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Density of Small Singular Values of the Shifted Real Ginibre Ensemble

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Abstract. We derive a precise asymptotic formula for the density of the small singular values of the real Ginibre matrix ensemble shifted by a complex parameter z as the dimension tends to infinity. For z away from the real axis the formula coincides with that for the complex Ginibre ensemble we derived earlier in Cipolloni et al. (Prob Math Phys 1:101–146, 2020). On the level of the one-point function of the low lying singular values we thus confirm the transition from real to complex Ginibre ensembles as the shift parameter z becomes genuinely complex; the analogous phenomenon has been well known for eigenvalues. We use the superbosonization formula (Littelmann et al. in Comm Math Phys 283:343–395, 2008) in a regime where the main contribution comes from a three dimensional saddle manifold.

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1. Introduction

The universality paradigm in random matrix theory asserts that the local eigenvalue statistics of large random matrices depend only on the basic symmetry class of the ensemble. In the Hermitian case, this dependence is usually investigated for the k-point functions starting from $k \geq 2$, while the one-point function is largely insensitive to the symmetry class apart from finite-size correction terms (see, for example, [26] for GOE/GUE). For non-Hermitian matrices; however, the real axis plays an interesting distinguishing role between the real and complex ensembles already on the level of the one-point function. In the simplest Gaussian case this phenomenon has been well known for the

eigenvalues; in this paper we investigate it for singular values where no explicit formulas are available.

We consider the *real* or *complex Ginibre ensemble* [20], i.e. large $N \times N$ random matrices X with independent identically distributed (i.i.d) real or complex Gaussian entries x_{ab}. The customary normalization, $\mathbf{E}\, x_{ab} = 0$, $\mathbf{E}\, |x_{ab}|^2 = N^{-1}$, guarantees that the density of eigenvalues of X converges to the uniform measure on the complex unit disk $\{z \,|\, |z| \leq 1\}$, known as the *circular law*, and that the spectral radius of X converges to 1 with very high probability (these results also hold for general non-Gaussian matrix elements, see, for example, [4–7,19,21,28]).

While the distribution of the complex Ginibre eigenvalues is clearly rotationally invariant, the real axis plays a special role for the real Ginibre ensemble, in particular there are typically $\sim \sqrt{N}$ real eigenvalues [16] (see also the exact formula for having precisely k real eigenvalues in [22]). In fact, all correlation functions of the Ginibre eigenvalues are explicitly known see [20] and [25] for the simpler complex case, and [8,15,17,23] for the more involved real case. The precise formulas reveal a remarkable phenomenon [8, Theorem 11]: the local *eigenvalue* statistics for real Ginibre matrices coincide with those for complex Ginibre matrices anywhere in the spectrum away from the real axis (see also [2]).

To what extent does this phenomenon hold for low lying *singular values* of X and their shifted version $X - z$ with a complex parameter z? While singular values may behave very differently than eigenvalues, intuitively the very small singular values of $X - z$ are still related to the eigenvalues of X near z, since z is an eigenvalue of X if and only if $X - z$ has a zero singular value. Hence, we expect that these small singular values of $X - z$ for z away from the real axis behave in the same way for real and complex Ginibre matrices. This was recently proven in [11, Theorem 2.8] for all k-point correlation functions and even for any i.i.d. (i.e. not necessarily Gaussian) distributed matrix elements but only in the regime $|\Im z| \sim 1$. In this paper we prove that this phenomenon holds down to very close to the imaginary axis, $|\Im z| \gg N^{-1/2}$, on the level of the density (or one-point function) of the singular values using supersymmetric (SUSY) techniques.

Singular values of $X - z$ coincide with the positive eigenvalues of

$$H^z := \begin{pmatrix} 0 & X - z \\ X^* - \bar{z} & 0 \end{pmatrix}. \tag{1}$$

Block matrix of this form with the same shift parameter z (interpreted as i-times the chemical potential) in both off-diagonal blocks is called the chiral random matrix ensemble [1] and is used to model massless Dirac operators in Stephanov's theory [27]. For real z the two models coincide; thus, the chiral ensemble with very small $\Im z$ can be considered as a non-Hermitian deformation of (1); thus, the density of eigenvalues of H^z is the starting point of a perturbative analysis. We remark that, independently of physics connections,
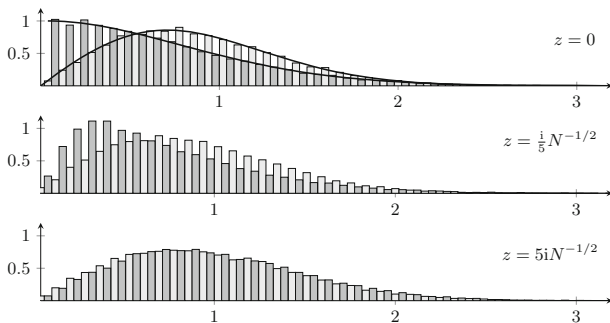
FIGURE 1. Histogram of rescaled smallest singular value $N\sigma_{\min}(X-z)$ of $X-z$ in the real (dark grey) and complex (light grey case). For $z=0$ the asymptotic densities $2xe^{-x^2}$ and $(1+x)e^{-x^2/2-x}$ have been computed by Edelman [14]

in our related paper [12] we also explore the power of our approach in numerical analysis by establishing new bounds on the eigenvector condition number and on the eigenvector overlaps [9, 10, 18].

More precisely, we find that in the large N limit the density of the low lying singular values of $X-z$ for a real Ginibre matrix coincides with that of the complex Ginibre matrix X as long as $|\Im z| \gg N^{-1/2}$, while it is different for $|\Im z| \sim N^{-1/2}$, c.f. Fig. 1.

This indicates a transition in the local singular value statistics of $X-z$ from real to complex as $|\Im z|$ increases beyond $N^{-1/2}$, similarly[1] to the local eigenvalue statistics of X.

Technically, we express the averaged trace of the resolvent of $(X-z)(X-z)^*$ in terms of contour integrals using the superbosonization formula [24] and perform the large N limit. This analysis has been carried out for the complex case in [13], now we handle the considerably more involved real case. The main additional complication stems from the structure of the superbosonization formula: the contour integration in the real case involves three integration variables, two of them are highly convolved and their contours cannot be deformed independently; while the complex case has only two variables and the phase function is decoupled in them. The entire analysis is done at the bottom of the spectrum of $(X-z)(X-z)^*$, at a distance comparable with the (square of the) local spacing of the singular values; hence, our result directly gives precise information on individual singular values. In this critical regime the answer does not come simply from a saddle point, but from a genuine three-fold integral even after the $N \to \infty$ limit is taken. With a careful choice of the interdependent deformations of the contours we achieve the negative sign in the real part of the phase function; hence, we can rigorously estimate the physically irrelevant highly oscillatory integration regimes. Note that the mere

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[1]We remark that [8, Theorem 11] did not explicitly state that the transition takes place for $|\Im z| \gg N^{-1/2}$, but it can be concluded from its proof.

existence of such deformation is not guaranteed by any physical principle, let alone finding them explicitly—this is what we achieve here. A further feature of our work is that we can handle the bulk, $|z| < 1$, as well as the edge regime, $|z| \approx 1$, where the scaling changes from N^{-1} to $N^{-3/4}$.

Notations and Conventions

For positive quantities f, g we write $f \lesssim g$ and $f \sim g$ if $f \leq Cg$ or $cg \leq f \leq Cg$, respectively, for some constants $c, C > 0$ which are independent of the basic parameters of the problem $N, \lambda, \widetilde{\eta}, \widetilde{\delta}$ in (2). For any two positive, possibly N-dependent, quantities f, g we write $f \ll g$ to denote that $f \lesssim N^{-\epsilon} g$, for some small $\epsilon > 0$ (however, this convention will be locally altered within the proof of Lemma 3.1). We abbreviate the minimum and maximum of real numbers by $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

2. Main Results

We consider the ensemble $Y^z := (X - z)(X - z)^*$ with $X \in \mathbf{R}^{N \times N}$ being a *real Ginibre* matrix, i.e. its entries x_{ab} are such that $\sqrt{N} x_{ab}$ are i.i.d. real standard Gaussian random variables, and $z \in \mathbf{C}$ is a fixed complex parameter such that $|z| \leq 1$. We compute the large N asymptotics for the spectral one-point function $\mathbf{E} \operatorname{Tr}(Y^z - w)^{-1}$, with $w = E + \mathrm{i}0$. The energy E is chosen to be comparable with the local eigenvalue spacing of Y^z, i.e. we study the small eigenvalues of Y^z. The imaginary part of $\mathbf{E} \operatorname{Tr}(Y^z - w)^{-1}$ is the density of states at the energy E. In particular, we focus on the transitional regime $|\Im z| \sim N^{-1/2}$ proving that $\mathbf{E} \operatorname{Tr}(Y^z - w)^{-1}$ exhibits a one-parameter family of behaviours depending on $N^{1/2} |\Im z|$. Additionally, we prove that $\mathbf{E} \operatorname{Tr}(Y^z - w)^{-1}$ behaves as in the case of complex Ginibre matrix X for $|\Im z| \gg N^{-1/2}$.

