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Asymptotic Scattering by Poissonian Thermostats

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Abstract. In the present paper, we consider an infinite chain of harmonic oscillators coupled with a Poisson thermostat attached at a point. The kinetic limit for the energy density of the chain, given by the Wigner distribution, satisfies a transport equation outside the thermostat location. A boundary condition emerges at this site, which describes the reflection-transmission-scattering of the wave energy scattered off by the thermostat. Formulas for the respective coefficients are obtained. Unlike the case of the Langevin thermostat studied in Komorowski et al. (Arch. Ration. Mech. Anal. 237, 497–543, 2020), the Poissonian thermostat scattering generates in the limit a continuous cloud of waves of frequencies different from that of the incident wave.

1. Introduction

In the present paper, we consider a one-dimensional infinite chain of harmonic oscillators, with a thermostat attached a point. The thermostat, maintained at a fixed temperature T, is usually modelled, at the microscopic level, by some stochastic process: for example, by the Langevin stochastic dynamics, or by the renewal of velocities at random times with Gaussian distributed velocities of variance T. The latter represents the interaction with an infinitely extended reservoir of independent particles in equilibrium at temperature T and uniform density. A natural question arises to describe the effect of a thermostat on the wave energy density propagation in the system in a large space-time scale limit. In the paper, we investigate this issue in the case of the kinetic (hyperbolic) space-time scaling. This question has been studied for a Langevin thermostat in the recent article [5]. The goal of this paper is to find out how other classes of thermostats, in particular of the Poisson type, influence the energy transport in the chain in the kinetic limit.

More specifically, consider an infinite one-dimensional chain of harmonic oscillators, where particles are labelled by the elements of the integer lattice $\mathbb{Z}$. The chain is coupled with a thermostat acting on the particle labelled 0. The thermostat is modelled by a random mechanism depending on two parameters: $\gamma > 0$, describing its strength, and $\mu \geq 1/2$, whose role is more technical as it describes an interpolation between Poisson and Gaussian mechanisms. At random times determined by a Poisson process of intensity $\gamma\mu$, the velocity p_0 of the particle 0 is changed to

$$p_0' = \left(1 - \frac{1}{\mu}\right) p_0 + \frac{\sqrt{2\mu - 1}}{\mu}\tilde{p},$$

where $\tilde{p}$ is a centred Gaussian random variable with variance T (the temperature of the thermostat). The case $\mu = 1/2$ corresponds to a velocity flip from $p_0 \mapsto -p_0$ at Poisson random times, $\mu = 1$ ensures complete renewal of p_0, replacing it at those times by a $\mathcal{N}(0, T)$ random variable $\tilde{p}$. Letting $\mu \to \infty$ the process described in the foregoing converges to the Langevin thermostat considered in [5](cf. (2.11)). In this sense, the parameter μ allows to interpolate between various models of thermostats: starting from the random flip process ($\mu = 1/2$), through the simple complete Poisson renewal ($\mu = 1$) and ending up at the Langevin thermostat ($\mu = +\infty$).

In the case $\mu = 1/2$ (the random velocity flip), the energy of the chain is conserved and there is no thermalization. On the other hand, when $\mu > 1/2$, the Gaussian distribution $\mathcal{N}(0, T)$ is the only stationary measure that is asymptotically stable for the process associated with the thermostat and the thermalization of the chain at temperature T occurs.

To describe the energy density distribution in the space and frequency domain, we use the Wigner distribution. When there is no thermostat present, the limit of the Wigner distribution, under the hyperbolic scaling, is the solution of a simple transport equation. It describes the evolution of the density of phonons, travelling independently of each other, with the group velocity $\bar{\omega}'(k) := \omega'(k)/(2\pi)$ corresponding to the phonon of wavenumber k. Here $\omega(k)$ is the dispersion relation of the harmonic chain and a wavenumber k belongs to $\mathbb{T}$ - the unit torus. Taking into account the presence of the thermostat the respective limit, see (2.53) below, can be decomposed into the parts that, besides the aforementioned free energy transport, correspond to the production, absorption, scattering, transmission and reflection of a phonon. More precisely, we show that when the dispersion relation is unimodal, see Sect. 2 for a precise definition, in the scaling limit, the thermostat at temperature $T > 0$ and corresponding to $\mu \geq 1/2$ enforces the following reflection-transmission and production conditions at $x = 0$: phonons of wavenumber ℓ are generated at the rate $p_{\mathrm{abs}}\mathfrak{g}(\ell)T$ and an incoming ℓ-phonon, arriving with velocity $\bar{\omega}'(\ell)$, is transmitted with probability $p_+(\ell)$, reflected with probability $p_-(\ell)$, scattered, as a k-phonon, with the outgoing velocity $\bar{\omega}'(k)$, according to the scattering kernel $\mathfrak{g}(\ell)p_{\mathrm{sc}}(k)$, and absorbed with probability $p_{\mathrm{abs}}\mathfrak{g}(\ell)$, see formulas (2.43) below. These coefficients are non-negative, depend on $\omega(\cdot)$, the parameters

$\gamma > 0$ and $\mu \geq 1/2$, and satisfy

$$p_+(\ell) + p_-(\ell) + p_{\rm abs}\mathfrak{J}(\ell) + \mathfrak{J}(\ell)\int_{\mathbb{T}} p_{\rm sc}(k)dk = 1, \quad \ell \in \mathbb{T}.$$

Coefficients $p_\pm(\ell), \mathfrak{J}(\ell)$ do not depend on μ. The coefficient $p_{\rm abs}$ is independent of ℓ and for $\mu \to +\infty$, $p_{\rm abs} \to 1$ and $p_{\rm sc}(k) \to 0$. With such boundary conditions the thermal equilibrium Wigner function $W(t, x, k) = T$ is a stationary solution of the transport equation for any $\mu > 1/2$.

Our result covers also the random flip of sign of p_0, i.e. $\mu = 1/2$. In this case there is no absorption of phonons: $p_{\rm abs} = 0$, and $\int_{\mathbb{T}} p_{\rm sc}(k)dk = 1$, i.e. all the energy that is not transmitted or reflected at the same frequency is scattered at various frequencies.

The thermostat corresponding to a finite value of μ plays a role of a "scatterer" of time-varying strength. At the macroscopic scale a wave incident on the thermostat produces reflected and transmitted waves at all frequencies. This is in stark contrast with the case of the Langevin thermostat ($\mu = +\infty$) considered in [5], where, after the scaling limit, the reflected and transmitted waves are of the same frequency as the incident wave ($p_{\rm sc}(k) = 0$).

Similar to [5] the presence of oscillatory integrals, responsible for the damping mechanism, presents the difficulty of the model and is dealt with using the Laplace transform of the Wigner distribution. An additional difficulty lies in the fact that, contrary to [5], the noise appearing in the dynamics (2.12) is multiplicative (rather than additive as in ibid.), which makes the computations much less explicit.

Introducing a rarefied random scattering in the bulk, in the same fashion as in [1], should lead to a similar transport equation with a linear scattering term, without modifying the conditions at the interface with the thermostat. Analogous case for the Langevin thermostat has been considered in [4].

2. Preliminaries and Formulation of the Main Result

2.1. Notation

We use the notation $\mathbb{T}_a = [-a/2, a/2]$ for the torus of size $a > 0$, with identified endpoints. In particular for $a = 1$ we write $\mathbb{T}$ instead of $\mathbb{T}_1$. We shall also write $\mathbb{T}_+ := [k \in \mathbb{T} : 0 < k < 1/2]$ and $\mathbb{T}_- := [k \in \mathbb{T} : -1/2 < k < 0]$.

The Fourier transform of a square integrable sequence (α_x) and the inverse Fourier transform of $\hat{\alpha} \in L^2(\mathbb{T})$ are defined as:

$$\hat{\alpha}(k) = \sum_{x\in\mathbb{Z}} \alpha_x \exp\{-2\pi i x k\}, \ \alpha_x = \int_{\mathbb{T}} \hat{\alpha}(k)\exp\{2\pi i x k\}dk, \quad x \in \mathbb{Z}, \ k \in \mathbb{T}. \tag{2.1}$$

Suppose that $f, g \in L^1[0, +\infty)$. Their convolution, also belonging to $L^1[0, +\infty)$, is given by

$$f \star g(t) := \int_0^t f(t-s)g(s)ds, \quad t \in [0, +\infty)$$

By $f^{\star,k}$ we denote the k-times convolution of f with itself, i.e. $f^{\star,1} := f$, $f^{\star,k+1} := f \star f^{\star,k}$, $k \geq 1$. We let $f^{\star,0} \star g := g$. We denote by

$$\tilde{f}(\lambda) = \int_0^{+\infty} e^{-\lambda t} f(t) dt, \quad \operatorname{Re} \lambda > 0,$$

the Laplace transform of f. We also use the notation

$$(a \star b)_y = \sum_{y' \in \mathbb{Z}} a_{y-y'} b_{y'}, \quad y \in \mathbb{Z} \tag{2.2}$$

for the convolution of two absolutely summable sequences $(a_y)_{y\in\mathbb{Z}}$, $(b_y)_{y\in\mathbb{Z}}$.

Given a function $G(x,k)$, we denote by $\tilde{G} : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$, $\hat{G} : \mathbb{R} \times \mathbb{T} \to \mathbb{C}$ the Fourier transforms of G in the k and x variables, respectively,

$$\begin{aligned} \tilde{G}(x,y) &:= \int_{\mathbb{T}} e^{-2\pi i k y} G(x,k) dk, \quad (x,y) \in \mathbb{R} \times \mathbb{Z}, \\ \hat{G}(\eta,k) &:= \int_{\mathbb{R}} e^{-2\pi i \eta x} G(x,k) dx, \quad (\eta,k) \in \mathbb{R} \times \mathbb{T}. \end{aligned} \tag{2.3}$$

Let us denote by $\mathcal{A}$ the Banach space obtained as the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{T})$ in the norm

$$\|G\|_{\mathcal{A}} := \int_{\mathbb{R}} \sup_{k \in \mathbb{T}} |\hat{G}(\eta,k)| d\eta \tag{2.4}$$

and by $\mathcal{A}'$ its dual.

2.2. Poisson Type Thermostat

The stochastic process describing a thermostat is a jump process, whose generator is given by

$$\begin{aligned} L_{\mu,\gamma} f(\mathfrak{p}) := \frac{\gamma\mu}{\sqrt{2\pi T}} \int_{\mathbb{R}} \left[f\left(\left(1 - \frac{1}{\mu}\right) \mathfrak{p} + \rho(\mu) \tilde{\mathfrak{p}} \right) - f(\mathfrak{p}) \right] \\ \exp\left\{ -\frac{\tilde{\mathfrak{p}}^2}{2T} \right\} d\tilde{\mathfrak{p}}, \quad f \in B_b(\mathbb{R}). \end{aligned} \tag{2.5}$$

Here, $B_b(\mathbb{R})$ denotes the space of all bounded and Borel measurable functions, $T, \gamma > 0$, $\mu \geq 1/2$ and

$$\rho(\mu) := \frac{\sqrt{2\mu - 1}}{\mu}. \tag{2.6}$$

It is easy to verify that the Gaussian measure $\mathcal{N}(0,T)$ is invariant under the dynamics of the process. In the case $\mu = 1/2$ Gaussian measure $\mathcal{N}(0,T')$ is invariant for each $T' \geq 0$.

The process $(\mathfrak{p}_t)_{t\geq 0}$ can be also described using the Itô stochastic differential equation, with a noise corresponding to a Poisson jump process, see, for example, [8, Chapter V],

$$\begin{aligned} d\mathfrak{p}(t) &= \left(\tilde{p}(t-) - \frac{1}{\mu} \mathfrak{p}(t-) \right) dN(\gamma\mu t), \quad t \geq 0, \\ \mathfrak{p}(0) &= \tilde{p}_0. \end{aligned} \tag{2.7}$$

Here $(N(t))_{t\geq 0}$ is a Poisson process of intensity 1 defined over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\tilde{p}(t))_{t\geq 0}$ is given by

$$\tilde{p}(t) := \rho(\mu)\tilde{p}_{N'(\gamma\mu t)}, \tag{2.8}$$

where $N'(t) = N(t) + 1$. We suppose that $(\tilde{p}_j)_{j\geq 0}$ are i.i.d. $\mathcal{N}(0, T)$ random variables over $(\Omega, \mathcal{F}, \mathbb{P})$.

The process $(\tilde{p}(t))_{t\geq 0}$ is, stationary and

$$\begin{aligned} &\mathbb{E}\,\tilde{p}(t) = 0, \\ &\mathbb{E}\,[\tilde{p}(t)\tilde{p}(t')] = \frac{2\mu - 1}{\mu^2} e^{-\gamma\mu|t-t'|} T, \quad t, t' \geq 0. \end{aligned} \tag{2.9}$$

From Eq. (2.7), we can see that in case $\mu = 1$ we have $\tilde{p}(t) = \tilde{p}_{N'(\gamma t)}$, $t \geq 0$. On the other hand, after a simple calculation, from (2.5), we conclude that for any $f \in C^2(\mathbb{R})$

$$\lim_{\mu\to+\infty} L_{\mu,\gamma} f(\mathfrak{p}) = L_{\infty,\gamma} f(\mathfrak{p}) := \gamma T \exp\left\{\frac{\mathfrak{p}^2}{2T}\right\} \frac{d}{d\mathfrak{p}} \left(\exp\left\{-\frac{\mathfrak{p}^2}{2T}\right\} \frac{df(\mathfrak{p})}{d\mathfrak{p}}\right). \tag{2.10}$$

The termostat corresponding to $\mu = +\infty$ can be therefore identified with the Langevin thermostat at temperature T, whose dynamics is described by the Itô stochastic differential equation, with an additive Gaussian white noise $dw(t)$:

$$\begin{aligned} &d\mathfrak{p}(t) = -\gamma\mathfrak{p}(t)dt + \sqrt{2\gamma T}dw(t), \quad t \geq 0, \\ &\mathfrak{p}(0) = \tilde{p}_0. \end{aligned} \tag{2.11}$$

This case has been considered in [5].

2.3. Harmonic Chain Coupled with a Point Thermostat

We couple the particle with label $y = 0$ with a thermostat described in Sect. 2.2. Then, the dynamics of the chain, with a stochastic source at $y = 0$, is governed by

$$\begin{aligned} &\dot{\mathfrak{q}}_y(t) = \mathfrak{p}_y(t), \\ &d\mathfrak{p}_y(t) = -(\alpha \star \mathfrak{q}(t))_y dt + \delta_{0,y} \left(\rho(\mu)\tilde{p}_{N(\gamma\mu t)} - \frac{1}{\mu}\mathfrak{p}_y(t-)\right) dN(\gamma\mu t), \quad y \in \mathbb{Z}. \end{aligned} \tag{2.12}$$

The convolution operator $\star$ is defined in (2.2). The coupling constants $(\alpha_y)_{y\in\mathbb{Z}}$ are even $\alpha_{-y} = \alpha_y$ for all $y \in \mathbb{Z}$ and real valued. In addition, we assume that they decay exponentially, i.e. there exists $C > 0$ so that

$$|\alpha_y| \leq Ce^{-|y|/C}, \text{ for all } y \in \mathbb{Z}, \tag{2.13}$$

and

$$\hat{\alpha}(k) := \sum_y \alpha_y \exp\{-2\pi iky\} > 0, \quad k \in \mathbb{T}_* := \mathbb{T} \setminus \{0\}. \tag{2.14}$$

Estimate (2.13) in particular implies that $\hat{\alpha} \in C^\infty(\mathbb{T})$. By $(\mathfrak{q}, \mathfrak{p}) = \big(\mathfrak{p}_y, \mathfrak{q}_y\big)_{y\in\mathbb{Z}}$ we denote the entire momentum-position configuration. Equation (2.12) possesses a unique (mild) cadlag solution taking values in the space of square summable sequences $(\mathfrak{q}, \mathfrak{p})$, see, for example, [7, Sect. 9.4].

2.3.1. The Dispersion Relation and its Basic Properties. Define the dispersion relation

$$\omega(k) := \sqrt{\hat{\alpha}(k)}, \quad k \in \mathbb{T}. \tag{2.15}$$

In light of (2.14), it is C^∞ regular when $\hat{\alpha}(0) > 0$. If, on the other hand $\hat{\alpha}(0) = 0$, the dispersion relation is a continuous function on $\mathbb{T}$ belonging to $C^\infty(\mathbb{T}_*)$, with the derivative possessing one sided limits at $k = 0$. The typical examples are provided by the *acoustic chains*, where $\omega(k) \sim |k|$ for $k \sim 0$, and the *optical chains* where $\omega'(k) \sim k$ for $k \sim 0$. We assume also that ω is *unimodal*, i.e. it is increasing on $[0, 1/2]$. Denote its unique minimum, attained at $k = 0$, by $\omega_{\min} \geq 0$ and its unique maximum, attained at $k = 1/2$, by $\omega_{\max}$. The two branches of the inverse of $\omega(\cdot)$ are denoted by $\omega_+ : [\omega_{\min}, \omega_{\max}] \to [0, 1/2]$ and $\omega_- = -\omega_+$.

