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Spectrum and Orthogonality of the Bethe Ansatz for the Periodic q-Difference Toda Chain on $\mathbb{Z}_{m+1}$

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Abstract. By means of a q-boson–q-Toda correspondence pointed out by Duval and Pasquier, the n-particle hamiltonian of the periodic quantum relativistic Toda chain on $\mathbb{Z}_{m+1}$ is mapped to the hamiltonian of a previously studied lattice discretization of the Lieb–Liniger model (which encodes the dynamics of $m+1$ q-bosons on $\mathbb{Z}_{n+1}$ in the center-of-mass frame). The map in question makes it possible to retrieve quantum integrals and an orthogonal eigenbasis of Bethe Ansatz wave functions given by Hall–Littlewood polynomials for the pertinent periodic q-difference Toda chain from the corresponding quantum integrals and eigenbasis for the lattice Lieb–Liniger model. This approach entails the spectrum of the periodic q-difference Toda chain in terms of the critical points of associated Yang–Yang type Morse functions and links the diagonalization via the algebraic Bethe Ansatz performed by Duval and Pasquier directly to the spectral analysis of the lattice Lieb–Liniger model.

Mathematics Subject Classification. Primary: 33D52; Secondary: 05E05, 81Q35, 81Q80.

1. Introduction

In seminal work, Ruijsenaars introduced an integrable one-parameter deformation of the n-particle Toda chain as a relativistic generalization [17]. At the level of classical mechanics, the corresponding n-particle dynamics was first integrated in the case of an open chain [17] and henceforth by Bruschi and Ragnisco in the situation of a closed (periodic) chain [4,16]. At the level of quantum mechanics, the hamiltonian of the relativistic Toda chain turns out to be given by a (q-)difference operator, for which explicit quantum integrals were presented in [17] in the form of n independent commuting difference operators.

Influential studies into the integration of this particle model within the context of the (quantum) Yang–Baxter equation were performed by Suris [18] (at the classical level) and by Kuznetsov and Tsyganov [14] (both at the classical and quantum level). Moreover, it was pointed out by Etingof [7] that quantum groups provide a natural representation-theoretical framework for the diagonalization of Ruijsenaars' q-difference Toda chain in terms of q-Whittaker functions. Important progress regarding the study of the eigenfunctions in question can be found in [1,10–12] and further references therein.

In [9], it was shown that—upon restricting the motion of the q-difference Toda particles to the integer lattice $\mathbb{Z}$—the model can be diagonalized by an eigenbasis given by q-Whittaker functions that arise as a (t-)parameter degeneration of the Macdonald polynomials. The pertinent discrete q-difference Toda dynamics turns out to be of interest in the theory of integrable probability through its connection with the q-Whittaker process [3]. Combinatorial constructions involving quantum groups for the q-Whittaker functions on $\mathbb{Z}$ can be found in [5,8], whereas in [21] it was observed that—in the presence of integrable boundary interactions—the dynamics of the q-difference Toda chain can be restricted to the nonnegative integer semi-lattice $\mathbb{Z}_+$; the corresponding q-difference Toda chain with boundary interactions is diagonalizable in turn by q-Whittaker functions involving a (t-)parameter degeneration of the Macdonald–Koornwinder polynomials.

The present note addresses the spectral problem for the n-particle q-difference Toda chain on the periodic integer lattice $\mathbb{Z}_{m+1}$. The main idea is to map the corresponding eigenvalue problem to that of an integrable lattice discretization of the Lieb–Liniger particle model introduced in [20]. Indeed, it can be gleaned from work of Korff [13] (cf. also [23]) that this lattice Lieb–Liniger model can be interpreted as describing the dynamics of $m+1$ q-bosons on $\mathbb{Z}_{n+1}$ [2,19] projected onto the center-of-mass frame. By virtue of a q-boson–q-Toda correspondence pointed out by Duval and Pasquier [6], the pertinent q-boson dynamics can be mapped to that of the n-particle q-difference Toda chain on the periodic integer lattice $\mathbb{Z}_{m+1}$. This state of affairs brings us in the position to retrieve both quantum integrals and an orthogonal eigenbasis of Bethe Ansatz wave functions given by Hall–Littlewood polynomials for the q-difference Toda chain on $\mathbb{Z}_{m+1}$ from the quantum integrals and the corresponding eigenbasis constructed in [20] for the diagonalization of the lattice Lieb–Liniger model. In this approach, the spectrum of the periodic q-difference Toda chain is obtained via the critical points of strictly convex Yang–Yang-type Morse functions stemming from the lattice Lieb–Liniger model. As a result, a direct link is established between the spectral analysis of the lattice Lieb–Liniger model in [20] and the diagonalization of the periodic q-difference Toda chain via the algebraic Bethe Ansatz in [6].

The material is organized as follows. Section 2 recalls the periodic q-difference Toda hamiltonian and verifies its self-adjointness with respect to a q-multinomial weight function on bounded partitions. Section 3 derives the Bethe Ansatz wave function for the periodic q-difference Toda hamiltonian in terms of Hall–Littlewood polynomials together with the pertinent system of Bethe Ansatz equations. Via the critical points of the associated family of strictly convex Yang–Yang-type Morse functions, an orthogonal eigenbasis of Bethe Ansatz wave functions is constructed in Sect. 4. A complete system of commuting quantum integrals for the periodic q-difference Toda chain is retrieved in Sect. 5 from the corresponding quantum integrals for the lattice Lieb–Liniger model. Remarkably, the q-difference Toda quantum integrals obtained this way differ from the well-known quantum integrals originating from the work of Ruijsenaars [17]. Section 6 closes the presentation by detailing the relation between Ruijsenaars' quantum integrals and those obtained from the lattice Lieb–Liniger model more precisely. It turns out that in both cases the eigenvalue equations for the quantum integrals of the periodic q-difference Toda chain correspond to affine Pieri rules for periodic Hall–Littlewood functions, cf. [22]. From this perspective, the quantum integrals retrieved from the lattice Lieb–Liniger model correspond to Pieri rules that add a column to the partition of the Hall–Littlewood polynomial (cf. Appendix A) whereas Ruijsenaars' quantum integrals for the q-difference Toda chain correspond to the Pieri rules that add a row to the partition.

2. Periodic q-Difference Toda Hamiltonian

Upon fixing $m, n \in \mathbb{N}$, we consider a system of n quantum particles hopping over the periodic integer lattice

$$\mathbb{Z}_{m+1} = \mathbb{Z}/(m+1)\mathbb{Z} \cong \{0, 1, 2, \ldots, m\}. \tag{2.1}$$

The positions of these particles are represented by a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ in the configuration space

$$\Lambda^{(n,m)} = \{\mu \in \mathbb{Z}^n \mid m \geq \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0\}. \tag{2.2}$$

For $q \in (-1, 1)$, the q-difference Toda dynamics on $\mathbb{Z}_{m+1}$ is generated by the quantum hamiltonian:

$$H = D + D^*, \tag{2.3a}$$

with

$$D = \sum_{1 \leq i \leq n} (1 - q^{\mu_{i-1} - \mu_i}) T_i \quad \text{and} \quad D^* = \sum_{1 \leq i \leq n} (1 - q^{\mu_i - \mu_{i+1}}) T_i^{-1}, \tag{2.3b}$$

subject to the periodicity convention

$$\mu_0 \equiv \mu_n + m + 1 \quad \text{and} \quad \mu_{n+1} \equiv \mu_1 - m - 1. \tag{2.3c}$$

The hopping operators T_i and T_i^{-1} act on n-particle wave functions $\psi(\mu_1, \ldots, \mu_n)$ via unit translations of the ith particle

$$(T_i^\epsilon \psi)(\mu_1, \ldots, \mu_n) = \psi(\mu_1, \ldots, \mu_{i-1}, \mu_i + \epsilon, \mu_{i+1}, \ldots, \mu_n) \quad (\epsilon \in \{1, -1\}), \tag{2.4}$$

where the periodicity of the integer lattice $\mathbb{Z}_{m+1}$ gives rise to the following boundary conditions for $\mu \in \Lambda^{(n,m)}$:

$$\psi(m+1, \mu_2, \mu_3, \ldots, \mu_n) \equiv \psi(\mu_2, \mu_3, \ldots, \mu_n, 0) \tag{2.5a}$$

and

$$\psi(\mu_1, \mu_2, \ldots, \mu_{n-1}, -1) \equiv \psi(m, \mu_1, \mu_2, \ldots, \mu_{n-1}). \tag{2.5b}$$

