 - n(x)) \log \left(\frac{1 - n(x)}{1 - g(x)} \right) + q(x)g(x) \right] - \mathfrak{F}_0[q] \right). \tag{69}$$

Proof. Using Eq. (68) and $\mathfrak{J}_{\text{ssep}}[n] = \max_{h(\cdot)} \left(\int_0^1 dx h(x)n(x) - \mathfrak{F}_{\text{ssep}}[h] \right)$, we write

$$\mathfrak{J}_{\text{ssep}}[n] = \max_{h(\cdot)} \left(\int_0^1 dx \left[h(x)n(x) - \log(1 + g(x)e(x)) + q(x)g(x) \right] - \mathfrak{F}_0[q] \right), \tag{70}$$

with $e(x) = e^{h(x)} - 1$. Here g and q are determined as the functions for which $\mathfrak{F}_{\text{ssep}}[h]$ takes its maximal value,

$$q(x) = \frac{e(x)}{1 + g(x)e(x)} \quad , \quad g(x) = \frac{\delta \mathfrak{F}_0[q]}{\delta q(x)}. \tag{71}$$

The maximum of (70) is attained for

$$h(x) = \log \left(\frac{n(x)(1 - g(x))}{g(x)(1 - n(x))} \right),$$

from which we get

$$1 + g(x)e(x) = \frac{1 - g(x)}{1 - n(x)}, \quad q(x) = \frac{n(x)}{g(x)} - \frac{1 - n(x)}{1 - g(x)}. \tag{72}$$

Inserting this back into the expression the rate function yields

$$\mathfrak{J}_{\text{ssep}}[n] = \int_0^1 dx \left[n \log \left(\frac{n}{g} \right) + (n - 1) \log \left(\frac{1 - g}{1 - n} \right) + qg \right] - \mathfrak{F}_0[q].$$

The two conditions (71) or (72) can be relaxed in writing this expression as a maximization problem (and simplifying the expression) as in Eq. (69). Indeed, one can check that the extremization condition (69) yields the same conditions for g and q as in (71) or (72). \square

4.5. Equivalence with the Previously Known Formulation

Finally, we present an explicit check that our new formula (68) for the SSEP large deviation function is identical to the previously known formula (49).

To prove this equivalence, we first formulate differently, but equivalently, the variational problem (49), as follows :

$$F_{\text{ssep}}[h] = \max_{g(\cdot); f(\cdot)} \widehat{F}[h; f, g], \widehat{F}[h; f, g] := \int_0^1 dx \left[\log(1 + g(x)e(x)) - f(x)g(x) \right] + V[f],$$

where the functional $V[f]$ is defined by

$$V[f] := \int_0^1 dx \log(w - \ell(x)) - w + 1, \quad \text{with} \quad \int_0^1 \frac{dy}{w - \ell(y)} = 1, \tag{73}$$

with $\ell(x) := - \int_x^1 dy f(y)$, so that $\ell'(x) = f(x)$ with $\ell(1) = 0$. We view w as a function of ℓ through the constraint $\int_0^1 \frac{dy}{w - \ell(y)} = 1$.

To prove the equivalence between the two variational problems, we first have to compute the functional derivative of $V[f]$. Chain rule implies

$$\frac{\delta V[f]}{\delta f(x)} = - \int_0^1 \left(\frac{\delta \ell(y)}{\delta f(x)} \right) \left(\frac{dy}{w - \ell(y)} \right) + \left(\frac{\delta w}{\delta f(x)} \right) \left(\int_0^1 \frac{dy}{w - \ell(y)} - 1 \right).$$

The second term vanishes due to the relation $\int_0^1 \frac{dy}{w - \ell(y)} = 1$. The definition of ℓ as $\ell(x) := - \int_x^1 dy f(y)$ implies $\frac{\delta \ell(y)}{\delta f(x)} = -\mathbb{I}_{\{x > y\}}$. Thus,

$$\frac{\delta V[f]}{\delta f(x)} = \int_0^x \frac{dy}{w - \ell(y)}. \tag{74}$$

The extremization conditions (49) read

$$f(x) = \frac{e(x)}{1 + e(x)g(x)}, \quad g(x) = \frac{\delta V[f]}{\delta f(x)} = \int_0^x \frac{dy}{w - \ell(y)}.$$

The relation $\int_0^1 \frac{dy}{w - \ell(y)} = 1$ implies the boundary conditions $g(1) = 1$. The last condition is equivalent to $1/g'(x) = w - \ell(x)$ with $g(0) = 0$, and hence to $(1/g'(x))' = -\ell'(x) = -f(x)$, with $f(x) = \frac{e(x)}{1 + e(x)g(x)}$, which is then equivalent to (50).