2.3.2. The Wave-Function. Define the complex-valued wave function

$$\psi_y(t) := (\tilde{\omega} \star \mathfrak{q}(t))_y + i\mathfrak{p}_y(t). \tag{2.16}$$

Here $\big(\tilde{\omega}_y\big)_{y\in\mathbb{Z}}$ is the inverse Fourier transform of the dispersion relation $\omega(k)$. The square of the wave function $|\psi_y(t)|^2$ describes the local energy of the chain at time t. The Fourier transform of $\big(\psi_y(t)\big)_{y\in\mathbb{Z}}$ is given by

$$\hat{\psi}(t, k) := \omega(k)\hat{\mathfrak{q}}(t, k) + i\hat{\mathfrak{p}}(t, k), \quad k \in \mathbb{T}. \tag{2.17}$$

We have

$$\hat{\mathfrak{p}}\,(t, k) = \frac{1}{2i}[\hat{\psi}(t, k) - \hat{\psi}^*(t, -k)] \quad \text{and} \quad \mathfrak{p}_0(t) = \int_{\mathbb{T}} \mathrm{Im}\,\hat{\psi}(t, k)dk.$$

Using (2.12), it is easy to check that the wave function evolves according to

$$d\hat{\psi}(t, k) = -i\omega(k)\hat{\psi}(t, k)dt + i\left(\tilde{p}(t-) - \frac{1}{\mu}\mathfrak{p}_0(t-)\right) dN(\gamma\mu t). \tag{2.18}$$

2.3.3. The Initial Conditions. Assume that for a given (small) value of the parameter $\varepsilon > 0$, the initial wave function is distributed randomly, according to a Borel probability measure μ_ε on the space of square summable configurations. We suppose that

$$\sup_{\varepsilon\in(0,1)} \sum_{y\in\mathbb{Z}} \varepsilon\langle|\psi_y|^2\rangle_{\mu_\varepsilon} = \sup_{\varepsilon\in(0,1)} \varepsilon\langle\|\hat{\psi}\|^2_{L^2(\mathbb{T})}\rangle_{\mu_\varepsilon} < \infty. \tag{2.19}$$

Here $\langle\cdot\rangle_{\mu_\varepsilon}$ denotes the expectation with respect to μ_ε. Assumption (2.19) guarantees that the energy density per unit length on the macroscopic scale $x \sim \varepsilon y$ stays finite, as $\varepsilon \to 0+$.

In addition, to simplify somewhat our ensuing calculations, we will also assume that

$$\langle\hat{\psi}(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon} = 0, \quad k, \ell \in \mathbb{T}, \tag{2.20}$$

The above hypothesis is of purely technical nature. It can be replaced by somewhat more general assumption that $\langle\hat{\psi}(k)\hat{\psi}(\ell)\rangle_{\mu_\varepsilon} \sim 0$, as $\varepsilon \to 0$, with no significant change in the main line our argument. However, the calculations would become more involved. Later on we shall also assume some additional hypothesis, see (2.27) below.

2.3.4. The Wigner Distributions. Denote the rescaled wave function $\psi_y^{(\varepsilon)}(t) = \psi_y(t/\varepsilon)$ and its Fourier transform $\hat{\psi}^{(\varepsilon)}(t,k)$. The (averaged) Wigner distributions $W_\pm^{(\varepsilon)}(t)$ and $Y_\pm^{(\varepsilon)}(t)$ are defined by their action on a test function $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$:

$$\begin{aligned}\langle G, W_\pm^{(\varepsilon)}(t)\rangle &= \int_{\mathbb{T}\times\mathbb{R}} \widehat{W}_{\varepsilon,\pm}(t,\eta,k)\hat{G}^*(\eta,k)d\eta dk,\\ \langle G, Y_\pm^{(\varepsilon)}(t)\rangle &= \int_{\mathbb{T}\times\mathbb{R}} \widehat{Y}_{\varepsilon,\pm}(t,\eta,k)\hat{G}^*(\eta,k)d\eta dk,\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}\widehat{W}_{\varepsilon,\pm}(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\left(\hat{\psi}^{(\varepsilon)}\right)^*\left(t,\pm k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,\pm k+\frac{\varepsilon\eta}{2}\right)\right],\\ \widehat{Y}_{\varepsilon,+}(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}^{(\varepsilon)}\left(t,k+\frac{\varepsilon\eta}{2}\right)\hat{\psi}^{(\varepsilon)}\left(t,-k+\frac{\varepsilon\eta}{2}\right)\right],\\ \widehat{Y}_{\varepsilon,-}(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\left(\hat{\psi}^{(\varepsilon)}\right)^\star\left(t,k-\frac{\varepsilon\eta}{2}\right)\left(\hat{\psi}^{(\varepsilon)}\right)^\star\left(t,-k-\frac{\varepsilon\eta}{2}\right)\right],\quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}\end{aligned} \tag{2.22}$$

are the respective Fourier–Wigner functions. Here, $\mathbb{E}$ is the expectation with respect to the product measure $\mu_\varepsilon \otimes \mathbb{P}$. To simplify the notation, we shall also write $\widehat{W}_\varepsilon(t,\eta,k)$ instead of $\widehat{W}_{\varepsilon,+}(t,\eta,k)$.

A straightforward calculation, using (2.18), shows that

$$\frac{d}{dt}\int_{\mathbb{T}}\mathbb{E}\big|\hat{\psi}^{(\varepsilon)}(t,k)\big|^2 dk = \frac{\gamma}{\varepsilon}\left(2-\frac{1}{\mu}\right)\left(T-\mathbb{E}[\mathfrak{p}_0^{(\varepsilon)}(t)]^2\right) \tag{2.23}$$

with $\mathfrak{p}_0^{(\varepsilon)}(t) := \mathfrak{p}_0(t/\varepsilon)$. As a result we get

$$\varepsilon\int_{\mathbb{T}}\mathbb{E}\big|\hat{\psi}^{(\varepsilon)}(t,k)\big|^2 dk \le \varepsilon\int_{\mathbb{T}}\mathbb{E}\big|\hat{\psi}^{(\varepsilon)}(0,k)\big|^2 dk + \left(2-\frac{1}{\mu}\right)\gamma T t,\quad t\ge 0. \tag{2.24}$$

Thus, we conclude from (2.24) that (see [2])

$$\sup_{t\in[0,\tau]}\|W^{(\varepsilon)}(t)\|_{\mathcal{A}'} < \infty,\ \text{for each}\quad \tau>0. \tag{2.25}$$

Hence, $W^{(\varepsilon)}(\cdot)$ is sequentially weak-$\star$ compact over $(L^1([0,\tau];\mathcal{A}))^\star$ for any $\tau>0$.

The initial Wigner distribution

$$\widehat{W}_\varepsilon(\eta,k) := \widehat{W}_\varepsilon(0,\eta,k),\quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T} \tag{2.26}$$

is assumed to converge $\star$-weakly, as $\varepsilon\to 0$, in $\mathcal{A}'$ to a non-negative function $W_0 \in L^1(\mathbb{R}\times\mathbb{T})\cap C(\mathbb{R}\times\mathbb{T})$. In addition, we suppose that there exist $C,\kappa>0$ such that

$$|\widehat{W}_\varepsilon(\eta,k)| \le C\varphi(\eta),\quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T},\ \varepsilon\in(0,1], \tag{2.27}$$

where

$$\varphi(\eta) := \frac{1}{(1+\eta^2)^{3/2+\kappa}}. \tag{2.28}$$

Define the Fourier–Laplace–Wigner functions

$$\widehat{w}_{\pm,\varepsilon}(\lambda,\eta,k) = \varepsilon \int_0^{+\infty} e^{-\lambda\varepsilon t}\widehat{W}_{\pm,\varepsilon}(t,\eta,k)dt, \tag{2.29}$$
$$\widehat{y}_{\pm,\varepsilon}(\lambda,\eta,k) = \varepsilon \int_0^{+\infty} e^{-\lambda\varepsilon t}\widehat{Y}_{\pm,\varepsilon}(t,\eta,k)dt,$$

where $\mathrm{Re}\,\lambda > 0$, $(\eta,k) \in \mathbb{T}_{2/\varepsilon} \times \mathbb{T}$. We shall also write $\widehat{w}_\varepsilon(\lambda,\eta,k)$ instead of $\widehat{w}_{+,\varepsilon}(\lambda,\eta,k)$.

2.4. Some Additional Notation

Define

$$J(t) = \int_{\mathbb{T}} \cos(\omega(k)t)\,dk, \quad t \in \mathbb{R}. \tag{2.30}$$

Its Laplace transform

$$\tilde{J}(\lambda) := \int_0^\infty e^{-\lambda t}J(t)dt = \int_{\mathbb{T}} \frac{\lambda}{\lambda^2+\omega^2(k)}dk, \quad \mathrm{Re}\,\lambda > 0. \tag{2.31}$$

One can easily see that

$$|\tilde{J}(\lambda)| < \frac{1}{\mathrm{Re}\,\lambda} \quad \text{for } \mathrm{Re}\,\lambda > 0. \tag{2.32}$$

Let

$$\tilde{g}(\lambda) := (1+\gamma\tilde{J}(\lambda))^{-1}. \tag{2.33}$$

We have $\mathrm{Re}\,\tilde{J}(\lambda) > 0$ for $\lambda \in \mathbb{C}_+ := [\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > 0]$, thus in consequence

$$|\tilde{g}(\lambda)| \le 1, \quad \lambda \in \mathbb{C}_+. \tag{2.34}$$

In addition, we have

$$(\tilde{g}\tilde{J})(\lambda) = \frac{1}{\gamma}(1-\tilde{g}(\lambda)) = \frac{\tilde{J}(\lambda)}{1+\gamma\tilde{J}(\lambda)} = \sum_{n=1}^{+\infty}(-\gamma)^{n-1}\tilde{J}(\lambda)^n. \tag{2.35}$$

The first two equalities in (2.35) hold for all $\lambda \in \mathbb{C}_+$, while the last one for $\mathrm{Re}\,\lambda > \gamma$ (cf (2.32)).

Since $|J(t)| \le 1$ we have $|J^{\star,n}(t)| \le t^{n-1}/(n-1)!$, as the n-th convolution power involves the integration over an $n-1$-dimensional simplex of size t. Therefore, the series

$$g_*(t) := \sum_{n=1}^{+\infty}(-\gamma)^n J^{\star,n}(t) \tag{2.36}$$

defines a C^∞ class function on $[0,+\infty)$ that satisfies the following growth condition: there exists $C > 0$ such that $|g_*(t)| \le Ce^{\gamma t}$, $t > 0$. In addition, comparing the Laplace transform of $g_*(t)$ with $(1-\tilde{g}(\lambda))/\gamma$, as expressed by the utmost right-hand side of (2.35), we conclude that

$$\tilde{g}_*(\lambda) = \tilde{g}(\lambda) - 1 = -\gamma(\tilde{g}\tilde{J})(\lambda), \quad \mathrm{Re}\,\lambda > \gamma. \tag{2.37}$$

Therefore, $\tilde{g}(\lambda)$, given by (2.33), is the Laplace transform of the signed measure $g(dt) := \delta_0(dt) + g_*(t)dt$. Combining (2.33), (2.36) and (2.37), we obtain

$$\gamma J \star g(t) = \sum_{n=1}^{+\infty} (-1)^{n-1} \gamma^n J^{\star,n}(t) = -g_*(t), \quad t \geq 0. \tag{2.38}$$

It turns out, see Lemma 1, that $J \star g \in L^2(\mathbb{R})$ and $\operatorname{supp} J \star g \subset [0, +\infty)$. This allows us to conclude the existence of $\tilde{g}_*$—the Laplace transform of $g_*(\cdot)$—and equality (2.37) for all $\lambda \in \mathbb{C}_+$.

2.5. Functions $\widetilde{g}$ and $\widetilde{J}$

Since the function $\tilde{g}(\cdot)$ is analytic on $\mathbb{C}_+$ we conclude, by the Fatou theorem, see, for example, p. 107 of [6], that

$$\tilde{g}(i\beta) := \lim_{\varepsilon \to +0} \tilde{g}(\varepsilon + i\beta), \quad \beta \in \mathbb{R} \tag{2.39}$$

exists a.e. In Sect. 6.1, we show the following.

Lemma 1. *The holomorphic function $\tilde{J}\tilde{g}$ belongs to the Hardy space $H^p(\mathbb{C}_+)$ for any $p \in (1, +\infty)$. The limit*

$$(\tilde{J}\tilde{g})(i\beta) := \lim_{\varepsilon \to +0} (\tilde{J}\tilde{g})(\varepsilon + i\beta), \quad \beta \in \mathbb{R} \tag{2.40}$$

exists both a.e. and in the $L^p(\mathbb{R})$ sense for $p \in (1, +\infty)$.

In addition, there exists

$$\nu(k) := \lim_{\varepsilon \to +0} \tilde{g}\big(\varepsilon + i\omega(k)\big), \quad k \in \Omega_*, \tag{2.41}$$

where $\Omega_* := [k \in \mathbb{T} : \omega'(k) = 0, \quad or \quad \omega(k) = 0]$. The function is continuous on $\mathbb{T} \setminus \Omega_*$. Moreover, for any $\delta > 0$ there exists $C > 0$ such that

$$\left| \tilde{g}\big(\varepsilon + i\omega(k)\big) - \nu(k) \right| \leq C\varepsilon, \quad \operatorname{dist}\big(k, \Omega_*\big) \geq \delta, \varepsilon \in (0, 1). \tag{2.42}$$

To state our main result, we need some additional notation. Define the group velocity

$$\bar{\omega}'(k) := \omega'(k)/(2\pi)$$

and

$$\wp(k) := \frac{\gamma\nu(k)}{2|\bar{\omega}'(k)|}, \quad \mathfrak{g}(k) := \frac{\gamma|\nu(k)|^2}{|\bar{\omega}'(k)|}, \quad p_+(k) := |1 - \wp(k)|^2, \quad p_-(k) := |\wp(k)|^2. \tag{2.43}$$

It has been shown in Section 10 of [5] that

$$\operatorname{Re} \nu(k) = \left(1 + \frac{\gamma}{2|\bar{\omega}'(k)|}\right) |\nu(k)|^2 \tag{2.44}$$

and

$$p_+(k) + p_-(k) = 1 - \mathfrak{g}(k) \leq 1, \tag{2.45}$$

so that, in particular, we have

$$0 \leq \mathfrak{g}(k) \leq 1, \quad k \in \mathbb{T}. \tag{2.46}$$

In the model considered in [5] the coefficients $p_+(k)$, $p_-(k)$ and $\mathfrak{g}(k)$ have expressed, see [5, Theorem 2.1], the probabilities of a phonon being transmitted, reflected and absorbed at the interface $[x = 0]$.

In our present situation, the absorption probability needs to be modified. In addition, the phonon can be also scattered at the interface with outgoing frequency ℓ with some scattering rate $r(k, \ell)$. To be more precise, we introduce the following notation

$$p_{\rm abs} := \frac{1}{1 - \Gamma/\mu}\left(1 - \frac{1}{2\mu}\right), \quad p_{\rm sc}(\ell) := \frac{1}{2\mu(1 - \Gamma/\mu)}|\nu(\ell)|^2, \tag{2.47}$$

where

$$\Gamma := \frac{\gamma}{2\pi}\int_{\mathbb{R}} |\tilde{J}\tilde{g}(i\beta')|^2 d\beta'. \tag{2.48}$$

The following result holds.

Lemma 2. *For any $\gamma > 0$ we have*

$$\Gamma + \frac{1}{2}\int_{\mathbb{T}} |\nu(\ell)|^2 d\ell = \frac{1}{2}. \tag{2.49}$$

In addition, if $\mu \geq 1/2$, then

$$p_{\rm abs} + \int_{\mathbb{T}} p_{\rm sc}(\ell) d\ell = 1. \tag{2.50}$$

The proof of the lemma is contained in Sect. 6.2.

Remark 1. It turns out, see [3, Theorem 4. part iii)], that for any unimodal dispersion relation we have $|\nu(\ell)| > 0$, except possibly $\ell = 0$, or $1/2$. Thanks to the identity (2.49) below, we have then

$$\Gamma < \frac{1}{2} \leq \mu. \tag{2.51}$$

Therefore, in particular, the coefficients defined in (2.47) are strictly positive for $\mu > 1/2$ and $\ell \notin \{0, 1/2\}$.

2.6. The Main Result

For brevity sake, we use the notation

$$[[0, a]] := \begin{cases} [0, a], & \text{if} \quad a > 0 \\ [a, 0], & \text{if } a < 0. \end{cases}$$

The main result of the paper can be formulated as follows.

