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On a Class of Quasilinear Equations Involving Critical Exponential Growth and Concave Terms in $\mathbb{R}^N$

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Abstract. In this work, we establish the existence and multiplicity of nonzero and non-negative solutions for a class of quasilinear elliptic equations, whose nonlinearity is allowed to enjoy the critical exponential growth with respect to a version of the Trudinger–Moser inequality, and it can also contain concave terms in $\mathbb{R}^N(N \geq 2)$. When $N = 2$ or $N = 3$, this equation is motivated for a physics application in fluid mechanics. In order to obtain our results, we combine variational arguments in a suitable subspace of a Orlicz–Sobolev space with a version of the Trudinger–Moser inequality and Ekeland variational principle. In a particular case, we show that the solution is a positive ground state.

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1. Introduction and Main Results

In this paper, we deal with existence and multiplicity of nonzero solutions for quasilinear elliptic equations of the form

$$-\Delta_\Phi u + V(x)\Phi'(|u|)\frac{u}{|u|} = h(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.1}$$

where $\Delta_\Phi u = \operatorname{div}(\Phi'(|\nabla u|)\nabla u/|\nabla u|)$, $N \geq 2$, $V : \mathbb{R}^N \to \mathbb{R}$ is non-negative and locally bounded potential, $h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying appropriate conditions and $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a $\mathcal{N}$–function satisfying some conditions which will be described later. For the definition and informations about $\mathcal{N}$–functions, see Sect. 2. Equation (1.1) appears in several physical

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contexts. As observed in [23] (and references [6–8] therein), equations of type (1.1) are related to concrete examples from fluid mechanics and plasticity theory. If Ω is a domain of $\mathbb{R}^N$, with $N = 2$ or $N = 3$, and

$$\Phi(t) = |t|^N + N \int_0^{|t|} s^{N-1} \operatorname{arcsinh}^\alpha s \, \mathrm{d}s, \tag{1.2}$$

$\alpha \in (0, N-1)$, the slow steady-state motion of a fluid of Prandtl–Eyring type in Ω can be modeled by the following set of equations

$$\begin{cases} \operatorname{div}\left(\Phi'(|Du|)Du/|Du|\right) + (\text{potential term}) = 0 \quad \text{in} \quad \Omega \\ \operatorname{div} u = 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $u : \Omega \to \mathbb{R}^N$ denotes the velocity field of a incompressible fluid and $Du := \frac{1}{2}(\nabla u + \nabla u^\perp)$ is the symmetric gradient of u (for more details see [8,22]).

In the literature, equations of type (1.1) in all $\mathbb{R}^N$ were studied by various authors; we can cite for instance [1,2,11,13,24,33] and references therein. In [13, Theorem 1.3], the author proves a version of the Moser–Trudinger inequality for Orlicz–Sobolev embedding into exponential and multiple exponential spaces on unbounded domains in $\mathbb{R}^N$,[1] $N \geq 2$, by assuming that $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a C^1-function fulfilling the following hypotheses:

(Φ_1) There exists $C \geq 1$ such that $t^N/C \leq \Phi(t) \leq Ct^N$ for all $t \in [0, 1/C)$;

(Φ_2) $\displaystyle\lim_{t\to+\infty} \frac{\Phi(t)}{t^N \log^\alpha t} = 1$, for some $\alpha \in [0, N-1)$.

As an application of the inequality and considering a third hypothesis on Φ [see condition (1.7) in [13]] he shows that (1.1) has a nonzero solution by assuming that the nonlinearity $h(x,t)$ has critical exponential growth with respect to this version of the Moser–Trudinger inequality and others appropriate assumptions. In [11], under the same conditions the author considers the non-homogeneous case and shows the existence of two distinct solutions.

In this paper, we are going to assume only the conditions (Φ_1) and (Φ_2). Next, let us make some comments about these hypotheses. As a consequence of (Φ_1), it follows that

$$\liminf_{t\to 0^+} \frac{\Phi(2t)}{\Phi(t)} \geq \frac{2^N}{C^2}.$$

Moreover, by (Φ_2) we have

$$\lim_{t\to+\infty} \frac{\Phi(2t)}{\Phi(t)} = 2^N.$$

Thus, there exists $K \geq 2^N$ such that $\Phi(2t) \leq K\Phi(t)$ for all $t \geq 0$, that is, Φ satisfies Δ_2-condition. Moreover, since Φ is convex we reach

$$(K-1)\Phi(t) \geq \Phi(2t) - \Phi(t) \geq \Phi'(t)t$$

[1] A version for bounded domains can be found for instance in [15,25].

and therefore

$$c_\alpha := \sup_{t>0} \frac{\Phi'(t)t}{\Phi(t)} \le K - 1. \tag{1.3}$$

On the other hand, given any $t_0 > 0$ by the mean value theorem there exists $s \in (0, t_0)$ verifying $\Phi(t_0) = \Phi'(s)t_0$ and since $\Phi'(t)$ is non-decreasing for $t > 0$ it follows that $\Phi(t_0) \le \Phi'(t_0)t_0$ and thus

$$1 \le m_\alpha := \inf_{t>0} \frac{\Phi'(t)t}{\Phi(t)}. \tag{1.4}$$

It is not difficult to verify that by (Φ_1) and (Φ_2) we must have $m_\alpha > 1$ (if $m_\alpha = 1$ we get a contradiction). Moreover, by deriving $\Phi(t)/t^{c_\alpha}$ we have that (1.3) implies that $\Phi(t)/t^{c_\alpha}$ is non-increasing for $t > 0$. Similarly, from (1.4) we deduce that $\Phi(t)/t^{m_\alpha}$ is non-decreasing for $t > 0$. Hence,

$$\Phi(1)t^{c_\alpha} \le \Phi(t) \le \Phi(1)t^{m_\alpha}, \quad \forall\, t \in [0, 1].$$

Combining these inequalities with condition (Φ_1), for all $t \in [0, 1/C)$ we reach

$$\Phi(1)t^{c_\alpha} \le \Phi(t) \le Ct^N \quad \text{and} \quad \frac{1}{C}t^N \le \Phi(t) \le \Phi(1)t^{m_\alpha},$$

which imply that $m_\alpha \le N \le c_\alpha$.

We observe that the function $\Phi : \mathbb{R} \to \mathbb{R}_+$, defined in (1.2), is a C^1-function satisfying conditions (Φ_1) and (Φ_2), and it is not difficult to verify that in this case $m_\alpha = N$ and $c_\alpha = N + \alpha$.

When $\alpha = 0$, $\Phi(t) = |t|^N$ satisfies conditions (Φ_1) and (Φ_2) (in this case $m_\alpha = c_\alpha = N$) and therefore (1.1) becomes a elliptic equation involving the N-Laplacian operator, namely,

$$-\Delta_N u + V(x)|u|^{N-2}u = \hat{h}(x, u) \quad \text{in} \quad \mathbb{R}^N,$$

where $\hat{h} = h/N$. This problem, with $\hat{h}(x, u)$ having critical exponential growth with respect to the Trudinger–Moser inequality, has been intensively investigated by many authors, see for example [16–21,26,28,30]. Motivated by theses aspects and papers [11,13], our main objective in the present paper is to obtain the existence and multiplicity of nonzero and non-negative solutions for the following class of equations:

$$-\Delta_\Phi u + V(x)\Phi'(|u|)\frac{u}{|u|} = f(x, u) + \lambda g(x, u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.5}$$

where $\lambda \ge 0$ is a parameter and $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying suitable conditions. Our intention is also to improve and complement some of the results cited above.

As in [11,13] we consider the maximal growth on the nonlinear term $f(x, t)$ which allows us to treat Eq. (1.5) variationally in an appropriate subspace of $W^{1,\Phi}(\mathbb{R}^N)$. We shall impose on $g(x, t)$ a growth of sublinear type. Here, $W^{1,\Phi}(\mathbb{R}^N)$ is the Orlicz–Sobolev space that consists of functions in $L_\Phi(\mathbb{R}^N)$ (the Orlicz space associated to the $\mathcal{N}$–function Φ) such that its weak derivatives exist and belong to $L_\Phi(\mathbb{R}^N)$. We regard $W^{1,\Phi}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{1,\Phi} = |\nabla u|_\Phi + |u|_\Phi,$$

where $|\cdot|_\Phi$ denotes the Luxemburg norm associated to $L_\Phi(\mathbb{R}^N)$. It is known that the Lorentz–Zygmund space $L^{N,N,\gamma}(\mathbb{R}^N)$, $\gamma = \alpha/N$, reproduces (up to equivalent norms) the Orlicz spaces $L_\Phi(\mathbb{R}^N)$ (see [7,29]). Thus, $W^{1,\Phi}(\mathbb{R}^N)$ is equivalent to $W^1L^{N,N,\gamma}(\mathbb{R}^N)$.