The boundary conditions in Eqs. (2.5a), (2.5b) render a well-defined action of H (2.3a)–(2.3c) on wave functions $\psi : \Lambda^{(n,m)} \to \mathbb{C}$ of the form

$$(H\psi)(\mu) = \sum_{1 \le i \le n} \left((1 - q^{\mu_{i-1} - \mu_i}) \psi(\mu + e_i) + (1 - q^{\mu_i - \mu_{i+1}}) \psi(\mu - e_i) \right), \tag{2.6}$$

where the vectors $e_1, \ldots, e_n$ represent the standard unit basis for $\mathbb{Z}^n$. Notice in this connection that for $1 < i \le n$ the coefficient of $\psi(\mu + e_i)$ vanishes for $\mu \in \Lambda^{(n,m)}$ such that $\mu + e_i \notin \Lambda^{(n,m)}$, while for $1 \le i < n$ the coefficient of $\psi(\mu - e_i)$ vanishes for $\mu \in \Lambda^{(n,m)}$ such that $\mu - e_i \notin \Lambda^{(n,m)}$.

Let $\ell^2(\Lambda^{(n,m)}, \Delta)$ denote the $\binom{n+m}{n}$-dimensional Hilbert space of complex functions $\psi : \Lambda^{(n,m)} \to \mathbb{C}$ equipped with an inner product

$$\langle \psi, \phi \rangle_\Delta = \sum_{\mu \in \Lambda^{(n,m)}} \psi(\mu) \overline{\phi(\mu)} \Delta_\mu \qquad \left(\psi, \phi \in \ell^2(\Lambda^{(n,m)}, \Delta) \right) \tag{2.7a}$$

governed by q-multinomials on $\Lambda^{(n,m)}$:

$$\Delta_\mu = \frac{(q;q)_{m+1}}{\prod_{1 \le i \le n} (q;q)_{\mu_i - \mu_{i+1}}} \tag{2.7b}$$

($\mu \in \Lambda^{(n,m)}$), where

$$(a;q)_l = \begin{cases} 1 & \text{if } l = 0 \\ (1-a)(1-aq)\cdots(1-aq^{l-1}) & \text{if } l = 1, 2, 3, \ldots \end{cases}$$

Proposition 1. (Self-adjointness) *For $q \in (-1, 1)$, the periodic q-difference Toda hamiltonian H (2.3a)–(2.3c) is self-adjoint in $\ell^2(\Lambda^{(n,m)}, \Delta)$, because*

$$\forall \psi, \phi \in \ell^2(\Lambda^{(n,m)}, \Delta) : \qquad \langle D\psi, \phi \rangle_\Delta = \langle \psi, D^* \phi \rangle_\Delta.$$

Proof. Elementary manipulations reveal that $\forall \psi, \phi \in \ell^2(\Lambda^{(n,m)}, \Delta)$:

$$\begin{aligned} \langle D\psi, \phi \rangle_\Delta &= \sum_{1 \le i \le n} \sum_{\mu \in \Lambda^{(n,m)}} (1 - q^{\mu_{i-1} - \mu_i}) \psi(\mu + e_i) \overline{\phi(\mu)} \Delta_\mu \\ &= \sum_{1 \le i \le n} \sum_{\tilde{\mu} \in \Lambda^{(n,m)}} (1 - q^{\tilde{\mu}_i - \tilde{\mu}_{i+1}}) \psi(\tilde{\mu}) \overline{\phi(\tilde{\mu} - e_i)} \Delta_{\tilde{\mu}} = \langle \psi, D^* \phi \rangle_\Delta. \end{aligned}$$

Indeed, for all $1 \le i \le n$ and $\mu \in \Lambda^{(n,m)}$ one has that:

$$\psi(\mu + e_i) \overline{\phi(\mu)} = \psi(\tilde{\mu}) \overline{\phi(\tilde{\mu} - e_i)} \quad \text{and} \quad (1 - q^{\mu_{i-1} - \mu_i}) \Delta_\mu = (1 - q^{\tilde{\mu}_i - \tilde{\mu}_{i+1}}) \Delta_{\tilde{\mu}}$$

if $\tilde{\mu} = \mu + e_i \in \Lambda^{(n,m)}$, whereas by virtue of the convention in Eq. (2.3c) and the periodic boundary conditions in Eqs. (2.5a), (2.5b):

$$\psi(\mu + e_1) \overline{\phi(\mu)} = \psi(\tilde{\mu}) \overline{\phi(\tilde{\mu} - e_n)} \quad \text{and} \quad (1 - q^{\mu_0 - \mu_1}) \Delta_\mu = (1 - q^{\tilde{\mu}_n - \tilde{\mu}_{n+1}}) \Delta_{\tilde{\mu}}$$

if $\mu = (m, \mu_2, \ldots, \mu_n)$ and $\tilde{\mu} = (\mu_2, \ldots, \mu_n, 0)$. □

Remark. The asymmetric q-difference operators D and D^* (2.3b) commute and thus constitute normal operators in $\ell^2(\Lambda^{(n,m)}, \Delta)$ by virtue of Proposition 1. While in the literature, the term q-difference Toda hamiltonian often refers to either D or D^*, here the symmetric q-difference operator H (2.3a) is singled out as the quantum hamiltonian so as to guarantee that our q-difference Toda dynamics is generated by a self-adjoint operator in the Hilbert space. It is instructive to rewrite the periodic q-difference Toda operators in the form

$$D = \sum_{1\leq i\leq n} (1 - q^{(m+1)\delta_{i-1}} q^{\mu_{[i-1]}-\mu_i}) T_i \tag{2.8a}$$

and

$$D^* = \sum_{1\leq i\leq n} (1 - q^{(m+1)\delta_{n-i}} q^{\mu_i-\mu_{[i+1]}}) T_i^{-1}, \tag{2.8b}$$

where for $l \in \mathbb{Z}$ and $0 \leq i \leq n+1$:

$$\delta_l = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad [i] = \begin{cases} n & \text{if } i = 0, \\ i & \text{if } 1 \leq i \leq n, \\ 1 & \text{if } i = n+1 \end{cases} \tag{2.8c}$$

(so $[i] = i \mod \mathbb{Z}_n \cong \{1, 2, \ldots, n\}$). From these formulas, it is immediate that formally the corresponding operators for the open n-particle q-difference Toda chain on $\mathbb{Z}$ are recovered in the limit $m \to +\infty$.

3. Bethe Ansatz Wave Function

For $\lambda \in \mathbb{Z}^{m+1}$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+1} \geq \lambda_1 - n$ and $\xi = (\xi_1, \ldots, \xi_{m+1})$ in

$$\mathbb{R}^{m+1}_{0,\text{reg}} = \{\xi \in \mathbb{R}^{m+1}_0 \mid \xi_j - \xi_k \notin 2\pi\mathbb{Z},\ \forall 1 \leq j < k \leq m+1\}, \tag{3.1}$$

where $\mathbb{R}^{m+1}_0 \equiv \{\xi \in \mathbb{R}^{m+1} \mid \xi_1 + \cdots + \xi_{m+1} = 0\}$, let

$$R_\lambda(\xi_1, \ldots, \xi_{m+1}) = \sum_{\sigma \in S_{m+1}} C(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(m+1)}) \exp(\mathrm{i}\xi_{\sigma(1)}\lambda_1 + \cdots + \mathrm{i}\xi_{\sigma(m+1)}\lambda_{m+1}) \tag{3.2}$$

with

$$C(\xi_1, \ldots, \xi_{m+1}) = \prod_{1\leq j<k\leq m+1} \frac{1 - qe^{-\mathrm{i}(\xi_j-\xi_k)}}{1 - e^{-\mathrm{i}(\xi_j-\xi_k)}}.$$

Here the summation runs over all permutations $\sigma = \begin{pmatrix} 1 & 2 & \cdots & m+1 \\ \sigma(1) & \sigma(2) & \cdots & \sigma(m+1) \end{pmatrix}$ of the symmetric group S_{m+1}.