Theorem 1. *Suppose that the initial conditions and the dispersion relation satisfy the above assumptions. Then, for any $\tau > 0$ and $G \in L^1([0, \tau]; \mathcal{A})$ we have*

$$\lim_{\varepsilon \to 0} \int_0^\tau \langle G(t), W_\varepsilon(t)\rangle dt = \int_0^\tau dt \int_{\mathbb{R}\times\mathbb{T}} G^*(t, x, k) W(t, x, k) dx dk, \tag{2.52}$$

where

$$\begin{aligned} W(t,x,k) =& W_0\left(x-\bar\omega'(k)t,k\right)1_{[[0,\bar\omega'(k)t]]^c}(x)+p_+(k)W_0\left(x-\bar\omega'(k)t,k\right)1_{[[0,\bar\omega'(k)t]]}(x)\\ &+p_-(k)W_0\left(-x+\bar\omega'(k)t,-k\right)1_{[[0,\bar\omega'(k)t]]}(x)\\ &+\mathfrak{g}(k)1_{[[0,\bar\omega'(k)t]]}(x)\int_{\mathbb{T}}W_0\left(\frac{\bar\omega'(\ell)}{\bar\omega'(k)}\left(x-\bar\omega'(k)t\right),\ell\right)p_{\rm sc}(\ell)d\ell\\ &+p_{\rm abs}\mathfrak{g}(k)T1_{[[0,\bar\omega'(k)t]]}(x). \end{aligned} \tag{2.53}$$

The proof of this result is given in Sect. 5.4.

The limit dynamics can be characterized as follows: $W(t,x,k)$ describes the energy density in (x,k) at time t of the phonons initially distributed according to $W_0(x,k)$. The first term corresponds then to the ballistic transport of those phonons which did not cross $\{x=0\}$ up to time t. The second and third terms correspond, respectively, to the transmission and reflection of the phonons at the boundary point $\{x=0\}$ with probabilities $p_+(k)$ and $p_-(k)$, respectively. The fourth term describes the phonon scattering that occurs at the interface. The phonon with frequency ℓ, arriving at the interface with the velocity $\bar\omega'(\ell)$ is scattered with frequency k at the rate $\mathfrak{g}(\ell)p_{\rm sc}(k)$ and moves away from the interface with the velocity $\bar\omega'(k)$. Finally, the last term in the right side of (2.53) describes the k-phonon production of the thermostat at the rate $p_{\rm abs}\mathfrak{g}(k)T$. From (2.45) and (2.50), we conclude that

$$1-p_+(\ell)-p_-(\ell)-\mathfrak{g}(\ell)\int_{\mathbb{T}}p_{\rm sc}(k)dk=p_{\rm abs}\mathfrak{g}(\ell),\quad \ell\in\mathbb{T}. \tag{2.54}$$

Therefore, the ℓ-phonon is absorbed by the thermostat with probability $p_{\rm abs}\mathfrak{g}(\ell)$. Note that in the special case when the thermostat operates by the flip of the momentum, which happens when $\mu=1/2$, there is no absorption, as according to (2.47) we have $p_{\rm abs}=0$. This is consistent with the fact that the total energy of the chain is then conserved, see (2.23).

Our result can be written as a boundary value problem. Note that $W(t,x,k)$ solves the homogeneous transport equation

$$\partial_t W(t,x,k)+\bar\omega'(k)\partial_x W(t,x,k)=0, \tag{2.55}$$

away from the boundary point $\{x=0\}$.

Let

$$W(t,0^\pm,k):=\lim_{x\to\pm 0}W(t,x,k).$$

If $k\in\mathbb{T}_+$ $(k>0)$, then

$$\begin{aligned} W\left(t,0^+,k\right)=&\,p_+(k)W\left(t,0^-,k\right)+p_-(k)W\left(t,0^+,-k\right)+p_{\rm abs}\,\mathfrak{g}(k)T\\ &+\mathfrak{g}(k)\int_{\mathbb{T}_+}W\left(t,0^-,\ell\right)p_{\rm sc}(\ell)d\ell\\ &+\mathfrak{g}(k)\int_{\mathbb{T}_+}W\left(t,0^+,-\ell\right)p_{\rm sc}(\ell)d\ell. \end{aligned} \tag{2.56}$$

If, on the other hand, $k \in \mathbb{T}_-$ $(k < 0)$, then

$$\begin{aligned} W\left(t, 0^-, k\right) = p_+(k) W\left(t, 0^+, k\right) + p_-(k) W\left(t, 0^-, -k\right) + p_{\mathrm{abs}}\, \mathfrak{g}(k) T \\ + \mathfrak{g}(k) \int_{\mathbb{T}_-} W\left(t, 0^+, \ell\right) p_{\mathrm{sc}}(\ell) d\ell \\ + \mathfrak{g}(k) \int_{\mathbb{T}_-} W\left(t, 0^-, -\ell\right) p_{\mathrm{sc}}(\ell) d\ell. \end{aligned}$$

3. The Solution of (2.18) and its Laplace–Fourier–Wigner Distribution

In this section, we obtain an explicit expression for the solution of the wave function (2.18). The mild formulation of the equation reads as follows:

$$\begin{aligned} \hat{\psi}(t,k) =& e^{-i\omega(k)t}\hat{\psi}(k) - \frac{i}{\mu}\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_0(s-)dN(\gamma\mu s) \\ &+ i\int_0^t e^{-i\omega(k)(t-s)}\tilde{p}(s-)dN(\gamma\mu s), \end{aligned} \tag{3.1}$$

where $\tilde{p}(t)$ is given by (2.8). Letting

$$\mathfrak{p}_0^0(t) := \mathrm{Im}\left(\int_{\mathbb{T}} e^{-i\omega(k)t}\hat{\psi}(k)dk\right) \tag{3.2}$$

we conclude the following closed equation on the momentum at $y = 0$:

$$\mathfrak{p}_0(t) = \mathfrak{p}_0^0(t) - \frac{1}{\mu}\int_0^t J(t-s)\mathfrak{p}_0(s-)dN(\gamma\mu s) + \int_0^t J(t-s)\tilde{p}(s-)dN(\gamma\mu s). \tag{3.3}$$

Equation (3.1) is linear, so its solution can be written as the sum of the solution $\hat{\psi}_1(t,k)$ corresponding to the null initial data $\hat{\psi}(k) \equiv 0$ and the solution $\hat{\psi}_2(t,k)$ of the homogeneous equation corresponding to $\tilde{p}(t) \equiv 0$.

More precisely, suppose that $\hat{\psi}_1(t,k)$ is the solution of

$$\begin{aligned} d\hat{\psi}_1(t,k) &= -i\omega(k)\hat{\psi}_1(t,k)dt + i\left(\tilde{p}(t-) - \frac{1}{\mu}\mathfrak{p}_{0,1}(t-)\right)dN(\gamma\mu t), \\ \hat{\psi}_1(0,k) &\equiv 0 \end{aligned} \tag{3.4}$$

and $\hat{\psi}_2(t,k)$ satisfies

$$\begin{aligned} d\hat{\psi}_2(t,k) &= -i\omega(k)\hat{\psi}_2(t,k) - \frac{i}{\mu}\mathfrak{p}_{0,2}(t-)dN(\gamma\mu t), \\ \hat{\psi}_2(0,k) &= \hat{\psi}(k). \end{aligned} \tag{3.5}$$

Here

$$\mathfrak{p}_{0,j}(t) := \mathrm{Im}\int_{\mathbb{T}} \hat{\psi}_j(t,k)dk, \quad j = 1,2.$$

Then

$$\hat{\psi}(t,k) = \hat{\psi}_1(t,k) + \hat{\psi}_2(t,k). \tag{3.6}$$

The respective Fourier–Wigner functions are defined as:

$$\widehat{W}_\varepsilon^{j_1,j_2}(t,\eta,k) := \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}_{j_1}^*\left(\frac{t}{\varepsilon},k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}_{j_2}\left(\frac{t}{\varepsilon},k+\frac{\varepsilon\eta}{2}\right)\right],\quad j_1,\,j_2\in\{1,2\}.$$

Since the process $\big(\tilde{p}(t)\big)_{t\geq 0}$ is independent of the initial data field $\big(\hat{\psi}(k)\big)_{k\in\mathbb{T}}$, we conclude easily that

$$\widehat{W}_\varepsilon^{j_1,j_2}(t,\eta,k)\equiv 0,\quad \text{if } j_1\neq j_2.$$

Therefore,

$$\begin{aligned}\widehat{W}_\varepsilon(t,\eta,k) &:= \frac{\varepsilon}{2}\mathbb{E}\left[\hat{\psi}^*\left(\frac{t}{\varepsilon},k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}\left(\frac{t}{\varepsilon},k+\frac{\varepsilon\eta}{2}\right)\right]\\ &= \widehat{W}_\varepsilon^{1,1}(t,\eta,k)+\widehat{W}_\varepsilon^{2,2}(t,\eta,k).\end{aligned}\tag{3.7}$$

Accordingly, the respective Laplace–Fourier–Wigner transforms satisfy

$$\widehat{w}_\varepsilon(\lambda,\eta,k)=\widehat{w}_\varepsilon^{1,1}(\lambda,\eta,k)+\widehat{w}_\varepsilon^{2,2}(\lambda,\eta,k),\tag{3.8}$$

where

$$\widehat{w}_\varepsilon(\lambda,\eta,k)=\int_0^{+\infty}e^{-\lambda t}\widehat{W}_\varepsilon(t,\eta,k)dt,\quad (\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}$$

and $\operatorname{Re}\lambda>0$. The definitions of $\widehat{w}_\varepsilon^{j,j}$, corresponding to $\widehat{W}_\varepsilon^{j,j}(t,\eta,k)$, $j=1,2$, are analogous.

3.1. Solving (2.18) for the Null Initial Data

We suppose that $\hat{\psi}(0,k)\equiv 0$. Let $s_0:=t$, $\Delta_1(t):=[0,t]$ and

$$\Delta_n(t):=[(s_1,\ldots,s_n):\ t>s_1>s_2>\ldots>s_n>0],\quad n\geq 2.$$

Iterating (3.3) and remembering that $\mathfrak{p}_0^0(t)\equiv 0$, we can write

$$\mathfrak{p}_{0,1}(t)=\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^{n-1}\int_{\Delta_n(t)}\prod_{j=1}^{n}J(s_{j-1}-s_j)\tilde{p}(s_n-)dN(\gamma\mu s_1)\ldots dN(\gamma\mu s_n),\tag{3.9}$$

with $s_0:=t$. Therefore, substituting for the momentum in the respective form of (3.1), we get

$$\hat{\psi}_1(t,k)=i\int_0^t e^{-i\omega(k)(t-s)}\left(\tilde{p}(s)-\frac{1}{\mu}\mathfrak{p}_0(s-)\right)dN(\gamma\mu s)=\sum_{n=1}^{+\infty}\hat{\psi}_{1,n}(t,k),\tag{3.10}$$

where

$$\begin{aligned}\hat\psi_{1,1}(t,k) &:= i\int_0^t e^{-i\omega(k)(t-s)}\tilde p(s-)dN(\gamma\mu s),\\ \hat\psi_{1,n}(t,k) &:= \left(-\frac{1}{\mu}\right)^{n-1} i\int_{\Delta_n(t)} e^{-i\omega(k)(t-s_1)}\\ &\quad\times\prod_{j=1}^{n-1} J(s_j-s_{j+1})\tilde p(s_n-)dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n),\quad n\geq 2.\end{aligned} \tag{3.11}$$

3.2. The Case $T = 0$ and Non-Zero Initial Data

The mild formulation of (3.5) is as follows:

$$\hat\psi_2(t,k) = e^{-i\omega(k)t}\hat\psi(k) - \frac{i}{\mu}\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_{0,2}(s-)dN(\gamma\mu s). \tag{3.12}$$

From here we conclude the following closed equation on the momentum at $y = 0$:

$$\mathfrak{p}_{0,2}(t) = \mathfrak{p}_0^0(t) - \frac{1}{\mu}\int_0^t J(t-s)\mathfrak{p}_{0,2}(s-)dN(\gamma\mu s), \tag{3.13}$$

where $\mathfrak{p}_0^0(t)$ is given by (3.2). The solution of (3.13) is given by

$$\begin{aligned}\mathfrak{p}_{0,2}(t) = \mathfrak{p}_0^0(t) + \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n &\int_{\Delta_n(t)} J(t-s_1)\dots J(s_{n-1}-s_n)\\ &\times\mathfrak{p}_0^0(s_n)dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n).\end{aligned} \tag{3.14}$$

Substituting in (3.12), we get

$$\hat\psi_2(t,k) = \sum_{n=0}^{+\infty}\hat\psi_{2,n}(t,k), \tag{3.15}$$

where

$$\begin{aligned}\hat\psi_{2,1}(t,k) &:= -\frac{i}{\mu}\int_0^t e^{-i\omega(k)(t-s)}\mathfrak{p}_0^0(s)dN(\gamma\mu s)\\ \hat\psi_{2,n}(t,k) &:= -i\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^{n+1}\int_{\Delta_n(t)} e^{-i\omega(k)(t-s_1)}\prod_{j=1}^{n-1}J(s_j-s_{j+1})\\ &\quad\times\mathfrak{p}_0^0(s_n)dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n),\quad n\geq 2.\end{aligned} \tag{3.16}$$

4. The Limit in Case of Null Initial Data: The Phonon Creation Term

Consider first the case when the null initial data, i.e. $\hat\psi_2(t,k)\equiv 0$. Then,

$$\widehat{w}_\varepsilon(\lambda,\eta,k) = \widehat{w}_\varepsilon^{(1,1)}(\lambda,\eta,k). \tag{4.1}$$

We wish to use the chaos expansion, corresponding to the Poisson process $(N(t))_{t\geq 0}$ to represent the Laplace–Fourier–Wigner function $\widehat{w}_\varepsilon(\lambda,\eta,k)$.

Lemma 3. *Suppose that $\mu > 1/2$. The following formula holds*

$$\widehat{w}_\varepsilon(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\int_0^{+\infty} e^{-\lambda\varepsilon s}\mathbb{E}\left[\hat{\chi}^*\left(s,k-\frac{\varepsilon\eta}{2}\right)\hat{\chi}\left(s,k+\frac{\varepsilon\eta}{2}\right)\right]ds \tag{4.2}$$

for any $\lambda\in\mathbb{C}_+$, $(\eta,k)\in\mathbb{T}_{2/\varepsilon}\times\mathbb{T}$ and $\varepsilon>0$. Here

$$\hat{\chi}(t,k) := \exp\{-i\omega(k)\,t\} + \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} \exp\{-i\omega(k)\,(t-s_1)\} \times \prod_{j=1}^{n} J(s_j-s_{j+1})dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n), \tag{4.3}$$

with $s_{n+1} := 0$.

If, on the other hand $\mu = 1/2$ and $\varepsilon,\gamma>0$, then

$$\widehat{w}_\varepsilon(\lambda,\eta,k) = 0, \quad \lambda\in\mathbb{C}_+,\ (\eta,k)\in\mathbb{R}\times\mathbb{T}. \tag{4.4}$$

Remark 2. Note that (4.4) is consistent with the physical interpretation of the model. Namely, we have assumed that initially the energy of the chain is null. On the other hand, the momentum flip mechanism of the thermostat, that corresponds to the case $\mu = 1/2$, conserves the total energy of the system.

Proof of Lemma 3

The series appearing on the right-hand side of (4.3) converges in the L^1 sense. Indeed, since $|J(t)|\le 1$ its terms are dominated by the respective terms of the series

$$\Theta(t) := 1 + \sum_{n=1}^{+\infty}\left(\frac{1}{\mu}\right)^n \int_{\Delta_n(t)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n). \tag{4.5}$$

The process $\Theta(t)$ is the unique solution of the stochastic differential equation $d\Theta(t) = \big(\Theta(t-)/\mu\big)dN(\gamma\mu t)$, $\Theta(0)=1$ and is given by the stochastic exponential, see, for example, [8, Theorem II.8.37, p. 84],

$$\Theta(t) = \exp\left\{N(\gamma\mu t)\log\left(1+\frac{1}{\mu}\right)\right\}.$$

We have $|\hat{\chi}(t,k)|\le\Theta(t)$, therefore

$$\mathbb{E}|\hat{\chi}(t,k)|^2 \le \mathbb{E}\Theta^2(t) = \exp\left\{\gamma\mu t\Big[\exp\left\{2\log\left(1+\frac{1}{\mu}\right)\right\}-1\Big]\right\}$$

and the right-hand side of (4.2) is well defined, as an element of $\mathcal{A}'$ (see (2.4)), at least for $\operatorname{Re}\lambda > 2\gamma\varepsilon^{-1}$. In what follows we show that equality (4.2) holds for this range of λ. Note that this implies the validity of (4.2) for all $\lambda\in\mathbb{C}_+$. Indeed, if $\mu = 1/2$, then by the analytic continuation we conclude that $\widehat{w}_\varepsilon(\lambda,\eta,k) = 0$ for all $\operatorname{Re}\lambda>0$ and the formula (4.4) follows.