Throughout this paper, we assumed that $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}_+$ is continuous and behaves like $\exp(b|t|^\gamma)$ as $t\to+\infty$, more precisely, we suppose the following growth condition on the non-linearity $f(x,t)$:

(f_1) there exist constants $C>0$ and $b>0$ such that

$$f(x,t) \le Ct^{N-1} + C\left[\exp(bt^\gamma) - S_{N,\alpha}(bt^\gamma)\right]$$

for all $t\ge 0$ and $x\in\mathbb{R}^N$, where $\gamma = N/(N-1-\alpha)$ and

$$S_{N,\alpha}(bt^\gamma) = \sum_{0\le j<\frac{N}{\gamma}} \frac{(bt^\gamma)^j}{j!}.$$

The main features of this class of problems, considered in this paper, are that it is defined in the whole $\mathbb{R}^N$, involves critical exponential growth (according to Lemma 3.3) and the non-homogenous generalized N-Laplacian operator. We will show that the energy functional associated to the problem verifies the Palais–Smale compactness condition in certain energy levels. By applying minimax methods combined with minimization arguments and the Ekeland variational principle, we obtain multiplicity of weak solutions for Eq. (1.5) in the subspace $X\subset W^{1,\Phi}(\mathbb{R}^N)$, given by

$$X := \left\{u\in W^{1,\Phi}(\mathbb{R}^N);\int_{\mathbb{R}^N} V(x)\Phi(u)\mathrm{d}x<\infty\right\}.$$

We say that $u:\mathbb{R}^N\to\mathbb{R}$ is a weak solution of the Eq. (1.5) if $u\in X$ and it holds

$$\int_{\mathbb{R}^N}\left[\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v + V(x)\Phi'(|u|)\frac{u}{|u|}v\right]\mathrm{d}x \\ -\int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x - \lambda\int_{\mathbb{R}^N} g(x,u)v\mathrm{d}x = 0,$$

for all $v\in X$. With respect to the potential $V(x)$, we require the following conditions:

(V_1) $V\in L^\infty_{loc}(\mathbb{R}^N)$ and there exists $V_0>0$ such that $V(x)\ge V_0$ for almost every $x\in\mathbb{R}^N$;

(V_2) the function $x\mapsto\tilde\Phi\left(\frac{1}{V(x)^{1/c_\alpha}}\right)$ belongs to $L^1(\mathbb{R}^N)$, where $\tilde\Phi$ is the conjugate N-function of Φ;

Here, we can assume without loss of generality that $V_0\le 1$. We emphasize that condition (V_2) is a generalization of the hypothesis

$$\int_{\mathbb{R}^N}\frac{1}{V(x)^{1/(N-1)}}\mathrm{d}x<\infty,$$

in the case $\Phi(t) = |t|^N$. This assumption was already considered in various papers dealing with the N-Laplacian operator, see for instance [19,26,34].

Since Φ satisfies Δ_2-condition, it is not difficult to see that X is a reflexive Banach space when endowed with the norm

$$\|u\| = |\nabla u|_\Phi + |u|_{\Phi,V},$$

where $|\cdot|_{\Phi,V}$ denotes the Luxemburg norm with relation to the measure $\mathrm{d}\mu = V(x)\mathrm{d}x$. For more details, see Sect. 2 and [5,32]. Condition (V_1) implies that the embedding $X \hookrightarrow W^{1,\Phi}(\mathbb{R}^N)$ is continuous. Moreover, under condition (V_2), we shall prove that the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for all $1 \le s < \infty$ (see Proposition 3.2).

Besides the condition (f_1) on the non-linearity $f(x,t)$, we consider the following assumptions:

(f_2) there exists $\sigma > c_\alpha$ verifying

$$\sigma F(x,t) \le f(x,t)t, \ \forall\, t \ge 0 \quad \text{and} \quad x \in \mathbb{R}^N,$$

where $F(x,t) := \int_0^t f(x,s)\mathrm{d}s$;

(f_3) there exist $\theta > m_\alpha$ and $\mu > 0$ such that

$$F(x,t) \ge \mu t^\theta, \forall\, t \in [0,1] \quad \text{and} \quad x \in \mathbb{R}^N.$$

We observe that for each $x \in \mathbb{R}^N$, deriving with respect to t the quotient $F(x,t)/t^\sigma$ we deduce from (f_2) that $F(x,t)/t^\sigma$ is non-decreasing for $t > 0$. Thus, by (f_1) we get

$$F(x,t) \le F(x,1)t^\sigma \le C_1 t^\sigma \quad \forall \quad t \in [0,1] \quad \text{and} \quad x \in \mathbb{R}^N.$$

Consequently, in view of (Φ_1), $F(x,t)/\Phi(t) \le C_1 C t^{\sigma - N}$ for all $t \in (0, 1/C)$ and therefore we obtain

$$\lim_{t \to 0^+} \frac{F(x,t)}{\Phi(t)} = 0, \quad \text{uniformly in} \quad x \in \mathbb{R}^N. \tag{1.6}$$

Assumption (f_3) is used to estimate the minimax level associated to the energy functional. Note that we require this condition only for $t \in [0,1]$.

With respect to the non-linearity $g : \mathbb{R}^N \times \mathbb{R}_+ \to \mathbb{R}_+$, we assume that g is continuous and satisfies the condition

(g_1) there exist $C_1 > 0$ and $q \in (1, m_\alpha)$ such that $g(x,t) \le C_1 t^{q-1}$, for all $t \ge 0$ and $x \in \mathbb{R}^N$.

Notice that if $m_\alpha = 2$ then $g(x,t)$ has sublinear growth. This is the case when $\alpha = 0$, $N = 2$ and $\Phi(t) = t^2$ because $m_\alpha = c_\alpha = 2$. From this and since $m_\alpha \le N$, we can interpret condition (g_1) as a generalization of the sublinear growth for this class of problems.

We emphasize that elliptic problems in bounded domains involving concave and convex terms have been studied extensively after the initial work of Ambrosetti–Brezis–Cerami [3] and little has been addressed for this type of problems in all $\mathbb{R}^N$. We are only aware of the works [6,9,14,16,27] which studied existence and multiplicity of solutions in $\mathbb{R}^N$ for some semilinear and quasilinear equations related to problem (1.5). So, one of our intentions in this paper was to consider non-linearities involving critical exponential growth and concave terms.

Now, we are ready to present our main results:

Theorem 1.1. *Assume that* Φ *is a* $\mathcal{N}-$*function verifying* (Φ_1)–(Φ_2). *Moreover, suppose that* $(V_1)-(V_2)$, (g_1), $(f_1)-(f_3)$ *are satisfied and, in* (f_3),

$$\mu \geq \max\left\{\mu_1, \left[\frac{|B_1|^{\frac{\theta-m_\alpha}{m_\alpha}}\left(\frac{\mu_1 m_\alpha}{\theta}\right)^{\frac{\theta}{\theta-m_\alpha}}}{\left(\frac{\sigma-c_\alpha}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}\right)}\right]^{\frac{\theta-m_\alpha}{m_\alpha}}\right\} =: \mu^*,$$

where $K_{N,\alpha} = B^{1/B} N \omega_{N-1}^{\gamma/N}$, $B = 1-\alpha/(N-1)$, ω_{N-1} *is the measure of the unit sphere in* $\mathbb{R}^N$, $\xi_0(t) = \min\{t^{m_\alpha}, t^{c_\alpha}\}$ *for* $t \geq 0$ *and*

$$\mu_1 := \frac{\Phi(1)|B_1|(1+\|V\|_{L^\infty(B_1)})}{|B_{1/2}|}.$$

Then, there exists $\lambda_* > 0$ *such that Eq.* (1.5) *has a nonzero and non-negative solution* u_λ *for all* $\lambda \in [0, \lambda_*]$. *Furthermore, if* $u_\lambda \in C^1(\mathbb{R}^N)$ *then* $u_\lambda(x) > 0$ *for all* $x \in \mathbb{R}^N$.

When $\lambda = 0$, the solution obtained in Theorem 1.1 is a positive *ground state*, as the next result shows.

Theorem 1.2. *Under the assumptions of Theorem* 1.1, *if we suppose in addition that* $\lambda = 0$, $\Phi'(t)/t^{c_\alpha-1}$ *is non-increasing for* $t > 0$ *and, for each* $x \in \mathbb{R}^N$, $f(x,t)/t^{c_\alpha-1}$ *is increasing for* $t > 0$, *then the solution obtained in Theorem* 1.1 *is a ground state. Furthermore, if this solution belongs to* $C^1(\mathbb{R}^N)$ *then it is positive in all* $\mathbb{R}^N$.

Notice that $\Phi(t) = |t|^N$ satisfies the above monotonicity condition since $c_\alpha = N$ in this case. Moreover, if $\Phi'(t)$ is differentiable and $\Phi''(t)t \leq (c_\alpha - 1)\Phi'(t)$ for $t > 0$, then $\Phi'(t)/t^{c_\alpha-1}$ is non-increasing for $t > 0$.

If we consider an additional hypothesis on $g(x,t)$, we are able to prove that problem (1.5) has a second nonzero solution.

Theorem 1.3. *In addition to the assumptions of Theorem* 1.1, *suppose that* $g(x,t)$ *satisfies the condition*

(g_2) *there exist constants* $\beta_1 > 0$, $q_1 \in (1, m_\alpha)$ *and* $t_1 > 0$ *such that for all* $(x,t) \in \mathbb{R}^N \times [0, t_1]$

$$g(x,t) \geq \beta_1 t^{q_1-1}.$$

Then, for all $\lambda > 0$, *problem* (1.5) *has a nonzero and non-negative solution* v_λ *which is different of* u_λ *when* $\lambda \in (0, \lambda_*)$.