On $\mathbb{R}^{m+1}_{0,\text{reg}}$ the function $R_\lambda(\xi)$ (3.2) coincides with a Hall–Littlewood polynomial in the variables $e^{\mathrm{i}\xi_1}, \ldots, e^{\mathrm{i}\xi_{m+1}}$ [15, Chapter III.1] and thus extends smoothly from $\xi \in \mathbb{R}^{m+1}_{0,\text{reg}}$ to $\xi \in \mathbb{R}^{m+1}_0$. The restriction of the Hall–Littlewood polynomials to the hyperplane $\mathbb{R}_0^{m+1}$ ensures the following translational invariance

$$\forall \xi \in \mathbb{R}_0^{m+1}: \quad R_{(\lambda_1+1,\ldots,\lambda_{m+1}+1)}(\xi) = R_{(\lambda_1,\ldots,\lambda_{m+1})}(\xi), \tag{3.3}$$

which in turn prompts a restriction of λ to the fundamental alcove

$$\Lambda_0^{(m+1,n)} = \{\lambda \in \mathbb{Z}^{m+1} \mid n \geq \lambda_1 \geq \cdots \geq \lambda_m \geq \lambda_{m+1} = 0\}. \tag{3.4}$$

Notice that $|\Lambda^{(n,m)}| = |\Lambda_0^{(m+1,n)}| = \frac{(n+m)!}{n!\,m!}$. By mapping $\mu \in \Lambda^{(n,m)}$ to its conjugate partition ('with the columns and rows interchanged'):

$$\mu' = \left(0^{m+1-\mu_1} 1^{\mu_1-\mu_2} 2^{\mu_2-\mu_3} \cdots (n-1)^{\mu_{n-1}-\mu_n} n^{\mu_n}\right) \in \Lambda_0^{(m+1,n)} \tag{3.5}$$

one establishes an explicit bijection from $\Lambda^{(n,m)}$ onto $\Lambda_0^{(m+1,n)}$. More specifically, $\mu' = (\mu_1', \mu_2', \ldots, \mu_{m+1}')$ denotes the (unique) partition in $\Lambda_0^{(m+1,n)}$ such that

$$\mathrm{m}_i(\mu') = \begin{cases} m+1-\mu_1 & \text{if } i=0, \\ \mu_i - \mu_{i+1} & \text{if } 0<i<n, \\ \mu_n & \text{if } i=n, \end{cases} \tag{3.6}$$

where for any $\lambda \in \Lambda_0^{(m+1,n)}$ and $0 \leq i \leq n$

$$\mathrm{m}_i(\lambda) = |\{1 \leq j \leq m+1 \mid \lambda_j = i\}| \tag{3.7}$$

counts the multiplicity of i in λ. Reversely, for $\lambda \in \Lambda_0^{(m+1,n)}$ we will write λ' for its conjugate in $\Lambda^{(n,m)}$ (given by the unique $\mu \in \Lambda^{(n,m)}$ such that $\mu' = \lambda$).

Proposition 2 (Bethe Ansatz wave function). *For $q \in (-1,1)$ and $\xi = (\xi_1, \ldots, \xi_{m+1}) \in \mathbb{R}_{0,\mathrm{reg}}^{m+1}$ (3.1), let $\psi_\xi \in \ell^2(\Lambda^{(n,m)}, \Delta)$ be given by*

$$\psi_\xi(\mu) = R_{\mu'}(\xi_1, \ldots, \xi_{m+1}) \qquad (\mu \in \Lambda^{(n,m)}). \tag{3.8a}$$

The Bethe Ansatz wave function ψ_ξ (3.8a) solves the following eigenvalue equation for the periodic q-difference Toda hamiltonian H (2.3a)–(2.3c)

$$H\psi_\xi = (1-q)E(\xi)\psi_\xi \quad \textit{with} \quad E(\xi) = 2\sum_{1\leq j\leq m} \cos(\xi_j), \tag{3.8b}$$

provided the spectral parameter $\xi \in \mathbb{R}_{0,\mathrm{reg}}^{m+1}$ obeys an algebraic system of Bethe Ansatz equations of the form

$$e^{\mathrm{i}n\xi_j} = \epsilon \prod_{\substack{1\leq k\leq m+1 \\ k\neq j}} \frac{1-qe^{\mathrm{i}(\xi_j-\xi_k)}}{e^{\mathrm{i}(\xi_j-\xi_k)}-q} \quad \textit{for} \quad j=1,\ldots,m+1, \tag{3.8c}$$

with $\epsilon^{m+1} = 1$.

Proof. The proof hinges on the affine Pieri rules for the Hall–Littlewood polynomials in Eqs. (A.2a), (A.2b) of Appendix A, which are valid when $\xi \in \mathbb{R}_{0,\mathrm{reg}}^{m+1}$ obeys the Bethe Ansatz equations (3.8c) (cf. the remark at the end of the appendix). By canceling common factors from the numerators and denominators of the expansion coefficients on the RHS, the Pieri rules in question can be rewritten as

$$(e^{\mathrm{i}\xi_1}+\cdots+e^{\mathrm{i}\xi_{m+1}})R_\lambda(\xi) = (1-q)^{-1}\sum_{\substack{1\le j\le m+1\\ \underline{\lambda+e_j}\in\Lambda_0^{(m+1,n)}}} R_{\underline{\lambda+e_j}}(\xi)\bigl(1-q^{\mathrm{m}_{\lambda_j}(\lambda)+\mathrm{m}_n(\lambda)\delta_{\lambda_j}}\bigr) \tag{3.9a}$$

with $\underline{\lambda+e_j}=\lambda+e_j-\delta_{m+1-j}(e_1+\cdots+e_{m+1})$, and

$$(e^{-\mathrm{i}\xi_1}+\cdots+e^{-\mathrm{i}\xi_{m+1}})R_\lambda(\xi) = (1-q)^{-1}\sum_{\substack{1\le j\le m+1\\ \underline{\lambda-e_j}\in\Lambda_0^{(m+1,n)}}} R_{\underline{\lambda-e_j}}(\xi)\bigl(1-q^{\mathrm{m}_{\lambda_j}(\lambda)+\mathrm{m}_0(\lambda)\delta_{n-\lambda_j}}\bigr) \tag{3.9b}$$

with $\underline{\lambda-e_j}=\lambda-e_j+\delta_{m+1-j}(e_1+\cdots+e_{m+1})$. (Here δ_l is defined as in Eq. (2.8c).)

Upon substituting $\lambda=\mu'$ (3.5) with $\mu\in\Lambda^{(n,m)}$, the affine Pieri rules (3.9a),(3.9b) take the form

$$(e^{\mathrm{i}\xi_1}+\cdots+e^{\mathrm{i}\xi_{m+1}})\psi_\xi(\mu)=(1-q)^{-1}\sum_{1\le i\le n}\psi_\xi(\mu+e_i)\bigl(1-q^{\mu_{i-1}-\mu_i}\bigr) \tag{3.10a}$$

and

$$(e^{-\mathrm{i}\xi_1}+\cdots+e^{-\mathrm{i}\xi_{m+1}})\psi_\xi(\mu)=(1-q)^{-1}\sum_{1\le i\le n}\psi_\xi(\mu-e_i)\bigl(1-q^{\mu_i-\mu_{i+1}}\bigr), \tag{3.10b}$$

respectively. Here we use that for any $\mu\in\Lambda^{(n,m)}$ and $j\in\{1,\ldots,m+1\}$ one has at $\lambda=\mu'$:

$$\underline{\lambda+e_j}\in\Lambda_0^{(m+1,n)} \Longleftrightarrow \forall\psi\in\ell^2(\Lambda^{(n,m)},\Delta): \quad \psi\bigl((\underline{\lambda+e_j})'\bigr)=\psi(\mu+e_i) \text{ with } i=\lambda_j+1\in\{1,\ldots,n\}$$

and

$$\underline{\lambda-e_j}\in\Lambda_0^{(m+1,n)} \Longleftrightarrow \forall\psi\in\ell^2(\Lambda^{(n,m)},\Delta): \quad \psi\bigl((\underline{\lambda-e_j})'\bigr)=\psi(\mu-e_i) \text{ with } i=\lambda_j+n\delta_{m+1-j}\in\{1,\ldots,n\},$$

by virtue of the boundary conditions in Eqs. (2.5a), (2.5b) (cf. Figs. 1, 2, 3 and 4). After summing Eqs. (3.10a) and (3.10b) the asserted eigenvalue equation for the periodic q-difference Toda hamiltonian follows. □

Proposition 2 agrees with an alternative construction of the Hall–Littlewood-type wave functions for the periodic q-difference chain by means of the algebraic Bethe Ansatz in [6, Sections 5, 6].

