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Spectral Properties of Relativistic Quantum Waveguides

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Abstract. We make a spectral analysis of the massive Dirac operator in a tubular neighbourhood of an unbounded planar curve, subject to infinite mass boundary conditions. Under general assumptions on the curvature, we locate the essential spectrum and derive an effective Hamiltonian on the base curve which approximates the original operator in the thin-strip limit. We also investigate the existence of bound states in the non-relativistic limit and give a geometric quantitative condition for the bound states to exist.

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1. Introduction

1.1. Motivations and State of the Art

Consider a massive particle in a guide modelled by a uniform tubular neighbourhood of an infinite planar curve. A classical particle, moving according to Newton's laws of motion with regular reflections on the boundary, will eventually leave any bounded set in a finite time, except for initial conditions of measure zero in the phase space corresponding to transverse oscillations. It came as a surprise in 1989 that the situation changes drastically for quantum particles modelled by the Schrödinger equation. In the pioneering paper [15] and further improvements [13, 16, 19], it was demonstrated that the quantum Hamiltonian identified with the Dirichlet Laplacian possesses discrete eigenvalues unless the base curve is a straight line. Roughly, and with a sharp contrast with the classical setting, the particle gets trapped in any non-trivially curved quantum waveguide. The existence and properties of the geometrically induced bound states have attracted a lot of attention in the last decades and the research field is still very active. We refer to the monograph [14] and the latest developments in [22] with further references.

The goal of the present paper is to consider relativistic counterparts of the quantum waveguides. Here, we model the relativistic quantum Hamiltonian by the Dirac operator in the same tubular neighbourhood as above, subject to infinite mass boundary conditions. The latter is probably the reason why the relativistic setting has escaped the attention of the community until now. Indeed, the self-adjointness of the Dirac operators on domains and the right replacement for the Dirichlet boundary conditions have been understood only recently [1–3,24].

There are four motivations for the present study. First, we would like to understand the influence of relativistic effects on spectral properties. Do the geometrically induced bound states exist independently of the mass of the particle? It is expected that they do exist for heavy particles because the Dirac operator converges, in a suitable sense involving an energy renormalization, to the Dirichlet Laplacian in the limit of large masses. For light particles, however, the answer is far from being obvious because it is well known that relativistic systems are less stable [25]. In this paper, we confirm the expectation by justifying the non-relativistic limit and provide partial (both qualitative and quantitative) answers for the whole ranges of masses.

Our second motivation is related to quantisation on submanifolds. It is well-known (see [21] for an overview with many references) that the non-relativistic quantum Hamiltonian converges to a one-dimensional Schrödinger operator on the base curve. (The convergence involving an energy renormalization can be understood either in a resolvent sense [12,20,21] or as an adiabatic limit [17,23,33].) It is remarkable that this non-relativistic effective operator is not the free quantum Hamiltonian on the submanifold but it contains an extrinsic geometric potential depending on the curvature of the base curve. In this paper, we find that the relativistic setting is very different, for the limiting operator describing the effective dynamics on the submanifold is just the free Dirac operator of the base curve.

Recently, the Dirac operator on metric graphs has been considered as a model for the transport of relativistic quasi-particles in branched structures [34], and the existence and transport of Dirac solitons in networks have been studied in [30]. Previous studies deal with the quantisation of graphs and spectral statistics for the Dirac operator [5], and self-adjoint extensions and scattering properties for different graph topologies [9]. Rigorous mathematical studies on linear and nonlinear Dirac equations on metric graphs recently appeared [6–8]. The result of the present paper can be understood as the first step towards a rigorous justification of the metric graph model as the limit of shrinking branched waveguides.

The last but not least motivation of this paper is that the present model is relevant for transport of quasi-particles in graphene nanostructures [27]. This makes our results not only interesting in the mathematical context of spectral geometry and in the physical concept of quantum relativity, but also directly accessible to laboratory experiments with the modern artificial materials. We hope that the present results will stimulate an experimental verification of the geometrically induced bound states in graphene waveguides.

1.2. Geometrical Setting and Standing Hypotheses

Before presenting our main results in more detail, let us specify the configuration space of the quantum system we are interested in.

Let $\Gamma \subset \mathbb{R}^2$ be a curve with an injective and C^3 arc-length parametrization $\gamma : \mathbb{R} \to \mathbb{R}^2$, i.e. $\gamma(\mathbb{R}) = \Gamma$. We define $\nu(s)$ the normal of Γ at the point $\gamma(s)$ chosen such that for all $s \in \mathbb{R}$ the couple $\big(\gamma'(s), \nu(s)\big)$ is a positive orthonormal basis of $\mathbb{R}^2$. The curvature of Γ at the point $\gamma(s)$, denoted $\kappa(s)$ is defined by the Frenet formula

$$\gamma''(s) = \kappa(s)\nu(s). \tag{1}$$

All along this paper, we make the following assumptions on the curvature κ:

(A) $\lim_{s\to\pm\infty} \kappa(s) = 0$,

(B) $\kappa' \in L^\infty(\mathbb{R})$.

Notice since we work with a C^3 curve, κ' is automatically continuous, so that assumption (B) implies it is also bounded.

Now, for $0 < \varepsilon < (\|\kappa\|_{L^\infty(\mathbb{R})})^{-1}$ (with the convention that the right-hand side equals $+\infty$ if $\kappa = 0$ identically), we define the tubular neighbourhood of radius ε of Γ in $\mathbb{R}^2$ as the domain

$$\Omega_\varepsilon := \{\gamma(s) + \varepsilon t\nu(s) : \ s \in \mathbb{R}, \ |t| < 1\}, \tag{2}$$

that is, Ω_ε is the planar strip of width 2ε along the curve Γ.

It is a well-known result of differential geometry that under these conditions

$$\Phi_\varepsilon : (s,t) \in \mathsf{Str} \mapsto \gamma(s) + \varepsilon t\nu(s) \in \mathbb{R}^2 \tag{3}$$

is a local C^2-diffeomorphism from the strip

$$\mathsf{Str} := \mathbb{R} \times (-1, 1) \tag{4}$$

to the set Ω_ε. In order to ensure that the map Φ_ε becomes a global diffeomorphism, we additionally assume that

(C) $0 < \varepsilon < (2\|\kappa\|_{L^\infty(\mathbb{R})})^{-1}$ and Φ_ε is injective.

Remark that in assumption (C), one could take $0 < \varepsilon < \|\kappa\|^{-1}_{L^\infty(\mathbb{R})}$ to guarantee that Φ_ε is a global diffeomorphism. However, for technical reasons, we need a more restrictive range of admissible width ε.

Remark 1. Despite being quite general, assumptions (A), (B) are probably not optimal. In [21], the authors deal with three-dimensional non-relativistic waveguides under minimal technical assumptions on the base curve (in particular, the curvature does not need to be differentiable), and then similar results can be expected in the present case. However, for ease of presentation, we prefer not to investigate this aspect here. The assumption on the size of ε in (C) is purely technical and allows to apply Kato's perturbation theory (see the proof of Theorem 2). We mention that another proof could be given adapting the general techniques developed in [28] for three-dimensional problems to our setting.

1.3. Main Results

We are interested in the relativistic quantum Hamiltonian of a (quasi-)particle of (effective) mass $m \geq 0$ described by the Dirac operator with infinite mass boundary conditions posed in the domain Ω_ε. Namely, we define the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ in the Hilbert space $L^2(\Omega_\varepsilon, \mathbb{C}^2)$ as

$$\begin{aligned} \operatorname{dom}(\mathcal{D}_\Gamma(\varepsilon, m)) &:= \{u \in H^1(\Omega_\varepsilon, \mathbb{C}^2) : -i\sigma_3 \sigma \cdot \nu_\varepsilon u = u \text{ on } \partial\Omega_\varepsilon\}, \\ \mathcal{D}_\Gamma(\varepsilon, m)u &:= -i\sigma \cdot \nabla u + m\sigma_3 u, \end{aligned} \tag{5}$$

where ν_ε is the outward pointing normal on $\partial\Omega_\varepsilon$.

In (5), we use the notation $\sigma \cdot v := \sigma_1 v + \sigma_2 v$, where $v \in \mathbb{C}^2$, and σ_k are the *Pauli matrices*

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6}$$

In particular, the action of the operator $\mathcal{D}_\Gamma$ is given by

$$\mathcal{D}_\Gamma = \begin{pmatrix} m & -i(\partial_1 - i\partial_2) \\ -i(\partial_1 + i\partial_2) & -m \end{pmatrix}.$$

As for the boundary conditions, their name is related to the folllowing fact, first recognized in [4].

Consider the Dirac operator on $L^2(\Omega, \mathbb{C}^2)$, acting as $T := -i\sigma \cdot \nabla$ and endowed with boundary conditions as in (5). In a sense specified in [2, Theorem 1.1], such operator is the (norm-resolvent) limit of Dirac operators on $L^2(\mathbb{R}^2, \mathbb{C}^2)$, of the form $T_M := -i\sigma \cdot \nabla + \chi_{\mathbb{R}^2 \setminus \Omega} M\sigma_3$, as $M \to +\infty$, with a mass term supported outside Ω (here χ denotes the characteristic function of a set). This justifies the name *infinite mass* boundary conditions.

In what follows, we denote by Sp the spectrum of an operator. Moreover, we shall distinguish between the *discrete spectrum* $\mathrm{Sp}_{\mathrm{dis}}$, namely the set of eigenvalues of finite multiplicity, and the *essential spectrum* $\mathrm{Sp}_{\mathrm{ess}} = \mathrm{Sp} \setminus \mathrm{Sp}_{\mathrm{dis}}$.

Our first result is about the self-adjointness and the structure of the spectrum of $\mathcal{D}_\Gamma(\varepsilon, m)$.

Theorem 2. *The operator $\mathcal{D}_\Gamma(\varepsilon, m)$ defined in (5) is self-adjoint. Its spectrum is symmetric with respect to the origin and there holds:*

$$\mathrm{Sp}_{\mathrm{ess}}(\mathcal{D}_\Gamma(\varepsilon, m)) = \left(-\infty, -\sqrt{\varepsilon^{-2}E_1(m\varepsilon) + m^2}\right] \cup \left[\sqrt{\varepsilon^{-2}E_1(m\varepsilon) + m^2}, +\infty\right),$$

where $E_1(m)$ is the unique root of the equation

$$m \sin(2\sqrt{E}) + \sqrt{E}\cos(2\sqrt{E}) = 0 \tag{7}$$

lying in the line segment $[\frac{\pi^2}{16}, \frac{\pi^2}{4})$.

In order to prove Theorem 2, a first step is to study the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ in the special case of Ω_ε being a straight strip. In this setting, a partial Fourier transform gives a fiber decomposition of the operator $\mathcal{D}_\Gamma(\varepsilon, m)$, and we are left with the investigation of one-dimensional operators which can be understood explicitly.

The second step is to show that in the case of general waveguides Ω_ε, $\mathcal{D}_\Gamma(\varepsilon, m)$ can be seen as a perturbation of the operator in the straight strip. To this aim, we will use the following proposition, which allows to work with the ε-independent Hilbert space $L^2(\mathsf{Str}, \mathbb{C}^2)$.

Proposition 3. *The operator $\mathcal{D}_\Gamma(\varepsilon, m)$ defined in (5) is unitarily equivalent to the operator $\mathcal{E}_\Gamma(\varepsilon, m)$ defined on $L^2(\mathsf{Str}, \mathbb{C}^2)$ as:*

$$\mathcal{E}_\Gamma(\varepsilon, m) := \frac{1}{1-\varepsilon t\kappa}(-i\sigma_1)\partial_s + \frac{1}{\varepsilon}(-i\sigma_2)\partial_t + \frac{\varepsilon t\kappa'}{2(1-\varepsilon t\kappa)^2}(-i\sigma_1) + m\sigma_3\,,$$
$$\mathrm{dom}\,(\mathcal{E}_\Gamma(\varepsilon, m)) := \{u = (u_1, u_2)^\top \in H^1(\mathsf{Str}, \mathbb{C}^2)\ :\ u_2(\cdot, \pm 1) = \mp u_1(\cdot, \pm 1)\}.$$

The main novelty here lies in a matrix-valued gauge transform involving the geometry of the base curve Γ in order to deal with the infinite mass boundary conditions. In particular, compared to similar strategies for non-relativistic waveguides, it allows to gauge out one part of the geometric induced potential.

The next two main results of this paper concern the study of the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ in the thin waveguide asymptotic regime $\varepsilon \to 0$ and the large mass regime $m \to +\infty$, respectively. It turns out that up to renormalization terms, both regimes are driven by effective operators but of very distinct kind. In the thin waveguide regime $\varepsilon \to 0$, the effective operator is a one-dimensional Dirac operator posed on the base curve Γ, while in the large mass regime $m \to +\infty$, the operator behaves as the Dirichlet Laplacian in the domain Ω_ε. Remark that in both regime Theorem 2 combined with the forthcoming Proposition 10 yields spectral gaps of orders ε^{-1} and m for the thin waveguide regime and the large mass regime, respectively.

Note that an interesting challenge would be to consider combined regimes in which $\varepsilon \to 0$ and $m \to +\infty$ at the same time, and to understand if other effective operators come into play. Finally, our last result is a quantitative result on the existence of bound states involving only the geometry of the domain Ω_ε.

1.3.1. Main Result in the Thin Waveguide Regime $\boldsymbol{\varepsilon \to 0}$. In this paragraph, we fix $m \geq 0$, and our result in the thin waveguide regime $\varepsilon \to 0$ deals with the existence at first order, up to a renormalization term, of an effective operator. This effective operator is the one-dimensional Dirac operator

$$\mathcal{D}_{1\mathrm{D}}(m)u := -i\sigma_1\partial_s u + m\sigma_3 u, \quad u \in \mathrm{dom}\,(\mathcal{D}_{1\mathrm{D}}(m)) := H^1(\mathbb{R}, \mathbb{C}^2). \tag{8}$$

It is well-known that $\mathcal{D}_{1\mathrm{D}}(m)$ is a self-adjoint operator with purely absolutely continuous spectrum $\mathrm{Sp}(\mathcal{D}_{1\mathrm{D}}) = (-\infty, -m] \cup [m, +\infty)$, as can be seen performing a Fourier transform (see [31, Thm. 1.1] for the analogue in dimension three).

Since this operator acts in $L^2(\mathbb{R}, \mathbb{C}^2)$, it is more convenient to work in the ε-independent Hilbert space $L^2(\mathsf{Str}, \mathbb{C}^2)$ and with the unitarily equivalent operator $\mathcal{E}_\Gamma(\varepsilon, m)$ introduced in Proposition 3.

Theorem 4. (Thin width limit) *There exists a closed subspace* $F \subset L^2(\mathsf{Str}, \mathbb{C}^2)$ *and a unitary map* V *such that* $V : L^2(\mathsf{Str}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2) \oplus F$ *and for* $\varepsilon \to 0$ *there holds*

$$V\left(\mathcal{E}_\Gamma(\varepsilon, m) - \frac{\pi}{4\varepsilon}(P^+ - P^-) - i\right)^{-1} V^{-1} = (\mathcal{D}_{1\mathrm{D}}(m_e) - i)^{-1} \oplus 0 + \mathcal{O}(\varepsilon), \tag{9}$$

in the operator norm, where $P^\pm$ *are explicit orthogonal projectors in* $L^2(\mathsf{Str}, \mathbb{C}^2)$ *and where the effective mass* m_e *is given by* $m_e := \frac{2}{\pi} m$.

The projectors $P^\pm$ in the renormalization term of Theorem 4 are projectors on positive and negative spectral subspaces of a one-dimensional transverse Dirac operator. It is remarkable that the geometry of the base curve Γ only appears at higher-order terms. We do not know if for ε small enough $\mathrm{Sp}_{\mathrm{dis}}(\mathcal{D}_\Gamma(\varepsilon, m)) \neq \emptyset$. In particular, it would be interesting to investigate further the remainder term in Theorem 4 to understand if the geometry can play a role in the creation of bound states.

Once again, the proof of Theorem 4 is divided in two steps. We first prove Theorem 4 in the special case of Ω_ε being a straight strip *via* a projection on the modes of a one-dimensional transverse Dirac operator. The obtained operator can be seen as a block operator 2×2 matrix, and the main difficulty here lies in the fact that if the mass m is nonzero, there are off-diagonal terms. They are handled using Schur's complement theory but a special care is needed in order to control the ε-dependence of each term.

In the second step, we use a perturbation argument to prove that the general waveguides Ω_ε can be seen as a perturbation of sufficiently high order of the special case of the straight strip. This step requires a thorough control in ε of the norm of the resolvent of some operators.

1.3.2. Large Mass Regime $m \to +\infty$. In order to state our results in the large mass regime $m \to +\infty$ we need a few notation and definition. First, all along the paper $\mathbb{N} := \{1, 2, \dots\}$ denotes the set of positive natural integers. We also recall the well-known definitions of the min–max values as well as the min–max principle (see [11, Thm. 4.5.1 & 4.5.2]).

Definition 5. Let $\mathfrak{q}$ be a closed lower semi-bounded below quadratic form with dense domain $\mathrm{dom}(\mathfrak{q})$ in a complex Hilbert space $\mathcal{H}$. For $n \in \mathbb{N}$, the n-th min–max value of $\mathfrak{q}$ is defined as

$$\mu_n(\mathfrak{q}) := \inf_{\substack{W \subset \mathrm{dom}(\mathfrak{q}) \\ \dim W = n}} \sup_{u \in W \setminus \{0\}} \frac{\mathfrak{q}(u)}{\|u\|_{\mathcal{H}}^2}. \tag{10}$$

We also denote by $\mathfrak{q}$ the associated sesquilinear form. If A is the unique self-adjoint operator acting on $\mathcal{H}$ associated with the sesquilinear form $\mathfrak{q}$ *via* Kato's first representation theorem (see [18, Ch. VI, Thm. 2.1])), we shall refer to (10) as the n-th min–max value of A and set $\mu_n(A) := \mu_n(\mathfrak{q})$.

Proposition 6. (min–max principle) *Let* $\mathfrak{q}$ *be a closed semi-bounded below quadratic form with dense domain in a Hilbert space* $\mathcal{H}$ *and let* A *be the unique* *self-adjoint operator associated with* $\mathfrak{q}$. *Then, for* $n \in \mathbb{N}$, *we have the following alternative:*

(1) *if* $\mu_n(A) < \inf \mathrm{Sp}_{\mathrm{ess}}(A)$ *then* $\mu_n(\mathfrak{q})$ *is the* n*-th eigenvalue of* A *(counted with multiplicity),*
(2) *if* $\mu_n(A) = \inf \mathrm{Sp}_{\mathrm{ess}}(A)$ *then for all* $k \geq n$ *there holds* $\mu_k(\mathfrak{q}) = \inf \mathrm{Sp}_{\mathrm{ess}}(A)$.

Now, we fix $\varepsilon > 0$ as we are interested in the large mass regime $m \to +\infty$. Up to an adequate renormalization, this limit can be interpreted as a non-relativistic limit and the Dirichlet Laplacian is expected to be the effective operator in this case (see [31, Sec. 6] for general remarks on this limit). To this aim, we introduce $\mathcal{L}_\Gamma(\varepsilon)$, the (spinorial) Dirichlet Laplacian in the waveguide Ω_ε, defined by

$$\mathcal{L}_\Gamma(\varepsilon) := -\Delta, \quad \mathrm{dom}\,(\mathcal{L}_\Gamma(\varepsilon)) := H_0^1(\Omega_\varepsilon, \mathbb{C}^2) \cap H^2(\Omega_\varepsilon, \mathbb{C}^2). \tag{11}$$

Observe that the Sobolev space $H_0^1(\Omega_\varepsilon, \mathbb{C}^2)$ of spinors consists of $\mathbb{C}^2$-valued functions $f = (f_1, f_1)^\top$ such that the components f_j belong to the ordinary (scalar-valued) Sobolev space $f \in H_0^1(\Omega)$. The same remarks, of course, applies to $H^2(\Omega_\varepsilon, \mathbb{C}^2)$ and to the other Sobolev spaces of spinors involved in the text.

The following proposition summarizes results established in [13, 19].

Proposition 7. $\mathcal{L}_\Gamma(\varepsilon)$ *is self-adjoint, and there holds*

$$\mathrm{Sp}_{\mathrm{ess}}(\mathcal{L}_\Gamma(\varepsilon)) = \Big[\frac{\pi^2}{4\varepsilon^2}, +\infty\Big).$$

Moreover, if Γ *is not a straight line, then there exists* $N_\Gamma \in \mathbb{N} \cup \{+\infty\}$ *such that*

$$\sharp\,\mathrm{Sp}_{\mathrm{dis}}(\mathcal{L}_\Gamma(\varepsilon)) = 2N_\Gamma. \tag{12}$$

The factor 2 in (12) comes from the fact that in (11), we consider the Dirichlet Laplacian acting on $\mathbb{C}^2$-valued functions instead of the usual scalar one. In particular, any eigenvalue of $\mathcal{L}_\Gamma(\varepsilon)$ has even multiplicity. Here, we use the convention that if $N_\Gamma = +\infty$ then $2N_\Gamma = +\infty$.