Here we improve and complement some of the works cited above. We treat a class of potentials which are not necessary coercive and we obtain, when $\lambda = 0$, a positive ground state solution. As far as we know, there are no papers which deal with Eq. (1.5) in the Orlicz context, where the potential $V(x)$ has these features and the non-linearities have critical exponentiall growth and involve *concave terms*. It is worthwhile to mention that the presence of the concave term brings additional difficulties to the problem, since the geometric structure of the associated functional becomes more delicate. Besides, to prove the existence of nonzero solution, we do not assume the conditions (1.7), (1.12), (1.13), (1.16) and (1.17) in [13] (see also similar assumptions in [11]). We also mention that we do not impose hypothesis on $F(x,t)$ at the origin ($t=0$) and we do not assume the condition

$$\exists\, M>0,\, t_0>0 \quad \text{such that} \quad F(x,t)\le Mf(x,t), \quad \forall\, t\ge t_0,$$

which is often used in semilinear and quasilinear problems involving exponential critical growth. In this direction, our paper improves and complements the works [11,13,16–20,26,28,34], which shows that our work is new even in the case $\Phi(t)=|t|^N$.

Example 1.4. Notice that, for $0\le\alpha<N-1$, the hypotheses of Theorem 1.3 are for example satisfied by $\Phi(t)$, $V(x)$, $f(x,t)$ and $g(x,t)$ given by:

(i) $\Phi_0(t)=|t|^N+N\int_0^{|t|} s^{N-1}\text{arcsinh}^{\alpha} s\, ds$;

(ii) It is not difficult to see that the conjugate N-function $\tilde{\Phi}_0$ of Φ_0 satisfies $\tilde{\Phi}_0(t)\le\tilde{\Phi}_0(1)t^{(N+\alpha)/(N+\alpha-1)}$ for all $t\in[0,1]$. Thus, an example of potential is given by

$$V(x)=\begin{cases}2, & \text{for } |x|\le 1,\\ 1+|x|^{\tau}, & \text{for } |x|\ge 2,\end{cases}$$

with $\tau>N(N+\alpha-1)$;

(iii) $f(x,t)=\partial F(x,t)/\partial t$ with $F(x,t)=\mu l(x)t^p\exp(bt^{\gamma})$, where $\mu\ge\mu^*$, $p>c_\alpha$, $b>0$, $\gamma=N/(N-1-\alpha)$ and $l\in C(\mathbb{R}^N,[1,2])$ is such that $l\equiv 2$ if $|x|\le 1$ and $l\equiv 1$ if $|x|\ge 2$;

(iv) $g(x,t)=t^{q-1}$, $t\ge 0$ and $q\in(1,m_\alpha)$.

Example 1.5. Another example of $g(x,t)$ satisfying conditions (g_1)–(g_2) which is not a pure power like above is given as follows. For $t_1=1$ and $1<q\le q_1<m_\alpha$, we define

$$g(x,t)=\begin{cases}\dfrac{1}{2}a(x)\left(t^{q_1-1}+t^{q-1}\right)\left[1+\cos\left(\dfrac{\pi}{2}t\right)\right], & \text{for } (x,t)\in\mathbb{R}^N\times[0,1],\\ a(x)\left(1+\sin(\pi t)\right)^{q-1}, & \text{for } (x,t)\in\mathbb{R}^N\times[1,+\infty),\end{cases}$$

where $a:\mathbb{R}^N\to\mathbb{R}$ is a continuous function satisfying

$$0<\inf_{x\in\mathbb{R}^N} a(x)\le\sup_{x\in\mathbb{R}^N} a(x)<\infty.$$

Indeed, to show this fact just to observe that the inequalities $1+\cos(\pi t/2)\ge 1$ and $t^{q-1}\ge t^{q_1-1}$ hold for $t\in[0,1]$. Moreover, $1\le t^{q-1}$ and $\sin(\pi t)\le\pi t$ hold for $t\ge 1$.

Remark 1.6. We emphasize that the approach used in this paper can be adapted with slight modifications to deal with non-homogeneous equations and to obtain similar results as in [17,19,26,34]. Furthermore, our approach works if we impose a more general condition than (Φ_2), namely,

$$\lim_{t\to+\infty}\frac{\Phi(t)}{t^N\left(\prod_{j=1}^{l-1}\log^{\alpha}_{[j]}(t)\right)\log^{\alpha}_{[l]}(t)}=1,\quad l\in\mathbb{N}\quad\text{and}\quad\alpha\in[0,N-1),$$

where $\log_{[k]}(t)=\log(\log_{[k-1]}(t))$ and $\log_{[1]}(t)=\log(t)$, which was considered in [11–13]. For the sake of simplicity, we prefer to treat only the case $l=1$.

This paper is organized as follows: in Sect. 2, we present some preliminary results about Orlicz spaces which are used in the work. In Sect. 3, we establish the variational framework for our problem and we obtain some embedding results involving our working space. Section 4 shows that the energy functional has the geometric structure of the Mountain Pass Theorem, and in Sect. 5 we prove that this functional satisfies the Palais–Smale condition in certain energy levels. Section 6 is devoted to the proof of Theorem 1.1, and in Sect. 7 we prove Theorem 1.2. Finally, by applying minimization arguments we obtain Theorem 1.3 in Sect. 8.

Throughout this paper, $W^{1,N}(\mathbb{R}^N)$ denotes the Sobolev space endowed with the norm

$$\|u\|_{1,N}=\int_{\mathbb{R}^N}(|\nabla u|^N+|u|^N)\mathrm{d}x,\quad u\in W^{1,N}(\mathbb{R}^N).$$

We use $|\cdot|_p$ to denote the norm of the Lebesgue space $L^p(\mathbb{R}^N)$, $1\le p\le\infty$. The symbols C, C_i, $i=0,1,2,\ldots$ will denote different (possibly) positive constants. We denote by B_r the ball centered at the origin with radius $r>0$ and $|D|$ the Lebesgue measure of a mensurable set of $\mathbb{R}^N$.

2. Preliminaries

In order to facilitate the understanding of the paper, in this section we present briefly some results about Orlicz spaces. For the proofs and more details see, for instance [5,32].

A function $A:\mathbb{R}\to[0,+\infty)$ is called $\mathcal{N}-$function if it is convex, even, $A(t)=0$ if and only if $t=0$, $A(t)/t\to 0$ as $t\to 0$ and $A(t)/t\to+\infty$ as $t\to+\infty$. In particular, we have $A'(0)=0$ and if A is differentiable then $A'(t)$ is non-decreasing for $t\ge 0$, which implies that $A(t)$ is increasing for $t>0$. For a $\mathcal{N}-$function A and an open set $\Omega\subset\mathbb{R}^N$, the Orlicz class is the set defined by

$$K_{A,\mu}(\Omega)=\left\{u:\Omega\to\mathbb{R};\ u\text{ is measurable and }\int_\Omega A(|u(x)|)\,\mathrm{d}\mu<\infty\right\}.$$

If μ is the Lebesgue measure, then we denote $K_{A,\mu}(\Omega)$ by $K_A(\Omega)$. The linear space $L_{A,\mu}(\Omega)$ generated by $K_{A,\mu}(\Omega)$ is called *Orlicz space*. When A satisfies the Δ_2-condition, namely, there exists a constant $k>0$ such that

$$A(2t) \leq kA(t), \quad \forall \quad t \geq 0,$$

the Orlicz class $K_{A,\mu}(\Omega)$ is a linear space, and hence equal to $L_{A,\mu}(\Omega)$. We consider the following norm (called of *Luxemburg's norm*) on $L_{A,\mu}(\Omega)$:

$$|u|_{A,\Omega} = \inf\left\{\lambda > 0; \int_{\Omega} A\left(\frac{|u(x)|}{\lambda}\right) \mathrm{d}\mu \leq 1\right\}.$$

It can be shown that $(L_{A,\mu}(\Omega), |\cdot|_{A,\Omega})$ is a Banach space (see [32]). In the case $\Omega = \mathbb{R}^N$, we denote $|\cdot|_{A,\mathbb{R}^N}$ by $|\cdot|_A$. The complement $\mathcal{N}-$function of A is defined by

$$\tilde{A}(t) = \sup_{s>0}\left\{ts - A(s)\right\}.$$

It is not difficult to verify that $\tilde{\tilde{A}} = A$. In the spaces $L_{A,\mu}(\Omega)$ and $L_{\tilde{A},\mu}(\Omega)$, an extension of the Hölder inequality holds, namely,

$$\left|\int_{\Omega} u(x)v(x)\,\mathrm{d}\mu\right| \leq 2|u|_{A,\Omega}|v|_{\tilde{A},\Omega}, \quad \forall\, u \in L_{A,\mu}(\Omega),\ v \in L_{\tilde{A},\mu}(\Omega). \tag{2.1}$$