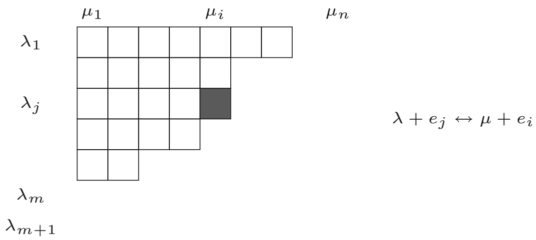


FIGURE 1. Adding a box to the partition

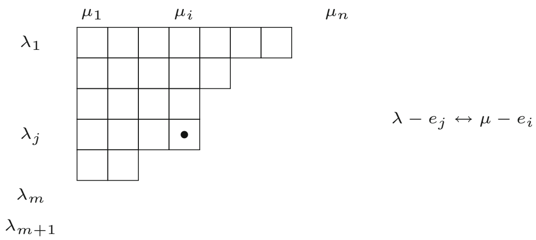


FIGURE 2. Deleting a box from the partition

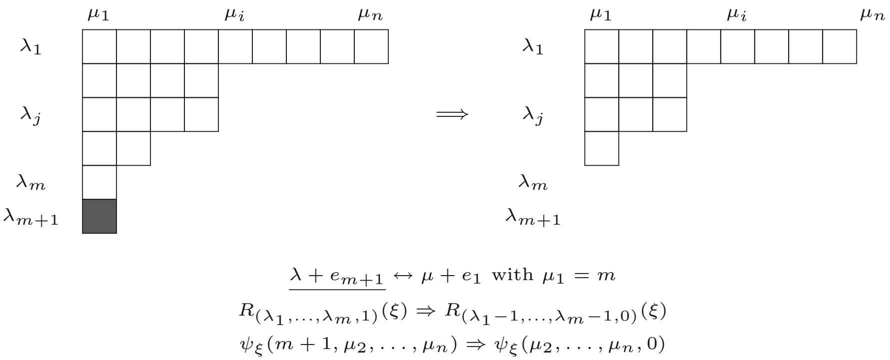
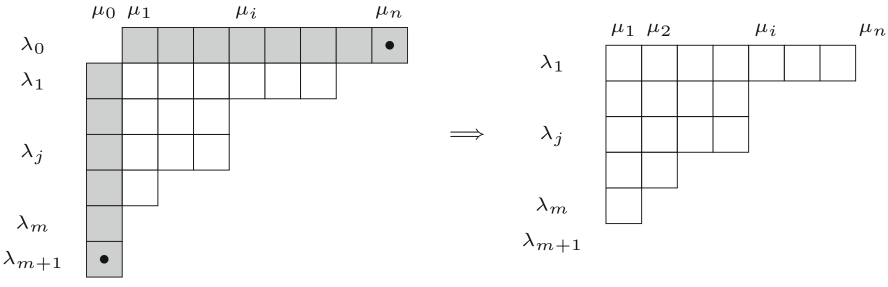


FIGURE 3. λ-Translational invariance vs μ-periodicity: column/row deletion

4. Spectrum and Orthogonal Eigenbasis

In [20, Section 4], the Bethe Ansatz equations of the previous section were recasted in a logarithmic form that describe the critical points of a Yang–Yang-type family of strictly convex Morse functions. Specifically, for any $\kappa \in \Lambda_0^{(m+1,n)}$ the following system of transcendental equations

$$n\xi_j + \sum_{\substack{1\leq k\leq m+1\\ k\neq j}} v_q(\xi_j - \xi_k) = 2\pi\left(\frac{m}{2} + 1 - j + \kappa_j - \frac{|\kappa|}{m+1}\right) \tag{4.1a}$$



$\underline{\lambda - e_{m+1}} \leftrightarrow \mu - e_n$ with $\mu_n = 0$ and $\lambda_0 \equiv \lambda_{m+1} + n$, $\mu_0 \equiv \mu_n + m + 1$

$R_{(\lambda_1,\ldots,\lambda_m,-1)}(\xi) \Rightarrow R_{(\lambda_1+1,\ldots,\lambda_m+1,0)}(\xi)$

$\psi_\xi(\mu_1,\ldots\mu_{n-1},-1) \Rightarrow \psi_\xi(m,\mu_1,\ldots,\mu_{n-1})$

FIGURE 4. λ-Translational invariance vs μ-periodicity: column/row addition

$j = 1,\ldots,m+1$ with

$$v_q(\vartheta) = \int_0^\vartheta \frac{(1-q^2)\,\mathrm{d}\theta}{1-2q\cos(\theta)+q^2} = \mathrm{i}\log\left(\frac{1-qe^{\mathrm{i}\vartheta}}{e^{\mathrm{i}\vartheta}-q}\right) \qquad (-1<q<1) \tag{4.1b}$$

provides a logarithmic form of the Bethe Ansatz equations (3.8c), with the value of the $(m+1)$th root of unity ϵ being equal to $(-1)^m e^{-\frac{2\pi\mathrm{i}|\kappa|}{m+1}}$, where

$$|\kappa| = \kappa_1 + \kappa_2 + \cdots + \kappa_{m+1}.$$

Indeed, upon multiplying Eq. (4.1a) by $\mathrm{i} = \sqrt{-1}$ and taking the exponential of both sides it is seen that any solution of Eq. (4.1a) gives rise to a solution of Eq. (3.8c) for the pertinent choice of ϵ. Solutions of Eq. (4.1a) are critical points of a smooth Morse function $V_\kappa : \mathbb{R}^{m+1} \to \mathbb{R}$ of the form

$$\begin{aligned} V_\kappa(\xi_1,\ldots,\xi_{m+1}) = &\sum_{1\le j<k\le m+1} \int_0^{\xi_j-\xi_k} v_q(\vartheta)\mathrm{d}\vartheta \\ &+ \sum_{1\le j\le m+1} \left(\frac{n}{2}\xi_j^2 - 2\pi\left(\frac{m}{2}+1-j+\kappa_j-\frac{|\kappa|}{m+1}\right)\xi_j\right). \end{aligned} \tag{4.2}$$

Since the Morse function in question is radially unbounded (i.e., $V_\kappa(\xi) \to +\infty$ if $\xi_1^2 + \cdots + \xi_{m+1}^2 \to +\infty$) and moreover strictly convex, given $\kappa \in \Lambda_0^{(m+1,n)}$ there exists only a single critical point corresponding to the global minimum of $V_\kappa(\xi_1,\ldots,\xi_{m+1})$.

For $q \in (-1,1)$ and any $\kappa \in \Lambda_0^{(m+1,n)}$, let $\boxed{\xi_\kappa \in \mathbb{R}^{m+1}}$ denote the unique global minimum of $V_\kappa(\xi_1,\ldots\xi_{m+1})$ (4.2). It is immediate from the analysis in [20, Section 4] that the solutions ξ_κ, $\kappa \in \Lambda_0^{(m+1,n)}$ for the Bethe Ansatz equations (3.8c) are all distinct and located within the open alcove

$$\mathbb{A}_0^{m+1} = \{(\xi_1, \ldots, \xi_{m+1}) \in \mathbb{R}_0^{m+1} \mid \xi_1 > \xi_2 > \cdots > \xi_{m+1} > \xi_1 - 2\pi\} \tag{4.3a}$$

($\subset \mathbb{R}^{m+1}_{0,\text{reg}}$) subject to the following constraints at $\xi = \xi_\kappa$:

$$\frac{2\pi(k-j+\kappa_j-\kappa_k)}{n+(m+1)\mathrm{K}_+} \leq \xi_j - \xi_k \leq \frac{2\pi(k-j+\kappa_j-\kappa_k)}{n+(m+1)\mathrm{K}_-} \tag{4.3b}$$

for $1 \leq j < k \leq m+1$, where $\mathrm{K}_\pm = \left(\frac{1+|q|}{1-|q|}\right)^{\pm 1}$. In particular, for $q \to 0$ the position of the minimum ξ_κ tends to

$$\xi_\kappa|_{q=0} = \frac{2\pi}{n+m+1}\biggl(\kappa_1 - \frac{|\kappa|}{m+1} + \frac{m}{2}, \ldots$$
$$\ldots, \kappa_j - \frac{|\kappa|}{m+1} + \frac{m}{2} + 1 - j, \ldots, \kappa_{m+1} - \frac{|\kappa|}{m+1} - \frac{m}{2}\biggr)$$

(where $\kappa_{m+1} = 0$).

