



Long-Time Dynamics in Quantum Spin Lattices: Ergodicity and Hydrodynamic Projections at All Frequencies and Wavelengths

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Dedicated to the memory of Krzysztof Gawędski.

Abstract. Obtaining rigorous and general results about the non-equilibrium dynamics of extended many-body systems is a difficult task. In quantum lattice models with short-range interactions, the Lieb–Robinson bound tells us that the spatial extent of operators grows at most linearly in time. But what happens within this light-cone? We discuss rigorous results on ergodicity and the emergence of the Euler hydrodynamic scale in correlation functions, which establish fundamental principles at the root of non-equilibrium physics. One key idea of the present work is that general structures of Euler hydrodynamics, obtained under ballistic scaling, follow *independently from the details of the microscopic dynamics*, and in particular do not necessitate chaos; they are consequences of “extensivity”. Another crucial observation is that these apply at *arbitrary frequencies and wavelengths*. That is, long-time, persistent oscillations of correlation functions over ballistic regions of spacetime, which may be of microscopic frequencies and wavelengths, are predicted by a general Euler-hydrodynamic theory that takes the same form as that for smoothed-out correlation functions. This involves a natural extension of notions of conserved quantities and hydrodynamic projection and shows that the Euler hydrodynamic paradigm covers the full frequency-wavelength plane.

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1. Introduction

The dynamics of many-body, extended, isolated, interacting quantum systems has been the focus of much recent theoretical and experimental research [1–3]. Of particular interest is their universal, large-scale behaviours away from equilibrium. Given the panoply of phenomena observed, it is crucial to rigorously establish the physical principles that guide them, such as ergodicity and hydrodynamics. Can we prove that certain forms of ergodicity occur, and that hydrodynamics emerges whereby a large amount of information is lost? Can we characterise the remaining degrees of freedom? For reversible, Hamiltonian dynamics in extended quantum systems, these are some of the deepest questions and remain largely open, forming an important part of Hilbert's sixth problem [4].

On the one hand, ergodicity is the idea that, over long times, the system covers uniformly enough the manifold of states, or at least the part of it that

is dynamically accessible, such as the energy shell. In the language of Gibbs' statistical mechanics, this translates into the equivalence of time averages with ensemble averages. In what situations does this happen in extended quantum systems? Conventional wisdom says that this requires the presence of chaos: the exponential separation of initially nearby trajectories, or, in quantum mechanics, certain conditions on the energies and matrix elements of macroscopic observables [5–7], or on out-of-time-ordered correlators (OTOC) [8–11]. Is there such a link between short-time (chaotic) and long-time (ergodic) behaviours?

On the other hand, hydrodynamics purports that the dynamics of the system reduces to that of a few emergent, slowly-decaying degrees of freedom. One stark realisation of this is the Boltzmann–Gibbs principle [12–14]: in hydrodynamic linear response, this is the idea that correlations due to local perturbations are carried by long-lived modes over which local perturbations “project”, and that propagate at hydrodynamic velocities. Indeed, the dominant correlations between a person's vocal chords and another's eardrum are carried by sound waves. The principle holds quite generally, including in extended Hamiltonian quantum models, and including with integrability [15–17]. What are the conditions for the emergence of such hydrodynamic projections, and what are the hydrodynamic modes?

Next, we discuss the phenomenon of hydrodynamic projections for Euler-scale correlation functions (that is, correlation functions obtained under the ballistic scaling of space and time). Using a form of almost-everywhere ergodicity, we prove hydrodynamic projection theorems in arbitrary dimension. We also explain how such results immediately extend to arbitrary frequencies f and wavenumbers \mathbf{k} (that is, to describe oscillations on ballistic-scale regions, of microscopic-scale frequencies and wavelengths). We give a precise definition of the space of hydrodynamic modes, as the kernel of the (in general f -oscillatory) unitary evolution operator on the Hilbert space of (in general \mathbf{k} -)extensive quantities. In general, this space depends on the state and on the chosen frequency and wavenumber. Again, results apply to all short-range hypercubic quantum lattices, and a large family of states.

Finally, oscillatory hydrodynamic projections give rise to the “oscillatory linearised Euler equation”, and we illustrate in the free-fermionic chain how this reproduces the oscillatory behaviours of two-point functions, with frequencies determined by the microscopic parameters of the model, at large space-time separations. This gives a “proof of principle” for the use of hydrodynamics to address long-time oscillatory behaviours of any frequencies and wavelengths.

From the physics perspective, our main messages are:

1. In contrast to the few-body case, the “right” notion of ergodicity in many-body systems *does not rely on the conventional idea of covering the energy shell*. Rather, in the thermodynamic limit, the fact that timelike averages of local observables become non-fluctuating on large times follows from the interplay between extensivity of the system and locality of the

observables. It happens on timescales which are much smaller than the Poincaré recurrence scale, or that for covering the energy shell. This is many-body ergodicity (related to what is sometimes called “typicality”). Interestingly, despite these physical differences, it is nevertheless the same mathematical principles from the basic ergodicity theory that underpins many-body and few-body ergodicity, including von Neumann’s ergodic theorem.

2. Euler-scale hydrodynamics hold for a wide family of systems, *independently from their specific dynamics* (integrable or chaotic, constrained or not, etc.). General structures, such as hydrodynamic projections, always emerge at large scales, and only specific properties that control the phenomenology are model dependent. The idea that hydrodynamics is a useful concept beyond its conventional field of applications, not requiring chaos, has come to the fore with the recent development of generalised hydrodynamics for integrable systems [18–21], and ties in with the idea that thermodynamic concepts should not rely on short-time behaviours, as emphasised recently in [22]. However, *rigorous* results about many-body dynamics, especially of such generality, are notoriously difficult to obtain.
3. The hydrodynamic principles and structures at the Euler scale (that is, under ballistic scaling of space and time) can be applied to *arbitrary frequencies and wavelengths* (not ballistically scaled). Thus, in D dimensions, there is a (f, \mathbf{k}) -hydrodynamic theory for every frequency $f \in \mathbb{R}$ and wavenumber $\mathbf{k} \in \mathbb{R}^D$. The (f, \mathbf{k}) -hydrodynamic modes control the (f, \mathbf{k}) -oscillatory behaviours of correlation functions on large regions of space-time. For any given (f, \mathbf{k}) , if the set of modes is non-empty, then such (f, \mathbf{k}) -oscillatory behaviours are stable under ballistic scaling of space and time. This means that the spatial \mathbf{k} -Fourier transform of the correlation function presents persistent oscillations of frequency f (has divergent temporal f -Fourier transform); and that at $\mathbf{k} + \delta\mathbf{k}$ with ballistic scaling $\delta\mathbf{k} = \boldsymbol{\kappa}/t$, the temporal f -average on large time intervals takes a universal form as a function of $\boldsymbol{\kappa}$ determined by (f, \mathbf{k}) -hydrodynamics. With the $\delta\mathbf{k}$ shift, what we are doing is essentially an oscillatory average on ballistic regions of spacetime; the hydrodynamic theory will then predict along which velocities nontrivial correlations (under such averaging) will occur, and their weights. See Fig. 1 for a pictorial representation. This goes beyond well-known effects such as the robustness of the sound dispersion relation at nonzero wavenumber. Effectively, this describes the singularity structures of correlation functions in the full frequency-wavenumber plane. Our construction shows that recent ideas on dynamical symmetries [23–27] can be extended to the hydrodynamic regime of correlation functions. In fact, the proofs we provide make it clear that the extension goes beyond simple phase oscillations, and include oscillations involving nontrivial internal transformations, such as spin rotations. This in principle allows hydrodynamics to be extended to describe large-scale

behaviours involving nontrivial microscopic internal structures, such as helicoidal spin structures.

From a more philosophical perspective, an important message is that *it is extremely fruitful to extract the relevant many-body physics out of equilibrium by analysing the operator algebra of local observables already in the thermodynamic limit*. This is to be contrasted with methods based, in quantum systems, on how the spectrum of large Hamiltonians behave as the thermodynamic limit is taken, or, in classical models, on kinetic equations. The operator algebra framework allows one to directly concentrate on the small part of the Hilbert space (or phase space in classical mechanics) that is significant in the thermodynamic limit. In particular, we emphasise how kernels of naturally defined unitary operators on Hilbert spaces of observables encode the relevant aspects of large-scale physics.

We believe most of these notions are valid in a large class of many-body systems, quantum or classical. In order to be precise, however, we will concentrate on translation-invariant, infinitely-extended D -dimensional hypercubic quantum lattices with finite local spaces and short-range interactions (interaction strength decaying at least exponentially with distance).

The paper is organised as follows. First, in Sect. 2, we discuss notions of ergodicity, and in particular many-body ergodicity, and “almost-everywhere ergodicity” established recently [28] within the context of the C^* algebraic formulation of quantum statistical mechanics. We contrast this with von Neumann’s ergodic theorem, and discuss application to the problem of return to equilibrium after local quenches. We then prove rigorously, in Sect. 3, the principle of hydrodynamic projection onto conserved modes in the Euler scaling limit, within the same context. This generalises the result of [29] to any dimension, although the proof requires the introduction of quite a few new ideas. Crucially, mathematical formalism makes it clear that the general principle can take into account oscillatory behaviours at any frequency and wavelength. We give an accurate definition of the full, complete space of hydrodynamic modes, which in general depends on the state, and on the frequency and wavelength. Finally, in Sect. 4 we review the linearised (oscillatory) Euler hydrodynamic equations established in [29] in one dimension and discuss how they recover oscillatory behaviours in free fermion models. The structures uncovered are pictorially illustrated in Fig. 1.

2. Ergodicity, Many-Body Ergodicity, Almost-Everywhere Ergodicity

Ergodicity plays a fundamental role in physics. In this section, we pedagogically review some concepts of ergodicity in many-body systems, one of the goals being to put in context some recent results by the authors. We first review the basic rigorous ergodicity result in classical mechanics from Birkhoff’s ergodic theorem. We then review von Neumann’s fundamental work on the quantum ergodic theorem, recasting the nice discussion in [6] into the language of local

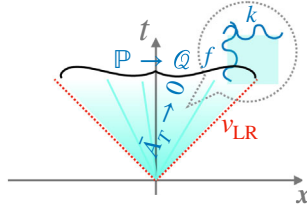


FIGURE 1. A pictorial representation of the mathematical results of [28], and of the hydrodynamic projection results of Sect. 3. Within the Lieb–Robinson cone $|\mathbf{x}| < v_{LR}t$, very little was known until now. We find that ray-averages \bar{A}_T of observables up to time T , with respect to every frequency f and wavelength k and at every velocity v , vanish at large times $T \rightarrow \infty$ (represented by the colour-gradient shaded region), except perhaps for a set of measure zero (represented by solid ray segments). Observables become thin as time evolve. The large-scale information that remains, after oscillatory fluid-cell means, is their projection onto a smaller space of degrees of freedom $\mathcal{Q}_{\omega,k}$, the slowly decaying hydrodynamic modes

observables in quantum spin lattices. We observe that the eigenstate thermalisation hypothesis (ETH) is in fact closely related to the conditions of von Neumann’s quantum ergodic theorem. Both the von Neumann quantum ergodic theorem and the ETH rely heavily on the structure of the spectrum, and how it behaves in the thermodynamic limit. Lastly, we avoid such difficulties by passing to the C^* -algebra formalism, in which context we discuss what many-body ergodicity may look like, and review the recent results [28], where almost-everywhere ergodicity is proven for every quantum spin model on (hyper-)cubic lattices with short-range interactions; we discuss some applications and relations with the problem of return to equilibrium.

We denote throughout this section the time average as

$$\overline{f(t)} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt f(t). \quad (1)$$

2.1. Ergodicity in Classical Mechanics

In the classical mechanics of interacting particles, the basic notion of ergodicity is the equivalence between time average and “ensemble average”, or average over the uniform measure on the energy shell.

Consider a system of N particles, with phase-space coordinates $(\mathbf{x}_n, \mathbf{p}_n) \in \mathbb{R}^{2D} : n = 1, \dots, N$ and time evolution generated by a Hamiltonian $H(\mathbf{x}, \mathbf{p})$ (we denote $(\mathbf{x}, \mathbf{p}) = ((\mathbf{x}_n, \mathbf{p}_n) : n = 1, \dots, N) \in \mathbb{R}^{2DN}$). The microcanonical ensemble average of an observable $A(\mathbf{x}, \mathbf{p})$ on the shell $\Omega_E = \{(\mathbf{x}, \mathbf{p}) : H(\mathbf{x}, \mathbf{p}) = E\}$ of energy E can be defined by a limit on the thickened shell

$S_{E_-,E_+} = \{(\mathbf{x}, \mathbf{p}) : E_- < H(\mathbf{x}, \mathbf{p}) < E_+\}$ as

$$\langle A \rangle_E := \lim_{E_-, E_+ \rightarrow E} \frac{1}{\text{vol } S_{E_-, E_+}} \int_{S_{E_-, E_+}} d^{DN}x d^{DN}p A(\mathbf{x}, \mathbf{p}) \quad (2)$$

where

$$\text{vol } S_{E_-, E_+} = \int_{S_{E_-, E_+}} d^{DN}x d^{DN}p. \quad (3)$$

Note that this naturally induces a measure μ_E on Ω_E .

Now consider points $(\mathbf{x}, \mathbf{p}) \in \Omega_E$. The ergodicity statement is that, if there is no nontrivial constant of the motion,¹ then, for every observable A , every $E \in \text{im } H$, and almost every $(\mathbf{x}, \mathbf{p}) \in \Omega_E$,

$$\overline{A(\mathbf{x}(t), \mathbf{p}(t))} = \langle A \rangle_E \quad (4)$$

where $(\mathbf{x}(t), \mathbf{p}(t))$ is the point (\mathbf{x}, \mathbf{p}) time-evolved by time t . This is equivalent to a more intuitive statement: the fraction of time that a trajectory $t \mapsto (\mathbf{x}(t), \mathbf{p}(t))$ spends in a neighbourhood $\Gamma \subset \Omega_E$ of a point $(\mathbf{x}', \mathbf{p}') \in \Gamma$ is, over long times, proportional to its “surface area” $\mu_E(\Gamma)$ on the energy shell, for almost every initial point (\mathbf{x}, \mathbf{p}) and for every $(\mathbf{x}', \mathbf{p}')$. In particular, over long times, the trajectory comes as close as desired to the point $(\mathbf{x}', \mathbf{p}')$.

The above ergodicity statement for classical mechanics follows from Birkhoff’s ergodic theorem [30]. Of course, proving ergodicity for any given system is difficult—the difficulty being in proving the absence of nontrivial constants of the motion. See also the discussion in [31].

2.2. Von Neumann’s Quantum Ergodic Theorem

It is natural to ask about the equivalent ergodicity notion for quantum systems. A problem arises: in quantum mechanics, phase space is essentially “discretised” as $dx dp \simeq \hbar/2$. Now we cannot ask a more precise question than if a trajectory will cross a neighbourhood of volume $\hbar/2$ of a point. Requiring that this happens for every point (in some precise way that can be worked out, see below) is a much weaker condition, that of “quantum recurrence”. Of course, at large values of the classical action $S \gg \hbar$, this quantisation is not seen, and one recovers classical mechanics. The above classical ergodicity notion, with the more stringent condition on the measure covered by trajectories, is then a good approximation. But how is this recovered directly from quantum mechanics? Large classical actions naturally occur in macroscopic systems. Can we formulate ergodicity for macroscopic quantum systems, and find the quantum properties that guarantee it? In particular, timescales for quantum recurrence are prohibitively large, while those for ergodicity of the corresponding classical system at $S \gg \hbar$ are short; how to account for this?

¹A nontrivial constant of the motion is a measurable function Q on phase space (an observable) that is invariant under the dynamics (constant of the motion), and that breaks the energy shells (nontrivial) in that for at least one E , there exist two non-intersecting subsets $\Gamma, \Gamma' \subset \Omega_E$ of nonzero μ_E -measure such that $Q(\Gamma) \cap Q(\Gamma') = \emptyset$.

Von Neumann’s idea is to look at “macroscopic observables”. Intuitively, such observables should behave like classical ones, but one asks what the microscopic quantum evolution says about these observables, and in particular their ergodicity properties. Perhaps the original idea was to consider a classical object (such as a tennis ball) as a macroscopic combination of quantum particles, and macroscopic observables for such objects. But in fact, it turns out that the abstract formulation applies quite generally. Making connections with modern developments, we will recast von Neumann’s line of thought to the context of many-body quantum systems with short-range interactions: what about ergodicity in large (macroscopic) quantum spin lattices?

Take a finite spin lattice: a quantum system with N -dimensional local spaces lying on $\Lambda_L := [-L, L]^D \cap \mathbb{Z}^D$, and Hilbert space $\mathcal{H} = (\mathbb{C}^N)^{\otimes \Lambda_L}$. Consider the Hamiltonian

$$H = \sum_{\mathbf{x} \in \Lambda_L} h_{\mathbf{x}} \quad (5)$$

where $h_{\mathbf{x}}$ is an operator supported on a finite, \mathbf{x} -independent number of sites near to \mathbf{x} . For simplicity we take the system periodic and translation invariant: there is a representation $\mathbf{y} \mapsto \iota_{\mathbf{y}}$ of the group $(\mathbb{Z}/(2L+1)\mathbb{Z})^D$ on the group of automorphisms of the operator algebra $\text{Aut}(\text{End } \mathcal{H})$, such that $\iota_{\mathbf{y}} h_{\mathbf{x}} = h_{\mathbf{x}+\mathbf{y}}$ (and $\iota_{\mathbf{y}} A$ is supported on the \mathbf{y} -translate of the support of A). The Hamiltonian has spectrum of eigenvalues $\{E_n\}$ and normalised eigenvectors $|n\rangle$.

Consider a microcanonical shell $S = \{n : |\Lambda_L|e_- < E_n < |\Lambda_L|e_+\}$, keeping the energy densities e_-, e_+ implicit for readability. The microcanonical ensemble is represented by the density matrix

$$\rho_{\text{mc}} = \frac{1}{|S|} \sum_{n \in S} |n\rangle\langle n|. \quad (6)$$

It can be shown that [32, 33], in the limit of large volume $L \rightarrow \infty$, followed by the limit of small shell $e_- \rightarrow e_+ \rightarrow e$, averages of local observables A (operators supported on finite numbers of sites) in this ensemble tend to averages in the Gibbs ensemble,

$$\lim_{e_-, e_+ \rightarrow e} \lim_{L \rightarrow \infty} \text{Tr } \rho_{\text{mc}} A = \lim_{L \rightarrow \infty} \text{Tr } \rho A, \quad \rho = \frac{e^{-\beta H}}{\text{Tr } e^{-\beta H}} \quad (7)$$

where β is chosen such that $\text{Tr } \rho h_{\mathbf{0}} = e$.

A number of ergodicity-like results can be obtained depending on the strength of the assumptions on the underlying system.

2.2.1. Assumption of Non-degeneracy and Quantum Recurrence. Before discussing von Neumann’s QET, let us first consider the simplest ideas of ergodicity, keeping L finite and the shell of finite thickness. Let us assume that $E_n \neq E_m$ if $n \neq m$. Take a state in the microcanonical shell,

$$|\psi\rangle = \sum_{n \in S} c_n |n\rangle. \quad (8)$$

We note that the time average of an operator A is simply expressed as a linear combinations on projections $P_n = |n\rangle\langle n|$:

$$\langle m|\overline{A(t)}|n\rangle = \overline{e^{i(E_m - E_n)t}} \langle m|A|n\rangle = \delta_{m,n} \langle n|A|n\rangle \quad \Rightarrow \quad \overline{A(t)} = \sum_n \langle n|A|n\rangle P_n. \quad (9)$$

From this we observe, on the one hand, that time averages in a generic state $|\psi\rangle$ of the microcanonical shell gives rise to the “diagonal ensemble”, the ensemble of density matrices that are diagonal in the energy eigenbasis:

$$\overline{\langle \psi(t)|A|\psi(t)\rangle} = \sum_{n \in S} |c_n|^2 \langle n|A|n\rangle. \quad (10)$$

The equivalent of the classical uniform shell-averaging is the uniform sum over n 's ($\text{Tr } \rho_{\text{mc}} A$), but here, the sum is not uniform, as the c_n 's may behave wildly as functions of n . Hence, this is certainly not ergodicity, even in the limit of large L and thin shell!

On the other hand, as an immediate consequence of (9), in any stationary state $|n\rangle$, the observable A is non-fluctuating:

$$\langle n|\overline{A(t)}^2|n\rangle = (\langle n|A|n\rangle)^2. \quad (11)$$

This is indeed a (weaker-looking) statement of ergodicity: “mean-square ergodicity”. The distribution of the random variable $\overline{A(t)}$ in the state $|n\rangle$ is supported on $\langle n|A|n\rangle$. But does this have a connection with the classical notion of ergodicity?

Indeed, *at the technical level*, mean-square ergodicity fully parallels the classical notion. In this parallel, first we consider the condition of non-degeneracy, which is equivalent to the condition that the space of all operators invariant under time evolution is spanned by the projections $P_m = |m\rangle\langle m|$. This parallels the classical condition that there be no nontrivial constants of the motion². Second, using this condition we see, in (9), that long-time averages of observables project onto $\text{span}(P_m)$, the subspace of all time-invariant operators. This parallels the result of Birkhoff's theorem for classical phase-space functions (under the condition of the absence of nontrivial constants of the motion). In fact, in evaluating time averages, instead of the direct calculation made in (9), one may use von Neumann's ergodic theorem from functional analysis [34, Theorem II.11] (not to be confused with von Neumann's quantum

²In this respect, it is sometimes stated in the literature that integrability—in the sense of the existence of large enough amount of nontrivial constants of the motion—for finite quantum spin lattices does not make sense, because there always are as many projections P_m , which commute with H , as the dimension of the Hilbert space $\dim \mathcal{H}$. This is misleading, as a similar statement could be made for classical systems: functions of the Hamiltonian are constants of the motion. But they are not “nontrivial”: they do not divide the energy shells. Similarly, P_m are not nontrivial constants of the motion. Only with degeneracies can new, nontrivial constants of the motion appear: the space of operators is of dimension $(\dim \mathcal{H})^2$, and integrability typically leads to enough degeneracies so that there are $\propto \log(\dim \mathcal{H}) \dim \mathcal{H}$ linearly independent projections commuting with H ; this is associated to the presence of a large enough amount of extensive conserved quantities.

ergodic theorem!). This quite directly parallels the use of Birkhoff's theorem from measure theory in the classical case. Indeed, von Neumann's ergodic theorem implies that, if U_t is a unitary one-parameter group on a Hilbert space, then

$$\overline{U_t} = \mathbb{P}_1 \quad (12)$$

where \mathbb{P}_1 the projector onto its unit-eigenvalue subspace. Then, we simply consider time evolution as a unitary operator acting on matrices (say under the Hilbert-Schmidt inner product), and its long-time average projects onto $\text{span}(P_m)$, the time-invariant operators. Finally, to complete the parallel, we evaluate the result within a single state $|n\rangle$. This parallels the taking of a fixed classical energy shell; we fix the state $|n\rangle$, because the limit of an infinitesimally thin shell, for fixed L , is just a single state. As $|n\rangle$ is stationary, the statement that $\overline{A(t)}$ be non-fluctuating and supported on $\langle n|A|n\rangle$, is the only natural ergodicity statement, and it is indeed the one we obtain.

But this parallel is essentially technical. The above simple statements, in fact, only relate to quantum recurrence, not many-body ergodicity. Indeed, the order of limits is simply incorrect for many-body systems: the limit in time $T \rightarrow \infty$, in the time averaging (1), is taken before the limit of large volumes $L \rightarrow \infty$ (which is in fact never taken in the mean-square ergodicity statement). This is thus valid, in realistic many-body systems, only at astronomically large times! As we will see in Sect. 2.3, however, with the right space of operators, a similar technical parallel can be made which becomes fully meaningful.

2.2.2. Assumptions on Maximal Deviations and von Neumann's QET. Assume that $\max_{n \in S} \left(\langle n|A|n\rangle - \text{Tr} \rho_{\text{mc}} A \right)^2$ is "small".