Our first result in the large mass regime reads as follows.

Proposition 8. *Let us assume that* Γ *is not a straight line, fix* $\varepsilon > 0$ *and let* $n \in \{1, \dots, N_\Gamma\}$. *There exists* $m_0 > 0$ *such that for all* $m > m_0$

$$\#\,\mathrm{Sp}_{dis}(\mathcal{D}_\Gamma(\varepsilon, m)) \geq 2n.$$

Proposition 8 is proved by comparing the quadratic forms of the renormalized operator $\mathcal{D}_\Gamma(\varepsilon, m)^2 - m^2$ to the quadratic form of $\mathcal{L}_\Gamma(\varepsilon)$, using the min–max principle (Proposition 6), the asymptotic behaviour of $E_1(m)$ when $m \to +\infty$ and Proposition 7.

Actually, one can show that all the min–max values of the renormalized operator $\mathcal{D}_\Gamma(\varepsilon, m)^2 - m^2$ converge to those of the Dirichlet Laplacian in the regime $m \to +\infty$. This is the purpose of the following theorem.

Theorem 9. (Large mass limit) *Let us assume additionally that* Γ *is of class* C^4 *and that* $\kappa'(s) \to 0$ *and* $\kappa''(s) \to 0$ *when* $|s| \to +\infty$. *Then for all* $n \in \mathbb{N}$, *there holds:*

$$\lim_{m\to+\infty} \left(\mu_n(\mathcal{D}^2_\Gamma(\varepsilon, m)) - m^2\right) = \mu_n(\mathcal{L}_\Gamma(\varepsilon)) . \tag{13}$$

In particular, consider the positive part of the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ defined by $\mathcal{D}^+_\Gamma(\varepsilon, m) := \mathbb{1}_{x>0}(\mathcal{D}_\Gamma(\varepsilon, m))$. Since the spectrum of $\mathcal{D}_\Gamma(\varepsilon, m)$ is symmetric with respect to zero, under the hypothesis of Theorem 9, we obtain for all $n \in \mathbb{N}$

$$\mu_n(\mathcal{D}^+_\Gamma(\varepsilon, m)) = m + \frac{1}{2m}\mu_{2n}(\mathcal{L}_\Gamma(\varepsilon)) + o\left(\frac{1}{m}\right), \quad m \to +\infty; \tag{14}$$

where we have taken into account that the spectrum of $\mathcal{L}_\Gamma(\varepsilon)$ has even multiplicity. Asymptotics (14) illustrates the physically expected fact that in the large mass regime $m \to +\infty$, the positive part of the Dirac operator with infinite mass boundary condition converges to the scalar Dirichlet Laplacian. The main novelty in Theorem 9 with respect to the previous work [1] is that we have to deal with the unbounded domain Ω_ε. This difficulty is overcome by a standard argument, approximating the min–max values of $\mathcal{D}^2_\Gamma(\varepsilon, m) - m^2$ by those of similar operators in bounded waveguides using the so-called IMS localization formula (see [10, Thm. 3.2]).

1.4. Outline of the Paper

Section 2 deals with the infinite mass Dirac operator in the straight strip and with the study of a one-dimensional Dirac operator on a finite interval, obtained by separating variables.

Then, in Sect. 3, we show that the Hamiltonian (5) is unitarily equivalent to a Dirac operator in a straight strip, perturbed by a term encoding the geometric properties of the waveguide. Using operator-theoretic methods, we are able to prove the self-adjointness and to locate the essential spectrum, as stated in Theorem 2.

Section 4 is devoted to the proof of Theorem 4, which is achieved in two steps. First, we deal with the case of the straight waveguide, and second, we add the perturbation induced by the curvature. A careful analysis of the resolvent operator allows to prove that after a suitable renormalization, the Hamiltonian (5) converges in the norm resolvent sense to that of a one-dimensional Dirac operator on the line.

Section 5 contains the proof of Theorem 9, showing that in the large mass regime the min–max values of the (renormalized) squared Hamiltonian converge to those of the vectorial Dirichlet Laplacian $\mathcal{L}_\Gamma(\varepsilon)$.

Finally, in Sect. 6, we obtain a quantitative condition for the existence of at least two bound states in the gap of the essential spectrum. Even though the existence of bound states can be obtained as a corollary of Proposition 8, we mention this alternative proof because here the condition is given by a simple inequality involving geometric properties of the waveguide Ω_ε.

2. Straight Waveguides

In this section, we collect results concerning some auxiliary one-dimensional operators that naturally appear in the study of the Hamiltonian (5) in the thin waveguide regime. In order to simplify the overall presentation, we postpone their proofs to "Appendix A".

2.1. The Transverse Dirac Operator

For $k \in \mathbb{R}$, consider the one-dimensional transverse Dirac operator

$$\begin{aligned} \mathcal{T}(k,m) &:= -i\sigma_2 \frac{d}{dt} + k\sigma_1 + m\sigma_3, \\ \operatorname{dom}(\mathcal{T}(k,m)) &:= \{u = (u_1,u_2)^\top \in H^1\big((-1,1),\mathbb{C}^2\big), u_2(\pm 1) = \mp u_1(\pm 1)\}. \end{aligned} \tag{15}$$

The following proposition holds true.

Proposition 10. *Let $k \in \mathbb{R}$, $m \geq 0$. The operator $\mathcal{T}(k,m)$ is self-adjoint and has compact resolvent. Moreover, the following holds:*

- (i) $\operatorname{Sp}\big(\mathcal{T}(k,m)\big) \cap \Big[-\sqrt{m^2+k^2}, \sqrt{m^2+k^2}\Big] = \emptyset$,
- (ii) *the spectrum of $\mathcal{T}(k,m)$ is symmetric with respect to zero and can be represented as $\operatorname{Sp}\big(\mathcal{T}(k,m)\big) = \bigcup_{p\geq 1}\{\pm\sqrt{m^2+k^2+E_p(m)}\}$, with $E_p(m) > 0$ for all $p \geq 1$,*
- (iii) *for all $p \in \mathbb{N}$, $E_p(m)$ is the only root lying in $\big[(2p-1)^2\frac{\pi^2}{16}, p^2\frac{\pi^2}{4}\big)$ of (7),*
- (iv) *there holds*

$$E_1(m) = \frac{\pi^2}{16} + m + \mathcal{O}(m^2), \quad \text{when } m \to 0,$$

- (v) *there holds*

$$E_1(m) = \frac{\pi^2}{4} - \frac{\pi^2}{4m} + \mathcal{O}(m^{-2}), \quad \text{when } m \to +\infty.$$

The proof of Proposition 10 will also yield the following corollary concerning the operator $\mathcal{T}_0 := \mathcal{T}(0,0)$, which is of crucial importance in the study of the regime $\varepsilon \to 0$.

Corollary 11. *The operator $\mathcal{T}_0$ is self-adjoint and has compact resolvent. Its spectrum is symmetric with respect to zero and verifies*

$$\operatorname{Sp}(\mathcal{T}_0) = \Big\{\pm k\frac{\pi}{4} : \ k \in \mathbb{N}\Big\}.$$

Corresponding normalized eigenfunctions are given by

$$u_k^{\pm}(t) := \frac{1}{2}\cos\Big(k\frac{\pi}{4}(t+1)\Big)\begin{pmatrix}1\\1\end{pmatrix} \pm \frac{1}{2}\sin\Big(k\frac{\pi}{4}(t+1)\Big)\begin{pmatrix}1\\-1\end{pmatrix}.$$

2.2. The Dirac Operator in the Straight Strip

As given in Sect. 3, Theorem 2 can be obtained *via* classical perturbation theory arguments. They rely on the fact that the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ can be seen as a perturbation of the operator $\mathcal{D}_{\Gamma_0}(\varepsilon, m)$ in the straight strip $\mathsf{Str}(\varepsilon) := \mathbb{R} \times (-\varepsilon, \varepsilon)$. Here the base curve $\Gamma_0 := \mathbb{R} \times \{0\}$ is a straight line, which we parametrize by $\gamma_0(s) := s(1, 0)$. The aim of this paragraph is to prove Theorem 2 in this special case.

Proposition 12. *Let $\varepsilon > 0$. The operator $\mathcal{D}_{\Gamma_0}(\varepsilon, m)$ is self-adjoint on its domain. Moreover, there holds*

$$\begin{aligned}\mathrm{Sp}\left(\mathcal{D}_{\Gamma_0}(\varepsilon, m)\right) &= \mathrm{Sp}_{\mathrm{ess}}\left(\mathcal{D}_{\Gamma_0}(\varepsilon, m)\right)\\ &= \left(-\infty, -\sqrt{\varepsilon^{-2}E_1(m\varepsilon) + m^2}\right] \cup \left[\sqrt{\varepsilon^{-2}E_1(m\varepsilon) + m^2}, +\infty\right),\end{aligned}$$

where $E_1(m)$ is defined in Theorem 2.

In order to work with operators defined on a fixed geometrical domain, we recall that $\mathsf{Str} = \mathsf{Str}(1)$ and consider the unitary map

$$U : L^2\left(\mathsf{Str}(\varepsilon), \mathbb{C}^2\right) \to L^2\left(\mathsf{Str}, \mathbb{C}^2\right), \quad (Uv)(x) := \sqrt{\varepsilon}v(x_1, \varepsilon x_2).$$

The operator $\mathcal{E}_0(\varepsilon, m) := U\mathcal{D}_{\Gamma_0}(\varepsilon, m)U^{-1}$ verifies

$$\mathcal{E}_0(\varepsilon, m) = -i\sigma_1\partial_s - i\varepsilon^{-1}\sigma_2\partial_t + m\sigma_3 \tag{16}$$

with domain $\mathrm{dom}\,(\mathcal{E}_0(\varepsilon, m)) = U\mathrm{dom}\,(\mathcal{D}_{\Gamma_0}(\varepsilon, m))$ which rewrites as

$$\mathrm{dom}\,(\mathcal{E}_0(\varepsilon, m)) = \{u = (u_1, u_2)^\top \in H^1(\mathsf{Str}, \mathbb{C}^2) : u_2(\cdot, \pm 1) = \mp u_1(\cdot, \pm 1)\} \tag{17}$$

In (16), we have used the new coordinates $(s, t) \in \mathsf{Str}$ defined by $s = x_1$ and $t = \varepsilon^{-1}x_2$.

Now, we are in a position to prove Proposition 12. We work with the unitarily equivalent operator $\mathcal{E}_0(\varepsilon, m)$ rather than the operator $\mathcal{D}_{\Gamma_0}(\varepsilon, m)$ and the proof relies on a direct integral decomposition of the operator $\mathcal{E}_0(\varepsilon, m)$ as presented, e.g. in [29, §XIII.16.].

Proof of Proposition 12. Consider the unitary partial Fourier transform in the s-variable

$$\mathcal{F} : L^2(\mathsf{Str}, \mathbb{C}^2) \to L^2(\mathsf{Str}, \mathbb{C}^2), \qquad (\mathcal{F}u)(k, t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-isk}u(s, t)\mathrm{d}s\,.$$

The operator $\mathcal{E}_0(\varepsilon, m)$ is unitarily equivalent to the direct integral

$$\mathcal{E}_0(\varepsilon, m) = \mathcal{F}^{-1}\widehat{\mathcal{E}_0}(\varepsilon, m)\mathcal{F}, \quad \widehat{\mathcal{E}_0}(\varepsilon, m) := \int_{\mathbb{R}}^{\oplus} \widehat{\mathcal{E}_0}(\varepsilon, m; k)\mathrm{d}k,$$

where $\mathrm{dom}\left(\widehat{\mathcal{E}_0}(\varepsilon, m)\right)$ is the subspace of functions $u = (u_1, u_2)^\top \in L^2(\mathsf{Str}, \mathbb{C}^2)$ such that for almost all $k \in \mathbb{R}$, we have $\partial_t u(k, \cdot) \in L^2((-1, 1), \mathbb{C}^2)$, $u_2(k, \pm 1) = \mp u_1(k, \pm 1)$ and for almost all $t \in (-1, 1)$ there holds $\int_{\mathbb{R}} k^2|u(k, t)|^2\mathrm{d}k < +\infty$.

One observes that $\widehat{\mathcal{E}_0}(\varepsilon, m; k)$ satisfies $\widehat{\mathcal{E}_0}(\varepsilon, m; k) = \frac{1}{\varepsilon}\mathcal{T}(k\varepsilon, m\varepsilon)$ where the operator $\mathcal{T}(\cdot, \cdot)$ is defined in (15). In particular, $\widehat{\mathcal{E}_0}(\varepsilon, m; k)$ is self-adjoint and so is $\widehat{\mathcal{E}_0}(\varepsilon, m)$ by [29, Thm. XIII.85 (a)]. In particular, we have proved that $\mathcal{E}_0(\varepsilon, m)$ is a self-adjoint operator.

By [29, Thm. XIII.85 (d)], there holds

$$\mathrm{Sp}(\mathcal{E}_0(\varepsilon, m)) = \bigcup_{k\in\mathbb{R}} \mathrm{Sp}(\widehat{\mathcal{E}_0}(\varepsilon, m; k)) . \tag{18}$$

Remark that we have

$$\mathrm{Sp}(\widehat{\mathcal{E}_0}(\varepsilon, m; k)) = \varepsilon^{-1}\,\mathrm{Sp}(\mathcal{T}(k\varepsilon, m\varepsilon)) = \bigcup_{p\in\mathbb{N}} \left\{ \pm\sqrt{m^2 + k^2 + \varepsilon^{-2} E_p(\varepsilon m)} \right\}.$$

By (iii) Proposition 10, for all $p \geq 1$, $E_p(\varepsilon m) \in \big[(2p-1)^2 \frac{\pi^2}{16}, p^2 \frac{\pi^2}{4}\big)$. In particular, there holds

$$\mathrm{Sp}\left(\mathcal{E}_0(\varepsilon, m)\right) = \Big(-\infty, -\sqrt{m^2 + \varepsilon^{-2}E_1(\varepsilon m)}\Big] \cup \Big[\sqrt{m^2 + \varepsilon^{-2}E_1(\varepsilon m)}, +\infty\Big).$$

It concludes the proof of Proposition 12. □

3. First Properties in Curved Waveguides

The main goal of this section is to prove Theorem 2. As mentioned before, the overall strategy consists in regarding the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ in the curved strip Ω_ε as a perturbation of the operator $\mathcal{D}_{\Gamma_0}(\varepsilon, m)$ in the straight strip $\mathsf{Str}(\varepsilon)$.

In the first paragraph of this section, we derive an operator in a straight waveguide, unitarily equivalent to $\mathcal{D}_\Gamma(\varepsilon, m)$, which is given by a Dirac-type operator in the horizontal strip $\mathsf{Str} = \mathbb{R} \times (-1, 1)$ perturbed by a curvature-induced potential. The second and third paragraphs deal with the self-adjointness and the invariance of the essential spectrum, respectively. The key arguments rely on perturbation theory.

3.1. Straightening the Waveguide

This paragraph is devoted to the proof of Proposition 3. The overall scheme is well-known in the study of non-relativistic waveguides and numerous works have taken advantage of such a reduction (see, e.g. [13]). However, we give a complete proof here because the algebraic structure of the Dirac operator allows to gauge out one part of the curvature-induced potential, which appears to be a new effect.

Proof of Proposition 3. The proof is divided into three steps. In the first one, we rewrite the problem in tubular coordinates in order to work in the strip Str. The resulting operator acts in a weighted L^2-space and we perform a unitary transform in order to work in a non-weighted L^2-space; this is the purpose of the second step. Finally, we build a unitary map in order to recover the same boundary condition as the one of the operator $\mathcal{E}_0(\varepsilon, m)$ investigated in Sect. 2.2. This last step partially simplifies the curvature-induced potential.

Step 1. Consider the unitary map

$$U_1 : L^2(\Omega_\varepsilon, \mathbb{C}^2) \longrightarrow L^2(\mathsf{Str}, \mathbb{C}^2; g ds dt), \quad (U_1 u)(s,t) := u(\Phi_\varepsilon(s,t)), \tag{19}$$

where Φ_ε is the parametrization of the waveguide given in (3) and where

$$g(s,t) := \varepsilon\big(1 - \varepsilon t\kappa(s)\big)\,.$$

Next, we consider the operator $\mathcal{D}_{\Gamma,1}(\varepsilon,m) := U_1\mathcal{D}_\Gamma(\varepsilon,m)U_1^{-1}$. One sees that its domain is

$$\begin{aligned}\mathrm{dom}\,(\mathcal{D}_{\Gamma,1}(\varepsilon,m)) &= U_1\mathrm{dom}\,(\mathcal{D}_\Gamma(\varepsilon,m))\\ &= \Big\{u=(u_1,u_2)^\top \in L^2(\mathsf{Str},\mathbb{C}^2; gdsdt):\\ &\qquad (1-\varepsilon t\kappa)^{-1}\partial_s u, \partial_t u \in L^2(\mathsf{Str},\mathbb{C}^2; gdsdt),\\ &\qquad \text{for all } s\in\mathbb{R}\ u_2(s,\pm1) = \pm i\mathbf{n}(s)u_1(s,\pm1)\Big\},\end{aligned}$$

where for $s\in\mathbb{R}$ we have set $\mathbf{n}(s) := \nu_1(s) + i\nu_2(s)$. The operator $\mathcal{D}_{\Gamma,1}(\varepsilon,m)$ acts on $u \in \mathrm{dom}\,(\mathcal{D}_{\Gamma,1}(\varepsilon,m))$ as

$$\mathcal{D}_{\Gamma,1}(\varepsilon,m)u = -\frac{i}{1-\varepsilon t\kappa}\sigma_{\gamma'}\partial_s u - \frac{i}{\varepsilon}\sigma_\nu\partial_t u + m\sigma_3 u,$$

where for $x=(x_1,x_2)\in\mathbb{R}^2$ we have set $\sigma_x := \sigma\cdot x$.

Step 2. In order to flatten the metric, consider the unitary map

$$U_2 : L^2(\mathsf{Str},\mathbb{C}^2; gdsdt) \longrightarrow L^2(\mathsf{Str},\mathbb{C}^2), \quad U_2 u := \sqrt{g}u\,. \tag{20}$$

Let $\mathcal{D}_{\Gamma,2}(\varepsilon,m) := U_2U_1\mathcal{D}_\Gamma(\varepsilon,m)U_1^{-1}U_2^{-1} = U_2\mathcal{D}_{\Gamma,1}(\varepsilon,m)U_2^{-1}$. The domain of $\mathcal{D}_{\Gamma,2}(\varepsilon,m)$ is given by

$$\begin{aligned}\mathrm{dom}\,(\mathcal{D}_{\Gamma,2}(\varepsilon,m)) &= U_2\mathrm{dom}\,(\mathcal{D}_{\Gamma,1}(\varepsilon,m))\\ &= \big\{u=(u_1,u_2)^\top \in H^1(\mathsf{Str},\mathbb{C}^2):\\ &\qquad \text{for all } s\in\mathbb{R}\ u_2(s,\pm1) = \pm i\mathbf{n}(s)u_1(s,\pm1)\big\},\end{aligned}$$

and for $u\in\mathrm{dom}\,(\mathcal{D}_{\Gamma,2}(\varepsilon,m))$ the operator $\mathcal{D}_{\Gamma,2}(\varepsilon,m)$ acts as

$$\begin{aligned}\mathcal{D}_{\Gamma,2}(\varepsilon,m)u &= \frac{1}{1-\varepsilon t\kappa}(-i\sigma_{\gamma'})\partial_s u + \frac{1}{\varepsilon}(-i\sigma_\nu)\partial_t u\\ &\quad + \frac{\varepsilon t\kappa'}{2(1-\varepsilon t\kappa)^2}(-i\sigma_{\gamma'})u + \frac{\kappa}{2(1-\varepsilon t\kappa)}(-i\sigma_\nu)u + m\sigma_3 u.\end{aligned}$$

Note that the $H^1(\mathsf{Str},\mathbb{C}^2)$ regularity of functions in $\mathrm{dom}\,(\mathcal{D}_{\Gamma,2}(\varepsilon,m))$ is a consequence of the regularity hypothesis on the curve Γ (see (A) and (B)).

