



The Absolutely Continuous Spectrum of Finitely Differentiable Quasi-Periodic Schrödinger Operators

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Abstract. We prove that the quasi-periodic Schrödinger operator with a finitely differentiable potential has purely absolutely continuous spectrum for all phases if the frequency is Diophantine and the potential is sufficiently small in the corresponding C^k topology. This extends the work of Eliasson [19] and Avila–Jitomirskaya [5] from the analytic topology to the finitely differentiable one which is much broader, revealing the interesting phenomenon that small oscillation of the potential leads to both zero Lyapunov exponent in the whole spectrum and purely absolutely continuous spectrum. Our result is based on a refined quantitative C^{k,k_0} almost reducibility theorem which only requires a quite low initial regularity “ $k > 14\tau + 2$ ” and much of the regularity “ $k_0 \leq k - 2\tau - 2$ ” is conserved in the end, where τ is the Diophantine constant of the frequency.

1. Introduction

In this paper, we shall consider the one-dimensional lattice quasi-periodic Schrödinger operator $H_{V,\alpha,\theta}$ with a finitely differentiable potential:

$$(H_{V,\alpha,\theta}x)_n = x_{n+1} + x_{n-1} + V(\theta + n\alpha)x_n, n \in \mathbb{Z}, \quad (1.1)$$

where $\theta \in \mathbb{T}^d = \mathbb{R}^d / (2\pi\mathbb{Z})^d$ (or $\mathbb{R}^d / \mathbb{Z}^d$ if preferable) is called the phase, $\alpha \in \mathbb{R}^d$ is called the frequency satisfying $\langle m, \alpha \rangle \notin 2\pi\mathbb{Z}$ for any $m \in \mathbb{Z}^d$ different from zero, and $V \in C^k(\mathbb{T}^d, \mathbb{R})$ is called the potential, $k, d \in \mathbb{N}^+$. A typical example is the almost Mathieu operator $H_{2\lambda \cos, \alpha, \theta}$:

$$(H_{2\lambda \cos, \alpha, \theta}x)_n = x_{n+1} + x_{n-1} + 2\lambda \cos(\theta + n\alpha)x_n, n \in \mathbb{Z},$$

where λ is called the coupling constant. There is a unified dynamical way to define the Schrödinger operator [16]. In our case, the base dynamics is simply given by an ergodic torus translation.

Schrödinger operators come from solid-state physics, showing the influence of an external magnetic field on the electrons of a crystal [26]. They arise naturally from the study of quasi-crystals such as graphene. Imagine that we have a one-dimensional discrete lattice with positive centers located at each integer point. We are mostly concerned about the fate of the electrons under the interaction with the lattice, i.e., whether they are localized (confined in some finite interval) or diffusing (escaping from any finite interval). As is very well known, the absolutely continuous spectrum corresponds to diffusion because the probability that we find the electron in any fixed interval is zero by the famous RAGE theorem [16]. In other words, the absolutely continuous spectrum of a Schrödinger operator is the set of energies at which the described physical system exhibits transport (like a conductor). Moreover, the absolutely continuous spectrum is the part of spectrum that has the best stability properties under small perturbation and its existence has strong implications, including an Oracle Theorem that predicts the potential, as shown by Remling [27]. It is thus an important and natural question to ask whether and when the Schrödinger operator has (even purely) absolutely continuous spectrum. In this paper, we give a positive answer to this question for weak-coupled finitely differentiable quasi-periodic Schrödinger operators with Diophantine base frequencies.

Recall that $\alpha \in \mathbb{R}^d$ is called *Diophantine* if there are $\kappa > 0$ and $\tau > d$ such that $\alpha \in \text{DC}(\kappa, \tau)$, where

$$\text{DC}(\kappa, \tau) := \left\{ \alpha \in \mathbb{R}^d : \inf_{j \in \mathbb{Z}} |\langle n, \alpha \rangle - 2\pi j| > \frac{\kappa}{|n|^\tau}, \quad \forall n \in \mathbb{Z}^d \setminus \{0\} \right\}. \quad (1.2)$$

Here we denote

$$|n| = |n_1| + |n_2| + \dots + |n_d|,$$

and

$$\langle n, \alpha \rangle = n_1\alpha_1 + n_2\alpha_2 + \dots + n_d\alpha_d.$$

Denote $\text{DC} = \bigcup_{\kappa, \tau} \text{DC}(\kappa, \tau)$, which is of full Lebesgue measure.

Our main result is the following:

Theorem 1.1. *Assume $\alpha \in \text{DC}(\kappa, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k > 35\tau + 2$. If λ is sufficiently small, then $H_{\lambda V, \alpha, \theta}$ has purely absolutely continuous spectrum for all $\theta \in \mathbb{T}^d$.*

Remark 1.1. It suffices to prove that the universal spectral measure $\mu_{univ} = \mu_{\delta_0} + \mu_{\delta_1}$ is absolutely continuous w.r.t. Lebesgue simply because any spectral measure is absolutely continuous to μ_{univ} . We do not claim the optimality of the lower bound of “ k ” but we point out that some kind of regularity is essential for the existence of (purely) absolutely continuous spectrum, as will be stated later. For some technical reason, we require “ k ” to be larger than $35\tau + 2$ instead of $14\tau + 2$.

It appears that the existence of (purely) absolutely continuous spectrum depends sensitively on the arithmetic properties of the frequency. Recently, Avila and Jitomirskaya [6] have constructed super-Liouvillean $\alpha \in \mathbb{T}^2$ such that for typical analytic potential, the corresponding quasi-periodic Schrödinger operator has no absolutely continuous spectrum. Relatively, Hou–Wang–Zhou [21] showed that there exists super-Liouvillean $\alpha \in \mathbb{T}^2$ such that for small analytic potential, the corresponding quasi-periodic Schrödinger operator has absolutely continuous spectrum. Moreover, they proved that if $\alpha \in \mathbb{T}^d$ with α being weak-Liouvillean and the potential is small enough, the absolutely continuous spectrum exists.

When $d = 1$, things can be characterized much more explicitly thanks to Avila’s fantastic global theory of analytic Schrödinger operators [2]. He showed that typical one-frequency operators have only point spectrum in the super-critical region and absolutely continuous spectrum in the subcritical region. It seems that the effect of α ’s arithmetic properties is weaker in one-frequency case, but we point out that the proofs of absolutely continuous spectrum are quite different according to the arithmetic assumptions on α . Focusing on the almost Mathieu operator $H_{2\lambda \cos, \alpha, \theta}$, there is a famous conjecture: Simon [29] (Problem 6) asked whether AMO has purely absolutely continuous spectrum for all $0 < |\lambda| < 1$, all phases and all frequencies. This conjecture was first proved for Diophantine α and almost every θ by Jitomirskaya [23], whose approach follows Aubry duality and localization theory. About ten years later, two key advances were achieved. On the one hand, Avila and Jitomirskaya [5] established so-called quantitative duality to prove Simon’s conjecture for Diophantine α and all θ . On the other hand, Avila and Damanik [4] used periodic approximation and Kotani theory to prove the conjecture for Liouvillean α and almost every θ . The complete solution to Simon’s problem was given by Avila [1]. He distinguished the whole proof into two parts: When $\beta = 0$ (the subexponential regime, see [1]), the proof relied on almost reducibility results developed in [5]; and when $\beta > 0$ (the exponential regime), he improved the periodic approximation method developed in [4]. Recall that β is defined as follows:

$$\beta = \beta(\alpha) = \limsup_{n \rightarrow \infty} \frac{\log q_{n+1}}{q_n},$$

where $\{q_n\}_{n \in \mathbb{N}}$ are the denominators of the continued fraction approximants $\{p_n/q_n\}_{n \in \mathbb{N}}$ to α .

Another very important factor which influences the existence of absolutely continuous spectrum is the regularity of the potential. In the analytic topology, Dinaburg and Sinai [18] proved that $H_{V, \alpha, \theta}$ has absolutely continuous spectrum component for all θ in the perturbative regime (V being analytically small and the smallness depends on α) by reducibility theory. Later, Eliasson [19] improved the KAM scheme and showed that $H_{V, \alpha, \theta}$ has purely absolutely continuous spectrum for all θ in the same setting. By “non-perturbative reduction to perturbative regime,” Eliasson’s result was extended

to the non-perturbative regime by Avila–Jitomirskaya [5] and Hou–You [22] for one-dimensional torus, $d = 1$.

However, when it comes to the finitely differentiable topology, there are few results in this direction. In this sense, our theorem is constructive and it mainly generalizes the result of Eliasson [19] and Avila–Jitomirskaya [5] to the finitely differentiable case. We emphasize that assuming enough regularity of the potential is necessary, not only for “purely” absolutely continuous spectrum, but also for the existence of an absolutely continuous spectrum component. In C^0 topology, Avila and Damanik [3] proved that for one-dimensional Schrödinger operators with ergodic continuous potentials, there exists a generic set of such potentials such that the corresponding operators have no absolutely continuous spectrum. Moreover, by Gordon’s Lemma, Boshernitzan and Damanik [12] proved that for generic ergodic continuous potentials, the corresponding operators have purely singular continuous spectrum.

Now, let us show the main strategy of our paper. The Schrödinger operator $H_{V,\alpha,\theta}$ is closely related to a Schrödinger cocycle (α, A) where

$$A(\theta) = S_E^V(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix},$$

since any formal solution (not necessarily ℓ^2) of $H_{V,\alpha,\theta}x = Ex$ satisfies

$$A(\theta + n\alpha) \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}.$$

Therefore, we can use reducibility method to analyze the dynamics of C^k quasi-periodic linear cocycle $(\alpha, A) \in \mathbb{T}^d \times C^k(\mathbb{T}^d, SL(2, \mathbb{R}))$ and then study the spectral properties of the corresponding operator. This approach, which was first developed in [19], has been proved to be very fruitful [1, 5, 7, 8, 25]. Readers are invited to consult You’s 2018 ICM survey [32] for more related achievements. In this paper, we acquire a finer quantitative C^k almost reducibility theorem by distinguishing resonances from non-resonances more precisely. For simplicity, let us introduce its qualitative version on the Schrödinger cocycle (for the quantitative one which works for general C^k $SL(2, \mathbb{R})$ -valued cocycles, see Theorem 3.2).

Theorem 1.2. *Let $\alpha \in DC(\kappa, \tau)$, $V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k > 14\tau + 2$. If λ is sufficiently small, then for any $E \in \mathbb{R}$, $(\alpha, S_E^{\lambda V})$ is C^{k,k_0} almost reducible with $k_0 \leq k - 2\tau - 2$.*

Remark 1.2. In particular, the Lyapunov exponent is constant zero among the spectrum. If we change the assumption into $k > 17\tau + 2$, we can further prove the $\frac{1}{2}$ -Hölder continuity of the Lyapunov exponent and the integrated density of states (see Theorem 3.3 and Theorem 3.4), which greatly reduces the initial regularity requirement of $k \geq 550\tau$ in [13]. Moreover, compared with [13], we obtain a much better upper bound of the remainder $k_0 \leq k - 2\tau - 2$ where the loss of regularity $2\tau + 2$ is independent of k !

To finish the introduction, we point out that our main idea of proving the purely absolutely continuous spectrum follows that of Avila [1] in the

subexponential regime. There are two important aspects. One is that we need to obtain a modified quantitative C^k almost reducibility theorem which was originally established by Cai–Chavaudret–You–Zhou [13]. It will provide us with finer estimates on the conjugation map, the constant matrix and the perturbation in each KAM step. The other is, we need to stratify the spectrum of $H_{V,\alpha,\theta}$ by the rotation number of (α, A) . Once they are done, we will be able to have a good control of the growth of the transfer matrix on each hierarchical part of the spectrum. These steps will be conducted in Sect. 3. The proofs left are standard by the theorems of Gilbert–Pearson [20] and Avila [1] in Sect. 4. As usual, we will introduce useful notions and definitions in Sect. 2.

2. Preliminaries

As an emphasis in the very beginning, it does not matter whether we define $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ or $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Everything follows exactly in the same way. We choose $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ just by preference.