As a consequence, for every $\tilde{u} \in L_{\tilde{A},\mu}(\Omega)$ there corresponds a continuous linear functional $f_{\tilde{u}} \in (L_{A,\mu}(\Omega))'$ given by $f_{\tilde{u}}(v) = \int_{\Omega} \tilde{u}(x)v(x)\,\mathrm{d}\mu,\ v \in L_{A,\mu}(\Omega)$. Thus, we can define

$$\|\tilde{u}\|_{\tilde{A},\Omega} = \sup_{|v|_{A,\Omega}\leq 1} \int_{\Omega} \tilde{u}(x)v(x)\,\mathrm{d}\mu$$

and $\|\cdot\|_{\tilde{A},\Omega}$ is called the Orlicz norm on the space $L_{\tilde{A},\mu}(\Omega)$. Similarly, we can define the Orlicz norm $\|\cdot\|_{A,\Omega}$ on $L_{A,\mu}(\Omega)$. The norms $|\cdot|_{A,\Omega}$ and $\|\cdot\|_{A,\Omega}$ are equivalent and satisfy the inequalities

$$|u|_{A,\Omega} \leq \|u\|_{A,\Omega} \leq 2|u|_{A,\Omega}.$$

We define the Orlicz–Sobolev space $W^{1,A}(\Omega)$ as follows

$$W^{1,A}(\Omega) = \{u : \Omega \to \mathbb{R} :\ u \text{ is measurable and } u, |\nabla u| \in L_A(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,A}(\Omega)} := |u|_{A,\Omega} + |\nabla u|_{A,\Omega},$$

where ∇u is the gradient of u and we are using its Euclidean norm in $\mathbb{R}^N$. An important property is that if A and $\tilde{A}$ verify the Δ_2-condition, then the spaces $L_A(\Omega)$ and $W^{1,A}(\Omega)$ are reflexive, separable and

$$\begin{aligned}(L_A(\Omega), |\cdot|_{A,\Omega})' = \left(L_{\tilde{A}}(\Omega), \|\cdot\|_{\tilde{A},\Omega}\right) \quad \text{and} \quad &\left(L_{\tilde{A}}(\Omega), |\cdot|_{\tilde{A},\Omega}\right)' \\ &= (L_A(\Omega), \|\cdot\|_{A,\Omega}).\end{aligned}$$

Next, we consider a lemma due to Fukagai, Ito and Narukawa (see [24, Lemma 2.1]) which will be used in our arguments.

Lemma 2.1. *Suppose that A is a differentiable $\mathcal{N}-$function satisfying*

$$m \leq \frac{A'(t)t}{A(t)} \leq M, \quad \forall\, t > 0, \tag{2.2}$$

for some $M \geq m > 0$. Defining, for $t \geq 0$, $\xi_0(t) = \min\{t^m, t^M\}$ and $\xi_1(t) = \max\{t^m, t^M\}$, one has

$$\xi_0(\rho)A(t) \leq A(\rho t) \leq \xi_1(\rho)A(t) \quad \text{for} \quad \rho, t \geq 0$$

and

$$\xi_0(|u|_A) \leq \int_\Omega A(|u|)\,\mathrm{d}\mu \leq \xi_1(|u|_A) \quad \text{for} \quad u \in L_{A,\mu}(\Omega).$$

Remark 2.2. By virtue of Lemma 2.1 with $m > 1$, if A satisfies Δ_2-condition then $\tilde{\Phi}$ also satisfies Δ_2-condition (see [24, Lemma 2.7]). Therefore, it can be shown that

$$u_n \to 0 \ \text{ in } \ L_{\Phi,\mu}(\mathbb{R}^N) \Longleftrightarrow \int_{\mathbb{R}^N} \Phi(|u_n|)\,\mathrm{d}\mu \to 0$$

and (u_n) is bounded in $L_{\Phi,\mu}(\mathbb{R}^N)$ if and only if $(\int_{\mathbb{R}^N} \Phi(|u_n|)\mathrm{d}\mu)$ is bounded. Moreover, as cited above $\big(L_{\Phi,\mu}(\mathbb{R}^N), |\cdot|_\Phi\big)$ is a separable and reflexive Banach space as well as $\big(W^{1,\Phi}(\mathbb{R}^N), |\cdot|_{1,\Phi}\big)$. As we saw in the Introduction, our $\mathcal{N}-$function Φ satisfies Δ_2-condition and assumption (2.2) with $m = m_\alpha$ and $M = c_\alpha$.

3. Variational Framework

In order to apply variational methods, as seen in the Introduction, we consider the following linear subspace of $W^{1,\Phi}(\mathbb{R}^N)$

$$X = \left\{ u \in W^{1,\Phi}(\mathbb{R}^N);\, \int_{\mathbb{R}^N} V(x)\Phi(|u|)\mathrm{d}x < \infty \right\}$$

which is a separable and reflexive Banach space with the norm $\|u\| := |\nabla u|_\Phi + |u|_{\Phi,V}$ (see Remark 2.2), where

$$|u|_{\Phi,V} = \inf\left\{ \lambda > 0;\, \int_{\mathbb{R}^N} V(x)\Phi\left(\frac{|u|}{\lambda}\right)\mathrm{d}x \leq 1 \right\}$$

and

$$|\nabla u|_\Phi = \inf\left\{ \lambda > 0;\, \int_{\mathbb{R}^N} \Phi\left(\frac{|\nabla u|}{\lambda}\right)\mathrm{d}x \leq 1 \right\}.$$

Moreover, we can see that $C_0^\infty(\mathbb{R}^N)$ is dense in X. The next lemma presents some embeddings which will be used in our arguments.

Lemma 3.1. *If (Φ_1)–(Φ_2) and (V_1) are satisfied, then the following embeddings are continuous:*

(a) $L_\Phi(\mathbb{R}^N) \hookrightarrow L^N(\mathbb{R}^N)$;
(b) $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow W^{1,N}(\mathbb{R}^N)$;

(c) $X \hookrightarrow W^{1,\Phi}(\mathbb{R}^N)$*;*
(d) $X \hookrightarrow L_\Phi(\mathbb{R}^N)$ *and* $X \hookrightarrow L^r(\mathbb{R}^N)$ *for any* $r \in [N, \infty)$*.*

Proof. We observe that by $(\Phi_1) - (\Phi_2)$ there exists $C_1 > 0$ such that $t^N \leq C_1\Phi(t)$ for all $t \geq 0$. Thus, if $u_n \to 0$ in $L_\Phi(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} |u_n|^N \mathrm{d}x \leq C_1 \int_{\mathbb{R}^N} \Phi(|u_n|)\mathrm{d}x \to 0$$

and item (a) is proved. Item (b) is a immediate consequence of (a). To prove (c), just see that

$$\int_{\mathbb{R}^N} \Phi(|u|)\mathrm{d}x \leq \frac{1}{V_0}\int_{\mathbb{R}^N} V(x)\Phi(|u|)\mathrm{d}x < \infty$$

for $u \in X$ and the proof of item $d)$ follows directly from $b)$, $c)$ and by the continuous embedding from $W^{1,N}(\mathbb{R}^N)$ into $L^r(\mathbb{R}^N)$ for any $r \in [N, \infty)$. □

Now, we prove a result of compact embedding, which will be crucial in the sequel.

Proposition 3.2. *Under conditions* $(V_1) - (V_2)$ *and* $(\Phi_1) - (\Phi_2)$*, the space* X *is compactly immersed into* $L^s(\mathbb{R}^N)$ *for all* $1 \leq s < \infty$*.*

Proof. Given $v \in X$, in view of (V_1) and by Lemma 2.1 we have

$$\begin{aligned}\int_{\mathbb{R}^N} \Phi(|v|V^{1/c_\alpha})\mathrm{d}x &= \int_{\{x;\ V_0 \leq V(x) < 1\}} \Phi(|v|V^{1/c_\alpha})\mathrm{d}x + \int_{\{x;\ V(x)\geq 1\}} \Phi(|v|V^{1/c_\alpha})\mathrm{d}x \\ &\leq \frac{1}{V_0}\int_{\{x;\ V_0 \leq V(x) < 1\}} V(x)\Phi(|v|)\mathrm{d}x + \int_{\{x;\ V(x)\geq 1\}} V(x)\Phi(|v|)\mathrm{d}x \\ &\leq \frac{1}{V_0}\int_{\mathbb{R}^N} V(x)\Phi(|v|)\mathrm{d}x < \infty,\end{aligned} \tag{3.1}$$

where we have used that Φ is increasing in $[0, +\infty)$. Thus, $vV^{1/c_\alpha} \in L_\Phi(\mathbb{R}^N)$ and by (V_2) and Hölder inequality (2.1) we obtain

$$\int_{\mathbb{R}^N} |v|\mathrm{d}x = \int_{\mathbb{R}^N} \frac{V(x)^{1/c_\alpha}}{V(x)^{1/c_\alpha}}|v|\mathrm{d}x \leq 2|V^{-1/c_\alpha}|_{\tilde{\Phi}}|vV^{1/c_\alpha}|_\Phi, \tag{3.2}$$

which shows that $v \in L^1(\mathbb{R}^N)$. Moreover, if $v_n \to 0$ in X then $\int_{\mathbb{R}^N} V(x)\Phi(|v_n|)\mathrm{d}x \to 0$. Hence, by using (3.1) we conclude that $\int_{\mathbb{R}^N} \Phi(|v_n|V^{1/c_\alpha})\mathrm{d}x \to 0$ and this proves that $|v_nV^{1/c_\alpha}|_\Phi \to 0$. Therefore, by (3.2) it follows that $\int_{\mathbb{R}^N} |v_n|\mathrm{d}x \to 0$ and the embedding $X \hookrightarrow L^1(\mathbb{R}^N)$ is continuous. By the interpolation inequality in the Lebesgue spaces and according to item $d)$ of Lemma 3.1, we conclude that the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for any $s \in [1, \infty)$.