For $\mu > 1/2$, the equality of the Laplace transforms, see (4.2), for $\operatorname{Re}\lambda > 2\gamma\varepsilon^{-1}$ implies in particular, when $\eta = 0$, that

$$T\gamma\left(1-\frac{1}{2\mu}\right)\mathbb{E}\left|\hat{\chi}\left(\frac{t}{\varepsilon},k\right)\right|^2 = \frac{\varepsilon}{2}\frac{d}{dt}\mathbb{E}\left|\hat{\psi}_1\left(\frac{t}{\varepsilon},k\right)\right|^2, \quad t\ge 0.$$

In light of (2.23), this allows us to extend the validity of (4.2) to all $\mathrm{Re}\,\lambda > 0$.

Now we proceed with the proof of (4.2) for $\mathrm{Re}\,\lambda > 2\gamma\varepsilon^{-1}$. Substituting from (3.11) we get

$$\widehat{w}_\varepsilon(\lambda,\eta,k) = \sum_{n,m=1}^{+\infty} \widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k), \tag{4.6}$$

where

$$\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k) := \frac{\varepsilon}{2}\int_0^{+\infty} e^{-\varepsilon\lambda t}\mathbb{E}\left[\hat{\psi}^\star_{1,n}\left(t,k-\frac{\varepsilon\eta}{2}\right)\hat{\psi}_{1,m}\left(t,k+\frac{\varepsilon\eta}{2}\right)\right]dt \quad n,m\geq 1.$$

The convergence of the series follows by the comparison with the series defining the stochastic exponential, see (4.5). Note that for $s > s'$

$$\mathbb{E}\Big[\tilde{p}(s-)\tilde{p}(s'-),\, N(\gamma\mu s-) - N(\gamma\mu s') \geq 1\Big] = 0.$$

The above implies that

$$\begin{aligned}
\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k) =& \frac{\varepsilon^2}{2}\left(-\frac{1}{\mu}\right)^{n+m}\int_0^{+\infty} e^{-\lambda\varepsilon t}dt\mathbb{E}\Bigg[\int_{\Delta_n(t)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_n)\\
&\int_{\Delta_m(t)} dN(\gamma\mu s'_1)\dots dN(\gamma\mu s'_m)\\
&\exp\left\{i\omega\left(k-\frac{\varepsilon\eta}{2}\right)(t-s_1)\right\}\exp\left\{-i\omega\left(k+\frac{\varepsilon\eta}{2}\right)(t-s'_1)\right\}\\
&\times\prod_{j=1}^{n-1}J(s_j-s_{j+1})\prod_{j=1}^{m-1}J(s'_j-s'_{j+1})\tilde{p}(s_n-)\tilde{p}(s'_m-)\Bigg]\\
=&\varepsilon^2 T\gamma\left(1-\frac{1}{2\mu}\right)\left(-\frac{1}{\mu}\right)^{n+m}\int_0^{+\infty} e^{-\lambda\varepsilon t}dt\int_0^t ds\\
&\times\mathbb{E}\Bigg[\int_{\Delta_{n-1}(t-s)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_{n-1})\\
&\times\int_{\Delta_{m-1}(t-s)} dN(\gamma\mu s'_1)\dots dN(\gamma\mu s'_{m-1})\\
&\times\exp\left\{i\omega\left(k-\frac{\varepsilon\eta}{2}\right)(t-s-s_1)\right\}\\
&\exp\left\{-i\omega\left(k+\frac{\varepsilon\eta}{2}\right)(t-s-s'_1)\right\}\\
&\prod_{j=1}^{n-1}J(s_j-s_{j+1})\prod_{j=1}^{m-1}J(s'_j-s'_{j+1})\Bigg].
\end{aligned}$$

Here $s_n = s'_m := 0$. Integrating out the t variable, we get

$$\begin{aligned}
\widehat{w}_{\varepsilon,n,m}(\lambda,\eta,k) =& \frac{\varepsilon\gamma T}{\lambda}\left(1-\frac{1}{2\mu}\right)\left(-\frac{1}{\mu}\right)^{n+m}\int_0^{+\infty} e^{-\lambda\varepsilon s}\\
&\exp\left\{i\left[\omega\left(k-\frac{\varepsilon\eta}{2}\right)-\omega\left(k+\frac{\varepsilon\eta}{2}\right)\right]s\right\}ds\\
&\times\mathbb{E}\Bigg[\int_{\Delta_{n-1}(s)} dN(\gamma\mu s_1)\dots dN(\gamma\mu s_{n-1})\\
&\times\int_{\Delta_{m-1}(s)} dN(\gamma\mu s'_1)\dots dN(\gamma\mu s'_{m-1})
\end{aligned}$$

$$\times \exp\left\{-i\omega\left(k-\frac{\varepsilon\eta}{2}\right)s_1\right\}\exp\left\{i\omega\left(k+\frac{\varepsilon\eta}{2}\right)s_1'\right\}$$
$$\prod_{j=1}^{n-1}J(s_j-s_{j+1})\prod_{j=1}^{m-1}J(s_j'-s_{j+1}')\Bigg]$$

for $n, m \geq 1$. Summing out over n, m we conclude (4.2). □

Next, we write the Poisson chaos decomposition of the random field $\hat{\chi}(t,k)$. Let

$$\phi(t,k) := \int_0^t e^{-i\omega(k)(t-s)}g(ds). \tag{4.7}$$

Define the cadlag martingale

$$\tilde{N}(t) := N(t) - t, \quad t \geq 0. \tag{4.8}$$

Lemma 4. *The following expansion holds*

$$\hat{\chi}(t,k) = \sum_{n=0}^{+\infty}\hat{\chi}_n(t,k), \tag{4.9}$$

where

$$\begin{aligned}
&\hat{\chi}_0(t,k) := \phi(t,k),\\
&\hat{\chi}_n(t,k) := \left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\phi(t-s_1,k)\prod_{j=1}^{n}J\star g(s_j-s_{j+1})\\
&\qquad\qquad \times d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n), \quad n\geq 1.
\end{aligned} \tag{4.10}$$

Proof. Writing $N(\gamma\mu t) = \tilde{N}(\gamma\mu t) + \gamma\mu t$, where $\left(\tilde{N}(\gamma\mu t)\right)_{t\geq 0}$ is a cadlag martingale, we obtain

$$\begin{aligned}
\hat{\chi}(t,k) = {}& \exp\{-i\omega(k)t\} + \sum_{n=1}^{+\infty}(-\gamma)^n\int_{\Delta_n(t)}\\
&\exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n}J(s_j-s_{j+1})ds_1\dots ds_n\\
&+\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\sum_{k=1}^{n-1}(\gamma\mu)^k\sum_{\mathbf{i}\in\mathcal{I}_k^n}\int_{\Delta_n(t)}\\
&\exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n}J(s_j-s_{j+1})d\mathbf{s_i}\prod_{j\notin\mathbf{i}}d\tilde{N}(\gamma\mu s_j)\\
&+\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\\
&\exp\{-i\omega(k)(t-s_1)\}\prod_{j=1}^{n}J(s_j-s_{j+1})d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n).
\end{aligned} \tag{4.11}$$

For $1 \le k \le n$ we denote by $\mathcal{I}_k^n$ the set of all ordered k-indices $\mathbf{i} : 1 \le i_1 < \ldots < i_k \le n$. We shall also use the abbreviation $d\mathbf{s_i} := \prod_{j\in\mathbf{i}} ds_j$.

Using (2.38), we can combine the first two terms in the right-hand side of (4.11) and obtain that they are equal to $\phi(t,k)$ (cf (4.7))

Changing the order of summation in the remaining two expressions in the right-hand side of (4.11), we conclude that their sum equals

$$\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \sum_{r_1=0}^{+\infty}\sum_{r_2,\ldots,r_n=1}^{+\infty}\int_{\Delta_n(t)} \exp\left\{-i\omega\left(k\right)\left(t-s_1\right)\right\}$$
$$\times \prod_{j=1}^{n}(-\gamma)^{r_j-1}J^{\star,r_j}(s_j-s_{j+1})d\tilde{N}(\gamma\mu s_1)\ldots d\tilde{N}(\gamma\mu s_n)$$

Using formula (2.38), the above expression can be rewritten in the form:

$$\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n \int_{\Delta_n(t)}\left(\int_0^{t-s_1}\exp\left\{-i\omega\left(k\right)\left(t-s_1-\sigma\right)\right\}g(d\sigma)\right)$$
$$\times\prod_{j=1}^{n}J\star g(s_j-s_{j+1})d\tilde{N}(\gamma\mu s_1)\ldots d\tilde{N}(\gamma\mu s_n)$$
$$=\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\phi(t-s_1,k)\prod_{j=1}^{n}J\star g(s_j-s_{j+1})d\tilde{N}(\gamma\mu s_1)\ldots d\tilde{N}(\gamma\mu s_n)$$

and (4.9), with (4.10) follow. □

Coming back to calculation of the asymptotics of $\widehat{w}_\varepsilon(\lambda,\eta,k)$ given by (4.1), we have the following result.

Proposition 1. *For any $\gamma > 0$ the parameter Γ, defined by (2.48), belongs to $(0,1/2)$. In addition, for any $\mu > 1/2$, $\gamma > 0$, $\lambda \in \mathbb{C}_+$ and $(\eta,k) \in \mathbb{R}\times\mathbb{T}$ we have*

$$\lim_{\varepsilon\to 0+}\widehat{w}_\varepsilon(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{(1-\Gamma/\mu)\lambda(\lambda+i\omega'(k)\eta)}\left(1-\frac{1}{2\mu}\right). \tag{4.12}$$

Proof. We can use the $L^2(\mathbb{P})$ orthogonality of the terms of the expansion (4.9), with (4.10). For $\mathrm{Re}\,\lambda > 0$ sufficiently large we get

$$\widehat{w}_\varepsilon(\lambda,\eta,k) = \sum_{n=0}^{+\infty}\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k), \tag{4.13}$$

where

$$\widehat{w}_\varepsilon^{(0)}(\lambda,\eta,k) := \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\int_0^{+\infty}e^{-\varepsilon\lambda t}\phi^\star\left(t,k-\frac{\varepsilon\eta}{2}\right)\phi\left(t,k+\frac{\varepsilon\eta}{2}\right)dt,$$
$$\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) := \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n$$
$$\int_0^{+\infty}e^{-\varepsilon\lambda t}dt\int_{\Delta_n(t)}\phi^\star\left(t-s_1,k-\frac{\varepsilon\eta}{2}\right)\phi\left(t-s_1,k+\frac{\varepsilon\eta}{2}\right) \tag{4.14}$$

$$\times \prod_{j=1}^{n} \left(J \star g(s_j - s_{j+1})\right)^2 ds_1 \dots ds_n, \quad n \geq 1$$

In what follows, see (4.23) below, we show that (4.13) in fact holds for all $\lambda \in \mathbb{C}_+$.

Computation of $\widehat{w}_\varepsilon^{(0)}(\lambda, \eta, k)$

Thanks to (4.10) and (4.14), we have

$$\widehat{w}_\varepsilon^{(0)}(\lambda, \eta, k) = \frac{\varepsilon T \gamma}{\lambda} \left(1 - \frac{1}{2\mu}\right) \int_0^{+\infty} \int_0^{+\infty} dt dt' e^{-\varepsilon\lambda(t+t')/2} \delta(t-t') \phi^\star \left(t, k - \frac{\varepsilon\eta}{2}\right) \phi \left(t', k + \frac{\varepsilon\eta}{2}\right) \quad (4.15)$$

Using

$$\delta(t-t') = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\beta(t-t')} d\beta, \quad (4.16)$$

we can write

$$\begin{aligned} \widehat{w}_\varepsilon^{(0)}(\lambda, \eta, k) = \frac{\varepsilon T \gamma}{(2\pi)\lambda} \left(1 - \frac{1}{2\mu}\right) \int_{\mathbb{R}} d\beta \int_0^{+\infty} e^{-(\varepsilon\lambda/2 - i\beta)t} dt \\ \int_0^t \exp\left\{ i\omega \left(k - \frac{\varepsilon\eta}{2}\right)(t-s) \right\} g(ds) \quad (4.17) \\ \times \int_0^{+\infty} e^{-(\varepsilon\lambda/2 + i\beta)t'} dt' \int_0^{t'} \exp\left\{ -i\omega \left(k + \frac{\varepsilon\eta}{2}\right)(t'-s') \right\} g(ds'). \end{aligned}$$

Remark 3. The use of formula (4.16) in derivation of (4.17) is a bit formal. To justify (4.17) rigorously one can modify (4.15) as follows: $\delta(\cdot)$ is replaced by its approximation, for example

$$N f_* \left(\frac{t-t'}{N}\right) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\beta(t-t')} \exp\left\{ -\frac{\beta^2}{2N} \right\} d\beta, \quad (4.18)$$

when $N \to +\infty$. Here $f_*(t) = (2\pi)^{-1/2} e^{-t^2/2}$ is the density of the standard normal distribution. Formula (4.17) is then a consequence of the passage with N to infinity and an application of the Lebesgue dominated convergence theorem.

Integrating out s, t and s', t' variables, we obtain

$$\begin{aligned} \widehat{w}_\varepsilon^{(0)}(\lambda, \eta, k) = \frac{\varepsilon T \gamma}{(2\pi)\lambda} \left(1 - \frac{1}{2\mu}\right) \int_{\mathbb{R}} \left\{ \varepsilon\lambda/2 - i\omega \left(k - \frac{\varepsilon\eta}{2}\right) - i\beta \right\}^{-1} \\ \left\{ i\omega \left(k + \frac{\varepsilon\eta}{2}\right) + \varepsilon\lambda/2 + i\beta \right\}^{-1} \tilde{g}(\varepsilon\lambda/2 - i\beta) \tilde{g}(\varepsilon\lambda/2 + i\beta) d\beta. \end{aligned}$$

Change variables $\varepsilon\beta' := \beta + \omega\left(k - \frac{\varepsilon\eta}{2}\right)$ and obtain, cf (2.41),

$$\begin{aligned} \widehat{w}_\varepsilon^{(0)}(\lambda, \eta, k) = \frac{T \gamma}{(2\pi)\lambda} \left(1 - \frac{1}{2\mu}\right) \int_{\mathbb{R}} \left\{ \lambda/2 - i\beta \right\}^{-1} \left\{ i\delta_\varepsilon \omega(k; \eta) + \lambda/2 + i\beta \right\}^{-1} \\ \times \tilde{g}\left(\varepsilon\lambda/2 - i\varepsilon\beta + i\omega\left(k - \frac{\varepsilon\eta}{2}\right)\right) \tilde{g}\left(\varepsilon\lambda/2 + i\varepsilon\beta - i\omega\left(k + \frac{\varepsilon\eta}{2}\right)\right) d\beta. \end{aligned}$$

Here

$$\delta_\varepsilon\omega(k;\eta) := \varepsilon^{-1}\Big[\omega\left(k+\frac{\varepsilon\eta}{2}\right)-\omega\left(k-\frac{\varepsilon\eta}{2}\right)\Big]. \tag{4.19}$$

Therefore

$$\lim_{\varepsilon\to 0+}\widehat{w}^{(0)}_\varepsilon(\lambda,\eta,k) = \frac{T\gamma|\nu(k)|^2}{(2\pi)\lambda}\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}}\Big\{\lambda/2-i\beta\Big\}^{-1} \times\Big\{i\omega'(k)\eta+\lambda/2+i\beta\Big\}^{-1}d\beta. \tag{4.20}$$

To integrate out the β variable, we use the Cauchy integral formula that in our context reads

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{f(i\beta)d\beta}{z-i\beta} = f(z), \quad z\in\mathbb{C}_+. \tag{4.21}$$

It is valid for any holomorphic function f on the right half-plane $\mathbb{C}_+$ that belongs to the Hardy class $H^p(\mathbb{C}_+)$ for some $p\geq 1$, see, for example, [6, p. 113]. Applying the formula we get

$$\lim_{\varepsilon\to 0+}\widehat{w}^{(0)}_\varepsilon(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{\lambda(\lambda+i\omega'(k)\eta)}\left(1-\frac{1}{2\mu}\right). \tag{4.22}$$

Computation of $\widehat{w}^{(n)}_\varepsilon(\lambda,\eta,k)$ for $n\geq 1$

Change variables

$$\tau_0 := t-s_1,\ldots,\tau_n := s_n - s_{n+1}(=s_n)$$

in (4.14). As a result, we get

$$\widehat{w}^{(n)}_\varepsilon(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n\int_0^{+\infty}e^{-\varepsilon\lambda t/2}dt\int_{[0,+\infty)^{n+1}}d\tau_{0,n} \exp\left\{-\varepsilon\lambda\left(\tau_0+\ldots+\tau_n\right)/2\right\}\times\delta\left(t-\tau_0-\ldots-\tau_n\right)\phi^\star\left(\tau_0,k-\frac{\varepsilon\eta}{2}\right) \times\phi\left(\tau_0,k+\frac{\varepsilon\eta}{2}\right)\prod_{j=1}^n\left(J\star g(\tau_j)\right)^2.$$

Here $d\tau_{0,n} := d\tau_0\ldots d\tau_n$. Using (4.16) for each variable t and τ_j, $j=0,\ldots,n$, we can further write

$$\widehat{w}^{(n)}_\varepsilon(\lambda,\eta,k) = \frac{\varepsilon T\gamma}{(2\pi)^{n+2}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}}d\beta\int_{\mathbb{R}^{n+1}}d\beta_{0,n}\int_{[0,+\infty)^{2n+2}}d\tau_{0,n}d\tau'_{0,n} \times\int_0^{+\infty}e^{-(\varepsilon\lambda/2-i\beta)t}dt\prod_{j=0}^n\exp\left\{-(\varepsilon\lambda/4+i\beta/2+i\beta_j)\tau_j\right\} \prod_{j=0}^n\exp\left\{-(\varepsilon\lambda/4+i\beta/2-i\beta_j)\tau'_j\right\} \times\phi^\star\left(\tau_0,k-\frac{\varepsilon\eta}{2}\right)\phi\left(\tau'_0,k+\frac{\varepsilon\eta}{2}\right)\prod_{j=1}^n\left(J\star g(\tau_j)\right)\prod_{j=1}^n\left(J\star g(\tau'_j)\right).$$

To abbreviate, we have used the notation $d\beta_{0,n} := d\beta_0\ldots d\beta_n$ and analogously for the remaining variables.