Theorem 3 (Diagonalization). *For $q \in (-1,1)$ the Bethe Ansatz wave functions ψ_{ξ_κ}, $\kappa \in \Lambda_0^{(m+1,n)}$ constitute an orthogonal eigenbasis for the periodic q-difference Toda hamiltonian H (2.3a)–(2.3c) in the Hilbert space $\ell(\Lambda^{(n,m)}, \Delta)$:*

$$\forall \kappa \in \Lambda_0^{(m+1,n)}: \qquad H\psi_{\xi_\kappa} = (1-q)E(\xi_\kappa)\psi_{\xi_\kappa} \tag{4.4a}$$

and

$$\forall \kappa, \nu \in \Lambda_0^{(m+1,n)}: \qquad \langle \psi_{\xi_\kappa}, \psi_{\xi_\nu}\rangle_\Delta = 0 \quad \text{if } \kappa \neq \nu. \tag{4.4b}$$

In particular, the spectrum of H thus consists of the eigenvalues $(1-q)E(\xi_\kappa)$, $\kappa \in \Lambda_0^{(m,n)}$ (with $E(\xi)$ taken from Eq. (3.8b)).

Proof. Since ξ_κ solves the Bethe Ansatz equations and $\psi_\xi(0^n) = R_{(0^{m+1})}(\xi) = \frac{(q;q)_{m+1}}{(1-q)^{m+1}} \neq 0$ (cf. Equation (1.4) in [15, Chapter III.1]), it is clear from Proposition 2 that ψ_{ξ_κ} provides a nontrivial solution of the eigenvalue equation (4.4a). Rewriting the inner product in Eq. (4.4b) in terms of Hall–Littlewood polynomials entails that:

$$\langle \psi_{\xi_\kappa}, \psi_{\xi_\nu}\rangle_\Delta = \sum_{\mu \in \Lambda^{(n,m)}} \psi_{\xi_\kappa}(\mu)\overline{\psi_{\xi_\nu}(\mu)}\Delta_\mu = \sum_{\lambda \in \Lambda_0^{(m+1,n)}} R_\lambda(\xi_\kappa)\overline{R_\lambda(\xi_\nu)}\Delta'_\lambda, \tag{4.5a}$$

where the weights Δ'_λ are such that $\Delta'_{\mu'} = \Delta_\mu$, i.e., (cf. Eqs. (2.3c), (2.7b) and (3.6)):

$$\begin{aligned}\Delta'_\lambda &= \frac{(q;q)_{m+1}}{(q;q)_{\mathrm{m}_0(\lambda)+\mathrm{m}_n(\lambda))}\prod_{0<i<n}(q;q)_{\mathrm{m}_i(\lambda)}} \\ &= \frac{(q;q)_{m+1}}{(1-q)^{m+1}} \prod_{\substack{1\leq j<k\leq m+1\\ \lambda_j=\lambda_k}} \frac{1-q^{k-j}}{1-q^{1+k-j}} \prod_{\substack{1\leq j<k\leq m+1\\ \lambda_j=\lambda_k+n}} \frac{1-q^{m+1-k+j}}{1-q^{m+2-k+j}}.\end{aligned} \tag{4.5b}$$

(Indeed, by canceling common factors in the numerator and the denominator of the expression for Δ'_λ on the second line one readily recovers the expression on the first line.) By comparing the formulas, Eqs. (4.5a) and (4.5b) reveal that the asserted orthogonality in Eq. (4.4b) amounts to an affine orthogonality relation for the Hall–Littlewood polynomials established in [20, Theorem 5.2]. □

5. Commuting Quantum Integrals

In the proof of Proposition 2, the periodic q-difference Toda hamiltonian H (2.3a)–(2.3c) was retrieved from the corresponding lattice Lieb–Liniger Hamiltonian in [20, Section 5] as a pullback with respect to the mapping $\mu \to \mu'$ from $\Lambda^{(n,m)}$ onto $\Lambda_0^{(m+1,n)}$. By pulling back the higher quantum integrals from [20, Theorem 5.1] in a similar fashion, one arrives in turn at closed formulas for the corresponding quantum integrals of the periodic q-difference Toda hamiltonian in $\ell^2(\Lambda^{(n,m)})$. To this end, it is convenient to employ horizontal strips (cf. e.g., [15, Chapter I.1]) by defining for any $\nu \in \Lambda^{(n+1,m+1)}$ and $\mu \in \Lambda^{(n,m)}$:

$$\nu \succeq \mu \Longleftrightarrow m+1 \geq \nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_n \geq \mu_n \geq \nu_{n+1} \geq 0 \\ \text{with } \mathrm{m}_0(\nu) + \mathrm{m}_{m+1}(\nu) > 0 \tag{5.1}$$

(so $\nu \succeq \mu$ iff $\nu - \mu$ is a horizontal strip with $\nu'_1 - \nu'_{m+1} \leq n$). We will also need q-binomials:

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{(q;q)_k}{(q;q)_l (q;q)_{k-l}} \quad (\text{for } l = 0, 1, \ldots, k).$$

Armed with these notational conventions, we are in the position to define—for any $1 \leq r \leq m$—the following operator $D_r : \ell^2(\Lambda^{(n,m)}, \Delta) \to \ell^2(\Lambda^{(n,m)}, \Delta)$ via its action on $\psi \in \ell^2(\Lambda^{(n,m)}, \Delta)$ evaluated at $\mu \in \Lambda^{(n,m)}$:

$$(D_r\psi)(\mu) = \sum_{\substack{\nu \in \Lambda^{(n+1,m+1)} \\ \nu \succeq \mu,\, |\nu| = |\mu| + r}} \begin{bmatrix} \mu_n - \mu_{n+1} \\ \nu_1 + \nu_{n+1} - \mu_1 \end{bmatrix}_q \prod_{1 \leq i < n} \begin{bmatrix} \mu_i - \mu_{i+1} \\ \nu_{i+1} - \mu_{i+1} \end{bmatrix}_q \psi(\nu), \tag{5.2a}$$

where

$$\psi(\nu) \equiv \begin{cases} \psi(\nu_1, \nu_2, \ldots, \nu_n) & \text{if } \nu_1 \leq m, \\ \psi(\nu_2, \ldots, \nu_n, \nu_{n+1}) & \text{if } \nu_1 = m+1. \end{cases} \tag{5.2b}$$

Up to normalization, one recovers for $r = 1$ and $r = m$ the difference operators D and D^* building H (2.3a)–(2.3c):

$$D = (1-q)D_1 \quad \text{and} \quad D^* = (1-q)D_m. \tag{5.3}$$

Theorem 4 (Quantum integrability). *Let $q \in (-1, 1)$ and $1 \leq r \leq m$.*

(*i*) *The operators $D_1, \ldots, D_m$ (5.2a)–(5.2b) commute in $\ell^2\big(\Lambda^{(n,m)}, \Delta\big)$ and are simultaneously diagonalized by the orthogonal basis of Bethe Ansatz wave functions ψ_{ξ_κ}, $\kappa \in \Lambda_0^{(m+1,n)}$:*

$$D_r \psi_{\xi_\kappa} = E_r(\xi_\kappa)\psi_{\xi_\kappa} \tag{5.4a}$$

with

$$E_r(\xi) = \sum_{1 \leq j_1 < j_2 < \cdots < j_r \leq m+1} \exp\big(\mathrm{i}\xi_{j_1} + \mathrm{i}\xi_{j_2} + \cdots + \mathrm{i}\xi_{j_r}\big). \tag{5.4b}$$