Step 3. Recall that $\mathbf{n} = \nu_1 + i\nu_2$, and by the Frenet formula (1), we have $\mathbf{n}' = i\kappa\mathbf{n}$. In particular, there holds

$$\mathbf{n}(s) = \exp\left(i\int_0^s \kappa(\xi)\mathrm{d}\xi\right)\mathbf{n_0}.$$

where we have set $\mathbf{n}_0 := \mathbf{n}(0)$. Moreover, there exists $\theta_0\in\mathbb{R}$ such that $\mathbf{n}_0 := e^{i\theta_0}$. By setting

$$\theta(s) := \theta_0 + \int_0^s \kappa(\xi)\mathrm{d}\xi\,, \tag{21}$$

we get $\mathbf{n}(s) = \exp(i\theta(s))$. For any fixed $s \in \mathbb{R}$, consider the unitary matrix

$$U_\theta(s) := \begin{pmatrix} \exp\left(i(\frac{\pi}{4} + \frac{1}{2}\theta(s))\right) & 0 \\ 0 & -\exp\left(-i(\frac{\pi}{4} + \frac{1}{2}\theta(s))\right) \end{pmatrix}.$$

Note that the mapping $s \in \mathbb{R} \mapsto U_\theta(s) \in \mathbb{C}^{2\times 2}$ is of class $C^2(\mathbb{R})$. In order to obtain a boundary condition independent of the normal vector ν, we introduce the unitary map

$$U_3 : L^2(\mathsf{Str}, \mathbb{C}^2) \longrightarrow L^2(\mathsf{Str}, \mathbb{C}^2), \quad U_3 u := U_\theta u. \tag{22}$$

The operator $\mathcal{D}_{\Gamma,3}(\varepsilon, m) := U_3 \mathcal{D}_{\Gamma,2}(\varepsilon, m) U_3^{-1}$ is unitarily equivalent to $\mathcal{D}_\Gamma(\varepsilon, m)$. As U_θ is a bounded and $C^2(\mathbb{R})$ function, its domain is given by

$$\begin{aligned} \mathrm{dom}\,(\mathcal{D}_{\Gamma,3}(\varepsilon, m)) &= U_3 \mathrm{dom}\,(\mathcal{D}_{\Gamma,2}(\varepsilon, m)) \\ &= \big\{u = (u_1, u_2)^\top \in H^1(\mathsf{Str}, \mathbb{C}^2) : \\ &\qquad \text{for all } s \in \mathbb{R}\ u_2(s, \pm 1) = \mp u_1(s, \pm 1)\big\}. \end{aligned}$$

Remark that other choices are possible for the matrices U_θ but the present one gives the same boundary condition as in the straight waveguide case. Moreover, for $u \in \mathrm{dom}\,(\mathcal{D}_{\Gamma,3}(\varepsilon, m))$, there holds

$$\begin{aligned} \mathcal{D}_{\Gamma,3}(\varepsilon, m) u = &\frac{1}{1 - \varepsilon t\kappa} U_\theta(-i\sigma_{\gamma'})\partial_s(U_\theta^* u) + \frac{1}{\varepsilon}(-i\sigma_2)\partial_t u \\ &+ \frac{\varepsilon t\kappa'}{2(1 - \varepsilon t\kappa)^2}(-i\sigma_1) u + \frac{\kappa}{2(1 - \varepsilon t\kappa)}(-i\sigma_2) u + m\sigma_3 u, \end{aligned}$$

where we have used the identities

$$U_\theta \sigma_{\gamma'} U_\theta^* = \sigma_1, \quad U_\theta \sigma_\nu U_\theta^* = \sigma_2, \quad U_\theta \sigma_3 U_\theta^* = \sigma_3.$$

One also obtains

$$U_\theta(-i\sigma_{\gamma'})\partial_s(U_\theta^*) = \frac{\kappa}{2}(i\sigma_2)$$

which finally gives

$$\begin{aligned} \mathcal{D}_{\Gamma,3}(\varepsilon, m) u = &\frac{1}{1 - \varepsilon t\kappa}(-i\sigma_1)\partial_s u + \frac{1}{\varepsilon}(-i\sigma_2)\partial_t u \\ &+ \frac{\varepsilon t\kappa'}{2(1 - \varepsilon t\kappa)^2}(-i\sigma_1) u + m\sigma_3 u. \end{aligned}$$

The proof is completed by setting $\mathcal{E}_\Gamma(\varepsilon, m) := \mathcal{D}_{\Gamma,3}(\varepsilon, m)$. □

3.2. Quadratic Form of the Square

This section contains an explicit expression of the quadratic form of the square of the operator $\mathcal{E}_\Gamma(\varepsilon, m)$ defined in Proposition 3. Throughout this section, we assume that Γ is of class C^4, in order to give a meaning to κ''.

Proposition 13. *Let us assume additionally that* Γ *is of class* C^4. *Then, for every* $u \in \operatorname{dom}(\mathcal{E}_\Gamma(\varepsilon, m))$, *there holds*

$$\begin{aligned}\|\mathcal{E}_\Gamma(\varepsilon,m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} = &\int_{\mathsf{Str}} \frac{1}{(1-\varepsilon t\kappa)^2}|\partial_s u - i\frac{\kappa}{2}\sigma_3 u|^2 \mathrm{d}s\mathrm{d}t + \frac{1}{\varepsilon^2}\int_{\mathsf{Str}} |\partial_t u|^2 \mathrm{d}s\mathrm{d}t \\ &+ \frac{m}{\varepsilon}\int_{\mathbb{R}} \left(|u(s,1)|^2 + |u(s,-1)|^2\right)\mathrm{d}s + m^2\|u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \\ &- \int_{\mathsf{Str}} \frac{\kappa^2}{4(1-\varepsilon t\kappa)^2}|u|^2\mathrm{d}s\mathrm{d}t - \frac{5}{4}\int_{\mathsf{Str}} \frac{(\varepsilon t\kappa')^2}{(1-\varepsilon t\kappa)^4}|u|^2\mathrm{d}s\mathrm{d}t \\ &- \frac{1}{2}\int_{\mathsf{Str}} \frac{\varepsilon t\kappa''}{(1-\varepsilon t\kappa)^3}|u|^2\mathrm{d}s\mathrm{d}t\,.\end{aligned}$$

The proof of Proposition 13 is omitted. It relies on the next lemma, whose proof follows arguing as in [3, Lemma 2.1]. Then, the quantity $\|\mathcal{E}_\Gamma(\varepsilon, m)u\|^2$ for $u \in \operatorname{dom}(\mathcal{E}_\Gamma(\varepsilon, m))$ can be simplified performing rather straightforward (but demanding) integration by parts.

Lemma 14. *The set* $C_0^\infty(\overline{\mathsf{Str}}, \mathbb{C}^2) \cap \operatorname{dom}(\mathcal{E}_0(\varepsilon, 0))$ *is dense in* $\operatorname{dom}(\mathcal{E}_0(\varepsilon, 0))$ *for the graph norm.*

3.3. Self-adjointness

In this paragraph, we prove that $\mathcal{D}_\Gamma(\varepsilon, m)$ is self-adjoint using the Kato-Rellich theorem (see, e.g. [18, Thm. 4.3.]).

Proposition 15. *The operator* $\mathcal{D}_\Gamma(\varepsilon, m)$ *is self-adjoint.*

Before going through the proof of Proposition 15, we need a few lemmata regarding the operator $\mathcal{E}_0(\varepsilon, m)$ introduced in (16). The first lemma is a consequence of Proposition 13, taking into account that in this special case $\kappa = 0$ and $m = 0$.

Lemma 16. *For all* $u \in \operatorname{dom}(\mathcal{E}_0(\varepsilon, 0))$, *there holds*

$$\|\mathcal{E}_0(\varepsilon,0)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} = \|\partial_s u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{1}{\varepsilon^2}\|\partial_t u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}.$$

The following Lemma is well-known and follows integrating by parts taking into account the boundary condition.

Lemma 17. *The operator* $\mathcal{D}_\Gamma(\varepsilon, m)$ *is symmetric.*

We are now ready to prove Proposition 15.

Proof of Proposition 15. Instead of working with the operator $\mathcal{D}_\Gamma(\varepsilon, m)$, we work with the unitarily equivalent operator $\mathcal{E}_\Gamma(\varepsilon, m)$ introduced in Proposition 3. Moreover, as the multiplication operator by σ_3 is bounded and self-adjoint in $L^2(\mathsf{Str}, \mathbb{C}^2)$, we set $m = 0$ without loss of generality.

Remark that $\operatorname{dom}(\mathcal{E}_\Gamma(\varepsilon, m)) = \operatorname{dom}(\mathcal{E}_0(\varepsilon, m))$ where $\mathcal{E}_0(\varepsilon, m)$ is defined in (16), and that for $u \in \operatorname{dom}(\mathcal{E}_\Gamma(\varepsilon, 0))$, there holds

$$\mathcal{E}_\Gamma(\varepsilon,0)u = \mathcal{E}_0(\varepsilon,0)u + V(\varepsilon),$$

where the perturbation operator $V(\varepsilon)$ is defined as

$$V(\varepsilon) := \frac{\varepsilon t\kappa}{1-\varepsilon t\kappa}(-i\sigma_1)\partial_s + \frac{\varepsilon t\kappa'}{2(1-\varepsilon t\kappa)^2}(-i\sigma_1),$$
$$\mathrm{dom}\,(V(\varepsilon)) := \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,0)). \tag{23}$$

Remark that $V(\varepsilon)$ is a symmetric operator because $V(\varepsilon)$ is the difference of two symmetric operators: $\mathcal{E}_0(\varepsilon,0)$ is self-adjoint thus symmetric (see Proposition 12) and $\mathcal{E}_\Gamma(\varepsilon,0)$ is symmetric because it is unitarily equivalent to a symmetric operator (see Lemma 17 and Proposition 3).

Now, remark that for $u \in C_0^\infty(\overline{\mathsf{Str}},\mathbb{C}^2)\cap \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,0))$, there holds

$$\|V(\varepsilon)u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} \le \frac{\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}{1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}\|\partial_s u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}$$
$$+\frac{\varepsilon\|\kappa'\|_{L^\infty(\mathbb{R})}}{2\big(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}\big)^2}\|u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}.$$

Using Lemma 16, we obtain

$$\|V(\varepsilon)u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} \le \frac{\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}{1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}\|\mathcal{E}_0(\varepsilon,0)u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}$$
$$+\frac{\varepsilon\|\kappa'\|_{L^\infty(\mathbb{R})}}{2\big(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}\big)^2}\|u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} \tag{24}$$

and by density of $C_0^\infty(\overline{\mathsf{Str}},\mathbb{C}^2)\cap \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,0))$ in $\mathrm{dom}\,(\mathcal{E}_0(\varepsilon,0))$ for the graph norm (see Lemma 14), (24) also holds for $u \in \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,0))$.

Remember that we assumed (C). Hence, we have

$$\frac{\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}{1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}} < 1.$$

As $V(\varepsilon)$ is symmetric and $\mathcal{E}_0(\varepsilon,0)$-bounded with $\mathcal{E}_0(\varepsilon,0)$-bound smaller than 1, we can apply [18, Thm. 4.3.] and $\mathcal{E}_\Gamma(\varepsilon,0)$ is self-adjoint. □

3.4. Invariance of the Essential Spectrum

In this paragraph, we prove that the essential spectrum of $\mathcal{E}_\Gamma(\varepsilon,m)$ is the same as the one of $\mathcal{E}_0(\varepsilon,m)$. This is the purpose of the following proposition.

Proposition 18. *There holds*

$$\mathrm{Sp}_{\mathrm{ess}}(\mathcal{D}_\Gamma(\varepsilon,m)) = \big(-\infty, -\sqrt{m^2+\varepsilon^{-2}E_1(m\varepsilon)}\big] \cup \big[\sqrt{m^2+\varepsilon^{-2}E_1(m\varepsilon)}, +\infty\big).$$

Proof of Proposition 18. Instead of working with the operator $\mathcal{D}_\Gamma(\varepsilon,m)$, we work with the unitarily equivalent operator $\mathcal{E}_\Gamma(\varepsilon,m)$. Our aim is to apply Weyl's criterion [29, Thm. XIII.14], and for this purpose, we define

$$\mathcal{W} := (\mathcal{E}_\Gamma(\varepsilon,m)+i)^{-1} - (\mathcal{E}_0(\varepsilon,m)+i)^{-1}$$

and by the second resolvent identity, one gets $\mathcal{W} = (\mathcal{E}_0 - i)^{-1}V(\varepsilon)(\mathcal{E}_\Gamma + i)^{-1}$, where the perturbation $V(\varepsilon)$ is defined in (23).

Observe that

$$V(\varepsilon) = a\partial_s + \partial_s a\,, \qquad \text{where} \qquad a := \frac{1}{2}\left(\frac{1}{1-\varepsilon\kappa t} - 1\right)(-i\sigma_1)\,.$$

Then, we get

$$\begin{aligned}-\mathcal{W} &= (\mathcal{E}_0 - i)^{-1} V(\varepsilon)(\mathcal{E}_\Gamma + i)^{-1} \\ &= (\mathcal{E}_0 - i)^{-1} a\ \partial_s(\mathcal{E}_\Gamma + i)^{-1} + (\mathcal{E}_0 - i)^{-1}\partial_s\ a(\mathcal{E}_\Gamma + i)^{-1}\,.\end{aligned}$$

Here, $a(\mathcal{E}_\Gamma + i)^{-1}$ and $(\mathcal{E}_0 - i)^{-1}a$ are compact operators in $L^2(\mathsf{Str}, \mathbb{C}^2)$ due to hypothesis (A) (the latter operator is compact because its adjoint $a(\mathcal{E}_0 + i)^{-1}$ is compact). At the same time, $\partial_s(\mathcal{E}_\Gamma + i)^{-1}$ and $(\mathcal{E}_0 - i)^{-1}\partial_s$ are bounded operators in $L^2(\mathsf{Str}, \mathbb{C}^2)$. (The latter operator is bounded because its adjoint $-\partial_s(\mathcal{E}_0 + i)^{-1}$ is bounded.) Then, the compactness of $\mathcal{W}$ follows by the well-known fact that compact operators are *-both-sided ideal in the space of bounded operators. □

3.5. Proof of Theorem 2

We are now in a good position to prove Theorem 2.

Proof of Theorem 2. Thanks to Proposition 15 and Proposition 18, the only thing left to prove is the symmetry of the spectrum of $\mathcal{D}_\Gamma(\varepsilon, m)$. It is a consequence of the invariance of the system under charge conjugation, corresponding to the operator

$$\mathfrak{C} := \sigma_1 C$$

where C is the complex conjugation operator. A straightforward computation shows that for all $u \in \operatorname{dom}(\mathcal{D}_\Gamma(\varepsilon, m))$, we have $\mathfrak{C}u \in \operatorname{dom}(\mathcal{D}_\Gamma(\varepsilon, m))$ and

$$\mathcal{D}_\Gamma(\varepsilon, m)(\mathfrak{C}u) = -\mathfrak{C}\mathcal{D}_\Gamma(\varepsilon, m)u.$$

In particular, any Weyl sequence $(u_n)_{n\in\mathbb{N}}$ associated with $\lambda \in Sp(\mathcal{D}_\Gamma(\varepsilon, m))$ corresponds to a Weyl sequence $(\mathfrak{C}u_n)_{n\in\mathbb{N}}$ associated with $-\lambda$ which proves that the spectrum of $\mathcal{D}_\Gamma(\varepsilon, m)$ is symmetric and concludes the proof of Theorem 2. □

4. Thin Waveguide Limit

In this section, we prove Theorem 4, which deals with the thin waveguide limit $\varepsilon \to 0$. We first show that up to a renormalization, the operator $\mathcal{E}_\Gamma(\varepsilon, m)$ defined in Proposition 3 converges to the one-dimensional Dirac operator (8) in the norm resolvent sense.

The proof is achieved in two different steps. First, in Sect. 4.1, we deal with the case of a straight strip and then, in Sect. 4.2, we consider the curved waveguide.

Roughly speaking, the main idea of the proof is to project onto the eigenfunctions of the transverse part of the operator. It turns out that after renormalization, all transverse modes converge to zero except the first positive and negative one. The operator $\mathcal{E}_\Gamma(\varepsilon, m)$ restricted to these two modes is unitarily equivalent to a one-dimensional Dirac operator as defined in (8).

4.1. Convergence for the Straight Strip

For $k \geq 1$, let π_k denote the projector in $L^2((-1,1),\mathbb{C}^2)$ on the vector space $\mathrm{span}(u_k^+, u_k^-)$, where $u_k^\pm$ are given in Corollary 11. Similarly, we consider the projectors in $L^2((-1,1),\mathbb{C}^2)$ defined by $p^\pm := \mathbb{1}_{\{\pm x>0\}}(\mathcal{T}_0)$ where $\mathcal{T}_0$ is defined in Sect. 2.1. These projectors can be extended to $L^2(\mathsf{Str},\mathbb{C}^2)$ setting for $u \in L^2(\mathsf{Str},\mathbb{C}^2)$

$$\Pi_k u := \pi_k u, \qquad P^\pm u := p^\pm u. \tag{25}$$

For further use, we renormalize the operator $\mathcal{E}_0(\varepsilon, m)$ as follows:

$$\mathcal{C}(\varepsilon,m) := \mathcal{E}_0(\varepsilon,m) - \frac{\pi}{4\varepsilon}\big(P^+ - P^-\big). \tag{26}$$

To investigate the behaviour of the resolvent operator $(\mathcal{C}(\varepsilon,m)-i)^{-1}$ in the thin waveguide regime $\varepsilon \to 0$, we consider the unitary map

$$U : L^2(\mathsf{Str},\mathbb{C}^2) \to \Pi_1 L^2(\mathsf{Str},\mathbb{C}^2) \times \Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2), \quad (Uv) := (\Pi_1 v, \Pi_1^\perp v)^\top, \tag{27}$$

and remark that there holds

$$U(\mathcal{C}(\varepsilon,m)-i)U^{-1} = \begin{pmatrix} \mathcal{C}_1(\varepsilon,m) - i & \Pi_1\mathcal{C}(\varepsilon,m)\Pi_1^\perp \\ \Pi_1^\perp\mathcal{C}(\varepsilon,m)\Pi_1 & \mathcal{C}_1^\perp(\varepsilon,m) - i \end{pmatrix}, \tag{28}$$

where we have set

$$\mathcal{C}_1^\perp(\varepsilon,m) := \Pi_1^\perp\mathcal{C}(\varepsilon,m)\Pi_1^\perp. \tag{29}$$

For further uses, for all $k \geq 1$, we introduce the operators

$$\mathcal{C}_k(\varepsilon,m) := \Pi_k\mathcal{C}(\varepsilon,m)\Pi_k. \tag{30}$$

In the remaining part of this paragraph, we will make an extensive use of block operator matrix theory to investigate (28) (see [32] for an extensive discussion).

4.1.1. A Few Lemmata. The first lemma is about the operators $\mathcal{C}_k(\varepsilon,m)$ defined in (30). It states that they are unitarily equivalent to one-dimensional Dirac operators (see (8)).