In order to study the transitional regime $|\Im z| \sim N^{-1/2}$, we introduce the rescaled variables

$$\lambda := N^{3/2}(1 \vee \widetilde{\delta})E, \qquad \widetilde{\eta} := N^{1/2}\Im z, \qquad \widetilde{\delta} := N^{1/2}\delta, \qquad (2)$$

with $\delta := 1 - |z|^2$. By [3, Sect. 5] it is easy to see that the level spacing of the eigenvalues of Y^z close to zero is of order

$$c(N, \widetilde{\delta}) := N^{-3/2} \cdot (1 \wedge \widetilde{\delta}^{-1}),$$

i.e. for $|z| < 1$ is given by $N^{-2}\delta^{-1}$ and for $|z| = 1$ by $N^{-3/2}$, which explains the scaling of λ. The unusual $N^{-3/2}$ scaling in the edge regime $|z| = 1$ originates from the fact that the density of eigenvalues of the Hermitized matrix H^z in (1) features a cubic cusp singularity that has a natural eigenvalue spacing $N^{-3/4}$.

We now state the main technical result on the large N asymptotics of the one-point function. The main conclusion of the paper will be given as its Corollary 2.2 afterwards. Note that the formulas (5) are considerably simplified when $\widetilde{\delta} = 0$, i.e. $|z| = 1$, in particular, the spectral scaling factor becomes $c(N, \widetilde{\delta} = 0) = N^{-3/2}$.

Theorem 2.1. *Let $C_0, C_1 > 0$ sufficiently large constants. For any $C_0^{-1} \leq \lambda \leq C_0$, for $\widetilde{\eta} = 0$ or $C_0^{-1} \leq |\widetilde{\eta}| \leq C_0$, and for $\widetilde{\delta} = 0$ or $\widetilde{\delta} \geq C_1$ it holds*

$$\mathbf{E} \frac{1}{N} \mathrm{Tr}(Y^z - \lambda c(N, \widetilde{\delta}) - \mathrm{i}0)^{-1} = N^{1/2} I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta}) + \mathcal{O}\left(1 + \widetilde{\delta}\right), \qquad (3)$$

where

$$I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta}) := \frac{1}{4\pi \mathrm{i}} \oint_\Gamma \mathrm{d}\xi \int_\Omega \mathrm{d}\tau \int_\Lambda \mathrm{d}a \, \frac{\xi^2 a}{\tau^{1/2}} e^{\mathfrak{f}(\xi, \lambda, \widetilde{\delta}) - \mathfrak{g}(a, \tau, \widetilde{\eta}, \lambda, \widetilde{\delta})} G(a, \tau, \xi, \widetilde{\eta}, \widetilde{\delta}), \quad (4)$$

with Γ any contour around 0 in a counter-clockwise direction, Λ any contour going out from 0 in the direction of $e^{\mathrm{i}\pi/6}$ for a while and then going to infinity in the direction $e^{3\pi \mathrm{i}/5}$, and Ω any contour in the fourth quadrant going out from zero in the direction $e^{-\mathrm{i}\pi/3}$ and ending in one with an angle $e^{\mathrm{i}\pi/3}$, see Fig. 2. Here

$$\mathfrak{f}(\xi, \lambda, \widetilde{\delta}) := -(1 \wedge \widetilde{\delta}^{-1})\lambda\xi + \frac{1}{2\xi^2} + \frac{\widetilde{\delta}}{\xi},$$

$$\mathfrak{g}(a, \tau, \widetilde{\eta}, \lambda, \widetilde{\delta}) := -(1 \wedge \widetilde{\delta}^{-1})\lambda a + \frac{2\widetilde{\eta}^2(1 - \tau)}{\tau} + \frac{2 - \tau}{2a^2\tau^2} + \frac{\widetilde{\delta}}{a\tau},$$

$$\begin{aligned}
G(a, \tau, \xi, \widetilde{\eta}, \widetilde{\delta}) := {}& \frac{1}{a^2\tau\xi^6} + \frac{2}{a^3\tau^2\xi^5} + \frac{4 - \tau}{a^4\tau^3\xi^4} \\
& + \frac{2}{a^5\tau^3\xi^3} + \frac{1}{a^6\tau^3\xi^2} + \frac{1}{a^2\tau\xi^4} + \frac{2}{a^3\tau^2\xi^3} \\
& + \frac{1}{a^4\tau^2\xi^2} + \frac{2\widetilde{\delta}}{a^2\tau\xi^5} + \frac{4\widetilde{\delta}}{a^3\tau^2\xi^4} + \frac{4\widetilde{\delta}}{a^4\tau^3\xi^3} \\
& + \frac{2\widetilde{\delta}}{a^5\tau^3\xi^2} + \frac{4\widetilde{\eta}^2}{a^2\tau^2\xi^4} + \frac{4\widetilde{\eta}^2}{a^3\tau^2\xi^3} \\
& + \frac{4\widetilde{\eta}^2}{a^4\tau^3\xi^2} + \frac{4\widetilde{\eta}^2\widetilde{\delta}}{a^2\tau^2\xi^3} + \frac{4\widetilde{\eta}^2\widetilde{\delta}}{a^3\tau^2\xi^2}.
\end{aligned} \qquad (5)$$

The implicit constant in $\mathcal{O}(\cdot)$ depends on C_0. Moreover, the integral $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$ is absolutely convergent and is bounded by $C(1 + \widetilde{\delta})$ with a constant that depends only on C_0 and C_1.

In Corollary 2.2 below we study the behaviour of $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$ in the large $|\widetilde{\eta}|$ regime and we show that, in the large $|\widetilde{\eta}|$ limit, $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$ agrees with the limiting one-point function $I^{(\mathbf{C})}(\lambda, \widetilde{\delta})$ of the complex Ginibre ensemble. We recall from [13, Eq. (13a)] that the limit analogous to (3) for the complex case is given by

$$I^{(\mathbf{C})}(\lambda, \widetilde{\delta}) := \frac{1}{2\pi \mathrm{i}} \int \mathrm{d}x \oint \mathrm{d}y \, e^{h(y) - h(x)} H(x, y) \qquad (6)$$

with

$$H(x, y) := \frac{1}{x^3} + \frac{1}{x^2 y} + \frac{1}{x y^2} + \frac{\widetilde{\delta}}{xy} + \frac{\widetilde{\delta}}{x^2}, \quad h(x) := -(1 \wedge \widetilde{\delta}^{-1})\lambda x + \frac{\widetilde{\delta}}{x} + \frac{1}{2x^2}. \qquad (7)$$
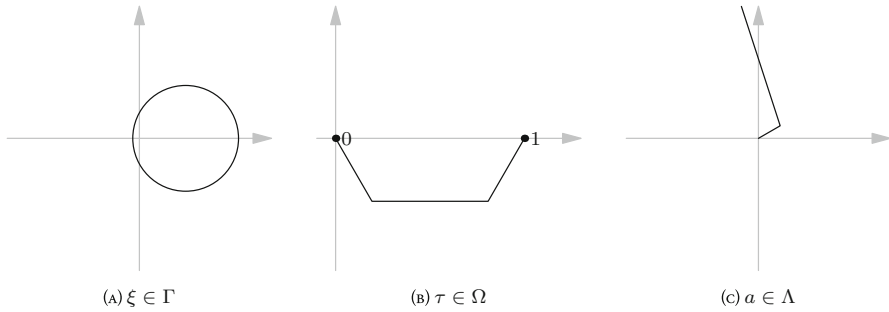
(A) $\xi \in \Gamma$ (B) $\tau \in \Omega$ (C) $a \in \Lambda$

FIGURE 2. Depiction of the chosen contours

The x-integration is over any contour from 0 to $e^{3\mathrm{i}\pi/4}\infty$, going out from 0 in the direction of the positive real axis, and the y-integration is over any contour around 0 in a counter-clockwise direction. It is easy to see that along such contours the integral is absolutely convergent. Note that the rhs. of (6) exactly agrees with [13, Eqs. (13a)–(b)] after the change of variables $\widetilde{z}_* x \to x$ and $\widetilde{z}_* y \to y$, using the notation therein.

Corollary 2.2. *Let $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$ be defined as in (4), then it holds*

$$\lim_{|\widetilde{\eta}| \to +\infty} I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta}) = I^{(\mathbf{C})}(\lambda, \widetilde{\delta}), \tag{8}$$

for any fixed $\lambda \in \mathbf{R}_+$ and $\widetilde{\delta} = 0$ or $\widetilde{\delta} \geq C_1$.

Remark 2.3. From our analysis in Sect. 3.5 (see (37) later) it actually follows that $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$ converges to $I^{(\mathbf{C})}(\lambda, \widetilde{\delta})$ with a rate $|\widetilde{\eta}|^{-1}$. Similarly to Theorem 2.1, the convergence in (8) is uniform in the entire range $\lambda \in [C_0^{-1}, C_0]$ and $\widetilde{\delta} \in \{0\} \cup [C_1, \infty)$ of the other two parameters.