A natural requirement would be that it is uniformly bounded in L (thus stays finite in the large volume limit), and this bound tends to zero in the limit of thin microcanonical shell. Of course, this cannot be expected to be valid for all observables A . But, at least for *local observables*, this should hold (certainly uniform boundedness in L is immediate by finiteness of the operator norm). A clear example is the energy density h_x . The states $|n\rangle$ may be assumed to diagonalise the translation operator, and therefore,

$$\langle n|h_x|n\rangle = \frac{1}{|\Lambda_L|} \sum_{x' \in \Lambda_L} \langle n|h_{x+x'}|n\rangle = \frac{1}{|\Lambda_L|} \langle n|H|n\rangle = \frac{E_n}{|\Lambda_L|}. \quad (13)$$

Thus,

$$e_- \leq \text{Tr} \rho_{\text{mc}} h_x \leq e_+ \quad (14)$$

and $\max_{n \in S} \left(\langle n|h_x|n\rangle - \text{Tr} \rho_{\text{mc}} h_x \right)^2 \leq (e_+ - e_-)^2$. The requirement is that such a smallness bound holds not just for the energy density, but for all (or at least for a large family of) local observables.

Then, as $\sum_n |c_n|^2 = 1$, we get from (10) that

$$\overline{\langle \psi(t)|A|\psi(t)\rangle} \text{ is near to } \text{Tr} \rho_{\text{mc}} A. \quad (15)$$

This looks much more like an ergodicity statement.

Making connection with more modern ideas, we note that the above assumption is in fact equivalent to a part of the eigenstate thermalisation hypothesis (ETH) [2]: that in the thermodynamic limit $L \rightarrow \infty$, averages of local observables in energy eigenstates vary continuously with the energy density.

One can get further with one more assumption:

Assume that $\max_{n,m} |\langle n|A|m\rangle|^2$ is small, and that the energy differences $E_m - E_n$ are non-degenerate (this implies, but is stronger than, non-degeneracy of energies).

Then, von Neumann proved that $\langle \psi(t)|A|\psi(t)\rangle$ has small time variance,

$$\overline{\left(\langle \psi(t)|A|\psi(t)\rangle - \text{Tr } \rho_{\text{mc}} A\right)^2} \text{ is small.} \quad (16)$$

This is (a loose formulation of) *von Neumann's quantum ergodic theorem*, proved in [5] (see [35] for English translation).

The assumption that $\max_{n,m} |\langle n|A|m\rangle|^2$ be small is implied by the full formulation of the ETH. Indeed, in the ETH, one equates off-diagonal matrix elements of local operators with an entropy factor that vanishes exponentially with the system size, times matrix elements from random matrix theory. Importantly, the assumption that $\max_{n,m} |\langle n|A|m\rangle|^2$ be small is in fact weaker. The energy-difference non-degeneracy condition would be paralleled by the assumed random-matrix distribution of eigenvalues in quantum chaos; although the full connection is not obvious.

One sees immediately two potential drawbacks from von Neumann's ergodic theorem:

- The assumptions on maximal deviations from the microcanonical averages are somewhat contrived; it is not clear, given a short-range interaction on a quantum lattice, how to show their validity. (The same can be said *a fortiori* of the ETH.) Is there something more universal that can be said? Are these assumptions really necessary to understand the structure of long-time dynamics in many-body quantum systems?
- Non-degeneracy assumptions are natural, but they are seemingly unrelated to the thermodynamic limit, where a continuum of states exists. Can something be said when there are degeneracies, most interestingly in the presence of a large number of nontrivial conserved quantities? Can this be formulated directly in the thermodynamic limit?

Remark 2.1. In von Neumann's work, instead of local observables, it is "macroscopic observables" that are the main focus. One should consider a family of macroscopic observables M_i , with the property that they almost mutually commute, $[M_i, M_j] \approx 0$, and thus can be almost simultaneously diagonalised. The associated subspaces correspond to different "macrostates", intuitively different cells in the classical phase space. Von Neumann's theory can be formulated in terms of projections on these subspaces, and conditions on these. One may take spatial averages of local observables, $\hat{A} := |\Lambda_L|^{-1} \sum_{\mathbf{x} \in \Lambda_L} A(\mathbf{x})$, which indeed have vanishing commutators, $||[\hat{A}, \hat{B}]|| \sim |\Lambda_L|^{-1}$. We find that

it is more transparent to forgo the abstract notion of macroscopic observables and their associated eigenspaces, and instead concentrate on local observables and their matrix elements.

2.3. Many-Body and Almost-Everywhere Ergodicity Theorems

Recall that in von Neumann’s quantum setup, we considered the “macroscopic”, or many-body, limit $L \rightarrow \infty$ in order to say something about ergodicity.

It is indeed commonly believed that, in typical interacting many-body systems and states $\langle \dots \rangle$ of physical importance, auto-correlation functions of local observables in the thermodynamic limit vanish at long times, $\langle A(\mathbf{0}, t)B(\mathbf{0}, 0) \rangle - \langle A \rangle \langle B \rangle \rightarrow 0$ ($t \rightarrow \infty$). This is the notion of mixing³: the effect of the small localised perturbation $B(\mathbf{0}, 0)$ mixes with its surrounding and, locally, decays in time. A weaker notion, implied by mixing, is that of ergodicity: $T^{-1} \int_0^T dt \langle A(\mathbf{0}, t)B(\mathbf{0}, 0) \rangle \rightarrow \langle A \rangle \langle B \rangle$ ($T \rightarrow \infty$). This is equivalent to mean-square ergodicity: the vanishing of the variance of the observable

$$\bar{A}_T := T^{-1} \int_0^T dt A(0, t) \quad (17)$$

at long times, $\langle \bar{A}_T^2 \rangle - \langle A \rangle^2 \rightarrow 0$. As explained above, mean-square ergodicity says that infinite-time averages of observables, in the state $\langle \dots \rangle$, do not fluctuate, and are simply equal to ensemble averages.

But an important observation, which we believe was mostly overlooked in the literature, is that the resulting question of ergodicity, in fact, has less to do with quantum mechanics, and much more with the **emergence of large-scale behaviours in many-body systems**. This is about the passage from microscopic laws to macroscopic dynamics. As we mentioned, in many-body quantum systems, **quantum recurrence**, as discussed in paragraph 2.2.1, is **irrelevant to this question**. But also, in many-body classical systems, **the uniform covering of the energy shell**, as discussed in Sect. 2.1, is **also irrelevant to this question**. In both cases, timescales are too large. The question of emergent ergodicity in many-body systems is of a different nature, and it brings out similar concepts both in quantum and classical many-body systems; “quantumness” does not play a fundamental role.

In this subsection, we take the C^* algebra viewpoint on statistical mechanics. This, we argue, is much better adapted to the thermodynamic limit and the question of many-body ergodicity. We discuss this question and in particular review results on almost-everywhere ergodicity established recently by the authors, which are extremely general and free from hard-to-check conditions. For clarity we still concentrate on quantum spin lattices; however, this viewpoint is much more easily adapted to other many-body systems, with only small differences between the quantum and classical cases.

³In quantum statistical mechanics, “mixing” is sometimes used to describe what we will refer to as spatial mixing, Eq. (18) below, and likewise “ergodicity” is used to describe a property related to spatial integrals. Here we reserve these words to the more physical notions involving time.

Interestingly, as we will see, the basic mathematical ideas behind quantum recurrence (especially as “mean-square ergodicity”) and classical ergodicity are still the right ones for many-body ergodicity, however only after **the space of observables has been reduced to that relevant for local physics in the thermodynamic limit**. It is the reduction to this space that allows these ideas to extract the correct form of ergodicity seen in many-body systems, associated to times that do not grow with the system size.

We note that the notions of ergodicity discussed in this subsection are not to be confused with what is sometimes called the “ergodic principle” (see, for example, [36], where a version of it is referred to as a hypothetical “strong ergodic theorem”). This principle stipulates that relaxation must lead to a state of the Gibbs form. What we discuss here is about correlation functions of local observables, and is associated to conserved quantities that are “localised”: time-invariant local observables. By contrast, thermalisation, the ergodic principle, and the Gibbs form are concepts related to extensive conserved quantities. Extensive conserved quantities are involved in the hydrodynamic projection theorem proved in Sect. 3, and in the thermalisation theorems shown in [37], where it is proposed that the Hilbert space of extensive conserved quantities is in fact the tangent space to the manifold of maximal entropy states. Localised and extensive conserved quantities span the kernels of different evolution operators, acting on different spaces of observables (local observable, and extensive observables, respectively).

We believe that analysing the structure of local observables in thermodynamic states is much more directly relevant to local physics than analysing the structure of spectra and how they behave as $L \rightarrow \infty$. But, perhaps most importantly, in the thermodynamic limit, more can be said, giving almost-everywhere ergodicity, which ultimately is at the basis of the emergence of hydrodynamic structures.

2.3.1. C^* Algebra Formulation and Basic Results. The strength of the C^* algebra formulation is that we can directly study dynamics in the thermodynamic limit: the limit $L \rightarrow \infty$ is already taken, and we concentrate on the remaining, relevant degrees of freedom. See Appendix A for a more detailed overview of the mathematical setup.

We thus consider a quantum spin model on an infinite lattice, with local spaces $\simeq \mathbb{C}^N$ on \mathbb{Z}^D ($N, D \geq 1$). The Hamiltonian is homogeneous (space-translation invariant) and with short-range interaction (exponentially decaying or faster, see Appendix A for the precise formulation); this generalises slightly the form (5), which was of finite-range interaction. Observables form a C^* algebra \mathfrak{U} , the norm-completion of the algebra $\mathfrak{U}_{\text{loc}}$ of local observables (supported on finite numbers of sites). As above, we denote $A(\mathbf{x}, t)$, $B(\mathbf{x}, t), \dots$ observables $A \in \mathfrak{U}$ translated to space-time points \mathbf{x}, t , and sometimes omit the time argument if $t = 0$. More precisely:

Definition 2.2 (*Dynamical System*). A dynamical system of a quantum spin lattice is a triple $(\mathfrak{U}, \iota, \tau)$ where \mathfrak{U} is a quasi-local C^* -algebra (as constructed in Appendix A), τ is a strongly continuous representation of the group \mathbb{R} by

*-automorphisms $\{\tau_t : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}\}_{t \in \mathbb{R}}$, and ι is a representation of the translation group \mathbb{Z}^D by *-automorphisms $\{\iota_{\mathbf{x}} : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}\}_{\mathbf{x} \in \mathbb{Z}^D}$, such that for any $\Lambda \in P_f(\mathbb{Z}^D)$: $A \in \mathfrak{U}_\Lambda \implies \iota_{\mathbf{x}}(A) \in \mathfrak{U}_{\Lambda+\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{Z}^D$. We further assume that τ is such that $\tau_t \iota_{\mathbf{x}} = \iota_{\mathbf{x}} \tau_t$, $\forall t \in \mathbb{R}$, $\mathbf{x} \in \mathbb{Z}^D$, i.e. time evolution is homogeneous. We use the notation $A(\mathbf{x}, t) := \iota_{\mathbf{x}} \tau_t A$.

A state⁴ ω is a continuous, positive linear functional on \mathfrak{U} , interpreted as giving ensemble averages, normalised to $\omega(\mathbf{1}) = 1$. It is assumed homogeneous and stationary, $\omega(A(\mathbf{x}, t)) = \omega(A) \forall \mathbf{x} \in \mathbb{Z}^D, t \in \mathbb{R}$.

Most importantly, the state is assumed to satisfy a property which we will simply refer to as “**spatial mixing**”. We consider two different implementations: “factoriality”, and a uniform-enough spatial mixing, see “Appendix A”. Both include, in particular, the clustering of correlations at large separations,

$$\lim_{x \rightarrow \infty} \omega(A(\mathbf{x})B) = \omega(A)\omega(B), \quad A, B \in \mathfrak{U}. \quad (18)$$

Results reviewed below on almost-everywhere ergodicity apply either if the state is factor [38, p. 81, Def. 2.4.8], or uniformly-enough spatially mixing (or both).

Any thermal state that is in a single thermodynamic phase (i.e. Kubo–Martin–Schwinger (KMS) state satisfying (18), see below) is an example of a state with all above properties: for $D = 1$ this is every thermal state at nonzero temperature as there are no thermal phase transitions, for $D > 1$ this includes every thermal state above a certain model-dependent temperature [39, Section 5.3].

Fundamental theorems have been established that guarantee that the infinite-volume limit of finite-volume thermal states exist and gives a state ω on \mathfrak{U} ; this is so at least at large enough temperatures if $D > 1$ (see, for example, [39, Section 5.3]), and at all temperatures if $D = 1$ [40]. For instance, for H of the form (5), there exists β_c , with $\beta_c = \infty$ if $D = 1$, such that for all $\beta < \beta_c$ and for all local observables $A \in \mathfrak{U}_{\text{loc}}$, the limit

$$\lim_{L \rightarrow \infty} \frac{\text{Tr} \left(e^{-\beta H} A \right)}{\text{Tr} \left(e^{-\beta H} \right)} = \omega(A) \quad (19)$$

exists and gives rise to a positive, continuous (with respect to the topology induced by the operator norm), normalised linear functional, that is invariant under space-time translations,

$$\omega(A^\dagger A) \geq 0, \quad |\omega(A)| \leq \|A\|, \quad \omega(\mathbf{1}) = 1, \quad \omega(A(\mathbf{x}, t)) = \omega(A). \quad (20)$$

The result is extended by continuity to \mathfrak{U} .

The thermodynamic limit point in Eq. (19) can be shown to be what’s called a KMS state [39, Proposition 6.2.15], which is defined as follows:

⁴Here we use the C^* algebra notation $\omega(\dots)$ for states, instead of the notation $\langle \dots \rangle$ from statistical physics or quantum mechanics.

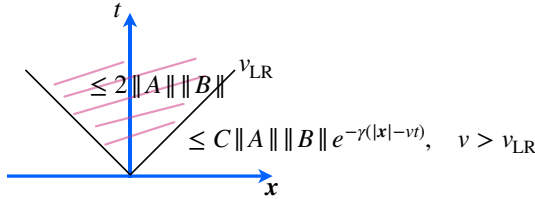


FIGURE 2. A pictorial representation of the basic bounds on correlation functions in space-time. The connected correlation function $\omega(A(\mathbf{x}, t)B) - \omega(A)\omega(B)$ is bounded by the operator norm in general, and by the sharper Lieb–Robinson bound outside of the Lieb–Robinson light-cone (here the latter is expressed more precisely than (22) by including the operator norm)

Definition 2.3 (*KMS state*). Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$, Definition (2.2) (see also Appendix A). A state ω is called a (τ, β) -KMS state, at inverse temperature β , if

$$\omega(A\tau_{i\beta}B) = \omega(BA) \tag{21}$$

for all A, B in a dense (with respect to the norm topology) τ -invariant *-subalgebra of \mathfrak{U}_τ , where \mathfrak{U}_τ is the set of entire analytic elements for τ .

At $D > 1$, there is a unique KMS state at high enough temperatures [39, Proposition 6.2.45], while at $D=1$ the KMS state is unique at all temperatures [40]. This unique state is factor, as an immediate consequence of [39, Theorem 5.3.30]. By the result of existence of space invariant KMS states [39, p. 296], this unique state will also be space invariant. Note that time invariance is an immediate consequence of the KMS condition, Eq. (21).

Further, it is guaranteed, again for $\beta < \beta_c$, that the state ω from Eq. (19) is exponentially clustering: there exists $\gamma > 0$ such that for every local $A, B \in \mathfrak{U}_{loc}$, there is $c > 0$ such that

$$|\omega(AB) - \omega(A)\omega(B)| \leq ce^{-\gamma \text{dist}(A,B)} \tag{22}$$

(the distance can be taken as the L_1 or L_2 distance—as chosen for convenience—on the square lattice between the supports of A and B). In particular, ω satisfies the spatial mixing condition. More generally, any Kubo–Martin–Schwinger (KMS) state satisfying (18) satisfies also (20) and the spatial mixing condition.

Finally, by the Lieb–Robinson bound [41], this exponential decay can be extended in space-time [28], to the full region outside of a Lieb–Robinson “light-cone”: there exists $v_{LR} > 0$ and $\gamma > 0$ such that for every local A, B and $v > v_{LR}$, there is $c > 0$ such that

$$|\omega(A(t)B) - \omega(A)\omega(B)| \leq ce^{-\gamma(\text{dist}(A,B)-vt)}. \tag{23}$$

The structure in space-time is depicted in Fig. 2.

We note that in the C^* algebra formulation, the concept of microcanonical shell, Eq. (6), does not make immediate sense, especially in view of the ensemble equivalence (7). Nevertheless, for the purposes of interpretation of the results, it will be convenient to consider any state ψ that is “near to” ω as being part of the microcanonical shell in the thermodynamic limit, $\psi \in \mathfrak{S}_{\text{micro}}$. Here, “near to” simply means “absolutely continuous” with respect to ω , that is, of the form

$$\psi \in \mathfrak{S}_{\text{micro}} \quad : \quad \psi(A) = \frac{\omega(B^\dagger AB)}{\omega(B^\dagger B)} \quad \forall A \in \mathfrak{U} \quad \text{for some } B \in \mathfrak{U}. \quad (24)$$

These states are still positive, continuous, normalised linear functionals, but not space-time translation invariant (which is of course as expected for states in the microcanonical shell). Further, by spatial mixing, the density of any extensive observable in ψ (averaged over space) is the same as it is in ω :

$$\lim_{L \rightarrow \infty} \frac{1}{|\Lambda_L|} \sum_{\mathbf{x} \in \Lambda_L} \psi(A(\mathbf{x})) = \omega(A). \quad (25)$$

In particular, the energy density is the same in ψ as in ω .

We justify referring to this as a “microcanonical shell”, by the idea that the spatially-averaged expectations of local conserved densities are the natural counterpart, in infinite volumes, to the total conserved quantities used to define the microcanonical shell in finite-volume settings. In the finite-volume quantum setting, one must take a shell of values of conserved densities, as the space should be spanned by a large number of states (proportional to the volume $|\Lambda_L|$, see Eq. (6)). Here, as the volume has been sent to infinity, the shell may be taken infinitesimally thin, and the number of states contained still may be infinitely large. In our choice of microcanonical shell, taking the shell infinitesimally thin is translated into concentrating on states that are absolutely continuous with respect to ω . These are the locally perturbed state $\psi \in \mathfrak{S}_{\text{micro}}$ defined above. Such a state ψ indeed has the same spatially-averaged expectations of densities of all extensive conserved quantities (in fact, of all extensive observables, conserved or not) as those of ω , and thus ψ is in the microcanonical shell of ω . With respect to the results that we will establish, we emphasise that the problem of the long-time limit from such a locally perturbed state is that of “return to equilibrium”, which is fundamentally different from that of “thermalisation” or “approach to equilibrium from nonequilibrium”. See the Remarks below.

Finally, in the theory of C^* algebras, a very important role is played by the Gelfand–Naimark–Segal (GNS) construction. In this construction, one associates to any state ω a Hilbert space \mathcal{H}_ω , constructed from the observables in \mathfrak{U} , with in particular the vector $|\Omega\rangle$ associated with the identity operator $\mathbf{1}$. This Hilbert space is such that $\omega(A) = \langle \Omega | A | \Omega \rangle$, and more generally $\omega(B^* AC) = \langle \Psi_B | A | \Psi_C \rangle$ [38, Chapter 2.3] where $|\Psi_B\rangle = B|\Omega\rangle$ (here we use the same notation for $A \in \mathfrak{U}$ and its representation as a bounded operator on \mathcal{H}_ω). More precisely, the Hilbert space is that arising from the completion of $\mathfrak{U}/\mathfrak{N}$ with respect to the inner product induced by the sesquilinear form

$(A, B) \mapsto \omega(A^*B)$ on the equivalence classes $A + \mathfrak{N} \in \mathfrak{U}/\mathfrak{N}$, with null space \mathfrak{N} (satisfying $\omega(B^*\mathfrak{N}) = 0 \forall B \in \mathfrak{U}$). Operators acting on \mathcal{H}_ω are sometimes called “superoperators” in the condensed matter literature. For ω being the infinite-temperature state or trace state (see above), the resulting norm is in fact (the infinite-dimensional completion of) the Hilbert–Schmidt norm, but the GNS construction is more general. Physically, in thermal states, \mathcal{H}_ω is the space of particle and hole excitations above the thermal “vacuum”.

Space and time translations can be extended to unitary group actions on the GNS representation, see also [38, Theorem 2.3.16, Corollary 2.3.17]. Here we express the related theorem.

Theorem 2.1 (GNS representation). *Given a state $\omega \in E_{\mathfrak{U}}$ of a unital C^* algebra \mathfrak{U} there exists a (unique, up to unitary equivalence) triple $(H_\omega, \pi_\omega, \Omega_\omega)$ where H_ω is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, π_ω is a representation of the C^* algebra by bounded operators acting on H_ω and Ω_ω is a cyclic vector for π_ω , i.e. the $\text{span}\{\pi_\omega(A)\Omega_\omega : A \in \mathfrak{U}\}$ is dense in H_ω , such that*

$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A)\Omega_\omega \rangle, \quad A \in \mathfrak{U}. \tag{26}$$

If additionally we have a group G of automorphisms $\{\tau_g\}_{g \in G}$ of \mathfrak{U} and ω is τ -invariant, then there exists a representation of G by unitary operators $U_\omega(g)$ acting on H_ω . This representation is uniquely determined by

$$U_\omega(g)\pi_\omega(A)U_\omega(g)^* = \pi_\omega(\tau_g(A)) \quad A \in \mathfrak{U}, g \in G \tag{27}$$

and invariance of the cyclic vector

$$U_\omega(g)\Omega_\omega = \Omega_\omega \forall g \in G. \tag{28}$$

See Appendix A for the full mathematical setup. In what follows, when ω is a time invariant state of the C^* algebra, we use the notation τ_t^ω instead of $U_\omega(t)$ for the unitary action of the time evolution on the GNS space.

2.3.2. Many-Body Ergodicity. In order to look at ergodicity, in the classical realm we needed to consider the energy shell to which the initial condition belonged, and in the von Neumann quantum setup, we needed a small quantum energy shell. So, in the C^* algebra context, it is natural to take a state ω assumed spacetime stationary and spatially mixing as described above, and states ψ in its microcanonical shell, as in (24).

In the C^* -operator algebra context, one can still consider time-averaging (1) of operators (either as elements of the C^* algebra, or bounded operators on \mathcal{H}_ω), where time integration is now to be understood as a Bochner integral [42]. For clarity, we will write explicitly the long-time limits, hence we will use (17).

The basic formulation of many-body ergodicity is conceptually similar to that of Sect. 2.1 (classical case), and paragraph 2.2.1 (quantum case):

Theorem 2.2 (Mean-square ergodicity). *Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$ and let ω be a spacetime stationary and spatially mixing state. Suppose that the unitary time evolution operator τ_t^ω on the GNS space \mathcal{H}_ω has trivial kernel,*

$\ker(\tau_s^\omega - 1) = \mathbb{C}|\Omega\rangle$ for some $s \in (0, \infty)$. Then for all integers $n \geq 1$ and any observable $A \in \mathfrak{A}$,

$$\lim_{T \rightarrow \infty} \omega \left((\bar{A}_T)^n \right) = \omega(A)^n, \text{ where } \bar{A}_T := T^{-1} \int_0^T dt A(0, t) \quad (29)$$

That is, we have mean-square ergodicity, and in fact the “random variable” \bar{A} is non-fluctuating within the state ω .

The theorem follows from von Neumann’s ergodic theorem for unitary operators on Hilbert spaces, as in paragraph 2.2.1, but now with the GNS space \mathcal{H}_ω as the Hilbert space. Using von Neumann’s ergodic theorem the time average gives a projection onto the subspace of \mathcal{H}_ω that is invariant under τ_s^ω , for some $s \in (0, \infty)$, see also [43, Theorem 3.1]. If then $\ker(\tau_s^\omega - 1) = \mathbb{C}|\Omega\rangle$, this projection is the rank one projection onto $|\Omega\rangle$, and the result is straightforward, as a direct application of the techniques used in [28] (also discussed in [44, Section 2]). Relevant results exist in various forms, including [45, Proposition 6.3.5], [46, 47].