For a bounded analytic (possibly matrix-valued) function $F(\theta)$ defined on $\mathcal{S}_h := \{\theta = (\theta_1, \dots, \theta_d) \in \mathbb{C}^d \mid \forall 1 \leq i \leq d, |\Im\theta_i| < h\}$, let $|F|_h = \sup_{\theta \in \mathcal{S}_h} \|F(\theta)\|$ and denote by $C_h^\omega(\mathbb{T}^d, *)$ the set of all these $*$ -valued functions ($*$ will usually denote $\mathbb{R}, sl(2, \mathbb{R}), SL(2, \mathbb{R})$). We denote $C^\omega(\mathbb{T}^d, *) = \cup_{h>0} C_h^\omega(\mathbb{T}^d, *)$ and set $C^k(\mathbb{T}^d, *)$ to be the space of k times differentiable with continuous k th derivatives functions. The norm is defined as

$$\|F\|_k = \sup_{|k'| \leq k, \theta \in \mathbb{T}^d} \|\partial^{k'} F(\theta)\|.$$

2.1. Conjugation and Reducibility

Given two cocycles $(\alpha, A_1), (\alpha, A_2) \in \mathbb{T}^d \times C^*(\mathbb{T}^d, SL(2, \mathbb{R}))$, “ $*$ ” stands for “ ω ” or “ k ,” one says that they are C^* conjugated if there exists $Z \in C^*(2\mathbb{T}^d, SL(2, \mathbb{R}))$, such that

$$Z(\theta + \alpha)A_1(\theta)Z^{-1}(\theta) = A_2(\theta).$$

Note that we need to define Z on the $2\mathbb{T}^d = \mathbb{R}^d/(4\pi\mathbb{Z})^d$ in order to make it still real-valued.

An analytic cocycle $(\alpha, A) \in \mathbb{T}^d \times C_h^\omega(\mathbb{T}^d, SL(2, \mathbb{R}))$ is said to be almost reducible if there exist a sequence of conjugations $Z_j \in C_{h_j}^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$, a sequence of constant matrices $A_j \in SL(2, \mathbb{R})$ and a sequence of small perturbation $f_j \in C_{h_j}^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$Z_j(\theta + \alpha)A(\theta)Z_j(\theta)^{-1} = A_j e^{f_j(\theta)}$$

with

$$|f_j(\theta)|_{h_j} \rightarrow 0, \quad j \rightarrow \infty.$$

Furthermore, we call it weak (C^ω) almost reducible if $h_j \rightarrow 0$ and we call it strong ($C_{h_j, h'}$) almost reducible if $h_j \rightarrow h' > 0$. We say (α, A) is $C_{h, h'}$ reducible

if there exist a conjugation map $\tilde{Z} \in C^{\omega}_{h'}(2\mathbb{T}^d, SL(2, \mathbb{R}))$ and a constant matrix $\tilde{A} \in SL(2, \mathbb{R})$ such that

$$\tilde{Z}(\theta + \alpha)A(\theta)\tilde{Z}(\theta)^{-1} = \tilde{A}(\theta).$$

In order to avoid repetition, we give an equivalent definition of C^k (almost) reducibility in the following.

A finitely differentiable cocycle (α, A) is said to be C^{k,k_1} almost reducible, if $A \in C^k(\mathbb{T}^d, SL(2, \mathbb{R}))$ and the C^{k_1} -closure of its C^{k_1} conjugacies contains a constant. Moreover, we say (α, A) is C^{k,k_1} reducible, if $A \in C^k(\mathbb{T}^d, SL(2, \mathbb{R}))$ and its C^{k_1} conjugacies contain a constant.

2.2. Rotation Number and Degree

Assume that $A \in C^0(\mathbb{T}^d, SL(2, \mathbb{R}))$ is homotopic to identity. It introduces the projective skew-product $F_A : \mathbb{T}^d \times \mathbb{S}^1 \rightarrow \mathbb{T}^d \times \mathbb{S}^1$ with

$$F_A(x, \omega) := \left(x + \alpha, \frac{A(x) \cdot \omega}{|A(x) \cdot \omega|} \right),$$

which is also homotopic to identity. Thus, we can lift F_A to a map $\tilde{F}_A : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{T}^d \times \mathbb{R}$ of the form $\tilde{F}_A(x, y) = (x + \alpha, y + \psi(x, y))$, where for every $x \in \mathbb{T}^d$, $\psi(x, y)$ is $2\pi\mathbb{Z}$ -periodic in y . The map $\psi : \mathbb{T}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is called a lift of A . Let μ be any probability measure on $\mathbb{T}^d \times \mathbb{R}$ which is invariant by \tilde{F}_A and whose projection on the first coordinate is given by Lebesgue measure. The number

$$\rho_{(\alpha, A)} := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d \times \mathbb{R}} \psi(x, y) d\mu(x, y) \text{ mod } 2\pi\mathbb{Z} \tag{2.1}$$

does not depend on the choices of the lift ψ or the measure μ . It is called the *fibred rotation number* of cocycle (α, A) (readers can consult [24] for more details).

Let

$$R_\phi := \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix},$$

if $A \in C^0(\mathbb{T}^d, SL(2, \mathbb{R}))$ is homotopic to $\theta \rightarrow R_{\langle n, \theta \rangle}$ for some $n \in \mathbb{Z}^d$, then we call n the *degree* of A and denote it by $\text{deg}A$. Moreover,

$$\text{deg}(AB) = \text{deg}A + \text{deg}B. \tag{2.2}$$

Note that the fibred rotation number is invariant under real conjugacies which are homotopic to identity. More generally, if the cocycle (α, A_1) is conjugated to (α, A_2) by $B \in C^0(2\mathbb{T}^d, SL(2, \mathbb{R}))$, i.e., $B(\cdot + \alpha)A_1(\cdot)B^{-1}(\cdot) = A_2(\cdot)$, then

$$\rho_{(\alpha, A_2)} = \rho_{(\alpha, A_1)} + \frac{\langle \text{deg}B, \alpha \rangle}{2}. \tag{2.3}$$

2.3. Integrated Density of States and Spectral Measure

Consider Schrödinger operators $H_{V,\alpha,\theta}$, an important concept is the integrated density of states (IDS), which is the function $N_{V,\alpha} : \mathbb{R} \rightarrow [0, 1]$ defined by

$$N_{V,\alpha}(E) = \int_{\mathbb{T}^d} \mu_{V,\alpha,\theta}(-\infty, E] d\theta,$$

where $\mu_{V,\alpha,\theta} = \mu_{V,\alpha,\theta}^{\delta_0} + \mu_{V,\alpha,\theta}^{\delta_1}$ is the universal spectral measure of $H_{V,\alpha,\theta}$ and $\{\delta_i\}_{i \in \mathbb{Z}}$ is the canonical basis of $\ell^2(\mathbb{Z})$. Here $\{\delta_0, \delta_1\}$ are called the pair of cyclic vectors of $H_{V,\alpha,\theta}$.

There are other ways of defining IDS by counting eigenvalues of the truncated Schrödinger operator. For more details, readers can refer to [9]. Moreover, $\rho(\alpha, S_E^V)$ relates to the IDS as follows:

$$N_{V,\alpha}(E) = 1 - 2\rho(\alpha, S_E^V) \pmod{\mathbb{Z}}. \tag{2.4}$$

For basic definitions of different types of spectral measures, one is invited to [16].

2.4. Analytic Approximation

Assume $f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$. By Zehnder [33], there exists a sequence $\{f_j\}_{j \geq 1}$, $f_j \in C^{\omega}_{\frac{1}{j}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ and a universal constant C' , such that

$$\begin{aligned} \|f_j - f\|_k &\rightarrow 0, \quad j \rightarrow +\infty, \\ |f_j|_{\frac{1}{j}} &\leq C' \|f\|_k, \\ |f_{j+1} - f_j|_{\frac{1}{j+1}} &\leq C' \left(\frac{1}{j}\right)^k \|f\|_k. \end{aligned} \tag{2.5}$$

Moreover, if $k \leq \tilde{k}$ and $f \in C^{\tilde{k}}$, then properties (2.5) hold with \tilde{k} instead of k . That means this sequence is obtained from f regardless of its regularity (since f_j is the convolution of f with a map which does not depend on k).

3. Dynamical Estimates: Almost Reducibility

In this section, we will establish the modified quantitative C^{k,k_0} almost reducibility for finitely differentiable quasi-periodic $SL(2, \mathbb{R})$ cocycles.

Set $A \in SL(2, \mathbb{R})$, $f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$, $d \in \mathbb{N}^+$ and $\alpha \in DC(\kappa, \tau)$. Our strategy is to analyze the approximating analytic cocycles $\{(\alpha, Ae^{f_j(\theta)})\}_{j \geq 1}$ first and then transfer the estimates to the targeted C^k cocycle $(\alpha, Ae^{f(\theta)})$ by analytic approximation.

3.1. Preparations

In the following subsections, parameters $\rho, \epsilon, N, \sigma$ will be fixed; one will refer to the situation where there exists n_* with $0 < |n_*| \leq N$ such that

$$\inf_{j \in \mathbb{Z}} |2\rho - \langle n_*, \alpha \rangle - 2\pi j| < \epsilon^\sigma,$$

as the “resonant case” (for simplicity, we just write “ $|2\rho - \langle n_*, \alpha \rangle|$ ” to represent the left side, same for $|\langle n_*, \alpha \rangle|$). The integer vector n_* will be referred to as a “resonant site”. This kind of small divisor problem naturally arises when we want to solve the cohomological equation in each KAM step. Resonances are linked to a useful decomposition of the space $\mathcal{B}_r := C_r^\omega(\mathbb{T}^d, su(1, 1))$, see Appendix the definition of $su(1, 1)$ and $SU(1, 1)$.

Assume that for given $\eta > 0$, $\alpha \in \mathbb{R}^d$ and $A \in SU(1, 1)$, we have a decomposition $\mathcal{B}_r = \mathcal{B}_r^{nre}(\eta) \oplus \mathcal{B}_r^{re}(\eta)$ satisfying that for any $Y \in \mathcal{B}_r^{nre}(\eta)$,

$$A^{-1}Y(\theta + \alpha)A \in \mathcal{B}_r^{nre}(\eta), |A^{-1}Y(\theta + \alpha)A - Y(\theta)|_r \geq \eta|Y(\theta)|_r. \tag{3.1}$$

And let $\mathbb{P}_{nre}, \mathbb{P}_{re}$ denote the standard projections from \mathcal{B}_r onto $\mathcal{B}_r^{nre}(\eta)$ and $\mathcal{B}_r^{re}(\eta)$, respectively.

Then we have the following crucial lemma which helps us remove all the non-resonant terms:

Lemma 3.1 [13, 22]. *Assume that $A \in SU(1, 1)$, $\epsilon \leq (4\|A\|)^{-4}$ and $\eta \geq 13\|A\|^2\epsilon^{\frac{1}{2}}$. For any $g \in \mathcal{B}_r$ with $|g|_r \leq \epsilon$, there exist $Y \in \mathcal{B}_r$ and $g^{re} \in \mathcal{B}_r^{re}(\eta)$ such that*

$$e^{Y(\theta+\alpha)}(Ae^{g(\theta)})e^{-Y(\theta)} = Ae^{g^{re}(\theta)},$$

with $|Y|_r \leq \epsilon^{\frac{1}{2}}$ and $|g^{re}|_r \leq 2\epsilon$.

Remark 3.1. In the inequality “ $\eta \geq 13\|A\|^2\epsilon^{\frac{1}{2}}$,” “ $\frac{1}{2}$ ” is sharp due to the quantitative implicit function theorem [11, 17]. The proof only relies on the fact that \mathcal{B}_r is a Banach space; thus, it also applies to C^k and C^0 topology. One can refer to the appendix of [13] for details.

3.2. Analytic KAM Theorem

According to the plan, we first establish KAM for the analytic quasi-periodic $SL(2, \mathbb{R})$ cocycle:

$$(\alpha, Ae^{f(\theta)}) : \mathbb{T}^d \times \mathbb{R}^2 \rightarrow \mathbb{T}^d \times \mathbb{R}^2; (\theta, v) \mapsto (\theta + \alpha, Ae^{f(\theta)} \cdot v),$$

where $A \in SL(2, \mathbb{R})$, $f \in C_r^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ with $r > 0, d \in \mathbb{Z}^+$, and $\alpha \in DC(\kappa, \tau)$. Note that A has eigenvalues $\{e^{i\rho}, e^{-i\rho}\}$ with $\rho \in \mathbb{R} \cup i\mathbb{R}$. We formulate our quantitative analytic KAM theorem as follows.