Remark 2.4. The limiting statement (8) follows by taking the $\widetilde{\eta}$ limit within the formula (4), i.e. after the $N \to \infty$ limit is taken. However, we believe that in the regime $|\widetilde{\eta}| \geq C$, using a computation similar to the ones in Sect. 3.5 and to the bound [13, Lemmas 6.2–6.4], but this time on the contours Λ, Ω, one may prove the following stronger result:

$$\begin{aligned}
\mathbf{E} \, \frac{1}{N} \, \mathrm{Tr}(Y^z - \lambda \cdot c(N, \widetilde{\delta}) - \mathrm{i}0)^{-1} &= N^{1/2} I^{(\mathbf{C})}(\lambda, \widetilde{\delta}) \\
&+ \mathcal{O}\left(\left[(1 \vee \widetilde{\delta}) + \frac{1}{|\widetilde{\eta}|}\right] \left(1 + |\log \lambda|\right)\right).
\end{aligned} \tag{9}$$

3. Derivation of the 1-Point Function

Supersymmetric methods, especially the superbosonization formula (see, for example, [24]), provide an explicit formula for $\mathbf{E} \, \mathrm{Tr}[Y^z - w]^{-1}$. This was derived

in [13, Eqs. (34)–(37)], and with the choice $w = E + \mathrm{i}\epsilon$ with $0 < \epsilon \ll E \ll 1$, we have that

$$\mathbf{E}\,\mathrm{Tr}[Y^z - w]^{-1} = \frac{N}{4\pi\mathrm{i}} \oint \mathrm{d}\xi \int_0^{\mathrm{i}\infty} \mathrm{d}a \int_0^1 \mathrm{d}\tau \frac{\xi^2 a}{\tau^{1/2}} e^{N[f(\xi,w)-g(a,\tau,\eta,w)]} G_N(a,\tau,\xi,\eta), \tag{10}$$

where the ξ-integration is over any counter-clockwise oriented contour around 0 that does not encircle -1, the a,τ-contours are straight lines, and, using the notation $\eta = \Im z$, the functions f and g are defined by

$$f(\xi, w) := -w\xi + \log(1 + \xi) - \log \xi - \frac{|z|^2}{1 + \xi}, \tag{11}$$

$$g(a, \tau, \eta, w) := -wa + \frac{1}{2} \log[1 + 2a + a^2\tau] - \log a - \frac{1}{2} \log \tau$$
$$- \frac{|z|^2(1 + a) - 2\eta^2 a^2(1 - \tau)}{1 + 2a + a^2\tau}. \tag{12}$$

The fact that the integral in (10) is absolutely convergent follows by the explicit expressions of f and g in (11), (12). Note that ϵ in $w = E + \mathrm{i}\epsilon$ is introduced only to make the a-integration on the imaginary axis absolutely convergent; hence, after the contours deformations described in Sect. 3.1 below, for all the practical purposes we can assume that $\epsilon = 0$ and so $w = E$. Indeed, after deforming the a-contour so that it ends in the second quadrant, i.e. in the region $\{a \in \mathbf{C} : \Re[a] < 0, \Im[a] > 0\}$, we can take the limit $\epsilon \to 0^+$ since the integral in (10) is absolutely convergent for $\epsilon = 0$. Note that $g(a, 1, \eta, w) = f(a, w)$; in particular, we remark that $g(a, 1, \eta, w)$ is independent of η for any $a \in \mathbf{C}$. Furthermore, the function

$$G_N(a, \tau, \xi, \eta) := G_{1,N}(a, \tau, \xi) + G_{2,N}(a, \tau, \xi, \eta) \tag{13}$$

is given by

$$G_{1,N} = \left(N^2 \frac{p_{2,0,0}}{a^2\xi^2(\xi+1)^2\tau} - N \frac{p_{1,0,0}}{a^2\xi^2(\xi+1)\tau} + \delta N^2 \frac{p_{2,0,1}}{a\xi(\xi+1)^2\tau} \right.$$
$$\left. - N\delta \frac{p_{1,0,1}}{a\xi(\xi+1)\tau} + N^2\delta^2 \frac{p_{2,0,2}}{(\xi+1)^2} \right) \times \left((a^2\tau + 2a + 1)^2(\xi+1)^2 \right)^{-1},$$
$$G_{2,N} = \left(N^2\eta^2 \frac{p_{2,2,0}}{a\xi(\xi+1)^3\tau} - N\eta^2 \frac{p_{1,2,0}}{a\xi\tau} + N^2\eta^2\delta \frac{p_{2,2,1}}{(\xi+1)} \right)$$
$$\times \left((a^2\tau + 2a + 1)^2(\xi+1)^2 \right)^{-1}, \tag{14}$$

where $p_{i,j,k} = p_{i,j,k}(a, \tau, \xi)$ are explicit polynomials in a, τ, ξ which we defer to Appendix B.2 and $\delta := 1 - |z|^2$. The indices i, j, k in the definition of $p_{i,j,k}$ denote the N, η and δ power, respectively. We split G_N as the sum of $G_{1,N}$ and $G_{2,N}$ since $G_{1,N}$ depends only on $|z|$, whilst $G_{2,N}$ depends explicitly on $\eta = \Im z$, in particular $G_{2,N} = 0$ if $z \in \mathbf{R}$.

3.1. Choice of the Integration Contours

From now on we only focus on the regime $\widetilde{\delta} = 0$, i.e. $|z| = 1$. The proof in the case $\widetilde{\delta} \geq C_1$ for some large $C_1 > 0$ only requires slightly different choice of

contours but otherwise the analysis of the integrals on them is analogous and so we omit the details.

3.1.1. Geometry of $\{\Re g > 0\}$ in the Regime $|\tau| \gtrsim 1$ (see Fig. 3).
In the regime $|\tau| \gtrsim 1$ there is a transition at $|1-\tau|\eta^2 = E^{2/3}$. In the regime $|1-\tau|\eta^2 \lesssim E^{2/3}$ there is only one relevant length scale of $E^{-1/3}$. On the contrary in the regime $|1 - \tau|\eta^2 \gg E^{2/3}$ there are two relevant length scales $E^{-1/3} \ll |1-\tau|\eta^2 E^{-1}$, the former describing the size of the two connected components of $\{\Re g > 0\}$ close to 0 and the latter describing the distance to the infinite connected component of $\{\Re g > 0\}$ in the direction $+\infty$. In Fig. 3 we present the level sets of $\Re g$ for various sizes of $|1 - \tau|$ and η.

3.1.2. Geometry of $\{\Re g > 0\}$ in the Regime $|\tau| \ll 1$ for $\tilde{\eta} = 0$ (see Fig. 4).
In the regime $|\tau| \ll 1$ for $\tilde{\eta} = 0$ there is a transitions around $|\tau| = E$. For $|\tau| \ll E$ there are two components of $\{\Re g < 0\}$, one unbounded at a distance of E^{-1} to the right of the origin, and a bounded one at a distance of $|\tau|^{-1}$ below the origin. As $|\tau|$ approaches E the two components merge but remain separated from the origin at a distance of $|\tau|^{-2/3}E^{-1/3}$, see Fig. 4 for an illustration.

3.1.3. Geometry of $\{\Re g > 0\}$ in the Regime $|\tau| \ll 1$ for $\tilde{\eta} > 0$ (see Fig. 5).
For $|\tau| \ll 1$ and $\tilde{\eta} > 0$ there are two components of $\{\Re g < 0\}$, one unbounded one at a distance of $|\tau|^{-1}E^{-1/3}$ to the right of the origin, and a bounded one at the bottom left of the origin. The bounded component is an approximate disk of diameter $|\tau|^{-1}$ for $|\tau| \ll E^{2/3}$ and is transformed into a "lying eight" of diameter $|\tau|^{-1/2}E^{-1/3}$ as $|\tau| \gg E^{2/3}$, see Fig. 5 for an illustration.

3.1.4. Deformation of Contours.
Now we explain how the contours in (10) can be deformed. The ξ-contour can be freely deformed as long as it does not cross 0 and -1. We can deform the τ-contour as long as $\Im[\tau] < 0$, then the a-contour has to be deformed accordingly to ensure the absolute convergence of the integral. The a-contour at infinity can be freely deformed, independently of τ, as long as it ends in the second quadrant; on the other hand the way how it goes out from zero depends on τ. Moreover, along the deformation of the a-contour we cannot cross the points $(-1 \pm \sqrt{1-\tau})\tau^{-1}$ which are the singularities of the term $a^2\tau + 2a + 1$ in g and G_N. In particular, note that the τ and a contours cannot be deformed independently: we first deform the τ-contour and then we deform the a-contour accordingly. In the remainder of this section we will always deform the integration contours as described above.