The last step consists in verifying that the extremum value coincide. Thanks to the extremum conditions, written as $g'(x)(w - \ell(x)) = 1$, and the boundary conditions on g and ℓ , we have

$$\int_0^1 dx f(x)g(x) = \int_0^1 dx \ell'(x)g(x) = - \int_0^1 dx \ell(x)g'(x) = 1 - w,$$

so that

$$\widehat{F}[h; f, g]|_{\text{ext}} - F[h; g]|_{\text{ext}} = - \int_0^1 dx f(x)g(x) - w + 1 = 0$$

It is now clear that the two extremization problems (49) and (68) are equivalent, with the correspondence is $w \rightsquigarrow z$, $\ell \rightsquigarrow -\mathbb{I}_{[a]}$, so that $\mathfrak{F}_{\text{ssep}}[h] = F_{\text{ssep}}[h]$.

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Appendices

A Formalities

A.1 Formal Power Series

If R is any commutative ring with unit, S is an arbitrary index set and $\lambda_\bullet := (\lambda_s)_{s \in S}$ a collection of variables, we denote by $R[\lambda_\bullet]$, the ring of polynomials and by $R(\lambda_\bullet)$ the ring of formal power series in the variables λ_\bullet with coefficients in R . We let $R(\lambda_\bullet)_{\geq 1}$ denote the ideal in $R(\lambda_\bullet)$ of formal power series with vanishing constant coefficient. The rings $R[\lambda_\bullet]$ and $R(\lambda_\bullet)$ are again commutative rings with unit, so if J is a new arbitrary index set and $z_\bullet := (z_i)_{i \in J}$, $\bar{z}_\bullet := (\bar{z}_i)_{i \in J}$ are new variables we may consider for instance $R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)$, the ring of formal power series in the variables λ_\bullet with coefficients in the ring $R[z_\bullet, \bar{z}_\bullet]$ of polynomials in $z_\bullet, \bar{z}_\bullet$. We note that $R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet) \subset R(z_\bullet, \bar{z}_\bullet, \lambda_\bullet)$. Notice that in $R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)$ the ‘‘constant’’ coefficient is now a polynomial in $R[z_\bullet, \bar{z}_\bullet]$ and this polynomial vanishes for members of $R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)_{\geq 1}$. If F is any formal power series in one variable and $L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) \in R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)_{\geq 1}$ then the composition $F \circ L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$ is well defined as an element of $R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)$. For instance, if $A_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) \in R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)$ then $A_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)e^{L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)} \in R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)$.

A.2 Formal Gaussian Integrals

From now on, the ground ring R is assumed to be a field of characteristic 0, say \mathbb{R} .

Our first aim is to make sense of

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) A_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$$

where \hbar is yet another formal variable and $A_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) \in R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)$. The result of integration will be a element of $R(\hbar, \hbar^{-1}, \lambda_\bullet)$ and in fact $A_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$ may contain \hbar explicitly (the notation is already heavy enough). It is useful to notice that contrary to the other formal variables involved, \hbar can be specialized to a numerical value without impact on most of the discussions that follow (the semiclassical expansion below being an important exception) and then the result of integration is in $R(\lambda_\bullet)$.

We define the integral by term by term integration of the λ_\bullet -formal power series expansion of $A_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$ so we are left with the task of defining

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) P(z_\bullet, \bar{z}_\bullet)$$

for $P(z_\bullet, \bar{z}_\bullet) \in R[z_\bullet, \bar{z}_\bullet]$. Imposing linearity, it is enough to deal with monomials in $R[z_\bullet, \bar{z}_\bullet]$, i.e., expressions of the form $\prod_{i \in J} z_i^{m_i} \bar{z}_i^{\bar{m}_i} =: z_\bullet^{m_\bullet} \bar{z}_\bullet^{\bar{m}_\bullet}$ where $m_i, \bar{m}_i \in \mathbb{N}$ and $\sum_{i \in J} m_i + \bar{m}_i$, called the degree of the monomial, is finite.

For such a monomial we set

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \prod_{i \in J} z_i^{m_i} \bar{z}_i^{\bar{m}_i} := \prod_{i \in J} \delta_{m_i, \bar{m}_i} \hbar^{m_i} m_i!,$$

a formula copied from the honest integral over the complex plane

$$\int_{\mathbb{C}} \frac{d\bar{z} \wedge dz}{2i\pi\hbar} \exp(-z\bar{z}/\hbar) z^m \bar{z}^{\bar{m}} = \delta_{m, \bar{m}} \hbar^m m!,$$

which holds for $m, \bar{m} \in \mathbb{N}$ and \hbar a complex number with strictly positive real part.

Our main interest lies in the computation of

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L_{\lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) / \hbar$$

where $L_{\lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) \in R[z_{\bullet}, \bar{z}_{\bullet}](\lambda_{\bullet})_{\geq 1}$. As $\exp L_{\lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) / \hbar$ belongs to $R[z_{\bullet}, \bar{z}_{\bullet}](\lambda_{\bullet})$ this is really a special case of the previous discussion.