Next, we show that $X \hookrightarrow L^1(\mathbb{R}^N)$ is compact. Let (v_n) be a bounded sequence in X. According to the norm in X, $(|v_n|_{\Phi,V})$ is bounded in $\mathbb{R}$ and by using Remark 2.2 it follows that $(\int_{\mathbb{R}^N} V(x)\Phi(|v_n|)\mathrm{d}x)$ is also bounded. Thus, from estimate (3.1) $(\int_{\mathbb{R}^N} \Phi(|v_n|V^{1/c_\alpha})\mathrm{d}x)$ is bounded. Applying again Remark 2.2, we deduce that there exists $C > 0$ such that $|V^{1/c_\alpha}v_n|_\Phi \leq C$ for all $n \in \mathbb{N}$. Since X is reflexive, up to a subsequence, $v_n \rightharpoonup v$ in X. For each $\varepsilon > 0$, by (V_2) there exists $R_0 > 0$ sufficiently large such that

$$|V^{-1/c_\alpha}|_{\tilde{\Phi}, B^c_{R_0}} < \frac{\varepsilon}{4(C + |V^{1/c_\alpha} v|_\Phi)},$$

where we have used that $\int_{B^c_R} \tilde{\Phi}\left(V^{-c_\alpha}\right) \mathrm{d}x \to 0$ as $R \to \infty$. Thus, by the Hölder inequality

$$\begin{aligned} \int_{B^c_{R_0}} |v_n - v| \mathrm{d}x &\leq 2|V^{-1/c_\alpha}|_{\tilde{\Phi}, B^c_{R_0}} |V^{1/c_\alpha}(v_n - v)|_\Phi \\ &< \frac{\varepsilon}{2(C + |V^{1/c_\alpha} v|_\Phi)}(C + |V^{1/c_\alpha} v|_\Phi) = \frac{\varepsilon}{2}. \end{aligned} \tag{3.3}$$

On the other hand, since the embedding $X \hookrightarrow W^{1,N}(\mathbb{R}^N)$ is continuous and $W^{1,N}(\mathbb{R}^N) \hookrightarrow L^1(B_{R_0})$ is compact, there exists $n_0 \in \mathbb{N}$ such that

$$\int_{B_{R_0}} |v_n - v| \mathrm{d}x < \frac{\varepsilon}{2}, \quad \forall\, n > n_0. \tag{3.4}$$

From (3.3) to (3.4), one has $\int_{\mathbb{R}^N} |v_n - v| \mathrm{d}x < \varepsilon$ for all $n > n_0$, which shows that $v_n \to v$ in $L^1(\mathbb{R}^N)$. Now, for $s > 1$ we have $|v_n - v|_{s+1}$ is bounded and by the interpolation inequality, for some $t \in (0, 1)$ we obtain

$$|v_n - v|_s \leq |v_n - v|_1^t |v_n - v|_{s+1}^{1-t} \leq C|v_n - v|_1^t \to 0,$$

which implies that the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for all $1 \leq s < \infty$ and the proof is complete. □

The next lemma presents a version of the Trudinger–Moser inequality for functions in $W^{1,\Phi}(\mathbb{R}^N)$, which was proved by Cerný in [13]. It is necessary to use variational methods to find solutions for problem (1.5) with non-linearities $f(x,t)$ satisfying the condition growth (f_1).

Lemma 3.3. *If $N \geq 2$, $K > 0$, $\alpha \in [0, N-1)$, Φ is a $\mathcal{N}-$function verifying $(\Phi_1) - (\Phi_2)$ and $u \in W^{1,\Phi}(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} \left[\exp(K|u|^\gamma) - S_{N,\alpha}(K|u|^\gamma)\right] \mathrm{d}x < \infty.$$

Furthermore, if $|\nabla u|_\Phi \leq 1$, $|u|_\Phi \leq M < \infty$ and $K < K_{N,\alpha}$, then there exists a constant $C = C(N, \alpha, M, \Phi, K) > 0$, which depends only N, α, M, Φ and K such that

$$\int_{\mathbb{R}^N} \left[\exp(K|u|^\gamma) - S_{N,\alpha}(K|u|^\gamma)\right] \mathrm{d}x \leq C,$$

where $K_{N,\alpha} = B^{1/B} N \omega_{N-1}^{\gamma/N}$, $B = 1 - \alpha/(N-1)$ and ω_{N-1} is the measure of the unit sphere in $\mathbb{R}^N$.

To finalize this section, we get two technique lemmas that will be necessary to show the regularity of the energy functional associated to our problem.

Lemma 3.4. *For each $p \geq 1$, there exists $C = C(p) > 0$ such that*

$$\left[\exp(t) - S_{N,\alpha}(t)\right]^p \leq C\left[\exp(pt) - S_{N,\alpha}(pt)\right], \ \forall\, t \geq 0.$$

Proof. It suffices to proof that the limits

$$\lim_{t\to 0}\frac{[\exp(t)-S_{N,\alpha}(t)]^p}{[\exp(pt)-S_{N,\alpha}(pt)]} \quad \text{and} \quad \lim_{t\to +\infty}\frac{[\exp(t)-S_{N,\alpha}(t)]^p}{[\exp(pt)-S_{N,\alpha}(pt)]}$$

are finite, which is a direct consequence of the L'Hospital Rule. □

Lemma 3.5. *Let (u_n) be a sequence in $W^{1,\Phi}(\mathbb{R}^N)$ strongly convergent. Then there exist a subsequence (u_{n_k}) of (u_n) and $v\in W^{1,\Phi}(\mathbb{R}^N)$ such that $u_{n_k}(x)\le v(x)$ almost everywhere in $x\in\mathbb{R}^N$.*

Proof. The arguments used to show this lemma follow the same lines of the proof of Proposition 1 of [19] with slight modifications and we omit the proof. □

The energy functional associated to Problem (1.5) is given by

$$J_\lambda(u)=\int_{\mathbb{R}^N}[\Phi(|\nabla u|)+V(x)\Phi(|u|)]\mathrm{d}x-\int_{\mathbb{R}^N}F(x,u)\mathrm{d}x-\lambda\int_{\mathbb{R}^N}G(x,u)\mathrm{d}x.$$

Notice that by (f_1), (g_1), Lemma 3.3, Proposition 3.2 and Lemma 3.5, J_λ is well defined on X and moreover by using standard computations (see [13, Proposition 4.1]), we can see that $J_\lambda\in C^1(X,\mathbb{R})$ and its derivative is given by

$$\begin{aligned}\langle J'_\lambda(u),v\rangle=&\int_{\mathbb{R}^N}\left[\Phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\nabla v+V(x)\Phi'(|u|)\frac{u}{|u|}v\right]\mathrm{d}x-\int_{\mathbb{R}^N}f(x,u)v\mathrm{d}x\\&-\lambda\int_{\mathbb{R}^N}g(x,u)v\mathrm{d}x,\end{aligned}$$

for $u,v\in X$. Consequently, critical points of J_λ are precisely the weak solutions of (1.5).

4. Mountain Pass Structure

In order to get Theorem 1.1, we shall use the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [4]:

Theorem 4.1. *Let X be a Banach space and $J\in C^1(X;\mathbb{R})$ with $J(0)=0$. Suppose that there exist $\rho,\tau>0$ and $e\in X$, with $\|e\|>\rho$, such that*

$$\inf_{\|u\|=\rho}J(u)\ge\tau \quad \text{and} \quad J(e)\le 0. \tag{4.1}$$

Then, J possesses a Palais–Smale sequence at level c characterized as

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}J(\gamma(t))\ge\tau,$$

where $\Gamma=\{\gamma\in C([0,1];X):\gamma(0)=0 \text{ and } \gamma(1)=e\}$. Moreover, if J satisfies the Palais-Smale condition at level c then J has a critical point u_0 such that $J(u_0)=c$.

The number c is called *Mountain Pass Level* or *Minimax Level* of the functional Φ.

In the sequel, we show that, for $\lambda > 0$ small, the functional J_λ has the Mountain Pass Geometry, condition (4.1) above. This is proved in the next lemmas.