Theorem 3.1 [13]. *Let $\alpha \in DC(\kappa, \tau)$, $\kappa, r > 0$, $\tau > d$, $\sigma < \frac{1}{6}$. Suppose that $A \in SL(2, \mathbb{R})$, $f \in C_r^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$. Then for any $r' \in (0, r)$, there exist constants $c = c(\kappa, \tau, d)$, $D > \frac{2}{\sigma}$ and $\tilde{D} = \tilde{D}(\sigma)$ such that if*

$$|f|_r \leq \epsilon \leq \frac{c}{\|A\|^{\tilde{D}}} (r - r')^{D\tau}, \tag{3.2}$$

then there exist $B \in C_{r'}^\omega(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_+ \in SL(2, \mathbb{R})$ and $f_+ \in C_{r'}^\omega(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$B(\theta + \alpha)(Ae^{f(\theta)})B^{-1}(\theta) = A_+e^{f_+(\theta)}.$$

More precisely, let $N = \frac{2}{r-r'}|\ln \epsilon|$, then we can distinguish two cases:

- (Non-resonant case) if for any $n \in \mathbb{Z}^d$ with $0 < |n| \leq N$, we have

$$|2\rho - \langle n, \alpha \rangle| \geq \epsilon^\sigma,$$

then

$$|B - Id|_{r'} \leq \epsilon^{\frac{1}{2}}, \quad |f_+|_{r'} \leq \epsilon^{3-\sigma}.$$

and

$$\|A_+ - A\| \leq 2\|A\|\epsilon.$$

- (Resonant case) if there exists n_* with $0 < |n_*| \leq N$ such that

$$|2\rho - \langle n_*, \alpha \rangle| < \epsilon^\sigma,$$

then

$$|B|_{r'} \leq 8 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{r-r'} |\ln \epsilon| \right)^{\frac{\tau}{2}} \times \epsilon^{\frac{-r'}{r-r'}},$$

$$\|B\|_0 \leq 8 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{r-r'} |\ln \epsilon| \right)^{\frac{\tau}{2}},$$

$$|f_+|_{r'} \leq \frac{2^{5+\tau} \|A\| |\ln \epsilon|^\tau}{\kappa (r-r')^\tau} \epsilon e^{-N'(r-r')} (N')^d e^{Nr'} \ll \epsilon^{100}, \quad N' > 2N^2.$$

Moreover, $A_+ = e^{A''}$ with $\|A''\| \leq 2\epsilon^\sigma$, $A'' \in sl(2, \mathbb{R})$. More accurately, we have

$$MA''M^{-1} = \begin{pmatrix} it & v \\ \bar{v} & -it \end{pmatrix}$$

with $|t| \leq \epsilon^\sigma$ and

$$|v| \leq \frac{2^{4+\tau} \|A\| |\ln \epsilon|^\tau}{\kappa (r-r')^\tau} \epsilon e^{-|n_*|r}.$$

Roughly speaking, if the KAM step is non-resonant, we have little cost and we can push the magnitude of the perturbation to better than its square. In the resonant case, we have relatively big cost but we can even push the magnitude of the perturbation to much smaller than its 100th power (it is straightforward to check that compared with the cost, the profit we have is still very much worthy: just compute the product of the norm of the conjugation map and the perturbation).

Since this theorem is not new, we put its proof in the Appendix just for self-containedness. Nevertheless, the novelty of this version is that it provides all possible choices of all parameters. For example, we may choose $\sigma = \frac{1}{10}$ and fix other parameters by concrete real numbers to make it simpler to read. However, this will lead to a larger initial regularity k in the end. In order to optimize our result in the widest possible topology, we must and have to reserve all the parameters as symbols instead of concrete real numbers. We apologize for this technicality.

Remark 3.2. The special structure of A_+ can give us a precise estimate of the upper triangular element of the parabolic constant matrix when the rotation number of the initial system is rational with respect to α , see [25]. However, it does not work for general case. Instead, $A_+ = e^{A''}$ with $\|A''\| \leq 2\epsilon^\sigma$ will work.

Remark 3.3. The limitation $\sigma < \frac{1}{6}$ is essential to making Lemma 3.1 applicable. The choice of D being larger than $\frac{2}{\sigma}$ is necessary for us to guarantee the arbitrariness of $r' \in (0, r)$ and the separation of resonant steps (see Claim 1).

3.3. C^k Almost Reducibility

As planned, we are going to establish the quantitative C^k almost reducibility via Theorem 3.1 and analytic approximation [33].

The crucial improvement here compared with [13] lies at the point where we are able to separate the resonant steps. In fact, Theorem 3.1 tells us that whenever we have a resonance, the norm of the conjugation map will be very large compared with the non-resonant case. If the resonant steps are far away from each other, we immediately have a much better control of the conjugation maps when we want to do the inductive argument. This, in return, will give us the best possible initial regularity k according to the technique.

Let $(f_j)_{j \geq 1}, f_j \in C^{\frac{\omega}{j}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ be the analytic sequence approximating $f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$ which satisfies (2.5).

For $0 < r' < r$, denote

$$\epsilon'_0(r, r') = \frac{c}{(2\|A\|)^{\bar{D}}} (r - r')^{D\tau}, \tag{3.3}$$

where c, D, \bar{D}, τ are defined in Theorem 3.1.

For $m \in \mathbb{Z}^+$, we define

$$\epsilon_m = \frac{c}{(2\|A\|)^{\bar{D}} m^{D\tau + \frac{1}{2}}}. \tag{3.4}$$

Then for any $0 < s \leq \frac{1}{6D\tau+3}$ fixed, there exists m_0 such that for any $m \geq m_0$ we have both

$$\epsilon_m \leq \epsilon'_0 \left(\frac{1}{m}, \frac{1}{m^{1+s}} \right), \tag{3.5}$$

and

$$\frac{1}{m^s - 1} \leq \frac{s}{4}.$$

We will start from $M > \max\{\frac{(2\|A\|)^{\bar{D}}}{c}, m_0\}$, $M \in \mathbb{N}^+$. Denote $l_j = M^{(1+s)^{j-1}}$, $j \in \mathbb{N}^+$. In case that l_j is not an integer, we just pick $[l_j] + 1$ instead of l_j .

Now, denote by $\Omega = \{l_{n_1}, l_{n_2}, l_{n_3}, \dots\}$ the sequence of all resonant steps. That is, the l_{n_j} th step is obtained by resonant case. Using analytic approximation (2.5) and Theorem 3.1 in every iteration step, we obtain the following almost reducibility result concerning each $(\alpha, Ae^{f_{l_j}(\theta)})$ by induction.

Proposition 3.1. *Let $\alpha \in DC(\kappa, \tau)$, $\sigma < \frac{1}{6}$. Assume that $A \in SL(2, \mathbb{R})$, $f \in C^k(\mathbb{T}^d, sl(2, \mathbb{R}))$ with $k > (D + 2)\tau + 2$ and $\{f_j\}_{j \geq 1}$ are defined above. There exists $\epsilon_0 = \epsilon_0(\kappa, \tau, d, k, \|A\|, \sigma)$ such that if $\|f\|_k \leq \epsilon_0$, then there exist $B_{l_j} \in C^{\omega_{\frac{1}{l_{j+1}}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_j} \in SL(2, \mathbb{R})$ and $f'_{l_j} \in C^{\omega_{\frac{1}{l_{j+1}}}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that*

$$B_{l_j}(\theta + \alpha)(Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{f'_{l_j}(\theta)},$$

with estimates

$$|B_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_j} - \frac{1}{l_{j+1}}} |\ln \epsilon_{l_j}| \right)^\tau \times \epsilon_{l_j}^{-\frac{2}{\frac{1}{l_j} - \frac{1}{l_{j+1}}}} \leq \epsilon_{l_j}^{-\frac{\sigma}{2} - s}, \tag{3.6}$$

$$\|B_{l_j}(\theta)\|_0 \leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_j} - \frac{1}{l_{j+1}}} |\ln \epsilon_{l_j}| \right)^\tau \leq \epsilon_{l_j}^{-\frac{\sigma}{2}}, \tag{3.7}$$

$$|f'_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \leq \epsilon_{l_j}^{3-\sigma}, \quad \|A_{l_j}\| \leq 2\|A\|. \tag{3.8}$$

Moreover, there exists unitary matrices $U_j \in SL(2, \mathbb{C})$ such that

$$U_j A_{l_j} U_j^{-1} = \begin{pmatrix} e^{\gamma_j} & c_j \\ 0 & e^{-\gamma_j} \end{pmatrix}$$

and

$$\|B_{l_j}(\theta)\|_0^2 |c_j| \leq 8\|A\|, \tag{3.9}$$

with $\gamma_j \in i\mathbb{R} \cup \mathbb{R}$ and $c_j \in \mathbb{C}$.

Proof. First step: Assume that

$$C' \|f(\theta)\|_k \leq \frac{c}{(2\|A\|)^{\bar{D}} l_1^{D\tau + \frac{1}{2}}},$$

then by (2.5) and (3.5) we have

$$|f_{l_1}(\theta)|_{\frac{1}{l_1}} \leq \epsilon_{l_1} \leq \epsilon'_0 \left(\frac{1}{l_1}, \frac{1}{l_2} \right).$$

Apply Theorem 3.1, we can find $B_{l_1} \in C^{\omega_{\frac{1}{l_2}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_1} \in SL(2, \mathbb{R})$ and $f'_{l_1} \in C^{\omega_{\frac{1}{l_2}}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$B_{l_1}(\theta + \alpha)(Ae^{f_{l_1}(\theta)})B_{l_1}^{-1}(\theta) = A_{l_1}e^{f'_{l_1}(\theta)}.$$

More precisely, we have two different cases:

- (Non-resonant case)

$$|B_{l_1}|_{\frac{1}{l_2}} \leq 1 + \epsilon_{l_1}^{\frac{1}{2}}, \quad |f'_{l_1}|_{\frac{1}{l_2}} \leq \epsilon_{l_1}^{3-\sigma},$$

and

$$\|A_{l_1} - A\| \leq 2\|A\|\epsilon_{l_1}.$$

- (Resonant case)

$$|B_{l_1}|_{\frac{1}{l_2}} \leq 8 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{\frac{1}{l_1} - \frac{1}{l_2}} |\ln \epsilon_{l_1}| \right)^{\frac{\tau}{2}} \times \epsilon_{l_1}^{-\frac{1}{\frac{1}{l_1} - \frac{1}{l_2}}},$$

$$|B_{l_1}|_0 \leq 8 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{\frac{1}{l_1} - \frac{1}{l_2}} |\ln \epsilon_{l_1}| \right)^{\frac{\tau}{2}}, \quad |f'_{l_1}|_{\frac{1}{l_2}} \leq \epsilon_{l_1}^{100}.$$

Moreover, $A_{l_1} = e^{A''_{l_1}}$ with $\|A''_{l_1}\| \leq 2\epsilon_{l_1}^\sigma$.

In both cases, it is clear that (3.6), (3.7), (3.8) and (3.9) are fulfilled.

Note that the first step is a little special as it does not involve the composition of conjugation maps. Therefore, in order to provide a more explicit iteration process, let us show one more step before induction.

Second step We have

$$B_{l_1}(\theta + \alpha)(Ae^{f_{l_2}})B_{l_1}^{-1}(\theta) = A_{l_1}e^{f'_{l_1}} + B_{l_1}(\theta + \alpha)(Ae^{f_{l_2}} - Ae^{f_{l_1}})B_{l_1}^{-1}(\theta).$$

We can rewrite that

$$A_{l_1}e^{f'_{l_1}(\theta)} + B_{l_1}(\theta + \alpha)(Ae^{f_{l_2}(\theta)} - Ae^{f_{l_1}(\theta)})B_{l_1}^{-1}(\theta) = A_{l_1}e^{\widetilde{f'_{l_1}(\theta)}}.$$

We pick $k > (D + 2)\tau + 2 \geq (1 + 3s)(D\tau + \frac{1}{2}) + 2\tau + 1$.

If the previous step is non-resonant,

$$\begin{aligned} |\widetilde{f'_{l_1}(\theta)}|_{\frac{1}{2}} &\leq |f'_{l_1}(\theta)|_{\frac{1}{2}} + \|A_{l_1}^{-1}\| |B_{l_1}(\theta + \alpha)(Ae^{f_{l_2}(\theta)} - Ae^{f_{l_1}(\theta)})B_{l_1}^{-1}(\theta)|_{\frac{1}{2}} \\ &\leq 4\epsilon_{l_1}^{3-\sigma} + 2\|A\|^2 \times 2 \times \frac{c}{(2\|A\|)^{\bar{D}} l_1^{D\tau + \frac{1}{2}} l_1^{k-1}} \\ &\leq \epsilon_{l_2} \\ &\leq \epsilon'_0 \left(\frac{1}{l_2}, \frac{1}{l_3} \right). \end{aligned}$$

Here the second inequality follows from the non-resonant estimates of the first step. The third inequality is due to the precise choice of our k and the definition of ϵ_m and l_j .

If the previous step is resonant,

$$\begin{aligned} |\widetilde{f'_{l_1}(\theta)}|_{\frac{1}{2}} &\leq |f'_{l_1}(\theta)|_{\frac{1}{2}} + \|A_{l_1}^{-1}\| |B_{l_1}(\theta + \alpha)(Ae^{f_{l_2}(\theta)} - Ae^{f_{l_1}(\theta)})B_{l_1}^{-1}(\theta)|_{\frac{1}{2}} \\ &\leq \epsilon_{l_1}^{100} + 128 \left(\frac{\|A\|^3}{\kappa} \right) \left(\frac{2}{\frac{1}{l_1} - \frac{1}{l_2}} |\ln \epsilon_{l_1}| \right)^\tau \epsilon_{l_1}^{-\frac{2}{\frac{1}{l_1} - \frac{1}{l_2}}} \frac{c}{(2\|A\|)^{\bar{D}} l_1^{D\tau + \frac{1}{2}} l_1^{k-1}} \\ &\leq \epsilon_{l_2} \\ &\leq \epsilon'_0 \left(\frac{1}{l_2}, \frac{1}{l_3} \right). \end{aligned}$$

Here the second inequality follows from the resonant estimates of the first step. The third inequality is again due to the precise choice of our k and the definition of ϵ_m and l_j .