(*ii*) *The quantum integrals D_r and D_{m+1-r} are each others adjoints in $\ell^2\big(\Lambda^{(n,m)},\Delta\big)$, i.e.,*

$$\forall \psi,\phi \in \ell^2\big(\Lambda^{(n,m)},\Delta\big): \qquad \langle D_r\psi,\phi\rangle_\Delta = \langle \psi, D_{m+1-r}\phi\rangle_\Delta. \tag{5.5}$$

(*iii*) *The quantum integrals $D_1,\ldots,D_m$ are complete in the sense that the full algebra of commuting quantum integrals for the periodic q-difference Toda chain*

$$\mathcal{I} \equiv \{I : \ell^2\big(\Lambda^{(n,m)},\Delta\big) \to \ell^2\big(\Lambda^{(n,m)},\Delta\big) \mid ID_r = D_rI \ \textit{for}\ r=1,\ldots,m\} \tag{5.6a}$$

is generated by the operators in question:

$$\mathcal{I} = \mathbb{C}[D_1,\ldots,D_m]. \tag{5.6b}$$

Proof. (*i*) It suffices to verify the eigenvalue equations (5.4a), (5.4b), because the commutativity then automatically follows from the fact that the Bethe Ansatz wave functions constitute a basis for $\ell^2\big(\Lambda^{(n,m)},\Delta\big)$ by virtue of Theorem 3. To infer the eigenvalue equations, we employ the affine Pieri rules in Eq. (A.2a) for $R_\lambda(\xi)$ at $\xi=\xi_\kappa$ (with $\lambda,\kappa\in\Lambda_0^{(m+1,n)}$):

$$E_r(\xi)R_\lambda(\xi) = \sum_{\substack{J\subset\{1,\ldots,m+1\},|J|=r\\ \underline{\lambda+e_J}\in\Lambda_0^{(m+1,n)}}} R_{\underline{\lambda+e_J}}(\xi)V_J(\lambda),$$

where $V_J(\lambda)$ is taken from Eq. (A.2b). For λ and $\underline{\lambda+e_J}$ in $\Lambda_0^{(m+1,n)}$, let

$$\Delta'_{\lambda,J} \equiv \frac{(q;q)_{|J|}}{(1-q)^{|J|}} \prod_{\substack{1\le j<k\le m+1\\ j,k\in J\\ \lambda_j=\lambda_k}} \frac{1-q^{k-j}}{1-q^{1+k-j}} \prod_{\substack{1\le j<k\le m+1\\ j,k\in J\\ \lambda_j=\lambda_k+n}} \frac{1-q^{m+1-k+j}}{1-q^{m+2-k+j}}$$
$$= \frac{(q;q)_{|J|}}{(q;q)_{\mathrm{m}_{0,J}(\lambda)+\mathrm{m}_{n,J}(\lambda))}\prod_{0<i<n}(q;q)_{\mathrm{m}_{i,J}(\lambda)}}$$

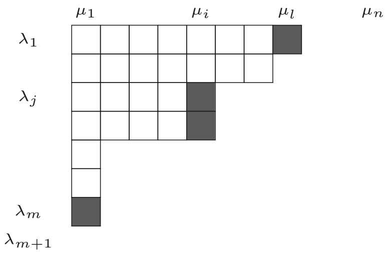
(cf. Equation (4.5b)), where $\mathrm{m}_{i,J}(\lambda)\equiv |\{j\in J\mid \lambda_j=i\}| = \sum_{0\le l\le i}\mathrm{m}_l(\lambda)-\mathrm{m}_l(\lambda+e_J)$. With the aid of $\Delta'_{\lambda,J}$, the coefficients $V_J(\lambda)$ (A.2b) are readily rewritten in terms of q-binomials as follows:

$$V_J(\lambda) = \begin{bmatrix} m+1\\ |J|\end{bmatrix}_q \frac{\Delta'_{\lambda,J}\Delta'_{\lambda,J^c}}{\Delta'_\lambda} = \begin{bmatrix} \mathrm{m}_0(\lambda)+\mathrm{m}_n(\lambda)\\ \mathrm{m}_{0,J}(\lambda)+\mathrm{m}_{n,J}(\lambda)\end{bmatrix}_q \prod_{0<i<n}\begin{bmatrix}\mathrm{m}_i(\lambda)\\ \mathrm{m}_{i,J}(\lambda)\end{bmatrix}_q.$$

Upon writing $\lambda=\mu'$ $(\in\Lambda_0^{(m+1,n)})$ with $\mu\in\Lambda^{(n,m)}$, and $\lambda+e_J = \nu'$ $(\in\Lambda^{(m+1,n+1)})$ with $\nu\in\Lambda^{(n+1,m+1)}$, we have—assuming $\underline{\lambda+e_J}\in\Lambda_0^{(m+1,n)}$—that $\nu\succeq\mu$, $R_{\underline{\lambda+e_J}}(\xi)=\psi_\xi(\nu)$ (cf. Figs. 5, 6), and

$$V_J(\lambda) = \begin{bmatrix}\mu_n-\mu_{n+1}\\ \nu_1+\nu_{n+1}-\mu_1\end{bmatrix}_q \prod_{1\le i<n}\begin{bmatrix}\mu_i-\mu_{i+1}\\ \nu_{i+1}-\mu_{i+1}\end{bmatrix}_q.$$

Hence, the affine Pieri rule passes over into the asserted eigenvalue equation (5.4a), (5.4b).



$$\nu' = (\lambda_1+1, \lambda_2, \ldots, \lambda_{j-1}, \lambda_j+1, \lambda_{j+1}+1, \lambda_{j+2}, \ldots, \lambda_{m-1}, \lambda_m+1, 0)$$

$$\updownarrow$$

$$\nu = (\mu_1+1, \mu_2, \ldots, \mu_{i-1}, \mu_i+2, \mu_{i+1}, \ldots, \mu_{l-1}, \mu_l+1, \mu_{l+1}, \ldots, \mu_n, 0)$$

FIGURE 5. Adding a strip to the partition

(ii) Upon exploiting once more that the Bethe Ansatz wave functions ψ_{ξ_κ}, $\kappa \in \Lambda_0^{(m+1,n)}$ constitute an orthogonal basis for $\ell^2\big(\Lambda^{(n,m)}, \Delta\big)$, the asserted adjointness $D_r^* = D_{m+1-r}$ in Eq. (5.5) is immediate from the diagonalization in the previous part *(i)* together with the elementary observation that

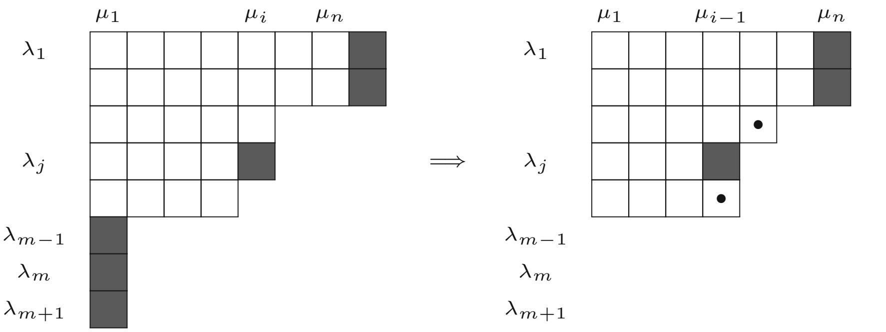
$$\overline{E_r(\xi)} = E_{m+1-r}(\xi) \quad \text{for } \xi \in \mathbb{R}_0^{m+1}.$$