We focus for a while on the case when the index set, J for the variables $(z_{\bullet}, \bar{z}_{\bullet})$ is fixed. We denote by $S(J)$ the set indexing monomials in $(z_{\bullet}, \bar{z}_{\bullet})$, i.e.,

$$S(J) := \left\{ (m_i, \bar{m}_i)_{i \in J}, m_i, \bar{m}_i \in \mathbb{N} \text{ for } i \in J, \sum_{i \in J} m_i + \bar{m}_i < +\infty \right\}.$$

The variables indexed by $S(J)$ are denoted by $\Lambda_{\bullet} := (\Lambda_{m_{\bullet}, \bar{m}_{\bullet}})_{(m_{\bullet}, \bar{m}_{\bullet}) \in S(J)}$ and our goal is the computation of

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L(J)_{\Lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) / \hbar,$$

where

$$L(J)_{\Lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) := \sum_{(m_{\bullet}, \bar{m}_{\bullet}) \in S(J)} \Lambda_{m_{\bullet}, \bar{m}_{\bullet}} \prod_{i \in J} \frac{z_i^{m_i} \bar{z}_i^{\bar{m}_i}}{m_i! \bar{m}_i!} \in R[z_{\bullet}, \bar{z}_{\bullet}](\Lambda_{\bullet})_{\geq 1}.$$

The appearance of factorials in the denominator will simplify the forthcoming formulæ.

We call this special case “the” master integral. It contains all the information to deal with the general case. Indeed, if B is an arbitrary set indexing variables $\lambda_{\bullet} := (\lambda_s)_{s \in B}$, any $L_{\lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) \in R[z_{\bullet}, \bar{z}_{\bullet}](\lambda_{\bullet})_{\geq 1}$ can be expanded in monomials:

$$L_{\lambda_{\bullet}}(z_{\bullet}, \bar{z}_{\bullet}) = \sum_{(m_{\bullet}, \bar{m}_{\bullet}) \in S(J)} \Lambda(B)_{m_{\bullet}, \bar{m}_{\bullet}} \prod_{i \in J} \frac{z_i^{m_i} \bar{z}_i^{\bar{m}_i}}{m_i! \bar{m}_i!}$$

where each $\Lambda(B)_{m_{\bullet}, \bar{m}_{\bullet}}$ belongs to $R(\lambda_{\bullet})_{\geq 1}$. Notice, however, that the formal power series $\Lambda(B)_{m_{\bullet}, \bar{m}_{\bullet}}$ are not arbitrary: a given λ_{\bullet} appears in only finitely

many $\Lambda(B)_{m_\bullet, \bar{m}_\bullet}$ so that its coefficient in the λ_\bullet expansion of $L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$ is indeed a polynomial in $z_\bullet, \bar{z}_\bullet$. Anyway,

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar$$

is recovered from

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar$$

by substitution, for each $(m_\bullet, \bar{m}_\bullet) \in S(J)$ of the formal power series $\Lambda(B)_{m_\bullet, \bar{m}_\bullet}$ for the formal variable $\Lambda_{m_\bullet, \bar{m}_\bullet}$.

A.3 Semiclassical Expansion

As a warm-up exercise, the reader is invited to check that, as a formal integral where z, \bar{z}, u and \bar{v} are formal variables,

$$\int \frac{d\bar{z} \wedge dz}{2i\pi\hbar} \exp(-(z + u)(\bar{z} + \bar{v}) / \hbar) = 1.$$

This simple identity, which can be read as translation invariance (independently over z, \bar{z} so even if it is valid for its honest integral avatar in the complex plane, this does not reduce to translation invariance of the Lebesgue measure), plays an important role in the manipulation of (formal) Gaussian integrals. The reader should also check the corollaries

$$\int \frac{d\bar{z} \wedge dz}{2i\pi\hbar} \exp(-z\bar{z} / \hbar) \exp(-(\bar{v}z + u\bar{z}) / \hbar) = e^{u\bar{v} / \hbar},$$

$$\int \frac{d\bar{z} \wedge dz}{2i\pi\hbar} \exp(-z\bar{z} / \hbar) \exp(((\bar{v} - \bar{v})z + (\mu - u)\bar{z} + \bar{v}u + \mu\bar{v} - u\bar{v}) / \hbar) = e^{\mu\bar{v} / \hbar},$$

whose expansion in the new formal variables μ, \bar{v} yields a more general version of translation invariance

$$\int \frac{d\bar{z} \wedge dz}{2i\pi\hbar} \exp(-(z + u)(\bar{z} + \bar{v}) / \hbar) (z + u)^m (\bar{z} + \bar{v})^{\bar{m}} = \delta_{m, \bar{m}} \hbar^m m!,$$

which also plays an important role in what follows.