Integrating the t, τ variables and their primed counterparts, we get

$$\begin{aligned}\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) =& \frac{\varepsilon T\gamma}{(2\pi)^{n+2}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{R}} \frac{d\beta}{\varepsilon\lambda/2 - i\beta} \int_{\mathbb{R}^{n+1}} d\beta_{0,n} \\ &\times \frac{\tilde{g}(\varepsilon\lambda/4 + i\beta_0 + i\beta/2)}{\varepsilon\lambda/4 + i\Big(\beta_0 + \beta/2 - \omega\left(k - \frac{\varepsilon\eta}{2}\right)\Big)} \\ &\times \frac{\tilde{g}(\varepsilon\lambda/4 - i\beta_0 + i\beta/2)}{\varepsilon\lambda/4 + i\Big(\beta/2 - \beta_0 + \omega\left(k + \frac{\varepsilon\eta}{2}\right)\Big)} \\ &\times \prod_{j=1}^n \tilde{J}\tilde{g}(\varepsilon\lambda/4 + i\beta/2 + i\beta_j) \prod_{j=1}^n \tilde{J}\tilde{g}(\varepsilon\lambda/4 + i\beta/2 - i\beta_j).\end{aligned}$$

We integrate the β variable using the Cauchy integral formula (4.21) and get

$$\begin{aligned}\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) =& \frac{\varepsilon T\gamma}{(2\pi)^{n+1}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{R}^{n+1}} d\beta_{0,n} \\ &\prod_{j=1}^n \tilde{J}\tilde{g}(\varepsilon\lambda/2 + i\beta_j) \prod_{j=1}^n \tilde{J}\tilde{g}(\varepsilon\lambda/2 - i\beta_j) \\ &\times \frac{\tilde{g}(\varepsilon\lambda/2 + i\beta_0)}{\varepsilon\lambda/2 + i\Big(\beta_0 - \omega\left(k - \frac{\varepsilon\eta}{2}\right)\Big)} \times \frac{\tilde{g}(\varepsilon\lambda/2 - i\beta_0)}{\varepsilon\lambda/2 + i\Big(-\beta_0 + \omega\left(k + \frac{\varepsilon\eta}{2}\right)\Big)}.\end{aligned}$$

Change of variables $\varepsilon\beta_0' := \beta_0 - \omega\left(k - \frac{\varepsilon\eta}{2}\right)$ and obtain

$$\begin{aligned}\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k) =& \frac{T\gamma}{(2\pi)^{n+1}\lambda}\left(1-\frac{1}{2\mu}\right)\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{R}^{n+1}} d\beta_{0,n} \prod_{j=1}^n \tilde{J}\tilde{g}(\varepsilon\lambda/2 + i\beta_j) \\ &\prod_{j=1}^n \tilde{J}\tilde{g}(\varepsilon\lambda/2 - i\beta_j) \times \frac{\tilde{g}(\varepsilon\lambda/2 + i\varepsilon\beta_0 + i\omega\left(k - \frac{\varepsilon\eta}{2}\right))}{\lambda/2 + i\beta_0} \\ &\times \frac{\tilde{g}(\varepsilon\lambda/2 - i\varepsilon\beta_0 - i\omega\left(k - \frac{\varepsilon\eta}{2}\right))}{\lambda/2 + i\left(-\beta_0 + \delta_\varepsilon\omega(k;\eta)\right)}.\end{aligned}$$

According to Lemma 1, we have $\tilde{J}\tilde{g} \in H^2(\mathbb{C}_+)$; therefore, see, for example, [9, Theorem 19.2],

$$\frac{\gamma}{2\pi}\int_{\mathbb{R}} |\tilde{J}\tilde{g}(\lambda + i\beta)|^2 d\beta \le \frac{\gamma}{2\pi}\int_{\mathbb{R}} |\tilde{J}\tilde{g}(i\beta)|^2 d\beta = \Gamma < \frac{1}{2} \le \mu, \quad \operatorname{Re}\lambda > 0.$$

The last estimate follows from (2.51). In particular, there exists a constant $C > 0$ such that

$$|\widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k)| \le C\left(\frac{\Gamma}{\mu}\right)^n, \quad n \ge 0,\ \varepsilon > 0,\ \lambda \in \mathbb{C}_+ \text{ and } (\eta,k) \in \mathbb{R}\times\mathbb{T}. \tag{4.23}$$

This proves that the validity of (4.13) for all $\lambda \in \mathbb{C}_+$.

Furthermore,

$$\widehat{w}^{(n)}(\lambda,\eta,k) := \lim_{\varepsilon\to 0+} \widehat{w}_\varepsilon^{(n)}(\lambda,\eta,k)$$

$$= \frac{\gamma T}{\lambda(2\pi)}\left(\frac{\Gamma}{\mu}\right)^n |\nu(k)|^2\left(1-\frac{1}{2\mu}\right)\int_{\mathbb{R}}\frac{d\beta_0}{(\lambda/2+i\beta_0)\{\lambda/2+i[-\beta_0+\omega'(k)\eta]\}}.$$

Here Γ is given by (2.48). Integrating the β_0 variable out, using again (4.21), we get

$$\widehat{w}^{(n)}(\lambda,\eta,k) = \frac{\gamma T|\nu(k)|^2}{\lambda(\lambda+i\omega'(k)\eta)}\left(\frac{\Gamma}{\mu}\right)^n\left(1-\frac{1}{2\mu}\right).$$

Using (4.23), by the dominated convergence theorem, we conclude that

$$\widehat{w}(\lambda,\eta,k) = \sum_{n=0}^{+\infty}\widehat{w}^{(n)}(\lambda,\eta,k) \tag{4.24}$$

and formula (4.12) follows. □

5. The Case $T = 0$ and Non-Zero Initial Data

Here, as in Sect. 3.2, we assume that $T = 0$ and the initial data need not be null, and satisfies the assumptions made in Sects. 2.3.3 and 2.3.4. The solution $\hat{\psi}(t,k)$ is then described by the expansion (3.14) and (3.16).

Using the same argument as in the proof of Lemma 4, we obtain the following Poisson chaos expansion for the momentum at $x = 0$ and the Fourier transform of the wave function

$$\begin{aligned}\mathfrak{p}_0(t) = g\star\mathfrak{p}_0^0(t) + \sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\\ \times\prod_{j=1}^{n}J\star g(s_{j-1}-s_j)g\star\mathfrak{p}_0^0(s_n)d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n),\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\hat{\psi}(t,k) = e^{-i\omega(k)t}\hat{\psi}(0,k) - i\gamma\int_0^t\phi(t-s,k)\mathfrak{p}_0^0(s)ds\\ + i\sum_{n=1}^{+\infty}\left(-\frac{1}{\mu}\right)^n\int_{\Delta_n(t)}\phi(t-s_1,k)\\ \prod_{j=1}^{n-1}J\star g(s_j-s_{j+1})g\star\mathfrak{p}_0^0(s_n)d\tilde{N}(\gamma\mu s_1)\dots d\tilde{N}(\gamma\mu s_n),\end{aligned} \tag{5.2}$$

where $\mathfrak{p}_0^0(\cdot)$ is given by (3.2). In light of (2.51), both of these expansions are valid for any $\mu \geq 1/2$.

On the other hand, from (2.18), with $\tilde{p}(t) \equiv 0$, we obtain the following equation on the Fourier–Wigner function $W_\varepsilon(t,\eta,k)$

$$\begin{aligned}\partial_t\widehat{W}_\varepsilon(t,\eta,k) + i\delta_\varepsilon\omega(k;\eta)\widehat{W}_\varepsilon(t,\eta,k) = \frac{\gamma}{2\mu}\mathbb{E}\left[\mathfrak{p}_0^2\left(\frac{t}{\varepsilon}\right)\right]\\ +\frac{i\gamma}{2}\left\{\mathbb{E}\left[\hat{\psi}\left(\frac{t}{\varepsilon},k+\frac{\varepsilon\eta}{2}\right)\mathfrak{p}_0\left(\frac{t}{\varepsilon}\right)\right] - \mathbb{E}\left[\hat{\psi}^*\left(\frac{t}{\varepsilon},k-\frac{\varepsilon\eta}{2}\right)\mathfrak{p}_0\left(\frac{t}{\varepsilon}\right)\right]\right\}\end{aligned} \tag{5.3}$$

Taking the Laplace transform on both sides, we arrive at

$$\begin{aligned}(\lambda + i\delta_\varepsilon\omega(k;\eta))\,\widehat{w}_\varepsilon(\lambda,\eta,k) &= W_\varepsilon(0,\eta,k)\\ &+ \frac{\gamma}{\mu}\mathfrak{e}_\varepsilon(\lambda) - \frac{\gamma}{2}\left[\mathfrak{d}_\varepsilon\left(\lambda, k-\frac{\varepsilon\eta}{2}\right) + \mathfrak{d}^\star_\varepsilon\left(\lambda, k+\frac{\varepsilon\eta}{2}\right)\right],\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\mathfrak{e}_\varepsilon(\lambda) &:= \frac{\varepsilon}{2}\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}\left[\mathfrak{p}_0^2(t)\right]dt \quad \text{and}\\ \mathfrak{d}_\varepsilon(\lambda,k) &:= i\varepsilon\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}\left[\hat{\psi}^*(t,k)\,\mathfrak{p}_0(t)\right]dt.\end{aligned} \tag{5.5}$$

In the present section, we show the following.

Proposition 2. *For any $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$ and $\operatorname{Re}\lambda > 0$ we have*

$$\int_{\mathbb{R}}\int_{\mathbb{T}} \widehat{w}(\lambda,\eta,k)G^*(\eta,k)d\eta dk = \lim_{\varepsilon\to 0+}\int_{\mathbb{R}}\int_{\mathbb{T}} \widehat{w}_\varepsilon(\lambda,\eta,k)G^*(\eta,k)d\eta dk,$$

where

$$\begin{aligned}\widehat{w}(\lambda,\eta,k) = \frac{\widehat{W}(0,\eta,k)}{\lambda+i\omega'(k)\eta} &+ \frac{\gamma|\nu(k)|^2}{2(1-\Gamma/\mu)(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}}\int_{\mathbb{T}}\frac{\widehat{W}(0,\eta,\ell)|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell\\ &- \frac{\gamma\operatorname{Re}[\nu(k)]}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',k)}{\lambda+i\omega'(k)\eta'}d\eta' + \frac{\gamma\mathfrak{g}(k)}{4(\lambda+i\omega'(k)\eta)}\\ &\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',k)d\eta'}{\lambda+i\omega'(k)\eta'} + \frac{\gamma\mathfrak{g}(k)}{4(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',-k)d\eta'}{\lambda-i\omega'(k)\eta'}.\end{aligned} \tag{5.6}$$

The proof of the proposition is carried out throughout Sects. 5.1–5.3.

5.1. Asymptotics of $\mathfrak{e}_\varepsilon(\lambda)$

Proposition 3. *Under the assumption about the initial data made in Sects. 2.3.3 and 2.3.4, we have*

$$\lim_{\varepsilon\to 0+}\mathfrak{e}_\varepsilon(\lambda) = \frac{1}{2(1-\Gamma/\mu)}\int_{\mathbb{R}}\int_{\mathbb{T}}\frac{\widehat{W}(0,\eta,\ell)|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell. \tag{5.7}$$

Proof. From (5.1) we get

$$\begin{aligned}\mathbb{E}\left[\mathfrak{p}_0^2(t)\right] &= \mathbb{E}[g\star\mathfrak{p}_0^0(t)]^2 + \sum_{n=1}^{+\infty}\left(\frac{\gamma}{\mu}\right)^n\int_{\Delta_n(t)}\\ &\prod_{j=1}^{n}(J\star g)^2(s_{j-1}-s_j)\mathbb{E}[g\star\mathfrak{p}_0^0(s_n)]^2 ds_1\ldots ds_n.\end{aligned} \tag{5.8}$$

Arguing as in the proof of Proposition 1, we conclude that for $\lambda \in \mathbb{C}_+$

$$\mathfrak{e}_\varepsilon(\lambda) = \sum_{n=0}^{+\infty} E_n^{(\varepsilon)}(\lambda), \quad \text{where}$$

$$E_0^{(\varepsilon)}(\lambda) := \frac{\varepsilon}{2}\int_0^{+\infty} e^{-\lambda\varepsilon t}\mathbb{E}[g\star\mathfrak{p}_0^0(t)]^2 dt,$$

$$E_n^{(\varepsilon)}(\lambda) := \frac{\varepsilon}{2}\left(\frac{\gamma}{\mu}\right)^n \int_0^{+\infty} e^{-\lambda\varepsilon t}dt \int_{\Delta_n(t)} (J\star g)^2(t-s_1)\prod_{j=1}^{n-1}(J\star g)^2(s_j-s_{j+1}) \times \mathbb{E}[g\star\mathfrak{p}_0^0(s_n)]^2 ds_1\dots ds_n. \tag{5.9}$$

Asymptotics of $E_0^{(\varepsilon)}(\lambda)$. Using (4.16) we can write

$$E_0^{(\varepsilon)}(\lambda) = -\frac{\varepsilon}{2^4\pi}\int_{\mathbb{T}^2} dkdk' \int_0^{+\infty} dt \int_0^{+\infty} dt' e^{-\lambda\varepsilon(t+t')/2} \int_0^t\int_0^{t'} g(d\sigma)g(d\sigma') \int_{\mathbb{R}} d\beta e^{i\beta(t-t')} \times \mathbb{E}\left\{\left\{e^{-i\omega(k)(t-\sigma)}\hat\psi(k) - e^{i\omega(k)(t-\sigma)}\hat\psi^*(k)\right\} \left\{e^{-i\omega(k')(t'-\sigma')}\hat\psi(k') - e^{i\omega(k')(t'-\sigma')}\hat\psi^*(k')\right\}\right\}.$$

Thanks to (2.20), we can write

$$E_0^{(\varepsilon)}(\lambda) = \frac{\varepsilon}{2^3\pi}\int_{\mathbb{T}^2} dkdk' \int_0^{+\infty} dt \int_0^{+\infty} dt' e^{-\lambda\varepsilon(t+t')/2} \int_0^t\int_0^{t'} g(d\sigma)g(d\sigma') \int_{\mathbb{R}} d\beta e^{i\beta(t-t')} \times \exp\{i\omega(k')(t'-\sigma') - i\omega(k)(t-\sigma)\}\,\mathbb{E}\left\{\hat\psi(k)\hat\psi^*(k')\right\}.$$

Integrating out the t and t' variables, we get

$$E_0^{(\varepsilon)}(\lambda) = \frac{\varepsilon}{2^3\pi}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{|\tilde g(\lambda\varepsilon/2 - i\beta)|^2}{\lambda\varepsilon/2 - i\beta + i\omega(k)} \cdot \frac{\mathbb{E}\left\{\hat\psi(k)\hat\psi^*(k')\right\}}{\lambda\varepsilon/2 + i\beta - i\omega(k')} dkdk'd\beta.$$

Next we change variables $\varepsilon\beta' := \beta - \omega(k')$, which leads to

$$E_0^{(\varepsilon)}(\lambda) = \frac{1}{2^3\pi}\int_{\mathbb{T}^2}\int_{\mathbb{R}} \frac{|\tilde g(\lambda\varepsilon/2 - i\varepsilon\beta + i\omega(k))|^2}{\lambda/2 - i\beta + i\varepsilon^{-1}[\omega(k)-\omega(k')]} \cdot \frac{\mathbb{E}\left\{\hat\psi(k)\hat\psi^*(k')\right\}}{\lambda/2 + i\beta} dkdk'd\beta. \tag{5.10}$$