Clearly, as we have taken already the limit $L \rightarrow \infty$, the timescales for convergence are not controlled by the volume of the system, contrary to the phenomenon of quantum recurrence as discussed in paragraph 2.2.1; in (29) convergence of long-time averages happens much before the recurrence time. Yet, the techniques are essentially the same. What has happened? The most important aspect, as mentioned, is that in the C^* algebra formulation, **we have reduced the set of degrees of freedom to the relevant ones for local physics**. Thus, in contrast to the condition of non-degeneracy, **the condition of triviality of the kernel of the time evolution properly encodes many-body ergodicity**.

In a formal sense, this kernel triviality condition is akin to asking that there be no nontrivial constants of the motion in the classical case, or to asking that there be no degeneracies in the quantum case. However, in a more physical sense, it is starkly different. In the classical context, for instance, integrability implies that there are nontrivial constants of the motion (as many as degrees of freedom). But in extensive systems, integrability is not related to the presence of nontrivial elements in $\ker(\tau_t^\omega - 1)$. Instead, it is related to the presence of extensive conserved quantities in addition to the Hamiltonian. These span the kernel of a different evolution operator, that which acts not on the GNS space built from the local observables, but on a Hilbert space of extensive observables, see Sect. 3. Extensive conserved quantities may exist, the system may be integrable, and yet many-body ergodicity may still hold. Thus, **we have separated the condition for many-body ergodicity to hold, from the requirement on nontrivial extensive conserved quantities, such as those found in integrable systems**.

The “nontrivial constants of the motion” that should not be present for many-body ergodicity to hold, are the **time-invariant local observables** (locality is here in the weak sense of elements of the C^* algebra). Typically, except in special systems such as those with dynamical constraints, there are no such

observables beyond (the trivial) multiples of $\mathbf{1}$. In fact, we note that time-invariant local observables play a role in the phenomenon of localisation—thus, many-body ergodicity occurs if there is no “localisation” (in the sense that there is no time-invariant local observables). As mentioned, extensive conserved quantities, instead, play a rôle in integrability. And in fact, most importantly for us, they are at the basis of the emergent hydrodynamic structures, which we explain in Sect. 3.

It turns out that one can go further if the state ω is also a KMS state, Eq. 21. In fact, it is not necessary for H in (19) to be the evolution Hamiltonian, or equivalently for τ_t in (21) to be the dynamics we take for time averaging. Any other H' , any other dynamics τ'_t , that keeps the state spacetime stationary will do—for instance, any other extensive conserved quantity, if available. Thus, in integrable model, any generalised Gibbs ensemble [48–50].

Theorem 2.3 (Ergodicity). *Under the same conditions as those of Theorem 2.2, if further ω is a τ' -KMS state, then, for all $A \in \mathfrak{A}$,*

$$\begin{aligned} \lim_{T \rightarrow \infty} \psi(\bar{A}_T) &= \omega(A) \quad \forall \psi \in \mathfrak{S}_{\text{micro}}, \quad \text{and} \\ \lim_{T \rightarrow \infty} \bar{A}_T &= A\mathbf{1} \quad \text{as operator on } \mathcal{H}_\omega, \text{ in the strong operator topology.} \end{aligned} \quad (30)$$

Again, the proof is omitted, but follows quite directly from the techniques developed in [28]. Recall that $C_T \rightarrow 0$ in the strong operator topology means $\|C_T|\psi\rangle\| \rightarrow 0$ for all $|\psi\rangle \in \mathcal{H}_\omega$.

Now the result is much stronger: this is truly an ergodicity property, stating that the time average of any observable, in any state within a micro-canonical shell surrounding ω , gives the average in ω . Time averaging gives back ensemble averaging. The operatorial equation, for A as an operator on the GNS space \mathcal{H}_ω , is in fact an equivalent statement.

Finally, an additional insight may be gained into many-body ergodicity by the following construction. Recall that in the classical case, ergodicity implies that every point in the energy shell is approached arbitrarily closely by the trajectory as time passes. Here, instead, with states playing the role of “points in phase space”, a trajectory starting from ψ approaches arbitrarily closely the final state ω : **for every $\epsilon > 0$, there is a finite time t such that the time-evolved state $\psi \circ \tau_t$ is a distance less than ϵ from ω , and in fact, this is true for a fraction of time that tends to 100% over long periods.**

This construction starts with the choice of a sequence of unit-norm, strictly local observables $A_n \in \mathfrak{A}_{\text{loc}}$, $\|A_n\| = 1$, $n = 1, 2, 3, \dots$ which spans $\mathfrak{A}_{\text{loc}}$ (every element of $\mathfrak{A}_{\text{loc}}$ may be written as a finite linear combination of these). Such a sequence certainly exists. Then, we define the following distance function between states:

$$d(\psi, \psi') = \sqrt{\sum_{n=1}^{\infty} 2^{-n} |\psi(A_n) - \psi'(A_n)|^2} \leq 2. \quad (31)$$

We note that there is a lot of freedom in defining the distance function; in particular, any finite set of C^* -algebra elements $A'_n \in \mathfrak{A}$, $n = 1, 2, \dots, N$ may

be adjoined to the sequence A_n . Thus, the general result on the distance function can be used to deduce results for any finite-dimensional subspace of such elements.

Equation (31) indeed defines a distance function. It is simple to see that $d(\psi, \psi'') \leq d(\psi, \psi') + d(\psi', \psi'')$, and that if $\psi = \psi'$ then $d(\psi, \psi') = 0$. Further, if $d(\psi, \psi') = 0$, then $\psi = \psi'$. This is shown as follows. Suppose $d(\psi, \psi') = 0$, and consider some $B \in \mathfrak{U}$. For every $\epsilon > 0$ we can find $A \in \mathfrak{U}_{\text{loc}}$ such that $\|B - A\| \leq \epsilon$. Therefore, $|\psi(B) - \psi'(B) - (\psi(A) - \psi'(A))| \leq 2\epsilon$. As $d(\psi, \psi') = 0$, then $\psi(A_n) = \psi'(A_n)$ for all n . As A_n 's span $\mathfrak{U}_{\text{loc}}$, then also $\psi(A) = \psi'(A)$. Therefore, by the triangle inequality, $|\psi(B) - \psi'(B)| \leq 2\epsilon$. As this holds for every $\epsilon > 0$, we conclude that $|\psi(B) - \psi'(B)| = 0$.

Now consider $d(\psi_t, \omega)^2$ where $\psi_t = \psi \circ \tau_t$ is the time-evolved state. We have

$$d(\psi_t, \omega)^2 = \sum_{n=1}^{\infty} 2^{-n} |\psi(\tau_t A_n) - \omega(A_n)|^2 \quad (32)$$

$$= \sum_{n=1}^{\infty} 2^{-n} (\psi^2(\tau_t A_n) - 2\psi(\tau_t A_n)\omega(A_n) + \omega^2(A_n)) \quad (33)$$

and we are interested in the long time limit of the time average:

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt d(\psi_t, \omega)^2 \\ &= \lim_{T \rightarrow \infty} \sum_{n=1}^{\infty} 2^{-n} \left(\frac{1}{T} \int_0^T \psi^2(\tau_t A_n) dt - 2\omega(A_n) \frac{1}{T} \int_0^T \psi(\tau_t A_n) dt + \omega^2(A_n) \right). \end{aligned} \quad (34)$$

It is easy to see, by the dominated convergence theorem, that we can move the $\lim_{T \rightarrow \infty}$ past the summation $\sum_{n=1}^{\infty}$. We then want to apply Theorem 2.3. We can immediately do this for the second term: $\lim_{T \rightarrow \infty} 2\omega(A_n) \frac{1}{T} \int_0^T \psi(\tau_t A_n) dt = 2\omega^2(A_n)$.

For the first term, $\frac{1}{T} \int_0^T \psi^2(\tau_t A_n) dt$, we notice that its limit coincides with the limit of $\omega(A) \frac{1}{T} \int_0^T \psi(\tau_t A_n) dt$, by the following argument:

$$\begin{aligned} & \left| \frac{1}{T} \int_0^T \psi^2(\tau_t A_n) dt - \omega(A) \frac{1}{T} \int_0^T \psi(\tau_t A_n) dt \right| \\ &= \left| \frac{1}{T} \int_0^T \psi(\tau_t A_n) (\psi(\tau_t A_n) - \omega(A)) dt \right| \end{aligned} \quad (35)$$

$$\leq \frac{1}{T} \int_0^T |\psi(\tau_t A_n)| |\psi(\tau_t A_n) - \omega(A)| dt \quad (36)$$

and $|\psi(\tau_t A_n)| \leq \|A\|$, while $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T |\psi(\tau_t A_n) - \omega(A)| dt = 0$, by Theorem 2.3. Thus,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \psi^2(\tau_t A_n) dt = \lim_{T \rightarrow \infty} \omega(A) \frac{1}{T} \int_0^T \psi(\tau_t A_n) dt = \omega^2(A_n) \quad \forall n \quad (37)$$

again by applying Theorem 2.3. Finally, applying the limits in Eq. (34), we show:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt d(\psi_t, \omega)^2 = 0. \quad (38)$$

This implies the following.

Theorem 2.4. *Let $\psi \in \mathfrak{S}_{\text{micro}}$ be a state in the microcanonical shell of ω . Under the conditions of Theorem 2.3, for every $\epsilon > 0$, there exists $t > 0$ such that $d(\psi_t, \omega) < \epsilon$. Moreover, $d(\psi_t, \omega) < \epsilon$ for 100% of the times t (that is, the ratio of the Lebesgue measure for the set $\{t \in [0, T] : d(\psi_t, \omega) \geq \epsilon\}$, to that for the set $\{t \in [0, T] : d(\psi_t, \omega) < \epsilon\}$, tends to zero as $T \rightarrow \infty$).*

Proof. We prove the last statement as it implies for first. Let us denote μ_T^+ and μ_T^- the measures for the sets $\{t \in [0, T] : d(\psi_t, \omega) \geq \epsilon\}$ and $\{t \in [0, T] : d(\psi_t, \omega) < \epsilon\}$, respectively. Note that $\mu_T^+ + \mu_T^- = T$. Then $\frac{1}{T} \int_0^T dt d(\psi_t, \omega)^2 \geq \epsilon^2 \frac{\mu_T^+}{T} = \epsilon^2 \left(1 + \frac{\mu_T^-}{\mu_T^+}\right)^{-1}$, hence by (38) we must have $\lim_{T \rightarrow \infty} \left(1 + \frac{\mu_T^-}{\mu_T^+}\right)^{-1} = 0$ and therefore $\lim_{T \rightarrow \infty} \frac{\mu_T^+}{\mu_T^-} = 0$. \square

In particular, from the above theorem we conclude that, for any $A \in \mathfrak{U}$, we have $|\psi(\tau_t A) - \omega(A)| < \epsilon$ for 100% of the times: the average in a state in a microcanonical shell, is almost always as near as desired to the average in the stationary state ω .

Some remarks about the context and meaning of these results are in order:

1. Because the results apply to the C^* algebra obtained from local observables, the results address *local relaxation*, a form of typicality (see, for example, [7]) which expresses *many-body ergodicity*.
2. Note how both Theorems 2.2 and 2.3 apply not just to thermal states, but to more general spacetime stationary states, such as generalised Gibbs ensembles. This therefore takes into account the possibility that the model possesses a large number of *extensive conserved quantities* similar to the Hamiltonian; these would lead, in the finite-volume setup, to a large number of degeneracies. Here, no condition arises on the presence or not of extensive conserved quantities.
3. The microcanonical shell represented by the states ψ should be understood as a being a shell around the microcanonical state *with respect to all extensive conserved quantities* (and in fact, all extensive observables), not just the energy: indeed all extensive observables have densities that are the same as in ω , Eq. (25). This is why ergodicity can hold in ψ without conditions on the presence or not of extensive conserved quantities.
4. One may wish to extend the definition of the microcanonical shell. Any state on the C^* algebra that, under long time evolution, converges (in an appropriate topology) to ω , might be argued to be in the microcanonical shell of ω in the infinite-volume setting. We have seen in what sense this is mathematically true for the states in $\mathfrak{S}_{\text{micro}}$. Because this is from a

local perturbation, this is the problem of “return to equilibrium” (see, for example, [51, 52] for results on this). However, there are many more such states, that are not necessarily absolutely continuous with respect to ω , hence that are “further away” from it. For instance, spatially-homogeneous states in which expectations of densities of all extensive conserved quantities, *but not necessarily of all extensive observables*, are equal to those in ω , are expected at long time to converge to ω . This is the problem of “thermalisation (or generalised thermalisation)” (see, for example, the review [2] and the results in [37]). Even more complicated, one may take a state that is not spatially homogeneous and extensively different from ω . There may not be any well-defined spatially-averaged conserved densities in such states. The long-time problem is now that of the emergence of hydrodynamics, and the solutions to the hydrodynamic equations. This is the more general problem of “approach to equilibrium (or approach to generalised equilibrium) from non-equilibrium”, and a typical example is the partitioning protocol (see, for example, the review [53]). Our definition of the microcanonical shell is perhaps restrictive, but convenient for the present discussion.

5. Requiring the KMS condition is not too strong a requirement. It is expected that most, or maybe all, spacetime stationary spatially mixing states are in fact KMS states. Perhaps the Tomita–Takesaki theory [38] could be used.
6. A similar C^* algebra construction can be made for classical systems, such as a classical gas, where the limit of infinitely many particles is taken. We expect that similar ergodicity statements can be obtained. In the classical case, the GNS space can also be constructed, and time evolution is again—at least in well-behaved systems—a one-parameter unitary group. Thus, von Neumann’s ergodic theorem for unitary operators applies. Again, as we have taken the limit of infinitely many particles, timescales are not those that would be necessary to cover the full energy shell—it is not true that over long times, the many-particle trajectory comes as close (in the conventional many-particle metric) as desired to a given point in the many-particle phase space. Particle numbers are so large that the phase space is immense, and “long times” are never enough.

Thus we have solved the problem of obtaining ergodicity in many-body systems with timescales below the quantum recurrence timescale (in a different way than in von Neumann’s QET, with different-looking conditions), or below the classical shell-covering timescales. We have also solved the problem of generalising to cases where nontrivial conservation laws exist (the extensive conserved quantities), and of having a formulation that is directly in the thermodynamic limit.

There remains the problem that the condition of triviality of the kernel of the time-evolution unitary one-parameter group on \mathcal{H}_ω is rather difficult to verify. It is also perhaps not too clear if the resulting ergodicity statement

is the one that is most physically relevant. These two problems are solved by *almost-everywhere ergodicity*, to which we now turn.

2.3.3. Almost-Everywhere Ergodicity. We now obtain ergodicity results *which are valid in every quantum spin lattice* as described above, and *in every spacetime stationary spatially mixing states*. No special, hard-to-check condition is required.

Such general results however are not valid for the standard notion of ergodicity, with integration purely in the time direction. Instead, we must *look in spacetime*. On the lattice, it is natural to restrict to the set of rational spatial directions $\mathbb{S}_{\mathbb{Q}}^{D-1} = \{\mathbf{x}/|\mathbf{x}| : \mathbf{x} \in \mathbb{Z}^D\}$ (this is dense on the sphere). Given $T > 0$, $v \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{S}_{\mathbb{Q}}^{D-1}$, we consider the average of $A \in \mathfrak{U}$ *along the ray* with velocity vector $\mathbf{v} = v\mathbf{n}$,

$$\bar{A}_T^v = \frac{1}{T} \int_0^T dt A(\lfloor \mathbf{v}t \rfloor, t) \tag{39}$$

where $\lfloor \mathbf{a} \rfloor = (\lfloor a_i \rfloor)_i$ is the vector of the integer parts.

In fact, the results apply more generally. That is, instead of considering the flat averages, we *may modulate the average with oscillating factors*. So, we consider instead

$$\bar{A}_T^v = \frac{1}{T} \int_0^T dt e^{ift - i\mathbf{k} \cdot \mathbf{v}t} A(\lfloor \mathbf{v}t \rfloor, t) \tag{40}$$

(keeping the wavenumber \mathbf{k} and frequency f implicit in the notation for the average, for lightness of notation). Let us also denote

$$E_A := \lim_{T \rightarrow \infty} \omega(\bar{A}_T^v) = \begin{cases} \omega(A) & (f - \mathbf{k} \cdot \mathbf{v} = 0) \\ 0 & (\text{otherwise}) \end{cases} \tag{41}$$

See Fig. 1 for a pictorial representation of what is going on. We show in [28]:

Theorem 2.5. *Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$ and a spacetime stationary and spatially mixing state ω . For every rational direction $\mathbf{n} \in \mathbb{S}_{\mathbb{Q}}^{D-1}$ and almost every $v \in \mathbb{R}$ with respect to the Lebesgue measure: (Mean-square ergodicity)*

$$\lim_{T \rightarrow \infty} \omega((\bar{A}_T^v)^m) = E_A^m \tag{42}$$

for every $A \in \mathfrak{U}$, $m \in \mathbb{N}$. Assume further, ω is a KMS state. Then (Ergodicity):

$$\lim_{T \rightarrow \infty} \psi(\bar{A}_T^v) = E_A, \tag{43}$$

for every $\psi \in \mathfrak{S}_{\text{micro}}$; and for any $A, B \in \mathfrak{U}$ we have in the GNS representation of ω :

$$\lim_{T \rightarrow \infty} \bar{A}_T^v |\Psi\rangle = E_A \mathbb{1} |\Psi\rangle, \quad \lim_{T \rightarrow \infty} [\bar{A}_T^v, B] |\Psi\rangle = 0 \tag{44}$$

for every $|\Psi\rangle \in \mathcal{H}_\omega$. That is, we have convergence in the strong operator topology of the GNS space \mathcal{H}_ω :

$$\bar{A}_T^v \xrightarrow{SOT} E_A \mathbb{1}, \quad [\bar{A}_T^v, B] \xrightarrow{SOT} 0 \tag{45}$$

Additionally, under the same assumptions of the Theorem above, and similarly to the derivation around Eq. (34), one can show that for any $\psi \in \mathfrak{S}_{\text{micro}}$ in the microcanonical shell of ω , the ray-averaged distance (with respect to the metric of Eq. (31)) between ψ and ω tends to zero, for almost every ray (of rational direction). Denoting $\psi_{vt} := \psi \circ \iota_{[vt]} \tau_t$, we have:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt d(\psi_{vt}, \omega)^2 = 0 \quad (46)$$

Taking the language of probability, Eq. (42) means “ $\bar{A}_T^v \rightarrow E_A$ in law”. This is *almost-everywhere ergodicity*: along almost velocity of rational direction, \bar{A}_T^v tends to a non-fluctuating quantity. As a consequence, by the Cauchy-Schwartz inequality, $\omega(\bar{A}_T^v B) \rightarrow E_A \langle B \rangle$ and $\omega([\bar{A}_T^v, B]) \rightarrow 0$ for every $A, B \in \mathfrak{U}$.

Equations (43), (44) are starker statements. They hold in spatially mixing, space-time translation invariant KMS states, including Gibbs and generalised Gibbs states. They imply that the limits in (43), (44) hold within any expectation value, multiplied by any other observables in any order. We thus obtain an extension of (a weak version of) the Lieb–Robinson bound *to within the Lieb–Robinson cone: the ray-averaged operator A_T^v becomes “thin” as $T \rightarrow \infty$, being un-observable by any B , at every frequency and wavelength, and at almost every velocity v of rational direction.*

Further, Theorem 2.5 means that **time-averages, at every frequency and wavelength and almost every velocity v of rational direction, converge to non-fluctuating (cluster out), classical (commutators vanish) variables.**

In particular, we have $\langle [\bar{A}_T^v, B]^2 \rangle \rightarrow 0$. At short times, the OTOC $\langle [A([\mathbf{v}t], t), B]^2 \rangle$ is expected to grow exponentially in chaotic systems [10, 54] (eventually reaching $O(1)$ values by boundedness of the state). The velocities where this happens are bounded by a state-dependent “butterfly velocity” v_B , $|\mathbf{v}| < v_B < v_{\text{LR}}$. We find that the *time-averaged* version of this quantity, $\langle [\bar{A}_T^v, B]^2 \rangle = T^{-2} \int_0^T dt dt' \langle [A([\mathbf{v}t], t), B] [A([\mathbf{v}t'], t'), B] \rangle$, in fact decays for almost every velocity of rational direction.

The results do not say anything about what happens in the pure time direction, $\mathbf{v} = \mathbf{0}$. For instance, there are non-interacting Hamiltonians for which the velocity $\mathbf{v} = \mathbf{0}$ is not ergodic; as mentioned, this is in fact related to localisation. However, all results hold for velocities as near as desired to $\mathbf{0}$.

Again, Theorem 2.5 is based on the von Neumann ergodic theorem [34, Theorem II.11], which relates time averages to projections onto invariant spaces. As far as we know, before our work there were no nontrivial results concerning the invariant subspace of \mathcal{H}_ω for unitary time evolution in quantum lattice models. We show it to be spanned by $|\Omega\rangle$ at almost every velocity. This happens because finite-dimensional local spaces (in fact, we believe that countable-dimensionality is sufficient) are too small to allow for operators to extend thickly on large distances over time: the set of speeds where the operator remains supported under averaging should be “small enough”.

Remarks 1–6 of paragraph 2.3.2 apply to almost-everywhere ergodicity results as well. We emphasise again that the results of Theorem 2.5 do not

require any nontrivial condition on the interaction or the dynamics (such as chaos, non-degenerate eigenvalues, absence of nontrivial constants of the motion, non-localisation condition, etc.). The results are **purely consequences of extensivity of the system**—that is, the fact that the interaction range is short enough, and that the thermodynamic, large-volume limit has been taken.

Thus we have now addressed the question of the conditions under which notions of ergodicity may hold: almost-everywhere ergodicity only requires extensivity. However, we have not yet addressed the question of the physical relevance of this notion. Surprisingly, it turns out that almost-everywhere ergodicity is in fact closely related to the hydrodynamic structure emerging on large scales. We now turn to this important result.

3. Hydrodynamic Projections and Spaces of Extensive Charges

Almost-everywhere ergodicity, or at least versions of it which differ slightly from what we have introduced above, can be utilised in order to prove a hydrodynamic projection theorem for Euler-scale correlation functions. This is probably the most interesting application of the ideas we have developed in the previous sections. In this section we start by giving an informal overview of the general idea, and proceed with the rigorous construction and proof. The main Theorem 3.1 and its proof are new results, extending to higher dimensions and to short-range interaction some of the results of [29].

What is a “hydrodynamic projection”? See for instance the short non-rigorous discussion in [21, Sect 2.6] as well as the review [17] for a modern introduction to this idea. The main point of the hydrodynamic projection theorem, that we will prove in generality in short-range quantum spin lattices, is that the Euler-scale correlation functions can be fully evaluated only by knowing the full set of conserved extensive charges of the system, and the “projection” onto these charges of the local operators involved in the correlation function. The physical idea is that the initial, dynamically complicated disturbance quickly relaxes and projects, at the Euler scale of long time and large distances, onto the conserved extensive quantities, that then carry correlations.

But there are a number of concepts in this statement that need clarifying and appropriate mathematical definitions. What is an Euler-scale correlation function? It heuristically involves the ballistic scaling of space and time, \mathbf{x}, t large with \mathbf{x}/t finite, but it requires a precise definition. What are the conserved extensive charges, and how do we define the full set of them? One needs a precise definition of an extensive quantity, and then of a conserved extensive quantity (or charge). We will define these as certain Hilbert spaces, in a way that is similar to, but different from, the GNS construction. Then, what does it mean to project a local operator onto such a conserved extensive quantity? This will be from the Hilbert space structure. Our main hydrodynamic projection theorem will make the connection between all these notions, the result being in agreement with the physical intuition.