Next, we describe how we concretely deform the integration contours in (10) in accordance with the rules just described. From now on we denote the ξ-contour by Γ, the τ-contour by Ω, and the a-contour by Λ. In particular, we choose
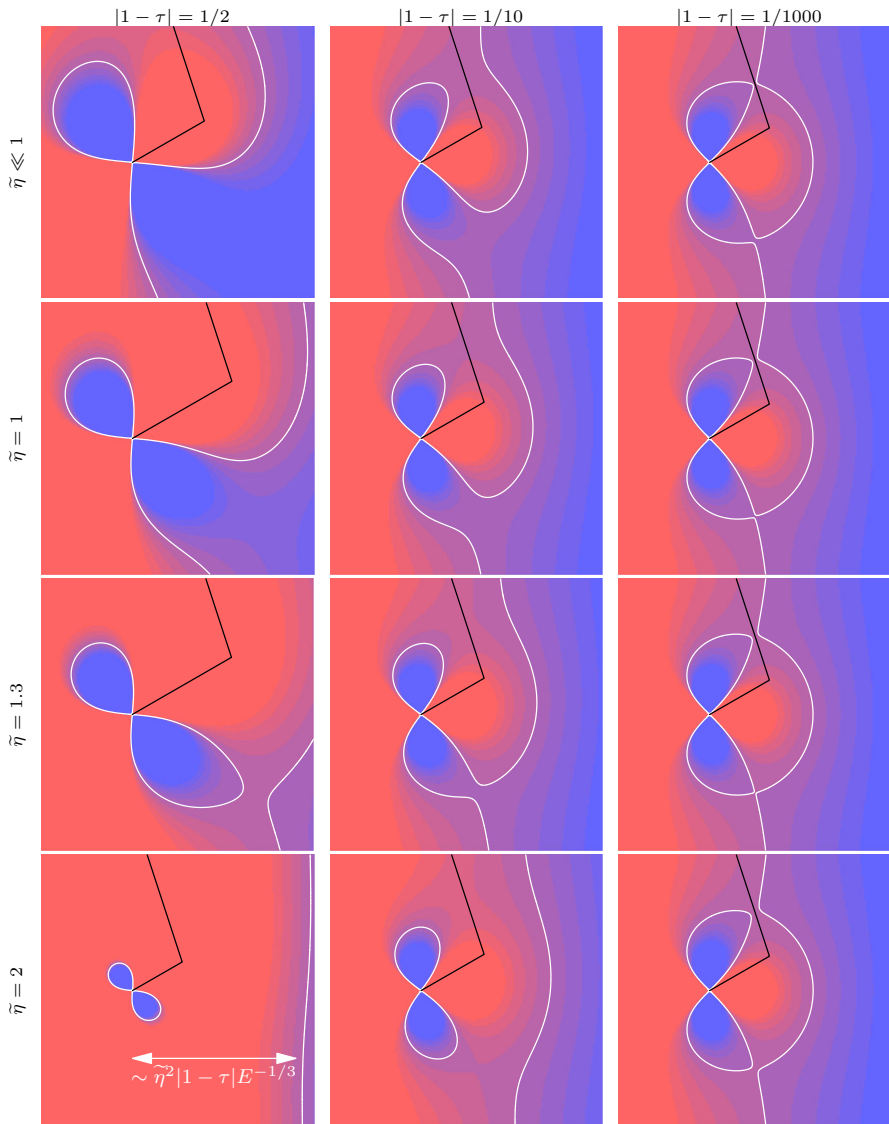
FIGURE 3. Contour plot of $\Re g(\cdot, \tau, \widetilde{\eta} E^{1/3}, E)$ for $E > 0$ for $\tau \in \Omega$ with $|1-\tau| \leq 1/2$. The white lines represent the level set $\Re g(\cdot, \tau, \eta, E) = 0$, while the black line represents the contour $r\Lambda$ for the a-integration. All figures are on the same scale $E^{-1/3}$, except for the bottom left figure which shows the larger scale $E^{-1/3} \widetilde{\eta}^2 |1 - \tau|$, in addition to the $E^{-1/3}$ length scale of the blue figure eight. The solid red colours are applied to regions where $\Re g > E^{2/3}$, while the solid blue colours are applied to regions where $\Re g < -E^{2/3}$
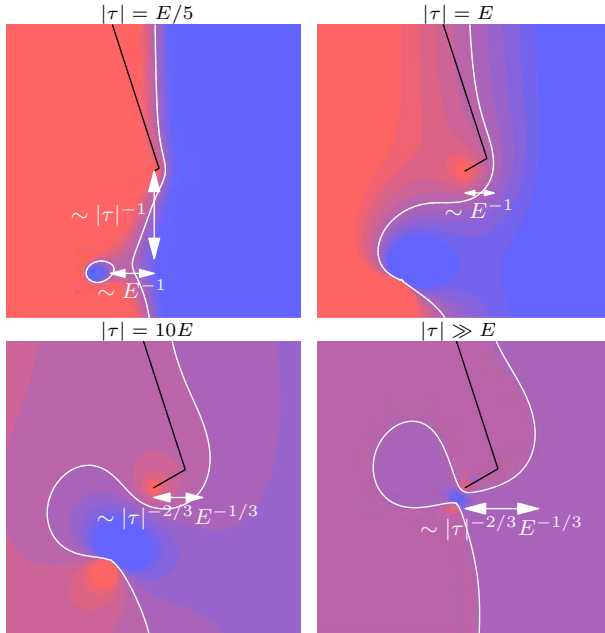
FIGURE 4. Contour plot of $\Re g(\cdot, \tau, 0, E)$ for $E > 0$ for $\tau \in \Omega$ with $|\tau| \ll 1$. The solid white lines represent the level set $\Re g(\cdot, \tau, 0, E) = 0$, while the solid black line represents the contour $r\Lambda$ for the a-integration. The solid red colours are applied to regions where $\Re g > 1$, while the solid blue colours are applied to regions where $\Re g < -1$

$$\Gamma := -\frac{1}{2} + E^{-1/3} + E^{-1/3}\partial \mathbf{D},$$

$$\Omega := \left[0, \frac{1}{3\sqrt{3}} - \frac{i}{3}\right] \cup \left(\frac{1}{3\sqrt{3}} - \frac{i}{3}, 1 - \frac{1}{3\sqrt{3}} - \frac{i}{3}\right] \cup \left(1 - \frac{1}{3\sqrt{3}} - \frac{i}{3}, 1\right], \quad (15)$$

$$\Lambda := \left[0, z_0\right] \cup \left(z_0, z_0 + e^{3i\pi/5}\infty\right), \qquad z_0 := \frac{\sin(4\pi/15)}{\sin(17\pi/30)} e^{i\pi/6},$$

and rescale $r\Lambda$ with a parameter $r > 0$ chosen later depending on $\widetilde{\eta}, \tau, E$. Here the interval $(0, e^{3i\pi/5}\infty)$ is understood as the open half-line going out to ∞ in the $e^{3i\pi/5}$ direction. Note that the contour Λ is designed such that $e^{i\pi/3} \in \Lambda$. According to the geometry of $\{\Re g < 0\}$ we choose the scaling parameter

$$r = r(\widetilde{\eta}, E, \tau) := \frac{1}{E^{1/3}}\left(\frac{1}{2} + \frac{1}{2(E \vee |\tau|)^{2/3}} + \frac{\widetilde{\eta}^2}{|\tau|}\right). \quad (16)$$

We note there is a lot of freedom in the choice of contours. In particular, it would not be necessary to choose the contour Γ mono-parametrically with a scaling factor depending only on E, and similarly the contour $r\Lambda$ with a single scaling factor depending on $E, \tau, \widetilde{\eta}$. For example, in certain parameter regimes it would be possible to have a Λ-contour going out from the origin directly
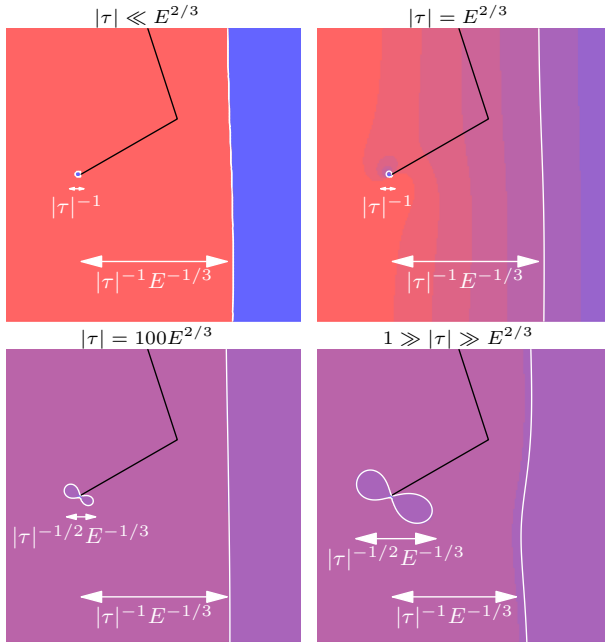
FIGURE 5. Contour plot of $\Re g(\cdot, \tau, \tilde{\eta}E^{1/3}, E)$ for $0 \leq E \ll 1$ and $\tilde{\eta} > 0$ for $\tau \in \Omega$ with $|\tau| \ll 1$. The solid white lines represent the level set $\Re g(\cdot, \tau, \tilde{\eta}E^{1/3}, E) = 0$, while the solid black line represents the contour $r\Lambda$ for the a-integration. The solid red colours are applied to regions where $\Re g > 1$, while the solid blue colours are applied to regions where $\Re g < -1$

in a north-western direction without the first segment in the north-eastern direction, c.f. Fig. 4. We nevertheless chose our contours in a mono-parametric way as this makes it easier to check the fact that $\Re g > 0$ on the entire contours by differentiation. We also note that there is some room in the chosen angles and lengths with the only hard constraint being imposed by the saddle in the right column of Fig. 3. The latter is ensured by the requirement that $e^{i\pi/3} \in \Lambda$ which explains the seemingly complicated choice of z_0 in (15).