We introduce collections $u_\bullet := (u_i)_{i \in J}, \bar{v}_\bullet := (\bar{v}_i)_{i \in J}$ of formal variables and use translation invariance:

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar$$

$$= \int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-(z_i + u_i)(\bar{z}_i + \bar{v}_i) / \hbar) \right)$$

$$\times \exp L_{\lambda_\bullet}(z_\bullet + u_\bullet, \bar{z}_\bullet + \bar{v}_\bullet) / \hbar.$$

We infer that if $U_\bullet := (U_i)_{i \in J}, \bar{V}_\bullet := (\bar{V}_i)_{i \in J}$ are arbitrary elements of $R(\lambda_\bullet)_{\geq 1}$ then

$$\begin{aligned} & \int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar \\ &= \int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-(z_i + U_i)(\bar{z}_i + \bar{V}_i) / \hbar) \right) \\ & \quad \times \exp L_{\lambda_\bullet}(z_\bullet + U_\bullet, \bar{z}_\bullet + \bar{V}_\bullet) / \hbar. \end{aligned}$$

We claim that for any given L_{λ_\bullet} , there are unique members $U_\bullet^*, \bar{V}_\bullet^*$ in $R(\lambda_\bullet)_{\geq 1}$ such that

$$\sum_{i \in J} (z_i + U_i^*)(\bar{z}_i + \bar{V}_i^*) - L_{\lambda_\bullet}(z_\bullet + U_\bullet^*, \bar{z}_\bullet + \bar{V}_\bullet^*)$$

has an extremum in z_\bullet and \bar{z}_\bullet at 0. Indeed the extremum equations are

$$\bar{V}_i = \frac{\partial L_{\lambda_\bullet}}{\partial z_i}(U_\bullet, \bar{V}_\bullet) \quad U_i = \frac{\partial L_{\lambda_\bullet}}{\partial \bar{z}_i}(U_\bullet, \bar{V}_\bullet)$$

for $i \in J$, and existence/uniqueness of $(U_\bullet^*, \bar{V}_\bullet^*)$ follow from a tedious but straightforward recursive argument on the degrees in the $(z_\bullet^*, \bar{z}_\bullet^*)$ of L_{λ_\bullet} .

Set $F^* := -\sum_{i \in J} U_i^* \bar{V}_i^* + L_{\lambda_\bullet}(U_\bullet^*, \bar{V}_\bullet^*)$, a member of $R(\lambda_\bullet)_{\geq 1}$. Then,

$$\begin{aligned} & -\sum_{i \in J} (z_i + U_i^*)(\bar{z}_i + \bar{V}_i^*) + L_{\lambda_\bullet}(z_\bullet + U_\bullet^*, \bar{z}_\bullet + \bar{V}_\bullet^*) - F_{\lambda_\bullet}^* \\ & \quad + \sum_{i \in J} z_i \bar{z}_i =: L_{\lambda_\bullet}^*(z_\bullet, \bar{z}_\bullet) \end{aligned}$$

belongs to $R[z_\bullet, \bar{z}_\bullet](\lambda_\bullet)_{\geq 1}$ (this fact concerns the λ_\bullet -expansion) and involves only terms of degree ≥ 2 in $(z_\bullet, \bar{z}_\bullet)$. Consequently, a simple power counting argument shows that

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L_{\lambda_\bullet}^*(z_\bullet, \bar{z}_\bullet) / \hbar$$

belongs to $R(\hbar, \lambda_\bullet)$ (no \hbar^{-1} involved) and that the constant term is 1. Putting everything together leads to

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L_{\lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar = \exp F_{\lambda_\bullet}^* / \hbar + \sum_{n \in \mathbb{N}} \hbar^n F_{\lambda_\bullet}^{(n)}$$

where each $F_{\lambda_\bullet}^{(n)}$ belongs to $R(\lambda_\bullet)_{\geq 1}$. This \hbar -expansion is the formal version of the saddle point expansion. We shall soon see a diagrammatic interpretation of this result.

B Feynman Graphs and Rules

We turn to graphical rules allowing the computation (in principle) of the master integral.

Consider a Λ_\bullet -monomial in the expansion of $\exp L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet)/\hbar$. It comes with:

- A coefficient which is proportional (the coefficient is in R) to a $(z_\bullet, \bar{z}_\bullet)$ monomial, and this monomial survives integration (gives a nonzero contribution) if and only if for each $i \in J$ the power of z_i and \bar{z}_i are equal, say n_i . In this case, we say that the given Λ_\bullet -monomial is balanced. Then, its integral is $\prod_{i \in J} n_i!$.
- A combinatorial denominator, a product of factorials. If $\Lambda_{m_\bullet, \bar{m}_\bullet}$ appears to the power k in the monomial, it contributes a multiplicative factor $k!(\prod_{i \in J} m_i! \bar{m}_i!)^k$ to the combinatorial denominator. The $k!$ comes from the expansion of the exponential function, and the other factor from the definition of $L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$.