Lemma 19. *Let $k \geq 1$ and consider the unitary map $U_k : \Pi_k L^2(\mathsf{Str},\mathbb{C}^2) \to L^2(\mathbb{R},\mathbb{C}^2)$ defined by $U_k v := \begin{pmatrix} \langle v, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)} \\ \langle v, u_k^-\rangle_{L^2((-1,1),\mathbb{C}^2)} \end{pmatrix}$. There holds*

$$U_k\mathcal{C}_k(\varepsilon,m)U_k^{-1} = \mathcal{D}_{1D}\big((k-1)\frac{\pi}{4\varepsilon} + m_{e,k}\big)$$

where $m_{e,k} := \begin{cases} 0 & \text{if } k \text{ is even,} \\ \frac{2}{k\pi}m & \text{if } k \text{ is odd.} \end{cases}$ In particular, there holds

$$\mathrm{Sp}(\mathcal{C}_k(\varepsilon,m)) = \big(-\infty, -(k-1)\frac{\pi}{4\varepsilon} - m_{e,k}\big] \cup \big[(k-1)\frac{\pi}{4\varepsilon} + m_{e,k}, +\infty\big).$$

Proof of Lemma 19. Let us pick $f = \begin{pmatrix} f^+ \\ f^- \end{pmatrix} \in H^1(\mathbb{R}, \mathbb{C}^2)$ and consider

$$\begin{aligned}
&\mathcal{C}_k(\varepsilon, m)U_k^{-1} f \\
&\quad = \mathcal{C}_k(\varepsilon, m)(f^+ u_k^+ + f^- u_k^-) \\
&\quad = \Pi_k((-i\sigma_1)\partial_s + \frac{1}{\varepsilon}(-i\sigma_2)\partial_t + m\sigma_3)(f^+ u_k^+ + f^- u_k^-) \\
&\quad = \big(-i(f^+)'\big)u_k^- + \big(-i(f^-)'\big)u_k^+ + (k-1)\frac{\pi}{4\varepsilon}\Big(f^+ u_k^+ - f^- u_k^-\Big) \\
&\qquad + m\Pi_k(f^+\sigma_3 u_k^+ + f^-\sigma_3 u_k^-) \\
&\quad = \big(-i(f^+)'\big)u_k^- + \big(-i(f^-)'\big)u_k^+ + (k-1)\frac{\pi}{4\varepsilon}\Big(f^+ u_k^+ - f^- u_k^-\Big) \\
&\qquad + mf^+(\langle\sigma_3 u_k^+, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)} u_k^+ + \langle\sigma_3 u_k^+, u_k^-\rangle_{L^2((-1,1),\mathbb{C}^2)} u_k^-) \\
&\qquad + mf^-(\langle\sigma_3 u_k^-, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)} u_k^+ + \langle\sigma_3 u_k^-, u_k^-\rangle_{L^2((-1,1),\mathbb{C}^2)} u_k^-).
\end{aligned}$$

However, using that $\sigma_1 u_k^\pm = u_k^\mp$ as well as the anti-commutation rules of the Pauli matrices, we get

$$\begin{aligned}
\langle\sigma_3 u_k^+, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)} &= -\langle\sigma_3 u_k^-, u_k^-\rangle_{L^2((-1,1),\mathbb{C}^2)}, \\
\langle\sigma_3 u_k^+, u_k^-\rangle_{L^2((-1,1),\mathbb{C}^2)} &= -\langle\sigma_3 u_k^-, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)}.
\end{aligned}$$

Now, a simple computation gives

$$\langle\sigma_3 u_k^+(t), u_k^-(t)\rangle_{\mathbb{C}^2} = 0, \quad \langle\sigma_3 u_k^+, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)} = \begin{cases} 0 & \text{if } k \text{ is even}, \\ \frac{2}{k\pi} & \text{if } k \text{ is odd}, \end{cases} \tag{31}$$

and we set $m_{e,k} := m\langle\sigma_3 u_k^+, u_k^+\rangle_{L^2((-1,1),\mathbb{C}^2)}$. In particular, there holds

$$\begin{aligned}
\mathcal{C}_k(\varepsilon, m)U_k^{-1} f &= \big(-i(f^+)'\big)u_k^- + \big(-i(f^-)'\big)u_k^+ + (k-1)\frac{\pi}{4\varepsilon}\Big(f^+ u_k^+ - f^- u_k^-\Big) \\
&\quad + m_{e,k} f^+ u_k^+ - m_{e,k} f^- u_k^+,
\end{aligned}$$

so that

$$\begin{aligned}
U_k \mathcal{C}_k(\varepsilon, m)U_k^{-1} f &= \Big(-i\sigma_1\frac{d}{ds} + (m_{e,k} + (k-1)\frac{\pi}{4\varepsilon})\sigma_3\Big) f \\
&= \mathcal{D}_{1D}(m_{e,k} + (k-1)\frac{\pi}{4\varepsilon}) f,
\end{aligned} \tag{32}$$

and the claim follows. □

Remark 20. Notice that for $k = 1$, the one-dimensional Dirac operator in (32) does not depend on ε, that is

$$U_1 \mathcal{C}_1(\varepsilon) U_1^{-1} = \Big(-i\sigma_1\frac{d}{ds} + \frac{2}{\pi}m\sigma_3\Big) = \mathcal{D}_{1D}(2\pi^{-1}m).$$

The next lemma concerns the off-diagonal operators $\Pi_j \mathcal{E}_0(\varepsilon, m)\Pi_k$ for $j, k \in \mathbb{N}$ and $j \neq k$.

Lemma 21. *Let $k, j \geq 1$ such that $j \neq k$. The operator $\Pi_j \mathcal{E}_0(\varepsilon, m)\Pi_k$ satisfies for all $u \in \operatorname{dom}(\mathcal{E}_0(\varepsilon, m))$:*

$$\Pi_j \mathcal{E}_0(\varepsilon, m)\Pi_k u = m\Pi_j \sigma_3 \Pi_k u.$$

Hence, $\Pi_j\mathcal{E}_0(\varepsilon,m)\Pi_k$ can be extended uniquely into a bounded operator in $L^2(\mathsf{Str},\mathbb{C}^2)$ with same operator norm.

Proof of Lemma 21. Let $v\in\mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m))$ and $k,j\geq 1$ such that $k\neq j$. Set $\Pi_k v = f^+u_k^+ + f^-u_k^-\in\mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m))$, there holds

$$\begin{aligned}\Pi_j\mathcal{E}_0(\varepsilon,m)\Pi_k v &= \Pi_j\Big((-i(f^+)')u_k^- + (-i(f^-)')u_k^+ + k\frac{\pi}{4\varepsilon}(f^+u_k^+ - f^-u_k^-)\Big)\\ &\quad + m\Pi_j\sigma_3\Pi_k v\\ &= m\Pi_j\sigma_3\Pi_k v.\end{aligned}$$

As $m\Pi_j\sigma_3\Pi_k$ is a bounded operator in $L^2(\mathsf{Str},\mathbb{C}^2)$ and $\mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m))$ is dense in $L^2(\mathsf{Str},\mathbb{C}^2)$, we deduce that $\Pi_j\mathcal{E}_0(\varepsilon,m)\Pi_k$ can be extended uniquely into a bounded operator in $L^2(\mathsf{Str},\mathbb{C}^2)$ and this operator acts as $m\Pi_j\sigma_3\Pi_k$. □

Proposition 22. *Let $\mathcal{C}_1^\perp(\varepsilon,m)$ be the operator defined in (29). The operator $\mathcal{C}_1^\perp(\varepsilon,m)-i$ acting in $\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)$ is boundedly invertible, and there exists $C>0$ and $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, there holds*

$$\|(\mathcal{C}_1^\perp(\varepsilon,m)-i)^{-1}\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))}\leq C\varepsilon.$$

Remark 23. In Proposition 22, we used the notation $\mathcal{B}(\mathcal{H})$ which for a complex Hilbert-space $\mathcal{H}$ stands for the space of bounded operators on $\mathcal{H}$. Similarly, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two complex Hilbert spaces, $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ denotes the set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

Proof of Proposition 22. First, remark that $\mathcal{C}_1^\perp(\varepsilon,m)$ is a self-adjoint operator when acting in $\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)$ with domain $\Pi_1^\perp\mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m))$. Hence, the operator $\mathcal{C}_1^\perp(\varepsilon,m)-i$ is boundedly invertible in $\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)$. Second, observe that on $\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)$, there holds

$$\begin{aligned}\mathcal{C}_1^\perp(\varepsilon,m) &= \big(\sum_{j\geq 2}\Pi_j\big)\mathcal{C}(\varepsilon,m)\big(\sum_{k\geq 2}\Pi_k\big)\\ &= \sum_{j\geq 2}(\Pi_j\mathcal{C}(\varepsilon,m)\Pi_j) + \sum_{\substack{j,k\geq 2\\ j\neq k}}(\Pi_j\mathcal{C}(\varepsilon,m)\Pi_k)\\ &= \underbrace{\sum_{j\geq 2}(\Pi_j\mathcal{C}(\varepsilon,m)\Pi_j)}_{:=\mathcal{G}(\varepsilon,m)} + m\underbrace{\sum_{\substack{j,k\geq 2\\ j\neq k}}\Pi_j\sigma_3\Pi_k}_{:=B},\end{aligned}$$

where we have used Lemma 21 in the last equation, observing that

$$\Pi_j\mathcal{C}(\varepsilon,m)\Pi_k = \Pi_j\mathcal{E}_0(\varepsilon,m)\Pi_k\,,\qquad \text{if } j\neq k.$$

Remark that as defined, the operator $\mathcal{G}(\varepsilon,m)$ is self-adjoint and $B\in\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))$. Indeed, we have

$$\begin{aligned}\sum_{\substack{j,k\geq 2\\ j\neq k}}\Pi_j\sigma_3\Pi_k &= \sum_{j\geq 2}\Pi_j\sigma_3\sum_{\substack{k\geq 2\\ k\neq j}}\Pi_k = \sum_{j\geq 2}\Pi_j\sigma_3(\Pi_1^\perp - \Pi_j)\\ &= \Pi_1^\perp\sigma_3\Pi_1^\perp - \sum_{j\geq 2}\Pi_j\sigma_3\Pi_j.\end{aligned}$$

Now, the first term on the right-hand side is a bounded operator in $\Pi_1^\perp L^2(\mathsf{Str}, \mathbb{C}^2)$, while for the second, we can argue as follows. Let $u \in \Pi_1^\perp L^2(\mathsf{Str}, \mathbb{C}^2)$, there holds

$$\begin{aligned}\Big\| \sum_{j\geq 2} \Pi_j \sigma_3 \Pi_j u \Big\|^2_{\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)} &= \sum_{j\geq 2} \|\Pi_j \sigma_3 \Pi_j u\|^2_{\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)} \\ &\leq \sum_{j\geq 2} \|\Pi_j u\|^2_{\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)} \\ &= \|u\|^2_{\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)}. \end{aligned} \tag{33}$$

Moreover, we have

$$(\mathcal{C}_1^\perp(\varepsilon, m) - i)^{-1} = (\mathcal{G}(\varepsilon, m) - i)^{-1} \big(1 + mB(\mathcal{G}(\varepsilon, m) - i)^{-1}\big)^{-1}. \tag{34}$$

Now, we need to estimate $\|(\mathcal{G}(\varepsilon, m) - i)^{-1}\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} = \mathrm{dist}(i, Sp\,(\mathcal{G}(\varepsilon, m)))^{-1}$. Recall that by construction, we have

$$\mathcal{G}(\varepsilon, m) = \bigoplus_{k\geq 2} \mathcal{C}_k(\varepsilon, m),$$

see [29, p. 268] for the definition of the direct sum of self-adjoint operators. In particular, by [29, Thm. XIII.85], there holds

$$\mathrm{Sp}(\mathcal{G}(\varepsilon, m)) = \bigcup_{k\geq 2} \mathrm{Sp}(\mathcal{C}_k(\varepsilon, m)) = \big(-\infty, -\frac{\pi}{4\varepsilon}\big] \;\cup\; \big[\frac{\pi}{4\varepsilon}, +\infty\big).$$

Indeed, thanks to Lemma 19 for all $k \geq 2$, there holds

$$\mathrm{Sp}(\mathcal{C}_k(\varepsilon, m)) = \big(-\infty, -(k-1)\frac{\pi}{4\varepsilon} - m_{e,k}\big] \cup \big[(k-1)\frac{\pi}{4\varepsilon} + m_{e,k}, +\infty\big),$$

and for all $k \geq 2$, we have

$$\inf_{k\geq 2} \Big\{ m_{e,k} + (k-1)\frac{\pi}{4\varepsilon} \Big\} = \frac{\pi}{4\varepsilon}.$$

Hence, we get $\mathrm{dist}(i, \mathrm{Sp}(\mathcal{G}(\varepsilon, m))) = \sqrt{1 + \frac{\pi^2}{16\varepsilon^2}}$, and we obtain

$$\|(\mathcal{G}(\varepsilon, m) - i)^{-1}\|_{B(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} = \frac{1}{\sqrt{1 + \frac{\pi^2}{16\varepsilon^2}}}.$$

In particular, we get

$$\|(\mathcal{G}(\varepsilon, m) - i)^{-1}\|_{B(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} = \frac{4}{\pi}\varepsilon + \mathcal{O}(\varepsilon^3), \quad \text{when } \varepsilon \to 0. \tag{35}$$

Next, remark that by (33), there holds $\|B\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} \leq 2$ and using a Neumann series, we arrive at

$$\|\big(1 + mB(\mathcal{G}(\varepsilon, m) - i)^{-1}\big)^{-1}\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} = 1 + \mathcal{O}(\varepsilon), \quad \text{when } \varepsilon \to 0. \tag{36}$$

Finally, combining (35) and (36), (34) yields

$$\|(\mathcal{C}_1^\perp - i)^{-1}\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} \leq \frac{4}{\pi}\varepsilon + \mathcal{O}(\varepsilon^2), \quad \text{when } \varepsilon \to 0.$$

It concludes the proof of Proposition 22. □

4.1.2. Proof of Theorem 4 in the Case of the Straight Strip. In this paragraph, we prove Theorem 4 in the special case of a straight strip but first, we need the next proposition whose proof is a direct application of block operator matrices theory.

Proposition 24. *Recall that U is the unitary map defined in (27). There holds*

$$U(\mathcal{C}(\varepsilon,m)-i)^{-1}U^{-1} = \begin{pmatrix} C_{1,1}(\varepsilon,m) & C_{1,2}(\varepsilon,m) \\ C_{2,1}(\varepsilon,m) & C_{2,2}(\varepsilon,m) \end{pmatrix}$$

where

$$\begin{aligned} C_{1,1}(\varepsilon,m) &:= (C_1(\varepsilon,m)-i)^{-1} \\ &\quad + m^2(C_1(\varepsilon,m)-i)^{-1}\Pi_1\sigma_3\Pi_1^{\perp}\mathcal{S}(i)^{-1}\Pi_1^{\perp}\sigma_3\Pi_1(C_1(\varepsilon,m)-i)^{-1}, \\ C_{1,2}(\varepsilon,m) &:= -m(\mathcal{C}_1(m,\varepsilon)-i)^{-1}\Pi_1\sigma_3\Pi_1^{\perp}\mathcal{S}(i)^{-1}, \\ C_{2,1}(\varepsilon,m) &:= -m\mathcal{S}(i)^{-1}\Pi_1^{\perp}\sigma_3\Pi_1(\mathcal{C}_1(\varepsilon,m)-i)^{-1}, \\ C_{2,2}(\varepsilon,m) &:= \mathcal{S}(i)^{-1}. \end{aligned}$$

Here, $\mathcal{S}(i)$ denotes the Schur complement:

$$\mathcal{S}(i) := \mathcal{C}_1^{\perp}(\varepsilon,m) - i - m^2\Pi_1^{\perp}\sigma_3\Pi_1(\mathcal{C}_1(\varepsilon,m)-i)^{-1}\Pi_1\sigma_3\Pi_1^{\perp}. \tag{37}$$

Proof of Proposition 24. According to the notation of [32, Thm. 2.3.3], we set

$$A := \mathcal{C}_1(\varepsilon,m), \quad B := m\Pi_1\sigma_3\Pi_1^{\perp}, \quad C := m\Pi_1^{\perp}\sigma_3\Pi_1, \quad D := \mathcal{C}_1^{\perp}(\varepsilon,m),$$

where we have used Lemma 21 to rewrite the operators B and C.

Now, we check all the hypothesis of [32, Thm. 2.3.3]:

- $\mathrm{dom}\,(A) = \Pi_1 \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m)) \subset \mathrm{dom}\,(C) = \Pi_1 L^2(\mathsf{Str},\mathbb{C}^2)$,
- A is self-adjoint as an operator acting in $\Pi_1 L^2(\mathsf{Str},\mathbb{C}^2)$ thus $i \notin \mathrm{Sp}(A)$,
- as A is self-adjoint and B is bounded, the operator $(A-i)^{-1}B$ is bounded in $\Pi_1^{\perp}L^2(\mathsf{Str},\mathbb{C}^2)$,
- the operator $\mathcal{S}(i)$ is closed because D is self-adjoint and the operator $\Pi_1^{\perp}\sigma_3\Pi_1(A-i)^{-1}\Pi_1\sigma_3\Pi_1^{\perp} \in \mathcal{B}(\Pi_1^{\perp}L^2(\mathsf{Str},\mathbb{C}^2))$ (hence both are closed).

Thus, [32, Thm. 2.3.3] yields

$$U(\mathcal{C}(\varepsilon,m)-i)^{-1}U^{-1} = \begin{pmatrix} \mathcal{C}_{1,1}(\varepsilon,m) & \mathcal{C}_{1,2}(\varepsilon,m) \\ \mathcal{C}_{2,1}(\varepsilon,m) & \mathcal{C}_{2,2}(\varepsilon,m) \end{pmatrix}$$

with

$$\begin{aligned} \mathcal{C}_{1,1}(\varepsilon,m) &:= (A-i)^{-1}\Big(\mathbb{1} + B\mathcal{S}(i)^{-1}C(A-i)^{-1}\Big) \\ \mathcal{C}_{1,2}(\varepsilon,m) &:= -(A-i)^{-1}B\mathcal{S}(i)^{-1} \\ \mathcal{C}_{2,1}(\varepsilon,m) &:= -\mathcal{S}(i)^{-1}C(A-i)^{-1} \\ \mathcal{C}_{2,2}(\varepsilon,m) &:= \mathcal{S}(i)^{-1}. \end{aligned}$$

This finishes the proof. □

We are now in a good position to prove (9) in Theorem 4 for the straight waveguide.

Proposition 25. *There exists a unitary map V such that $V : L^2(\mathsf{Str},\mathbb{C}^2) \to L^2(\mathbb{R},\mathbb{C}^2) \oplus \Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2)$, and there holds*

$$V\big(\mathcal{E}_0(\varepsilon,m) - \frac{\pi}{4\varepsilon}(P^+ - P^-) - i\big)^{-1} V^{-1} = \big(\mathcal{D}_{1D}(2\pi^{-1}m) - i\big)^{-1} \oplus 0 + \mathcal{O}(\varepsilon)\,,$$

in the operator norm, where $P^\pm$ are the projectors defined in (25).

Proof of Proposition 25. The proof is performed in three steps. In the first two steps, we estimate the norm of the bounded operators $(\mathcal{C}_1(\varepsilon,m)-i)^{-1}$ and the Schur complement $\mathcal{S}(i)^{-1}$ (defined in (37)). In the last step, we use Proposition 24 to obtain an asymptotic expansion of the operator $U(\mathcal{C}(\varepsilon,m)-i)^{-1}U^{-1}$.
Step 1. Thanks to Lemma 19, we know that $\mathrm{Sp}(\mathcal{C}_1(\varepsilon,m)) = (-\infty, -\frac{2}{\pi}m] \cup [\frac{2}{\pi}m, +\infty)$. In particular, there holds

$$\|(\mathcal{C}_1(\varepsilon,m)-i)^{-1}\|_{\mathcal{B}(\Pi_1 L^2(\mathsf{Str},\mathbb{C}^2))} = \frac{1}{\mathrm{dist}(i, \mathrm{Sp}(\mathcal{C}_1(\varepsilon,m)))} = \frac{1}{\sqrt{1+\frac{4}{\pi^2}m^2}}. \tag{38}$$

Step 2. Remark that there holds

$$\mathcal{S}(i)^{-1} = \big(\mathbb{1} - m^2(\mathcal{C}_1^\perp(\varepsilon,m)-i)^{-1}\Pi_1^\perp\sigma_3\Pi_1(\mathcal{C}_1(\varepsilon,m)-i)^{-1}\Pi_1\sigma_3\Pi_1^\perp\big)^{-1} (\mathcal{C}_1^\perp(\varepsilon,m)-i)^{-1},$$

and in particular, we have

$$\begin{aligned}
&\|(\mathcal{C}_1^\perp(\varepsilon,m)-i)^{-1}\Pi_1^\perp\sigma_3\Pi_1(\mathcal{C}_1(\varepsilon,m)-i)^{-1}\Pi_1\sigma_3\Pi_1^\perp\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))}\\
&\quad\le \|(\mathcal{C}_1^\perp(\varepsilon,m)-i)^{-1}\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))}\|(\mathcal{C}_1(\varepsilon,m)-i)^{-1}\|_{\mathcal{B}(\Pi_1 L^2(\mathsf{Str},\mathbb{C}^2))}\\
&\quad\le \frac{C}{\sqrt{1+\frac{4}{\pi^2}m^2}}\varepsilon := \tilde{C}\varepsilon, \quad \text{when } \varepsilon \to 0.
\end{aligned}$$

Here, the first inequality is obtained using that σ_3 is a unitary operator from $L^2(\mathsf{Str},\mathbb{C}^2)$ onto itself and that Π_1 and $\Pi_1^\perp$, being orthogonal projectors, are bounded operators with norm smaller than 1. The second inequality is a consequence of (38) and Proposition 22. In particular, using a Neumann series and Proposition 22, it yields the existence of $C' > 0$ and $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0,\varepsilon_1)$, there holds

$$\|\mathcal{S}(i)^{-1}\|_{B(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} \le C'\varepsilon. \tag{39}$$

Step 3. Thanks to Proposition 24, there holds

$$U(\mathcal{C}(\varepsilon,m)-i)^{-1}U^{-1} = \begin{pmatrix} (\mathcal{C}_1(\varepsilon,m)-i)^{-1} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix} R_{1,1}(\varepsilon,m) & C_{1,2}(\varepsilon,m)\\ C_{2,1}(\varepsilon,m) & C_{2,2}(\varepsilon,m)\end{pmatrix},$$

where we have set

$$R_{1,1}(\varepsilon,m) = m^2(C_1(\varepsilon,m)-i)^{-1}\Pi_1\sigma_3\Pi_1^\perp\mathcal{S}(i)^{-1}\Pi_1^\perp\sigma_3\Pi_1(C_1(\varepsilon,m)-i)^{-1}.$$

Now, we examine the norm of each bounded operator appearing in the second block matrix on the right-hand side. Remark that by (38) and (39), for all $\varepsilon \in (0,\varepsilon_1)$, there holds

$$\|R_{1,1}(\varepsilon,m)\|_{\mathcal{B}(\Pi_1 L^2(\mathsf{Str},\mathbb{C}^2))}$$

$$\leq m^2 \|(\mathcal{C}_1(\varepsilon, m) - i)^{-1}\|^2_{\mathcal{B}(\Pi_1 L^2(\mathsf{Str},\mathbb{C}^2))} \|\mathcal{S}(i)^{-1}\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))}$$
$$\leq m^2 \frac{C'}{1 + \frac{4}{\pi^2} m^2} \varepsilon. \tag{40}$$

Similarly, for $\varepsilon \in (0, \varepsilon_1)$, there holds

$$\|C_{1,2}(\varepsilon, m)\|_{\mathcal{B}(\Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2), \Pi_1 L^2(\mathsf{Str},\mathbb{C}^2))} \leq m \frac{\tilde{C}}{\sqrt{1 + \frac{4}{\pi^2} m^2}} \varepsilon \tag{41}$$

and

$$\|C_{2,1}(\varepsilon, m)\|_{\mathcal{B}(\Pi_1 L^2(\mathsf{Str},\mathbb{C}^2), \Pi_1^\perp L^2(\mathsf{Str},\mathbb{C}^2))} \leq m \frac{\tilde{C}}{\sqrt{1 + \frac{4}{\pi^2} m^2}} \varepsilon. \tag{42}$$

Gathering (40), (41), (42) and (39), we get

$$U(\mathcal{C}(\varepsilon, m) - i)^{-1} U^{-1} = \begin{pmatrix} (\mathcal{C}_1(\varepsilon, m) - i)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon), \quad \text{when } \varepsilon \to 0.$$

To conclude, we introduce the unitary map

$$V : L^2(\mathsf{Str}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2) \oplus \Pi_1^\perp(L^2(\mathsf{Str}, \mathbb{C}^2)), \quad (Vu) := \left(U_1 \Pi_1 u, \Pi_1^\perp u\right),$$

where the unitary map U_1 is defined in Lemma 19. When $\varepsilon \to 0$, there holds

$$\begin{aligned} V(\mathcal{C}(\varepsilon, m) - i)^{-1} V^{-1} &= \begin{pmatrix} U_1(\mathcal{C}_1(\varepsilon, m) - i)^{-1} U_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon) \\ &= \begin{pmatrix} (\mathcal{D}_{1D}(2\pi^{-1} m) - i)^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}(\varepsilon) \\ &= (\mathcal{D}_{1D}(2\pi^{-1} m) - i)^{-1} \oplus 0 + \mathcal{O}(\varepsilon). \end{aligned}$$

□

4.2. Convergence for the Curved Waveguide

This paragraph is devoted to the proof of Theorem 4. Once again, we use a perturbation argument. We start with a few auxiliary results.