Change variables $(k,k') \mapsto (\eta,\ell)$, by letting

$$k := \ell + \frac{\varepsilon\eta}{2}, \qquad k' := \ell - \frac{\varepsilon\eta}{2}. \tag{5.11}$$

The image of $\mathbb{T}^2$ under this mapping is

$$T_\varepsilon^2 := \left[(\eta,\ell) :\ |\eta| \le \frac{1}{\varepsilon},\ |\ell| \le \frac{1-\varepsilon|\eta|}{2}\right] \subset \mathbb{T}_{2/\varepsilon}\times\mathbb{T}. \tag{5.12}$$

Then, cf (4.19),

$$E_0^{(\varepsilon)}(\lambda)=\frac{1}{2^2\pi}\int_{T_\varepsilon^2}\widehat{W}_\varepsilon(0,\eta,\ell)d\eta d\ell\int_{\mathbb{R}}d\beta\frac{\left|\tilde{g}\left(\lambda\varepsilon/2-i\varepsilon\beta+i\omega(\ell+\frac{\varepsilon\eta}{2})\right)\right|^2}{(\lambda/2-i\beta+i\delta_\varepsilon\omega(\ell,\eta))\,(\lambda/2+i\beta)}.$$

Using estimates (2.27), (2.28) and the Cauchy formula (4.21), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0+}E_0^{(\varepsilon)}(\lambda)&=\frac{1}{2^2\pi}\int_{\mathbb{R}}d\eta\int_{\mathbb{T}}d\ell\int_{\mathbb{R}}d\beta\frac{|\nu(\ell)|^2\,\widehat{W}(0,\eta,\ell)}{(\lambda/2-i\beta+i\omega'(\ell)\eta)\,(\lambda/2+i\beta)}\\&=\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{T}}\frac{\widehat{W}(0,\eta,\ell)\,|\nu(\ell)|^2}{\lambda+i\omega'(\ell)\eta}d\eta d\ell.\end{aligned}\tag{5.13}$$

Asymptotics of $E_n^{(\varepsilon)}(\lambda)$ for $n\geq 1$. Using (3.2) and (2.20), we get

$$\begin{aligned}E_n^{(\varepsilon)}(\lambda)=\frac{\varepsilon}{2^2}\left(\frac{\gamma}{\mu}\right)^n&\int_{\mathbb{T}}dk\int_{\mathbb{T}}dk'\mathbb{E}\left\{\hat{\psi}(k)\hat{\psi}^*(k')\right\}\int_0^{+\infty}e^{-\lambda\varepsilon t}dt\int_{\Delta_n(t)}(J\star g)^2(t-s_1)dt\\&\times\prod_{j=1}^{n-1}(J\star g)^2(s_j-s_{j+1})\int_0^{s_n}\int_0^{s_n}g(d\sigma_1)g(d\sigma_1')\\&\times\exp\left\{i\omega(k')(s_n-\sigma_1')-i\omega(k)(s_n-\sigma_1)\right\}.\end{aligned}$$

We substitute $\tau_j:=s_j-s_{j+1}$, $j=0,\dots,n$, with $s_0:=t$ and $s_{n+1}:=0$, and then use (4.16) to double variables τ_j and τ_j'. In this way we obtain

$$\begin{aligned}E_n^{(\varepsilon)}(\lambda)=\frac{\varepsilon}{2^2(2\pi)^{n+3}}&\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}^2}d\beta d\beta'\int_{\mathbb{T}^2}dkdk'\\&\int_{(0,+\infty)^2}dtdt'e^{-\lambda\varepsilon(t+t')/4}\mathbb{E}\left\{\hat{\psi}(k)\hat{\psi}^*(k')\right\}\\&\times\int_{(0,+\infty)^{n+1}}d\tau_{0,n}\int_{(0,+\infty)^{n+1}}d\tau_{0,n}'\int_{\mathbb{R}^{n+1}}d\beta_{0,n}\prod_{j=0}^{n}e^{i\beta_j(\tau_j-\tau_j')}\\&\times\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^{n}\tau_j\right)/4\right\}\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^{n}\tau_j'\right)/4\right\}\\&\exp\left\{i\beta\left(t-\sum_{j=0}^{n}\tau_j\right)\right\}\exp\left\{i\beta'\left(t'-\sum_{j=0}^{n}\tau_j'\right)\right\}\\&\times\prod_{j=0}^{n-1}(J\star g)(\tau_j)\prod_{j=0}^{n-1}(J\star g)(\tau_j')\int_0^{\tau_n}\\&\int_0^{\tau_n'}g(d\sigma)g(d\sigma')\exp\left\{i\omega(k')(\tau_n'-\sigma')-i\omega(k)(\tau_n-\sigma)\right\}.\end{aligned}$$

To abbreviate we have used the notation $d\tau_{0,n}=d\tau_0\dots d\tau_n$, $d\beta_{0,n}=d\beta_0\dots d\beta_n$ and similarly for the prime variables. Integrating out the t, τ variables and their prime counterparts, we get

$$E_n^{(\varepsilon)}(\lambda) = \frac{\varepsilon}{2^2(2\pi)^{n+3}}\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{R}^2} d\beta d\beta' \int_{\mathbb{T}^2} dk dk' \int_{\mathbb{R}^{n+1}} d\beta_{0,n}\mathbb{E}\left\{\hat{\psi}(k)\hat{\psi}^*(k')\right\}$$
$$\times \prod_{j=0}^{n-1}\left(\tilde{J}\tilde{g}\right)(\lambda\varepsilon/4 - i\beta_j + i\beta) \prod_{j=0}^{n-1}\left(\tilde{J}\tilde{g}\right)(\lambda\varepsilon/4 + i\beta_j + i\beta')$$
$$\frac{1}{\lambda\varepsilon/4 - i\beta}\cdot\frac{1}{\lambda\varepsilon/4 - i\beta'}\cdot\frac{\tilde{g}(\lambda\varepsilon/4 - i\beta_n + i\beta)}{\lambda\varepsilon/4 - i\beta_n + i\beta + i\omega(k)}\cdot\frac{\tilde{g}(\lambda\varepsilon/4 + i\beta_n + i\beta')}{\lambda\varepsilon/4 + i\beta_n + i\beta' - i\omega(k')}. \tag{5.14}$$

Change variables k, k' according to (5.11) and

$$\varepsilon\beta_n := \beta_n - \omega(k'), \quad \varepsilon\beta := \beta, \quad \varepsilon\beta' := \beta'$$

we obtain

$$E_n^{(\varepsilon)}(\lambda) = \frac{1}{2(2\pi)^{n+3}}\left(\frac{\gamma}{\mu}\right)^n \int_{\mathbb{R}^2} \frac{d\beta d\beta'}{(\lambda/4 - i\beta)(\lambda/4 - i\beta')} \int_{T_\varepsilon^2} \widehat{W}_\varepsilon(0,\eta,\ell) d\eta d\ell$$
$$\times \int_{\mathbb{R}^{n+1}} d\beta_{0,n} \prod_{j=0}^{n-1}\left(\tilde{J}\tilde{g}\right)(\lambda\varepsilon/4 - i\beta_j + i\varepsilon\beta) \prod_{j=0}^{n-1}\left(\tilde{J}\tilde{g}\right)(\lambda\varepsilon/4 + i\beta_j - i\varepsilon\beta')$$
$$\times \frac{\tilde{g}(\lambda\varepsilon/4 - i\varepsilon\beta_n - i\omega(\ell - \varepsilon\eta/2) + i\varepsilon\beta)}{\lambda/4 - i\beta_n + i\beta + i\delta_\varepsilon\omega(\ell,\eta)}\cdot\frac{\tilde{g}(\lambda\varepsilon/4 + i\varepsilon\beta_n + i\omega(\ell - \varepsilon\eta/2) + i\varepsilon\beta')}{\lambda/4 + i\beta_n + i\beta'}.$$

Hence

$$\lim_{\varepsilon\to 0+} E_n^{(\varepsilon)}(\lambda) = \frac{\Gamma^n}{2\mu^n}\int_{\mathbb{R}}\int_{\mathbb{T}} \frac{\widehat{W}(0,\eta,\ell)|\nu(\ell)|^2}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.15}$$

The conclusion of the proposition then follows from an application of the dominated convergence theorem to the series appearing in (5.9), as $\Gamma/\mu \in (0,1)$. □

5.2. Asymptotics of the Term Involving $\mathfrak{d}_\varepsilon(\boldsymbol{\lambda})$

Invoking (5.4) we wish to calculate the limit $\lim_{\varepsilon\to 0+}\mathfrak{L}_\varepsilon$, where

$$\mathfrak{L}_\varepsilon := \int_{\mathbb{T}}\int_{\mathbb{R}}\left[\mathfrak{d}_\varepsilon\left(\lambda, k - \frac{\varepsilon\eta}{2}\right) + \mathfrak{d}_\varepsilon^\star\left(\lambda, k + \frac{\varepsilon\eta}{2}\right)\right]\frac{G^\star(\eta,k)d\eta dk}{\lambda + i\delta_\varepsilon\omega(k;\eta)}, \tag{5.16}$$

for any $G \in \mathcal{S}(\mathbb{R}\times\mathbb{T})$.

Taking into account (5.1) and (5.2), we get

$$\mathfrak{d}_\varepsilon(\lambda,k) = \sum_{n=0}^{+\infty} D_n^\varepsilon(\lambda,k), \tag{5.17}$$

where

$$D_0^\varepsilon(\lambda,k) = D_{0,1}^\varepsilon(\lambda,k) + D_{0,2}^\varepsilon(\lambda,k) \tag{5.18}$$

and

$$D_{0,1}^\varepsilon(\lambda,k) := i\varepsilon\int_0^{+\infty} e^{-\lambda\varepsilon t}e^{i\omega(k)t}\mathbb{E}\left[\hat{\psi}^\star(0,k)g\star\mathfrak{p}_0^0(t)\right]dt,$$
$$D_{0,2}^\varepsilon(\lambda,k) := -\varepsilon\gamma\int_0^{+\infty} e^{-\lambda\varepsilon t}dt\int_0^t \phi^\star(t-s,k)\mathbb{E}\left[\mathfrak{p}_0^0(s)g\star\mathfrak{p}_0^0(t)\right]ds,$$

$$D_n^\varepsilon(\lambda,k) := \varepsilon\left(\frac{\gamma}{\mu}\right)^n \int_0^{+\infty} e^{-\lambda t}dt \int_{\Delta_n(t)} \phi^*(t-s_1,k)(J\star g)(t-s_1) \quad (5.19)$$
$$\times \prod_{j=1}^{n-1}(J\star g)^2(s_j-s_{j+1})\mathbb{E}\left[\left(g\star \mathfrak{p}_0^0(s_n)\right)^2\right] ds_1\dots ds_n,\ n\geq 1.$$

Accordingly we can write $\mathfrak{L}_\varepsilon = \sum_{n=0}^{+\infty}\mathfrak{L}_\varepsilon^{(n)}$, where

$$\mathfrak{L}_\varepsilon^{(n)} := \int_{\mathbb{R}}\int_{\mathbb{T}}\left[D_n^\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+(D_n^\varepsilon)^*\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]\frac{G^\star(\eta,k)}{\lambda+i\delta_\varepsilon\omega(k;\eta)}d\eta dk. \quad (5.20)$$

5.2.1. Computation of $D_{0,1}^\varepsilon(\lambda,k)$. The term $D_{0,1}^\varepsilon(\lambda,k)$ coincides with $\mathfrak{d}_\varepsilon^1(\lambda,k)$ defined in [5, formulas (5.6) and (5.7)]. Therefore, see [5, Lemma 5.1], we have the following result.

Lemma 1. *For any test function $G\in\mathcal{S}(\mathbb{R}\times\mathbb{T})$ and $\lambda>0$, we have*

$$-\frac{\gamma}{2}\lim_{\varepsilon\to 0+}\int_{\mathbb{R}\times\mathbb{T}}\frac{G^*(\eta,k)}{\lambda+i\delta_\varepsilon\omega(k,\eta)}\left\{D_{0,1}^\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+(D_{0,1}^\varepsilon)^*\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right\}d\eta dk \quad (5.21)$$
$$=-\gamma\int_{\mathbb{R}\times\mathbb{T}}\mathrm{Re}[\nu(k)]\frac{\widehat{W}(0,\eta',k)}{\lambda+i\omega'(k)\eta'}\left\{\int_{\mathbb{R}}\frac{G^*(\eta,k)}{\lambda+i\omega'(k)\eta}d\eta\right\}dkd\eta'.$$

5.2.2. Asymptotics of $D_{0,2}^\varepsilon(\lambda,k)$. Using (4.7) we can write

$$D_{0,2}^\varepsilon(\lambda,k) = -\varepsilon\gamma\int_0^{+\infty}e^{-\lambda\varepsilon t}dt\int_0^t ds\exp\{i\omega(k)(t-s)\}\,\mathbb{E}\left[g\star\mathfrak{p}_0^0(s)g\star\mathfrak{p}_0^0(t)\right]$$

The expression for $D_{0,2}^\varepsilon(\lambda,k)$ is therefore identical with $\mathfrak{d}_\varepsilon^2(\lambda,k)$ defined by [5, formulas (5.6) and (5.7)]. We have therefore, see [5, Lemma 5.2].

Lemma 5. *For any $\lambda>0$ and $G\in\mathcal{S}(\mathbb{R}\times\mathbb{T})$ we have*

$$-\frac{\gamma}{2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}\times\mathbb{T}}\left[D_{0,2}^\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+\left(D_{0,2}^\varepsilon\right)^*\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]\frac{\hat{G}^*(\eta,k)d\eta dk}{\lambda+i\delta_\varepsilon\omega(k,\eta)}$$
$$=\frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{f}(k)\widehat{W}(0,\eta',k)d\eta' dk}{\lambda+i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)d\eta}{\lambda+i\omega'(k)\eta}$$
$$+\frac{\gamma}{4}\int_{\mathbb{R}\times\mathbb{T}}\frac{\mathfrak{f}(k)\widehat{W}(0,\eta',-k)d\eta' dk}{\lambda-i\omega'(k)\eta'}\int_{\mathbb{R}}\frac{\hat{G}^*(\eta,k)d\eta}{\lambda+i\omega'(k)\eta}. \quad (5.22)$$

Summarizing, taking into account definitions (2.43), we have

$$-\frac{\gamma}{2}\lim_{\varepsilon\to 0}\mathfrak{L}_\varepsilon^{(0)} = \frac{(p_+(k)-1)|\bar{\omega}'(k)|}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}\times\mathbb{T}}\frac{\widehat{W}(0,\eta',k)d\eta'}{\lambda+i\omega'(k)\eta'}$$
$$+\frac{p_-(k)|\bar{\omega}'(k)|}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}}\frac{\widehat{W}(0,\eta',-k)d\eta'}{\lambda-i\omega'(k)\eta'}. \quad (5.23)$$

5.2.3. Asymptotics of $\sum_{n=1}^{+\infty} D_n^\varepsilon(\lambda,k)$. We prove the following.

Lemma 6. *For any $\lambda>0$ we have*

$$-\frac{\gamma}{2}\lim_{\varepsilon\to 0}\sum_{n=1}^{+\infty}\int_{\mathbb{R}\times\mathbb{T}}\left[D_n^\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+(D_n^\varepsilon)^*\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]\frac{\hat G^*(\eta,k)d\eta dk}{\lambda+i\delta_\varepsilon\omega(k,\eta)}$$
$$=-\frac{\gamma}{2\mu(1-\Gamma/\mu)}\int_{\mathbb{R}\times\mathbb{T}}\frac{G^*(\eta,k)[1-|\nu(k)|^2]d\eta dk}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}\times\mathbb{T}}\frac{|\nu(\ell)|^2\widehat{W}(0,\eta',\ell)d\eta' d\ell}{\lambda+i\omega'(\ell)\eta'}. \tag{5.24}$$

The proof of the lemma is presented in Sect. 5.2.5. It requires some auxiliary calculations that are done in Sect. 5.2.4.