We translate the first fact as follows. For $(m_\bullet, \bar{m}_\bullet) \in S(J)$, we represent $\Lambda_{m_\bullet, \bar{m}_\bullet} \prod_{i \in J} z_i^{m_i} \bar{z}_i^{\bar{m}_i}$ as a vertex with m_i outgoing edges and \bar{m}_i incoming edges carrying the tag i . Thus, a term in the expansion of $\exp L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet)/\hbar$ is represented as a collection of vertices with pending edges. We represent the integration as a black box in which all the edges meet:

- If the number of incoming edges and outgoing edges carrying the tag $i \in J$ are not equal, the integration yields 0.
- If both equal n_i , integration yields a factor $n_i!$. The obvious but crucial observation is: $n_i!$ is precisely the number of ways to pair the incoming edges of type i to the outgoing edges of type i , that is, the number of ways to organize the inside of the black box so that no pending edge remains. For each such choice of pairing, opening the black box reveals an oriented graph, in which each edges carries a type $i \in J$. Conversely, given such a graph, breaking each edge in two pending edges (keeping track of the type and the orientation) one reconstructs a term in the expansion of $\exp L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet)/\hbar$.

We stress that the graphs the appear in this graphical representation have their edges oriented and tagged but:

- They are not simple. For instance several edges with the same tag and orientation may join two vertices. Moreover they may have tadpoles.
- Neither the vertices nor the edges are labeled.

So we pause for a moment to establish the definitions. A (labeled) oriented graph with edges tagged by J is given by the following data: finite sets V, E^\pm , arbitrary maps $t^\pm: E^\pm \rightarrow J$, $a^\pm: E^\pm \rightarrow V$, and a bijection $b: E^+ \rightarrow E^-$ such that $t^- \circ b = t^+$. Though there are slightly simpler (but equivalent) definitions, this one is well suited to make contact with the graphical representation of the master integral. If $v \in V$ and $e^\pm \in E^\pm$, we view v as a vertex, e^+ as an outgoing edge tagged with $t^+(e^+)$ pending at vertex $a^+(e^+)$ and e^- as an incoming edge tagged with $t^-(e^-)$ pending at vertex $a^-(e^-)$ and the (e^+, e^-) is an oriented edge (made by two pending edges) tagged with $i \in J$ if $e^- = b(e^+)$ and $t^+(e^+) = i$ (in which case $t^-(e^-) = i$ as well). The definitions ensure that, for each $i \in J$, $(t^+)^{-1}(i)$ and $(t^-)^{-1}(i)$ have the same size.

To (V, E^\pm, t^\pm, a^\pm) (i.e., forgetting b for a moment), we may associate a balanced Λ_\bullet -monomial (one whose integral gives a nonzero result): if vertex v has m_i outgoing pending edges and \bar{m}_i incoming pending edges tagged with i , it contributes a factor $\Lambda_{m_\bullet, \bar{m}_\bullet}$ to the Λ_\bullet -monomial. Then, b describes a wiring inside the integration black box.

Two graphs $(V_1, E_1^\pm, t_1^\pm, a_1^\pm, b_1)$ and $(V_2, E_2^\pm, t_2^\pm, a_2^\pm, b_2)$ are equivalent if there are bijections $\omega: V_1 \rightarrow V_2$ and $\phi^\pm: E_1^\pm \rightarrow E_2^\pm$ such that $t_2^\pm \circ \phi^\pm = t_1^\pm$, $a_2^\pm \circ \phi^\pm = \omega \circ a_1^\pm$ and $b_2 \circ \phi^+ = \phi^- \circ b_1$. Clearly two equivalent graphs yield the same associated balanced Λ_\bullet -monomial. Notice that, tautologically, if $(V_1, E_1^\pm, t_1^\pm, a_1^\pm, b_1)$ and there are bijections $\omega: V_1 \rightarrow V_2$ and $\phi^\pm: E_1^\pm \rightarrow E_2^\pm$, then $(V_2, E_2^\pm, t_1^\pm \circ (\phi^\pm)^{-1}, \omega \circ a_1^\pm \circ (\phi^\pm)^{-1}, \phi^- \circ b_1 \circ (\phi^+)^{-1})$ is an equivalent graph.