The first lemma deals with the quadratic form for the *transverse part* of the operator.

$$\begin{aligned} \tau_m(u) &:= \|(-i\sigma_2) u'\|^2_{L^2((-1,1),\mathbb{C}^2)} + m\left(|u(1)|^2 + |u(-1)|^2\right), \\ \mathrm{dom}\,(\tau_m) &:= \{u = (u_1, u_2)^\top \in H^1((-1,1), \mathbb{C}^2) : u_2(\pm 1) = \mp u_1(\pm 1)\}. \end{aligned} \tag{43}$$

Remark that τ_m is the quadratic form associated with the operator $\mathcal{T}(0, m)^2 - m^2$, where $\mathcal{T}(0, m)$ is defined in (15), as can be seen in (69) and below.

Lemma 26. *Let $u \in \mathrm{dom}\,(\tau_m)$, there holds*

$$\tau_m(u) \geq E_1(m) \|\pi_1 u\|^2_{L^2((-1,1),\mathbb{C}^2)} + \tau_0(\pi_1^\perp u),$$

where the projector π_1 is defined in Sect. 4.1 and where we have set $\pi_1^\perp = \mathbb{1} - \pi_1$.

Proof of Lemma 26. Let $u \in \mathrm{dom}\,(\tau_m)$, there holds

$$\begin{aligned}\tau_m(u) = \tau_m(\pi_1 u + \pi_1^\perp u) &= \tau_m(\pi_1 u) + \tau_m(\pi_1^\perp u) + 2\Re(\tau_m(\pi_1 u, \pi_1^\perp u))\\ &\geq E_1(m)\|\pi_1 u\|^2_{L^2((-1,1),\mathbb{C}^2)} + \tau_0(\pi_1^\perp u)\\ &\quad + 2\Re(\tau_m(\pi_1 u, \pi_1^\perp u)), \end{aligned} \tag{44}$$

where we have used the min–max principle (Proposition 6) and bounded from below the quadratic form τ_m by τ_0. Now, remark that for all $v \in \mathrm{dom}\,(\tau_m)$, there holds

$$\tau_m(v) = \|(-i\sigma_2)v' + m\sigma_3 v\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} - m^2\|v\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}.$$

In particular, for the associated sesquilinear form, it gives

$$\begin{aligned}\tau_m(\pi_1 v, \pi_1^\perp v) &= \langle\big((-i\sigma_2)\frac{d}{dt} + m\sigma_3\big)\pi_1 v, \big((-i\sigma_2)\frac{d}{dt} + m\sigma_3\big)\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)}\\ &\quad - m^2\langle \pi_1 v, \pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)}\\ &= m\Big(\langle \mathcal{T}_0\pi_1 v, \sigma_3\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)} + \langle \sigma_3\pi_1 v, \mathcal{T}_0\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)}\Big).\end{aligned}$$

Now, remark that

$$\begin{aligned}\langle \mathcal{T}_0\pi_1 v, \sigma_3\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)} &= \langle \mathcal{T}_0 v, \pi_1\sigma_3\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)},\\ \langle \sigma_3\pi_1 v, \mathcal{T}_0\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)} &= \langle \pi_1^\perp\sigma_3\pi_1 v, \mathcal{T}_0 u\rangle_{L^2((-1,1),\mathbb{C}^2)}.\end{aligned}$$

If $v \in \mathrm{dom}\,(\tau_m) = \mathrm{dom}\,(\mathcal{T}_0)$, then $\pi_1^\perp\sigma_3\pi_1 u \in \mathrm{dom}\,(\mathcal{T}_0)$ and as $\mathcal{T}_0$ is self-adjoint there holds

$$\begin{aligned}\langle \mathcal{T}_0 v, \pi_1\sigma_3\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)} &= \langle v, \mathcal{T}_0\pi_1\sigma_3\pi_1^\perp v\rangle_{L^2((-1,1),\mathbb{C}^2)}\\ &= -\langle \pi_1^\perp\sigma_3\pi_1 v, \mathcal{T}_0 v\rangle_{L^2((-1,1),\mathbb{C}^2)},\end{aligned}$$

where we have used that $\mathcal{T}_0$ commutes with π_1 and $\pi_1^\perp$ and that σ_2 anti-commutes with σ_3. In particular, we obtain that $\tau_m(\pi_1 u, \pi_1^\perp u) = 0$ which combined with equation (44) yields

$$\tau_m(u) \geq E_1(m)\|\pi_1 u\|^2_{L^2((-1,1),\mathbb{C}^2)} + \tau_0(\pi_1^\perp u),$$

which is precisely Lemma 26. □

Remark 27. Observe that the quadratic form τ_m in (43) is *a priori* defined for functions of $t \in (-1,1)$. With an abuse of notation, in what follows, we extend it to functions defined on the strip Str acting only on the transverse variable. More precisely, there holds $u(s,\cdot) \in \mathrm{dom}\,(\tau_m)$ for a.e. $s \in \mathbb{R}$, if $u \in \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m))$.

We now state a technical result, of crucial importance in the proof of Theorem 4, whose proof is postponed to "Appendix A" for simplicity.

Lemma 28. *Let $u \in \mathrm{dom}\,(\mathcal{E}_0(\varepsilon,m))$, there exists $\varepsilon_0 > 0$ and $K > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, we have*

$$\|(-i\sigma_1)\partial_s u + m\sigma_3 u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} \leq \|\mathcal{C}(\varepsilon,m)u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} + K\|u\|_{L^2(\mathsf{Str},\mathbb{C}^2)},$$

where the operator $\mathcal{C}(\varepsilon,m)$ is defined in (26).

We are now in a good position to prove Theorem 4.

Proof of Theorem 4. Let us set

$$\mathcal{C}_\Gamma(\varepsilon, m) := \mathcal{E}_\Gamma(\varepsilon, m) - \frac{\pi}{4\varepsilon}(P^+ - P^-).$$

and remark that

$$\mathcal{C}_\Gamma(\varepsilon, m) = \mathcal{C}(\varepsilon, m) + V(\varepsilon),$$

where $\mathcal{C}(\varepsilon, m)$ is defined in (26) and the symmetric operator $V(\varepsilon)$ is defined in (23).

Consider the operator

$$\begin{aligned}(\mathcal{C}_\Gamma(\varepsilon, m) - i)^{-1} &= (\mathcal{C}(\varepsilon, m) - i + V(\varepsilon))^{-1}\\ &= (\mathcal{C}(\varepsilon, m) - i)^{-1}\big(\mathbb{1} + V(\varepsilon)(\mathcal{C}(\varepsilon, m) - i)^{-1}\big)^{-1}.\end{aligned}$$

We claim that there exists $\varepsilon_0 > 0$ and $K' > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there holds

$$\|V(\varepsilon)(\mathcal{C}(\varepsilon, m) - i)^{-1}\|_{\mathcal{B}(L^2(\mathsf{Str}, \mathbb{C}^2))} \leq K'\varepsilon.$$

Indeed, for $u \in L^2(\mathsf{Str}, \mathbb{C}^2)$, there holds

$$\begin{aligned}&\|V(\varepsilon)(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\quad\leq \frac{\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}{1 - \varepsilon\|\kappa\|_{L^\infty(\mathbb{R})}}\|(-i\sigma_1)\partial_s(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\qquad+ \frac{\varepsilon\|\kappa'\|_{L^\infty(\mathbb{R})}}{2(1 - \varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})^2}\|(\mathcal{C}(\varepsilon, m) - i)^{-1}\|_{\mathcal{B}(L^2(\mathsf{Str}, \mathbb{C}^2))}\|u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}.\end{aligned}$$

One remarks that

$$\begin{aligned}&\|(-i\sigma_1)\partial_s(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\quad= \|\big((-i\sigma_1)\partial_s + m\sigma_3 - m\sigma_3\big)(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\quad\leq \|\big((-i\sigma_1)\partial_s + m\sigma_3\big)(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\qquad+ m\|(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}.\end{aligned}$$

Hence, by Lemma 28, there exists $K > 0$ and $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, there holds:

$$\begin{aligned}&\|(-i\sigma_1)\partial_s(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\quad\leq \|\mathcal{C}(\varepsilon, m)(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\qquad+ (m + K)\|(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\quad\leq \|(\mathcal{C}(\varepsilon, m) - i)(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\qquad+ (1 + m + K)\|(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\quad\leq \|u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}\\ &\qquad+ (m + 1 + K)\|(\mathcal{C}(\varepsilon, m) - i)^{-1}\|_{\mathcal{B}(L^2(\mathsf{Str}, \mathbb{C}^2))}\|u\|_{L^2(\mathsf{Str}, \mathbb{C}^2)}.\end{aligned}$$

Remarking that

$$\|(\mathcal{C}(\varepsilon, m) - i)^{-1}\|_{\mathcal{B}(L^2(\mathsf{Str}, \mathbb{C}^2))} = \mathrm{dist}(i, Sp(\mathcal{C}(\varepsilon, m)))^{-1} \leq 1 \tag{45}$$

, we obtain that there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that

$$\|V(\varepsilon)(\mathcal{C}(\varepsilon, m) - i)^{-1}u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} \leq K'\varepsilon\|u\|_{L^2(\mathsf{Str},\mathbb{C}^2)},$$

for some constant $K' > 0$. Thus, developing in Neumann series and using (45), we get

$$(\mathcal{C}_\Gamma(\varepsilon, m) - i)^{-1} = (\mathcal{C}(\varepsilon, m) - i)^{-1} + \mathcal{O}(\varepsilon)$$

and the theorem is proved applying Proposition 25. □

5. Non-relativistic Limit

This section is devoted to the proof of Proposition 8 and Theorem 9. In the sequel, we will assume $\varepsilon > 0$ to be fixed, as we are only interested in the regime $m \to +\infty$. We start by proving Proposition 8 before turning to the proof of Theorem 9.

5.1. Proof of Proposition 8

Our starting point is the expression of the quadratic form associated with the operator $\mathcal{D}_\Gamma(\varepsilon, m)^2$ that can be computed arguing as in [26, Prop. 14].

Lemma 29. *Given $u \in \mathrm{dom}\,(\mathcal{D}_\Gamma(\varepsilon, m))$, there holds*

$$\|\mathcal{D}_\Gamma(\varepsilon, m)u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} = \|\nabla u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} + m^2\|u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} + \int_{\partial\Omega_\varepsilon}(m - \frac{\kappa_\varepsilon}{2})|u|^2 \mathrm{d}s\,,$$

where κ_ε is the signed curvature of the boundary $\partial\Omega_\varepsilon$ with respect to the outer normal ν_ε.

Let us introduce the quadratic forms

$$q_m(u) := \|\mathcal{D}_\Gamma(\varepsilon, m)u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} - m^2\|u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}, \quad \mathrm{dom}\,(q_m) := \mathrm{dom}\,(\mathcal{D}_\Gamma)\,,$$

and

$$q_\infty(u) := \|\nabla u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^4)}, \quad \mathrm{dom}\,(q_\infty) := H^1_0(\Omega_\varepsilon, \mathbb{C}^2).$$

of the Dirichlet Laplacian $\mathcal{L}_\Gamma(\varepsilon)$ defined in (11). In the following, we shall consider the min–max values of the forms above, as introduced in Definition 5. We are now in a good position to prove Proposition 8.

Proof of Proposition 8. Observe that $\mathrm{dom}\,(q_\infty) \subset \mathrm{dom}\,(\mathcal{D}_\Gamma(\varepsilon, m))$ and that by Lemma 29 if $u \in \mathrm{dom}\,(q_\infty)$ we have $q_\infty(u) = q_m(u)$. Then, by Proposition 6, we immediately get for all $j \in \mathbb{N}$:

$$\mu_j(q_m) \leq \mu_j(q_\infty). \tag{46}$$

Recall that by Theorem 2, $\varepsilon^{-2}E_1(m\varepsilon)$ is the bottom of the essential spectrum of $\mathcal{D}_\Gamma(\varepsilon, m)^2 - m^2$. Now, fix $j_0 \in \mathbb{N}$ with $j_0 < N_\Gamma + 1$ (with the convention that $N_\Gamma + 1 = +\infty$ if $N_\Gamma = +\infty$). Then, by Proposition 6, (v) of Proposition 10 and Proposition 7, we get for all $j \in \{1, \ldots, 2j_0\}$:

$$\mu_j(q_m) - \frac{E_1(m\varepsilon)}{\varepsilon^2} \leq \mu_j(q_\infty) - \frac{E_1(m\varepsilon)}{\varepsilon^2} \leq \underbrace{\mu_{2j_0}(q_\infty) - \frac{\pi^2}{4\varepsilon^2}}_{<0} + \frac{C}{m}\,,$$

for some constant $C > 0$. Then, the claim follows taking m large enough. □

5.2. Finite Waveguides

In our proof of Theorem 9, we need to investigate the min–max values of quadratic forms in finite waveguides. To this aim, for $R > 0$, we split the waveguide Ω_ε into the following three domains:

$$\Omega_\varepsilon^R := \{\gamma(s) + \varepsilon t\nu(s) : |s| < R,\ t \in (-1,1)\},$$
$$\Omega_\varepsilon^{R,\pm} := \{\gamma(s) + \varepsilon t\nu(s) : \pm s > R,\ t \in (-1,1)\},$$

and consider the following four forms:

$$q_\infty^R(u) := \|\nabla u\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^4)},$$
$$\mathrm{dom}\left(q_\infty^R\right) := H_0^1(\Omega_\varepsilon^R, \mathbb{C}^2),$$
$$q_m^R(u) := \|\mathcal{D}_\Gamma(\varepsilon,m)u\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)} - m^2\|u\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)},$$
$$\mathrm{dom}\left(q_m^R\right) := \big\{u \in H^1(\Omega_\varepsilon^R, \mathbb{C}^2) : -i\sigma_3\sigma\cdot\nu_\varepsilon u = u \text{ on } \partial\Omega_\varepsilon^R \cap \partial\Omega_\varepsilon,$$
$$u = 0 \text{ on } \partial\Omega_\varepsilon^R \setminus \partial\Omega_\varepsilon\big\},$$
$$q_m^{R,\pm}(u) := \|\mathcal{D}_\Gamma(\varepsilon,m)u\|^2_{L^2(\Omega_\varepsilon^{R,\pm},\mathbb{C}^2)} - m^2\|u\|^2_{L^2(\Omega_\varepsilon^{R,\pm},\mathbb{C}^2)},$$
$$\mathrm{dom}\left(q_m^{R,\pm}\right) := \big\{u \in H^1(\Omega_\varepsilon^{R,\pm}, \mathbb{C}^2) : -i\sigma_3\sigma\cdot\nu_\varepsilon u = u \text{ on } \partial\Omega_\varepsilon^{R,\pm} \cap \partial\Omega_\varepsilon,$$
$$u = 0 \text{ on } \partial\Omega_\varepsilon^{R,\pm} \setminus \partial\Omega_\varepsilon\big\}.$$

In the following, we shall consider the min–max values of the above forms as introduced in Definition 5.

The same compactness argument as in [1, Prop. 2.1] allows to prove the following local convergence result, whose proof is omitted.

Lemma 30. *For all $R > 0$ and $j \in \mathbb{N}$, there holds*

$$\lim_{m\to+\infty} \mu_j(q_m^R) = \mu_j(q_\infty^R)\,.$$

For further use, we need the following lemma which is proved using the well-known IMS formula.

Lemma 31. *For all $j \in \mathbb{N}$ there holds*

$$\lim_{R\to+\infty} \mu_j(q_\infty^R) = \mu_j(q_\infty)\,.$$

Proof of Lemma 31. Fix $j \in \mathbb{N}$ and observe that thanks to a Dirichlet bracketing argument one gets $\mu_j(q_\infty^R) \geq \mu_j(q_\infty)$, for all $R > 0$. Then,

$$\liminf_{R\to\infty} \mu_j(q_\infty^R) \geq \mu_j(q_\infty)\,. \tag{47}$$

Now, we need to prove the opposite inequality

$$\limsup_{R\to\infty} \mu_j(q_\infty^R) \leq \mu_j(q_\infty)\,. \tag{48}$$

Take a cut-off function $\theta \in C_0^\infty(\mathbb{R})$ such that $0 \leq \theta \leq 1$, $\theta(s) = 1$ for $|s| \leq \frac{1}{2}$ and $\theta(s) = 0$ for $|s| \geq 1$. Given $R > 0$, define

$$\theta_R(s) := \theta(R^{-1}s)\,, \qquad s \in \mathbb{R}\,.$$

We introduce

$$\chi_R := (U_1)^{-1}\theta_R,$$

where U_1 is the unitary map (19). For further use, we compute $\nabla\chi_R$.