Lemma 4.2. *Assume* (f_1), (f_2) *and* (g_1). *Then, there exist* $\lambda_0, \rho_0, \tau_0 > 0$ *such that, for all* $\lambda \in [0, \lambda_0]$

$$J_\lambda(u) \geq \tau_0, \quad \forall \quad \|u\| = \rho_0.$$

Proof. Given $\varepsilon \in (0, V_0)$, by virtue of (1.6) there exists $\delta > 0$ verifying

$$F(x,t) \leq \varepsilon\Phi(t), \quad \forall \quad |t| \leq \delta \quad \text{and} \quad x \in \mathbb{R}^N.$$

On the other hand, by using (f_1) and taking $p > c_\alpha$ we have

$$F(x,t) \leq C|t|^p[\exp(b|t|^\gamma) - S_{N,\alpha}(b|t|^\gamma)], \quad \forall\ |t| > \delta \quad \text{and} \quad x \in \mathbb{R}^N,$$

for some $C = C(\delta, p) > 0$. Therefore,

$$F(x,t) \leq \varepsilon\Phi(t) + C|t|^p[\exp(b|t|^\gamma) - S_{N,\alpha}(b|t|^\gamma)], \quad \forall\ t \geq 0 \quad \text{and} \quad x \in \mathbb{R}^N. \tag{4.2}$$

According to (V_1), (g_1), Eq. (4.2), Hölder inequality, Lemma 3.4 and Proposition 3.2, we reach

$$\begin{aligned} J_\lambda(u) \geq & \left(1 - \frac{\varepsilon}{V_0}\right) \int_{\mathbb{R}^N} [\Phi(|\nabla u|) + V(x)\Phi(u)]\mathrm{d}x \\ & - C_1 \left\{ \int_{\mathbb{R}^N} [\exp(2b|u|^\gamma) - S_{N,\alpha}(2b|u|^\gamma)]\mathrm{d}x \right\}^{\frac{1}{2}} |u|_{2p}^p - \lambda C_2 |u|_q^q \\ \geq & \left(\frac{V_0 - \varepsilon}{V_0}\right) \int_{\mathbb{R}^N} [\Phi(|\nabla u|) + V(x)\Phi(|u|)]\mathrm{d}x \\ & - C_3 \left\{ \int_{\mathbb{R}^N} \left[\exp\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right) \right.\right. \\ & \left.\left. - S_{N,\alpha}\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right)\right] \mathrm{d}x \right\}^{\frac{1}{2}} \|u\|^p \\ & - \lambda C_4 \|u\|^q. \end{aligned}$$

Now, if $\rho > 0$ is such that $2b\rho^\gamma < K_{N,\alpha}$ then for $\|u\| = \rho$ we obtain $2b|\nabla u|_\Phi^\gamma \leq 2b\rho^\gamma < K_{N,\alpha}$. Thus, by Lemma 3.3

$$\left\{ \int_{\mathbb{R}^N} \left[\exp\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right) - S_{N,\alpha}\left(2b|\nabla u|_\Phi^\gamma \left(\frac{|u|}{|\nabla u|_\Phi}\right)^\gamma\right)\right] \mathrm{d}x \right\}^{\frac{1}{2}} \leq C.$$

Consequently, for $\|u\| = \rho$ with $\rho \leq 1$, $2b\rho^\gamma < K_{N,\alpha}$, $\varepsilon = V_0/2$ and by Remark 2.2 we get

$$\begin{aligned} J_\lambda(u) & \geq \frac{1}{2}\left(|\nabla u|_\Phi^{c_\alpha} + |u|_{\Phi,V}^{c_\alpha}\right) - C_5\rho^p - \lambda C_4 \rho^q \\ & \geq \frac{1}{2} 2^{(1-c_\alpha)}\rho^{c_\alpha} - C_5\rho^p - \lambda C_4\rho^q \\ & = \rho^{c_\alpha}\left(2^{-c_\alpha} - C_5\rho^{p-c_\alpha}\right) - \lambda C_4\rho^q. \end{aligned}$$

Moreover, choosing $\rho = \rho_0 > 0$ so that $2^{-c_\alpha} - C_5\rho_0^{p-c_\alpha} =: \beta_0 > 0$, we conclude

$$J_\lambda(u) \geq \tau_0, \quad \forall \quad \lambda \in [0, \lambda_0] \quad \text{and} \quad \|u\| = \rho_0,$$

where $\lambda_0 = \beta_0\rho_0^{c_\alpha - q}/(2C_4)$ and $\tau_0 = \beta_0\rho_0^{c_\alpha}/2 > 0$. □

Lemma 4.3. *There exists $v_0 \in X$ with $\|v_0\| > \rho_0$ such that $J_\lambda(v_0) < 0$.*

Proof. Let $u_0 \in X \setminus \{0\}$ be such that $u_0 \geq 0$ and has compact support. Setting $\mathcal{K} = supp(u_0)$, by (f_2) we know that there exist $C_1, C_2 > 0$ such that

$$F(x,t) \geq C_1 t^\sigma - C_2, \quad \forall \quad t \geq 0 \quad \text{and} \quad x \in \mathcal{K}.$$

Consequently, for $t > 1$ and using Lemma 2.1, we obtain

$$\begin{aligned} J_\lambda(tu_0) &\leq \xi_1(t)\int_{\mathbb{R}^N}[\Phi(|\nabla u_0|) + V(x)\Phi(u_0)]\mathrm{d}x - c_1 t^\sigma \int_{\mathcal{K}} |u_0|^\sigma \mathrm{d}x + c_2|\mathcal{K}| \\ &= C_0 t^{c_\alpha} - C_3 t^\sigma + C_4 \longrightarrow -\infty \quad \text{as} \quad t \to +\infty, \end{aligned}$$

since $\sigma > c_\alpha$ and $C_3 > 0$. Taking $v_0 := t_0 u_0$, with t_0 large enough, the proof is finished. □

5. On Palais–Smale Sequences

First we recall that $(u_n) \subset X$ is a Palais–Smale ($(PS)_c$ for short) sequence at level $c \in \mathbb{R}$ for the functional J_λ if $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ in the dual space X'. We say that J_λ satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence has a convergent subsequence. In this section, our main objective is to prove the $(PS)_c$ condition for J_λ with c in a convenient interval.

Lemma 5.1. *If $(u_n) \subset X$ is a $(PS)_c$ sequence associated to J_λ, then (u_n) is bounded in X for all $\lambda \in [0, \lambda_0]$.*

Proof. By condition (f_2), (g_1) and since $\Phi'(t)t \leq c_\alpha\Phi(t)$, we have

$$\begin{aligned} J_\lambda(u_n) - \frac{1}{\sigma}\langle J'_\lambda(u_n), u_n\rangle \geq \left(1 - \frac{c_\alpha}{\sigma}\right)\int_{\mathbb{R}^N}[\Phi(|\nabla u_n|) + V(x)\Phi(|u_n|)]\mathrm{d}x \\ - \lambda C_1 \int_{\mathbb{R}^N}|u_n|^q \mathrm{d}x. \end{aligned} \tag{5.1}$$

On the other hand,

$$J_\lambda(u_n) - \frac{1}{\sigma}\langle J'_\lambda(u_n), u_n\rangle \leq c + o_n(1) + o_n(1)\|u_n\|. \tag{5.2}$$

Combining (5.1) and (5.2), using the continuous embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ and Lemma 2.1, we obtain

$$c + o_n(1) + o_n(1)\|u_n\| + \lambda C_2\|u_n\|^q \geq \left(\frac{\sigma - c_\alpha}{\sigma}\right)[\xi_0(|\nabla u_n|_\Phi) + \xi_0(|u_n|_{\Phi,V})] \tag{5.3}$$

Now, we argue by contradiction. Suppose that, up to a subsequence, $\|u_n\| \to \infty$. We have three possibilities to consider:

(i) $|\nabla u_n|_\Phi \to \infty$ and $(|u_n|_{\Phi,V})$ is bounded;
(ii) $(|\nabla u_n|_\Phi)$ is bounded and $|u_n|_{\Phi,V} \to \infty$;

(iii) $|\nabla u_n|_\Phi \to \infty$ and $|u_n|_{\Phi,V} \to \infty$.
If item (i) holds then there exists $n_1 \in \mathbb{N}$ such that $|\nabla u_n|_\Phi > 1$ for all $n > n_1$. Thus, by the definition of ξ_0 and inequality (5.3) we get

$$c+o_n(1)+o_n(1)|\nabla u_n|_\Phi+2^{q-1}\lambda_0 C_2\left(|\nabla u_n|_\Phi^q + C_3^q\right) \geq \left(\frac{\sigma - c_\alpha}{\sigma}\right)|\nabla u_n|_\Phi^{m_\alpha}, \quad \forall\, n > n_1, \tag{5.4}$$

where $|u_n|_{\Phi,V} \leq C_3$. Dividing this estimate by $|\nabla u_n|_\Phi^{m_\alpha}$ and since $m_\alpha > q > 1$ we get a contradiction doing $n \to \infty$. Thus, (i) does not happen. Similarly, we can show that items (ii) and (iii) do not happen as well. Therefore, (u_n) must be bounded in X and the proof is finalized. □

Remark 5.2. We observe by the previous proof that the boundedness of the sequence (u_n) is uniform in $\lambda \in [0, \lambda_0]$ according to estimate (5.4).