We can thus rewrite (10) as

$$\mathbf{E}\,\mathrm{Tr}[Y^z - w]^{-1} = \frac{N}{4\pi i} \oint_\Gamma d\xi \int_\Omega d\tau \int_{r\Lambda} da \frac{\xi^2 a}{\tau^{1/2}} e^{N[f(\xi, w) - g(a, \tau, \eta, w)]} G_N(a, \tau, \xi, \eta).$$

(17)

In the following we split the computation of the leading term of (17) into two parts: (i) in Sect. 3.2 we deal with the regime when either $|\xi| \leq N^\omega$ or $|a\tau| \leq N^{2\omega}$ or $|\tau| \leq N^{-\omega}$, for some small fixed $\omega > 0$, (ii) in Sect. 3.3 we deal with the complementary regime when $|\xi|$ and $|a\tau|$ are bigger than N^ω and $|\tau| > N^{-\omega}$.

3.2. Small $|\xi|$ or small $|a\tau|$ or small $|\tau|$ regime

By the explicit form of the phase functions $f(\xi, E)$ and $g(a, \tau, \eta, E)$ in (11), (12) it follows that the contribution to (10) of the small $|\xi|$ and $|a\tau|$ regimes is negligible; in particular, the smallness comes from the logarithmic factors in the phase functions. This is made rigorous in Lemma 3.1. For this purpose we define the contours

$$\widetilde{\Gamma} := \{\xi \in \Gamma : |\xi| \leq N^\omega\},$$
$$\widetilde{\Lambda} := \{a \in r\Lambda : |a| \leq N^{2\omega}|\tau|^{-1}\},$$
$$\widetilde{\Omega} := \{\tau \in \Omega : |\tau| \leq N^{-\omega}\}, \tag{18}$$

for some small fixed $\omega > 0$. Note that $r\Lambda \setminus \widetilde{\Lambda}$ is always connected.

Lemma 3.1. *Let f, g, G_N be defined in (11)–(14), and let $\widetilde{\Gamma}, \widetilde{\Omega}, \widetilde{\Lambda}$ be the contours defined in (18), then for any large constant $C_4 > 0$, for any $E = \lambda N^{-3/2}$, with $C_4^{-1} \leq \lambda \leq C_4$, and for any $\widetilde{\eta} = 0$ or $C_4^{-1} \leq |\widetilde{\eta}| \leq C_4$, we have that*

$$\left| \left(\oint_\Gamma d\xi \int_\Omega d\tau \int_{r\Lambda} da - \oint_{\Gamma \setminus \widetilde{\Gamma}} d\xi \int_{\Omega \setminus \widetilde{\Omega}} d\tau \int_{r\Lambda \setminus \widetilde{\Lambda}} da \right) \right.$$
$$\left. \left[e^{N[f(\xi, E) - g(a, \tau, \eta, E)]} \frac{a\xi^2}{\tau^{1/2}} G_N(a, \tau, \xi, z) \right] \right| \leq C e^{-N^{\omega/10}}. \tag{19}$$

The constant $C > 0$ only depends on C_4.

Proof. The proof relies on two quantitative lower bounds on $\Re g$ outlined in the following lemmata, the proofs of which we defer to Appendix A. Within these Lemmas and their proofs we deviate from our general convention and the notation $f \ll g$ means that $f \leq cg$ for a sufficiently small N-independent constant c.

Lemma 3.2. *For $|\tau| \leq N^{-\epsilon}$ we have the following lower bound on $\Re g$ which for clarity we formulate separately depending on the relative sizes of τ, E, a and whether $\widetilde{\eta} = 0$ or $\neq 0$.*

1. $|\tau| \lesssim E$ and $\tilde{\eta} = 0$, hence $r \sim E^{-1}$.
   a) $|a| \leq |\tau|^{-1} : \Re g \gtrsim 1 - \log|a\tau|$
   b) $|\tau|^{-1} < |a| \lesssim E^{-1} : \Re g \gtrsim 1$
   c) $|a| \gg E^{-1} : \Re g \gtrsim E|a|$.
2. $|\tau| \lesssim E$ and $\tilde{\eta} \neq 0$, hence $r \sim E^{-1/3}|\tau|^{-1}$.
   a) $|a| \lesssim E^{-1} \wedge |\tau|^{-1} : \Re g \gtrsim 1 - \log|a\tau|$
   b) $E^{-1} \wedge |\tau|^{-1} \ll |a| \leq |\tau|^{-1} : \Re g \gtrsim E^{2/3}\widetilde{\eta}^2|a| - \log|a\tau|$
   c) $|\tau|^{-1} < |a| \lesssim |\tau|^{-1}E^{-1/3} : \Re g \gtrsim E^{2/3}\widetilde{\eta}^2|\tau|^{-1}$
   d) $|a| \gg E^{-1/3}|\tau|^{-1} : \Re g \gtrsim E|a|$.
3. $E \ll |\tau| \leq N^{-\epsilon}$ and $\widetilde{\eta} = 0$, hence $r \sim E^{-1/3}|\tau|^{-2/3}$.
   a) $|a| \ll |\tau|^{-1} : \Re g \gtrsim -\log|a\tau|$
   b) $|\tau|^{-1} \lesssim |a| \lesssim E^{-1/3}|\tau|^{-2/3} : \Re g \gtrsim E^{2/3}|\tau|^{-2/3}$
   c) $|a| \gg E^{-1/3}|\tau|^{-2/3} : \Re g \gtrsim E|a|$.

4. $E \ll |\tau| \le N^{-\epsilon}$ and $\widetilde{\eta} \ne 0$, hence $r \sim E^{-1/3}|\tau|^{-1}$.
    a) $|a| \ll |\tau|^{-1}$:$\Re g \gtrsim -\log|a\tau|$
    b) $|\tau|^{-1} \lesssim |a| \lesssim E^{-1/3}|\tau|^{-1}$:$\Re g \gtrsim E^{2/3}\widetilde{\eta}^2|\tau|^{-1}$
    c) $|a| \gg E^{-1/3}|\tau|^{-1}$:$\Re g \gtrsim E|a|$.

Lemma 3.3. *For any $1 \ge |\tau| \gg E$ with $\tau \in \Omega$ the function*

$$x \mapsto \Re g(xe^{i\pi/6}, \tau, 0, E) \tag{20}$$

is monotonically decreasing in x for $0 \le x \ll E^{-1/3}$. Moreover, for any $\eta \ge 0$, and any $1 \ge |\tau| \gg E$, $0 \le x \ll E^{-1/3}$ we have

$$\Re g(xe^{i\pi/6}, \tau, \eta, E) \ge \Re g(xe^{i\pi/6}, \tau, 0, E). \tag{21}$$

We now split the proof of (19) into three parts, we first prove that the contribution to (17) in the regime $\tau \in \widetilde{\Omega}$ is exponentially small uniformly in $\xi \in \Gamma$ and $a \in r\Lambda$. Then we prove that the regime $a \in \widetilde{\Lambda}$ is also exponentially small for any $\xi \in \Gamma$ and $\tau \in \Omega \setminus \widetilde{\Omega}$. Finally, we conclude that also the contribution for $\xi \in \widetilde{\Gamma}$ is negligible.

We start with the regime $\tau \in \widetilde{\Omega}$. Similarly to [13, Eq. (97)], using that $|1 + 2a + a^2\tau| \gtrsim 1$, we have that

$$\left| \oint_\Gamma G_N(a, \tau, \xi, z)\xi^2 e^{Nf(\xi)} \,\mathrm{d}\xi \right| \lesssim N^3 \left(1 + \frac{1}{|a|} + \frac{1}{|a|^2|\tau|} \right). \tag{22}$$

Then, given, the lower bounds for $\Re g$ in 1.a–4.c by simple computations we conclude the following lemma.

Lemma 3.4. *For any $\alpha, \gamma \in \mathbf{R}$ it holds*

$$\int_{\widetilde{\Omega}} |\,\mathrm{d}\tau| \int_{r\Lambda} |\,\mathrm{d}a| \, |a|^\alpha |\tau|^{-\gamma} e^{-\Re g(a,\tau,\eta,E)} \le N^{C(\alpha,\gamma)} e^{-N^{\omega/10}}, \tag{23}$$

for some N-independent constant $C(\alpha,\gamma) > 0$.

Using the bound in (23) we readily conclude that the contribution of the regime $\tau \in \widetilde{\Omega}$ is exponentially small and so negligible.

We now consider the regime $a \in \widetilde{\Lambda}$. We split this regime into two cases: (i) $|a| \ge N^{-10}$, (ii) $|a| \le N^{-10}$. For $|a| \ge N^{-10}$, by (22) and Lemma 3.3, we readily conclude that

$$\int_{\Omega \setminus \widetilde{\Omega}} |\,\mathrm{d}\tau| \int_{\widetilde{\Lambda}} |\,\mathrm{d}a| \, |a|^\alpha |\tau|^{-\gamma} e^{-\Re g(a,\tau,\eta,E)} \le N^{C(\alpha,\gamma)} e^{-N^{1-2\omega}}. \tag{24}$$

In the regime $|a| \le N^{-10}$ we conclude a bound as in (24) using the explicit form of g in (12) and that $|\tau| \le 1$. This proves that also the regime $a \in \widetilde{\Lambda}$ is negligible.