Since $\chi_R(\gamma(s)+\varepsilon t\nu(s)) = \theta(R^{-1}s)$, we get

$$\begin{cases} \partial_s\chi_R = \gamma_1'(1-\varepsilon t\kappa)\partial_1\chi_R + \gamma_2'(1-\varepsilon t\kappa)\partial_2\chi_R = R^{-1}\theta'(R^{-1}s)\,, \\ \partial_t\chi_R = \varepsilon\nu_1\partial_1\chi_R + \varepsilon\nu_2\partial_2\chi_R = 0\,, \end{cases} \tag{49}$$

where $\gamma' = (\gamma_1,\gamma_2)^\top$ and $\nu = (\nu_1,\nu_2)^\top = (-\gamma_2',\gamma_1')^\top$. Then, (49) can be rewritten as

$$\begin{pmatrix} \gamma_1'(1-\varepsilon t\kappa) & \gamma_2'(1-\varepsilon t\kappa) \\ -\varepsilon\gamma_2' & \varepsilon\gamma_1' \end{pmatrix}\begin{pmatrix} \partial_1\chi_R \\ \partial_2\chi_R \end{pmatrix} = \begin{pmatrix} R^{-1}\theta'(R^{-1}s) \\ 0 \end{pmatrix}, \tag{50}$$

so that, inverting the matrix in (50) and after straightforward computations one finds for $x = \gamma(s)+\varepsilon t\nu(s)$:

$$\nabla\chi_R(x) = \nabla\chi_R(\gamma(s)+\varepsilon t\nu(s)) = \frac{\theta'(R^{-1}s)}{R(1-\varepsilon t\kappa(s))}\gamma'(s). \tag{51}$$

Take $u = (u_1,u_2)^\top \in \operatorname{dom}(q_\infty)$. As chosen, we have $\chi_R u \in \operatorname{dom}\left(q_\infty^R\right)$. Thus, we find

$$q_\infty(\chi_R u) = q_\infty^R(\chi_R u). \tag{52}$$

On the other hand, we have

$$\begin{aligned} q_\infty(\chi_R u) = \sum_{k=1}^{2}\Big(&\underbrace{\|\chi_R\nabla u_k\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}}_{:=a_k} + \underbrace{\|u_k\nabla\chi_R\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}}_{:=b_k} \\ &+ \underbrace{2\Re\big(\langle\chi_R\nabla u_k, u_k\nabla\chi_R\rangle_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}\big)}_{:=c_k}\Big). \end{aligned} \tag{53}$$

Let $k\in\{1,2\}$, we get

$$a_k \le \|\nabla u_k\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}.$$

By (51), the second term b_k can be estimated as

$$b_k \le \frac{\|\theta'\|^2_{L^\infty(\mathbb{R})}}{R^2(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})^2}\|u_k\|^2_{L^2(\Omega_\varepsilon)}\,.$$

Similarly, we obtain

$$\begin{aligned} c_k &\le 2\|(\nabla\chi_R)u_k\|_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}\|\chi_R\nabla u_k\|_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} \\ &\le \frac{2\|\theta'\|_{L^\infty(\mathbb{R})}}{R(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})}\|u_k\|_{L^2(\Omega_\varepsilon)}\|\nabla u_k\|_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} \\ &\le \frac{\|\theta'\|_{L^\infty(\mathbb{R})}}{R(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})}(\|\nabla u_k\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)} + \|u_k\|^2_{L^2(\Omega_\varepsilon)}). \end{aligned}$$

Combining the above estimates with (52) and (53), we obtain that there exists $C > 0$ such that for all $R > 0$, there holds

$$q_\infty^R(\chi_R u) \leq \Big(1 + \frac{C}{R}\Big) q_\infty(u) + \frac{C}{R}\|u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}. \tag{54}$$

Now, by Definition 5, for $\eta > 0$, there exists $W_\eta \subset \mathrm{dom}\,(q_\infty)$ a j-th dimensional vector space such that

$$\mu_j(q_\infty) \leq \sup_{u\in W_\eta\setminus\{0\}} \frac{q_\infty(u)}{\|u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}} \leq \mu_j(q_\infty) + \eta. \tag{55}$$

Remark that if $(u_1^\eta, \dots, u_j^\eta)$ is an orthonormal basis of W_η, then there exists $R_0 := R_0(\eta) > 0$ such that for all $R > R_0$ the family $(\chi_R u_1^\eta, \dots, \chi_R u_j^\eta)$ is a basis in $L^2(\Omega_\varepsilon^R, \mathbb{C}^2)$ of the vector space $W_\eta^R := \{\chi_R u : u \in \mathrm{span}(u_1^\eta, \dots, u_j^\eta)\}$. Indeed, for all $k, p \in \{1, \dots, j\}$, there holds

$$\langle \chi_R u_k^\eta, \chi_R u_p^\eta\rangle_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)} = \delta_{k,p} - \int_{\Omega_\varepsilon^R} (1-\chi_R^2)\langle u_k^\eta, u_p^\eta\rangle_{\mathbb{C}^2}\mathrm{d}x.$$

Hence, by the dominated convergence theorem, the second term on the right-hand side of the above equation converges to 0 as $R \to +\infty$ and there exists $R_0 > 0$ such that for all $R > R_0$ there holds $\dim(W_\eta^R) = j$.

Now, pick a $u_\star \in W_\eta \setminus \{0\}$ such that

$$\frac{q_\infty^R(\chi_R u_\star)}{\|\chi_R u_\star\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)}} = \sup_{u\in W_\eta^R\setminus\{0\}} \frac{q_\infty^R(u)}{\|u\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)}} \geq \mu_j(q_\infty^R).$$

Consequently, as $W_\eta^R \subset \mathrm{dom}\,(q_\infty^R)$, the min–max principle (Proposition 6), (54) and (55) give

$$\mu_j(q_\infty^R)\frac{\|\chi_R u_\star\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)}}{\|u_\star\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}} \leq (1+\frac{C}{R})\frac{q_\infty(u_\star)}{\|u_\star\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}} + \frac{C}{R} \leq (1+\frac{C}{R})(\mu_j(q_\infty)+\eta) + \frac{C}{R}. \tag{56}$$

Observe that by dominated convergence, one also gets $\|\chi_R u_\star\|_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)} \to \|u_\star\|_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}$, as $R \to \infty$. Thus, letting $R \to \infty$ in (56), we obtain the inequality

$$\limsup_{R\to\infty} \mu_j(q_\infty^R) \leq \mu_j(q_\infty) + \eta\,.$$

As this is true for all $\eta > 0$, we get (48) and the proof is concluded. □

We conclude this paragraph with the following lemma.

Lemma 32. *Let us assume additionally that* Γ *is of class* C^4*, that* $\kappa'(s) \to 0$ *and* $\kappa''(s) \to 0$ *when* $|s| \to +\infty$ *and let* $R > 0$*. For all* $u \in \mathrm{dom}\,(q_m^{R,\pm})$*, there holds*

$$\mu_1(q_m^{R,\pm}) \geq \frac{E_1(m\varepsilon)}{\varepsilon^2} - \eta^\pm(R),$$

where $\eta^\pm \geq 0$ *does not depend on* m *and verifies* $\eta^\pm(R) \to 0$ *when* $R \to +\infty$*.*

Proof of Lemma 32. Let $u \in \operatorname{dom}\left(q_m^{R,\pm}\right)$ and consider u_0 its extension by 0 to the whole waveguide Ω_ε. Remark that $u_0 \in \operatorname{dom}(q_m)$ and set $v_0 = (U_3 U_2 U_1) u_0$ where the unitary maps U_1, U_2 and U_3 are defined in (19), (20) and (22), respectively. By Proposition 13, and using the min–max principle on the operator acting in the t-variable, we get

$$\begin{aligned} q_m^{R,\pm}(u) = q_m(u_0) \geq{}& \frac{E_1(m\varepsilon)}{\varepsilon^2}\|v_0\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} - \int_{\mathsf{Str}} \frac{\kappa^2}{4(1-\varepsilon t\kappa)^2}|v_0|^2 \mathrm{d}s\mathrm{d}t \\ & - \frac{5}{4}\int_{\mathsf{Str}} \frac{(\varepsilon t \kappa')^2}{(1-\varepsilon t\kappa)^4}|u|^2\mathrm{d}s\mathrm{d}t - \frac{1}{2}\int_{\mathsf{Str}} \frac{\varepsilon t\kappa''}{(1-\varepsilon t\kappa)^3}|u|^2 \mathrm{d}s\mathrm{d}t \\ ={}& \frac{E_1(m\varepsilon)}{\varepsilon^2}\|v_0\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} - \int_{\mathsf{Str}^{R,\pm}} \frac{\kappa^2}{4(1-\varepsilon t\kappa)^2}|v_0|^2 \mathrm{d}s\mathrm{d}t \\ & - \frac{5}{4}\int_{\mathsf{Str}^{R,\pm}} \frac{(\varepsilon t \kappa')^2}{(1-\varepsilon t\kappa)^4}|v_0|^2\mathrm{d}s\mathrm{d}t \\ & - \frac{1}{2}\int_{\mathsf{Str}^{R,\pm}} \frac{\varepsilon t\kappa''}{(1-\varepsilon t\kappa)^3}|v_0|^2 \mathrm{d}s\mathrm{d}t, \end{aligned}$$

where we have taken into account that v_0 is supported in $\mathsf{Str}^{R,\pm} := \{(s,t) \in \mathbb{R}^2 : \pm s > R, t \in (-1,1)\}$. This last equality gives

$$q_m^{R,\pm}(u) \geq \frac{E_1(m\varepsilon)}{\varepsilon^2}\|u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} - \eta^\pm(R)\|u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}$$

with

$$\begin{aligned} &\eta^\pm(R) \\ &:= \sup_{\{\pm s > R\}} \left\{ \frac{\kappa^2(s)}{4(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})} + \frac{5}{4}\frac{\varepsilon^2\kappa'(s)^2}{(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})^4} + \frac{1}{2}\frac{\varepsilon|\kappa''(s)|}{(1-\varepsilon\|\kappa\|_{L^\infty(\mathbb{R})})^3} \right)\Big\}. \end{aligned}$$

By (A) and by the additional assumptions on κ' and κ'', we get $\eta^\pm(R) \to 0$ when $R \to +\infty$ and the Lemma is proved applying the min–max principle (Proposition 6). □

5.3. Convergence of Min–Max Values for $m \to +\infty$

Combining the results of the previous paragraph, we can prove the convergence of the min–max values in the large mass limit. This proof relies on the well-established IMS formula.

Proof of Theorem 9. In this proof, we assume that Γ is of class C^4, $\kappa'(s) \to 0$ and $\kappa''(s) \to 0$ when $|s| \to +\infty$.

Consider a partition of unity given by cut-off functions $\theta_1, \theta_2, \theta_3 \in C^\infty(\mathbb{R})$, with $0 \leq \theta_k \leq 1$, $k = 1, 2, 3$, and such that $\theta_1^2 + \theta_2^2 + \theta_3^2 = 1$. We also assume that

$$\begin{cases} \theta_1(s) = 0 & \text{if } s \geq -\dfrac{1}{2}\,, \\ \theta_2(s) = 0 & \text{if } s \leq \dfrac{1}{2}\,, \\ \theta_3(s) = 0 & \text{if } |s| \geq 1\,. \end{cases}$$

Recall that U_1 is the unitary map defined in (19) and for $k \in \{1,2,3\}$, define

$$\chi_{k,R} := (U_1^{-1}\theta_{k,R}),$$

where for $s \in \mathbb{R}$ we have set $\theta_{k,R}(s) := \theta_k(R^{-1}s)$. In particular, arguing as in (51), we get for all $x = \gamma(s) + t\varepsilon\nu(s) \in \Omega_\varepsilon$:

$$\nabla\chi_{k,R}(x) = \frac{\theta'_{k,R}(R^{-1}s)}{R(1-\varepsilon t\kappa)}\gamma'(s). \tag{57}$$

Let $u = (u_1, u_2)^\top \in \operatorname{dom}(q_m)$, then by Lemma 29 and the fact that $\chi_{1,R}^2 + \chi_{2,R}^2 + \chi_{3,R}^2 = 1$, we have

$$q_m(u) = \sum_{k=1}^{3}\left(\int_{\Omega_\varepsilon}|\chi_{k,R}\nabla u|^2\,\mathrm{d}x + \int_{\partial\Omega_\varepsilon}(m - \frac{\kappa_\varepsilon}{2})|\chi_{k,R}u|^2\,\mathrm{d}s\right). \tag{58}$$

Let us rewrite the first integral in (58). We have

$$\begin{aligned}\int_{\Omega_\varepsilon}|\chi_{k,R}\nabla u|^2\,\mathrm{d}x &= \sum_{j=1}^{2}\int_{\Omega_\varepsilon}|\nabla(\chi_{k,R}u_j) - u_j\nabla\chi_{k,R}|^2\,\mathrm{d}x\\ &= \sum_{j=1}^{2}\Bigg\{\int_{\Omega_\varepsilon}|\nabla(\chi_{k,R}u_j)|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon}|u_j|^2|\nabla\chi_{k,R}|^2\,\mathrm{d}x\\ &\qquad - 2\Re\left(\int_{\Omega_\varepsilon}\langle\nabla(\chi_{k,R}u_j), u_j\nabla\chi_{k,R}\rangle\mathrm{d}x\right)\Bigg\}.\end{aligned}$$

Moreover for $j \in \{1,2\}$, there holds

$$\begin{aligned}2\Re\left(\int_{\Omega_\varepsilon}\langle\nabla(\chi_{k,R}u_j), u_j\nabla\chi_{k,R}\rangle\mathrm{d}x\right) =& 2\int_{\Omega_\varepsilon}|u_j|^2|\nabla\chi_{k,R}|^2\,\mathrm{d}x\\ &+ \frac{1}{2}\int_{\Omega_\varepsilon}\langle\nabla(\chi_{k,R}^2), \nabla(|u_j|^2)\rangle\mathrm{d}x\,.\end{aligned}$$

Recall that $\sum_{k=1}^{3}\chi_{k,R}^2 = 1$, so that, summing up with respect to $k \in \{1,2,3\}$, the last term in the above formula vanishes. Thus, we find the following IMS formula :

$$q_m(u) = q_m^{\frac{R}{2},-}(\chi_{1,R}u) + q_m^{\frac{R}{2},+}(\chi_{2,R}u) + q_m^{R}(\chi_{3,R}u) - \int_{\Omega_\varepsilon}W_R|u|^2\,\mathrm{d}x\,, \tag{59}$$

where $W_R := \sum_{k=1}^{3}|\nabla\chi_{k,R}|^2$ and $\|W_R\|_{L^\infty(\Omega_\varepsilon)} \le \frac{C}{R^2}$, for some constant $C > 0$, by (57).

Now, fix $j \in \mathbb{N}$ and consider the isometry

$$\mathcal{I} : L^2(\Omega_\varepsilon, \mathbb{C}^2) \to L^2(\Omega_\varepsilon^{\frac{R}{2},-}, \mathbb{C}^2) \times L^2(\Omega_\varepsilon^{\frac{R}{2},-}, \mathbb{C}^2) \times L^2(\Omega_\varepsilon^{R}, \mathbb{C}^2)$$

defined by $\mathcal{I}u = (\chi_{1,R}u, \chi_{2,R}u, \chi_{3,R}u)$. Let $W \subset \operatorname{dom}(q_m)$ be a vector space of dimension j, by (59), there holds

$$\left(\sup_{u\in W\setminus\{0\}} \frac{q_m(u)}{\|u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}}\right) + \frac{C}{R^2}$$

$$\geq \sup_{v=(v_1,v_2,v_3)\in(\mathcal{I}W)\setminus\{0\}} \frac{q_m^{\frac{R}{2},-}(v_1) + q_m^{\frac{R}{2},+}(v_2) + q_m^R(v_3)}{\|v_1\|^2_{L^2(\Omega_\varepsilon^{\frac{R}{2},-},\mathbb{C}^2)} + \|v_2\|^2_{L^2(\Omega_\varepsilon^{\frac{R}{2},+},\mathbb{C}^2)} + \|v_3\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)}}.$$

As $\mathcal{I}$ is an isometry, we get $\dim(\mathcal{I}W) = j$ and by definition of the cut-off functions $\chi_{k,R}$ $(k \in \{1,2,3\})$, we also have $(\mathcal{I}W) \subset \mathfrak{D} := \operatorname{dom}\left(q_m^{\frac{R}{2},-}\right) \times \operatorname{dom}\left(q_m^{\frac{R}{2},+}\right) \times \operatorname{dom}\left(q_m^R\right)$. In particular, there holds

$$\left(\sup_{u\in W\setminus\{0\}} \frac{q_m(u)}{\|u\|^2_{L^2(\Omega_\varepsilon,\mathbb{C}^2)}}\right) + \frac{C}{R^2}$$

$$\geq \inf_{\substack{V\subset\mathfrak{D}\\ \dim(V)=j}} \sup_{v=(v_1,v_2,v_3)\in V\setminus\{0\}} \frac{q_m^{\frac{R}{2},-}(v_1) + q_m^{\frac{R}{2},+}(v_2) + q_m^R(v_3)}{\|v_1\|^2_{L^2(\Omega_\varepsilon^{\frac{R}{2},-},\mathbb{C}^2)} + \|v_2\|^2_{L^2(\Omega_\varepsilon^{\frac{R}{2},+},\mathbb{C}^2)} + \|v_3\|^2_{L^2(\Omega_\varepsilon^R,\mathbb{C}^2)}}.$$

Now, taking the infimum over all vector spaces $W \subset \operatorname{dom}(q_m)$ of dimension j and noting that the right-hand side is the j-th min–max value of the quadratic form of the tensor product of the three self-adjoint operators associated with the quadratic forms $q_m^{\frac{R}{2},-}$, $q_m^{\frac{R}{2},+}$ and q_m^R, respectively, the min–max principle (Proposition 6) yields:

$$\mu_j(q_m) + \frac{C}{R^2} \geq \; j\text{-th smallest element of the set}$$
$$\{\mu_j(q_m^R)\}_{j\in\mathbb{N}} \bigcup \{\mu_j(q_m^{\frac{R}{2},+})\}_{j\in\mathbb{N}} \bigcup \{\mu_j(q_m^{\frac{R}{2},-})\}_{j\in\mathbb{N}}.$$

First, remark that by the min–max principle for all $j \in \mathbb{N}$, $m \mapsto \mu_j(q_m)$ is a non-decreasing function on $[0,+\infty)$ and such that $\mu_j(q_m) \leq \mu_j(q_\infty)$. In particular, $\mu_j(q_m)$ has a limit when $m \to +\infty$.

Now, pick $j_0 \in \mathbb{N}$ such that $j_0 < N_\Gamma + 1$ (with the convention that $N_\Gamma + 1 = +\infty$ if $N_\Gamma = +\infty$). Recall that by Proposition 7 $\mu_j(q_\infty) < \frac{\pi^2}{4\varepsilon^2}$ for all $j \in \{1,\dots,2j_0\}$. For all $k \in \mathbb{N}$, by Lemma 32, there holds

$$\mu_k\left(q_m^{\frac{R}{2},\pm}\right) \geq \mu_1\left(q_m^{\frac{R}{2},\pm}\right) \geq \frac{E_1(m\varepsilon)}{\varepsilon^2} - \eta^\pm(R),$$

and $\eta^\pm$ does not depend on m and $\eta^\pm(R) \to 0$ when $R \to +\infty$. In particular, if one fixes $\alpha > 0$, there exists $R_0 > 0$ such that for all $R > R_0$, there holds $\eta^\pm(R) < \frac{\alpha}{2}$. Now, using (v) of Proposition 10, there exists $m_0 > 0$ such that for all $m > m_0$, there holds $\frac{E_1(m\varepsilon)}{\varepsilon^2} \geq \frac{\pi^2}{4\varepsilon^2} - \frac{\alpha}{2}$. Choosing $\alpha = \frac{1}{4}\big(\frac{\pi^2}{4\varepsilon^2} - \mu_{2j_0}(q_\infty)\big)$ it gives

$$\mu_1(q_m^{\frac{R}{2},\pm}) \geq \frac{\pi^2}{4\varepsilon^2} - \frac{1}{4}\left(\frac{\pi^2}{4\varepsilon^2} - \mu_{2j_0}(q_\infty)\right), \tag{60}$$

and by Lemma 31, there exists $m_1 > 0$ such that for all $m \geq m_1$, there holds

$$\mu_j(q_m^R) \leq \mu_j(q_\infty^R) \leq \mu_j(q_\infty) + \frac{1}{4}\left(\frac{\pi^2}{4\varepsilon^2} - \mu_{2j_0}(q_\infty)\right) \leq \mu_{2j_0}(q_\infty)$$

$$+\frac{1}{4}\big(\frac{\pi^2}{4\varepsilon^2}-\mu_{2j_0}(q_\infty)\big). \tag{61}$$

As there holds

$$\mu_{2j_0}(q_\infty)+\frac{1}{4}\left(\frac{\pi^2}{4\varepsilon^2}-\mu_{2j_0}(q_\infty)\right)<\frac{\pi^2}{4\varepsilon^2}-\frac{1}{4}\left(\frac{\pi^2}{4\varepsilon^2}-\mu_{2j_0}(q_\infty)\right),$$

(60) and (61) give that for all $m>\max(m_0,m_1)$ and all $R>R_0$, there holds

$$\mu_j(q_m)+\frac{C}{R^2}\geq\mu_j(q_m^R).$$

Hence, taking the limit $m\to+\infty$ then $R\to+\infty$ in the last equation, by Lemma 30 and Lemma 31, we obtain

$$\lim_{m\to+\infty}\mu_j(q_m)\geq\mu_j\left(q_\infty\right).$$

In particular, if $N_\Gamma=+\infty$, the proof is completed. Now assume that $N_\Gamma<+\infty$ and let $j\geq 2N_\Gamma+1$. Let us prove that $\mu_j(q_m)$ converges to $\frac{\pi^2}{4\varepsilon^2}$. By Proposition 6 and Proposition 7, there holds

$$\mu_j(q_m)\leq\mu_j(q_\infty)=\frac{\pi^2}{4\varepsilon^2}.$$

In particular, let us consider the j-th smallest element of the set

$$\{\mu_j(q_m^R)\}_{j\in\mathbb{N}}\bigcup\{\mu_j(q_m^{\frac{R}{2},+})\}_{j\in\mathbb{N}}\bigcup\{\mu_j(q_m^{\frac{R}{2},-})\}_{j\in\mathbb{N}}.$$

Either there exists $k_0\geq 2N_\Gamma+1$ such that this element is $\mu_{k_0}(q_m^R)$ or $p_0\in\mathbb{N}$ such that this element is $\mu_{p_0}(q_m^{\frac{R}{2},\pm})$. In the first case, there holds:

$$-\frac{C}{R^2}\leq\frac{\pi^2}{4\varepsilon^2}-(\mu_j(q_m)+\frac{C}{R^2})\leq\frac{\pi^2}{4\varepsilon^2}-\mu_{k_0}(q_m^R)\leq\frac{\pi^2}{4\varepsilon^2}-\mu_{2N_\Gamma+1}(q_m^R).$$

Now, in the second case, there holds

$$\begin{aligned}-\frac{C}{R^2}\leq\frac{\pi^2}{4\varepsilon^2}-(\mu_j(q_m)+\frac{C}{R}^2)\leq\frac{\pi^2}{4\varepsilon^2}-\mu_{p_0}(q_m^{\frac{R}{2},\pm})&\leq\frac{\pi^2}{4\varepsilon^2}-\mu_1(q_m^{\frac{R}{2},\pm})\\&\leq\frac{\pi^2}{4\varepsilon^2}-\frac{E_1(m\varepsilon)}{\varepsilon^2}+\eta^\pm(R),\end{aligned}$$

where we have used Lemma 32. These two inequalities yield

$$\begin{aligned}-\frac{C}{R^2}&\leq\frac{\pi^2}{4\varepsilon^2}-(\mu_j(q_m)+\frac{C}{R}^2)\\&\leq\min\Big(\frac{\pi^2}{4\varepsilon^2}-\mu_{2N_\Gamma+1}(q_m^R),\frac{\pi^2}{4\varepsilon^2}-\frac{E_1(m\varepsilon)}{\varepsilon^2}+\eta^\pm(R)\Big)\end{aligned}$$

Now, taking the limit $m\to+\infty$ and then $R\to+\infty$ by Lemma 30, Lemma 31, (v) of Proposition 10 and Lemma 32, we get

$$\lim_{m\to+\infty}\mu_j(q_m)=\frac{\pi^2}{4\varepsilon^2}$$

and Theorem 9 is proved. ☐

6. A Quantitative Condition for the Existence of Bound States

The goal of this section is to obtain an explicit geometric condition on the curvature of the base curve Γ which ensures that the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ has at least two bound states.