Corollary 5.3. *If $(u_n) \subset X$ is a $(PS)_c$ sequence for J_λ, then there exists $C_1 > 0$, independent of λ, such that*

$$\xi_0(|\nabla u_n|_\Phi) \leq \left(\frac{\sigma}{\sigma - c_\alpha}\right) c + \lambda C_1 + o_n(1), \quad \forall\, \lambda \in [0, \lambda_0].$$

Proof. By Lemma 5.1 and Remark 5.2, $\|u_n\| \leq C$ for all $n \in \mathbb{N}$, where C does not depend on λ. Hence, as a direct consequence of (5.3) we have the desired result. □

Before to show that J_λ satisfies the Palais–Smale condition in a convenient interval, we shall need of the following convergence result:

Lemma 5.4. *Let (u_n) be a Palais–Smale sequence for the functional J_λ at any level $c \in \mathbb{R}$ such that*

$$c < \left(\frac{\sigma - c_\alpha}{\sigma}\right) \xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}\right).$$

Then, there exists $\lambda_1 > 0$ such that for each $\lambda \in [0, \lambda_1]$ we obtain $u_\lambda \in X$ verifying

$$\int_{\mathbb{R}^N} f(x, u_n)(u_n - u_\lambda)\mathrm{d}x \to 0.$$

Proof. By Corollary 5.3, for some $C_1 > 0$ independent of λ, we have

$$\xi_0(|\nabla u_n|_\Phi) \leq \left(\frac{\sigma}{\sigma - c_\alpha}\right) c + \lambda C_1 + o_n(1), \quad \forall\, \lambda \in [0, \lambda_0].$$

Thus, since $\xi_0(t)$ is increasing for $t \geq 0$ and $\sigma c/(\sigma - c_\alpha) + \lambda C_1 + o_n(1) \to \sigma c/(\sigma - c_\alpha)$ as $\lambda \to 0^+$ and $n \to \infty$, we can obtain $0 < \lambda_* \leq \lambda_0$ and $n_1 \in \mathbb{N}$ such that

$$|\nabla u_n|_\Phi \leq \frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}} - \delta, \quad \forall \quad \lambda \in [0, \lambda_*],\ \forall\, n > n_1,$$

for some $\delta > 0$ sufficiently small. Choosing still $r > 1$ close to 1, we obtain

$$rb|\nabla u_n|_\Phi^\gamma \leq K_{N,\alpha} - \delta_1, \quad \forall \quad \lambda \in [0, \lambda_*],\ \forall\, n > n_1, \tag{5.5}$$

for some appropriate $\delta_1 > 0$. Now, by Lemma 5.1, up to a subsequence, $u_n \rightharpoonup u_\lambda \in X$ for each $\lambda \in [0, \lambda_1]$. By assumption (f_1), Hölder inequality and Lemma 3.4, it follows that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N} f(x,u_n)(u_n - u_\lambda)\mathrm{d}x\Big| \\
&\le C\int_{\mathbb{R}^N} |u_n|^{N-1}|u_n - u_\lambda|\mathrm{d}x + C\int_{\mathbb{R}^N} [\exp(b|u_n|^\gamma) - S_{N,\alpha}(b|u_n|^\gamma)]| \\
&\qquad u_n - u_\lambda|\mathrm{d}x \\
&\le C|u_n|_N^{N-1}|u_n - u_\lambda|_N + C_1\left(\int_{\mathbb{R}^N}[\exp(rb|u_n|^\gamma) - S_{N,\alpha}(rb|u_n|^\gamma)]\mathrm{d}x\right)^{\frac{1}{r}} \\
&\qquad |u_n - u_\lambda|_{r'}.
\end{aligned}$$

Since the embedding $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \ge 1$, we have $|u_n|_N^N|u_n - u_\lambda|_N \to 0$ and $|u_n - u_\lambda|_{r'} \to 0$. Hence, to finalize the proof, we have to show that

$$\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^N}[\exp(rb|u_n|^\gamma) - S_{N,\alpha}(rb|u_n|^\gamma)]\mathrm{d}x < \infty.$$

Indeed, we can write this integral as

$$\int_{\mathbb{R}^N}\left[\exp\left(rb|\nabla u_n|_\Phi^\gamma\left(\frac{|u_n|}{|\nabla u_n|_\Phi}\right)^\gamma\right) - S_{N,\alpha}\left(rb|\nabla u_n|_\Phi^\gamma\left(\frac{|u_n|}{|\nabla u_n|_\Phi}\right)^\gamma\right)\right]\mathrm{d}x$$

and by (5.5) we have $rb|\nabla u_n|_\Phi^\gamma < K_{N,\alpha} - \delta_1 < K_{N,\alpha}$. Therefore, invoking Lemma 3.3 we conclude that the above supreme is finite and the proof is complete. □

Lemma 5.5. *For each $\lambda \in [0, \lambda_1]$, the funcional J_λ satisfies the $(PS)_c$ condition for all*

$$c < \left(\frac{\sigma - c_\alpha}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}\right). \tag{5.6}$$

Proof. Let (u_n) be in X such that $J_\lambda(u_n) \to c$ and $J'_\lambda(u_n) \to 0$ in X' with c satisfying (5.6). By Lemma 5.1, (u_n) is bounded in X and therefore, up to a subsequence, $u_n \rightharpoonup u_\lambda$ in X. Since the functional $I(u) := \int_{\mathbb{R}^N}[\Phi(|\nabla u|) + V(x)\Phi(|u|)]\mathrm{d}x$ is convex, we get

$$\begin{aligned}
\int_{\mathbb{R}^N}[\Phi(|\nabla u_\lambda|) &+ V(x)\Phi(|u_\lambda|)]\mathrm{d}x - \int_{\mathbb{R}^N}[\Phi(|\nabla u_n|) + V(x)\Phi(|u_n|)]\mathrm{d}x \\
&\ge \int_{\mathbb{R}^N}\Phi'(|\nabla u_n|)\frac{\nabla u_n}{|\nabla u_n|}(\nabla u_\lambda - \nabla u_n)\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N}V(x)\Phi'(|u_n|)\frac{u_n}{|u_n|}(u_\lambda - u_n)\mathrm{d}x \\
&= \langle J'_\lambda(u_n), u_\lambda - u_n\rangle + \int_{\mathbb{R}^N} f(x,u_n)(u_\lambda - u_n)\mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^N} g(x,u_n)(u_\lambda - u_n)\mathrm{d}x.
\end{aligned} \tag{5.7}$$

According to Lemma 5.4, we know that $\int_{\mathbb{R}^N} f(x,u_n)(u_\lambda - u_n)\mathrm{d}x \to 0$. Moreover, by condition (g_1), Hölder inequality and compact embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ it follows that

$$\left|\int_{\mathbb{R}^N} g(x,u_n)(u_\lambda - u_n)\mathrm{d}x\right| \le C_1|u_n|_q^{q-1}|u_\lambda - u_n|_q \to 0.$$

Thus, by (5.7) one has

$$\int_{\mathbb{R}^N}[\Phi(|\nabla u_\lambda|) + V(x)\Phi(|u_\lambda|)]\mathrm{d}x \ge \int_{\mathbb{R}^N}[\Phi(|\nabla u_n|) + V(x)\Phi(|u_n|)]\mathrm{d}x + o_n(1)$$

and consequently

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}[\Phi(|\nabla u_n|) + V(x)\Phi(|u_n|)]\mathrm{d}x \le \int_{\mathbb{R}^N}[\Phi(|\nabla u_\lambda|) + V(x)\Phi(|u_\lambda|)]\mathrm{d}x. \tag{5.8}$$

Since $I_1(u) := \int_{\mathbb{R}^N}\Phi(|\nabla u|)\mathrm{d}x$ is continuous and convex on X, the weak convergence $u_n \rightharpoonup u_\lambda$ in X implies that

$$\int_{\mathbb{R}^N}\Phi(|\nabla u_\lambda|)\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^N}\Phi(|\nabla u_n|)\mathrm{d}x. \tag{5.9}$$

Similarly, one has

$$\int_{\mathbb{R}^N}V(x)\Phi(|u_\lambda|)\mathrm{d}x \le \liminf_{n\to\infty}\int_{\mathbb{R}^N}V(x)\Phi(|u_n|)\mathrm{d}x. \tag{5.10}$$

By virtue of (5.8), we must have the equality in (5.9) and (5.10). Hence, up to subsequences,

$$\int_{\mathbb{R}^N}\Phi(|\nabla u_n|)\mathrm{d}x \to \int_{\mathbb{R}^N}\Phi(|\nabla u_\lambda|)\mathrm{d}x$$

and

$$\int_{\mathbb{R}^N}V(x)\Phi(|u_n|)\mathrm{d}x \to \int_{\mathbb{R}^N}V(x)\Phi(|u_\lambda|)\mathrm{d}x.$$

Now, arguing as in [13, Lemma 6.2] we can see that, up to a subsequence, $\nabla u_n \to \nabla u$ almost everywhere in $\mathbb{R}^N$. Thus, since Φ is convex and $\Phi(|\nabla u_n - \nabla u_\lambda|)$ is bounded in $L^1(\mathbb{R}^N)$, by using the Brezis–Lieb Lemma (see [10, Thereom 2]), we conclude

$$\int_{\mathbb{R}^N}\Phi(|\nabla u_n - \nabla u_\lambda|)\mathrm{d}x \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N}V(x)\Phi(|u_n - u_\lambda|)\mathrm{d}x \to 0.$$