For the Hilbert spaces \mathcal{H}_0 of extensive quantities, to every local observable $A \in \mathfrak{U}_{\text{loc}}$ we will associate, in a manner similar to the GNS construction, a respective “extensive quantity”, which we will denote as ΣA , defined as an appropriate equivalence class of the observable A . As the notation suggests, ΣA will have the interpretation as the sum of all translates, “ $\Sigma A = \sum_{\mathbf{x} \in \mathbb{Z}^D} A(\mathbf{x})$ ”, of the observable A , and A as a density of ΣA . The density of an extensive observable is not unique—one can add “total derivatives” for instance—and ΣA can be seen as the equivalence class of all densities of the extensive quantity of which A is a density. ΣA will indeed contain all spatial translates of A . Naturally, the infinite sum itself, $\sum_{\mathbf{x} \in \mathbb{Z}^D} A(\mathbf{x})$, does not converge in \mathfrak{U} ; but we will have convergence in a different topology. Note that, unlike the GNS construction where the observables of the C^* algebra are represented by operators acting on a Hilbert space, here instead we will form a Hilbert space of extensive observables—*not a representation as operators*. On this Hilbert space, the C^* -algebra time evolution τ_t induces a time evolution τ_t^0 , which is unitary. The conserved extensive charges \mathcal{Q}_0 are just the elements of \mathcal{H}_0 that are invariant under τ_t^0 . This is a closed subspace of \mathcal{H}_0 , thus allowing us to define an orthogonal projection $\mathbb{P} : \mathcal{H}_0 \rightarrow \mathcal{Q}_0$ of extensive quantities to the conserved ones.

The objects of interest are Euler-scale correlation functions. Denoting $(A, B) = \omega(A^\dagger B) - \omega(A^\dagger)\omega(B)$ the sesquilinear connected correlation, we would like to study the connected correlation function $(A(\mathbf{x}, t), B)$ for \mathbf{x}, t large and \mathbf{x}/t finite. In fact, it is conventional to talk about the Fourier transform of the space coordinate, at small momentum \mathbf{k} , with $\mathbf{k}t$ fixed, see, for example, [12, 17, 21]. Setting $\mathbf{k} = \boldsymbol{\kappa}/t$, we thus are looking for $\sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\boldsymbol{\kappa} \cdot \mathbf{x}/t} (A(\mathbf{x}, t), B)$ in the limit $t \rightarrow \infty$ with $\boldsymbol{\kappa}$ fixed. In order to obtain rigorous results, especially using appropriate almost-everywhere ergodicity ideas, it is convenient to take the time average of that (see [29]); of course, if the limit $t \rightarrow \infty$ exists then this does not change anything. Thus, we are looking at

$$S_{\Sigma A, \Sigma B}(\boldsymbol{\kappa}) := \widetilde{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\boldsymbol{\kappa} \cdot \mathbf{x}/t} (A(\mathbf{x}, t), B). \quad (47)$$

We do not know how to show the existence of the limit $\lim_{T \rightarrow \infty}$, but we can show that $\sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\boldsymbol{\kappa} \cdot \mathbf{x}/t} (A(\mathbf{x}, t), B)$ is uniformly bounded. This allows us to use the notion of a Banach limit $\widetilde{\lim}_{T \rightarrow \infty}$, see [55, Chapter III.7] and [29, Appendix A]; if the limit exists, then the Banach limit agrees with it. We can also show that the result is indeed a function not merely of A, B , but in fact of the equivalence classes $\Sigma A, \Sigma B$, as the notation suggests. The resulting function $S_{\Sigma A, \Sigma B}$ is what we define as the Euler-scale correlation function.

As mentioned, the essence of the hydrodynamic projection is that, in Euler-scale correlation functions, every extensive observable will project onto the conserved charges, irrespectively of the choice of a Banach limit:

$$S_{\Sigma A, \Sigma B}(\boldsymbol{\kappa}) = S_{\mathbb{P}(\Sigma A), \mathbb{P}(\Sigma B)}(\boldsymbol{\kappa}). \quad (48)$$

This is in fact an expression of the Boltzmann-Gibbs principle [15–17]: the reduction of the number of degrees of freedom at large space-time separations by projections onto hydrodynamic modes. This occurs thanks to almost-everywhere ergodicity, as is made apparent in the proof of Theorem 3.1 in Sect. 3.4 (see [29] for a proof in $D=1$).

We believe this is the first general, rigorous result concerning the Boltzmann-Gibbs principle in deterministic interacting systems of arbitrary dimensions. The principle is applicable to the large class of quantum lattices, going beyond interacting particle systems conventionally studied in statistical physics and hydrodynamics. It provides further support to the idea that the basic principles of hydrodynamics hold independently from the details of the microscopic dynamics.

Surprisingly, the hydrodynamic projection can be generalised to describe oscillatory behaviours: to any frequency $f \in \mathbb{R}$, and wavenumber $\mathbf{k} \in \mathbb{R}^D$. We show that the projection still takes place for \mathbf{k} -extensive quantities, formally “ $\Sigma^{\mathbf{k}} A = \sum e^{-i\mathbf{k} \cdot \mathbf{x}} A(\mathbf{x})$ ”, which will project to (f, \mathbf{k}) -conserved charges, defined as those $\Sigma^{\mathbf{k}} A$ such that their time evolution takes the form $\Sigma^{\mathbf{k}} A(t) = e^{if t} \Sigma^{\mathbf{k}} A$. This, we believe, is an observation that had not been made in full before, and which has potential for many new results.

Oscillatory hydrodynamic projection describes oscillatory behaviours that emerge at large space and time separations in correlation functions. One in general expects that (f, \mathbf{k}) -conserved charges only exist for certain frequency-wavenumber pairs (f, \mathbf{k}) , depending on the specific model. A simple example is the free fermionic lattice, with some dispersion relation $E(\mathbf{k})$. In this system, creation and annihilation operators at momentum \mathbf{k} are $(E(\mathbf{k}), \mathbf{k})$ -extensive conserved quantities. In Sect. 4, in order to illustrate the phenomenon we will explain (in a non-rigorous fashion) how oscillatory hydrodynamic projection recovers the oscillatory algebraic decay of fermion two-point functions in a free fermionic quantum chain, in agreement with a saddle-point analysis.

In the present section, we concentrate on the rigorous and general results of hydrodynamic projections in quantum lattice models.

3.1. Clustering and Basic Assumptions

Throughout all of Sect. 3 we consider a dynamical system $(\mathfrak{U}, \iota, \tau)$, see Definition 2.2 and Appendix A for more details. We use the notation $A(\mathbf{x}, t)$ for space-time translated observables $\iota_{\mathbf{x}} \tau_t A$. We also denote $A(\mathbf{x}) = A(\mathbf{x}, 0)$, while we keep the notation $\tau_t A$ for pure time translations. The system will be in a space-time invariant state ω , Definition A.2, which will be assumed to have clustering properties with respect to space translations. The almost-everywhere ergodicity theorems shown in [28], and discussed in Sect. 2.3, require the state ω to be clustering in space, in particular $\lim_{\mathbf{x} \rightarrow \infty} (\omega(A(\mathbf{x})B) - \omega(A)\omega(B)) = 0$ for any $A, B \in \mathfrak{U}$. Almost-everywhere ergodicity holds no matter how fast the connected correlations of observables decay at large spatial separations. For the hydrodynamic projection to occur stronger clustering assumptions need to be imposed. In particular, clustering of correlations at large space separations will need to happen faster than $|\mathbf{x}|^p$, for p large

enough:

$$\lim_{\mathbf{x} \rightarrow \infty} |\mathbf{x}|^p (\omega(A(\mathbf{x})B) - \omega(A)\omega(B)) = 0. \quad (49)$$

Additionally we will need to control the dependence on the size of the supports of the observables in the clustering bound; the support of a local observable A is defined as $\Lambda_A := \bigcap \{X \subset \mathbb{Z}^D : A \in \mathfrak{U}_X\}$. That is because time evolving observables “spreads out” their support. A state satisfying this form of clustering will be called r -sizeably p -clustering:

Definition 3.1 (*p-clustering*). A state ω of a dynamical system $(\mathfrak{U}, \iota, \tau)$ is called r -sizeably p -clustering for $p > r > 0$ if there exists a constant u such that for any $A, B \in \mathfrak{U}_{\text{loc}}$

$$|\omega(A(\mathbf{x})B) - \omega(A)\omega(B)| \leq C_{A,B}(1 + |\mathbf{x}|)^{-p}, \text{ for all } \mathbf{x} \in \mathbb{Z}^D \quad (50)$$

and $C_{A,B}$ is bounded as:

$$C_{A,B} \leq u \|A\| \|B\| |\Lambda_A|^r |\Lambda_B|^r. \quad (51)$$

A subset $\mathfrak{C} \subset \mathfrak{U}$ is called uniformly p -clustering for $p > D$ if there exist a $C > 0$ s.t. for all $A, B \in \mathfrak{C}$:

$$|\omega(A(\mathbf{x})B) - \omega(A)\omega(B)| \leq C(1 + |\mathbf{x}|)^{-p}, \quad \forall \mathbf{x} \in \mathbb{Z}^D. \quad (52)$$

Of course $C_{A,B}$ will, in general, depend on the pair of observables, i.e. on their norm and the size of their support. In $D = 1$ quantum spin chains, [29], the state is assumed to be r -sizeably p -clustering; this notion originates from [37, Definition 4.2]. The definition of p -clustering, for every $p > 0$, includes high temperature KMS states, which are exponentially clustering in space, as shown in [56, 57].

Definition 3.2. A state ω of a dynamical system $(\mathfrak{U}, \iota, \tau)$ is called exponentially clustering if there exist constants $u, r, \lambda > 0$ such that for any $A, B \in \mathfrak{U}_{\text{loc}}$

$$|\omega(A(\mathbf{x})B) - \omega(A)\omega(B)| \leq k_{A,B} e^{-\lambda|\mathbf{x}|}, \text{ for all } \mathbf{x} \in \mathbb{Z}^D. \quad (53)$$

and

$$k_{A,B} < u \|A\| \|B\| |\Lambda_A|^r |\Lambda_B|^r \quad (54)$$

Uniform (exponential) clustering of a subset $\mathfrak{C} \subset \mathfrak{U}$ is defined similarly.

In order for the time evolution to be well defined as a unitary group action on the Hilbert spaces of extensive quantities, we need to assume that the time translations of local observables cluster in a uniform enough manner. First, we require that any element A in $\text{span}\{\tau_t A : A \in \mathfrak{U}_{\text{loc}}, t \in \mathbb{R}\}$ is approximated by a sequence $\sigma_n A$ of local elements. We will show that this follows from the Lieb–Robinson bound. Additionally, we require that for any local A, B the set of pairs $\{(\sigma_n A, \sigma_n B)\}$ is uniformly clustering in space, as per Eq. (52).

Combining spatial clustering of the state ω with the Lieb–Robinson bound, one can obtain space-like p_c clustering, defined as follows:

Definition 3.3. The dynamical system $(\mathfrak{U}, \iota, \tau)$ in the state ω is called space-like p_c -clustering with velocity v_c , if

1. $\forall A \in \hat{\mathfrak{U}}_{\text{loc}} = \text{span}\{\tau_t A : A \in \mathfrak{U}_{\text{loc}}, t \in \mathbb{R}\}$ there exists a sequence $\sigma_n A \in \mathfrak{U}_{\text{loc}}, n \in \mathbb{N}$, such that $\lim \sigma_n A = A$. For any $A \in \mathfrak{U}_{\text{loc}}$ we define $\sigma_n A = A, \forall n$.
2. $\forall A, B \in \hat{\mathfrak{U}}_{\text{loc}}$ the set of pairs $\{(\sigma_n A, \sigma_n B)\}$ is uniformly p -clustering for some $p > p_c$.
3. $\forall A, B \in \mathfrak{U}_{\text{loc}}$ there exist $p > p_c, 0 < V < v_c$ and $C_{A,B} > 0$ such that

$$|\omega(A(\mathbf{x}, t)B) - \omega(A)\omega(B)| \leq \frac{C_{A,B}}{(|\mathbf{x}| + 1)^p} \tag{55}$$

for all $\mathbf{x} \in \mathbb{Z}^D, |\mathbf{x}| \geq V|t|$.

Similarly, we define exponential space-like clustering by the same conditions, with Condition 2 replaced by uniform exponential clustering and Eq. (55) replaced by $|\omega(A(\mathbf{x}, t), B) - \omega(A)\omega(B)| \leq C_{A,B}e^{-\lambda|\mathbf{x}|}$, for some $\lambda > 0$.

Space-like p_c -clustering for $p_c > D$, where D is the lattice dimension, **is a sufficient condition for showing the hydrodynamic projection theorem.** In turn, for space-like p_c -clustering to hold, it is sufficient to have the following two conditions:

1. fast decaying interactions, satisfying Eq. (130), so that the Lieb–Robinson bound holds
2. a p -clustering (spatially mixing) state ω , Definition 3.1, with appropriately controlled growth of $C_{A,B}$ with respect to $|\Lambda_A|, |\Lambda_B|$.

This is realised, for example, in any quantum spin lattice with nearest-neighbour, finite range, or exponentially decaying two-body interactions that is in a high temperature thermal state. This is because high temperature thermal states are exponentially clustering⁵, [56, Theorem 3.2] [57, Theorem 2]. In the case of r -sizeably p -clustering states [37, Definition 4.2], we can extend the arguments in [29, Section 8] to arbitrary lattice dimension D , to show that whenever $p > D(r + 1)$, a r -sizeably p -clustering state will be space-like p_c -clustering for $p_c > D$, in any system with interaction that satisfies Eq. (130). This is slightly more technical in $D > 1$, but it largely follows the same ideas as in the $D = 1$ proof in [29], hence we omit it. Instead, in Appendix C we show how exponential space clustering and the Lieb–Robinson bound lead to exponential space-like clustering.

3.2. Spaces of Extensive Charges

We start by constructing Hilbert spaces of extensive quantities from the dynamical system $(\mathfrak{U}, \iota, \tau)$. For each wavenumber $\mathbf{k} \in \mathbb{R}^D$ we define the positive-semidefinite sesquilinear form

$$\langle A, B \rangle_{\mathbf{k}} = \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\mathbf{k} \cdot \mathbf{x}} (A(\mathbf{x}), B) \quad A, B \in \mathfrak{U}_{\text{loc}}, \tag{56}$$

where we recall that $(A, B) = \omega(A^\dagger B) - \omega(A^\dagger)\omega(B)$ is the sesquilinear connected correlation. We define the equivalence relation $A \sim^{\mathbf{k}} A'$ on $\mathfrak{U}_{\text{loc}}$ by

⁵Note that the constant $C_{A,B}$ grows linearly with $|\Lambda_A|, |\Lambda_B|$ in the case of exponential clustering of KMS states.

$\langle A' - A, A' - A \rangle_{\mathbf{k}} = 0$. The Hilbert spaces of extensive quantities are the norm-completion of the quotient spaces formed of the set of equivalence classes of $\sim^{\mathbf{k}}$. That is, the equivalence class of $A \in \mathfrak{U}_{\text{loc}}$ is $\Sigma^{\mathbf{k}} A := \{A' \in \mathfrak{U}_{\text{loc}} : A' \sim^{\mathbf{k}} A\}$. This is to be understood as the \mathbf{k} -extensive observable associated to the density A , and it contains all other densities A' that lead to the same \mathbf{k} -extensive observable. For instance, it is clear that $A' = e^{i\mathbf{k} \cdot \mathbf{x}'} A(\mathbf{x}') \in \Sigma^{\mathbf{k}} A$ for any fixed $\mathbf{x}' \in \mathbb{Z}^D$: the \mathbf{k} -translate of a density A still is a density for the same extensive observable. Formally, we can associate $\Sigma^{\mathbf{k}} A$ with the infinite series “ $\Sigma^{\mathbf{k}} A = \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{-i\mathbf{k} \cdot \mathbf{x}} A(\mathbf{x})$ ”, the “total” A on the full quantum lattice, as is intuitively suggested by the (56). Of course this series does not converge within \mathfrak{U} ; the quantity $\Sigma^{\mathbf{k}} A$ is defined as an equivalence class. The set of equivalence classes is $\mathcal{V}_{\mathbf{k}} := \mathfrak{U}_{\text{loc}} / \sim^{\mathbf{k}} = \{\Sigma^{\mathbf{k}} A : A \in \mathfrak{U}_{\text{loc}}\}$ and its Cauchy completion gives the Hilbert space of \mathbf{k} -extensive observables $\mathcal{H}_{\mathbf{k}} = \overline{\mathcal{V}_{\mathbf{k}}}$.

Remark 3.4. The quantity $\Sigma^{\mathbf{k}} A$ is not to be confused with the spatially-averaged quantity $\lim_{\Lambda \rightarrow \mathbb{Z}^D} \frac{1}{|\Lambda|} \sum_{\mathbf{x} \in \Lambda} A(\mathbf{x})$. Here, we are studying the total, extensive observable itself $\lim_{\Lambda \rightarrow \mathbb{Z}^D} \sum_{\mathbf{x} \in \Lambda} A(\mathbf{x})$, as an element of some Hilbert space. The information that is kept of the extensive observable, as encoded within the Hilbert space inner product, are its convergent connected correlation functions with local observables, while of course the spatially-averaged quantities have zero connected correlation functions with local observables. Note that connected correlation functions of conserved extensive observables with local observables have the physical interpretation as susceptibilities. For example, consider a chain with spin 1/2 at each site. Suppose that the Hamiltonian conserved the total spin $\sum_{x \in \mathbb{Z}} \sigma_3(x)$. Then, in a thermal state ω with inverse temperature β and magnetic field h , we have $\partial_{\omega}(A)/\partial h = \beta \langle \sigma_3, A \rangle_0$. Similarly, a staggered magnetic field would correspond to the case $k = \pi$.

Remark 3.5. As noted in the introduction of this Section, the construction of the Hilbert spaces $\mathcal{H}_{\mathbf{k}}$ should not be confused with the GNS representation. The GNS representation is a representation of the elements of a C^* algebras by bounded operators $B(H_{\omega})$ acting on a Hilbert space H_{ω} . Instead, we directly form Hilbert spaces by Cauchy completion of the quotient spaces $\mathfrak{U}_{\text{loc}} / \sim^{\mathbf{k}}$; these are not representations of the C^* algebra.

In order to examine the long time dynamics of these extensive observables we have to extend the action of time evolution τ_t , $t \in \mathbb{R}$ to the Hilbert spaces $\mathcal{H}_{\mathbf{k}}$. This is done rigorously in [29, Section 5.3] for $D = 1$ (spin chains) and can be immediately extended to arbitrary dimension D . The key in order to be able to do this is the uniform clustering condition 2 in Definition 3.3. Thus, we can show that time evolution acts as a unitary operator $\tau_t^{\mathbf{k}}$ on $\mathcal{H}_{\mathbf{k}}$. For lightness of notation, we omit the superscript and simply write τ_t for the unitary action of time evolution on these spaces.

With this construction, we can define the subspace of conserved extensive charges as those elements of $\mathcal{H}_{\mathbf{k}}$ that are invariant under τ_t :

$$\mathcal{Q}_{\mathbf{k}} = \{q \in \mathcal{H}_{\mathbf{k}} : \tau_t q = q, \forall t \in \mathbb{R}\}. \quad (57)$$

In fact, we can go one step further and define the subspace of (f, \mathbf{k}) -extensive conserved quantities (or f -oscillatory \mathbf{k} -extensive charges) as:

$$\mathcal{Q}_{(f,\mathbf{k})} = \{\lrcorner \in \mathcal{H}_{\mathbf{k}} : \tau_t \lrcorner = e^{-ift} \lrcorner, \forall t \in \mathbb{R}\}. \tag{58}$$

See Sect. 4 for a discussion of these charges and examples in a free fermion chain. It is immediate that $\mathcal{Q}_{(f,\mathbf{k})}$ is a closed subspace of $\mathcal{H}_{\mathbf{k}}$, hence there is an orthogonal projection

$$\mathbb{P}_{(f,\mathbf{k})} : \mathcal{H}_{\mathbf{k}} \rightarrow \mathcal{Q}_{(f,\mathbf{k})}. \tag{59}$$

3.3. A Hydrodynamic Projection Theorem

The correlation functions with (f, \mathbf{k}) -fluid-cell averaging are given by

$$S_{\Sigma^{\mathbf{k}} A, \Sigma^{\mathbf{k}} B}^{(f,\mathbf{k})}(\boldsymbol{\kappa}) := \widetilde{\lim}_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\mathbf{k} \cdot \mathbf{x} - ift} e^{i\boldsymbol{\kappa} \cdot \mathbf{x}/t} (A(\mathbf{x}, t), B). \tag{60}$$

Note how one extracts, thanks to the time integral and factor e^{-ift} , the time-oscillatory behaviour of the correlation function with frequency f . Note also how the full wavenumber is $\mathbf{k} + \boldsymbol{\kappa}/t$, representing, in the large-time limit, a long-wavelength modulation of a \mathbf{k} -oscillatory factor; this extracts the space-oscillatory behaviour with wavenumber \mathbf{k} .

The limit on T is in general a Banach limit [55, Chapter III.7], [29, Appendix A]. The result will hold irrespectively of the choice of Banach limit, and all the aspects of the proof regarding the Banach limit are the same as in $D = 1$, as shown in [29, Section 6]. As the notation implies, the result only depends on the equivalence classes $\Sigma^{\mathbf{k}} A, \Sigma^{\mathbf{k}} B$, which we'll denote with the respective lowercase letters a, β . In fact this defines a continuous sesquilinear form on $\mathcal{H}_{\mathbf{k}}$. The hydrodynamic projection theorem is rigorously stated as follows:

Theorem 3.1 (Hydrodynamic Projection). *Consider a dynamical system $(\mathfrak{A}, \iota, \tau)$ with interactions satisfying Eq. (130), in an r -sizeably $D(r+1)$ -clustering state ω , as per Definition 3.1. For every frequency-wavenumber pair $(f, \mathbf{k}) \in \mathbb{R} \times \mathbb{R}^D$, rational vector $\boldsymbol{\kappa} \in \mathbb{R}^D : \exists r \in \mathbb{R} \mid r\boldsymbol{\kappa} \in \mathbb{Z}^D$, and any \mathbf{k} -extensive elements $a, b \in \mathcal{H}_{\mathbf{k}}$,*

$$S_{a,b}^{(f,\mathbf{k})}(\boldsymbol{\kappa}) = S_{\mathbb{P}_{(f,\mathbf{k})} \lrcorner, \mathbb{P}_{(f,\mathbf{k})} b}^{(f,\mathbf{k})}(\boldsymbol{\kappa}). \tag{61}$$

Specifically for the simple case $(f, \mathbf{k}) = (0, \mathbf{0})$ we have

$$S_{a,b}(\boldsymbol{\kappa}) = S_{\mathbb{P}_a, \mathbb{P}_b}(\boldsymbol{\kappa}). \tag{62}$$

The case $\boldsymbol{\kappa} = 0$ is of special interest, and is known at the (oscillatory version of the) Drude weight:

Corollary 3.2. *The oscillatory Drude weight*

$$D_{\Sigma^{\mathbf{k}} A, \Sigma^{\mathbf{k}} B}^{(f,\mathbf{k})} := \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\mathbf{k} \cdot \mathbf{x} - ift} (A(\mathbf{x}, t), B) \tag{63}$$

satisfies the projection formula

$$D_{-1, \lfloor}^{(f, \mathbf{k})} = D_{\mathbb{P}_{(f, \mathbf{k})}^{-1}, \mathbb{P}_{(f, \mathbf{k})} \rfloor}^{(f, \mathbf{k})} \quad (64)$$

Here the limit is in fact an ordinary limit. The projection formula, when written explicitly in a basis decomposition, is exactly (an oscillatory version of) the so-called Mazur bound, here shown rigorously to be saturated. This is a very immediate extension to the oscillatory realm, and to arbitrary dimensions, of the result [29, Thm 6.1], and can be proven quite straightforwardly without the extensive machinery we develop below for dealing with the case $\kappa \neq 0$, see the proof of [29, Thm 6.1] and the discussion there for more details.