Now for $(\alpha, A_{l_1} e^{\widetilde{f_{l_1}}(\theta)})$, we can apply Theorem 3.1 again to get $\tilde{B}_{l_1} \in C^{\omega}_{\frac{1}{l_3}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_2} \in SL(2, \mathbb{R})$ and $f'_{l_2} \in C^{\omega}_{\frac{1}{l_3}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$\tilde{B}_{l_1}(\theta + \alpha)(A_{l_1} e^{\widetilde{f_{l_1}}(\theta)})\tilde{B}_{l_1}^{-1}(\theta) = A_{l_2} e^{f'_{l_2}(\theta)}, \tag{3.10}$$

which gives

$$B_{l_2}(\theta + \alpha)(A e^{f_{l_2}(\theta)})B_{l_2}^{-1}(\theta) = A_{l_2} e^{f'_{l_2}(\theta)}. \tag{3.11}$$

Here $B_{l_2}(\theta) = \tilde{B}_{l_1}(\theta) \circ B_{l_1}(\theta)$. □

Before giving the precise estimates of each term, let us first introduce a Claim showing that all the resonant steps are separated.

Claim 1. *We have $\epsilon_{l_{n_j+1}} < \epsilon_{l_{n_j}}^2, \forall j \in \mathbb{Z}^+$.*

Proof. It is enough to prove for $j = 1$. By definition, we have l_{n_1} th step is obtained by resonant case. Thus, Theorem 3.1 implies $\rho(A_{l_{n_1}}) \leq 2\epsilon_{l_{n_1}}^\sigma$ and $\rho(A_{l_{n_2-1}}) \leq 4\epsilon_{l_{n_1}}^\sigma$ since every step between l_{n_1} and l_{n_2} is non-resonant.

By the resonant condition of l_{n_2} th step, there exists a unique n with $0 < |n| \leq N_{l_{n_2}}$ such that

$$|2\rho(A_{l_{n_2-1}}) - \langle n, \alpha \rangle| \leq \epsilon_{l_{n_2}}^\sigma. \tag{3.12}$$

However, by the Diophantine condition of α and condition (3.2), we have

$$|\langle n, \alpha \rangle| \geq \frac{\kappa}{|N_{l_{n_2}}|^\tau} \geq 10\epsilon_{l_{n_2}}^{\frac{\sigma}{2}}.$$

If $\epsilon_{l_{n_2}} \geq \epsilon_{l_{n_1}}^2$, then (3.12) yields contradiction. □

Now by Claim 1, we have (in the worst case scenario)

$$\begin{aligned} |B_{l_2}(\theta)|_{\frac{1}{l_3}} &\leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_2} - \frac{1}{l_3}} |\ln \epsilon_{l_2}| \right)^\tau \times \epsilon_{l_2}^{-\frac{\frac{2}{l_3}}{\frac{1}{l_2} - \frac{1}{l_3}}} \leq \epsilon_{l_2}^{-\frac{\sigma}{2} - s}, \\ \|B_{l_2}(\theta)\|_0 &\leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_2} - \frac{1}{l_3}} |\ln \epsilon_{l_2}| \right)^\tau \leq \epsilon_{l_2}^{-\frac{\sigma}{2}}, \\ |f'_{l_2}(\theta)|_{\frac{1}{l_3}} &\leq \epsilon_{l_2}^{3-\sigma}, \quad \|A_{l_j}\| \leq 2\|A\|. \end{aligned}$$

Thus, (3.6), (3.7) and (3.8) are fulfilled again. For (3.9), all three cases (1. there is no resonance, 2. the first step is resonant, 3. the second step is resonant) satisfy it. The discussion is similar to that in the induction step below, so we omit it here for simplicity.

Induction step Assume that for $l_n, n \leq \tilde{n}$, we already have (3.9) and

$$B_{l_n}(\theta + \alpha)(A e^{f_{l_n}(\theta)})B_{l_n}^{-1}(\theta) = A_{l_n} e^{f'_{l_n}(\theta)}, \tag{3.13}$$

with

$$|B_{l_n}(\theta)|_{\frac{1}{l_{n+1}}} \leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_n} - \frac{1}{l_{n+1}}} |\ln \epsilon_{l_n}| \right)^\tau \epsilon_{l_n}^{-\frac{2}{\frac{1}{l_n} - \frac{1}{l_{n+1}}}} \triangleq \zeta_n^2 \leq \epsilon_{l_n}^{-\frac{\sigma}{2} - s}, \tag{3.14}$$

$$\|B_{l_n}(\theta)\|_0 \leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_n} - \frac{1}{l_{n+1}}} |\ln \epsilon_{l_n}| \right)^\tau \leq \epsilon_{l_n}^{-\frac{\sigma}{2}}, \quad |f'_{l_n}(\theta)|_{\frac{1}{l_{n+1}}} \leq \epsilon_{l_n}^{3-\sigma}, \tag{3.15}$$

and

$$\|A_{l_n}\| \leq 2\|A\|. \tag{3.16}$$

Moreover, if the n th step is obtained by the resonant case, we have

$$A_{l_n} = e^{A''_{l_n}}, \quad \|A''_{l_n}\| < 2\epsilon_{l_n}^\sigma. \tag{3.17}$$

Note that this is an inductive step that follows from Theorem 3.1.

If the n th step is obtained by the non-resonant case, we have

$$\|A_{l_n} - A_{l_{n-1}}\| \leq 2\|A_{l_{n-1}}\| \epsilon_{l_n} \tag{3.18}$$

and

$$\|B_{l_n}\|_{\frac{1}{l_{n+1}}} \leq (1 + \epsilon_{l_n}^{\frac{1}{2}}) \|B_{l_{n-1}}\|_{\frac{1}{l_n}}, \quad \|B_{l_n}\|_0 \leq (1 + \epsilon_{l_n}^{\frac{1}{2}}) \|B_{l_{n-1}}\|_0. \tag{3.19}$$

Now by (3.13), for $l_{n+1}, n = \tilde{n}$, we have

$$B_{l_n}(\theta + \alpha)(Ae^{f_{l_{n+1}}})B_{l_n}^{-1}(\theta) = A_{l_n}e^{f'_{l_n}} + B_{l_n}(\theta + \alpha)(Ae^{f_{l_{n+1}}} - Ae^{f_{l_n}})B_{l_n}^{-1}(\theta).$$

In the last part of (2.5), taking a telescoping sum for j from l_n to $l_{n+1} - 1$, we have

$$|f_{l_{n+1}}(\theta) - f_{l_n}(\theta)|_{\frac{1}{l_{n+1}}} \leq \frac{c}{(2\|A\|)\bar{D}l_1^{D\tau + \frac{1}{2}}l_n^{k-1}}. \tag{3.20}$$

Moreover, (2.5) also gives us

$$|f_{l_{n+1}}(\theta)|_{\frac{1}{l_{n+1}}} + |f_{l_n}(\theta)|_{\frac{1}{l_{n+1}}} \leq \frac{2c}{(2\|A\|)\bar{D}M^{D\tau + \frac{1}{2}}}. \tag{3.21}$$

Thus, if we rewrite that

$$A_{l_n}e^{f'_{l_n}(\theta)} + B_{l_n}(\theta + \alpha)(Ae^{f_{l_{n+1}}(\theta)} - Ae^{f_{l_n}(\theta)})B_{l_n}^{-1}(\theta) = A_{l_n}e^{\widetilde{f_{l_n}(\theta)}},$$

by (3.14), (3.15), (3.16), (3.20) and (3.21) we obtain

$$\begin{aligned} |\widetilde{f_{l_n}(\theta)}|_{\frac{1}{l_{n+1}}} &\leq |f'_{l_n}(\theta)|_{\frac{1}{l_{n+1}}} + \|A_{l_n}^{-1}\| \|B_{l_n}(\theta + \alpha)(Ae^{f_{l_{n+1}}(\theta)} - Ae^{f_{l_n}(\theta)})B_{l_n}^{-1}(\theta)|_{\frac{1}{l_{n+1}}} \\ &\leq \epsilon_{l_n}^{3-\sigma} + \left(\frac{\|A\|^4}{\kappa^2} \right) \left(\frac{2}{\frac{1}{l_n} - \frac{1}{l_{n+1}}} |\ln \epsilon_{l_n}| \right)^{2\tau} \epsilon_{l_n}^{-\frac{4}{\frac{1}{l_n} - \frac{1}{l_{n+1}}}} \frac{2 \times 64 \times 64c}{(2\|A\|)\bar{D}l_1^{D\tau + \frac{1}{2}}l_n^{k-1}} \\ &\leq \epsilon_{l_{n+1}} \\ &\leq \epsilon'_0 \left(\frac{1}{l_{n+1}}, \frac{1}{l_{n+2}} \right). \end{aligned}$$

Here the second inequality follows from the estimates of the n th step by assumption and also from the telescoping sum estimates of the analytic approximants. The third inequality follows from the precise choice of our k and the definition of ϵ_m and l_j .

Now for $(\alpha, A_{l_n} e^{\widetilde{f_{l_n}}(\theta)})$, we can apply Theorem 3.1 again to get $\widetilde{B}_{l_n} \in C^{\omega_{\frac{1}{l_{n+2}}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$, $A_{l_{n+1}} \in SL(2, \mathbb{R})$ and $f'_{l_{n+1}} \in C^{\omega_{\frac{1}{l_{n+2}}}}(\mathbb{T}^d, sl(2, \mathbb{R}))$ such that

$$\widetilde{B}_{l_n}(\theta + \alpha)(A_{l_n} e^{\widetilde{f_{l_n}}(\theta)})\widetilde{B}_{l_n}^{-1}(\theta) = A_{l_{n+1}} e^{f'_{l_{n+1}}(\theta)},$$

with

$$|f'_{l_{n+1}}(\theta)|_{\frac{1}{l_{n+2}}} \leq \epsilon_{l_{n+1}}^{3-\sigma}.$$

Denote $B_{l_{n+1}} := \widetilde{B}_{l_n} B_{l_n} \in C^{\omega_{\frac{1}{l_{n+2}}}}(2\mathbb{T}^d, SL(2, \mathbb{R}))$. By Claim 1, using the notation ζ_n from (3.14) we have (in the worst situation)

$$\begin{aligned} |B_{l_{n+1}}(\theta)|_{\frac{1}{l_{n+2}}} &\leq \zeta_{n+1}^{1+\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+\dots+(\frac{1}{2})^n+\dots} \\ &\leq \zeta_{n+1}^2 \\ &= 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_{n+1}} - \frac{1}{l_{n+2}}} |\ln \epsilon_{l_{n+1}}| \right)^\tau \times \epsilon_{l_{n+1}}^{-\frac{\frac{2}{l_{n+2}}}{\frac{1}{l_{n+1}} - \frac{1}{l_{n+2}}}} \\ &\leq \epsilon_{l_{n+1}}^{-\frac{\sigma}{2}-s} \end{aligned}$$

and

$$\|B_{l_{n+1}}(\theta)\|_0 \leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_{n+1}} - \frac{1}{l_{n+2}}} |\ln \epsilon_{l_{n+1}}| \right)^\tau \leq \epsilon_{l_{n+1}}^{-\frac{\sigma}{2}},$$

where this estimate follows from the C^0 norm estimates in Theorem 3.1.

For the remaining estimates, we distinguish two cases.

If the $(n + 1)$ th step is in the resonant case, we have

$$A_{l_{n+1}} = e^{A''_{l_{n+1}}}, \quad \|A''_{l_{n+1}}\| < 2\epsilon_{l_{n+1}}^\sigma, \quad \|A_{l_{n+1}}\| \leq 1 + 2\epsilon_{l_{n+1}}^\sigma \leq 2\|A\|.$$

Then there exists unitary $U \in SL(2, \mathbb{C})$ such that

$$U A_{l_{n+1}} U^{-1} = \begin{pmatrix} e^{\gamma_{n+1}} & c_{n+1} \\ 0 & e^{-\gamma_{n+1}} \end{pmatrix}, \tag{3.22}$$

with $|c_{n+1}| \leq 2\|A''_{l_{n+1}}\| \leq 4\epsilon_{l_{n+1}}^\sigma$. Thus, (3.9) is fulfilled because

$$\|B_{l_{n+1}}(\theta)\|_0^2 |c_{n+1}| \leq 4. \tag{3.23}$$

If it is in the non-resonant case, one traces back to the resonant step j which is closest to $n + 1$.