Finally, the fact that the regime $\xi \in \widetilde{\Gamma}$ is exponentially small, given that both the regimes $\tau \in \widetilde{\Omega}$ and $a \in \widetilde{\Lambda}$ are removed, follows exactly as in the proof [13, Lemma 6.4]. $\qquad\square$

3.3. The regime where $|\xi|$, $|a\tau|$ and $|\tau|$ are all large

In the remainder of this section we focus on the regime when $|\xi| \geq N^\omega$, $|a| \geq N^{2\omega}|\tau|^{-1}$ and $|\tau| \leq N^{-\omega}$, and in this regime we expand $f(\cdot, E)$, $g(\cdot, \tau, \eta, E)$, G_N similarly to Eq. (75)-(77) of [13]. By Taylor expansion for large $|\xi|$ and large $|a\tau|$ we have

$$f(\xi, E) = \left[-E\xi + \frac{1}{2\xi^2}\right] \times \left(1 + \mathcal{O}\left(|\xi|^{-1}\right)\right)$$

$$g(a, \tau, \eta, E) = \left[-Ea - \frac{2\eta^2(\tau-1)}{\tau} + \frac{\delta}{a\tau} + \frac{(2-\tau)}{2a^2\tau^2}\right.$$
$$\left. + \frac{2\eta^2(\tau-5)}{a^2\tau^2}\right] \times \left(1 + \mathcal{O}\left(|a\tau|^{-1}\right)\right), \tag{25}$$

and[2]

$$G_{1,N}(a, \tau, \xi, |z|) = \left[\sum_{\substack{a,\beta \geq 2,\, \alpha+\beta=8, \\ \gamma=\min\{\alpha-1,3\}}} \frac{c_{1,\alpha,\beta,\gamma}N^2}{a^\alpha \tau^\gamma \xi^\beta}\right.$$

$$- \frac{N}{a^4\tau^2\xi^4} + \sum_{\substack{\alpha,\beta \geq 2,\, \alpha+\beta=7, \\ \gamma=\min\{\alpha-1,3\}}} \frac{c_{2,\alpha,\beta,\gamma}N^2\delta}{a^\alpha \tau^\gamma \xi^\beta}$$

$$\left. + \sum_{\substack{a,\beta \geq 2,\, \alpha+\beta=6, \\ \gamma=\min\{\alpha-1,2\}}} \frac{c_{3,\alpha,\beta,\gamma}N}{a^\alpha \tau^\gamma \xi^\beta} + \sum_{\substack{\alpha,\beta \geq 2, \alpha+\beta=6 \\ \gamma=\min\{\alpha-1,2\}}} \frac{c_{4,\alpha,\beta,\gamma}N^2\delta^2}{a^\alpha \tau^\gamma \xi^\beta}\right]$$

$$\times \left[1 + \mathcal{O}\left(|a\tau|^{-1} + |\xi|^{-1}\right)\right], \tag{26}$$

$$G_{2,N}(a, \tau, \xi, z) = \left[\sum_{\substack{\alpha,\beta \geq 2,\, \alpha+\beta=6, \\ \gamma=\max\{\alpha-1,2\}}} \frac{4N^2\eta^2}{a^\alpha \tau^\gamma \xi^\beta} + \sum_{\substack{\alpha,\beta \geq 2, \alpha+\beta=5, \\ \gamma=\max\{\alpha-1,2\}}} \frac{4N^2\eta^2\delta}{a^\alpha \tau^\gamma \xi^\beta}\right.$$

$$\left. + \sum_{\substack{\alpha,\beta=2,3,\, \alpha+\beta=5 \\ \gamma=\max\{\alpha-1,2\}}} \frac{4N\eta^2}{a^\alpha \tau^\gamma \xi^\beta}\right] \times \left[1 + \mathcal{O}\left(|a\tau|^{-1} + |\xi|^{-1}\right)\right], \tag{27}$$

where $c_{i,\alpha,\beta,\gamma} \in \mathbf{R}$ are defined as in Appendix B.1.

3.4. Proof of Theorem 2.1

We recall that we only prove the case $\widetilde{\delta} = 0$; the case $\widetilde{\delta} \geq C_1$ is completely analogous and so omitted. By Lemma 3.1 and (17) we conclude that

$$\mathbf{E}\,\mathrm{Tr}\,[Y^z - w]^{-1} = \frac{N}{4\pi i}\oint_{\Gamma\setminus\widetilde{\Gamma}} \text{\dh}\xi \int_{\Omega\setminus\widetilde{\Omega}} \mathrm{d}\tau$$

$$\int_{r\Lambda\setminus\widetilde{\Lambda}} \mathrm{d}a \frac{\xi^2 a}{\tau^{1/2}} e^{N[f(\xi,w)-g(a,\tau,\eta,w)]} G_N(a, \tau, \xi, \eta) \tag{28}$$

---

[2]Note that the term $-\frac{N^2}{a^4\tau^2\xi^4}$ was erroneously missing in the expansion in [13, Eq. (76)].

up to an exponentially small error that we will always ignore in the sequel. In order to compute the leading order of (28) as N goes to infinity, we use the change of variables

$$\widetilde{\eta} = N^{1/2}\eta, \quad \lambda = N^{3/2}E, \quad a' = aN^{-1/2}, \quad \xi' = \xi N^{-1/2}, \tag{29}$$

where a', ξ' are the new integration variables. We get that (omitting the primes, i.e. using the notation a, ξ for the new variables as well to make the notation simpler)

$$
\begin{aligned}
&\mathbf{E} \, \mathrm{Tr}\, [Y^z - w]^{-1} \\
&= \frac{N^{3/2}}{4\pi\mathrm{i}} \int_{N^{-1/2}(\Gamma\backslash\widetilde{\Gamma})} \mathrm{d}\xi \int_{\Omega\backslash\widetilde{\Omega}} \mathrm{d}\tau \\
&\int_{N^{-1/2}(r\Lambda\backslash\widetilde{\Lambda})} \mathrm{d}a \frac{\xi^2 a}{\tau^{1/2}} e^{N[\mathfrak{f}(\xi,w) - \mathfrak{g}(a,\tau,\eta,w)]} G(a,\tau,\xi,\widetilde{\eta}) \\
&+ \mathcal{O}(N).
\end{aligned}
\tag{30}
$$

Here we used the asymptotic relations

$$
\begin{aligned}
f(\sqrt{N}\xi, \lambda) &= N^{-1}\mathfrak{f}(\xi,\lambda)\left(1 + \mathcal{O}\left(\frac{1}{N^{1/2}|\xi|}\right)\right), \\
g(\sqrt{N}a, \tau, \sqrt{N}\widetilde{\eta}, \lambda) &= N^{-1}\mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)\left(1 + \mathcal{O}\left(\frac{1}{N^{1/2}|a\tau|}\right)\right),
\end{aligned}
\tag{31}
$$

with $\mathfrak{f}$, $\mathfrak{g}$, and G defined in (5). The pre-factor $N^{3/2}$ in the leading term of (3) follows by a simple power counting: $a \sim N^{1/2}$, $\xi \sim N^{1/2}$, $\eta \sim N^{-1/2}$, the volume factor from the Jacobian of the change of variables (29) gives a factor of N. In order to bound the error term in (31) we also used the following lemma.

Lemma 3.5. *Let $\mathfrak{f}$ and $\mathfrak{g}$ be the functions defined in* (5). *Then for any fixed $\alpha, \beta, \gamma \in \mathbf{R}$ it holds*

$$\int_{N^{-1/2}(\Gamma\backslash\widetilde{\Gamma})} |\mathrm{d}\xi| \int_{\Omega\backslash\widetilde{\Omega}} |\mathrm{d}\tau| \int_{N^{-1/2}(r\Lambda\backslash\widetilde{\Lambda})} |\mathrm{d}a| \left| \frac{1}{a^\alpha \tau^\gamma \xi^\beta} e^{\mathfrak{f}(\xi,\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)} \right| \le C, \tag{32}$$

for some constant $C < \infty$ which depends only on α, β, γ and on the control parameters C_0, C_1 from Theorem 2.1.

Proof. The bound in (32) directly follows from the explicit form of $\mathfrak{f}$ and $\mathfrak{g}$ in (5) and by the fact that on the chosen contours Γ, Ω, $r\Lambda$ it holds $\Re\mathfrak{g} > 0$, $\Re\mathfrak{f} < 0$. $\qquad\square$

Using Lemma 3.1 once more, we can add back the regimes $\xi \in N^{-1/2}\widetilde{\Gamma}$, $\tau \in \widetilde{\Omega}$, $a \in N^{-1/2}\widetilde{\Lambda}$ to (31). Hence, using that we can deform the integration contours by holomorphicity, we conclude (3), (4). The absolute convergence of $I^{(\mathbf{R})}(\lambda, \widetilde{\eta})$ follows from Lemma 3.5.

3.5. The limit $|\widetilde{\eta}| \to +\infty$.

The main goal of this section is to study the asymptotic of $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$, defined in (4), in the limit $|\widetilde{\eta}| \to +\infty$; in particular we prove that $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta})$ converges to the 1-point function of the shifted complex Ginibre ensemble $I^{(\mathbf{C})}(\lambda, \widetilde{\delta})$, which is defined in (6). To make the presentation clearer, also in this case we present the proof only for the case $\widetilde{\delta} = 0$ and denote $I^{(\mathbf{R})}(\lambda, \widetilde{\eta}) := I^{(\mathbf{R})}(\lambda, \widetilde{\eta}, \widetilde{\delta} = 0)$.