We proceed with the proof of the hydrodynamic projection theorem. We will first prove it for $(f, \mathbf{k}) = (0, \mathbf{0})$, and then easily generalise the proof to arbitrary (f, \mathbf{k}) . We note again that for $\kappa = \mathbf{0}$, the theorem is a straightforward application of von Neumann’s ergodic theorem for unitary operators. We thus concentrate on the case $\kappa \neq \mathbf{0}$.

3.4. Proof of Hydrodynamic Projection

The main idea for the proof of the hydrodynamic projection formula in quantum spin lattices is this: By using an appropriate geometric construction, we recast the D -dimensional problem of hydrodynamic projection into a 1-dimensional problem. This is done by identifying the summation over space coordinates in a plane perpendicular to the wavenumber direction, with a sesquilinear form and its new associated Hilbert space, which is to play the role of the sesquilinear correlation in an effective one-dimensional problem. Once this is done, the proof follows that of [29] done for $D = 1$. The schematics of our proof are as follows: We assume a dynamical system in a p -clustering state ω , Definition 3.1 and with sufficiently fast decaying interactions, Eq. (130), so that the Lieb–Robinson bound, Eq. (132), holds. We combine p -clustering with the Lieb–Robinson bound (proof in Appendix C) in order to obtain space-like p_c -clustering, Definition 3.3. This leads to the special Property 3.3, below, for the dynamical system. Using this property we can prove Lemma 3.4, which forms the basis for the hydrodynamic projection formula, Theorem 3.1.

The most important difference from the $D = 1$ case is the new geometric construction, which we now explain. We define the rational unit sphere in D -dimensional space as $\mathbb{S}_{\mathbb{Q}}^{D-1} = \{\kappa/|\kappa| : \kappa \in \mathbb{Z}^D\}$. We also denote the set of vectors in rational directions in $\mathbb{R}^D \setminus \{\mathbf{0}\}$ by $\mathbb{R}_{\mathbb{Q}}^D = \{\kappa \in \mathbb{R}^D \setminus \{\mathbf{0}\} : \exists r \in \mathbb{R} \mid r\kappa \in \mathbb{Z}^D\}$. Clearly, by definition, $\mathbb{S}_{\mathbb{Q}}^{D-1} \subset \mathbb{R}_{\mathbb{Q}}^D$. For any such vector $\kappa \in \mathbb{R}_{\mathbb{Q}}^D$, there is a unique unit vector $\widehat{\kappa}' = \pm\kappa/|\kappa| \in \mathbb{S}_{\mathbb{Q}}^{D-1}$ that lies on the half-unit sphere: such that $\widehat{\kappa}'_1 > 0$, or $\widehat{\kappa}'_1 = 0, \widehat{\kappa}'_2 > 0$, or \dots , or $\widehat{\kappa}'_1 = 0, \dots, \widehat{\kappa}'_{D-1} = 0, \widehat{\kappa}'_D > 0$. Then, given such a unit vector $\widehat{\kappa}'$, we denote by $\widehat{\kappa} = r\widehat{\kappa}'$ the “smallest” integer vector associated to it, where $r > 0$ is the smallest positive number such that $\widehat{\kappa} \in \mathbb{Z}^D$ ($\widehat{\kappa}$ exists because $\widehat{\kappa}' \in \mathbb{R}_{\mathbb{Q}}^D$). Clearly $r\widehat{\kappa} = \widehat{\kappa}$ for every $r \in \mathbb{R}$, and in fact, $\widehat{\kappa}$ is an element of \mathbb{Z}^D that identifies the “rational ray” on which κ lies: there is a bijection between the set of $\widehat{\kappa}$ ’s and the set of $\{r\kappa : r \in \mathbb{R}\}$ ’s for $\kappa \in \mathbb{R}_{\mathbb{Q}}^D$. Below we will refer to $\widehat{\kappa}$ as a rational ray, and it will be important

that it lies in \mathbb{Z}^D . For any rational ray $\widehat{\boldsymbol{\kappa}}$, we may build its perpendicular plane in \mathbb{Z}^D : a sublattice of \mathbb{Z}^D that is isomorphic to \mathbb{Z}^{D-1} , and whose vectors have vanishing vector dot product with $\widehat{\boldsymbol{\kappa}}$. Given $\widehat{\boldsymbol{\kappa}}$, there is a set (possibly empty) of zero components $\zeta = \{i : \widehat{\boldsymbol{\kappa}}_i = 0\}$ and its complement (always non-empty) $\tilde{\zeta} = \{1, \dots, D\} \setminus \zeta = \{j_1, \dots, j_d\}$ for unique $1 \leq d = |\tilde{\zeta}| \leq D$ and $j_1 < \dots < j_d$. We construct the vectors

$$\begin{aligned} \mathbf{h}^{(i)} \in \mathbb{Z}^D : h_m^{(i)} &= \delta_{i,m} \quad (m \in \{1, \dots, D\}, i \in \zeta) \\ h_{j_l}^{(j_l)} &= \widehat{\boldsymbol{\kappa}}_{j_{l+1}}, h_{j_{l+1}}^{(j_l)} \\ &= -\widehat{\boldsymbol{\kappa}}_{j_l}, h_m^{(j_l)} = 0 \quad (l \in \{1, \dots, d-1\}, \\ & m \in \{1, \dots, D\} \setminus \{j_l, j_{l+1}\}). \end{aligned} \tag{65}$$

This defines $D - 1$ vectors: all $\mathbf{h}^{(i)}, i = 1, \dots, D$, except $\mathbf{h}^{(j_d)}$ which has not been defined. If $j_d \neq D$, then $D \in \zeta$, and for convenience, in this case, we simply define $\mathbf{h}^{(j_d)} = \mathbf{h}^{(D)}$ as well as $\tilde{\zeta} = \zeta \setminus \{D\} \cup \{j_d\}$ (otherwise $\tilde{\zeta} = \zeta$). Thus we may always concentrate on $\mathbf{h}^{(i)} : i = 1, \dots, D - 1$.

By construction, the set $\{\mathbf{h}^{(i)} : i = 1, \dots, D - 1\}$ is linearly independent, and satisfy $\mathbf{h}^{(i)} \cdot \widehat{\boldsymbol{\kappa}} = \mathbf{h}^{(i)} \cdot \boldsymbol{\kappa} = 0$. The perpendicular plane is $\mathbb{Z}^D \supset \mathbb{H}_{\widehat{\boldsymbol{\kappa}}}^{D-1} := \text{span}_{\mathbb{Z}}(\mathbf{h}^{(1)}, \dots, \mathbf{h}^{(D-1)}) \simeq \mathbb{Z}^{D-1}$, i.e. $\mathbb{H}_{\widehat{\boldsymbol{\kappa}}}^{D-1}$ is the \mathbb{Z} -module freely generated by the $\mathbf{h}^{(i)}$. The isomorphism between \mathbb{Z}^{D-1} and $\mathbb{H}_{\widehat{\boldsymbol{\kappa}}}^{D-1}$, as \mathbb{Z} -modules, is clear by construction, in particular

$$\mathbb{H}_{\widehat{\boldsymbol{\kappa}}}^{D-1} = \{x_i \mathbf{h}^{(i)} : \mathbf{x} \in \mathbb{Z}^{D-1}\} \tag{66}$$

(with implied summation over repeated indices). Of course, we could have taken away the greatest common divisor of the components of $\mathbf{h}^{(i)}$ in order to have “denser” planes, but this is not necessary in the following construction.

By stacking the perpendicular planes parallelly, we obtain a sublattice of \mathbb{Z}^D which is itself isomorphic to \mathbb{Z}^D , defined as the \mathbb{Z} -module freely generated by the set $\{\mathbf{h}^{(i)}, \widehat{\boldsymbol{\kappa}}\}$:

$$\mathbb{Z}_{\widehat{\boldsymbol{\kappa}}}^D := \{x_i \mathbf{h}^{(i)} + z \widehat{\boldsymbol{\kappa}} : (\mathbf{x}, z) \in \mathbb{Z}^D\}. \tag{67}$$

In-between the points of this lattice lie “fundamental cells”. That is, consider the region $R_{\widehat{\boldsymbol{\kappa}}} = \{x_i \mathbf{h}^{(i)} + z \widehat{\boldsymbol{\kappa}} : x_i \in [0, 1] \forall i, z \in [0, 1)\} \subset \mathbb{R}^D$ and the cell in \mathbb{Z}^D given by

$$\Lambda_{\widehat{\boldsymbol{\kappa}}} = \mathbb{Z}^D \cap R_{\widehat{\boldsymbol{\kappa}}} = \mathbb{Z}^D \cap \{x_i \mathbf{h}^{(i)} + z \widehat{\boldsymbol{\kappa}} : x_i \in [0, 1] \forall i, z \in [0, 1)\}. \tag{68}$$

Then the union of all unit cells shifted by the sublattice gives back \mathbb{Z}^D :

$$\mathbb{Z}^D = \bigcup_{\mathbf{r} \in \mathbb{Z}_{\widehat{\boldsymbol{\kappa}}}^D} (\Lambda_{\widehat{\boldsymbol{\kappa}}} + \mathbf{r}). \tag{69}$$

Indeed, by linear independence and the fact that the Jacobian J is finite and nonzero, every $\mathbf{u} \in \mathbb{R}^D$ may be written in a unique way as $\mathbf{u} = x_i \mathbf{h}^{(i)} + z \widehat{\boldsymbol{\kappa}}$ for some $(\mathbf{x}, z) \in \mathbb{R}^D$, and thus $\mathbf{u} = \lfloor x_i \rfloor \mathbf{h}^{(i)} + \lfloor z \rfloor \widehat{\boldsymbol{\kappa}} + \boldsymbol{\lambda}$ where $\boldsymbol{\lambda} \in R_{\widehat{\boldsymbol{\kappa}}}$. This holds in particular for $\mathbf{u} \in \mathbb{Z}^D$, and since $\lfloor x_i \rfloor \mathbf{h}^{(i)} + \lfloor z \rfloor \widehat{\boldsymbol{\kappa}} \in \mathbb{Z}_{\widehat{\boldsymbol{\kappa}}}^D$ it gives (69).

We define for any wavenumber $\boldsymbol{\kappa} \in \mathbb{R}_{\mathbb{Q}}^D$ and $\boldsymbol{\kappa}' \in \mathbb{R}^D$, some positive-semidefinite sesquilinear forms, and associated Hilbert spaces, in a way similar

to that done above: first by summation over the perpendicular planes $\mathbb{H}_{\widehat{\kappa}}^{D-1}$, and second by summation over the full sublattices $\mathbb{Z}_{\widehat{\kappa}}^D := \{x_i \mathbf{h}^{(i)} + z \widehat{\kappa} : (\mathbf{x}, z) \in \mathbb{Z}^D\}$. Let

$$(A, B)^{\widehat{\kappa}, \perp} := \sum_{\mathbf{x} \in \mathbb{H}_{\widehat{\kappa}}^{D-1}} (A(\mathbf{x}), B), \quad (A, B)_{\widehat{\kappa}'} := \sum_{\mathbf{x} \in \mathbb{Z}_{\widehat{\kappa}'}^D} e^{i\widehat{\kappa}' \cdot \mathbf{x}} (A(\mathbf{x}), B),$$

$$A, B \in \mathfrak{U}_{\text{loc}}. \quad (70)$$

Note that these are well defined by clustering of the connected correlator $(A, B) = \omega(A^\dagger B) - \omega(A^\dagger)\omega(B)$. Similarly to the construction of the spaces \mathcal{H}_{κ} , we define the Hilbert spaces $\mathcal{H}^{\widehat{\kappa}, \perp}$ and $\mathcal{H}_{\widehat{\kappa}'}$, as the norm completions of $\mathcal{V}^{\widehat{\kappa}, \perp} := \mathfrak{U}_{\text{loc}} / \sim^{\widehat{\kappa}, \perp}$ and $\mathcal{V}_{\widehat{\kappa}'} := \mathfrak{U}_{\text{loc}} / \sim_{\widehat{\kappa}'}$, respectively, where $A \sim^{\widehat{\kappa}, \perp} A' \Leftrightarrow (A - A', A - A')^{\widehat{\kappa}, \perp} = 0$ and $A \sim_{\widehat{\kappa}'} A' \Leftrightarrow (A - A', A - A')_{\widehat{\kappa}'} = 0$. The equivalence class of $A \in \mathfrak{U}_{\text{loc}}$ is denoted by the respective lowercase a and it is immediate that $(\cdot, \cdot)^{\widehat{\kappa}, \perp}$ and $(\cdot, \cdot)_{\widehat{\kappa}'}$ are inner products on their respective Hilbert spaces, and thus satisfies the Cauchy–Schwarz inequality. See Appendix D for proofs of basic properties. It can be established that $\iota_x := \iota_{x\widehat{\kappa}}$ ($x \in \mathbb{Z}$) and τ_t ($t \in \mathbb{R}$) act unitarily on $\mathcal{H}^{\widehat{\kappa}, \perp}$ as representations of the groups \mathbb{Z} and \mathbb{R} respectively, and that τ_t ($t \in \mathbb{R}$) act unitarily on $\mathcal{H}_{\widehat{\kappa}'}$ as a representation of the group \mathbb{R} . Details and other basic results are discussed in Appendix E. We denote space-time translations of elements $a \in \mathcal{H}^{\widehat{\kappa}, \perp}$ by $\iota_z \tau_t a$, time translations of $a \in \mathcal{H}_{\widehat{\kappa}'}$ by $\tau_t a$, while keeping the notation $A(\mathbf{x}, t)$ for $A \in \mathfrak{U}$.

Property 3.3 Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$. (Definition 2.2) that is space-like $D + 1$ -clustering (Definition 3.3). For every $\kappa \in \mathbb{R}_{\mathbb{Q}}^D$, it follows that:

1. $\forall a, b \in \mathcal{H}^{\widehat{\kappa}, \perp}$, for almost all $v \in \mathbb{R}$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\iota_{\lfloor vt \rfloor} \tau_t a, b)^{\widehat{\kappa}, \perp} dt = 0. \quad (71)$$

2. $\forall a, b \in \mathcal{V}^{\widehat{\kappa}, \perp}$, there exist $T > 0$ (independent of $\widehat{\kappa}$) and a Lebesgue measurable function $f : \mathbb{R} \mapsto \mathbb{R}_+$ such that:

$$|(\iota_{\lfloor vt \rfloor} \tau_t a, b)^{\widehat{\kappa}, \perp}| \leq f(v) \quad \forall v \in \mathbb{R}, \quad t \geq T \text{ satisfying } \int_{-\infty}^{\infty} dv (|v| + 1) f(v) < \infty. \quad (72)$$

Proof. Property 3.3.1 is a consequence of the almost-everywhere ergodicity theorem, [28, Theorem 3.2]. One can see this by proving that $(\iota_{\lfloor vt \rfloor} \tau_t a, b)^{\widehat{\kappa}, \perp}$ is space-like ergodic, [28, Definition 3.1], see Appendix B for the general idea. That is, first, note that our assumptions immediately imply that the state ω is space-like ergodic:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=0}^{N-1} \left(A(m\mathbf{n}, mv^{-1}|\mathbf{n}|), B \right) = 0 \quad v > v_{LR} \quad A, B \in \mathfrak{U}_{\text{loc}}. \quad (73)$$

Consider $a, b \in \mathcal{V}^{\hat{\mathbf{k}}, \perp}$ and class representatives $A, B \in \mathfrak{U}_{\text{loc}}$ respectively. Then,

$$\lim_N \frac{1}{N} \sum_{m=0}^{N-1} (\iota_{\lfloor mz \rfloor} \tau_{mv^{-1}z} a, b)^{\hat{\mathbf{k}}, \perp} = \lim_N \frac{1}{N} \sum_{m=0}^{N-1} \sum_{\mathbf{x} \in \mathbb{H}_{\hat{\mathbf{k}}}^{D-1}} \left(A(\lfloor mz \rfloor \hat{\mathbf{k}} + \mathbf{x}, mv^{-1}z), B \right). \tag{74}$$

Using space-like clustering, we can see for any $z \in \mathbb{R}$, $v > v_c$ and $A, B \in \mathfrak{U}_{\text{loc}}$

$$\left| \frac{1}{N} \sum_{m=0}^N \left(A(\lfloor mz \rfloor \hat{\mathbf{k}} + \mathbf{x}, mv^{-1}z), B \right) \right| \leq \frac{1}{N} \sum_{m=0}^N \frac{c}{(\sqrt{\lfloor mz \rfloor^2 |\hat{\mathbf{k}}|^2 + |\mathbf{x}|^2} + 1)^p} \tag{75}$$

where the summand is N -independent, \mathbf{x} -summable, and uniformly bounded by a m -summable function, allowing us to apply the dominated convergence theorem in order to move \lim_N inside the sum $\sum_{\mathbf{x} \in \mathbb{H}_{\hat{\mathbf{k}}}^{D-1}}$ in Eq. (74). Thus,

$$\lim_N \frac{1}{N} \sum_{m=0}^{N-1} (\iota_{\lfloor mz \rfloor} \tau_{mv^{-1}z} a, b)^{\hat{\mathbf{k}}, \perp} = 0, \text{ for } v > v_{LR}. \tag{76}$$

Applying the almost-everywhere ergodicity theorem for the system $(\mathcal{H}^{\hat{\mathbf{k}}, \perp}, \iota, \tau)$ (see Appendix B), we get Property 1.

For Property 3.3.2: Choose a V as per the definition of space-like clustering, Definition 3.3, choose $T > 0$, and consider all $t \geq T$. Note that the choice of T and V does not depend on $\hat{\mathbf{k}}$.

First consider the case $|v| \geq V/|\hat{\mathbf{k}}| + T^{-1}$. Note that $(|v|t - 1)|\hat{\mathbf{k}}| \geq Vt$ and $(|v|t - 1)|\hat{\mathbf{k}}| \geq (|v|T - 1)|\hat{\mathbf{k}}| \geq VT$. We have, for some $p > D + 1$ (each step is explained below):

$$\begin{aligned} |(\iota_{\lfloor vt \rfloor} \tau_t a, b)^{\hat{\mathbf{k}}, \perp}| &\leq \sum_{\mathbf{x} \in \mathbb{Z}^{D-1}} \left| \left(A(\lfloor vt \rfloor \hat{\mathbf{k}} + x_i \mathbf{h}^{(i)}, t), B \right) \right| \\ &\leq \sum_{\mathbf{x} \in \mathbb{Z}^{D-1}} \frac{c}{\left(\sqrt{(|v|t - 1)^2 |\hat{\mathbf{k}}|^2 + |x_i \mathbf{h}^{(i)}|^2} + 1 \right)^p} \\ &\leq \sum_{\mathbf{x} \in \mathbb{Z}^{D-1}} \frac{c}{((|v|t - 1)^2 |\hat{\mathbf{k}}|^2 + |x_i \mathbf{h}^{(i)}|^2)^{p/2}}. \end{aligned} \tag{77}$$

Note that the right-hand side never gets infinite, since $\hat{\mathbf{k}}$ is fixed and $(|v|t - 1)|\hat{\mathbf{k}}| \geq VT > 0$. In the second line we used the fact that

$$\begin{aligned} &| |vt \rfloor \hat{\mathbf{k}} + x_i \mathbf{h}^{(i)}|^2 \\ &= | |vt \rfloor|^2 |\hat{\mathbf{k}}|^2 + |x_i \mathbf{h}^{(i)}|^2 \quad (\text{by orthogonality and the } L^2 \text{ norm}) \\ &\geq (|v|t - 1)^2 |\hat{\mathbf{k}}|^2 + |x_i \mathbf{h}^{(i)}|^2 \quad (\text{as } \lfloor vt \rfloor > vt - 1 \text{ and } |vt - 1| \geq |v|t - 1) \\ &\geq (Vt)^2 \quad (\text{as } (|v|t - 1)|\hat{\mathbf{k}}| \geq Vt) \end{aligned} \tag{78}$$

which allowed us to use space-like p -clustering. We now use the fact that $\sum_{\mathbf{x} \in \mathbb{Z}^{D-1}} F(\mathbf{x}) = \int d^{D-1}x F(\lfloor \mathbf{x} \rfloor)$, hence we'll have to deal with $|\lfloor x_i \rfloor \mathbf{h}^{(i)}|$. The idea is to do the following transformation:

$$\begin{aligned} |x_i \mathbf{h}^{(i)}|^2 &= \sum_{m \in \zeta} |x_m|^2 + |x_{j_1} \widehat{\kappa}_{j_2}|^2 + |x_{j_2} \widehat{\kappa}_{j_3} - x_{j_1} \widehat{\kappa}_{j_2}|^2 + \dots + |x_{j_{d-1}} \widehat{\kappa}_{j_d} - x_{j_{d-2}} \widehat{\kappa}_{j_{d-1}}|^2 \\ &= \sum_{m=1}^{D-1} |y_m|^2 = |\mathbf{y}|^2, \quad y_m = \begin{cases} x_m & (m \in \zeta) \\ x_{j_1} \widehat{\kappa}_{j_2} & (m = j_1) \\ x_{j_l} \widehat{\kappa}_{j_{l+1}} - x_{j_{l-1}} \widehat{\kappa}_{j_l} & (m = j_l \in \{j_2, \dots, j_{d-1}\}). \end{cases} \end{aligned} \quad (79)$$

The finite Jacobian of the change of variable above is (nonzero)

$$J = \left| \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right| = \prod_{l=1}^{d-1} |\widehat{\kappa}_{j_{l+1}}|, \quad 0 < J < \infty. \quad (80)$$

We have by the triangle inequality:

$$|\lfloor x_i \rfloor \mathbf{h}^{(i)}| = |(x_i - \varepsilon_i) \mathbf{h}^{(i)}| \geq |x_i \mathbf{h}^{(i)}| - |\varepsilon_i \mathbf{h}^{(i)}| \geq |x_i \mathbf{h}^{(i)}| - h \quad (81)$$

where we used $\lfloor x_i \rfloor = x_i - \varepsilon_i$ for some $\varepsilon_i \in [0, 1)$, and where $h = \sum_i |\mathbf{h}^{(i)}|$. Hence, doing the transformation $\mathbf{x} \mapsto \mathbf{y}$, defined in Eq. (79):

$$\begin{aligned} |(\ell_{\lfloor vt \rfloor} \tau_t a, b)^{\widehat{\kappa}, \perp}| &\leq \int_{\mathbb{R}^{D-1}} d^{D-1}x \frac{c}{((|v|T-1)|\widehat{\kappa}|^2 + (|x_i \mathbf{h}^{(i)}| - h)^2)^{p/2}} \\ &= J^{-1} \int_{\mathbb{R}^{D-1}} d^{D-1}y \frac{c}{((|v|T-1)|\widehat{\kappa}|^2 + (|\mathbf{y}| - h)^2)^{p/2}} \\ &= J^{-1} (|v|T-1)|\widehat{\kappa}|^{D-1-p} \int_{\mathbb{R}^{D-1}} d^{D-1}y \frac{c}{\left(1 + \left(|\mathbf{y}| - \frac{h}{(|v|T-1)|\widehat{\kappa}|}\right)^2\right)^{p/2}} \\ &\leq cIJ^{-1} (|v|T-1)|\widehat{\kappa}|^{D-1-p} \quad (I \text{ is defined in (83)}). \end{aligned} \quad (82)$$

In the third line we used the fact that $(|v|T-1)|\widehat{\kappa}| > 0$. In the final line, we used $(|v|T-1)|\widehat{\kappa}| \geq VT$ and, for every $\ell \in (0, h/VT]$,

$$\begin{aligned} &\int_{\mathbb{R}^{D-1}} d^{D-1}y \frac{1}{\left(1 + (|\mathbf{y}| - \ell)^2\right)^{p/2}} \\ &\leq \int_{\mathbb{R}^{D-1}, |\mathbf{y}| \geq \frac{h}{VT}} d^{D-1}y \frac{1}{\left(1 + \left(|\mathbf{y}| - \frac{h}{VT}\right)^2\right)^{p/2}} + \int_{\mathbb{R}^{D-1}, |\mathbf{y}| < \frac{h}{VT}} d^{D-1}y 1 =: I \end{aligned} \quad (83)$$

where I , as defined by the right-hand side of the inequality, is finite, as $p > D + 1 > D - 1$, and only depends on $h/(VT)$ and D .