If j exists, by (3.14) and (3.17) we have

$$\begin{aligned} |B_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} &\leq \epsilon_{l_j}^{-\frac{\sigma}{2}-s}, \quad \|B_{l_j}(\theta)\|_0 \leq \epsilon_{l_j}^{-\frac{\sigma}{2}}, \\ A_{l_j} &= e^{A''_{l_j}}, \quad \|A''_{l_j}\| < 2\epsilon_{l_j}^\sigma, \quad \|A_{l_j}\| \leq 1 + 2\epsilon_{l_j}^\sigma. \end{aligned}$$

By our choice of j , from j to $n + 1$, every step is non-resonant. Thus, by (3.18) we obtain

$$\|A_{l_{n+1}} - A_{l_j}\| \leq 2\epsilon_{l_j}^{\frac{1}{2}}, \tag{3.24}$$

so

$$\|A_{l_{n+1}}\| \leq 1 + 2\epsilon_{l_j}^\sigma + 2\epsilon_{l_j}^{\frac{1}{2}} \leq 2\|A\|.$$

Estimate (3.24) implies that if we rewrite $A_{l_{n+1}} = e^{A''_{l_{n+1}}}$, then

$$\|A''_{l_{n+1}}\| \leq 4\epsilon_{l_j}^\sigma.$$

Moreover, by (3.19), we have

$$\|B_{l_{n+1}}(\theta)\|_0 \leq \sqrt{2}|B_{l_j}(\theta)|_0 \leq \sqrt{2}\epsilon_{l_j}^{-\frac{\sigma}{2}}.$$

Similarly to the process of (3.22), (3.9) is fulfilled because

$$\|B_{l_{n+1}}(\theta)\|_0^2 |c_{n+1}| \leq 8. \tag{3.25}$$

If j does not exist, it immediately implies that from 1 to $n + 1$, each step is non-resonant. In this case, $\|A_{l_{n+1}}\| \leq 2\|A\|$ and the estimate (3.9) is naturally satisfied as

$$\|B_{l_{n+1}}(\theta)\|_{\frac{1}{l_{n+1}}} \leq 2.$$

With Proposition 3.1 in hand, we are ready to transfer all the estimates from $(\alpha, Ae^{f_{l_j}(\theta)})$ to $(\alpha, Ae^{f(\theta)})$ by analytic approximation. We establish our quantitative C^k almost reducibility theorem as follows.

Theorem 3.2. *Let $\alpha \in \text{DC}(\kappa, \tau)$, $\sigma < \frac{1}{6}$, $A \in \text{SL}(2, \mathbb{R})$, $f \in C^k(\mathbb{T}^d, \text{sl}(2, \mathbb{R}))$ with $k > (D+2)\tau+2$, there exists $\epsilon_1 = \epsilon_1(\kappa, \tau, d, k, \|A\|, \sigma)$ such that if $\|f\|_k \leq \epsilon_1$ then $(\alpha, Ae^{f(\theta)})$ is C^{k, k_0} almost reducible with $k_0 \in \mathbb{N}$, $k_0 \leq \frac{k-2\tau-1.5}{1+s}$. Moreover, if we further assume $(\alpha, Ae^{f(\theta)})$ is not uniformly hyperbolic, then there exists $B_{l_j} \in C^{\omega_{\frac{1}{l_j+1}}}(\mathbb{2T}^d, \text{SL}(2, \mathbb{C}))$, $A_{l_j} \in \text{SL}(2, \mathbb{C})$, $\tilde{F}'_{l_j} \in C^k(\mathbb{T}^d, \text{gl}(2, \mathbb{C}))$, such that*

$$B_{l_j}(\theta + \alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j} + \tilde{F}'_{l_j}(\theta)$$

with

$$\|B_{l_j}(\theta)\|_0 \leq \epsilon_{l_j}^{-\frac{\sigma}{2}}, \quad \|\tilde{F}'_{l_j}(\theta)\|_0 \leq \epsilon_{l_j}^{\frac{1}{4}}$$

and $A_{l_j} = \begin{pmatrix} e^{\gamma_j} & c_j \\ 0 & e^{-\gamma_j} \end{pmatrix}$ with estimate

$$\|B_{l_j}(\theta)\|_0^2 |c_j| \leq 8\|A\|, \tag{3.26}$$

where $\gamma_j \in i\mathbb{R}$ and $c_j \in \mathbb{C}$.

Proof. We first deal with the C^0 estimate. By Proposition 3.1, we have for any $l_j, j \in \mathbb{N}^+$:

$$B_{l_j}(\theta + \alpha)(Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{f'_{l_j}(\theta)};$$

thus,

$$B_{l_j}(\theta + \alpha)(Ae^{f(\theta)})B_{l_j}^{-1}(\theta) = A_{l_j}e^{f'_{l_j}(\theta)} + B_{l_j}(\theta + \alpha)(Ae^{f(\theta)} - Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta).$$

Denote

$$A_{l_j} + \tilde{F}_{l_j}(\theta) = A_{l_j}e^{f'_{l_j}(\theta)} + B_{l_j}(\theta + \alpha)(Ae^{f(\theta)} - Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta). \tag{3.27}$$

In the last part of (2.5), taking a telescoping sum from l_j to $+\infty$, we get

$$\|f(\theta) - f_{l_j}(\theta)\|_0 \leq \frac{c}{(2\|A\|)\bar{D}l_1^{D\tau+\frac{1}{2}}l_j^{k-1}}, \tag{3.28}$$

and

$$\|f(\theta)\|_0 + \|f_{l_j}(\theta)\|_0 \leq \frac{c}{(2\|A\|)\bar{D}M^{D\tau+\frac{1}{2}}} + \frac{c}{(2\|A\|)\bar{D}C'M^{D\tau+\frac{1}{2}}}. \tag{3.29}$$

Proposition 3.1 also gives the estimates

$$\|B_{l_j}(\theta)\|_0 \leq 64 \left(\frac{\|A\|}{\kappa} \right) \left(\frac{2}{\frac{1}{l_j} - \frac{1}{l_{j+1}}} |\ln \epsilon_{l_j}| \right)^\tau \leq \epsilon_{l_j}^{-\frac{\sigma}{2}}, \tag{3.30}$$

$$|f'_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \leq \epsilon_{l_j}^{3-\sigma}, \tag{3.31}$$

and

$$\|A_{l_j}\| \leq 2\|A\|. \tag{3.32}$$

Thus, by (3.27)–(3.31), we have

$$\begin{aligned} \|\tilde{F}_{l_j}(\theta)\|_0 &\leq \|A_{l_j}f'_{l_j}(\theta)\|_0 + \|B_{l_j}(\theta + \alpha)(Ae^{f(\theta)} - Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta)\|_0 \\ &\leq \|A\|\epsilon_{l_j}^{3-\sigma} + \left(\frac{\|A\|^3}{\kappa^2} \right) \left(\frac{2}{\frac{1}{l_j} - \frac{1}{l_{j+1}}} |\ln \epsilon_{l_j}| \right)^{2\tau} \times \frac{64 \times 64c}{(2\|A\|)\bar{D}l_1^{D\tau+\frac{1}{2}}l_j^{k-1}} \\ &\leq \epsilon_{l_j}^{1+s}. \end{aligned} \tag{3.33}$$

Now let us prove estimate (3.26). Note that Proposition 3.1 implies that we only need to rule out the possibility that $\gamma_j \in \mathbb{R} \setminus \{0\}$.

Assume that $\text{spec}(A_{l_j}) = \{e^{\lambda_j}, e^{-\lambda_j}\}$, $\lambda_j \in \mathbb{R} \setminus \{0\}$, then there exists $P \in SO(2, \mathbb{R})$ such that

$$PA_{l_j}P^{-1} = \begin{pmatrix} e^{\lambda_j} & c_j \\ 0 & e^{-\lambda_j} \end{pmatrix},$$

with $|c_j| \leq \|A_{l_j}\| \leq 2\|A\|$.

If $|\lambda_j| > \epsilon_{l_j}^{\frac{1}{4}}$, set $B = \text{diag}\{\|4A\|^{-3}\epsilon_{l_j}^{\frac{1}{4}}, \|4A\|^3\epsilon_{l_j}^{-\frac{1}{4}}\}$, then

$$BP(A_{l_j} + \tilde{F}_{l_j}(\theta))P^{-1}B^{-1} = \begin{pmatrix} e^{\lambda_j} & 0 \\ 0 & e^{-\lambda_j} \end{pmatrix} + F(\theta), \tag{3.34}$$

where $\|F(\theta)\|_0 \leq \frac{\epsilon_{l_j}^{\frac{1}{2}}}{C\|A\|^5}$. We rewrite

$$\begin{pmatrix} e^{\lambda_j} & 0 \\ 0 & e^{-\lambda_j} \end{pmatrix} + F(\theta) = \begin{pmatrix} e^{\lambda_j} & 0 \\ 0 & e^{-\lambda_j} \end{pmatrix} e^{\tilde{f}(\theta)}$$

with $\|\tilde{f}(\theta)\|_0 \leq \frac{\epsilon_{l_j}^{\frac{1}{2}}}{C\|A\|^4}$. Then by Remark 3.1 and Corollary 3.1 of [22], one can conjugate (3.34) to

$$\begin{pmatrix} e^{\lambda_j} & 0 \\ 0 & e^{-\lambda_j} \end{pmatrix} \begin{pmatrix} e^{\tilde{f}^{re}(\theta)} & 0 \\ 0 & e^{-\tilde{f}^{re}(\theta)} \end{pmatrix} = \begin{pmatrix} e^{\lambda_j} e^{\tilde{f}^{re}(\theta)} & 0 \\ 0 & e^{-\lambda_j} e^{-\tilde{f}^{re}(\theta)} \end{pmatrix} \quad (3.35)$$

with $\|\tilde{f}^{re}(\theta)\|_0 \leq \frac{\epsilon_{l_j}^{\frac{1}{2}}}{C\|A\|^2}$. Therefore, $(\alpha, Ae^{f(\theta)})$ is uniformly hyperbolic, which contradicts our assumption. Now we only need to consider $|\lambda_j| \leq \epsilon_{l_j}^{\frac{1}{4}}$. In this case, we put λ_j into the perturbation so that the new perturbation satisfies $\|\tilde{F}'_{l_j}\|_0 \leq \epsilon_{l_j}^{\frac{1}{4}}$ and

$$A_{l_j} = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}.$$

Now let us deal with the differentiable almost reducibility. By Cauchy integral formula, for $k_0 \in \mathbb{N}$ with $k_0 \leq k$ and $n \in \mathbb{N}$ with $n \geq j$, we have

$$\begin{aligned} & \|B_{l_j}(\theta + \alpha)(Ae^{f_{l_{n+1}}(\theta)} - Ae^{f_{l_n}(\theta)})B_{l_j}^{-1}(\theta)\|_{k_0} \\ & \leq \sup_{|l| \leq k_0, \theta \in \mathbb{T}^d} \|(\partial_{\theta_1}^{l_1} \cdots \partial_{\theta_d}^{l_d})(B_{l_j}(\theta + \alpha)(Ae^{f_{l_{n+1}}(\theta)} - Ae^{f_{l_n}(\theta)})B_{l_j}^{-1}(\theta))\| \\ & \leq (k_0)!(l_{n+1})^{k_0} |B_{l_j}(\theta + \alpha)(Ae^{f_{l_{n+1}}(\theta)} - Ae^{f_{l_n}(\theta)})B_{l_j}^{-1}(\theta)|_{\frac{1}{l_{n+1}}} \\ & \leq (k_0)!(l_n)^{(1+s)k_0} \left(\frac{\|A\|^3}{\kappa^2}\right) \left(\frac{2}{\frac{1}{l_n} - \frac{1}{l_{n+1}}} |\ln \epsilon_{l_n}|\right)^{2\tau} \epsilon_{l_n}^{-\frac{4}{\frac{1}{l_n} - \frac{1}{l_{n+1}}}} \\ & \frac{64 \times 64c}{(2\|A\|)^{\tilde{D}} l_1^{D\tau + \frac{1}{2}} l_n^{k-1}} \leq \frac{C_1}{l_n^{k-(1+s)k_0 - 2\tau - 2s(D\tau + \frac{1}{2}) - 1}} \end{aligned}$$

where C_1 is independent of j .

Taking a telescoping sum from l_j to $+\infty$, we get

$$\|B_{l_j}(\theta + \alpha)(Ae^{f(\theta)} - Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta)\|_{k_0} \leq \frac{2C_1}{l_j^{k-(1+s)k_0 - 2\tau - 2s(D\tau + \frac{1}{2}) - 1}}.$$

Similarly by Cauchy integral formula, we have

$$\begin{aligned} \|f'_{l_j}(\theta)\|_{k_0} & \leq (k_0)!(l_{j+1})^{k_0} |f'_{l_j}(\theta)|_{\frac{1}{l_{j+1}}} \\ & \leq (k_0)!(l_j)^{(1+s)k_0} \times \epsilon_{l_j}^{3-\sigma} \\ & \leq (k_0)!(l_j)^{(1+s)k_0} \times \left(\frac{c}{(2\|A\|)^{\tilde{D}} l_j^{D\tau + \frac{1}{2}}}\right)^{3-\sigma} \\ & \leq \frac{C_2}{l_j^{(D\tau + \frac{1}{2})(3-\sigma) - (1+s)k_0}} \end{aligned}$$

where C_2 is independent of j .