Take a graph $(V, E^\pm, t^\pm, a^\pm, b)$. Consider triples (ω, ϕ^\pm) , where $\omega: V \rightarrow V$ and $\phi^\pm: E^\pm \rightarrow E^\pm$ are bijections such that $t^\pm \circ \phi^\pm = t^\pm$ and $a^\pm \circ \phi^\pm = \omega \circ a^\pm$. Such triples form a group under composition. The conditions that (ω, ϕ^\pm) must satisfy mean that one may

- Arbitrarily permute the pending edges at a given vertex if they have the same orientation and tag
- Arbitrarily permute vertices (with their pending edges) if they have the same environment, that is if for each orientation and tag they have the same number of pending edges of that orientation and tag.

Thus, the order of this group is seen to match exactly the combinatorial denominator that comes with the balanced Λ_\bullet -monomial associated to the graph.

Moreover by construction $(V, E^\pm, t^\pm, a^\pm, b)$ and $(V, E^\pm, t^\pm, a^\pm, \phi^- \circ b \circ (\phi^+)^{-1})$ are equivalent graphs. If $\phi^- \circ b = b \circ \phi^+$ the action of the triple (ω, ϕ^\pm) fixes the graph. This condition defines a subgroup of the above group. This subgroup is (called) the automorphism group of the graph.

Forgetting about set theoretic issues, an unlabeled graph is informally an equivalence class of labeled graphs. More concretely, given the size m of V and n of E^\pm , we may take arbitrary bijections from V to $\llbracket 1, m \rrbracket$ (we define $\llbracket 1, 0 \rrbracket := \emptyset$) and from E^\pm to $\llbracket 1, n \rrbracket$ to define an equivalent graph, so we may for instance define an unlabeled graph as a graph with vertex set $\llbracket 1, m \rrbracket$ and edge set $\llbracket 1, n \rrbracket$ for some (m, n) , modulo equivalence.

Back to our graph counting problem. Given a balanced Λ_\bullet -monomial (one whose integral gives a nonzero result) of degree m in the expansion of $\exp L(J)_\Lambda(z_\bullet, \bar{z}_\bullet)/\hbar$, with degree $2n$ in $(z_\bullet, \bar{z}_\bullet)$ we may take $V = \llbracket 1, m \rrbracket$ to label the m “ Λ_\bullet ” factors, and $E^\pm = \llbracket 1, n \rrbracket$ to label the n “ z_\bullet ” and “ \bar{z}_\bullet ” factors. This defines the maps t^\pm, a^\pm . The wiring inside the integration black box defines the bijection b and we end with a (labeled) oriented graph with edges tagged by J . In general, different wirings may yield equivalent graphs (i.e., the same unlabeled graph) and it is more economical to group systematically such equivalent wirings. By the above discussion, this replaces the combinatorial denominator by the size of the automorphism group. In what follows, and unless obvious from the context, the term “graph” means “unlabeled graph.”

If G is such a graph, we denote by $\text{Aut } G$ its automorphisms group, whose cardinal is $|\text{Aut } G|$.

To summarize, we have obtained the following result:

- The master integral has a graphical expansion in formal power series

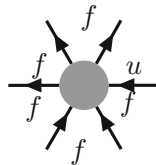
$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar = \sum_G w(G)$$

where the sum is over all unlabeled oriented graphs whose edges carry a type $i \in J$.

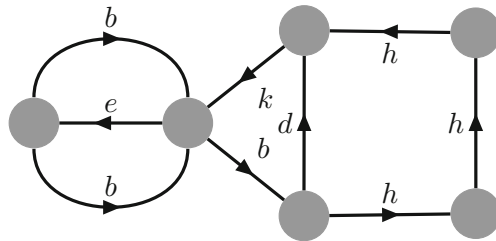
- The weight $w(G)$ of a graph is computed as follows:
 - The edges incident at a vertex define a vertex type $(m_\bullet, \bar{m}_\bullet) \in S(J)$ counting how many incoming and outgoing edges of each type are incident, leading to a factor $\Lambda_{m_\bullet, \bar{m}_\bullet} \hbar^{-1}$ in $w(G)$.
 - Each edge contributes a factor \hbar in $w(G)$.
 - There is an overall factor $1/|\text{Aut } G|$ in $w(G)$.

We give a few examples. For concreteness we take J to be the standard alphabet $J := \{a, b, \dots, y, z\}$ (of which we use only a subset in the examples!).

- Example: A vertex with 3 outgoing and 2 incoming edges of type f , and an incoming edge of type u . It is convenient to use a compact notation, additive or multiplicative for instance, and denote the corresponding coupling by $\Lambda_{3f, 2\bar{f}+\bar{u}}$ (additive, chosen below) or $\Lambda_{f^3 \bar{f}^2 \bar{u}}$ (multiplicative):



- Example: A diagram



with weight

$$\frac{1}{2!} \Lambda_{2b, \bar{e}} \Lambda_{b+e, 2\bar{b}+\bar{k}} \Lambda_{k, \bar{d}} \Lambda_{d+h, \bar{b}} \Lambda_{h, \bar{h}}^2.$$

The symmetry number $1/2!$ comes from the two equivalent outgoing type b edges at the vertex on the left.