We recall that by Theorem 2.1 we have

$$I^{(\mathbf{R})}(\lambda, \widetilde{\eta}) = \frac{1}{4\pi \mathrm{i}} \oint_{\Gamma} \mathrm{d}\xi \int_{\Omega} \mathrm{d}\tau \int_{\Lambda} \mathrm{d}a \frac{\xi^2 a}{\tau^{1/2}} e^{\mathfrak{f}(\xi,\lambda) - \mathfrak{f}(a,\lambda)} e^{\mathfrak{g}(a,1,\widetilde{\eta},\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)} G(a, \tau, \xi, \widetilde{\eta}),$$
$$(33)$$

with Γ, Ω, Λ from Theorem 2.1, where we used that $\mathfrak{g}(a, 1, \widetilde{\eta}, \lambda) = \mathfrak{f}(a, \lambda)$ for any $a \in \mathbf{C}$.

Proof of Corollary 2.2. In this proof we use the notation

$$\widetilde{\Omega} := \{\tau \in \Omega : |\tau| \le C|\widetilde{\eta}|^{-1/2}\}, \quad \widetilde{\Lambda} := \{a \in \Lambda : |a| \le |\widetilde{\eta}|^{-1/2}\},$$

for some large constant $C > 0$ (note that $\widetilde{\Omega}$, $\widetilde{\Lambda}$ have already been used in (18) to denote different segments). Then, similarly to the proof of Lemma 3.1, it is easy to see that the integral in the regime when either $\tau \in \widetilde{\Omega}$ or $a \in \widetilde{\Lambda}$ is bounded by $e^{-c|\widetilde{\eta}|^{1/4}}$, for some small fixed $c > 0$. In particular, by (33) we get that

$$I^{(\mathbf{R})}(\lambda, \widetilde{\eta}) = \frac{1}{4\pi \mathrm{i}} \oint_{\Gamma} \mathrm{d}\xi \int_{\Lambda \setminus \widetilde{\Lambda}} \mathrm{d}a \int_{\Omega \setminus \widetilde{\Omega}} \mathrm{d}\tau \frac{\xi^2 a}{\tau^{1/2}} e^{\mathfrak{f}(\xi,\lambda) - \mathfrak{f}(a,\lambda)} e^{\mathfrak{g}(a,1,\widetilde{\eta},\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)} G(a, \tau, \xi, \widetilde{\eta})$$
$$+ \mathcal{O}\left(e^{-c|\widetilde{\eta}|^{1/4}}\right).$$
$$(34)$$

Note that by the definition of G in (5) the ξ-integral and the (a, τ)-integral factorize; hence, from now on we will consider only the (a, τ)-integral.

Then, to prove (8), in the following lemma, whose proof is postponed to the end of this section, we compute the leading order term of the τ-integral in (34).

Lemma 3.6. *For any large constant $C_0 > 0$, and for any fix $\gamma \in \mathbf{R}$, $C_0^{-1} \le \lambda \le C_0$ it holds*

$$\int_{\Omega \setminus \widetilde{\Omega}} \tau^{-\gamma} e^{\mathfrak{g}(a,1,\widetilde{\eta},\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)} \, \mathrm{d}\tau = \frac{1}{2\widetilde{\eta}^2} + \mathcal{O}\left(|\widetilde{\eta}|^{-3}\right), \tag{35}$$

uniformly in $a \in \Lambda \setminus \widetilde{\Lambda}$. The implicit constant in $\mathcal{O}(\cdot)$ depends on C_0.

Next, using (34) and Lemma 3.6, we conclude the proof of Corollary 2.2. First of all we notice that the leading term in (35) does not depend on γ, hence after performing the τ-integration the power of τ that appears in $G(a, \tau, \xi, \widetilde{\eta})$ does not matter. For this reason after the τ-integration we consider $G(a, 1, \xi, \widetilde{\eta})$, i.e. for convenience we evaluate G at $\tau = 1$. More precisely,

by Lemma 3.6 it follows that

$$
\int_{\Omega \setminus \widetilde{\Omega}} \frac{1}{\tau^{1/2}} e^{-\mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)} G(a,\tau,\xi,\widetilde{\eta}) \, \mathrm{d}\tau
$$
$$
= e^{-\mathfrak{g}(a,1,\widetilde{\eta},\lambda)} G(a,1,\xi,\widetilde{\eta}) \left(\frac{1}{2\widetilde{\eta}^2} + \mathcal{O}\left(\frac{1}{|\widetilde{\eta}|^3} \right) \right). \tag{36}
$$

Then, by (34) together with (36), it follows that

$$
I^{(\mathbf{R})}(\lambda, \widetilde{\eta}) = \frac{1}{8\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\xi \int_{\Lambda \setminus \widetilde{\Lambda}} \mathrm{d}a \frac{\xi^2 a}{\widetilde{\eta}^2} e^{\mathfrak{f}(\xi,\lambda) - \mathfrak{f}(a,\lambda)} G(a,1,\xi,\widetilde{\eta}) + \mathcal{O}\left(|\widetilde{\eta}|^{-1} \right)
$$
$$
= \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \mathrm{d}\xi \int_\Lambda \mathrm{d}a \left(\frac{1}{a\xi^2} + \frac{1}{a^2\xi} + \frac{1}{a^3} \right) e^{\mathfrak{f}(\xi,\lambda) - \mathfrak{f}(a,\lambda)} + \mathcal{O}\left(|\widetilde{\eta}|^{-1} \right), \tag{37}
$$

where in the second equality we used the explicit form of G from (5), and that we can add back the regime $a \in \widetilde{\Lambda}$ at the price of a negligible error. This concludes the proof of (8). $\qquad \square$

We now present the proof of Lemma 3.6.

Proof of Lemma 3.6. From now on we assume that $|\widetilde{\eta}| \geq C$, for some large constant $C > 0$, since we are interested in the asymptotics for $|\widetilde{\eta}| \to +\infty$. Additionally, since

$$
\mathfrak{g}(a,1,\widetilde{\eta},\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda) = -\frac{2\widetilde{\eta}^2(1-\tau)}{\tau} + \frac{\tau^2 + \tau - 2}{2a^2\tau^2} \tag{38}
$$

depends only on $\widetilde{\eta}^2$, without loss of generality we assume that $\widetilde{\eta} > 0$.

Next we split the τ-integral in (35) into two parts: $|\tau| \in [0, 1 - \widetilde{\eta}^{-3/2})$ and $|\tau| \in [1 - \widetilde{\eta}^{-3/2}, 1]$, which we denote by Ω_1 and Ω_2, respectively. It is easy to see that

$$
\left| \int_{\Omega_1 \setminus \widetilde{\Omega}} \tau^{-\gamma} e^{\mathfrak{g}(a,1,\widetilde{\eta},\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)} \, \mathrm{d}\tau \right| \lesssim \int_{\Omega_1 \setminus \widetilde{\Omega}} \frac{e^{-c\widetilde{\eta}^{1/2}\Re[\tau^{-1}]}}{|\tau|^\gamma} \, d\tau \lesssim e^{-c\widetilde{\eta}^{1/2}}, \tag{39}
$$

for some small fixed $c > 0$, where we used that by (38) we have

$$
\Re[\mathfrak{g}(a,1,\widetilde{\eta},\lambda) - \mathfrak{g}(a,\tau,\widetilde{\eta},\lambda)] = -\Re\left[\frac{2\widetilde{\eta}^2(1-\tau)}{\tau} \left(1 + \frac{2+\tau}{4a^2\tau\widetilde{\eta}^2} \right) \right] \leq -c\Re\left[\frac{\widetilde{\eta}^{1/2}}{\tau} \right],
$$

for any $a \in \Lambda \setminus \widetilde{\Lambda}$ and $\tau \in \Omega_1 \setminus \widetilde{\Omega}$. Hence, in order to conclude the proof, we are left only with the regime $\tau \in \Omega_2$ and $a \in \Lambda \setminus \widetilde{\Lambda}$.

Define $t(\tau) := -e^{4\mathrm{i}\pi/3} 2\widetilde{\eta}^2(1-\tau)$, hence $\tau = \tau(t) = 1 + te^{-4\mathrm{i}\pi/3}/(2\widetilde{\eta}^2)$, then we have that

$$
g(a,1,\widetilde{\eta},\lambda) - g(a,\tau(t),\widetilde{\eta},\lambda) = te^{-4\mathrm{i}\pi/3} + \mathcal{O}\left(\frac{t}{\widetilde{\eta}^2} + \frac{t}{|a|^2\widetilde{\eta}^2} \right),
$$

and so that

$$\int_{\Omega_2} e^{g(a,1,\widetilde{\eta},\lambda)-g(a,\tau,\widetilde{\eta},\lambda)} \, \mathrm{d}\tau = \frac{e^{-4\mathrm{i}\pi/3}}{2\widetilde{\eta}^2} \int_0^{t(\tau_0)} e^{te^{-4\mathrm{i}\pi/3}} \left[1 + \mathcal{O}\left(\frac{t}{\widetilde{\eta}^2} + \frac{t}{|a|^2\widetilde{\eta}^2}\right)\right] \mathrm{d}t \tag{40}$$

$$= \frac{1}{2\widetilde{\eta}^2} + \mathcal{O}\left(\frac{1}{|\widetilde{\eta}|^3}\right),$$

where $\tau_0 := \{|\tau| = 1 - \widetilde{\eta}^{-3/2}\} \cap \Omega$, and in the last equality we used that $|a|^{-2} \leq \widetilde{\eta}$. Combining (39), (40) we conclude (35). $\qquad\square$

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Appendix A. Additional Technical Results

Proof of Lemma 3.2. The proofs of all lower bounds are similar, hence we will not provide details for all of them. For definiteness we will prove cases 1.a 1.b and 1.c since those already demonstrate the qualitatively different $|a\tau| \ll 1$, $|a\tau| \sim 1$ and $|a\tau| \gg 1$ regimes.