To state it, whenever Γ is of class C^4, we introduce the well-known geometric potential (cf. [15, Eq. (3.9)])

$$V_\varepsilon(s,t) := -\frac{1}{4}\,\frac{\kappa(s)^2}{(1-\varepsilon t\kappa(s))^2} - \frac{1}{2}\,\frac{\kappa''(s)\,\varepsilon t}{(1-\varepsilon t\kappa(s))^3} - \frac{5}{4}\,\frac{\kappa'(s)^2\,\varepsilon^2 t^2}{(1-\varepsilon t\kappa(s))^4}\,.$$

It depends on the geometry of the waveguide Ω_ε through the curvature κ of the base curve Γ, its two derivatives and the radius ε of the tubular neighbourhood.

The sufficient condition we obtain reads as follows.

Proposition 33. (Quantitative existence of bound states) *Let us assume additionally that Γ is of class C^4 and that* $\operatorname{supp}\kappa \subset (-L, L)$ *with $L > 0$. If*

$$I_\varepsilon := -\int_{\mathbb{R}}\int_{-1}^{1} V_\varepsilon(s,t)\,\cos^2\Big(\frac{\pi}{2}\,t\Big)\,\mathrm{d}t\,\mathrm{d}s > 0\,, \tag{62}$$

then there exists $m_0 \in \mathbb{R}$ such that for every $m > m_0$,

$$\mathrm{Sp}_{\mathrm{dis}}(\mathcal{D}_\Gamma(\varepsilon, m)) \neq \emptyset. \tag{63}$$

Moreover, there holds

$$m_0 \leq \frac{1}{2\varepsilon}\Big[\frac{1}{I_\varepsilon^2}\Big(\frac{4\pi^2 L}{3\varepsilon^2} + \frac{2}{L}\Big)^2 - 1\Big] \tag{64}$$

Remark that if (63) holds, due to charge conjugation symmetry, we have $\#Sp_{dis}(\mathcal{D}_\Gamma(\varepsilon, m)) \geq 2$.

Note that the integral I_ε is independent of m. Since $V_\varepsilon(s,t) \to -\frac{1}{4}\kappa(s)^2$ as $\varepsilon \to 0$, uniformly in $(s,t) \in \mathbb{R} \times (-1,1)$, the sufficient condition (62) is always satisfied whenever the curvature κ is not identically equal to zero and ε is small enough.

Compared to Proposition 8, Proposition 33 gives a quantitative geometric bound control on m_0 to obtain the existence of bound states.

We work with the square of the operator $\mathcal{D}_\Gamma(\varepsilon, m)$ studying the min–max value $\mu_1(\mathcal{D}_\Gamma(\varepsilon, m)^2)$ following the notation introduced in Definition 5. The main idea is that thanks to Proposition 3 and Proposition 13, we have

$$\mu_1(\mathcal{D}_\Gamma(\varepsilon, m)^2) = \inf_{u \in \mathrm{dom}(\mathcal{E}_\Gamma(\varepsilon,m))\setminus\{0\}} \frac{\|\mathcal{E}_\Gamma(\varepsilon, m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}}{\|u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}}. \tag{65}$$

Proof of Proposition 33. In view of (65) and the symmetry of the spectrum of $\mathcal{D}_\Gamma(\varepsilon, m)$ (see Theorem 2), it is enough to find a test function $u \in \mathrm{dom}\,(\mathcal{E}_\Gamma(\varepsilon, m))$ such that

$$q(u) := \|\mathcal{E}_\Gamma(\varepsilon, m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} - \big(m^2 + \varepsilon^{-2}E_1(m\varepsilon)\big)\,\|u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} < 0\,, \tag{66}$$

with $\mathrm{dom}\,(q) := \mathrm{dom}\,(\mathcal{E}_\Gamma(\varepsilon, m))$. Then, necessarily we have $\mu_1(q) < 0$.

Fix $\eta > 0$, and define

$$u_\eta(s,t) := \frac{1}{\sqrt{2}}\varphi_\eta(s)\cos\left(\frac{\pi}{2}t\right)\begin{pmatrix} e^{i\frac{\theta(s)}{2}} \\ e^{-i\frac{\theta(s)}{2}} \end{pmatrix},$$

where θ is defined in (21) and, for every $\eta \in \mathbb{R}$,

$$\varphi_\eta(s) := \begin{cases} 1 & \text{if} \quad |s| \le \eta\,, \\ \dfrac{2\eta - |s|}{\eta} & \text{if} \quad \eta < |s| < 2\eta\,, \\ 0 & \text{if} \quad |s| \ge 2\eta\,. \end{cases}$$

Remark that $u_\eta \in H_0^1(\mathsf{Str}, \mathbb{C}^2) \subset \operatorname{dom}(q)$ and $\|u_\eta\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} = \|\varphi_\eta\|^2_{L^2(\mathbb{R})} = \frac{8}{3}\eta$. Using the boundary condition, one easily checks the identity

$$\frac{1}{\varepsilon^2}\int_{\mathsf{Str}} |\partial_t u_\eta(s,t)|^2\, \mathrm{d}s\, dt + \varepsilon m \int_{\mathbb{R}} |u_\eta(s,-1)|^2\, \mathrm{d}s + \varepsilon m \int_{\mathbb{R}} |u_\eta(s,1)|^2\, \mathrm{d}s = \frac{\pi^2}{4\varepsilon^2}\|u_\eta\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\,.$$

Consequently, there holds

$$\begin{aligned} q(u_\eta) =& \varepsilon^{-2}\Big(\frac{\pi^2}{4} - E_1(m\varepsilon)\Big)\|u_\eta\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \int_{\mathsf{Str}} \frac{|(\partial_s - i\frac{\kappa}{2}\sigma_3)u_\eta(s,t)|^2}{(1-\varepsilon t\kappa(s))^2}\, \mathrm{d}s\, \mathrm{d}t \\ &+ \int_{\mathsf{Str}} V_\varepsilon(s,t)\, |u_\eta(s,t)|^2\, \mathrm{d}s\, \mathrm{d}t\,. \end{aligned} \tag{67}$$

To deal with the second term on the right-hand side of (67), we set $v_\eta := e^{-i\frac{\theta}{2}\sigma_3}u_\eta$ and remark that for all $(s,t) \in \mathsf{Str}$ there holds

$$v_\eta(s,t) = \frac{1}{\sqrt{2}}\varphi_\eta(s)\cos\left(\frac{\pi}{2}t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \|v_\eta(s,t)\|_{\mathbb{C}^2} = \|u_\eta(s,t)\|_{\mathbb{C}^2}.$$

In particular, we remark that

$$e^{-i\frac{\theta}{2}\sigma_3}(\partial_s - i\frac{\kappa}{2}\sigma_3)u_\eta(s,t) = (\partial_s v_\eta)(s,t) = \frac{1}{\sqrt{2}}\varphi'_\eta(s)\cos\left(\frac{\pi}{2}t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Consequently, we obtain

$$\begin{aligned} q(u_\eta) &= \varepsilon^{-2}\Big(\frac{\pi^2}{4} - E_1(m\varepsilon)\Big)\|\varphi_\eta\|^2_{L^2(\mathbb{R})} \\ &\quad + \int_{\mathbb{R}} |\varphi'_\eta(s)|^2 \int_{-1}^{1} \frac{1}{(1-\kappa(s)\,\varepsilon t)^2}\, \cos^2\left(\frac{\pi}{2}\,t\right)\, \mathrm{d}t\, \mathrm{d}s \\ &\quad + \int_{\mathbb{R}} |\varphi_\eta(s)|^2\, V_\varepsilon(s,t)\, \cos^2\left(\frac{\pi}{2}\,t\right)\, \mathrm{d}t\, \mathrm{d}s. \end{aligned} \tag{68}$$

Now, we employ the hypothesis that the curvature κ (and therefore also its derivatives κ' and κ'') is compactly supported and choose $\eta \ge L$. Then, the last line equals $-I_\varepsilon$ and the second line equals $\|\varphi'_\eta\|^2_{L^2(\mathbb{R})} = \frac{2}{\eta}$. In summary,

$$q(u_\eta) = \varepsilon^{-2}\Big(\frac{\pi^2}{4} - E_1(m\varepsilon)\Big)\frac{8}{3}\eta + \frac{2}{\eta} - I_\varepsilon\,.$$

Using in (73) the elementary bound $\tan(x) \leq x - \pi$ valid for every $x \in (\frac{\pi}{2}, \pi]$, we get the estimate

$$\sqrt{E_1(m\varepsilon)} \geq \frac{\pi}{2}\frac{2m\varepsilon}{1+2m\varepsilon}\,.$$

Remark that this lower bound on $E_1(m\varepsilon)$ holds for all masses $m \geq 0$ but there is no reason for it to be optimal for small masses. Consequently, using the elementary inequality $(1+4m\varepsilon) \leq 2(1+2m\varepsilon)$, we get

$$q(u_\eta) \leq \frac{\pi^2}{4\varepsilon^2}\frac{1+4m\varepsilon}{(1+2m\varepsilon)^2}\frac{8}{3}\eta + \frac{2}{\eta} - I_\varepsilon \leq \frac{\pi^2}{2\varepsilon^2}\frac{1}{(1+2m\varepsilon)}\frac{8}{3}\eta + \frac{2}{\eta} - I_\varepsilon\,.$$

Setting $\eta := L\sqrt{1+2m\varepsilon} \geq L$, we find

$$q(u_\eta) \leq \left(\frac{4\pi^2 L}{3\varepsilon^2} + \frac{2}{L}\right)\frac{1}{\sqrt{1+2m\varepsilon}} - I_\varepsilon\,.$$

Therefore, if $I_\varepsilon > 0$, we see that $q(u_\eta)$ is negative whenever $m \geq \tilde{m}_0$, where $\tilde{m}_0$ coincides with the right-hand-side of (64). It concludes the proof of Proposition 33. □

Remark 34. The hypothesis that κ is compactly supported is apparently just a technical condition in order to simplify the expression (68).

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Appendix A. Proof of some technical results

In this section, we collect the proofs of some technical results stated in the paper, in order to simplify the overall presentation.

Proof of Proposition 10 and Corollary 11. The multiplication operators by σ_1 and σ_3 are bounded and self-adjoint in $L^2\big((-1,1),\mathbb{C}^2\big)$ thus $\mathcal{T}(k,m)$ is self-adjoint if and only if $\mathcal{T}_0$ is self-adjoint. An integration by parts easily yields that $\mathcal{T}_0$ is symmetric and by definition, one has

$$\mathrm{dom}\,(\mathcal{T}_0^*) = \Big\{u \in L^2\big((-1,1),\mathbb{C}^2\big) : \exists\, w \in L^2\big((-1,1),\mathbb{C}^2\big) \text{ such that}$$
$$\forall\, v \in \mathrm{dom}\,(\mathcal{T}_0)\,,\ \langle u, \mathcal{T}_0 v\rangle_{L^2((-1,1),\mathbb{C}^2)} = \langle w, v\rangle_{L^2((-1,1),\mathbb{C}^2)}\Big\}.$$

For every $v \in \mathcal{D} := C_0^\infty\big((-1,1),\mathbb{C}^2\big)$ and $u \in \mathrm{dom}\,(\mathcal{T}_0^*)$, there holds

$$\begin{aligned}\langle \mathcal{T}_0^* u, v\rangle_{L^2((-1,1),\mathbb{C}^2)} = \langle u, \mathcal{T}_0 v\rangle_{L^2((-1,1),\mathbb{C}^2)} &= \langle u, -i\sigma_2 v'\rangle_{L^2((-1,1),\mathbb{C}^2)}\\ &= \langle u, i\overline{\sigma_2 v'}\rangle_{\mathcal{D}',\mathcal{D}}\\ &= \langle -i\sigma_2 u', \overline{v}\rangle_{\mathcal{D}',\mathcal{D}}\\ &= \langle \mathcal{T}_0^* u, \overline{v}\rangle_{\mathcal{D}',\mathcal{D}},\end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathcal{D}',\mathcal{D}}$ is the duality bracket of distributions. In particular, we know that $\mathcal{T}_0^* u = -i\sigma_2 u' \in L^2\big((-1,1),\mathbb{C}^2\big)$ thus we get $u \in H^1\big((-1,1),\mathbb{C}^2\big)$. Moreover, if $v \in \mathrm{dom}\,(\mathcal{T}_0)$, there holds

$$\begin{aligned}\langle \mathcal{T}_0^* u, v\rangle_{L^2((-1,1),\mathbb{C}^2)} &= \langle -i\sigma_2 u', v\rangle_{L^2((-1,1),\mathbb{C}^2)}\\ &= \langle u, -i\sigma_2 v'\rangle_{L^2((-1,1),\mathbb{C}^2)} + \Big[\langle -i\sigma_2 u, v\rangle_{\mathbb{C}^2}\Big]_{-1}^{1}\\ &= \langle u, \mathcal{T}_0 v\rangle_{L^2((-1,1),\mathbb{C}^2)} - u_2(1)\overline{v_1}(1) + u_1(1)\overline{v_2}(1)\\ &\quad + u_2(-1)\overline{v_1}(-1) - u_1(-1)\overline{v_2}(-1).\end{aligned}$$

Since $v \in \mathrm{dom}\,(\mathcal{T}_0)$, we obtain

$$0 = -(u_2(1) + u_1(1))\overline{v_1}(1) + (u_2(-1) - u_1(-1))\overline{v_1}(-1).$$

This holds for any $v \in \mathrm{dom}\,(\mathcal{T}_0)$, so that $u_2(\pm 1) = \mp u_1(\pm 1)$ and $v \in \mathrm{dom}\,(\mathcal{T}_0)$. In particular $\mathcal{T}_0^* = \mathcal{T}_0$. Observe that, by the closed graph theorem, $\mathrm{dom}\,(\mathcal{T}(k,m))$ is continuously embedded in $H^1\big((-1,1),\mathbb{C}^2\big)$ which itself is compactly embedded in $L^2\big((-1,1),\mathbb{C}^2\big)$. Thus, $\mathcal{T}(k,m)$ has compact resolvent.

Let us prove Point (i) by picking $u \in \mathrm{dom}\,(\mathcal{T}(k,m))$ and considering

$$\begin{aligned}\|\mathcal{T}(k,m)u\|^2 = \|u'\|^2_{L^2((-1,1),\mathbb{C}^2)} &+ (m^2+k^2)\|u\|^2_{L^2((-1,1),\mathbb{C}^2)}\\ &+ 2mk\Re\big(\langle \sigma_3 u, \sigma_1 u\rangle_{L^2((-1,1),\mathbb{C}^2)}\big)\\ &+ 2m\Re\big(\langle -i\sigma_2 u', \sigma_3 u\rangle_{L^2((-1,1),\mathbb{C}^2)}\big)\\ &+ 2k\Re\big(\langle -i\sigma_2 u', \sigma_1 u\rangle_{L^2((-1,1),\mathbb{C}^2)}\big).\end{aligned} \tag{69}$$

We rewrite (69), arguing as follows. Using the anti-commutation rules of Pauli matrices and the boundary condition, we get

$$2\Re\big(\langle \sigma_3 u, \sigma_1 u\rangle_{L^2((-1,1),\mathbb{C}^2)}\big) = 2\Re\big(\langle -i\sigma_2 u', \sigma_1 u\rangle_{L^2((-1,1),\mathbb{C}^2)}\big) = 0$$

and

$$2\Re\big(\langle -i\sigma_2 u', \sigma_3 u\rangle_{L^2((-1,1),\mathbb{C}^2)}\big) = \|u(1)\|^2_{\mathbb{C}^2} + \|u(-1)\|^2_{\mathbb{C}^2}. \tag{70}$$

In particular, we obtain

$$\begin{aligned}\|\mathcal{T}(k,m)u\|^2_{L^2((-1,1),\mathbb{C}^2)} &= \|u'\|^2_{L^2((-1,1),\mathbb{C}^2)} + (m^2+k^2)\|u\|^2_{L^2((-1,1),\mathbb{C}^2)}\\ &\quad + m(\|u(1)\|^2_{\mathbb{C}^2} + \|u(-1)\|^2_{\mathbb{C}^2})\\ &\ge (m^2+k^2)\|u\|^2_{L^2((-1,1),\mathbb{C}^2)}.\end{aligned}$$

Hence, by the min–max principle (see Proposition 6), if $\lambda \in Sp(\mathcal{T}(k,m))$, we get $|\lambda| \ge \sqrt{m^2+k^2}$. Moreover, the last inequality is strict. Indeed, if u is an eigenfunction of $\mathcal{T}(k,m)$ associated with an eigenvalue λ such that $|\lambda| = \sqrt{m^2+k^2}$ we necessarily get that u is a constant $\mathbb{C}^2$-valued function on $(-1,1)$ satisfying the boundary conditions given in (15). It is a contradiction because it implies that $u = 0$ identically. Hence, $\mathrm{Sp}(\mathcal{T}(k,m)) \cap [-\sqrt{m^2+k^2}, \sqrt{m^2+k^2}] = \emptyset$ and Point (i) is proved.

Now, let $\lambda \in \mathrm{Sp}(\mathcal{T}(k,m))$ and pick an associated eigenfunction $u = (u_1, u_2)^\top \in \mathrm{dom}\,(\mathcal{T}(k,m))$. There holds

$$\begin{cases} mu_1 + ku_2 - u_2' = \lambda u_1\,, \\ ku_1 + u_1' - mu_2 = \lambda u_2\,. \end{cases} \tag{71}$$

The second equation gives $(m+\lambda)u_2 = ku_1 + u_1'$ and multiplying the first line by $(\lambda + m)$, we get

$$-u_1'' = Eu_1\,, \qquad E := \lambda^2 - (m^2 + k^2)\,.$$

Recall that $m \geq 0$ and that by Point (i), we have $E > 0$ for all $k \in \mathbb{R}$. Thus, we find

$$u_1(t) = \alpha \cos\left(\sqrt{E}(t+1)\right) + \beta \sin\left(\sqrt{E}(t+1)\right),$$

for some constants $\alpha, \beta \in \mathbb{C}$ and as $m + \lambda \neq 0$ we get

$$u_2(t) = \frac{1}{\lambda+m} \cos\left(\sqrt{E}(t+1)\right)\left(k\alpha + \sqrt{E}\beta\right) + \frac{1}{\lambda+m} \sin\left(\sqrt{E}(t+1)\right)\left(k\beta - \sqrt{E}\alpha\right).$$

The boundary condition at $t = -1$ gives

$$(m + \lambda - k)\alpha - \sqrt{E}\beta = 0\,.$$

The boundary condition at $t = 1$ gives

$$\left((m+\lambda+k)\cos(2\sqrt{E}) - \sqrt{E}\sin(2\sqrt{E})\right)\alpha + \left((m+\lambda+k)\sin(2\sqrt{E}) + \sqrt{E}\cos(2\sqrt{E})\right)\beta = 0.$$

To obtain a nonzero eigenfunction u, there has to hold

$$0 = \begin{vmatrix} m+\lambda-k & -\sqrt{E} \\ (m+\lambda+k)\cos(2\sqrt{E}) - \sqrt{E}\sin(2\sqrt{E}) & (m+\lambda+k)\sin(2\sqrt{E}) + \sqrt{E}\cos(2\sqrt{E}) \end{vmatrix}.$$

Computing the determinant, we are left with the implicit equation

$$m \sin(2\sqrt{E}) + \sqrt{E}\cos(2\sqrt{E}) = 0. \tag{72}$$

In particular, it yields that the spectrum of $\mathcal{T}(k,m)$ is symmetric with respect to the origin and we remark that when $m = k = 0$, we necessarily have $\sqrt{E} = |\lambda| = p\frac{\pi}{4}$ (with $p \in \mathbb{N}$), and that in this case, a normalized eigenfunction associated with $\lambda = \pm\frac{\pi}{4}$ is given by

$$u_k^\pm(t) = \frac{1}{2}\cos\left(k\frac{\pi}{4}(t+1)\right)\begin{pmatrix}1\\1\end{pmatrix} \pm \frac{1}{2}\sin\left(k\frac{\pi}{4}(t+1)\right)\begin{pmatrix}1\\-1\end{pmatrix},$$

which proves Corollary 11.