Thus, we have

$$\begin{aligned} \|\tilde{F}_{l_j}(\theta)\|_{k_0} &\leq \|A_{l_j} f'_{l_j}(\theta)\|_{k_0} + \|B_{l_j}(\theta + \alpha)(Ae^{f(\theta)} - Ae^{f_{l_j}(\theta)})B_{l_j}^{-1}(\theta)\|_{k_0} \\ &\leq \frac{C_3}{l_j^{(D\tau + \frac{1}{2})(3-\sigma) - (1+s)k_0}} + \frac{2C_1}{l_j^{k - (1+s)k_0 - 2\tau - 2s(D\tau + \frac{1}{2}) - 1}}. \end{aligned} \tag{3.36}$$

Note that although we write $k > (D + 2)\tau + 2$ in the assumption, we simply choose $k = [(D + 2)\tau + 2] + 1$ in the actual operation process. So the quantity of k is fully determined by D . Here “[x]” stands for the integer part of x .

So if $k_0 \leq \frac{k - 2\tau - \frac{3}{2}}{1+s}$, then

$$\|\tilde{F}_{l_j}(\theta)\|_{k_0} \leq \frac{C_4}{l_j^{\frac{1}{6}}},$$

which immediately shows that

$$\lim_{j \rightarrow +\infty} \|\tilde{F}_{l_j}(\theta)\|_{k_0} = 0.$$

It means precisely that $(\alpha, Ae^{f(\theta)})$ is C^{k, k_0} almost reducible. This finishes the proof of Theorem 3.2. □

Remark 3.4. In view of Corollary 3.1 in [13], we have $L(\alpha, Ae^{f(\theta)}) = 0$. Moreover, estimate (3.26) is essential to proving 1/2 Hölder continuity of the IDS, which is furthermore essential to proving purely absolutely continuous spectrum, see Remark 4.1.

Remark 3.5. One novelty of our quantitative C^k almost reducibility in this version is that the norm of the conjugation map B_{l_j} can be adjusted easily by the variation of the parameters. More precisely, if we assume $D > \frac{t}{\sigma}$ with $t \geq 2$, then by the proof of Claim 1 we have $\epsilon_{l_{n_j+1}} < \epsilon_{l_{n_j}}^t, \forall j \in \mathbb{Z}^+$ and

$$\|B_{l_j}(\theta)\|_0 \leq \epsilon_{l_j}^{-\frac{\sigma}{2t} \sum_{j=0}^{\infty} \frac{1}{i^j}}.$$

This is quite useful for spectral applications. In certain cases, we need to reduce the norm of B_{l_j} at the cost of enlarging D , i.e., the initial regularity k increases.

In order to obtain the $\frac{1}{2}$ -Hölder continuity of the Lyapunov exponent, we have to assume $D > \frac{5}{2\sigma}$ so that $\|B_{l_j}(\theta)\|_0 \leq \epsilon_{l_j}^{-\frac{\sigma}{3}}$. Recall that the restriction on σ is “ $\sigma < \frac{1}{6}$.” In order to make the initial regularity k small, we need to fix σ sufficiently close to $\frac{1}{6}$ in the beginning.

By Theorem 3.2 and Theorem 1.1 in [13], we have

Theorem 3.3. *Let $\alpha \in \text{DC}(\kappa, \tau)$, if (α, A) is $C^{k', k}$ almost reducible with $k' > k > 17\tau + 2$, then for any continuous map $B : \mathbb{T}^d \rightarrow SL(2, \mathbb{C})$, we have*

$$|L(\alpha, A) - L(\alpha, B)| \leq C \|B - A\|_0^{\frac{1}{2}}, \tag{3.37}$$

where C is a constant depending on d, κ, τ, A, k .

Proof. The proof is almost the same as that in [13], with the only difference that the covering interval I_j becomes: $C\epsilon_{l_j}^{\frac{1}{4}} \leq \epsilon \leq c\epsilon_{l_j}^{\frac{2}{3}}$. Here C, c are two constants depending on d, κ, τ, A . It is clear that all the small ϵ tending to zero can be covered by the interval $\{I_j\}_{j \geq 1}$ since $l_{j+1} = l_j^{1+s}, 0 < s \leq \frac{1}{6D\tau+3}$. \square

Similarly, by Theorem 3.2 and the Proof of Theorem 1.2 in [13], we have

Theorem 3.4. *Let $\alpha \in DC(\kappa, \tau), V \in C^k(\mathbb{T}^d, \mathbb{R})$ with $k > 17\tau + 2$, then there exists λ_0 depending on V, d, κ, τ, k such that if $\lambda < \lambda_0$, then we have the following:*

- (1) *For any $E \in \mathbb{R}, (\alpha, S_E^{\lambda V})$ is C^{k, k_0} almost reducible with $k_0 \leq k - 2\tau - 2$.*
- (2) *$N_{\lambda V, \alpha}$ is $1/2$ -Hölder continuous:*

$$N(E + \epsilon) - N(E - \epsilon) \leq C_0 \epsilon^{\frac{1}{2}}, \forall \epsilon > 0, \forall E \in \mathbb{R},$$

where C_0 depends only on d, κ, τ, k .

Remark 3.6. In fact for (1), by Theorem 3.2 we only require k to be larger than $14\tau + 2$ since $D > \frac{2}{\sigma}$ is enough for almost reducibility, which gives Theorem 1.2.

3.4. Stratification via rotation number

For our purpose, we will apply our C^k almost reducibility theorem on a special type of C^k quasi-periodic $SL(2, \mathbb{R})$ cocycles: Schrödinger cocycle $(\alpha, S_E^{\lambda V})$, where

$$S_E^{\lambda V}(\theta) = \begin{pmatrix} E - \lambda V(\theta) & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.38}$$

For the sake of unification, let us rewrite $S_E^{\lambda V}(\theta) = Ae^{f(\theta)}$ where

$$A = \begin{pmatrix} E & -1 \\ 1 & 0 \end{pmatrix}.$$

In the following, the assumptions of Theorem 3.2 are always fulfilled by assuming the small condition on λ in Theorem 1.1. It gives that $(\alpha, S_E^{\lambda V})$ is C^{k, k_0} almost reducible for all $E \in \mathbb{R}$, particularly for $E \in \Sigma$ where Σ is the spectrum of the corresponding Schrödinger operator. Now let us divide Σ into countable sets of energy E in the following way. For $m \in \mathbb{Z}^+$, define

$$K_m = \{E \in \Sigma \mid (\alpha, S_E^{\lambda V}) \text{ has a resonance at } m\text{-th step}\}. \tag{3.39}$$

Moreover, define

$$K_0 = \{E \in \Sigma \mid (\alpha, S_E^{\lambda V}) \text{ is reducible}\},$$

then we have

$$\Sigma = \bigcup_{m=1}^{\infty} K_m.$$

When the resonance occurs, we can depict each K_m more precisely by the rotation number of $(\alpha, S_E^{\lambda V})$ and we denoted it by $\rho(E)$ for convenience.

Lemma 3.2. *Assume $E \in K_m$, $m \geq 1$, there exists $n \in \mathbb{Z}^d$ with $0 < |n| \leq N_m$ such that*

$$|2\rho(E) - \langle n, \alpha \rangle|_{\mathbb{T}} \leq 5\epsilon_{l_m}^\sigma$$

where $N_m = 5l_m \ln \frac{1}{\epsilon_{l_m}}$.

Proof. If $E \in K_m$, by the resonant case of Theorem 3.1 and Theorem 3.2, we have

$$|2\rho(\alpha, A_{l_{m-1}}) - \langle n', \alpha \rangle|_{\mathbb{T}} \leq \epsilon_{l_m}^\sigma, \tag{3.40}$$

for some $n' \in \mathbb{Z}^d$ satisfying $0 < |n'| \leq N = \frac{2}{\frac{1}{l_m} - \frac{1}{l_{m+1}}} |\ln \epsilon_{l_m}|$ (note that $A_{l_0} = A$). In addition, after doing this resonant step, by (5.28) we have

$$|\rho(\alpha, A_{l_m})| \leq \|A''_{l_m}\| \leq 2\epsilon_{l_m}^\sigma. \tag{3.41}$$

Moreover, formula (2.3) gives

$$\rho(E) + \frac{\langle \deg B_{l_m}, \alpha \rangle}{2} = \rho(\alpha, A_{l_m} + \tilde{F}_{l_m}(\theta)). \tag{3.42}$$

By the properties of rotation number and (3.33), there exists a numerical constant c such that

$$|\rho(\alpha, A_{l_m} + \tilde{F}_{l_m}(\theta)) - \rho(\alpha, A_{l_m})| \leq c \|\tilde{F}_{l_m}(\theta)\|_0^{\frac{1}{2}} \leq c\epsilon_{l_m}^{\frac{1+s}{2}}. \tag{3.43}$$

Denote $\rho(E) + \frac{\langle \deg B_{l_m}, \alpha \rangle}{2} - \rho(\alpha, A_{l_m}) = *$, then by (3.40)-(3.43), we have

$$\begin{aligned} & |\rho(E) + \frac{\langle \deg B_{l_m}, \alpha \rangle}{2}|_{\mathbb{T}} \\ &= |\rho(E) + \frac{\langle \deg B_{l_m}, \alpha \rangle}{2} - \rho(\alpha, A_{l_m}) + \rho(\alpha, A_{l_m})|_{\mathbb{T}} \\ &= |* + \rho(\alpha, A_{l_m})|_{\mathbb{T}} \\ &\leq c\epsilon_{l_m}^{\frac{1+s}{2}} + 2\epsilon_{l_m}^\sigma \\ &\leq \frac{5}{2}\epsilon_{l_m}^\sigma. \end{aligned}$$

By Claim 1 and Remark 3.5, for $t \geq 2$, we have

$$|\deg B_{l_m}| \leq N \times \sum_{j=0}^{\infty} \frac{1}{t^j} \leq 5l_m \ln \frac{1}{\epsilon_{l_m}},$$

if we denote $N_m = 5l_m \ln \frac{1}{\epsilon_{l_m}}$, the result follows immediately. □

In order to make more preparations, we denote the transfer matrices by

$$A_n(E, \theta) = \prod_{j=n-1}^0 S_E^{\lambda V}(\theta + j\alpha). \tag{3.44}$$

We will show that nice quantitative almost reducibility indicates nice control on the growth of A_n on each K_m , $m \geq 1$ (as will be shown in Section 4, we do not need to estimate things on K_0 since it is transcendentially excluded in the proof of purely absolutely continuous spectrum).

Lemma 3.3. *Assume $k > 35\tau + 2$, then for every $E \in K_m, m \geq 1$, we have $\sup_{0 \leq s \leq c\epsilon_{l_m}^{-1+\frac{\sigma}{2}}} \|A_s\|_0 \leq \epsilon_{l_m}^{-\frac{2\sigma}{9}}$, where c is a universal constant.*

Proof. For $E \in K_m, m \geq 1$, $(\alpha, S_E^{\lambda V})$ has a resonance at m th step. By the resonant estimates (5.13)-(5.26) of Theorem 3.1, Theorem 3.2 and Remark 3.5, if $D > \frac{11}{2\sigma}$ (thus, we assume $k > 35\tau + 2$), then there exist $B_{l_m} \in C^{\frac{\omega}{l_{m+1}}}(2\mathbb{T}^d, SL(2, \mathbb{C}))$, $A_{l_m} \in SL(2, \mathbb{C})$ and $\tilde{F}'_{l_m} \in C^{k_0}(\mathbb{T}^d, gl(2, \mathbb{C}))$ such that

$$B_{l_m}(\theta + \alpha)(Ae^{f(\theta)})B_{l_m}^{-1}(\theta) = A_{l_m} + \tilde{F}'_{l_m}(\theta),$$

with

$$\|B_{l_m}(\theta)\|_0 \leq \epsilon_{l_m}^{-\frac{\sigma}{9}}, A_{l_m} = \begin{pmatrix} e^{\gamma_m} & 0 \\ 0 & e^{-\gamma_m} \end{pmatrix}, \|\tilde{F}'_{l_m}(\theta)\|_0 \leq \epsilon_{l_m}^{1-\frac{\sigma}{2}} + \epsilon_{l_m}^{1+s},$$

where “ $\epsilon_{l_m}^{1-\frac{\sigma}{2}}$ ” is the upper bound of the off-diagonal’s norm, see (5.23). And “ $\epsilon_{l_m}^{1+s}$ ” corresponds to the quantity of the perturbation, see (3.33). Moreover, we have $\gamma_m \in i\mathbb{R}$.