(iii) The elementary symmetric polynomials $E_1(\xi), \ldots, E_m(\xi)$ (5.4b) separate the points of the fundamental alcove $\mathbb{A}_0^{m+1}$ (4.3a) and hence the points of the spectral variety $X_0^{(m+1,n)} \equiv \{\xi_\kappa \mid \kappa \in \Lambda_0^{(m+1,n)}\} \subset \mathbb{A}_0^{m+1}$. Any operator $I : \ell^2\big(\Lambda^{(n,m)}, \Delta\big) \to \ell^2\big(\Lambda^{(n,m)}, \Delta\big)$ commuting with $D_1, \ldots, D_m$ is therefore diagonalized by our basis of Bethe Ansatz wave functions, i.e.,

$$\forall \kappa \in \Lambda_0^{(m+1,n)} : \quad I\psi_{\xi_\kappa} = E_I(\xi_\kappa)\psi_{\xi_\kappa}$$

for a certain function $E_I : X_0^{(m+1,n)} \to \mathbb{C}$ collecting the corresponding eigenvalues. It is clear that the Hall–Littlewood polynomials $R_\lambda(\xi)$, $\lambda \in \Lambda_0^{(m+1,n)}$ restrict on $X_0^{(m+1,n)}$ to a basis for the $\binom{n+m}{m}$-dimensional space of complex functions on the spectral variety $X_0^{(m+1,n)}$ (again because ψ_{ξ_κ}, $\kappa \in \Lambda_0^{(m+1,n)}$ is a basis for $\ell^2\big(\Lambda^{(n,m)}, \Delta\big)$, so the matrix $[R_\lambda(\xi_\kappa)]_{\lambda,\kappa\in\Lambda_0^{(m+1,n)}}$ is of full rank $\binom{n+m}{m}$). In other words, $E_I : X_0^{(m+1,n)} \to \mathbb{C}$ can be written as a linear combination of Hall–Littlewood polynomials restricted to $X_0^{(m+1,n)}$. Since the elementary symmetric polynomials $E_1(\xi), \ldots, E_m(\xi)$ generate the space of symmetric polynomials in $e^{i\xi_1}, \ldots, e^{i\xi_{m+1}}$ on the hyperplane $\mathbb{R}_0^{m+1}$, it follows that $I \in \mathbb{C}[D_1, \ldots, D_m]$. □

Theorem 4 links the spectral analysis of the lattice Lieb–Liniger model in [20] to an alternative construction of the commuting quantum integrals and the Hall–Littlewood eigenbasis for the periodic q-difference Toda chain by means of the algebraic Bethe Ansatz in [6, Sections 5,6].



$$\nu' = (\lambda_1+1, \lambda_2+1, \lambda_3, \ldots, \lambda_{j-1}, \lambda_j+1, \lambda_{j+1}, \ldots, \lambda_{m-2}, \lambda_{m-1}+1, \lambda_m+1, 1)$$
$$R_{\nu'}(\xi) \Rightarrow R_{(\lambda_1,\lambda_2,\lambda_3-1,\ldots,\lambda_{j-1}-1,\lambda_j,\lambda_{j+1}-1,\ldots,\lambda_{m-2}-1,\lambda_{m-1},\lambda_m,0)}(\xi)$$

$$\nu = (m+1, \mu_2, \ldots, \mu_{i-1}, \mu_i+1, \mu_{i+1}, \ldots, \mu_n, 2)$$
$$\psi(\nu) \Rightarrow \psi(\mu_2, \ldots, \mu_{i-2}, \mu_{i-1}+1, \mu_i, \ldots, \mu_n, 2)$$

FIGURE 6. Adding a strip to the partition: column/row deletion

6. Epilogue

The operators D_r (5.2a), (5.2b) differ from Ruijsenaars' commuting quantum integrals for the q-difference Toda chain, cf. [17, Equation (2.13)]. Specifically, the restriction of Ruijsenaars' operators to lattice functions on $\Lambda^{(n,m)}$ gives rise to the following generalization of the q-difference Toda operator D (2.3b) $(= \mathcal{D}_1)$:

$$\mathcal{D}_l = \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=l}} \left(\prod_{\substack{i \in I \\ [i-1] \notin I}} (1 - q^{\mu_{i-1}-\mu_i}) \right) \prod_{i \in I} T_i \tag{6.1}$$

for $l = 1, \ldots, n$ (where the periodicity conventions in Eqs. (2.3c) and (2.8c) are assumed). Here the Ruijsenaars operators have been gauged (cf. e.g., Equations (2.73)–(2.76) in [10]) such that the adjoint of $\mathcal{D}_l$ in $\ell^2(\Lambda^{(n,m)}, \Delta)$ is given by

$$\mathcal{D}_l^* = \sum_{\substack{I \subset \{1,\ldots,n\} \\ |I|=l}} \left(\prod_{\substack{i \in I \\ [i+1] \notin I}} (1 - q^{\mu_i-\mu_{i+1}}) \right) \prod_{i \in I} T_i^{-1}, \tag{6.2}$$

where the following iterated generalization of the periodic boundary conditions in Eqs. (2.5a), (2.5b) is employed:

$$\psi(m+1, \cdots, m+1, \mu_{i+1}, \mu_{i+2}, \ldots, \mu_n) \equiv \psi(\mu_{i+1}, \mu_{i+2}, \ldots, \mu_n, 0, \ldots, 0) \tag{6.3a}$$

and

$$\psi(\mu_1,\mu_2,\ldots,\mu_{i-1},-1,\cdots,-1)\equiv\psi(m,\ldots,m,\mu_1,\mu_2,\ldots,\mu_{i-1}) \tag{6.3b}$$

(for all $\mu\in\Lambda^{(n,m)}$ and $1\leq i\leq n$). The commutativity of Ruijsenaars' operators is manifestly inherited by their lattice discretizations $\mathcal{D}_1,\ldots,\mathcal{D}_n$ (6.1) and these discrete q-difference operators are moreover normal in $\ell^2\big(\Lambda^{(n,m)},\Delta\big)$ since $\mathcal{D}_l^*=\mathcal{D}_{n-l}\mathcal{D}_n^{-1}$ (with the convention that $\mathcal{D}_0$ equals the identity operator). Hence, if for a given $\kappa\in\Lambda_0^{(m+1,n)}$ the eigenvalue $E_1(\xi_\kappa)$ (5.4b) of $D=\mathcal{D}_1$ is simple, then the corresponding Bethe wave function ψ_{ξ_κ} (3.8a) provides a joint eigenfunction for $\mathcal{D}_1,\ldots,\mathcal{D}_n$:

$$\mathcal{D}_l\psi_{\xi_\kappa}=\mathcal{E}_l(\xi_\kappa)\psi_{\xi_\kappa}. \tag{6.4a}$$

To compute the pertinent eigenvalue, it suffices to evaluate both sides at $\mu=(0^n)$:

$$\mathcal{E}_l(\xi_\kappa)=\frac{(\mathcal{D}_l\psi_{\xi_\kappa})(0^n)}{\psi_{\xi_\kappa}(0^n)}=\frac{(1-q^{m+1})^{1-\delta_{n-l}}\psi_{\xi_\kappa}(0^{n-l}1^l)}{\psi_{\xi_\kappa}(0^n)}=\frac{Q_{(0^ml)}(\xi_\kappa)}{(1-q^{m+1})^{\delta_{n-l}}} \tag{6.4b}$$

(cf. Eq. (3.5)), with

$$\begin{aligned}Q_{(0^ml)}(\xi)&=\frac{(1-q^{m+1})R_{(0^ml)}(\xi)}{R_{(0^{m+1})}(\xi)}=\frac{(1-q)^{m+1}}{(q;q)_m}R_{(0^ml)}(\xi)\\&=(1-q)\sum_{1\leq j\leq m+1}e^{il\xi_j}\prod_{\substack{1\leq k\leq m+1\\k\neq j}}\frac{1-qe^{-\mathrm{i}(\xi_j-\xi_k)}}{1-e^{-\mathrm{i}(\xi_j-\xi_k)}}\end{aligned} \tag{6.4c}$$

(cf. e.g., Equation (2.9) in [15, Chapter III.2]).