For $|v| < V/|\widehat{\kappa}| + T^{-1}$, we instead use the fact that

$$|(\ell_{\lfloor vt \rfloor} \tau_t a, b)^{\widehat{\kappa}, \perp}| \leq \|a\|^{\widehat{\kappa}, \perp} \|b\|^{\widehat{\kappa}, \perp} \quad (84)$$

by the Cauchy–Schwartz inequality and space-time translation invariance.

Thus we set

$$f(v) = \begin{cases} cIJ^{-1}(|v|T-1)|\widehat{\kappa}|)^{D-1-p} & (|v| \geq V/|\widehat{\kappa}| + T^{-1}) \\ \|a\|_{\widehat{\kappa}, \perp} \|b\|_{\widehat{\kappa}, \perp} & (|v| < V/|\widehat{\kappa}| + T^{-1}). \end{cases} \quad (85)$$

We see that, for $p > D + 1$, this indeed satisfies the right properties, and in particular this lower bound on p is necessary for the integral of $(|v| + 1)f(v)$ to exist on \mathbb{R} . \square

From this, we obtain the following crucial lemma:

Lemma 3.4. *If the dynamical system satisfies Property 3.3, then $\forall A, B \in \mathfrak{U}_{\text{loc}}$, $\forall s \in \mathbb{R}$, there exists $T_0 > 0$ such that for every $\kappa \in \mathbb{R}_{\mathbb{Q}}^D$, the following holds:*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{T_0}^T dt g(t) = 0 \quad g(t) = \sum_{\mathbf{x} \in \mathbb{Z}^D} (e^{i\kappa \mathbf{x}/t} - e^{i\kappa \mathbf{x}/(t+s)}) (A(\mathbf{x}, t), B). \quad (86)$$

Proof. First we write

$$g(t) = G_0(t) - G_s(t), \quad G_s(t) = \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{i\kappa \mathbf{x}/(t+s)} (A(\mathbf{x}, t), B). \quad (87)$$

Let us first simplify $G_s(t)$ for arbitrary s . Writing (in a unique fashion) $\mathbf{x} = \mathbf{r} + \boldsymbol{\lambda}$ where $\mathbf{r} \in \mathbb{Z}_{\widehat{\kappa}}^D$ and $\boldsymbol{\lambda} \in \Lambda_{\widehat{\kappa}}$, we get

$$G_s(t) = \sum_{\mathbf{r} \in \mathbb{Z}_{\widehat{\kappa}}^D} e^{i\kappa \mathbf{r}/(t+s)} (\tilde{A}_{t+s}(\mathbf{r}, t), B) = (\tau_t \tilde{a}_{t+s}, b)_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}} \quad (88)$$

where (recall that $\Lambda_{\widehat{\kappa}}$ is finite)

$$\tilde{A}_u = \sum_{\boldsymbol{\lambda} \in \Lambda_{\widehat{\kappa}}} e^{i\kappa \boldsymbol{\lambda}/u} \iota_{\boldsymbol{\lambda}} A \quad (89)$$

and the lowercase $\tilde{a}_u, b \in \mathcal{V}_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}}$ are the respective equivalence classes of \tilde{A}_u, B (see Equation 70 and discussion below). We note that

$$\begin{aligned} \lim_{t \rightarrow \infty} |G_s(t) - (\tau_t \tilde{a}_{\infty}, b)_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}}| &\leq \lim_{t \rightarrow \infty} \|\tau_t(\tilde{a}_{t+s} - \tilde{a}_{\infty})\|_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}} \|b\|_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}} \\ &\leq \lim_{t \rightarrow \infty} \max\{|\sin(\kappa \boldsymbol{\lambda}/(2(t+s)))| : \boldsymbol{\lambda} \in \Lambda_{\widehat{\kappa}}\} \|\tilde{a}_{\infty}\|_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}} \|b\|_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}} \\ &= 0 \end{aligned} \quad (90)$$

where we used the fact that $\lim_{\kappa' \rightarrow \mathbf{0}} \|c\|_{\widehat{\kappa}'}^{\widehat{\kappa}'} = \|c\|_{\mathbf{0}}^{\widehat{\kappa}}$ exists for any c . Therefore, using the form $\mathbf{r} = x_i \mathbf{h}^{(i)} + z \widehat{\kappa} \in \mathbb{Z}_{\widehat{\kappa}}^D$,

$$\begin{aligned} &\lim_{T \rightarrow \infty} \frac{1}{T} \int_{T_0}^T dt g(t) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{T_0}^T dt \left((\tau_t \tilde{a}_{\infty}, b)_{\widehat{\kappa}/t}^{\widehat{\kappa}} - (\tau_t \tilde{a}_{\infty}, b)_{\widehat{\kappa}/(t+s)}^{\widehat{\kappa}} \right) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{T_0}^T dt \sum_{z \in \mathbb{Z}} \left(e^{i|\kappa| \|\widehat{\kappa}\| z/t} - e^{i|\kappa| \|\widehat{\kappa}\| z/(t+s)} \right) (\iota_z \tau_t \tilde{a}_{\infty}, b)_{\widehat{\kappa}, \perp}^{\widehat{\kappa}, \perp}. \end{aligned} \quad (91)$$

At this point, we have managed to recast the problem into an effectively one-dimensional problem. The proof now broadly follows that in $D = 1$, [29]. In order to show the Lemma, i.e. show $\lim_{T \rightarrow \infty} \frac{1}{T} \int_{T_0}^T dt g(t) = 0$, we want to commute the limit and the time integral past the summation $\sum_{z \in \mathbb{Z}} g(t)$. The result would then follow by applying Property 3.3.1. We proceed to uniformly (in t) bound the summand in $g(t)$. As in the proof of Lemma 6.6 in [29], we write (denoting $\kappa = |\boldsymbol{\kappa}| \hat{\boldsymbol{\kappa}}$):

$$\begin{aligned} g(t) &= \sum_{z \in \mathbb{Z}} 2i \exp \frac{i\kappa z}{2} \left(\frac{1}{t} + \frac{1}{t+s} \right) \sin \left[\frac{\kappa z}{2} \left(\frac{1}{t} - \frac{1}{t+s} \right) \right] (\iota_z \tau_t \tilde{a}_\infty, b)^{\kappa, \perp} \\ &= \sum_{v \in t^{-1}\mathbb{Z}} \frac{i\kappa v}{2} \exp \frac{i\kappa v}{2} \left(1 + \frac{t}{t+s} \right) \frac{2t}{\kappa v} \sin \left[\frac{\kappa v}{2t} \left(t - \frac{t^2}{t+s} \right) \right] (\iota_{v_t} \tau_t \tilde{a}_\infty, b)^{\kappa, \perp} \\ &= \int_{\mathbb{R}} i\kappa v_t \exp \frac{i\kappa v_t}{2} \left(1 + \frac{t}{t+s} \right) \frac{2t}{\kappa v_t} \sin \left(\frac{\kappa v_t}{2t} \left(t - \frac{t^2}{t+s} \right) \right) (\iota_{\lfloor v_t \rfloor} \tau_t \tilde{a}_\infty, b)^{\kappa, \perp} dv \end{aligned} \quad (92)$$

where $v_t = t^{-1} \lfloor vt \rfloor$. Having established Property 3.3, the rest of the proof follows exactly as in [29, Lemma 6.6]. \square

Finally, having shown Lemma 86, we have for all $A, B \in \mathfrak{U}_{\text{loc}}$, any rational vector $\boldsymbol{\kappa} \in \mathbb{R}^D$, $s \in \mathbb{R}$ and any choice of Banach Limit $\lim_{t \rightarrow \infty}$:

$$\begin{aligned} 0 &= \widetilde{\lim}_{t \rightarrow \infty} \frac{1}{T} \int_0^T \left(\langle \tau_t a, b \rangle_{\boldsymbol{\kappa}/t} - \langle \tau_t a, b \rangle_{\boldsymbol{\kappa}/(t+s)} \right) dt \\ &= \widetilde{\lim}_{t \rightarrow \infty} \frac{1}{T} \int_0^T \langle \tau_t a, b \rangle_{\boldsymbol{\kappa}/t} dt - \widetilde{\lim}_{t \rightarrow \infty} \frac{1}{T} \int_0^T \langle \tau_t a, b \rangle_{\boldsymbol{\kappa}/(t+s)} dt \end{aligned} \quad (93)$$

where $a, b \in \mathcal{V}_{\boldsymbol{\kappa}/t}$ are the respective equivalence classes (see Eq. (56) and below). We then proceed exactly as in the proof [29, Theorem 6.7], which completes the proof of the hydrodynamic projection theorem for any dimension D and for zero wavenumber and frequency.

The proofs can easily be generalised to all frequencies and wavelengths, as all the bounds remain essentially unchanged. We consider different representations of the groups of space and time translations on the C^* -algebra \mathfrak{U} , for each frequency-wavenumber pair, defined as

$$\tilde{\iota}_{\mathbf{x}} := e^{i\mathbf{k}\mathbf{x}} \iota_{\mathbf{x}}, \quad \tilde{\tau}_t := e^{ift} \tau_t \quad \text{for } \mathbf{k} \in \mathbb{R}^D, f \in \mathbb{R}. \quad (94)$$

These form representations of the groups \mathbb{Z}^D, \mathbb{R} respectively, by bijective linear maps on \mathfrak{U} . They are not $*$ -automorphisms like $\iota_{\mathbf{x}}$ and τ_t . However, linearity and unitarity on the Hilbert spaces defined from the state are all that is used in our proofs (and of course the group representation properties). Hence we can relax the Definition of a dynamical system, Definition 2.2, to a triplet $(\mathfrak{U}, \tilde{\iota}, \tilde{\tau})$ where $\tilde{\tau}$ is a strongly continuous representation of the group \mathbb{R} by bijective linear maps $\{\tilde{\tau}_t : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}\}_{t \in \mathbb{R}}$, and $\tilde{\iota}$ is a representation of the translation group \mathbb{Z}^D by bijective linear maps $\{\tilde{\iota}_{\mathbf{x}} : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}\}_{\mathbf{x} \in \mathbb{Z}^D}$, and the state is required to have the invariance property $(\tilde{\iota}_{\mathbf{x}} \tilde{\tau}_t(A), \tilde{\iota}_{\mathbf{x}} \tilde{\tau}_t(B)) = (A, B)$. It is clear that if uniform p -clustering holds for a subset of elements in $\mathfrak{U}_{\text{loc}}$ under $\iota_{\mathbf{x}}$, then it

also holds under $\tilde{\iota}_x$. Under this new representation of space translations we can construct the Hilbert space of extensive quantities as

$$\tilde{\mathcal{H}}_0 = \mathcal{H}_k. \quad (95)$$

Further, it is immediate that the Lieb–Robinson bound also holds for $\tilde{\tau}_t = e^{ift}\tau_t$. Thus, by the Lieb–Robinson bound, in this case $\tilde{\tau}_t$ is also a one-parameter unitary group on $\tilde{\mathcal{H}}_0$. We then can define the space of $\tilde{\tau}_t$ -invariants $\tilde{\mathcal{Q}}$,

$$\tilde{\mathcal{Q}} = \{a \in \mathcal{H}_k : \tau_t a = e^{-ift} a \forall t \in \mathbb{R}\}, \quad (96)$$

and the orthogonal projection

$$\tilde{\mathbb{P}} : \tilde{\mathcal{H}}_0 \rightarrow \tilde{\mathcal{Q}}. \quad (97)$$

Finally, it is easily seen (by p -clustering and the Lieb–Robinson bound) that if the dynamical system $(\mathfrak{U}, \iota, \tau)$ and state ω is space-like p_c clustering according to Definition 3.3, then so is $(\mathfrak{U}, \tilde{\iota}, \tilde{\tau})$, ω . From there on, the results we proved above also hold for these new quantities:

$$\tilde{S}_{a,b}(\boldsymbol{\kappa}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{-ift} \langle \tau_t a, b \rangle_{\mathbf{k}+\boldsymbol{\kappa}/t} dt. \quad (98)$$

4. Linearised Oscillatory Euler Equation in One Dimension

Linearised Euler equations are equations for two-point functions of local conserved densities, in the limit of large times and large wavelengths. They can be heuristically obtained by linearising the Euler equation around a chosen spacetime-stationary state. See [12] for a general discussion, and [17, 21] for modern viewpoints.

In [29], it is shown, in one-dimensional quantum lattices (quantum chains), that **the theorem of hydrodynamic projection can be used to rigorously obtain linearised Euler equations**. There aren't many rigorous results for the linearised Euler equation in interacting systems. Besides [29], we know only of strong results in gases of particles at low densities within the context of the Boltzmann equation, see for instance the survey book [58] and the recent paper [59]; and in the hard-rod gas [60]. In the previous sections, we have seen how both the concepts of ergodicity and that of hydrodynamic projections hold as well with respect to any wavenumber \mathbf{k} and frequency f . In [29], it was commented that the linearised Euler equation will also hold at arbitrary wavenumber and frequency. Thus, this allows us to **extend the principles of hydrodynamic correlations to predict oscillatory behaviours**. This is perhaps the most important consequence of the observation that large-scale concepts in fact hold with oscillatory factors. In essence, we obtain the precise mathematical structure to describe the hydrodynamic expansion near a nonzero point in the (f, \mathbf{k}) -plane. Currently, besides the comment in [29], we are not aware of any other rigorous results for the oscillatory linearised Euler equation.

In this section, we illustrate this principle in a simple quantum chain, in a non-rigorous fashion. Recall $\mathcal{Q}_{f,\mathbf{k}}$, Eq. (58): this is the f -oscillatory

closed subspace of the Hilbert space $\mathcal{H}_{\mathbf{k}}$ generated by the \mathbf{k} -extensive observables. One may consider the corresponding (f, \mathbf{k}) -conserved densities and currents $A, B \in \mathfrak{U}_{\text{loc}}$: the density gives rise to the \mathbf{k} -extensive charge $\Sigma^{\mathbf{k}} A = \sum_{\mathbf{x} \in \mathbb{Z}^D} e^{-i\mathbf{k} \cdot \mathbf{x}} A(\mathbf{x}) \in \mathcal{Q}_{f, \mathbf{k}}$ which is f -oscillatory $\tau_t(\Sigma^{\mathbf{k}} A) = e^{-ift} \Sigma^{\mathbf{k}} A$. Such densities and currents should satisfy the oscillatory continuity equation

$$0 = \frac{d}{dt} A(\mathbf{x}, t) + ifA(\mathbf{x}, t) + \sum_i \left(B(\mathbf{x} + \mathbf{e}_i, t) - B(\mathbf{x}, t) + (e^{-i\mathbf{k} \cdot \mathbf{e}_i} - 1) B(\mathbf{x} + \mathbf{e}_i, t) \right). \quad (99)$$

In terms of the infinitesimal generator $i[H, \cdot]$ for τ_t , the \mathbf{k} -extensive charge satisfies $[H, \Sigma^{\mathbf{k}} A] + f\Sigma^{\mathbf{k}} A = 0$. If such a density and current exist, then two-point correlation functions of observables that have nonzero overlap with this density (or more precisely, with $\Sigma^{\mathbf{k}} A$), should present oscillatory behaviours at large scales of space and time. Indeed, under the (f, \mathbf{k}) -fluid-cell mean, defined, as on the r.h.s. of (60), with a factor $e^{i\mathbf{k} \cdot \mathbf{x} - ift}$, correlation functions at large scales project onto $\mathcal{Q}_{f, \mathbf{k}}$, as per the projection Theorem 3.1. As this holds for the oscillatory fluid-cell mean, that is under averaging with oscillatory factors, this extracts the oscillatory behaviour of the correlation function.

For any given model, one expects only certain values of (f, \mathbf{k}) that would give nontrivial $\mathcal{Q}_{f, \mathbf{k}} \neq \{0\}$. That is, only certain type of oscillatory behaviour can be observed (if any). A simple example is the free fermionic lattice, with some dispersion relation $E(\mathbf{k})$. In this system, creation and annihilation operators at momentum \mathbf{k} are $(E(\mathbf{k}), \mathbf{k})$ -extensive conserved quantities, and oscillatory hydrodynamic projection corresponds to the oscillatory algebraic decay of fermion two-point functions conventionally obtained by a saddle-point analysis. In fact, it is convenient to go beyond just the hydrodynamic projection principle, and get the oscillatory linearised Euler hydrodynamic equations, as in [29]. This gives the explicit oscillatory behaviours of correlation functions of conserved densities. Below we discuss this, and illustrate the phenomenon in a free fermion model. We concentrate on the one-dimensional case

$$D = 1. \quad (100)$$

(f, \mathbf{k}) -hydrodynamic projection is a hydrodynamic expansion, of linear-response type, in the neighbourhood of arbitrary (f, \mathbf{k}) , which extends the paradigm of hydrodynamics to oscillatory behaviours. It generalises ideas in recent works on time crystals, where dynamical symmetries are used to explain persistent oscillations in Drude weights [23, 25].

4.1. Oscillatory Linearised Euler Hydrodynamics

As mentioned, in one-dimensional models, the results of [29] are stronger than those reported in Sect. 3, as the linearised Euler equation is also proven. This makes the asymptotic of correlation functions more explicit. At a non-rigorous level, the formulae obtained are reviewed in [17] in the usual (non-oscillatory) case. For our purpose, in the non-oscillatory case, it is sufficient to recall that if A_i, B_i are pairs of conserved densities and currents, such that the extensive

charges $\neg_i = \Sigma A_i$ form a basis for the space of conserved quantities \mathcal{Q} , then the connected correlation function behaves as

$$\overline{(A_i(x, t), A_j(0, 0))} \sim \ell^{-1} [\delta(\bar{x} - \mathbf{A}\bar{t})\mathbf{C}]_{ij} \quad (x = \ell\bar{x}, t = \ell\bar{t}, \ell \rightarrow \infty) \quad (101)$$

where the flux Jacobian is

$$A_i^j = \sum_l \langle B_i, A_l \rangle_0 C^{lj}, \quad (102)$$

the static susceptibility matrix is

$$C_{ij} = \langle A_i, A_j \rangle_0 \quad (\text{with } C^{ij} \text{ the inverse matrix, } \sum_j C^{ij} C_{jl} = \delta_l^i) \quad (103)$$

and it is expected that, for the results to hold, the fluid-cell mean can be taken in the more intuitive form of an average over a cell in space-time:

$$\overline{A(x, t)} = \frac{1}{L^2} \sum_{y=-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} A(x+y, t+s) ds. \quad (104)$$

Here the mesoscopic length can be taken as $L = L(\ell)$ with $L \rightarrow \infty$ fast enough as $\ell \rightarrow \infty$, and $L/\ell \rightarrow 0$ (or as $L = \epsilon\ell$, then taking $\epsilon \rightarrow 0^+$ on the asymptotic large- ℓ result).

Three remarks are in order, see the explanations in [17]:

- (i) The right-hand side of Eq. (101) is obtained from the right-hand side of Eq. (48) by (1) expressing the continuity equation relating $(A_i(x, t), A_j(0, 0))$ and $(B_i(x, t), A_j(0, 0))$; (2) using the projection formula, Eq. (48), expressing $S_{\Sigma B_i, \Sigma A_j}(\kappa)$ in terms of $S_{\Sigma A_i, \Sigma A_j}(\kappa)$, Fourier transforming back to real space to get $(B_i(x, t), A_j(0, 0))$ in terms of $(A_l(x, t), A_j(0, 0))$; and (3) solving the resulting continuity equation for the matrix of correlators $(A_i(x, t), A_j(0, 0))$, with appropriate initial condition. The solution is (101).
- (ii) The fluid-cell mean (104) is different from that in (47). The latter is used to express the rigorous projection result, but the result is expected to hold for a variety of possible definitions of fluid-cell means. The fluid-cell mean chosen above is more physically transparent and more convenient for our purposes.
- (iii) If there is a finite set of basis charges (a finite set of indices i), the right-hand side of (101) is a ‘‘generalised function’’. Its meaning is that for \bar{x}/\bar{t} equal to an eigenvalue of \mathbf{A} , the large- ℓ asymptotic decays more slowly than $1/\ell$, while for other velocities, it decays more rapidly. The delta-function comes about from the normalisation condition:

$$C_{ij} = \sum_{x \in \mathbb{Z}} (A_i(x, t), A_j(0, 0)) = \ell \int d\bar{x} \overline{(A_i(x, t), A_j(0, 0))} \quad (105)$$

where we use $1 = \ell d\bar{x}$ and the fact that $\sum_{x \in \mathbb{Z}} A_i(x, t)$ is independent of time. If there is a continuous set of basis charges, as is typical in integrable

models, then the large- ℓ asymptotics is exactly $1/\ell$, and the right-hand side of (101) is an ordinary function. See [16, 17].

The observation made in [29] is that all formulae apply equally well in the oscillatory case: with the space-time translation group defined as

$$\eta_{x,t}^{\omega,k} : A \mapsto e^{i\omega t - ikx} A(x, t). \quad (106)$$

This is because $\eta_{x,t}^{\omega,k}$ still is a unitary operator on the GNS space. Thus we have a space of k -extensive observables, the equivalence classes $\Sigma^k A$ built using the positive semidefinite sesquilinear form

$$\langle A, B \rangle_k = \sum_{k \in \mathbb{Z}} e^{ikx} (A(x), B(0)), \quad (107)$$

and from it we may construct, formally, the (f, k) -conserved charges, $\mathcal{Q}_{f,k}$.

Thus, for frequency-wavenumber f, k , if A_i, B_i are pairs of oscillatory conserved densities and currents, Eq. (99), such that $\bar{\cdot}_i = \Sigma^k A_i$ form a basis for $\mathcal{Q}_{f,k}$, then

$$\begin{aligned} \overline{(A_i(x, t))^{f,k}}, A_j(0, 0) \rangle &\sim e^{ift - ikx} \ell^{-1} [\delta(\bar{x} - \mathbf{A}^{f,k} \bar{t}) \mathbf{C}^{f,k}]_{ij} \\ (x = \ell \bar{x}, t = \ell \bar{t}, \ell \rightarrow \infty) \end{aligned} \quad (108)$$

where the oscillatory flux Jacobian is

$$[\mathbf{A}^{f,k}]_i^j = \sum_l \langle B_i, A_l \rangle_k [\mathbf{C}^{f,k}]^{lj}, \quad (109)$$

the static susceptibility matrix is

$$[\mathbf{C}^{f,k}]_{ij} = \langle A_i, A_j \rangle_k \quad (110)$$

(note that these depend on f because the basis A_i 's does), and the fluid-cell mean is

$$\overline{A(x, t)}^{f,k} = \frac{1}{L^2} \sum_{y=-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{ifs - iky} A(x + y, t + s) ds, \quad (111)$$

4.2. Oscillatory Behaviour from a Saddle-Point Analysis

For simplicity, and in order to verify the ideas, we consider a one-dimensional quadratic model, where asymptotics of correlation functions can be obtained by an elementary saddle-point analysis. Such models are integrable, and the usual hydrodynamic projection principle has been studied widely in the context of generalised hydrodynamics [15–17, 21], and verified to agree with a saddle-point analysis [61].