Then we easily conclude

$$\sup_{0 \leq s \leq c\epsilon_{l_m}^{-1+\frac{\sigma}{2}}} \|A_s\|_0 \leq \|B_{l_m}(\theta)\|_0^2 \leq \epsilon_{l_m}^{-\frac{2\sigma}{9}}.$$

□

4. Spectral application: absolutely continuous spectrum

With the dynamical estimates in hand, we can prove our main theorem easily. Let us cite the well known result shown by Gilbert–Pearson [20].

Theorem 4.1 [20]. *Let \mathcal{B} be the set of $E \in \mathbb{R}$ such that the cocycle (α, S_E^V) is bounded (i.e., the norm of its transfer matrix is uniformly bounded in $n \in \mathbb{N}$ and $\theta \in \mathbb{T}^d$). Then $\mu_{univ, (V, \alpha, \theta)}|_{\mathcal{B}}$ is absolutely continuous for all $\theta \in \mathbb{T}^d$ where $\mu_{univ} = \mu_{\delta_0} + \mu_{\delta_1}$.*

Besides, let us recall two convenient results proved by Avila [1] (in the following, μ stands for $\mu_{univ, (\lambda V, \alpha, \theta)}$).

Theorem 4.2 [1]. *We have $\mu(E - \epsilon, E + \epsilon) \leq C\epsilon \sup_{0 \leq s \leq C\epsilon^{-1}} \|A_s\|_0^2$, where $C > 0$ is a universal constant.*

Theorem 4.3 [1]. *If $E \in \Sigma$ then for $0 < \epsilon < 1$, $N(E + \epsilon) - N(E - \epsilon) \geq c\epsilon^{\frac{3}{2}}$, where $c > 0$ is a universal constant.*

Remark 4.1. The proof of Theorem 4.3 requires the $\frac{1}{2}$ -Hölder continuity of the integrated density of states in our case, which is ensured by Theorem 3.4.

4.1. Proof of Theorem 1.1

Proof. Denote \mathcal{B} the set of $E \in \Sigma$ such that $(\alpha, S_E^{\lambda V})$ is bounded. Denote \mathcal{R} be the set of $E \in \Sigma$ such that $(\alpha, S_E^{\lambda V})$ is reducible. Then Theorem 4.1 ensure that we only need to prove $\mu(\Sigma \setminus \mathcal{B}) = 0$.

Note that $\mathcal{R} \setminus \mathcal{B}$ has only E such that $(\alpha, S_E^{\lambda V})$ is reducible to parabolic. By the Gap Labeling Theorem [10, 24], for any $E \in \mathcal{R} \setminus \mathcal{B}$, there exists a unique $m \in \mathbb{Z}^d$ such that $2\rho(\alpha, S_E^{\lambda V}) \equiv \langle m, \alpha \rangle \pmod{2\pi\mathbb{Z}}$, which shows $\mathcal{R} \setminus \mathcal{B}$ is countable. Moreover, if $E \in \mathcal{R}$, then any nonzero solution $H_{\lambda V, \alpha, \theta} u = Eu$ satisfies $\inf_{n \in \mathbb{Z}} [|u_n|^2 + |u_{n+1}|^2] > 0$. So there are no eigenvalues in \mathcal{R} and $\mu(\mathcal{R} \setminus \mathcal{B}) = 0$. Therefore, it is enough to prove $\mu(\Sigma \setminus \mathcal{R}) = 0$.

Define $K_m, m \in \mathbb{Z}^+$ as in (3.39). By the definition of K_m , we have $\Sigma \setminus \mathcal{R} \subset \limsup K_m$.

For every $E \in K_m$, let $J_m(E)$ be an open $\epsilon_m = C\epsilon_{l_m}^{\frac{2\sigma}{3}}$ neighborhood of E . By Lemma 3.3,

$$\sup_{0 \leq s \leq C\epsilon_m^{-1}} \|A_s\|_0^2 \leq \epsilon_{l_m}^{-\frac{4\sigma}{9}}.$$

Moreover, by Theorem 4.2

$$\mu(J_m(E)) \leq C\epsilon_{l_m}^{-\frac{4\sigma}{9}} |J_m(E)|,$$

where “ $|\cdot|$ ” stands for the Lebesgue measure. By compactness we can take a finite subcover $\overline{K_m} \subset \bigcup_{j=0}^{r_0} J_m(E_j)$. By refining this subcover, we can assume that any $x \in \mathbb{R}$ is contained in at most 2 different $J_m(E_j)$ (the number of subcovers may grow due to refinement, but it is finite because of compactness and we denote it by r). By the Borel–Cantelli lemma, it is sufficient for us to prove the measure of $\bigcup_{j=0}^r J_m(E_j)$ has some good decay with respect to m so that $\sum_m \mu(\overline{K_m}) < \infty$.

By Theorem 4.3, we have

$$|N(J_m(E))| \geq c|J_m(E)|^{\frac{3}{2}}.$$

By Lemma 3.2, if $E \in K_m$, then

$$|2\pi N(E) - \langle n, \alpha \rangle|_{\mathbb{T}} \leq 5\epsilon_{l_m}^{\sigma}$$

for some $|n| \leq 5l_m \ln \frac{1}{\epsilon_{l_m}}$. This shows that $N(K_m)$ can be covered by $(10l_m \ln \frac{1}{\epsilon_{l_m}} + 1)^d$ intervals I_s of length $5\epsilon_{l_m}^{\sigma}$. Since $|I_s| \leq C|N(J_m(E))|$ for any s and $E \in K_m$, there are at most $2C + 4$ intervals $J_m(E_j)$ such that $N(J_m(E_j))$ intersects I_s . We conclude that there are at most $C(10l_m \ln \frac{1}{\epsilon_{l_m}} + 1)^d$ intervals of $J_m(E_j)$. Then

$$\mu(\overline{K_m}) \leq \sum_{j=0}^r \mu(J_m(E_j)) \leq C \left(10l_m \ln \frac{1}{\epsilon_{l_m}} + 1 \right)^d \times C\epsilon_{l_m}^{-\frac{4\sigma}{9}} \times C\epsilon_{l_m}^{\frac{2\sigma}{3}} \leq l_m^{-\frac{\tau}{5}},$$

which gives

$$\sum_m \mu(\overline{K_m}) \leq C.$$

This finishes the proof. \square

4.2. Further comments

The technical requirement of “ $k > 35\tau + 2$ ” is really due to the necessity of ensuring the slower growth of the conjugation maps (less cost) as well as the faster decay of the perturbations (more profit). For “ $k > 14\tau + 2$,” although we have the almost reducibility, there is not enough quantitative control of the conjugacy and the perturbation. Indeed, understanding the competition between these two terms is our core of using dynamical estimates to derive spectral results.

There is something also interesting to mention. Of course, the smallness of the coupling constant λ ensures the zero Lyapunov exponent and purely absolutely continuous spectrum in our context. But perhaps it would be more accurate to say that it is the smallness of the oscillation of the potential that does the job. In fact, Wang and You [30, 31] created examples of smooth Schrödinger operators with large coupling constant which have zero Lyapunov exponent for some energy in the spectrum. The tricky part is that the potential has some flat region which corresponds to small oscillation.

Finally, in an ongoing project named “Mixed Random-quasiperiodic Cocycles” [14, 15] with Pedro Duarte and Silviu Klein, our purely ac spectrum result will serve as a fundamental example to show the metal–insulator transition between quasi-periodicity and randomness.

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Declarations

Conflicts of interest There is no conflict of interest

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5. Appendix

Proof of Theorem 3.1. For readers who are quite familiar with the analytic KAM scheme, this proof can be skipped since the structure is similar to that

in [13]. But the estimates here are sharp compared with those in [13] (see Remark 3.3), so we prefer to provide the detailed proof for self-containedness.

Recall that $sl(2, \mathbb{R})$ is isomorphic to $su(1, 1)$, which consists of matrices of the form

$$\begin{pmatrix} it & v \\ \bar{v} & -it \end{pmatrix}$$

with $t \in \mathbb{R}, v \in \mathbb{C}$. The isomorphism between them is given by $A \rightarrow MAM^{-1}$, where

$$M = \frac{1}{1+i} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

and a simple calculation yields

$$M \begin{pmatrix} x & y+z \\ y-z & -x \end{pmatrix} M^{-1} = \begin{pmatrix} iz & x-iy \\ x+iy & -iz \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. $SU(1, 1)$ is the corresponding Lie group of $su(1, 1)$. We will prove this theorem in $SU(1, 1)$, which is isomorphic to $SL(2, \mathbb{R})$. \square

We distinguish two cases:

Non-resonant case For $0 < |n| \leq N = \frac{2}{r-r'} |\ln \epsilon|$, we have

$$|2\rho - \langle n, \alpha \rangle| \geq \epsilon^\sigma; \tag{5.1}$$

by (3.2) with $D > \frac{2}{\sigma}$, we have

$$|\langle n, \alpha \rangle| \geq \frac{\kappa}{|n|^\tau} \geq \frac{\kappa}{|N|^\tau} \geq \epsilon^{\frac{\sigma}{2}} \geq \epsilon^\sigma. \tag{5.2}$$

It is well known that (5.1) and (5.2) are the conditions which are used to overcome the small denominator problem in KAM theory.

Define

$$\Lambda_N = \left\{ f \in C_r^\omega(\mathbb{T}^d, su(1, 1)) \mid f(\theta) = \sum_{k \in \mathbb{Z}^d, 0 < |k| < N} \hat{f}(k) e^{i\langle k, \theta \rangle} \right\}. \tag{5.3}$$

Our goal is to solve the cohomological equation

$$Y(\theta + \alpha)A - AY(\theta) = A(-\mathcal{T}_N f(\theta) + \hat{f}(0)),$$

i.e.,

$$A^{-1}Y(\theta + \alpha)A - Y(\theta) = -\mathcal{T}_N f(\theta) + \hat{f}(0). \tag{5.4}$$

Here \mathcal{T}_N is the truncation operator such that

$$(\mathcal{T}_N f)(\theta) = \sum_{k \in \mathbb{Z}^d, |k| < N} \hat{f}(k) e^{i\langle k, \theta \rangle}.$$

Take the Fourier transform for (5.4) and compare the corresponding Fourier coefficients of the two sides. By (5.1) (apply it twice to solve the off-diagonal) along with (5.2) (apply it once to solve the diagonal), we obtain that if $Y \in \Lambda_N$, then

$$|Y(\theta)|_r \leq \epsilon^{-3\sigma} |\mathcal{T}_N f(\theta) - \hat{f}(0)|_r,$$

which gives

$$|A^{-1}Y(\theta + \alpha)A - Y(\theta)|_r \geq \epsilon^{3\sigma}|Y(\theta)|_r. \tag{5.5}$$

Moreover, we have $A^{-1}Y(\theta + \alpha)A \in \Lambda_N$ by (5.3). For $\eta = \epsilon^{3\sigma}$, we define $\mathcal{B}_r^{nre}(\epsilon^{3\sigma})$ by (3.1), then we have $\Lambda_N \subset \mathcal{B}_r^{nre}(\epsilon^{3\sigma})$.

Since $\epsilon^{3\sigma} \geq 13\|A\|^2\epsilon^{\frac{1}{2}}$ (it holds by σ being smaller than $\frac{1}{6}$ and \tilde{D} depending on σ), by Lemma 3.1 we have $Y \in \mathcal{B}_r$ and $f^{re} \in \mathcal{B}_r^{re}(\epsilon^{3\sigma})$ such that

$$e^{Y(\theta+\alpha)}(Ae^{f(\theta)})e^{-Y(\theta)} = Ae^{f^{re}(\theta)},$$

with $|Y|_r \leq \epsilon^{\frac{1}{2}}$ and

$$|f^{re}|_r \leq 2\epsilon. \tag{5.6}$$

By (5.3)

$$(\mathcal{I}_N f^{re})(\theta) = \hat{f}^{re}(0), \quad \|\hat{f}^{re}(0)\| \leq 2\epsilon,$$

and

$$\begin{aligned} |(\mathcal{R}_N f^{re})(\theta)|_{r'} &= \left| \sum_{|n|>N} \hat{f}^{re}(n)e^{i\langle n, \theta \rangle} \right|_{r'} \\ &\leq 2\epsilon e^{-N(r-r')}(N)^d \\ &\leq 2\epsilon \cdot \epsilon^2 \cdot \frac{1}{4}\epsilon^{-\sigma} \\ &= \frac{1}{2}\epsilon^{3-\sigma}. \end{aligned} \tag{5.7}$$

Moreover, we can compute that

$$e^{\hat{f}^{re}(0)+\mathcal{R}_N f^{re}(\theta)} = e^{\hat{f}^{re}(0)}(Id + e^{-\hat{f}^{re}(0)}\mathcal{O}(\mathcal{R}_N f^{re})) = e^{\hat{f}^{re}(0)}e^{f_+(\theta)},$$

by (5.7), we have

$$|f_+(\theta)|_{r'} \leq 2|\mathcal{R}_N f^{re}(\theta)|_{r'} \leq \epsilon^{3-\sigma}.$$

Finally, if we denote

$$A_+ = Ae^{\hat{f}^{re}(0)},$$

then we have

$$\|A_+ - A\| \leq \|A\| \|Id - e^{\hat{f}^{re}(0)}\| \leq 2\|A\|\epsilon.$$

Resonant case In fact, we only need to consider the case in which A is elliptic with eigenvalues $\{e^{i\rho}, e^{-i\rho}\}$ for $\rho \in \mathbb{R} \setminus \{0\}$ since if $\rho \in i\mathbb{R}$, then the non-resonant condition is always satisfied due to the Diophantine condition on α and then it actually belongs to the non-resonant case.