Rewritten in terms of Hall–Littlewood polynomials the eigenvalue equation (6.4a) becomes $\forall\lambda\in\Lambda_0^{(m+1,n)}$:

$$\begin{aligned}&Q_{(0^ml)}(\xi_\kappa)R_\lambda(\xi_\kappa)\\&\quad=(1-q^{m+1})^{\delta_{n-l}}\sum_\nu R_{\underline{\nu}}(\xi_\kappa)\prod_{\substack{0\leq i<n\\ \mathrm{m}_i(\nu)+\delta_i\mathrm{m}_n(\nu)=\\ \mathrm{m}_i(\lambda)+\delta_i\mathrm{m}_n(\lambda)-1}}(1-q^{\mathrm{m}_i(\lambda)+\delta_i\mathrm{m}_n(\lambda)}),\end{aligned} \tag{6.5}$$

where

$$\underline{\nu}=(\nu_1-\nu_{m+1},\nu_2-\nu_{m+1},\ldots,\nu_m-\nu_{m+1},0).$$

In Eq. (6.5), the sum is over all $\nu\in\Lambda^{(m+1,n)}$ such that $\nu-\lambda$ is a horizontal l-strip, i.e., $|\nu|=|\lambda|+l$ and

$$n\geq\nu_1\geq\lambda_1\geq\nu_2\geq\lambda_2\geq\cdots\geq\lambda_m\geq\nu_{m+1}\geq\lambda_{m+1}=0.$$

To establish the identity in equation (6.5) for any $\kappa\in\Lambda_0^{(m+1,n)}$ as an affine Pieri relation (therewith including possible cases in which the eigenvalues $E_1(\xi_\kappa)$ (5.4b) have a multiplicity >1) requires checking that $\mathcal{D}_l\in\mathcal{I}$ (5.6a) (as expected). For $l > 1$ this is, unfortunately, beyond the scope of the present note (but cf. [13, Corollary 7.4]). In the absence of this check, however, one can in principle still rely on the spectral theorem for commuting normal operators so as to construct a joint orthogonal eigenbasis diagonalizing $\mathcal{D}_1, \ldots, \mathcal{D}_n$ in $\ell^2\big(\Lambda^{(n,m)}, \Delta\big)$ from the Bethe-Ansatz basis ψ_{ξ_κ}, $\kappa \in \Lambda_0^{(m+1,n)}$ by means of suitable orthogonal linear combinations of Bethe Ansatz wave functions pertaining to the same eigenvalue of $\mathcal{D}_1$.

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Appendix A: Affine Pieri Rules for Periodic Hall–Littlewood Functions

This appendix recalls affine Pieri rules for periodic Hall–Littlewood functions [22, Section 1] that originate from the diagonalization of the lattice Lieb–Liniger model in [20] (cf. also [13, Corollary 7.4]). The affine Pieri rules in question are *on-shell* in the sense that the relations hold *provided* the polynomial variable satisfies the Bethe Ansatz equations. This is in contrast with the conventional Pieri rules for the Hall–Littlewood functions [15, Chapter III.3] which hold as identities between symmetric functions (and are therefore also valid *off-shell*).

Let us first recall that $E_r(\xi)$ denotes the rth elementary symmetric polynomial in the variables $e^{i\xi_1}, \ldots, e^{i\xi_{m+1}}$ (cf. Equation (5.4b)). By [20, Theorem 5.1], the Hall–Littlewood polynomials $R_\lambda(\xi)$, $\lambda \in \Lambda_0^{(m+1,n)}$ satisfy the following affine Pieri rule at $\xi = \xi_\kappa$, $\kappa \in \Lambda_0^{(m+1,n)}$ for $r = 1, \ldots, m$:

$$E_r(\xi) R_\lambda(\xi) = \sum_{\substack{J \subset \{1,\ldots,m+1\}, |J|=r \\ \underline{\lambda + e_J} \in \Lambda_0^{(m+1,n)}}} R_{\underline{\lambda + e_J}}(\xi) V_J(\lambda), \tag{A.1a}$$

with

$$V_J(\lambda) = \prod_{\substack{1\le j<k\le m+1\\ j\in J,\, k\in J^c\\ \lambda_j=\lambda_k}} \frac{1-q^{1+k-j}}{1-q^{k-j}} \prod_{\substack{1\le j<k\le m+1\\ j\in J^c,\, k\in J\\ \lambda_j=\lambda_k+n}} \frac{1-q^{m+2-k+j}}{1-q^{m+1-k+j}} \tag{A.1b}$$

(cf. [22, Eq. (1.2)]). Here $|J|$ denotes the cardinality of $J \subset \{1,\ldots,m+1\}$, $e_J = \sum_{j\in J} e_j$, $J^c = \{1,\ldots,m+1\}\backslash J$, and

$$\underline{\lambda + e_J} \equiv \begin{cases} \lambda + e_J & \text{if } m+1 \in J^c,\\ \lambda - e_{J^c} & \text{if } m+1 \in J. \end{cases}$$

For $r=1$ and $r=m$, the affine Pieri rule reads, respectively (cf. [20, Equations (5.5a)–(5.8d)]):

$$(e^{\mathrm{i}\xi_1}+\cdots+e^{\mathrm{i}\xi_{m+1}})R_\lambda(\xi) = \sum_{\substack{1\le j\le m+1\\ \underline{\lambda+e_j}\in\Lambda_0^{(m+1,n)}}} R_{\underline{\lambda+e_j}}(\xi) \prod_{\substack{j<k\le m+1\\ \lambda_k=\lambda_j}} \frac{1-q^{1+k-j}}{1-q^{k-j}} \prod_{\substack{1\le k<j\\ \lambda_k=\lambda_j+n}} \frac{1-q^{m+2-j+k}}{1-q^{m+1-j+k}} \tag{A.2a}$$

and (upon recalling that $E_m(\xi) = E_1(-\xi)$ for $\xi \in \mathbb{R}_0^{m+1}$)

$$(e^{-\mathrm{i}\xi_1}+\cdots+e^{-\mathrm{i}\xi_{m+1}})R_\lambda(\xi) = \sum_{\substack{1\le j\le m+1\\ \underline{\lambda-e_j}\in\Lambda_0^{(m+1,n)}}} R_{\underline{\lambda-e_j}}(\xi) \prod_{\substack{1\le k<j\\ \lambda_k=\lambda_j}} \frac{1-q^{1+j-k}}{1-q^{j-k}} \prod_{\substack{j<k\le m+1\\ \lambda_k=\lambda_j-n}} \frac{1-q^{m+2-k+j}}{1-q^{m+1-k+j}}, \tag{A.2b}$$

with

$$\underline{\lambda + e_j} = \begin{cases} (\lambda_1,\ldots,\lambda_{j-1},\lambda_j+1,\lambda_{j+1},\ldots,\lambda_m,0) & \text{if } 1\le j\le m,\\ (\lambda_1-1,\ldots,\lambda_m-1,0) & \text{if } j=m+1, \end{cases}$$

and

$$\underline{\lambda - e_j} = \begin{cases} (\lambda_1,\ldots,\lambda_{j-1},\lambda_j-1,\lambda_{j+1},\ldots,\lambda_m,0) & \text{if } 1\le j\le m,\\ (\lambda_1+1,\ldots,\lambda_m+1,0) & \text{if } j=m+1. \end{cases}$$

Remark. While the affine Pieri rules at $\xi=\xi_\kappa$, $\kappa\in\Lambda_0^{(m+1,n)}$ simply reproduce the eigenvalue equations in [20, Theorem 5.1], it is readily seen from the proof in [20] for the theorem in question that these affine Pieri rules actually hold more generally for any $\xi \in \mathbb{R}^{m+1}_{0,\mathrm{reg}}$ obeying Bethe Ansatz equations of the form

$$e^{in(\xi_j-\xi_k)} = \prod_{\substack{1\le \ell\le m+1\\ \ell\neq j}} \frac{1-qe^{i(\xi_j-\xi_\ell)}}{e^{i(\xi_j-\xi_\ell)}-q} \prod_{\substack{1\le \ell\le m+1\\ \ell\neq k}} \frac{1-qe^{i(\xi_\ell-\xi_k)}}{e^{i(\xi_\ell-\xi_k)}-q} \tag{A.3}$$

($\forall 1 \leq j \neq k \leq m+1$). In particular, since any solution $\xi \in \mathbb{R}^{m+1}_{0,\text{reg}}$ of the Bethe Ansatz equations (3.8c) automatically satisfies Eq. (A.3), the affine Pieri rules stated above are valid when ξ is restricted to such solutions.

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