5.2.4. Auxiliary Calculations. We suppose that $n\geq 1$. Using the change of variables $\tau_j:=s_j-s_{j+1}$, $j=0,\dots,n$, with $s_0:=t$ and $s_{n+1}:=0$ in the last formula of (5.19) and then (4.16) we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{2\pi}\left(\frac{\gamma}{\mu}\right)^n\int_0^{+\infty}e^{-\lambda\varepsilon t/2}dt\int_{\mathbb{R}}d\beta\int_{(0,+\infty)^{n+1}}d\tau_{0,n}$$
$$\times\exp\left\{i\beta\left(t-\sum_{j=0}^n\tau_j\right)\right\}\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j\right)/2\right\}\phi^*(\tau_0,k)(J\star g)(\tau_0) \tag{5.25}$$
$$\prod_{j=1}^{n-1}(J\star g)^2(\tau_j)\mathbb{E}\left[\left(g\star\mathfrak{p}_0^0(\tau_n)\right)^2\right],\ n\geq 1.$$

Doubling the τ_j variables, via (4.16), we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{(2\pi)^{n+2}}\left(\frac{\gamma}{\mu}\right)^n\int_0^{+\infty}dt\int_{\mathbb{R}^{n+2}}d\beta_{0,n}d\beta\int_{(0,+\infty)^{n+1}}d\tau_{0,n}\int_{(0,+\infty)^{n+1}}d\tau'_{0,n}$$
$$\times e^{-\lambda\varepsilon t/2}\prod_{j=0}^n\exp\left\{i\beta_j(\tau_j-\tau'_j)\right\}\exp\left\{i\beta\left(t-\frac{1}{2}\sum_{j=0}^n\tau_j-\frac{1}{2}\sum_{j=0}^n\tau'_j\right)\right\}$$
$$\times\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau_j\right)/4\right\}\exp\left\{-\lambda\varepsilon\left(\sum_{j=0}^n\tau'_j\right)/4\right\}$$
$$\times\phi^*(\tau'_0,k)(J\star g)(\tau_0)\prod_{j=1}^{n-1}(J\star g)(\tau_j)\prod_{j=1}^{n-1}(J\star g)(\tau'_j)\mathbb{E}\left[\left(g\star\mathfrak{p}_0^0(\tau_n)\right)\left(g\star\mathfrak{p}_0^0(\tau'_n)\right)\right].$$

Integrating out the t, τ and τ' variables, we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{(2\pi)^{n+2}}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{R}}\frac{d\beta}{\lambda\varepsilon/2-i\beta} \tag{5.26}$$
$$\int_{\mathbb{R}^{n+1}}d\beta_{0,n}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_0+i\beta/2\right)\tilde\phi^*(\lambda\varepsilon/4-i\beta_0-i\beta/2,k)$$
$$\times\prod_{j=1}^{n-1}(\tilde J\tilde g)\left(\lambda\varepsilon/4-i\beta_j+i\beta/2\right)$$

$$\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})\left(\lambda\varepsilon/4+i\beta_j+i\beta/2\right)\tilde{g}(\lambda\varepsilon/4-i\beta_n+i\beta/2)\tilde{g}(\lambda\varepsilon/4+i\beta_n+i\beta/2)$$
$$\times\,\mathbb{E}\left[\tilde{\mathfrak{p}}_0^0(\lambda\varepsilon/4-i\beta_n+i\beta/2)\tilde{\mathfrak{p}}_0^0(\lambda\varepsilon/4+i\beta_n+i\beta/2)\right].$$

Here

$$\tilde{\phi}(\lambda,k)=\frac{\tilde{g}(\lambda)}{\lambda+i\omega(k)}$$

and

$$\tilde{\mathfrak{p}}_0^0(\lambda)=\frac{1}{2i}\int_{\mathbb{T}}\left\{\frac{\hat{\psi}(\ell)}{\lambda+i\omega(\ell)}-\frac{\hat{\psi}^*(\ell)}{\lambda-i\omega(\ell)}\right\}d\ell$$

are the Laplace transforms of $\phi(t,k)$ and $\mathfrak{p}_0^0(t)$, respectively.

Thanks to (2.20), we have

$$\mathbb{E}\left[\tilde{\mathfrak{p}}_0^0(\lambda_1)\tilde{\mathfrak{p}}_0^0(\lambda_2)\right]=\frac{1}{2^2}\int_{\mathbb{T}}d\ell\int_{\mathbb{T}}d\ell'$$
$$\left\{\frac{\mathbb{E}\left[\hat{\psi}(\ell)\hat{\psi}^*(\ell')\right]}{\left(\lambda_1+i\omega(\ell)\right)\left(\lambda_2-i\omega(\ell')\right)}+\frac{\mathbb{E}\left[\hat{\psi}(\ell')\hat{\psi}^*(\ell)\right]}{\left(\lambda_1-i\omega(\ell)\right)\left(\lambda_2+i\omega(\ell')\right)}\right\}$$

Substituting in (5.26), we get

$$D_n^\varepsilon(\lambda,k)=\frac{\varepsilon}{2^2(2\pi)^{n+2}}\left(\frac{\gamma}{\mu}\right)^n\int_{\mathbb{T}^2}d\ell d\ell'\int_{\mathbb{R}}\frac{d\beta}{\lambda\varepsilon/2-i\beta}$$
$$\int_{\mathbb{R}^{n+1}}d\beta_{0,n}(\tilde{J}\tilde{g})\left(\lambda\varepsilon/4-i\beta_0+i\beta/2\right)$$
$$\times\frac{\tilde{g}(\lambda\varepsilon/4+i\beta_0+i\beta/2)}{\lambda\varepsilon/4+i\beta_0+i\beta/2-i\omega(k)}\tilde{g}(\lambda\varepsilon/4-i\beta_n+i\beta/2)\tilde{g}(\lambda\varepsilon/4+i\beta_n+i\beta/2)$$
$$\times\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})\left(\lambda\varepsilon/4-i\beta_j+i\beta/2\right)\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})\left(\lambda\varepsilon/4+i\beta_j+i\beta/2\right)$$
$$\times\left\{\frac{\mathbb{E}\left[\hat{\psi}(\ell)\hat{\psi}^*(\ell')\right]}{[\lambda\varepsilon/4-i\beta_n+i\beta/2+i\omega(\ell)][\lambda\varepsilon/4+i\beta_n+i\beta/2-i\omega(\ell')]}\right.$$
$$\left.+\frac{\mathbb{E}\left[\hat{\psi}^*(\ell)\hat{\psi}(\ell')\right]}{[\lambda\varepsilon/4-i\beta_n+i\beta/2-i\omega(\ell)][\lambda\varepsilon/4+i\beta_n+i\beta/2+i\omega(\ell')]}\right\}.$$

Change variables $\beta_j':=\beta_j+\beta/2$, $j=0,\dots,n$ and integrate out the β variable, using (4.21). We can write then

$$D_n^\varepsilon(\lambda,k)=\frac{1}{4\mu^n}\left(\frac{\gamma}{2\pi}\right)^{n-1}I_\varepsilon II_\varepsilon\int_{\mathbb{R}^n}d\beta_{1,n}\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})\left(3\lambda\varepsilon/4-i\beta_j\right)$$
$$\times\prod_{j=1}^{n-1}(\tilde{J}\tilde{g})\left(\lambda\varepsilon/4+i\beta_j\right),\tag{5.27}$$

where

$$I_\varepsilon := \frac{\gamma}{2\pi}\int_{\mathbb{R}} (\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_0)\,\frac{\tilde{g}(\lambda\varepsilon/4 + i\beta_0)}{\lambda\varepsilon/4 + i\beta_0 - i\omega(k)}d\beta_0 \tag{5.28}$$

and

$$\begin{aligned} II_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} \tilde{g}(3\lambda\varepsilon/4 - i\beta_n)\tilde{g}(\lambda\varepsilon/4 + i\beta_n) \\ \times\left\{\frac{\mathbb{E}\left[\hat{\psi}(\ell)\hat{\psi}^*(\ell')\right]}{[3\lambda\varepsilon/4 - i\beta_n + i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n - i\omega(\ell')]}\right. \\ \left. + \frac{\mathbb{E}\left[\hat{\psi}^*(\ell)\hat{\psi}(\ell')\right]}{[3\lambda\varepsilon/4 - i\beta_n - i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n + i\omega(\ell')]}\right\} d\beta_n. \end{aligned} \tag{5.29}$$

5.2.5. The End of the Proof of Lemma 6. Using formula (5.27) we conclude, cf (5.20) and (2.48), that

$$\lim_{\varepsilon\to 0+} \mathfrak{L}_\varepsilon^{(n)} = \lim_{\varepsilon\to 0+} \bar{\mathfrak{L}}_\varepsilon^{(n)}, \tag{5.30}$$

where

$$\bar{\mathfrak{L}}_\varepsilon^{(n)} := 2\int_{\mathbb{R}}\int_{\mathbb{T}} \mathrm{Re}\,\tilde{D}_n^\varepsilon(\lambda,k)\,\frac{G^\star(\eta,k)}{\lambda + i\omega'(k)\eta}d\eta dk. \tag{5.31}$$

Here

$$\tilde{D}_n^\varepsilon(\lambda,k) := \frac{\Gamma^{n-1}}{4\mu^n} I_\varepsilon\, II_\varepsilon.$$

The calculation of the limit (5.30) reduces therefore to computing the limits of I_ε and II_ε.

Computation of $\lim_{\varepsilon\to 0+} I_\varepsilon$. Since $\tilde{g}(\lambda) = 1 - \gamma\tilde{J}\tilde{g}(\lambda)$, we can write $I_\varepsilon = I_\varepsilon^1 + I_\varepsilon^2$, where

$$\begin{aligned} I_\varepsilon^1 &:= \frac{\gamma}{2\pi}\int_{\mathbb{R}} \frac{(\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_0)}{\lambda\varepsilon/4 + i\beta_0 - i\omega(k)}d\beta_0 \\ I_\varepsilon^2 &:= -\frac{\gamma^2}{2\pi}\int_{\mathbb{R}} (\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_0)\,\frac{(\tilde{J}\tilde{g})(\lambda\varepsilon/4 + i\beta_0)}{\lambda\varepsilon/4 + i\beta_0 - i\omega(k)}d\beta_0. \end{aligned}$$

Using (4.21) we get

$$I_\varepsilon^1 = \frac{\gamma}{2\pi}\int_{\mathbb{R}} \frac{(\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_0)}{\lambda\varepsilon/4 + i\beta_0 - i\omega(k)}d\beta_0 = \gamma(\tilde{J}\tilde{g})\,(\lambda\varepsilon - i\omega(k))\,.$$

Therefore

$$\lim_{\varepsilon\to 0+} I_\varepsilon^1 = 1 - \nu(k). \tag{5.32}$$

On the other hand

$$\lim_{\varepsilon\to 0}(\tilde{J}\tilde{g})\,(3\lambda\varepsilon/4 - i\beta_0)\,(\tilde{J}\tilde{g})(\lambda\varepsilon/4 + i\beta_0) = |(\tilde{J}\tilde{g})|^2(i\beta_0)$$

in any $L^p(\mathbb{R})$, $p \in (1, +\infty)$ and pointwise. Therefore,

$$\lim_{\varepsilon\to 0+} I_\varepsilon^2 = -\frac{\gamma^2}{2\pi} \lim_{\varepsilon\to 0+} \left\{ \int_{\mathbb{R}} \frac{|(\tilde{Jg})\,(i\beta_0)\,|^2 d\beta_0}{\lambda\varepsilon/4 + i\beta_0 - i\omega(k)} \right\}.$$

Since $j(\beta_0) := |(\tilde{Jg})\,(i\beta_0)\,|^2$ belongs to any $L^p(\mathbb{R})$ for $p \in [1, +\infty)$, by the multiplier theorem, see, for example, [11, Corollary of Theorem 3, p. 96]

$$\lim_{\varepsilon\to 0+} \int_{\mathbb{R}} \frac{j(\beta_0)d\beta_0}{\lambda\varepsilon/4 + i\beta_0 - i\beta} = \mathfrak{j}(\beta) := 2\pi \int_{-\infty}^{0} e^{2\pi i\eta\beta} \hat{j}(\eta) d\eta,$$

in the $L^p(\mathbb{R})$ sense, for any $p \in (1, +\infty)$. Here

$$\hat{j}(\eta) := \int_{\mathbb{R}} e^{-2\pi i\eta\beta} j(\beta) d\beta$$

is the Fourier transform of j.

We have $\omega_+^{-1}(\omega_{\min}) = 0$, $\omega_+^{-1}(\omega_{\max}) = 1/2$. In the case $\omega \in C^\infty(\mathbb{T})$:

$$\left(\omega_\pm^{-1}\right)'(w) = \pm(w - \omega_{\min})^{-1/2} \rho_*(w),\ w - \omega_{\min} \ll 1, \tag{5.33}$$

and

$$\left(\omega_\pm^{-1}\right)'(w) = \pm(\omega_{\max} - w)^{-1/2} \rho^*(w),\ \omega_{\max} - w \ll 1, \tag{5.34}$$

with $\rho_*, \rho^* \in C^\infty(\mathbb{T})$ that are strictly positive. When ω is not differentiable at 0 (the acoustic case), condition (5.34) does not change but then

$$\left(\omega_\pm^{-1}\right)'(w) = \pm\rho_*(w),\ w - \omega_{\min} \ll 1. \tag{5.35}$$

In consequence,

$$\lim_{\varepsilon\to 0+} I_\varepsilon^2 = -\frac{\gamma^2}{2\pi} \mathfrak{j}(\omega(k)) \tag{5.36}$$

in the $L^p(\mathbb{T})$ sense for any $p \in [1, 2)$. We have shown therefore that

$$\lim_{\varepsilon\to 0+} I_\varepsilon = I := 1 - \nu(k) - \frac{\gamma^2}{2\pi} \mathfrak{j}(\omega(k)) \tag{5.37}$$

in the $L^p(\mathbb{T})$ sense for any $p \in [1, 2)$. Since j is real valued we have

$$\frac{1}{2\pi} \mathrm{Re}\, \mathfrak{j}(\beta) = \frac{1}{2} j(\beta) \tag{5.38}$$

and

$$\frac{1}{2\pi} \mathrm{Re}\, \mathfrak{j}(\omega(k)) = \frac{1}{2} |(\tilde{Jg})\,(i\omega(k))\,|^2.$$

Thus, using the relation

$$\gamma(\tilde{Jg})(\lambda) = 1 - \tilde{g}(\lambda),$$

we conclude that

$$\begin{aligned} \mathrm{Re}\, I &:= 1 - \mathrm{Re}\, \nu(k) - \frac{\gamma^2}{2} |(\tilde{Jg})\,(i\omega(k))\,|^2 \\ &= 1 - \mathrm{Re}\, \nu(k) - \frac{1}{2} |1 - \nu(k)|^2 = \frac{1}{2} \left(1 - |\nu(k)|^2\right). \end{aligned} \tag{5.39}$$

Computation of $\lim_{\varepsilon\to 0+} I\!I_\varepsilon$. We have $I\!I_\varepsilon = I\!I^1_\varepsilon + I\!I^2_\varepsilon$, where

$$I\!I^1_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} d\beta_n \frac{\tilde g(3\lambda\varepsilon/4 - i\beta_n)\tilde g(\lambda\varepsilon/4 + i\beta_n)\mathbb{E}\left[\hat\psi(\ell)\hat\psi^*(\ell')\right]}{[3\lambda\varepsilon/4 - i\beta_n + i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n - i\omega(\ell')]},$$

$$I\!I^2_\varepsilon := \frac{\varepsilon}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} d\beta_n \frac{\tilde g(3\lambda\varepsilon/4 - i\beta_n)\tilde g(\lambda\varepsilon/4 + i\beta_n)\mathbb{E}\left[\hat\psi^*(\ell)\hat\psi(\ell')\right]}{[3\lambda\varepsilon/4 - i\beta_n - i\omega(\ell)][\lambda\varepsilon/4 + i\beta_n + i\omega(\ell')]}.$$

Changing variables $\varepsilon\beta'_n := \beta_n - \omega(\ell')$, we obtain

$$I\!I^1_\varepsilon := \frac{1}{2\pi}\int_{\mathbb{T}^2} d\ell d\ell' \int_{\mathbb{R}} d\beta_n \times \frac{\tilde g(3\lambda\varepsilon/4 - i\varepsilon\beta_n - i\omega(\ell))\tilde g(\lambda\varepsilon/4 + i\varepsilon\beta_n + i\omega(\ell))\mathbb{E}\left[\hat\psi(\ell)\hat\psi^*(\ell')\right]}{[3\lambda/4 - i\beta_n + i\varepsilon^{-1}\big(\omega(\ell) - \omega(\ell')\big)][\lambda/4 + i\beta_n]}.$$

Therefore

$$\lim_{\varepsilon\to 0+} I\!I^1_\varepsilon = \frac{1}{2\pi}\lim_{\varepsilon\to 0+}\int_{\mathbb{T}^2} \frac{|\nu(\ell)|^2\mathbb{E}\left[\hat\psi(\ell)\hat\psi^*(\ell')\right]}{\lambda + i\varepsilon^{-1}\big(\omega(\ell) - \omega(\ell')\big)} d\ell d\ell'.$$

Changing again variables

$$\ell = \tilde\ell + \frac{\varepsilon\eta}{2}, \quad \ell' = \tilde\ell - \frac{\varepsilon\eta}{2}$$

we conclude that

$$\lim_{\varepsilon\to 0+} I\!I^1_\varepsilon = 2\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta,\ell)}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.40}$$

A similar calculation proves that also

$$\lim_{\varepsilon\to 0+} I\!I^2_\varepsilon = 2\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta,\ell)}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.41}$$

We conclude therefore

$$I\!I = \lim_{\varepsilon\to 0+} I\!I_\varepsilon = 4\int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta,\ell)}{\lambda + i\omega'(\ell)\eta} d\eta d\ell. \tag{5.42}$$