Before we do this, we note that the mathematical results reported in the main text apply to quantum lattice models with finite local spaces, and are based on the bosonic version of the C^* algebra formulation of quantum statistical mechanics, where local operators commute with each other (thus, we are describing spins). Hence, they do not cover free bosonic chains (infinite-dimensional local space), nor free fermionic chains (fermionic formulation). Nevertheless, most results are expected to hold in both cases, as they only

rely on general properties of correlation functions. For the case of an infinite-dimensional local space, some work may be needed in order to check this statement, as some of the techniques and results we use are specific to finite-dimensional local spaces (e.g. the Lieb–Robinson bound, in the way we have expressed it). For the case of fermionic chains, the only differences will come from exchanging the positions of operators, where one would need to use $AB = (-1)^{d(A)+d(B)}$ if A and B are supported on disjoint subsets of \mathbb{Z}^D , where $d(A)$ is the fermionic degree of the operator A . It would be a matter of keeping track of signs at various points in the calculations and proofs; but properties of correlation functions are well established for fermionic theories [39]. Here, therefore, we simply take the example of a free fermionic chain, and assume that the main hydrodynamic projection results hold. We consider the Hamiltonian

$$H = -\frac{J}{2} \sum_{x \in \mathbb{Z}} (c_{x+1}^\dagger c_x + c_x^\dagger c_{x+1}), \quad \{c_x^\dagger, c_y\} = \delta_{x,y}. \quad (112)$$

We are looking to evaluate the asymptotic form of the thermal correlation function

$$(c_x(t), c_0(0)) = \omega(c_x^\dagger(t)c_0(0)) = \frac{\text{Tr} \left(e^{-\beta H} c_x^\dagger(t) c_0(0) \right)}{\text{Tr} \left(e^{-\beta H} \right)}, \quad c_x(t) = e^{iHt} c_x e^{-iHt} \quad (113)$$

as $x, t \rightarrow \infty$ (as one-point functions of fermions are zero, this is the connected correlator). The Hamiltonian (112) is diagonalised by the Fourier transform

$$c_x = \int_{-\pi}^{\pi} \frac{dk}{\sqrt{2\pi}} e^{ikx} a(k), \quad \{a^\dagger(k), a(k')\} = \delta(k - k') \quad (114)$$

as

$$H = - \int_{-\pi}^{\pi} dk J \cos k a^\dagger(k) a(k) \quad (115)$$

with energy spectrum $E(k) = -J \cos k$ and

$$a(k, t) = e^{iHt} a(k) e^{-iHt} = e^{-iE(k)t} a(k). \quad (116)$$

Therefore, we obtain the usual expression for the correlator

$$(c_x(t), c_0(0)) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} n(k) e^{iE(k)t - ikx}, \quad n(k) = \frac{1}{1 + e^{\beta E(k)}}. \quad (117)$$

A similar analysis as that below can be performed in a generalised Gibbs ensemble (GGE), where $n(k)$ takes an arbitrary form that characterises the GGE.

The asymptotic behaviour is easily obtained by a saddle point analysis. With k_\pm solving $v(k_\pm) = x/t$ where $v(k) = E'(k) = J \sin k$ and $k_+ \in [-\pi/2, \pi/2]$, $k_- = \text{sgn}(k_+) (\pi - k_+)$, we obtain

$$(c_x(t), c_0(0)) \sim \sum_{\pm} \frac{n(k_\pm)}{\sqrt{2\pi i t E''(k_\pm)}} e^{iE(k_\pm)t - ik_\pm x} \quad (118)$$

as $t \rightarrow \infty$ with $x/t = \xi$ fixed. The decay in $1/\sqrt{t}$, which is slower than $1/t$, indicates that the hydrodynamic projection formula should give a generalised function. As all factors explicitly written as functions of k_{\pm} are slowly varying, they may be assumed to be constant within the fluid cell. Then we see that the support of the (f, k) -fluid-cell mean (111) is on $k = k_{\pm}$ and $f = E(k_{\pm})$, which is

$$\xi = v(k), \quad f = E(k). \quad (119)$$

In fact, this saddle point analysis does not provide the full information that we need about the shape of the correlation function around the velocity $\xi = v(k)$. However, because we know the support, we may immediately write, as a consequence of this saddle-point result,

$$(\overline{c_x(t)}^{f,k}, c_0(0)) \sim e^{ift-ikx} \ell^{-1} \delta(\bar{x} - v(k)\bar{t}) R(k). \quad (120)$$

The normalisation $R(k)$ is obtained by evaluating

$$\frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} ds \sum_{x \in \mathbb{Z}} e^{ikx-ifs} (c_x(t+s), c_0(0)) \sim \ell \int d\bar{x} e^{ikx} (\overline{c_x(t)}^{f,k}, c_0(0)) \quad (121)$$

where again we use $1 = \ell d\bar{x}$. The left-hand side is found to be $e^{ift} n(k)$ from (117), while the right-hand side is found to be $e^{ift} R(k)$ from (120), giving

$$R(k) = n(k). \quad (122)$$

4.3. Oscillatory Behaviour from Oscillatory Linearised Euler Hydrodynamics

We note that

$$e^{iE(k)t} a(k, t) \quad (123)$$

is independent of time. Further,

$$a(k) = \frac{1}{\sqrt{2\pi}} \sum_{x \in \mathbb{Z}} e^{-ikx} c_x. \quad (124)$$

Therefore, $a(k)$ is a k -extensive, f -oscillatory charge, $a(k) \in \mathcal{Q}_{f,k}$, for $f = E(k)$, with local density

$$A = \frac{c_0}{\sqrt{2\pi}}. \quad (125)$$

It is a simple matter to verify that the associated current has the form

$$B = \frac{iJ}{2\sqrt{2\pi}} (c_{-1} - e^{ik} c_0) \quad (126)$$

in that $A(x, t)$ and $B(x, t)$ satisfy eq. (99). Then we evaluate

$$\mathcal{C}^{f,k} = \frac{1}{\sqrt{2\pi}} (a(k), c_0) = \frac{n(k)}{2\pi}, \quad \langle B, A \rangle_k = (B, a(k)) = \frac{n(k)v(k)}{2\pi} \quad (127)$$

and thus

$$A^{\omega,k} = v(k). \quad (128)$$

Hence, the prediction from hydrodynamic projections is

$$(\overline{c_x(t)}^{f,k}, c_0(0)) \sim e^{ift-ikx} \ell^{-1} \delta(\bar{x} - v(k)\bar{t}) n(k) \quad (129)$$

in agreement with (120) and (122).

5. Conclusion

We have analysed the large-time, long-wavelength behaviours of short-range quantum spin models on \mathbb{Z}^D .

We have first considered many-body ergodicity, which is ergodicity as viewed from the viewpoint of local physics in the thermodynamic limit. We contrasted it with the standard notion of ergodicity in systems with finite degrees of freedom, and von Neumann’s quantum ergodic theorem. In particular, we have obtained strong results in KMS states. We have shown that, under a condition of non-localisation, time averages of local observables become classical, taking the values of their ensemble averages—this is a natural notion of ergodicity. We have also shown, under the same condition, that a time-evolved state within a “microcanonical shell” of the KMS state—defined as perturbations by elements of the C^* algebra—is arbitrarily close to the KMS state for 100% of the times $t \in [0, T]$ when looking at arbitrarily large intervals of time $T \rightarrow \infty$, under a natural metric based on local observables. These results show that the long-time evolution locally loses a large amount of information, in agreement with expected ergodicity.

The non-localisation condition is that the kernel of the time-evolution operator on the GNS space should be “trivial” (one-dimensional): there should not be non-trivial observables (elements of the C^* algebra different from $\mathbf{1}$) that come back to themselves after some time. This is in general hard to check, and it would be interesting to study this condition in explicit models.

Only the new aspects of the proofs of the above many-body ergodicity statements were presented, as the main parts follow directly from the techniques presented in [28].

We have then reviewed the notion of almost-everywhere ergodicity, introduced in [28, 29]. This is ergodicity, as above, for almost every ray of rational direction in space-time. It is shown in [28] to hold *for all short-range quantum lattices on \mathbb{Z}^D* . That is, when considering displacements not just in time, but in space-time, *there is no need for non-localisation conditions in order for ergodicity to hold almost everywhere*. This includes the parallel result on the distance of states in the microcanonical shell, under space-time displacements. This strongly constrains the structure of time-evolved local observables at large times: essentially, local observables become “thin” over time. Intuitively, this is because the finite-dimensional local space of quantum spin lattices does not allow to have operators nontrivially supported over growing regions—only a countable set of space-time rays (of rational direction) may remain where the observable is nontrivial.

We have also observed that all these ergodicity results in fact hold *with arbitrary oscillatory factors*, going beyond conventional ergodicity to strong constraints on how observables may oscillate over long times.

Almost everywhere ergodicity was then used to obtain new rigorous results on *hydrodynamic projections*. We have shown that the principle of hydrodynamic projection—by which correlation functions, at large wavelengths and long times, project onto the extensive conserved quantities admitted by the model—hold in every short-range quantum spin model on \mathbb{Z}^D , and at every frequency f and wavenumber \mathbf{k} . Projection occurs on f, \mathbf{k} -dependent spaces, which are kernels of the time evolution operator on f, \mathbf{k} -dependent Hilbert spaces, different from, but constructed in a similar way to, the GNS space.

The hydrodynamic projection theorem naturally generalises that shown in [29] to arbitrary dimension $D > 1$. However, the proof is a nontrivial extension of that presented there for $D = 1$, requiring the construction of new Hilbert spaces and the analysis of their properties. The hydrodynamic projection theorem was presented in its most general form, involving arbitrary frequency f and wavenumber \mathbf{k} .

The almost-everywhere ergodicity and hydrodynamic projection results underline the large universality of general ergodic and hydrodynamic principles: they emerge solely from *the separation of scales, the large gap between extensivity and locality*. Further, the mathematical construction based on Hilbert spaces built from local observables and their correlations shows the inherent flexibility in defining notions of extensivity (or homogeneity) and stationarity, allowing for oscillations.

Finally, using this flexibility, we have considered the oscillatory linearised Euler equation: the linearised Euler equation, but obtained from oscillatory hydrodynamic projections instead of the conventional ones. This was proposed in [29], where the linearised Euler equation was shown for finite-range quantum spin chains. The equation describes the oscillatory behaviours of correlation functions at large space-time separations. We have illustrated it, in a non-rigorous fashion, on a free-fermion chain, showing that the oscillatory hydrodynamic principle correctly reproduces the oscillatory behaviour that can be inferred from a simple saddle-point evaluation of two-point fermion correlation functions at large space-time separations. This gives a proof-of-principle for the idea that Euler-scale hydrodynamics can in fact describe oscillatory behaviours seen at large space-time, with frequencies that are far from 0 and wavelengths that are microscopic.

The results presented open many doors for further studies. Applying the results to more examples, including specific structures inside the LR light-cone, the kernel of the GNS evolution operator and the related many-body ergodicity, and oscillatory hydrodynamic projections in interacting systems generalising the free-fermion example, would be very interesting. It is likely that, in integrable systems, oscillatory hydrodynamic projection would connect with the finite-density form factor expansions of quantum models; this may

also provide a way of defining and studying such form factor expansions in classical models.

The results illustrate how the choice of Hilbert space relates to the large-scale physics of interest; the important elements always lie within the kernel of the evolution operator, but the latter can be taken as acting on local observables, or extensive observables. The idea that different Hilbert spaces relate to different scales was introduced in [62], and it would be interesting to extend the rigorous analysis presented here to the “diffusive” Hilbert space constructed there.

The (f, \mathbf{k}) -hydrodynamic projection result paves the way for a full (f, \mathbf{k}) -hydrodynamics, a subject which should help uncover new universal dynamics and which we hope to investigate in the near future. In fact, going beyond simple oscillatory phases, it is also possible to re-define space- and time-translation operators by adjoining internal motions of local observables (such as spin rotations). It is clear that all proofs provided immediately generalise to this case whenever the internal motion is an extensive internal symmetry transformation (it commutes with time evolution and acts tensorially on individual spatial cells). It would be extremely interesting to study the related hydrodynamics.

Finally, it would be interesting to go beyond clean spin lattices, towards quantum and classical gases of particles, and also systems with disorder. We believe that the general methods of operator algebras, and in particular the principles presented here, give an alternative set of tools to the traditional ones (such as kinetic or Boltzmann equations), which have the potential to provide new fruitful ways of understanding the emergence of large-scale behaviours from the microscopic physics.

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Appendix A: Mathematical Set-Up

We consider a hypercubic lattice \mathbb{Z}^D and to each site $\mathbf{x} \in \mathbb{Z}^D$ we associate a quantum spin described by the (matrix) algebra of observables $\mathfrak{U}_{\mathbf{x}} := \mathbb{C}^{N_{\mathbf{x}}}$, $N_{\mathbf{x}} \in \mathbb{N}$, with the operator norm $\|\cdot\|$, and for all lattice points $\mathbf{x} \in \mathbb{Z}^D$ the dimension of the spin matrix algebra is uniformly bounded, $N_{\mathbf{x}} \leq N$ for some $N \in \mathbb{N}$. To each finite $\Lambda \subset \mathbb{Z}^D$ we have the algebra $\mathfrak{U}_{\Lambda} := \bigotimes_{\mathbf{x} \in \Lambda} \mathfrak{U}_{\mathbf{x}}$ and the algebra of local observables is defined as the direct limit of the increasing net of algebras $\{\mathfrak{U}_{\Lambda}\}_{\Lambda \in P_f(\mathbb{Z}^D)}$, where $P_f(\mathbb{Z}^D)$ denotes the set of finite subsets of \mathbb{Z}^D : $\mathfrak{U}_{\text{loc}} := \varinjlim \mathfrak{U}_{\Lambda}$. The norm completion of $\mathfrak{U}_{\text{loc}}$ defines the quasi-local C^* -algebra of the quantum spin lattice: $\mathfrak{U} := \overline{\mathfrak{U}_{\text{loc}}}$. For details on the definition of quasi-local C^* -algebras see [63, Chapter 3.2.3].

Space translations are naturally defined on $\mathfrak{U}_{\text{loc}}$ and extended (by continuity) to \mathfrak{U} , see [63, Chapter 3.2.6]. Time evolution is defined explicitly from the interaction of the quantum spin lattice, see [39, Chapter 6.2]. An interaction is defined as a map $\Phi : P_f(\mathbb{Z}^D) \rightarrow \mathfrak{U}_{\text{loc}}$, s.t. $\Phi(\Lambda) \in \mathfrak{U}_{\Lambda}$ and $\Phi(\Lambda) = \Phi^*(\Lambda)$, $\forall \Lambda \in P_f(\mathbb{Z}^D)$. The Hamiltonian associated with any $\Lambda \in P_f(\mathbb{Z}^D)$ is $H_{\Lambda} := \sum_{X \subset \Lambda} \Phi(X)$ which defines the local time evolution as $\tau_t^{\Lambda}(A) := e^{itH_{\Lambda}} A e^{-itH_{\Lambda}}$, $A \in \mathfrak{U}_{\Lambda}$, $t \in \mathbb{R}$. Time evolution of the infinite system is defined when the limit $\lim_{\Lambda \rightarrow \infty} \tau_t^{\Lambda}(A)$ exists in the norm for all $A \in \mathfrak{U}_{\text{loc}}$ and can be uniquely extended to a strongly continuous $*$ -automorphism τ_t of \mathfrak{U} . This can be proved for a large class of interactions, including finite range and exponentially decaying ones [39, Theorem 6.2.11]. In particular, this holds for dynamical systems $(\mathfrak{U}, \iota, \tau)$ with interaction that satisfies for some $\lambda > 0$:

$$\|\Phi\|_{\lambda} := \sup_{n \in \mathbb{Z}^D} \sum_{X \ni n} \|\Phi(X)\| |X| N^{2|X|} e^{\lambda \text{diam}(X)} < \infty. \tag{130}$$

where $\text{diam}(X) = \max\{|x - y| : x, y \in X\}$. This is the type of interactions that we consider: Φ can include m -body interactions for any m , as long as the interaction drops at least exponentially with m and at least exponentially with the distance.

We thus define a dynamical system of a D -dimensional quantum spin lattice model, as defined in the main text Definition 2.2:

Definition A.1 (Dynamical System). A dynamical system of a quantum spin lattice is a triple $(\mathfrak{U}, \iota, \tau)$ where \mathfrak{U} is a quasi-local C^* -algebra (as constructed above), τ is a strongly continuous representation of the group \mathbb{R} by $*$ -automorphisms $\{\tau_t : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}\}_{t \in \mathbb{R}}$, and ι is a representation of the translation group \mathbb{Z}^D by $*$ -automorphisms $\{\iota_{\mathbf{x}} : \mathfrak{U} \xrightarrow{\sim} \mathfrak{U}\}_{\mathbf{x} \in \mathbb{Z}^D}$, such that for any $\Lambda \in P_f(\mathbb{Z}^D)$: $A \in \mathfrak{U}_{\Lambda} \implies \iota_{\mathbf{x}}(A) \in \mathfrak{U}_{\Lambda+\mathbf{x}}$ for all $\mathbf{x} \in \mathbb{Z}^D$. We further assume

that τ is such that $\tau_t \iota_{\mathbf{x}} = \iota_{\mathbf{x}} \tau_t, \forall t \in \mathbb{R}, \mathbf{x} \in \mathbb{Z}^D$, i.e. time evolution is homogeneous. We use the notation $A(\mathbf{x}, t) := \iota_{\mathbf{x}} \tau_t A$.

A state of the dynamical system is defined as:

Definition A.2 (States, Invariance). A state of a dynamical system $(\mathfrak{U}, \iota, \tau)$ is a positive linear functional $\omega : \mathfrak{U} \rightarrow \mathbb{C}$ such that $\|\omega\| = 1$. The set of states is denoted by $E_{\mathfrak{U}}$. A state $\omega \in E_{\mathfrak{U}}$ is called space invariant if $\omega(\iota_{\mathbf{n}}(A)) = \omega(A), \forall A \in \mathfrak{U}, \mathbf{n} \in \mathbb{Z}^D$ and time invariant if $\omega(\tau_t(A)) = \omega(A), \forall A \in \mathfrak{U}, t \in \mathbb{R}$. We will refer to a space and time invariant state simply as invariant.

Of great importance in all our results is the Lieb–Robinson bound, which we state as in [63, Corollary 4.3.3]:

Theorem A.1 (Lieb–Robinson Bound). Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$ (Definition 2.2) with interaction Φ that satisfies Equation (130), that is

$$\|\Phi\|_{\lambda} := \sup_{\mathbf{n} \in \mathbb{Z}^D} \sum_{X \ni \mathbf{n}} \|\Phi(X)\| |X| N^{2|X|} e^{\lambda \text{diam}(X)} < \infty \tag{131}$$

for some $\lambda > 0$. Then, there exists a $v_{LR} > 0$ such that for all local $A, B \in \mathfrak{U}_{\text{loc}}$ with supports $\text{supp}(A) = \Lambda_A \in P_f(\mathbb{Z}^D), \text{supp}(B) = \Lambda_B \in P_f(\mathbb{Z}^D)$ respectively, and all $t \in \mathbb{R}$ we have the bound

$$\|[\tau_t(A), B]\| \leq 4\|A\| \|B\| |\Lambda_A| |\Lambda_B| N^{2|\Lambda_A|} \exp(-\lambda(\text{dist}(A, B) - v_{LR}|t|)) \tag{132}$$

with $v_{LR} = 2 \frac{\|\Phi\|_{\lambda}}{\lambda}$ called the Lieb–Robinson velocity.

Appendix B: Ergodicity in the Various Completions of the Space of Local Observables

Consider any positive-semidefinite sesquilinear form $\langle \cdot, \cdot \rangle$ defined on elements of $\mathfrak{U}_{\text{loc}}$. For example, this can be the connected correlator $(A, B) = \omega(A^\dagger B) - \omega(A^\dagger)\omega(B)$, or other forms defined in the main text, such as $(A, B)^{\hat{\mathbf{k}}, \perp}$. We define, by the usual construction, a Hilbert space \mathcal{H} from $\langle \cdot, \cdot \rangle$. On it, we assume that we have space-time translation group actions, $\iota_{\mathbf{x}}$ (for $\mathbf{x} \in \mathbb{Z}^d$) and τ_t . For the case of $(A, B)^{\hat{\mathbf{k}}, \perp}$, as summation over the perpendicular plane is done, there remain only a one-dimensional translation group, $d = 1$. We say that we have the system $(\mathcal{H}, \iota, \tau)$.

We can generalise the definitions of clustering of Sect. 3.1 for any such forms, so that we define: a subset $\mathfrak{C} \subset \mathfrak{U}_{\text{loc}}$ to be called uniformly p -clustering, with respect to $\langle \cdot, \cdot \rangle$, for $p > D$ if there exist a $C > 0$ s.t. for all $A, B \in \mathfrak{C}$:

$$|\langle \iota_{\mathbf{x}} A, B \rangle| \leq C(1 + |\mathbf{x}|)^{-p}, \quad \forall \mathbf{x} \in \mathbb{Z}^d. \tag{133}$$

Likewise, we can adapt Definition 3.3 of space-like p_c clustering to $\langle \cdot, \cdot \rangle$, and we define, as per [28, Def 3.1, Theorem 3.2], $\langle \cdot, \cdot \rangle$ to be space-like ergodic when there exists a v_c such that for every $A, B \in \mathfrak{U}_{\text{loc}}$ and $v > v_c$:

$$\lim_N \frac{1}{N} \sum_{m=0}^{N-1} \langle \iota_{\mathbf{n}}^m \tau_{v^{-1}|\mathbf{n}|}^m(A), B \rangle = 0. \tag{134}$$

The proof of the almost everywhere ergodicity theorem [28, Theorem 3.2] can now be adapted to the system $(\mathcal{H}, \iota, \tau)$, as long as space-like ergodicity holds.

Further, with space-like p_c -clustering for $p_c > d$, we can define an equivalence relation and similarly to the construction of the spaces of extensive quantities, we can construct a Hilbert space by the completion of the space of equivalence classes. We can also extend the action of ι, τ on the Hilbert space and it will be unitary. This is a matter of replicating the methods in [29, Section 5].

Appendix C: Space-Like Clustering Proof

Throughout this appendix we use the notation $\iota_x \tau_t A$ for space-time translations of observables $A \in \mathfrak{U}$, in order to maintain visual clarity of expressions. We prove the following proposition:

Theorem C.1. *Consider a dynamical system $(\mathfrak{U}, \iota, \tau)$ with interaction satisfying Equation (130), in an exponentially clustering state (Definition 3.2). It follows that the system is space-like exponentially clustering with respect to ω , Definition 3.3, for the Lieb–Robinson velocity v_{LR} .*

First, we show the first condition of Definition 3.3. The Lieb–Robinson bound allows us to approximate the time evolved observables by local ones, by projecting the time evolved $\tau_t(A)$, $A \in \mathfrak{U}_{loc}$ onto local ones $\sigma_\Lambda(\tau_t(A))$ supported on finite $\Lambda \subset \mathbb{Z}^D$. This is done by using the result [64, Corollary 4.4] and satisfies the first condition of Definition 3.3:

Lemma C.2. *Let $A \in \mathfrak{U}$ and consider a finite $\Lambda \subset \mathbb{Z}^D$. If there is an $\varepsilon > 0$ such that*

$$s\|[A, B]\| \leq \varepsilon \|A\| \|B\| \quad \forall B \in \mathfrak{U}_{\mathbb{Z}^D \setminus \Lambda} \tag{135}$$

then we can approximate A by a strictly local $\sigma_\Lambda(A) \in \mathfrak{U}_\Lambda$:

$$\|\sigma_\Lambda(A) - A\| \leq 2\varepsilon \|A\|. \tag{136}$$

This Lemma states that if an observable $A \in \mathfrak{U}$ almost commutes with every $B \in \mathfrak{U}_{\mathbb{Z}^D \setminus \Lambda}$ supported outside a finite Λ , then it can be well approximated by a local $\sigma_\Lambda(A) \in \mathfrak{U}_\Lambda$. Combined with the Lieb–Robinson bound Equation (132) we can prove (this is also described in [63, Chapter 4.3]):

Theorem C.3. *Consider a dynamical system with exponentially decaying interactions, the time evolution $\tau_t A$ of a local $A \in \mathfrak{U}_\Lambda$ and the finite sets $\Lambda_r = \cup_{\mathbf{x} \in \Lambda} B_{\mathbf{x}}(r)$, $r = 1, 2, 3, \dots$, where $B_{\mathbf{x}}(r)$ is the ball of radius $r > 0$ around \mathbf{x} , i.e. Λ_r is Λ extended by a distance r around all of its points. Then, we can approximate $\tau_t A$ by the local $\sigma_{\Lambda_r}(\tau_t A) \in \mathfrak{U}_{\Lambda_r}$:*

$$\|\sigma_{\Lambda_r}(\tau_t(A)) - \tau_t(A)\| \leq 2\varepsilon_r \|A\| \tag{137}$$

with $\varepsilon_r = C|\Lambda|N^{2|\Lambda|} \exp -\lambda(r - v_{LR}|t|)$, $r = 1, 2, 3, \dots$.