Letting $n_v(G)$, $n_e(G)$, $n_l(G)$ and $n_c(G)$ denote, respectively, the number of vertices, edges, loops and connected components⁵ of G , we observe that the overall power of \hbar in $w(G)$ is $n_e(G) - n_v(G)$, which is also $n_l(G) - n_c(G)$ by a standard topological relation. At this point, we may pause to ask why it is worth to bother going from the plain “black box expansion” version of the master integral to the “graphical” version: after all, one black box gives rise to a whole family of graphs. One of the advantages of the graphical representation is that it behaves very nicely from the combinatorial viewpoint: the weight $w(G)$ factors nicely over connected components, so that

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet) / \hbar = \exp \sum_G w(G)$$

where now the sum is over all *connected* oriented graphs whose edges carry a type $i \in J$, the weight being computed as before. The power of \hbar in $w(G)$ is simply $n_l(G) - 1$ if G is connected, and comparing with the semiclassical expansion in Sect. **A.3 we infer that for the master integral the $(z_\bullet, \bar{z}_\bullet)$ -extremum $F(J)_{\Lambda_\bullet}^* := F(J)_{\Lambda_\bullet}^{(-1)}$ of $\sum_{i \in J} z_i \bar{z}_i - L(J)_{\Lambda_\bullet}(z_\bullet, \bar{z}_\bullet)$, and the corrections $F(J)_{\Lambda_\bullet}^{(n)}$, $n = 0, 1, \dots$ are given by

$$F(J)_{\Lambda_\bullet}^{(n)} = \hbar^{-n} \sum_G w(G) \text{ for } n \geq -1,$$

where the sum is over connected oriented graphs with $n + 1$ loops whose edges carry a type $i \in J$.

B.1 A Special Case

We turn to a special case—one that suffices for our main interest—and change notations accordingly. We denote by S and \bar{S} arbitrary sets, and introduce formal variables $\bar{\lambda}_\bullet := (\bar{\lambda}_s)_{s \in S}$ and $\lambda_\bullet := (\lambda_s)_{s \in \bar{S}}$. The formal Gaussian integral of interest is now

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp(L_{\bar{\lambda}_\bullet}(z_\bullet) + \bar{L}_{\lambda_\bullet}(\bar{z}_\bullet)) / \hbar,$$

where $L_{\bar{\lambda}_\bullet}(z_\bullet)$ and $\bar{L}_{\lambda_\bullet}(\bar{z}_\bullet)$ belong to $R[z_\bullet](\bar{\lambda}_\bullet)_{\geq 1}$ and $R[\bar{z}_\bullet](\lambda_\bullet)_{\geq 1}$, respectively. Thus, the “function” that we integrate against the Gaussian measure splits as a product of a “holomorphic function” and an “anti-holomorphic function,” and we are really dealing with a special case of the general discussion.

The semiclassical expansion carries through. The splitting between z_\bullet and \bar{z}_\bullet leads to the extremum equations

$$\bar{V}_i = \frac{\partial L_{\bar{\lambda}_\bullet}}{\partial z_i}(U_\bullet) \quad U_i = \frac{\partial \bar{L}_{\lambda_\bullet}}{\partial \bar{z}_i}(\bar{V}_\bullet)$$

with solution $(U_\bullet^*, \bar{V}_\bullet^*)$ and extremal value $F^* := \sum_{i \in J} U_i^* \bar{V}_i^* - L_{\bar{\lambda}_\bullet}(U_\bullet^*) - \bar{L}_{\lambda_\bullet}(\bar{V}_\bullet^*)$

⁵Loops and connected components are defined by discarding the orientation. For instance, connected means there is a path, possibly not respecting the orientation of edges, between any two vertices.

There is also a (restricted) master integral version adapted to the splitting. The index set J for the variables $(z_\bullet, \bar{z}_\bullet)$ is fixed but we consider only monomials purely in z_\bullet or in \bar{z}_\bullet . We denote now by $S(J)$ the set indexing monomials in z_\bullet so

$$S(J) := \left\{ (m_i)_{i \in J}, m_i \in \mathbb{N} \text{ for } i \in J, \sum_{i \in J} m_i < +\infty \right\}.$$

The variables indexed by $S(J)$ are denoted by $\bar{\Lambda}_\bullet := (\bar{\Lambda}_{m_\bullet})_{(m_\bullet) \in S(J)}$. The set indexing monomials in \bar{z}_\bullet is denoted by $\bar{S}(J)$ and the variables it indexes by Λ_\bullet . We specialize the general graphical rules to compute the restricted master integral