Claim 2. n_* is the unique resonant site with

$$0 < |n_*| \leq N = \frac{2}{r-r'}|\ln \epsilon|.$$

Proof. Indeed, if there exists $n'_* \neq n_*$ satisfying $|2\rho - \langle n'_*, \alpha \rangle| < \epsilon^\sigma$, then by the Diophantine condition of α , we have

$$\frac{\kappa}{|n'_* - n_*|^\tau} \leq |\langle n'_* - n_*, \alpha \rangle| < 2\epsilon^\sigma,$$

which implies that $|n'_*| > 2^{-\frac{1}{\tau}} \kappa^{\frac{1}{\tau}} \epsilon^{-\frac{\sigma}{\tau}} - N > 2N^2$. □

Since we have

$$|2\rho - \langle n_*, \alpha \rangle| < \epsilon^\sigma, \tag{5.8}$$

the smallness condition on ϵ implies that

$$|\ln \epsilon|^\tau \epsilon^\sigma \leq \frac{\kappa(r - r')^\tau}{2^{\tau+1}}.$$

Thus,

$$\frac{\kappa}{|n_*|^\tau} \leq |\langle n_*, \alpha \rangle| \leq \epsilon^\sigma + 2|\rho| \leq \frac{\kappa}{2|n_*|^\tau} + 2|\rho|,$$

which implies that

$$|\rho| \geq \frac{\kappa}{4|n_*|^\tau}.$$

Then by Lemma 8.1 of Hou–You [22], one can find $P \in SU(1, 1)$ with

$$\|P\| \leq 2 \left(\frac{\|A\|}{|\rho|} \right)^{\frac{1}{2}} \leq 4 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} |n_*|^{\frac{\tau}{2}},$$

such that

$$PAP^{-1} = \begin{pmatrix} e^{i\rho} & 0 \\ 0 & e^{-i\rho} \end{pmatrix} = A'.$$

Denote $g = PfP^{-1}$, by (3.2) we have:

$$\|P\| \leq 4 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} |N|^{\frac{\tau}{2}} \leq 4 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{r - r'} |\ln \epsilon| \right)^{\frac{\tau}{2}}, \tag{5.9}$$

$$|g|_r \leq \|P\|^2 |f|_r \leq \frac{2^{4+\tau} \|A\| |\ln \epsilon|^\tau}{\kappa(r - r')^\tau} \times \epsilon := \epsilon'. \tag{5.10}$$

Now we define

$$\begin{aligned} \Lambda_1(\epsilon^\sigma) &= \{n \in \mathbb{Z}^d : |\langle n, \alpha \rangle| \geq \epsilon^\sigma\}, \\ \Lambda_2(\epsilon^\sigma) &= \{n \in \mathbb{Z}^d : |2\rho - \langle n, \alpha \rangle| \geq \epsilon^\sigma\}. \end{aligned}$$

For $\eta = \epsilon^\sigma$, we define the decomposition $\mathcal{B}_r = \mathcal{B}_r^{nre}(\epsilon^\sigma) \oplus \mathcal{B}_r^{re}(\epsilon^\sigma)$ as in (3.1) with A substituted by A' . Recall that $su(1, 1)$ consists of matrices of the form

$$\begin{pmatrix} it & v \\ \bar{v} & -it \end{pmatrix}$$

with $t \in \mathbb{R}$, $v \in \mathbb{C}$. Direct computation shows that any $Y \in \mathcal{B}_r^{nre}(\epsilon^\sigma)$ takes the precise form:

$$\begin{aligned}
 Y(\theta) &= \sum_{n \in \Lambda_1(\epsilon^\sigma)} \begin{pmatrix} i\hat{t}(n) & 0 \\ 0 & -i\hat{t}(n) \end{pmatrix} e^{i\langle n, \theta \rangle} \\
 &+ \sum_{n \in \Lambda_2(\epsilon^\sigma)} \begin{pmatrix} 0 & \hat{v}(n)e^{i\langle n, \theta \rangle} \\ \hat{v}(n)e^{-i\langle n, \theta \rangle} & 0 \end{pmatrix}.
 \end{aligned}
 \tag{5.11}$$

Since $\epsilon^\sigma \geq 13\|A'\|^2(\epsilon')^{\frac{1}{2}}$, we can apply Lemma 3.1 to remove all the non-resonant terms of g , which means there exist $Y \in \mathcal{B}_r$ and $g^{re} \in \mathcal{B}_r^{re}(\eta)$ such that

$$e^{Y(\theta+\alpha)}(A'e^{g(\theta)})e^{-Y(\theta)} = A'e^{g^{re}(\theta)},$$

with $|Y|_r \leq (\epsilon')^{\frac{1}{2}}$ and $|g^{re}|_r \leq 2\epsilon'$.

Combining with the Diophantine condition on the frequency α and the Claim 2, we have:

$$\begin{aligned}
 &\{\mathbb{Z}^d \setminus \Lambda_1(\epsilon^\sigma)\} \cap \{n \in \mathbb{Z}^d : |n| \leq \kappa^{\frac{1}{\tau}} \epsilon^{-\frac{\sigma}{\tau}}\} = \{0\}, \\
 &\{\mathbb{Z}^d \setminus \Lambda_2(\epsilon^\sigma)\} \cap \{n \in \mathbb{Z}^d : |n| \leq 2^{-\frac{1}{\tau}} \kappa^{\frac{1}{\tau}} \epsilon^{-\frac{\sigma}{\tau}} - N\} = \{n_*\}.
 \end{aligned}$$

Let $N' := 2^{-\frac{1}{\tau}} \kappa^{\frac{1}{\tau}} \epsilon^{-\frac{\sigma}{\tau}} - N$, then we can rewrite $g^{re}(\theta)$ as

$$\begin{aligned}
 g^{re}(\theta) &= g_0^{re} + g_1^{re}(\theta) + g_2^{re}(\theta) \\
 &= \begin{pmatrix} i\hat{t}(0) & 0 \\ 0 & -i\hat{t}(0) \end{pmatrix} + \begin{pmatrix} 0 & \hat{v}(n_*)e^{i\langle n_*, \theta \rangle} \\ \hat{v}(n_*)e^{-i\langle n_*, \theta \rangle} & 0 \end{pmatrix} \\
 &+ \sum_{|n| > N'} \hat{g}^{re}(n)e^{i\langle n, \theta \rangle}.
 \end{aligned}$$

Define the $4\pi\mathbb{Z}^d$ -periodic rotation $Q(\theta)$ as below:

$$Q(\theta) = \begin{pmatrix} e^{-\frac{\langle n_*, \theta \rangle}{2}i} & 0 \\ 0 & e^{\frac{\langle n_*, \theta \rangle}{2}i} \end{pmatrix}.$$

So we have

$$|Q(\theta)|_{r'} \leq e^{\frac{1}{2}Nr'} \leq \epsilon^{\frac{-r'}{r-r'}}. \tag{5.12}$$

One can also show that

$$Q(\theta + \alpha)(A'e^{g^{re}(\theta)})Q^{-1}(\theta) = \tilde{A}e^{\tilde{g}(\theta)},$$

where

$$\tilde{A} = Q(\theta + \alpha)A'Q^{-1}(\theta) = \begin{pmatrix} e^{i(\rho - \frac{\langle n_*, \alpha \rangle}{2})} & 0 \\ 0 & e^{-i(\rho - \frac{\langle n_*, \alpha \rangle}{2})} \end{pmatrix} \tag{5.13}$$

and

$$\tilde{g}(\theta) = Qg^{re}(\theta)Q^{-1} = Qg_0^{re}Q^{-1} + Qg_1^{re}(\theta)Q^{-1} + Qg_2^{re}(\theta)Q^{-1}.$$

Moreover,

$$Qg_0^{re}Q^{-1} = g_0^{re} = \begin{pmatrix} i\hat{t}(0) & 0 \\ 0 & -i\hat{t}(0) \end{pmatrix} \in su(1, 1), \tag{5.14}$$

$$Qg_1^{re}(\theta)Q^{-1} = \begin{pmatrix} 0 & \hat{v}(n_*) \\ \hat{v}(n_*) & 0 \end{pmatrix} \in su(1, 1). \tag{5.15}$$

Now we return back from $su(1, 1)$ to $sl(2, \mathbb{R})$. Denote

$$L = M^{-1}(Qg_0^{re}Q^{-1} + Qg_1^{re}(\theta)Q^{-1})M, \tag{5.16}$$

$$F = M^{-1}Qg_2^{re}(\theta)Q^{-1}M, \tag{5.17}$$

$$B = M^{-1}(Q \circ e^Y \circ P)M, \tag{5.18}$$

$$\tilde{A}' = M^{-1}\tilde{A}M, \tag{5.19}$$

then we have:

$$B(\theta + \alpha)(Ae^{f(\theta)})B^{-1}(\theta) = \tilde{A}'e^{L+F(\theta)}. \tag{5.20}$$

By (5.9) and (5.12), we have the following estimates:

$$\|B\|_0 \leq |e^Y|_r \|P\| \leq 8 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{r - r'} |\ln \epsilon| \right)^{\frac{\tau}{2}}, \tag{5.21}$$

$$|B|_{r'} \leq 8 \left(\frac{\|A\|}{\kappa} \right)^{\frac{1}{2}} \left(\frac{2}{r - r'} |\ln \epsilon| \right)^{\frac{\tau}{2}} \times \epsilon^{\frac{-r'}{r-r'}}, \tag{5.22}$$

$$\|L\| \leq \|Qg_0^{re}Q^{-1}\| + \|Qg_1^{re}(\theta)Q^{-1}\| \leq \epsilon' + \epsilon'e^{-|n_*|r}, \tag{5.23}$$

$$|F|_{r'} \leq |Qg_2^{re}(\theta)Q^{-1}|_{r'} \leq \frac{2^{4+\tau}\|A\|\|\ln \epsilon\|^\tau}{\kappa(r - r')^\tau} \epsilon e^{-N'(r-r')} (N')^d e^{Nr'}. \tag{5.24}$$

By (5.23) and (5.24), direct computation shows that

$$e^{L+F(\theta)} = e^L + \mathcal{O}(F(\theta)) = e^L(Id + e^{-L}\mathcal{O}(F(\theta))) = e^L e^{f+(\theta)}. \tag{5.25}$$

It immediately implies that

$$|f_+(\theta)|_{r'} \leq 2|F(\theta)|_{r'} \leq \frac{2^{5+\tau}\|A\|\|\ln \epsilon\|^\tau}{\kappa(r - r')^\tau} \epsilon e^{-N'(r-r')} (N')^d e^{Nr'} \ll \epsilon^{100}.$$

Thus, we can rewrite (5.20) as

$$B(\theta + \alpha)(Ae^{f(\theta)})B^{-1}(\theta) = A_+ e^{f+(\theta)},$$

with

$$A_+ = \tilde{A}' e^L = e^{A''}, \quad A'' \in sl(2, \mathbb{R}). \tag{5.26}$$

Now recall that Baker–Campbell–Hausdorff formula [28] says that

$$\ln(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \dots, \tag{5.27}$$

where $[X, Y] = XY - YX$ denotes the Lie Bracket and \dots denotes the sum of higher order terms. Using this formula and by a simple calculation, (5.26) gives

$$MA''M^{-1} = \begin{pmatrix} it & v \\ \bar{v} & -it \end{pmatrix}$$

where

$$t = \rho - \frac{\langle n_*, \alpha \rangle}{2} + \hat{t}(0) + \text{higher order terms}$$

and

$$v = \hat{v}(n_*) + \text{higher order terms.}$$

By (5.8) and (5.23), we obtain $|t| \leq \epsilon^\sigma$ and

$$|v| \leq \frac{2^{4+\tau} \|A\| |\ln \epsilon|^\tau}{\kappa(r-r')^\tau} \epsilon e^{-|n_*|r}.$$

Finally, the following estimate is straightforward:

$$\|A''\| \leq 2(|\rho - \frac{\langle n_*, \alpha \rangle}{2}| + \|Qg_0^{re}Q^{-1}\| + \|Qg_1^{re}(\theta)Q^{-1}\|) \leq 2\epsilon^\sigma. \quad (5.28)$$

This finishes the proof of Theorem 3.1. \square

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