The right-hand side of (5.42) is real valued. Gathering all the facts proven above, we conclude that

$$\lim_{\varepsilon\to 0+} \bar{\mathfrak{L}}^{(n)}_\varepsilon = \frac{\Gamma^{n-1}}{2\mu^n}\int_{\mathbb{R}\times\mathbb{T}} I\!I \mathrm{Re}\, I \frac{G^\star(\eta,k)}{\lambda + i\omega'(k)\eta} d\eta dk \tag{5.43}$$

$$= \frac{\Gamma^{n-1}}{\mu^n}\int_{\mathbb{R}\times\mathbb{T}} \frac{\big(1 - |\nu(k)|^2\big)G^\star(\eta,k)}{\lambda + i\omega'(k)\eta} d\eta dk \int_{\mathbb{R}\times\mathbb{T}} \frac{|\nu(\ell)|^2\widehat W(0,\eta',\ell)}{\lambda + i\omega'(\ell)\eta'} d\eta' d\ell.$$

Combining this with formula (5.31), we conclude the proof of Lemma 6. □

5.3. Proof of Proposition 2

According to (5.4) for any we have

$$\int_{\mathbb{R}\times\mathbb{T}} \widehat{w}_\varepsilon(\lambda,\eta,k)G^*(\eta,k)d\eta dk = \sum_{j=1}^{3}\mathcal{W}_j^{(\varepsilon)}, \quad \text{where}$$
$$\begin{aligned}
\mathcal{W}_1^{(\varepsilon)} &:= \int_{\mathbb{R}\times\mathbb{T}} \frac{W_\varepsilon(0,\eta,k)G^*(\eta,k)}{\lambda+i\delta_\varepsilon\omega(k;\eta)}d\eta dk\\
\mathcal{W}_2^{(\varepsilon)} &:= \frac{\gamma\mathfrak{e}_\varepsilon(\lambda)}{\mu}\int_{\mathbb{R}\times\mathbb{T}} \frac{G^*(\eta,k)}{\lambda+i\delta_\varepsilon\omega(k;\eta)}d\eta dk\\
\mathcal{W}_3^{(\varepsilon)} &:= -\frac{\gamma}{2}\int_{\mathbb{R}\times\mathbb{T}} \frac{G^*(\eta,k)}{\lambda+i\delta_\varepsilon\omega(k;\eta)}\left[\mathfrak{d}_\varepsilon\left(\lambda,k-\frac{\varepsilon\eta}{2}\right)+\mathfrak{d}_\varepsilon^\star\left(\lambda,k+\frac{\varepsilon\eta}{2}\right)\right]d\eta dk.
\end{aligned} \tag{5.44}$$

It is easy to see that the limit of $\mathcal{W}_1^{(\varepsilon)}$, as $\varepsilon\to 0+$, corresponds to the first term in the right hand side of (5.6). Using Proposition 3, we conclude that the limit of $\mathcal{W}_2^{(\varepsilon)}$ matches the second term there. Finally, $\mathcal{W}_3^{(\varepsilon)} = -\frac{\gamma}{2}\sum_{n=0}^{+\infty}\mathfrak{L}_\varepsilon^{(n)}$ and the respective limit is a consequence of Lemmas 1, 5 and 6. This ends the proof of the proposition. □

5.4. The End of the Proof of Theorem 1

Using the equality (3.8) and the results of Proposition 1 (for $\mu>1/2$), Lemma 3 (for $\mu=1/2$) and Proposition 1, together with formula (5.6) we conclude that for any $\lambda\in\mathbb{C}_+$ the Laplace–Fourier–Wigner functions $\widehat{w}_\varepsilon(\lambda,\eta,k)$ converge, as $\varepsilon\to 0+$, in $\mathcal{A}'$, in the $\star$-weak topology to

$$\begin{aligned}
\widehat{w}(\lambda,\eta,k) &= \frac{\widehat{W}(0,\eta,k)}{\lambda+i\omega'(k)\eta} + \frac{\gamma T|\nu(k)|^2}{(1-\Gamma/\mu)\lambda(\lambda+i\omega'(k)\eta)}\left(1-\frac{1}{2\mu}\right)\\
&+ \frac{\gamma|\nu(k)|^2}{2\mu[\lambda+i\omega'(k)\eta](1-\Gamma/\mu)}\int_{\mathbb{R}\times\mathbb{T}}\frac{|\nu(\ell)|^2\widehat{W}(0,\eta',\ell)}{\lambda+i\omega'(\ell)\eta'}d\eta' d\ell\\
&- \frac{\gamma\mathrm{Re}[\nu(k)]}{\lambda+i\omega'(k)\eta}\int_{\mathbb{R}}\frac{\widehat{W}(0,\eta',k)}{\lambda+i\omega'(k)\eta'}d\eta'\\
&+ \frac{\gamma\mathfrak{J}(k)}{4(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}}\frac{\widehat{W}(0,\eta',k)d\eta'}{\lambda+i\omega'(k)\eta'} + \frac{\gamma\mathfrak{J}(k)}{4(\lambda+i\omega'(k)\eta)}\int_{\mathbb{R}}\frac{\widehat{W}(0,\eta',-k)d\eta'}{\lambda-i\omega'(k)\eta'}.
\end{aligned} \tag{5.45}$$

Inverting both the Laplace transform in t and Fourier transform in x we obtain (2.53), which ends the proof of the theorem. □

6. Proofs of Lemmas 1 and 2

6.1. Proof of Lemma 1

We have

$$\tilde{J}(\lambda) = G(\lambda) + H(\lambda), \tag{6.1}$$

where

$$G(\lambda) := \frac{1}{2}\int_{\mathbb{T}_+}\frac{d\ell}{\lambda+i\omega(\ell)} \qquad H(\lambda) := \frac{1}{2}\int_{\mathbb{T}_+}\frac{d\ell}{\lambda-i\omega(\ell)}. \tag{6.2}$$

Thanks to (6.1) and (2.34), we conclude that

$$|(\tilde{g}\tilde{J})(\lambda)| \leq \frac{1}{|\lambda| - \omega_{\max}}, \quad |\lambda| > \omega_{\max}, \operatorname{Re}\lambda > 0. \tag{6.3}$$

On the other hand, thanks to (2.34) and (2.35), we have also

$$|(\tilde{g}\tilde{J})(\lambda)| \leq \frac{2}{\gamma}, \quad \operatorname{Re}\lambda > 0. \tag{6.4}$$

As a result $\tilde{g}\tilde{J} \in H^p(\mathbb{C}_+)$ for any $p \in (1, +\infty)$. The limits in (2.40) and (2.41) can be substantiated by the results of Sections A and B of Chapter 6 of [6].

Recall that $\omega_+^{-1}(\cdot)$ is the inverse of the restriction $\omega_{[0,1/2]}$. From (6.2) we get

$$G(\varepsilon + i\omega(k)) = \frac{1}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{dv}{\omega'(\omega_+^{-1}(v))[\varepsilon + i(v + \omega(k))]}$$

To simplify assume that $k \in [0, 1/2]$. It is clear that

$$\lim_{\varepsilon \to 0+} G(\varepsilon + i\omega(k)) = G(i\omega(k)) = -\frac{i}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{dv}{\omega'(\omega_+^{-1}(v))(v + \omega(k))}$$

and there exists $C > 0$ such that

$$\Big|G(\varepsilon + i\omega(k)) - G(i\omega(k))\Big| \leq C\varepsilon, \quad k \in \Omega_*^{(\delta)}, \varepsilon \in (0, 1), \tag{6.5}$$

where $\Omega_*^{(\delta)} := [k \in \mathbb{T} : \operatorname{dist}\big(k, \Omega_*\big) \geq \delta]$. Concerning $H(\cdot)$ we have

$$H(\varepsilon + i\omega(k)) = \frac{1}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{dv}{\omega'(\omega_+^{-1}(v))[\varepsilon + i(\omega(k) - v)]}$$

A simple calculation leads to

$$H(i\omega(k)) := \lim_{\varepsilon \to 0+} H(\varepsilon + i\omega(k)) = \frac{1}{2\omega'(k)}\left[\pi + i\log\left(\frac{\omega_{\max} - \omega(k)}{\omega(k) - \omega_{\min}}\right)\right]$$
$$+ \frac{i}{2}\int_{\omega_{\min}}^{\omega_{\max}} \frac{[\omega'(k) - \omega'(\omega_+^{-1}(v))]dv}{\omega'(\omega_+^{-1}(v))\omega'(k)(\omega(k) - v)}.$$

Since $\omega'(\cdot)$ is Lipschitz the integral in the right hand side makes sense. A straightforward calculation implies the existence of $C > 0$ such that

$$\Big|H(\varepsilon + i\omega(k)) - H(i\omega(k))\Big| \leq C\varepsilon, \quad k \in \Omega_*^{(\delta)}, \varepsilon \in (0, 1). \tag{6.6}$$

From (6.5) and (6.6) we conclude (2.42). In addition we infer also the continuity of $\nu(k)$ on $\mathbb{T} \setminus \Omega_*$. □

6.2. Proof of Lemma 2

For a given $f \in L^1(\mathbb{R})$ such that $f \geq 0$ a.e. we let

$$M(z) := \int_{\mathbb{R}} \frac{f(\alpha)d\alpha}{z + i\alpha}, \quad z \in \mathbb{C}_+. \tag{6.7}$$

The function is holomorphic and $\operatorname{Re} M(z) > 0$ for $z \in \mathbb{C}_+$. In addition, for any $\rho > 0$ we have

$$M(\rho + i\beta) := \int_{\mathbb{R}} \frac{\rho f(\alpha) d\alpha}{\rho^2 + (\beta+\alpha)^2} - i \int_{\mathbb{R}} \frac{(\beta+\alpha) f(\alpha) d\alpha}{\rho^2 + (\beta+\alpha)^2}, \quad \beta \in \mathbb{R}. \tag{6.8}$$

Suppose also that $f \in L^p(\mathbb{R})$ for some $p > 1$. By [11, Corollary of Theorem 3, p. 96] we conclude that

$$M_+(\beta) := \lim_{\rho \to 0+} M(\rho + i\beta) = \pi f(-\beta) - i\mathcal{H}[f](\beta), \quad \beta \in \mathbb{R}, \tag{6.9}$$

where

$$\mathcal{H}[f](\beta) := \lim_{\rho \to 0+} \int_{\mathbb{R}} \frac{(\beta+\alpha) f(\alpha) d\alpha}{\rho^2 + (\beta+\alpha)^2}, \quad \beta \in \mathbb{R}, \tag{6.10}$$

and the limits in (6.9) and (6.10) are understood in the L^p sense.

We shall prove the following result.

Proposition 4. *Suppose that $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ for some $p > 1$ and $f \geq 0$ a.e. Then, for any $\gamma > 0$ the following identity holds*

$$\frac{\gamma}{2\pi} \int_{\mathbb{R}} \frac{|M_+(\beta)|^2 d\beta}{|1 + \gamma M_+(\beta)|^2} + \frac{1}{2} \int_{\mathbb{R}} \frac{f(-\beta) d\beta}{|1 + \gamma M_+(\beta)|^2} = \frac{1}{2} \int_{\mathbb{R}} f(\beta) d\beta. \tag{6.11}$$

Before proving the proposition, which we are going to do momentarily, let us first apply it to show how, with its help, to finish the proof of Lemma 2.

6.2.1. Proof of Lemma 2. From (2.31) we get

$$\tilde{J}(\lambda) = \int_0^{1/2} \frac{dk}{\lambda + i\omega(k)} + \int_0^{1/2} \frac{dk}{\lambda - i\omega(k)} = \int_{\mathbb{R}} \frac{f_*(v) dv}{\lambda + iv}, \tag{6.12}$$

where

$$f_*(v) := \frac{1_{[\omega_{\min}, \omega_{\max}]}(|v|)}{\omega'\big(\omega_+^{-1}(|v|)\big)}, \quad v \in \mathbb{R}. \tag{6.13}$$

Recalling that

$$\omega'\big(\omega_+^{-1}(v)\big) \sim (\omega_{\max} - v)^{1/2}, \quad \omega_{\max} - v \ll 1,$$

see (5.33), and

$$\omega'\big(\omega_+^{-1}(v)\big) \sim (v - \omega_{\min})^{1/2}, \quad v - \omega_{\min} \ll 1$$

in the optical case (see (5.34)), and $|\omega'\big(\omega_+^{-1}(v)\big)| \sim 1$, $v \ll 1$ in the acoustic one we conclude that $f_* \in L^p(\mathbb{R})$ for any $p \in [1, 2)$ and $\int_{\mathbb{R}} f_*(v) dv = 1$. It is easy to see from (6.12) and (6.13) that

$$J^\star(\lambda) = J(\lambda^\star), \quad \lambda \in \mathbb{C}_+. \tag{6.14}$$

Recall that $\tilde{J}(i\omega(k)) = \lim_{\varepsilon \to 0+} \tilde{J}(\varepsilon + i\omega(k))$, cf (2.40), therefore

$$\int_{\mathbb{T}} |\nu(\ell)|^2 d\ell = \int_{\mathbb{T}} \frac{d\ell}{|1 + \gamma \tilde{J}(i\omega(\ell))|^2}$$
$$= \int_0^{1/2} \frac{d\ell}{|1 + \gamma \tilde{J}(i\omega(\ell))|^2} + \int_0^{1/2} \frac{d\ell}{|1 + \gamma \tilde{J}^\star(i\omega(\ell))|^2}$$

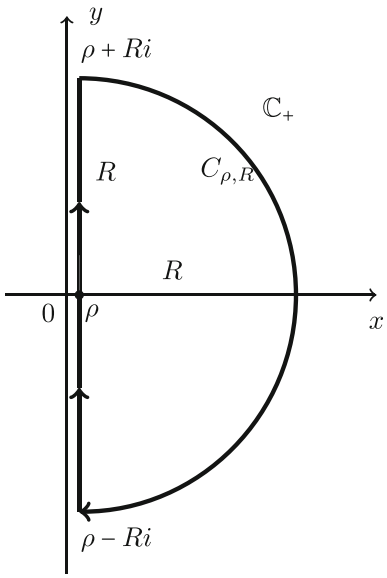

FIGURE 1. Contour of integration

$$= \int_0^{1/2} \frac{d\ell}{|1+\gamma\tilde{J}(i\omega(\ell))|^2} + \int_0^{1/2} \frac{d\ell}{|1+\gamma\tilde{J}(-i\omega(\ell))|^2} = \int_{\mathbb{R}} \frac{f_*(v)dv}{|1+\gamma\tilde{J}(iv)|^2}.$$

Formula (2.49) is then a direct consequence of (6.11). Equality (2.49) is in fact equivalent with

$$1 = \frac{1}{2(1-\Gamma)} + \frac{1}{2(1-\Gamma)} \int_{\mathbb{T}} |\nu(\ell)|^2 d\ell, \tag{6.15}$$

which in turn yields (2.50). □

6.2.2. Proof of Proposition 4. Suppose that $\rho, R > 0$. Consider the contour $C_{\rho,R}$, cf Figure 1, made of the line segment from $\rho - Ri$ to $\rho + Ri$ and the semicircle centred at ρ of radius R, oriented clockwise. Since $M(z)$ is analytic in $\mathbb{C}_+$, we have

$$\int_{C_{\rho,R}} \frac{M(z)dz}{1+\gamma M(z)} = 0. \tag{6.16}$$

The above equality yields

$$\int_{-R}^{R} \frac{M(\rho+i\beta)d\beta}{1+\gamma M(\rho+i\beta)} = \int_{-\pi/2}^{\pi/2} \frac{M(\rho+Re^{i\theta})Re^{i\theta}d\theta}{1+\gamma M(\rho+Re^{i\theta})}. \tag{6.17}$$

Letting first $\rho \to 0+$ and then $R \to +\infty$, in this order, we conclude, thanks to the definition of $M(z)$ and the fact that the expression under the integral is bounded, that

$$\lim_{R\to+\infty} \int_{-R}^{R} \frac{M_+(\beta)d\beta}{1+\gamma M_+(\beta)} = \pi \int_{\mathbb{R}} f(\beta)d\beta. \tag{6.18}$$

Taking complex conjugation on both sides

$$\lim_{R\to+\infty}\int_{-R}^{R}\frac{M_+^\star(\beta)d\beta}{1+\gamma M_+^\star(\beta)}=\pi\int_{\mathbb{R}}f(\beta)d\beta. \tag{6.19}$$

Adding (6.18) and (6.19) sideways, and using (6.9) we get

$$\begin{aligned}&2\gamma\lim_{R\to+\infty}\int_{-R}^{R}\frac{|M_+(\beta)|^2d\beta}{|1+\gamma M_+(\beta)|^2}+2\pi\lim_{R\to+\infty}\int_{-R}^{R}\frac{f(-\beta)d\beta}{|1+\gamma M_+(\beta)|^2}\\&=2\pi\int_{\mathbb{R}}f(\beta)d\beta.\end{aligned} \tag{6.20}$$

This ends the proof of the proposition. □

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