Using this result we can show the first condition of space-like clustering:

Proof of 3.3.1. The projection σ allows us to define the sequences $\sigma_r(A) := \sigma_{\Lambda_r}A$ required by condition 1 of Definition 3.3. Consider $\mathbf{n} \in \mathbb{Z}^D$ and $A, B \in \mathfrak{U}_{loc}$ with supports

$$\text{supp}(A) := \Lambda_A \quad \text{supp}(B) := \Lambda \quad \text{and} \quad \text{supp}(\iota_{\mathbf{n}}A) := \Lambda_A + \mathbf{n} \quad (138)$$

Let $r \in \mathbb{N}$ and by Theorem C.3 consider the local approximation of $\tau_t B$ by $\sigma_{\Lambda_r}(\tau_t B)$, supported in $\Lambda_r = \cup_{x \in \Lambda} B_x(r)$:

$$\|\sigma_{\Lambda_r} \tau_t B - \tau_t B\| \leq 2C|\Lambda|N^{2|\Lambda|}\|B\| \exp(-\lambda(r - v_{LR}|t|)) \quad (139)$$

Using this we can approximate the time evolution of observables by a local sequence $\sigma_r(\tau_t A) := \sigma_{\Lambda_r}(\tau_t A)$ so that $\lim_r \sigma_r(\tau_t A) = \tau_t A$. \square

We show the uniformity condition 2 of the elements $\sigma_r(\tau_t A)$ in the end of the appendix, we first proceed to show the third condition of Definition 3.3.

Proof of 3.3.2. Consider the quantity $I := |\omega(\iota_{\mathbf{n}}(A)\sigma_{\Lambda_r}(\tau_t B)) - \omega(A)\omega(\sigma_{\Lambda_r}(\tau_t B))|$, $r \in \mathbb{N}$ where we are interested in the large r limit. By linearity and the triangle inequality, it holds for any $r, l \in \mathbb{N}$:

$$\begin{aligned} I &= \left| \omega\left(\iota_{\mathbf{n}}(A)(\sigma_{\Lambda_r}(\tau_t B) + \sigma_{\Lambda_l}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B))\right) \right. \\ &\quad \left. - \omega(A)\omega(\sigma_{\Lambda_r}(\tau_t B) + \sigma_{\Lambda_l}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B)) \right| \\ &\leq \left| \omega(\iota_{\mathbf{n}}(A)\sigma_{\Lambda_l}(\tau_t B)) - \omega(A)\omega(\sigma_{\Lambda_l}(\tau_t B)) \right| + \\ &\quad + \left| \omega(\iota_{\mathbf{n}}(A)(\sigma_{\Lambda_r}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B))) - \omega(A)\omega(\sigma_{\Lambda_r}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B)) \right| \end{aligned} \quad (140)$$

The idea is to control the first part using exponential clustering and the second one using the approximation of time evolution. We now estimate the quantities $T_1 = |\omega(\iota_{\mathbf{n}}(A)\sigma_{\Lambda_l}(\tau_t B)) - \omega(A)\omega(\sigma_{\Lambda_l}(\tau_t B))|$ and $T_2 = |\omega(\iota_{\mathbf{n}}(A)(\sigma_{\Lambda_r}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B))) - \omega(A)\omega(\sigma_{\Lambda_r}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B))|$. Consider r large enough and

$$l = \lceil \varepsilon \text{dist}(\Lambda_A + \mathbf{n}, \Lambda) \rceil + \lceil \varepsilon \text{diam}(\Lambda_A \cup \Lambda) \rceil + 2 \quad \text{for some } \frac{1}{2} < \varepsilon < 1 \quad (141)$$

We first estimate T_1 , using exponential clustering

$$T_1 \leq k_{\Lambda_A + \mathbf{n}, \Lambda_l} e^{-\lambda \text{dist}(\Lambda_A + \mathbf{n}, \Lambda_l)} \quad (142)$$

Where we have the following bound for $k(\Lambda_A + \mathbf{n}, \Lambda_l)$:

$$k_{\Lambda_A + \mathbf{n}, \Lambda_l} \leq u\|A\|\|\sigma_{\Lambda_l}(\tau_t B)\|\|\Lambda_A\|^r |\Lambda_l|^r \quad (143)$$

Obviously $|\Lambda_A + \mathbf{n}| = |\Lambda_A|$ and $|\Lambda_l| \leq |B_0(l)||\Lambda|$, where $|B_0(l)|$ is the number of lattice points in the D-ball of radius l , which is a polynomial in l of degree D , hence it can be bounded by Kl^D for some constant $K > 0, \forall l \geq 1$. Using this, the chosen value for l (Eq. (141)) and the triangle inequality for $\text{dist}(\Lambda_A + \mathbf{n}, \Lambda_l)$:

$$\begin{aligned} |\Lambda_l| &\leq |\Lambda|K(\lceil \varepsilon \text{dist}(\Lambda_A + \mathbf{n}, \Lambda) \rceil + \lceil \varepsilon \text{diam}(\Lambda_A \cup \Lambda) \rceil + 2)^D \\ &\leq |\Lambda|K\varepsilon^D (\text{dist}(\Lambda_A + \mathbf{n}, \Lambda) + \text{diam}(\Lambda_A \cup \Lambda) + 4)^D \\ &\leq |\Lambda|K\varepsilon^D (|\mathbf{n}| + \text{dist}(\Lambda_A, \Lambda) + \text{diam}(\Lambda_A \cup \Lambda) + 4)^D \end{aligned} \quad (144)$$

Since $(|\mathbf{n}| + \text{dist}(\Lambda_A, \Lambda) + \text{diam}(\Lambda_A \cup \Lambda) + 2)^D$ is a polynomial in $|\mathbf{n}|$ of degree D it can be bounded by $L|\mathbf{n}|^D$ for some constant $L > 0$ (depending on $\text{dist}(\Lambda_A, \Lambda)$, $\text{diam}(\Lambda_A \cup \Lambda)$), $\forall |\mathbf{n}| \geq 1$. Hence:

$$|\Lambda_l| \leq |\Lambda| K L \varepsilon^D |\mathbf{n}|^D \quad (145)$$

By the explicit construction of the projection σ in [64] it holds that $\|\sigma_{\Lambda_l} C\| \leq \|C\|$, $\forall C \in \mathfrak{U}$. Additionally, since τ_t is a *-automorphism for all $t \in \mathbb{R}$, it is norm preserving, hence

$$\|\sigma_{\Lambda_l}(\tau_t B)\| \leq \|B\| \quad (146)$$

Next, we can estimate $\text{dist}(\Lambda_A + \mathbf{n}, \Lambda_l)$ with respect to $|\mathbf{n}|$ by simple geometric arguments:

$$\begin{aligned} \text{dist}(\Lambda_A + \mathbf{n}, \Lambda_l) &\geq \text{dist}(\Lambda_A + \mathbf{n}, \Lambda) - l \geq (1 - \varepsilon) \text{dist}(\Lambda_A + \mathbf{n}, \Lambda) \\ &\quad - \varepsilon \text{diam}(\Lambda_A \cup \Lambda) - 2 \end{aligned} \quad (147)$$

and

$$\begin{aligned} \text{dist}(\Lambda_A + \mathbf{n}, \Lambda) &= \min\{|\mathbf{y} - \mathbf{z}| : \mathbf{y} \in \Lambda_A + \mathbf{n}, \mathbf{z} \in \Lambda\} \\ &= \min\{|\mathbf{y} + \mathbf{n} - \mathbf{z}| : \mathbf{y} \in \Lambda_A, \mathbf{z} \in \Lambda\} \\ &\geq \min\{|\mathbf{n}| - |\mathbf{y} - \mathbf{z}| : \mathbf{y} \in \Lambda_A, \mathbf{z} \in \Lambda\} \\ &= |\mathbf{n}| - \max\{|\mathbf{y} - \mathbf{z}| : \mathbf{y} \in \Lambda_A, \mathbf{z} \in \Lambda\} \\ &\geq |\mathbf{n}| - \text{diam}(\Lambda_A \cup \Lambda) \end{aligned} \quad (148)$$

Putting these together we get

$$e^{-\lambda \text{dist}(\Lambda_A + \mathbf{n}, \Lambda)} \leq e^{2\lambda} e^{\lambda \text{diam}(\Lambda_A \cup \Lambda)} e^{-\lambda(1-\varepsilon)|\mathbf{n}|} \quad (149)$$

and combining all the estimates for the terms in T_1 , inequality (142) becomes

$$T_1 \leq u e^{2\lambda} K^r L^r |\Lambda_A|^r |\Lambda|^r \varepsilon^{rD} \|A\| \|B\| e^{\lambda \text{diam}(\Lambda_A \cup \Lambda)} |\mathbf{n}|^{rD} e^{-\lambda(1-\varepsilon)|\mathbf{n}|} \quad (150)$$

To estimate T_2 , we use the triangle inequality and note that ω has norm 1:

$$T_2 \leq \|\iota_{\mathbf{n}} A\| \|\sigma_{\Lambda_r}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B)\| + \|A\| \|\sigma_{\Lambda_r}(\tau_t B) - \sigma_{\Lambda_l}(\tau_t B)\| \quad (151)$$

where $\|\iota_{\mathbf{n}} A\| = \|A\|$. Using the approximation of $\tau_t B$, inequality (139), we get

$$T_2 \leq 2\|A\| 2C |\Lambda| N^{2|\Lambda|} \|B\| (\exp\{-\lambda(r - v_{LR}|t|)\} + \exp\{-\lambda(l - v_{LR}|t|)\}) \quad (152)$$

We consider a compact subset $T := \varepsilon' v_{LR}^{-1} [-|\mathbf{n}|, |\mathbf{n}|]$ for some $0 < \varepsilon' < \varepsilon$, and let $t \in T$. Then, for the value (141) of l , since $\lfloor y \rfloor \geq y - 1, \forall y > 0$, we have:

$$\begin{aligned} l - v_{LR}|t| &\geq \varepsilon \text{dist}(\Lambda_A + \mathbf{n}, \Lambda) - 1 + \varepsilon \text{diam}(\Lambda_A \cup \Lambda) - 1 + 2 - \varepsilon' |\mathbf{n}| \\ &\geq \varepsilon |\mathbf{n}| - \varepsilon \text{diam}(\Lambda_A \cup \Lambda) + \varepsilon \text{diam}(\Lambda_A \cup \Lambda) - \varepsilon' |\mathbf{n}| \\ &= (\varepsilon - \varepsilon') |\mathbf{n}| > 0 \end{aligned} \quad (153)$$

where in the second line we used relation (148). Hence:

$$\exp\{-\lambda(l - v_{LR}|t|)\} \leq \exp\{-\lambda(\varepsilon - \varepsilon')|\mathbf{n}|\} \quad \forall t \in T, 0 < \varepsilon' < \varepsilon < 1 \quad (154)$$

and also $\lim_r \exp\{-\lambda(r - v_{LR}|t|)\} = 0, \forall t \in T$. We can now take \lim_r of inequality 140, where the left-hand side becomes $\lim_r I = |\omega(\iota_{\mathbf{n}} A \tau_t B) -$

$\omega(A)\omega(B)$ by continuity. The right-hand side is bounded by the estimates (150), (151), (154), to get:

$$\begin{aligned} & |\omega(\iota_n A \tau_t B) - \omega(A)\omega(B)| \leq \\ & ue^{2\lambda} K^r L^r |\Lambda_A|^r |\Lambda|^r \varepsilon^{rD} \|A\| \|B\| e^{\lambda \text{diam}(\Lambda_A \cup \Lambda)} |\mathbf{n}|^{rD} e^{-\lambda(1-\varepsilon)|\mathbf{n}|} \\ & + 4CN^{2|\Lambda|} |\Lambda| \|A\| \|B\| e^{-\lambda(\varepsilon-\varepsilon')|\mathbf{n}|} \end{aligned} \quad (155)$$

Concluding the proof, note that for any $0 < \tilde{\lambda} < \lambda(1 - \varepsilon)$ there exists a large enough $G > 0$ s.t. $|\mathbf{n}|^{rD} e^{-\lambda(1-\varepsilon)|\mathbf{n}|} \leq G e^{-\tilde{\lambda}|\mathbf{n}|}$. Hence we arrive at the result by defining $\mu := \min(\tilde{\lambda}, \lambda(\varepsilon - \varepsilon')) > 0$ and

$$c = \|A\| \|B\| |\Lambda| (ue^{-2} K^r L^r |\Lambda_A|^r |\Lambda|^{r-1} \varepsilon^{rD} e^{\lambda \text{diam}(\Lambda_A \cup \Lambda)} G + 4CN^{2|\Lambda|}) \quad (156)$$

to get that there exist $c > 0$ and a $\mu > 0$ s.t.

$$|\omega(\iota_n A \tau_t B) - \omega(A)\omega(B)| \leq ce^{-\mu|\mathbf{n}|} \quad (157)$$

for all $\mathbf{n} \in \mathbb{Z}^D$ and for any $t \in T = \varepsilon' v_{LR}[-|\mathbf{n}|, |\mathbf{n}|]$, with any choice $0 < \varepsilon' < \varepsilon$ and $1/2 < \varepsilon < 1$. Hence, for any $v = v_{LR}/\varepsilon' > v_{LR}$ and choosing $t = -v^{-1}|\mathbf{n}|$, we have

$$|\omega(\iota_n A \tau_{-v^{-1}|\mathbf{n}|} B) - \omega(A)\omega(B)| \leq ce^{-\mu|\mathbf{n}|} \quad \forall \mathbf{n} \in \mathbb{Z}^D \quad (158)$$

and taking advantage of the time invariance of ω , we have established exponential space-like clustering, and as a consequence space-like p -clustering for all p . \square

It remains to prove the uniformity condition, i.e. that the set $\{(\sigma_n \tau_t a, \sigma_m \tau_s b) : n, m \in \mathbb{N}\}$ is uniformly clustering, Equation (52) in Definition 3.1.

Proof of 3.3.3. We start with $I := |\omega(\iota_n(A)\sigma_{\Lambda_r}(\tau_t B)) - \omega(A)\omega(\sigma_{\Lambda_r}(\tau_t B))|$ and choose $l = \text{dist}(\Lambda_A + \mathbf{n}, \Lambda)$. If $r < l$ then we bound I by the space clustering property, as in Equation (142):

$$I_{r < l} \leq u |\Lambda_A|^r |\Lambda_l|^r \|A\| \|B\| e^{-\lambda \text{dist}(\Lambda_A + \mathbf{n}, \Lambda_l)} \quad (159)$$

and $e^{-\lambda \text{dist}(\Lambda_A + \mathbf{n}, \Lambda_r)} \leq e^{-\lambda|\mathbf{n}|} e^{\lambda \text{diam}(\Lambda_A \cup \Lambda) - \lambda l} = e^{-2\lambda|\mathbf{n}|} e^{2\lambda \text{diam}(\Lambda_A \cup \Lambda)}$. Thus, we get

$$I_{r < l} \leq u |\Lambda_A|^r |\Lambda|^r \|A\| \|B\| e^{2\lambda \text{diam}(\Lambda_A \cup \Lambda)} e^{-2\lambda|\mathbf{n}|} \quad (160)$$

Now, if $r > l$ we proceed as in Eq. 140 to get

$$\begin{aligned} I & \leq I_{r < l} + 2 \|A\| 2C |\Lambda| N^{2|\Lambda|} \|B\| (\exp\{-\lambda(r - v_{LR}|t|)\} + \exp\{-\lambda(l - v_{LR}|t|)\}) \\ & \leq I_{r < l} + 4 \|A\| 2C |\Lambda| N^{2|\Lambda|} \|B\| \exp\{-\lambda(l - v_{LR}t)\} \\ & \leq I_{r < l} + 4 \|A\| 2C |\Lambda| N^{2|\Lambda|} \|B\| e^{v_{LR}t} e^{-\lambda \text{dist}(\Lambda_A + \mathbf{n}, \Lambda)} \\ & \leq I_{r < l} + 4 \|A\| 2C |\Lambda| N^{2|\Lambda|} \|B\| e^{v_{LR}t} e^{\lambda \text{diam}(\Lambda_A \cup \Lambda)} e^{-\lambda|\mathbf{n}|} \end{aligned} \quad (161)$$

Hence, we can overall obtain a uniform in r exponential clustering. Similarly we can repeat the process for the replacement $\iota_n A \rightarrow \sigma_{r'} \tau_s \iota_n A$, which will finally yield uniform exponential clustering for the set $\{(\sigma_{r'} \tau_s a, \sigma_r \tau_t b) : r, r' \in \mathbb{N}\}$ for any $t, s \in \mathbb{R}$. This obviously implies uniform p -clustering for all p . \square

Appendix D: Sesquilinear Forms Properties

Here we prove some basic properties for the sesquilinear forms used throughout the main text.

Lemma D.1. *For any $\mathbf{k} \in \mathbb{R}^D$ and $A \in \mathfrak{A}_{\text{loc}}$ in a p -clustering with $p > D$ dynamical system, it follows $\langle A, A \rangle_{\mathbf{k}} \geq 0$.*

Proof. Define $B = \sum_{\mathbf{x} \in [0, L]^D} e^{i\mathbf{k} \cdot \mathbf{x}} A(\mathbf{x}) \in \mathfrak{A}_{\text{loc}}$, for $L \in \mathbb{N}$. Then

$$\begin{aligned} 0 \leq (B, B) &= \sum_{\mathbf{x}, \mathbf{y} \in [0, L]^D} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} (A(\mathbf{x} - \mathbf{y}), A) \\ &= \sum_{\mathbf{z} \in [-L, L]^D} \left(\prod_{j=1}^D (L + 1 - |z_j|) \right) e^{i\mathbf{k} \cdot \mathbf{z}} (A(\mathbf{z}), A), \\ &\quad \mathbf{z} = (z_1, z_2, \dots, z_D), \\ &= (L + 1)^D \sum_{\mathbf{z} \in [-L, L]^D} e^{i\mathbf{k} \cdot \mathbf{z}} (A(\mathbf{z}), A) + o((L + 1)^{D-1}). \end{aligned} \tag{162}$$

We divide both sides of the inequality by $(L + 1)^D$ and the Lemma follows by taking the limit $L \rightarrow \infty$. That is because the first term becomes $\langle A, A \rangle_{\mathbf{k}}$, while the term $o((L + 1)^{D-1})$ tends to 0 for $p > D$, after diving by $(L + 1)^D$ and taking $L \rightarrow \infty$. This can be seen when writing $o((L + 1)^D)$ as a sum over all permutations σ of z_1, z_2, \dots, z_D and all $k = 1, 2, \dots, D$, of terms of the form:

$$(L + 1)^{D-k} \sum_{\mathbf{z} \in [-L, L]^D} \prod |\sigma(z_j)| e^{i\mathbf{k} \cdot \mathbf{z}} (A(\mathbf{z}), A) \quad k = 1, 2, \dots, D \tag{163}$$

where the product is taken over a choice of k elements of z_1, z_2, \dots, z_D . We can bound the absolute value of any such term as follows, where we denote $S_D(d) = |\{\mathbf{x} \in \mathbb{Z}^D : |\mathbf{x}| = d\}|$ the number of lattice points on a circle of radius d :

$$\begin{aligned} &(L + 1)^{D-k} \sum_{\mathbf{z} \in [-L, L]^D} |\mathbf{z}|^k \frac{1}{(|\mathbf{z}| + 1)^p} \leq (L + 1)^{D-k} \sum_{\mathbf{z} \in [-L, L]^D} \frac{(|\mathbf{z}| + 1)^k}{(|\mathbf{z}| + 1)^p} \\ &\leq (L + 1)^{D-k} \sum_{d=0}^{DL} \frac{1}{(d + 1)^{p-k}} S_D(d) \\ &\leq (L + 1)^{D-k} S_D(0) + C(L + 1)^{D-k} \sum_{d=1}^{DL} \frac{d^{D-1}}{(d + 1)^{p-k}} \quad , \text{ for } C > 0 \\ &\leq O(L^{D-k}) + C(L + 1)^{D-k} \sum_{d=1}^{DL} \frac{(d + 1)^{D-1}}{(d + 1)^{p-k}} \\ &\leq O(L^{D-k}) + C(L + 1)^{D-k} \int_0^{DL} dx (x + 1)^{D-p+k-1} \\ &= O(L^{D-k}) + C(L + 1)^{D-k} \left((DL + 1)^{D-p+k} - 1 \right). \end{aligned} \tag{164}$$

Dividing by $(L + 1)^D$ and taking $L \rightarrow \infty$ we see that the right-hand side vanishes for all $k = 1, 2, \dots, D$ when $p > D$. \square

A similar result can be obtained for the forms $(\cdot, \cdot)^{\widehat{k}, \perp}$ and $(\cdot, \cdot)_{\widehat{k}'}$:

Lemma D.2. *For any $\mathbf{k} \in \mathbb{R}_Q^D, \mathbf{k}' \in \mathbb{R}^D$ and $A \in \mathfrak{U}_{loc}$ in a p -clustering dynamical system with $p > D$, it follows that $(A, A)^{\widehat{k}, \perp} \geq 0$ and $(A, A)_{\widehat{k}'} \geq 0$.*

Proof. The important point about the proof of Lemma D.1 is that the form is obtained by a summation over all elements of a (sub-)group of \mathbb{Z}^D , of the action of these elements on one factor of the form; the proof works for any subgroup as clustering gives *a fortiori* a strong enough decay. Specifically, to see that $(A, A)^{\widehat{k}, \perp}$ is non-negative we write

$$B = \sum_{\mathbf{x} \in [0, L]^{D-1}} A(x_i \mathbf{h}^{(i)}) \tag{165}$$

and proceed as in Lemma D.1 from $\langle B, B \rangle \geq 0$. Similarly for $(A, A)_{\widehat{k}'}$. \square

Lemma D.3. *For any $A, B \in \mathfrak{U}$ that are p -clustering for $p > D$, $\langle A, B \rangle_k$ is uniformly bounded for $k \in \mathbb{R}^D$, and*

$$\lim_{k \rightarrow 0} \langle A, B \rangle_k = \langle A, B \rangle_0. \tag{166}$$

The same holds for $(a, a)_{\widehat{k}'}$.

Proof. The proof is an immediate generalisation of the proof of [29, Lemma 5.6] for $p > D$. \square

Appendix E: Unitarity of Space-Time Translations on Various Completions of \mathfrak{U}_{loc}

Consider the sesquilinear forms $(\cdot, \cdot)^{\widehat{k}, \perp}$ and $(\cdot, \cdot)_{\widehat{k}'}$ defined in Lemma 70. We have to establish that $\iota_x := \iota_{x\widehat{k}}$ ($x \in \mathbb{Z}$) and τ_t ($t \in \mathbb{R}$) act unitarily on $\mathbb{H}_{\widehat{k}}^{D-1}$ as representations of the groups \mathbb{Z} and \mathbb{R} respectively, and that τ_t ($t \in \mathbb{R}$) act unitarily on $\mathcal{H}_{\widehat{k}'}$.

We need to first show that these actions are indeed well defined on the equivalence classes. If $A \sim_{\widehat{k}, \perp} A'$ then also $\iota_z A \sim_{\widehat{k}, \perp} \iota_z A', \forall z \in \mathbb{Z}$, since

$$(\iota_z A - \iota_z A', \iota_z A - \iota_z A')^{\widehat{k}, \perp} = \sum_{\mathbf{x} \in \mathbb{H}_{\widehat{k}}^{D-1}} (\iota_{z\widehat{k}} \iota_{\mathbf{x}}(A - A'), \iota_{z\widehat{k}} \iota_{\mathbf{x}}(A - A')) \tag{167}$$

and ι acts unitarily inside the inner product (\cdot, \cdot) defined by the connected correlation, by space-time invariance of the state. Unitarity is also immediately established from this fact. Finally space translations are extended to the full Hilbert space $\mathcal{H}_{\widehat{k}, \perp}$ by continuous extension.

For τ_t the situation is more complicated, as it maps local elements to non-local ones. However, the treatment is the same as is in $D = 1$, see [29, Theorem 5.11].

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