Remark that for $m > 0$, a solution E to (72) verifies $\cos(2\sqrt{E}) \neq 0$, and we obtain

$$\tan(2\sqrt{E}) + \frac{\sqrt{E}}{m} = 0. \tag{73}$$

Now, for $p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, define the line segments $I_0 := [0, \frac{\pi}{2})$ and $I_{p+1} = ((2p+1)\frac{\pi}{2}, (2p+3)\frac{\pi}{2})$

$$g_p : I_p \to \mathbb{R}, \quad g_p(x) = \tan(2x) + \frac{x}{m}. \tag{74}$$

Remark that $g_p'(x) > 0$, and in particular, the only solution to $g_0(x) = 0$ is $x = 0$. For all $p \geq 1$, we have

$$\lim_{x \to (2p-1)\frac{\pi}{2}^+} g_p(x) = -\infty, \quad g_p(p\pi) = p\frac{\pi}{m} > 0.$$

In particular, for all $p \geq 1$, there is a unique solution $x_p \in I_p$ to $g_p(x) = 0$. Moreover, it satisfies $x_p \in \big((2p-1)\frac{\pi}{2}, p\pi\big)$. Hence, for $p \geq 1$, $E_p(m)$ is defined as the unique solution E to $g_p(2\sqrt{E}) = 0$. In particular, $E_p(m) \in ((2p-1)^2\frac{\pi^2}{16}, p^2\frac{\pi^2}{4})$ which proves Points (ii) and (iii).

Now, we prove (iv). Guided by (72), we define the C^∞ function

$$F : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (\mu, m) \mapsto 2m\sin(\mu) + \mu\cos(\mu) \end{cases}.$$

One remarks that $F(\frac{\pi}{2}, 0) = 0$ and $\partial_\mu F(\frac{\pi}{2}, 0) = \frac{\pi}{2}$. Hence, by the implicit function theorem, there exists $\delta_1, \delta_2 > 0$ and a C^∞ function $\mu : (-\delta_1, \delta_1) \to (\frac{\pi}{2} - \delta_2, \frac{\pi}{2} + \delta_2)$ verifying $\mu(0) = \frac{\pi}{2}$ and such that for all $|m| < \delta_1$, there holds $F(\mu(m), m) = 0$. Moreover, when $m \to 0$, there holds

$$\mu(m) = \mu(0) + \mu'(0)m + \mathcal{O}(m^2) = \frac{\pi}{2} + \frac{4}{\pi}m + \mathcal{O}(m^2).$$

Necessarily, for $m > 0$ sufficiently small, there holds $E_1(m) = \frac{1}{4}\mu(m)^2$. Hence, when $m \to 0$, there holds

$$E_1(m) = \frac{\pi^2}{16} + m + \mathcal{O}(m^2),$$

which is precisely Point (iv).

Finally, we prove (v). Once again, guided by (72), we define the C^∞ function

$$G : \begin{cases} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (\mu, \nu) \mapsto 2\sin(\mu) + \mu\nu\cos(\mu) \end{cases}.$$

One remarks that $G(\pi, 0) = 0$ and $\partial_\mu G(\pi, 0) = -2$. Hence, by the implicit function theorem, there exists $\delta_1, \delta_2 > 0$ and a C^∞ function $\mu : (-\delta_1, \delta_1) \to (\pi - \delta_2, \pi + \delta_2)$ verifying $\mu(0) = \pi$ and such that for all $|\nu| < \delta_1$ there holds $G(\mu(\nu), \nu) = 0$. Moreover, when $\nu \to 0$, there holds

$$\mu(\nu) = \mu(0) + \mu'(0)\nu + \mathcal{O}(\nu^2) = \pi - \frac{\pi}{2}\nu + \mathcal{O}(\nu^2).$$

Necessarily, for $m > 0$ sufficiently large, there holds $E_1(m) = \frac{1}{4}\mu(m^{-1})^2$. Hence, when $m \to +\infty$, there holds

$$E_1(m) = \frac{\pi^2}{4} - \frac{\pi^2}{4m} + O(m^{-2}),$$

which gives (v). □

Proof of Lemma 28. Let $u \in \mathrm{dom}\,(\mathcal{E}_0(\varepsilon, m))$ and remark that there holds

$$\begin{aligned}
\|\mathcal{C}(\varepsilon,m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} &= \|(-i\sigma_1)\partial_s u + m\sigma_3 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \\
&\quad + \frac{1}{\varepsilon^2}\underbrace{\|(-i\sigma_2)\partial_t u - \frac{\pi}{4}(P^+ - P^-)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}}_{:=A} \\
&\quad + \frac{1}{\varepsilon}\underbrace{2\Re(\langle(-i\sigma_1)\partial_s u, (-i\sigma_2)\partial_t u - \frac{\pi}{4}(P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)})}_{:=B} \\
&\quad + \frac{m}{\varepsilon}\underbrace{2\Re(\langle\sigma_3 u, (-i\sigma_2)\partial_t u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)})}_{:=C} \\
&\quad - \frac{m\pi}{4\varepsilon}\underbrace{2\Re(\langle\sigma_3 u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)})}_{:=D}.
\end{aligned} \tag{75}$$

Now, we deal with each term appearing on the right-hand side of (75). For further use, for all $k \geq 1$, we set $f_k^{\pm} := \langle u, u_k^{\pm}\rangle_{L^2((-1,1),\mathbb{C}^2)}$ and recall that Π_k denotes the projector defined in (25). In particular, for all $k \geq 1$, there holds

$$\|\Pi_k u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} = \int_{\mathbb{R}} \Big(|f_k^+(s)|^2 + |f_k^-(s)|^2\Big)\mathrm{d}s.$$

Step 1. In this step, we analyze the term A appearing in (75). We remark that

$$(-i\sigma_2\partial_t - \frac{\pi}{4}(P^+ - P^-))u = \sum_{k\geq 2}\frac{(k-1)\pi}{4}(f_k^+ u_k^+ - f_k^- u_k^-). \tag{76}$$

In particular, it gives

$$A = \frac{\pi^2}{16}\sum_{k\geq 2}(k-1)^2\|\Pi_k u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}. \tag{77}$$

Step 2. A straightforward computation gives

$$-i\sigma_1\partial_s u = \sum_{k\geq 1} -i(f_k^-)' u_k^+ - i(f_k^+)' u_k^-.$$

In particular, using (76), there holds

$$\begin{aligned}
&\langle -i\sigma_1\partial_s u, \big(-i\sigma_2\partial_t - \frac{\pi}{4}(P^+ - P^-)\big)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)} \\
&= \frac{\pi}{4}\sum_{k\geq 2}(k-1)\big(-i\int_{\mathbb{R}}(f_k^-)'(s)\overline{f_k^+(s)}\mathrm{d}s + i\int_{\mathbb{R}}(f_k^+)'(s)\overline{f_k^-(s)}\mathrm{d}s\big).
\end{aligned} \tag{78}$$

Integrating by parts, we find

$$\begin{aligned}
&\overline{-i\int_{\mathbb{R}}(f_k^-)'(s)\overline{f_k^+(s)}ds + i\int_{\mathbb{R}}(f_k^+)'\overline{f_k^-(s)}\mathrm{d}s} \\
&\quad = i\int_{\mathbb{R}}\overline{(f_k^-)'(s)}f_k^+(s)\mathrm{d}s - i\int_{\mathbb{R}}\overline{(f_k^+)'(s)}f_k^-(s)\mathrm{d}s \\
&\quad = -\left(-i\int_{\mathbb{R}}(f_k^-)'(s)\overline{f_k^+(s)}ds + i\int_{\mathbb{R}}(f_k^+)'(s)\overline{f_k^-(s)}\mathrm{d}s\right),
\end{aligned}$$

and then using (78), we get

$$\begin{aligned}&\langle -i\sigma_1\partial_s u, \big(-i\sigma_2\partial_t - \frac{\pi}{4}(P^+ - P^-)\big)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}\\ &\quad = -\langle\big(-i\sigma_2\partial_t - \frac{\pi}{4}(P^+ - P^-)\big)u, -i\sigma_1\partial_s u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}.\end{aligned}$$

In particular, we obtain

$$B = 2\Re(\langle -i\sigma_1\partial_s u, \big(-i\sigma_2\partial_t - \frac{\pi}{4}(P^+ - P^-)\big)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}) = 0. \tag{79}$$

Step 3. In this step, we deal with the term C. Integrating by parts as in (70), we obtain:

$$C = \int_{\mathbb{R}} |u(s,1)|^2 + |u(s,-1)|^2 \mathrm{d}s. \tag{80}$$

Step 4 It remains to deal with the term D. To do so, we remark that:

$$\begin{aligned}\langle\sigma_3 u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)} &= \underbrace{\langle\Pi_1\sigma_3\Pi_1 u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}}_{:=\alpha}\\ &\quad + \underbrace{\langle\Pi_1^\perp\sigma_3\Pi_1^\perp u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}}_{:=\beta}\\ &\quad + \underbrace{\langle\Pi_1\sigma_3\Pi_1^\perp u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}}_{:=\gamma}\\ &\quad + \underbrace{\langle\Pi_1^\perp\sigma_3\Pi_1 u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}}_{:=\delta}.\end{aligned} \tag{81}$$

Now, in each of the next substep, we deal with the terms appearing on the right-hand side of (81).

Substep 4.1 Remark that there holds

$$\begin{aligned}\alpha &= \langle f_1^+\sigma_3 u_1^+ + f_1^-\sigma_3 u_1^-, f_1^+ u_1^+ - f_1^- u_1^-\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}\\ &= \langle\sigma_3 u_1^+, u_1^+\rangle_{L^2((-1,1),\mathbb{C}^2)}\|f_1^+\|^2_{L^2(\mathbb{R})} - \langle\sigma_3 u_1^-, u_1^-\rangle_{L^2((-1,1),\mathbb{C}^2)}\|f_1^-\|_{L^2(\mathbb{R})}\\ &\quad - \langle\sigma_3 u_1^+, u_1^-\rangle_{L^2((-1,1),\mathbb{C}^2)}\langle f_1^+, f_1^-\rangle_{L^2(\mathbb{R})}\\ &\quad + \langle\sigma_3 u_1^-, u_1^+\rangle_{L^2((-1,1),\mathbb{C}^2)}\langle f_1^-, f_1^+\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

Thanks to (31), we get

$$\alpha = \frac{2}{\pi}\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}. \tag{82}$$

Substep 4.2 We handle the term β by obtaining the following upper-bound thanks to the Cauchy–Schwarz inequality:

$$|\beta| = |\langle\Pi_1^\perp\sigma_3\Pi_1^\perp u, (P^+ - P^-)u\rangle_{L^2(\mathsf{Str},\mathbb{C}^2)}| \leq \|\Pi_1^\perp u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}. \tag{83}$$

Substep 4.3 Now, let us focus on the two off-diagonal terms γ and δ. A direct computation shows that

$$\langle\sigma_3 u_k^-, u_1^+\rangle_{\mathbb{C}^2} = -\langle\sigma_3 u_k^+, u_1^-\rangle_{\mathbb{C}^2}, \quad \langle\sigma_3 u_k^-, u_1^-\rangle_{\mathbb{C}^2} = -\langle\sigma_3 u_k^+, u_1^+\rangle_{\mathbb{C}^2}.$$

Then, we get

$$\Pi_1\sigma_3\Pi_1^{\perp}u = \left(\sum_{k\geq 2} a_k f_k^+ - b_k f_k^-\right) u_1^+ + \left(\sum_{k\geq 2} b_k f_k^+ - a_k f_k^-\right) u_1^-,$$

where we have set for $k \geq 2$

$$\begin{aligned} a_k &:= \langle \sigma_3 u_k^+, u_1^+\rangle_{L^2((-1,1),\mathbb{C}^2)} = \frac{4}{\pi}\frac{\sin^2(\frac{\pi}{4}(k+1))}{(k+1)}, \\ b_k &:= \langle \sigma_3 u_k^+, u_1^-\rangle_{L^2((-1,1),\mathbb{C}^2)} = \frac{4}{\pi}\frac{\sin^2(\frac{\pi}{4}(k-1))}{(k-1)}. \end{aligned} \tag{84}$$

Thus, we find

$$\gamma = \sum_{k\geq 2}\int_{\mathbb{R}} \langle (a_k - \sigma_1 b_k)\begin{pmatrix} f_k^+ \\ f_k^- \end{pmatrix}, \begin{pmatrix} f_1^+ \\ f_1^- \end{pmatrix}\rangle_{\mathbb{C}^2}\mathrm{d}s. \tag{85}$$

A similar computation gives

$$\delta = \sum_{k\geq 2}\int_{\mathbb{R}} \langle \begin{pmatrix} f_1^+ \\ f_1^- \end{pmatrix}, (a_k + \sigma_1 b_k)\begin{pmatrix} f_k^+ \\ f_k^- \end{pmatrix}\rangle_{\mathbb{C}^2}\mathrm{d}s. \tag{86}$$

In particular, using (85) and (86), we get

$$\begin{aligned} \gamma + \delta = 2\Re\Big(\sum_{k\geq 2} a_k \int_{\mathbb{R}} \langle \begin{pmatrix} f_1^+ \\ f_1^- \end{pmatrix}, \begin{pmatrix} f_k^+ \\ f_k^- \end{pmatrix}\rangle_{\mathbb{C}^2}\mathrm{d}s\Big) \\ + 2i\Im\Big(\sum_{k\geq 2} b_k \int_{\mathbb{R}} \langle \begin{pmatrix} f_1^+ \\ f_1^- \end{pmatrix}, \sigma_1\begin{pmatrix} f_k^+ \\ f_k^- \end{pmatrix}\rangle_{\mathbb{C}^2}\mathrm{d}s\Big). \end{aligned} \tag{87}$$

Using (81), (82) and (87), we obtain

$$D = \frac{4}{\pi}\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + 2\Re(\beta) + 4\Re\Big(\sum_{k\geq 2} a_k \int_{\mathbb{R}} \langle \begin{pmatrix} f_1^+ \\ f_1^- \end{pmatrix}, \begin{pmatrix} f_k^+ \\ f_k^- \end{pmatrix}\rangle_{\mathbb{C}^2}\mathrm{d}s\Big).$$

In particular, using the Cauchy–Schwartz inequality, we get

$$D \leq \frac{4}{\pi}\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + 2|\beta| + 4\sum_{k\geq 2}\Big(|a_k|\|\Pi_1 u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}\|\Pi_k u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}\Big). \tag{88}$$

Now, let us fix $c > 0$ to be chosen later. For all $a, b \in \mathbb{R}$ and $\varepsilon > 0$, we recall the elementary inequality $ab \leq \frac{c\varepsilon}{2}a^2 + \frac{1}{2c\varepsilon}b^2$ that we use to get for all $k \geq 2$:

$$|a_k|\|\Pi_1 u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}\|\Pi_k u\|_{L^2(\mathsf{Str},\mathbb{C}^2)} \leq \frac{c\varepsilon}{2}a_k^2\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{1}{2c\varepsilon}\|\Pi_k u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}.$$

Then, summing up for $k \geq 2$, we get

$$\begin{aligned} \sum_{k\geq 2}\Big(|a_k|\|\Pi_1 u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}\|\Pi_k u\|_{L^2(\mathsf{Str},\mathbb{C}^2)}\Big) \leq \frac{1}{2}c\varepsilon S\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \\ + \frac{1}{2c\varepsilon}\|\Pi_1^{\perp} u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}, \end{aligned} \tag{89}$$

where we have set $S = \sum_{k\geq 2} a_k^2 < +\infty$ because $a_k^2 = \mathcal{O}(k^{-2})$ when $k \to +\infty$ by (84). Taking into account (83) and (89), (88) gives

$$D \leq (\frac{4}{\pi} + 2cS\varepsilon)\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + 2(1+\frac{1}{c\varepsilon})\|\Pi_1^\perp u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}. \tag{90}$$

Step 5. In this step, we conclude the proof. Using (77), (79) and (80), (75) becomes

$$\begin{aligned}
&\|\mathcal{C}(\varepsilon,m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&\quad= \|(-i\sigma_1\partial_s + m\sigma_3)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{\pi^2}{16\varepsilon^2}\sum_{k\geq 2}(k-1)^2\|\Pi_k u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&\qquad + \frac{m}{\varepsilon}\int_{\mathbb{R}}\Big(|u(s,1)|^2 + |u(s,-1)|^2\Big)\mathrm{d}s - \frac{m\pi}{4\varepsilon}D\\
&\quad= \|(-i\sigma_1\partial_s + m\sigma_3)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{\pi^2}{16\varepsilon^2}\sum_{k\geq 2}(k-1)^2\|\Pi_k u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&\qquad + \frac{1}{\varepsilon^2}\int_{\mathbb{R}}(\tau_{m\varepsilon}(u)(s) - \tau_0(u)(s))\mathrm{d}s - \frac{m\pi}{4\varepsilon}D\\
&\quad= \|(-i\sigma_1\partial_s + m\sigma_3)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{\pi^2}{16\varepsilon^2}\sum_{k\geq 2}(k-1)^2\|\Pi_k u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&\qquad - \frac{\pi^2}{16\varepsilon^2}\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{1}{\varepsilon^2}\int_{\mathbb{R}}(\tau_{m\varepsilon}(u)(s) - \tau_0(\Pi_1^\perp u)(s))\mathrm{d}s - \frac{m\pi}{4\varepsilon}D,
\end{aligned}$$

where the quadratic forms $\tau_{\varepsilon m}$ and τ_0 are defined in (43). Notice that in the above formula, we used the fact that

$$\begin{aligned}
\tau_0(u)(s) &= \tau_0(\Pi_1 u)(s) + \tau_0(\Pi_1^\perp u)(s)\\
&= \frac{\pi^2}{16}\|(\Pi_1 u)(s)\|^2_{L^2(-1,1,\mathbb{C}^2)} + \tau_0(\Pi_1^\perp u)(s)\,, \qquad s\in\mathbb{R}
\end{aligned}$$

and

$$\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} = \int_{\mathbb{R}}\|(\Pi_1 u)(s)\|^2_{L^2((-1,1),\mathbb{C}^2)}\mathrm{d}s\,.$$

Using Lemma (26), this last inequality becomes

$$\begin{aligned}
\|\mathcal{C}(\varepsilon,m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \geq{}& \|(-i\sigma_1\partial_s + m\sigma_3)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \frac{\pi^2}{16\varepsilon^2}\|\Pi_1^\perp u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&+ \frac{1}{\varepsilon^2}\Big(E_1(m\varepsilon) - \frac{\pi^2}{16}\Big)\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} - \frac{m\pi}{4\varepsilon}D
\end{aligned}$$

and (90) yields

$$\begin{aligned}
\|\mathcal{C}(\varepsilon,m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \geq{}& \|(-i\sigma_1\partial_s + m\sigma_3)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&+ \frac{1}{\varepsilon^2}\Big(\frac{\pi^2}{16} - \frac{m\pi}{2c} - \frac{m\pi}{2}\varepsilon\Big)\|\Pi_1^\perp u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}\\
&+ \frac{1}{\varepsilon^2}\Big(E_1(m\varepsilon) - \frac{\pi^2}{16} - m\varepsilon - \frac{m\pi cS}{2}\varepsilon^2\Big)\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}.
\end{aligned} \tag{91}$$

Now, we choose $c > \frac{8m}{\pi}$ and remark that there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in (0, \varepsilon_1)$, there holds

$$\frac{\pi^2}{16} - \frac{m\pi}{2c} - \frac{m\pi}{2}\varepsilon > 0. \tag{92}$$

Moreover, thanks to (iv) of Proposition 10, there exists ε_2 and $K > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$

$$E_1(m\varepsilon) - \frac{\pi^2}{16} - m\varepsilon - \frac{m\pi c S}{2}\varepsilon^2 > -K\varepsilon^2. \tag{93}$$

Setting $\varepsilon_0 := \min(\varepsilon_1, \varepsilon_2)$ and taking into account (92) and (93) in (91) we obtain that for all $\varepsilon \in (0, \varepsilon_0)$, there holds

$$K\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} + \|\mathcal{C}(\varepsilon, m)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \geq \|(-i\sigma_1\partial_s + m\sigma_3)u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}.$$

The proof of Lemma 28 is completed remarking that $\|\Pi_1 u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)} \leq \|u\|^2_{L^2(\mathsf{Str},\mathbb{C}^2)}$. □

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