Proof of 1.a-1.b. First consider the $|a| \ll |\tau|^{-1}$ case (note that this relation is necessarily fulfilled if $|\tau| \ll E$ since then $|a| \lesssim E^{-1} \ll |\tau|^{-1}$). We have to prove $\Re g \gtrsim -\log|a\tau|$ which follows from

$$\Re g \approx -E\Re a + \Re\left[\frac{1}{2}\log\frac{1+2a}{a^2\tau} - \frac{1+a}{1+2a}\right]$$

$$\gtrsim -E\Re a + \frac{1}{2}\log\left|\frac{1+2a}{a}\right| + \frac{1}{2}\log\frac{1}{|a\tau|} - 1 \gtrsim -\log|a\tau|.$$

Thus, we are only left with the $|a|^{-1} \sim |\tau| \sim E$ case in which we introduce the parametrization $\tau = te^{-\mathrm{i}\pi/3}E$ with a real $t \sim 1$, so that

$$r = \frac{1 \wedge t^{-2/3}}{2E}\left(1 + \mathcal{O}(E^{2/3})\right).$$

For $|a| \leq r$ we have $\arg(a) = \pi/6$ and parametrize $a\tau = se^{-i\pi/6}$ with

$$s \in \left[0, \frac{t \wedge t^{1/3}}{2} \frac{\sin(4\pi/15)}{\sin(17\pi/30)}\right]. \tag{41}$$

Using that $Ea = st^{-1}e^{i\pi/6}$ and that $|a| \gg 1$, the claim is thus equivalent to showing

$$\Re g \approx -\frac{s}{t}\Re e^{i\pi/6} + \Re\left[\frac{1}{2}\log\frac{2 + se^{-i\pi/6}}{se^{-i\pi/6}} - \frac{1}{2 + se^{-i\pi/6}}\right]$$

$$= \frac{1}{52}\left(-28 + 6\sqrt{3} - \frac{26\sqrt{3}s}{t} + 13\log\left(\frac{4}{s^2} + \frac{2\sqrt{3}}{s} + 1\right)\right) \gtrsim 1 + (\log s^{-1})_+$$

where the last inequality is valid for s as in (41) and any $t \leq 100$.

We turn to the case $|a| > r$, i.e. to the second segment of the contour $r\Lambda$ where we parametrize

$$a = E^{-1}\left(\frac{1 \wedge t^{-2/3}}{2}\frac{\sin(4\pi/15)}{\sin(17\pi/30)}e^{i\pi/6} + se^{3i\pi/5}\right) =: E^{-1}\tilde{a}.$$

with $s \in [0, \infty)$. We express $\Re g$ in terms of $\tilde{a} = \tilde{a}(s)$ and differentiate it as a function of s we see that that function

$$s \mapsto \Re\left[-\tilde{a} - \frac{1}{2 + \tilde{a}te^{-i\pi/3}} + \frac{1}{2}\log\left(1 + \frac{2e^{i\pi/3}}{\tilde{a}t}\right)\right]$$

has a local minimum of size $\sim t^{-2/3}$ at $s \sim t^{-2/3}$ and thus for $t \lesssim 1$ we obtain $\Re g \gtrsim 1 + \log(s^{-1})_+$ also in this final case, completing the proof. $\qquad\square$

Proof of 1.c. For $|a| \gg E^{-1} \sim r$ it follows that $-E\Re a \sim E|a|$ by the choice of contour Λ. Therefore it is sufficient to prove

$$\Re g = -E\Re a + \Re\left[\frac{1}{2}\log\frac{1 + 2a + a^2\tau}{a^2\tau} - \frac{1 + a}{1 + 2a + a^2\tau}\right] \gtrsim E|a|. \tag{42}$$

If $|a| \gtrsim |\tau|^{-1}$ then we estimate

$$\left|\Re\left[\frac{1}{2}\log\frac{1 + 2a + a^2\tau}{a^2\tau} - \frac{1 + a}{1 + 2a + a^2\tau}\right]\right| \approx \left|\Re\left[\frac{1}{2}\log\frac{2 + a\tau}{a\tau} - \frac{1}{2 + a\tau}\right]\right|$$

$$\lesssim \left|\Re\left[\frac{1}{a^2\tau^2}\right]\right| \sim \frac{1}{|a\tau|^2} \ll E|a|,$$

where in the first step we used that $|a| \gg 1$ and in the last step we used $|a| \gg |\tau|^{-1} \gtrsim |\tau|^{-2/3}E^{-1/3}$, confirming (42). On the other hand, if $|a| \ll |\tau|^{-1}$ (but still $|a| \gg 1$) then by Taylor expansion in $|a\tau| \gg 1$ we obtain

$$\Re\left[\frac{1}{2}\log\frac{1 + 2a + a^2\tau}{a^2\tau} - \frac{1 + a}{1 + 2a + a^2\tau}\right] \gtrsim \Re\frac{1}{2}\log\frac{1}{a\tau} > 0,$$

trivially confirming (42). $\qquad\square$

The remaining cases 2.a–4.c may be estimated by similar elementary considerations. $\qquad\square$

Proof of Lemma 3.3. The first assertion follows from elementary calculations resulting in

$$\frac{\mathrm{d}\Re g(xe^{\mathrm{i}\pi/6}, \tau, 0, E)}{\mathrm{d}x} \lesssim -\Re\frac{1}{xe^{\mathrm{i}\pi/6}} < 0. \tag{43}$$

and the second assertion from

$$\Re\left[\frac{x^2 e^{\mathrm{i}\pi/3}(1-\tau)}{1 + 2xe^{\mathrm{i}\pi/6} + x^2 e^{\mathrm{i}\pi/3}\tau}\right] \geq 0 \tag{44}$$

using the definition of the τ-contour Ω. $\qquad\square$

B Lists of coefficients

B.1. Explicit coefficients for the real 1-point function integral representation

Here we collect the explicit coefficients in (26):

$$c_{1,2,6,1} = c_{1,6,2,3} = 1, \quad c_{1,3,5,2} = c_{1,5,3,3} = 2, \quad c_{1,4,4,3} = 4,$$
$$c_{2,2,5,1} = c_{2,5,2,3} = 2, \quad c_{2,3,4,2} = c_{2,4,3,3} = 4,$$
$$c_{3,2,4,1} = c_{3,4,2,2} = 1, \quad c_{3,3,3,2} = 2,$$
$$c_{4,2,4,1} = c_{4,4,2,2} = 1, \quad c_{4,3,3,2} = 2.$$

B.2. Explicit formulas for the real symmetric integral representation

Here we collect the explicit formulas for the polynomials of a, ξ, τ in the definition of G_N in (14).

$$p_{2,0,0} := a^4\tau^2 + 2a^3\xi\tau + 4a^3\tau - a^2\xi^2\tau + 4a^2\xi^2 + 8a^2\xi + 2a^2\tau$$
$$+ 4a^2 + 2a\xi^3 + 8a\xi^2 + 10a\xi + 4a + \xi^4 + 4\xi^3 + 6\xi^2 + 4\xi + 1,$$
$$p_{1,0,0} := -a^4\xi\tau^2 + a^4\tau^2 - 2a^3\xi^2\tau - 2a^3\xi\tau + 4a^3\tau - a^2\xi^3\tau - 3a^2\xi^2\tau$$
$$- 2a^2\xi\tau + 4a^2\xi + 2a^2\tau + 4a^2 + 2a\xi^2 + 6a\xi + 4a + \xi^3 + 3\xi^2 + 3\xi + 1,$$
$$p_{2,2,0} := 4(a+1)\left(a^2\tau + a\xi\tau + 2a\tau + \xi^2 + 2\xi + 1\right),$$
$$p_{1,2,0} := 4(a+1)\left(a^2\tau + a\xi\tau + 2a\tau + \xi + 1\right),$$
$$p_{2,0,1} := 2\big(a^3\tau^2 + 2a^2\xi\tau + 4a^2\tau + 2a\xi^2 + 2a\xi\tau$$
$$+ 4a\xi + 3a\tau + 2a + \xi^3 + 4\xi^2 + 5\xi + 2\big)$$
$$p_{1,0,1} := 2\big(a^3\tau^2 + 2a^2\xi\tau + 4a^2\tau + a\xi^2\tau + 3a\xi\tau$$
$$+ 2a\xi + 3a\tau + 2a + \xi^2 + 3\xi + 2\big),$$
$$p_{2,2,1} := 4(a+1)(a+\xi+2),$$
$$p_{2,0,2} := a^2\tau + 2a\xi + 4a + \xi^2 + 4\xi + 4.$$

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