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp(L(J)_{\bar{\Lambda}_\bullet}(z_\bullet) + \bar{L}(J)_{\Lambda_\bullet}(\bar{z}_\bullet)) / \hbar,$$

where

$$L(J)_{\bar{\Lambda}_\bullet}(z_\bullet) := \sum_{m_\bullet \in S(J)} \bar{\Lambda}_{m_\bullet} \prod_{i \in J} \frac{z_i^{m_i}}{m_i!} \in R[z_\bullet]_{(\bar{\Lambda}_\bullet)_{\geq 1}},$$

and

$$\bar{L}(J)_{\Lambda_\bullet}(\bar{z}_\bullet) := \sum_{\bar{m}_\bullet \in \bar{S}(J)} \Lambda_{\bar{m}_\bullet} \prod_{i \in J} \frac{\bar{z}_i^{\bar{m}_i}}{\bar{m}_i!} \in R[\bar{z}_\bullet]_{(\Lambda_\bullet)_{\geq 1}}.$$

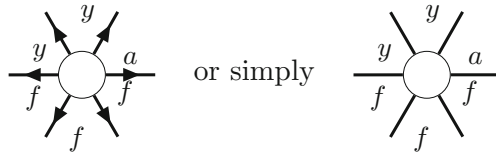
Instead of one general family of vertices, we are led to consider two special families of vertices. For $m_\bullet \in S(J)$, we represent $\bar{\Lambda}_{m_\bullet} \prod_{i \in J} z_i^{m_i}$ as a *white* vertex with m_i outgoing edges carrying the tag i , while for $\bar{m}_\bullet \in \bar{S}(J)$ we represent $\Lambda_{\bar{m}_\bullet} \prod_{i \in J} \bar{z}_i^{\bar{m}_i}$ as a *black* vertex with \bar{m}_i incoming edges carrying the tag i . Thus, there is an expansion of the restricted master integral as

$$\int \left(\prod_{i \in J} \frac{d\bar{z}_i \wedge dz_i}{2i\pi\hbar} \exp(-z_i \bar{z}_i / \hbar) \right) \exp(L(J)_{\bar{\Lambda}_\bullet}(z_\bullet) + \bar{L}(J)_{\Lambda_\bullet}(\bar{z}_\bullet)) / \hbar = \sum_G w(G)$$

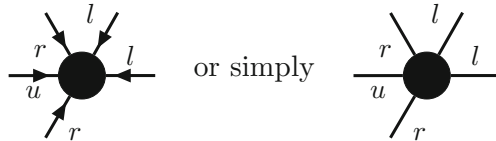
where the sum is over all bicolored (white,black) graphs whose edges carry a tag $i \in J$. The weight $w(G)$ of a graph is computed as before, though in this restricted case slightly simpler rules could be given to compute the automorphism group. Note that the colors of vertices allow to reconstruct the orientation, edges going from white to black vertices. The restricted master integral exponentiates as $\exp \sum_G w(G)$ where now the sum is over all connected bicolored graphs whose edges carry a tag $i \in J$.

We give a few examples. We take again J to be the standard alphabet $J := \{a, b, \dots, y, z\}$.

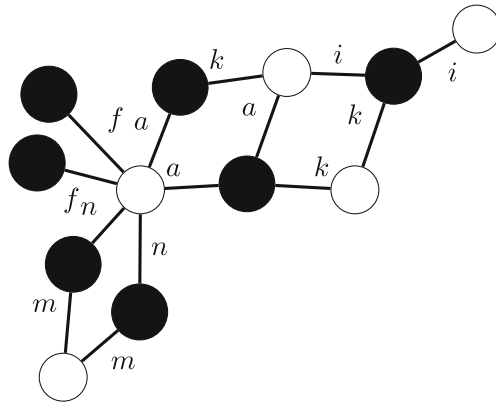
– Example: A white vertex with 6 (outgoing) edges, 3 with tag f , 2 with tag y and 1 with tag a , with weight $\bar{\Lambda}_{a+3f+2y}$:



– Example: A black vertex with 5 (incoming) edges, 2 with tag r , 2 with tag l and 1 with tag u , with weight $\Lambda_{2\bar{l}+2\bar{r}+\bar{u}}$:



– Example: A diagram



with weight (reading more or less from left to right, in that simple case symmetries are “local” on the graph)

$$\frac{1}{2!} \Lambda_f^2 \frac{1}{2!} \bar{\Lambda}_{2m} \Lambda_{m+n}^2 \bar{\Lambda}_{2a+2f+2n} \Lambda_{\bar{a}+\bar{k}} \Lambda_{2\bar{a}+\bar{k}} \bar{\Lambda}_{a+i+k} \bar{\Lambda}_{2k} \Lambda_{2\bar{i}+\bar{k}} \bar{\Lambda}_i.$$

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