According to Remark 2.2 it follows that $\|u_n - u_\lambda\| = |\nabla u_n - \nabla u_\lambda|_\Phi + |u_n - u_\lambda|_{\Phi,V} \to 0$ and the proof is finalized. □

6. Proof of Theorem 1.1

In order to apply Theorem 4.1 to find a nonzero critical for J_λ, we need to estimate the minimax level c_λ of J_λ where

$$c_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{0\le t\le 1} J_\lambda(\gamma(t)) \quad \text{and} \quad \Gamma_\lambda := \{\gamma \in C([0,1];X); \gamma(0)=0 \ \text{ and } \ J_\lambda(\gamma(1))<0\}.$$

First, we are going consider a function $\varphi_0 \in C_0^\infty(\mathbb{R}^N)$ given by $\varphi_0(x)=1$ if $|x|\le 1/2$, $\varphi_0(x)=0$ if $|x|\ge 1$, $0\le\varphi_0(x)\le 1$ for all $x\in\mathbb{R}^N$ and $|\nabla\varphi_0(x)|\le 1$ for all $x\in\mathbb{R}^N$. Introducing the number

$$\mu_1 := \frac{\Phi(1)|B_1|(1+\|V\|_{L^\infty(B_1)})}{|B_{1/2}|},$$

by (f_4) we infer that if $\mu\ge\mu_1$ then

$$\begin{aligned} J_\lambda(\varphi_0) &\le \int_{B_1}[\Phi(|\nabla\varphi_0|)+V(x)\Phi(|\varphi_0|)]\mathrm{d}x - \mu_1\int_{B_1}|\varphi_0|^\theta \mathrm{d}x \\ &< \Phi(1)|B_1|(1+\|V\|_{L^\infty(B_1)}) - \mu_1|B_{1/2}| \\ &= 0. \end{aligned}$$

In particular,

$$\int_{B_1}[\Phi(|\nabla\varphi_0|)+V(x)\Phi(|\varphi_0|)]\mathrm{d}x < \mu_1\int_{B_1}|\varphi_0|^\theta \mathrm{d}x. \tag{6.1}$$

Lemma 6.1. (Minimax Estimate) *If condition (f_4) holds with*

$$\mu \ge \max\left\{\mu_1, \left[\frac{|B_1|\frac{\theta-m_\alpha}{m_\alpha}\left(\frac{\mu_1 m_\alpha}{\theta}\right)^{\frac{\theta}{\theta-m_\alpha}}}{\left(\frac{\sigma-c_\alpha}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}\right)}\right]^{\frac{\theta-m_\alpha}{m_\alpha}}\right\} =: \mu^*, \tag{6.2}$$

then

$$c_\lambda < \left(\frac{\sigma-c_\alpha}{\sigma}\right)\xi_0\left(\frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}\right), \quad \forall\ \lambda\in[0,\lambda_0].$$

Proof. For $\lambda\in[0,\lambda_0]$, Lemmas 4.2 and 4.3 imply that $c_\lambda>0$ and by definition of c_λ, (f_4) and (6.1) one has

$$\begin{aligned} c_\lambda &\le \max_{t\in[0,1]} J_\lambda(t\varphi_0) \\ &\le \max_{t\in[0,1]}\left[t^{m_\alpha}\int_{B_1}[\Phi(|\nabla\varphi_0|)+V(x)\Phi(|\varphi_0|)]\mathrm{d}x - \mu t^\theta\int_{B_1}|\varphi_0|^\theta\mathrm{d}x\right] \\ &\le \max_{t\ge 0}\left[\mu_1 t^{m_\alpha}-\mu t^\theta\right]\int_{B_1}|\varphi_0|^\theta\mathrm{d}x. \end{aligned}$$

A straightforward calculation shows that

$$\max_{t\ge 0}\left[\mu_1 t^{m_\alpha}-\mu t^\theta\right] = \frac{1}{\mu^{\frac{m_\alpha}{\theta-m_\alpha}}}\frac{\theta-m_\alpha}{m_\alpha}\left(\frac{\mu_1 m_\alpha}{\theta}\right)^{\frac{\theta}{\theta-m_\alpha}}.$$

and therefore

$$c_\lambda < \frac{|B_1|}{\mu^{\frac{m_\alpha}{\theta - m_\alpha}}} \frac{\theta - m_\alpha}{m_\alpha} \left(\frac{\mu_1 m_\alpha}{\theta}\right)^{\frac{\theta}{\theta - m_\alpha}}.$$

Thus, by using (6.2) we reach the estimate

$$c_\lambda < \left(\frac{\sigma - c_\alpha}{\sigma}\right) \xi_0 \left(\frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}\right), \quad \forall\, \lambda \in [0, \lambda_0].$$

and the proof is finalized. □

Finalizing the proof of Theorem 1.1: According to Lemmas 5.5 and 6.1 , J_λ satisfies $(PS)_{c_\lambda}$ condition. Moreover, J_λ has the Mountain Pass Geometry for $\lambda \in [0, \lambda_*]$, invoking Mountain Pass Theorem we conclude that there exists a nonzero critical $u_\lambda \in X$ such that $J_\lambda(u_\lambda) = c_\lambda$. Now, since $J_\lambda'(u_\lambda).(-u_\lambda^-) = 0$ and $\int_{\mathbb{R}^N} f(x, u_\lambda)(-u_\lambda^-)\mathrm{d}x = \int_{\mathbb{R}^N} g(x, u_\lambda)(-u_\lambda^-)\mathrm{d}x = 0$, we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \left[\Phi'(|\nabla u_\lambda|)\frac{|\nabla u_\lambda^-|^2}{|\nabla u_\lambda|} + V(x)\Phi'(|u_\lambda|)\frac{(u_\lambda^-)^2}{|u_\lambda|}\right] \mathrm{d}x \\ &\geq m_\alpha \min\{1, V_0\} \int_{\mathbb{R}^N} \left[\Phi(|\nabla u_\lambda|)\frac{|\nabla u_\lambda^-|^2}{|\nabla u_\lambda|^2} + \Phi(|u_\lambda|)\frac{(u_\lambda^-)^2}{u_\lambda^2}\right] \mathrm{d}x \\ &= m_\alpha \min\{1, V_0\} \int_{\mathbb{R}^N} \left[\Phi(|\nabla u_\lambda^-|) + \Phi(|u_\lambda^-|)\right] \mathrm{d}x, \end{aligned}$$

which implies that $u_\lambda^- = 0$ and therefore $u_\lambda = u_\lambda^+ \geq 0$.

Next, supposing that $u_\lambda \in C^1(\mathbb{R}^N)$ we are going to prove that $u_\lambda(x) > 0$ for all $x \in \mathbb{R}^N$. Suppose by contradiction that there exists $y \in \mathbb{R}^N$ such that $u_\lambda(y) = 0$ and consider an arbitrary ball $B_R(y)$. By virtue of (f_1), in the ball $B_R(y)$ we obtain (in the distribution sense) that

$$\begin{aligned} \operatorname{div}\left(\Phi'(|\nabla u_\lambda|)\frac{\nabla u_\lambda}{|\nabla u_\lambda|}\right) &= V(x)\Phi'(|u_\lambda|)\frac{u_\lambda}{|u_\lambda|} - f(x, u_\lambda) - \lambda g(x, u_\lambda) \\ &\leq M_1 \Phi'(|u_\lambda|), \end{aligned}$$

where $M_1 = \sup_{x \in B_R(y)} V(x)$. For $t > 0$, we define $\widehat{A}(t) = \Phi'(t)/t$ and $\widehat{f}(t) = M_1 \Phi'(t)$. Since Φ is a $\mathcal{N}-$function of class C^1 fulfilling condition (Φ_1) and the property $1 < m_\alpha \leq \Phi'(t)t/\Phi(t) \leq c_\alpha$, it is not difficult to verify that the hypotheses of Theorem 1 (Strong Maximum Principle) in [31] are satisfied for $\widehat{A}(t)$ and $\widehat{f}(t)$. Hence, $u_\lambda \equiv 0$ in $B_R(y)$ which is an absurd since $B_R(y)$ is arbitrary. Therefore, we have $u_\lambda(x) > 0$ for all $x \in \mathbb{R}^N$ and this completes the proof of Theorem 1.1.

7. Proof of Theorem 1.2

In this section, we present the proof of Theorem 1.2. For $\lambda = 0$, let us to denote the functional J_λ by J and the Mountain Pass Level c_λ by c. Setting

$$S = \{u \in X \backslash \{0\} : J'(u) = 0\} \quad \text{and} \quad m = \inf_{u \in S} J(u),$$

since the solution u_0 obtained in Theorem 1.1 satisfies $J(u_0) = c$, it is enough to prove that $c \leq m$. Let $u \in S$ and define $h : (0, \infty) \to \mathbb{R}$ by $h(t) = J(tu)$. We have that h is differentiable and

$$h'(t) = J'(tu)u = \int_{\mathbb{R}^N} \Phi'(t|\nabla u|)|\nabla u| \mathrm{d}x + \int_{\mathbb{R}^N} V(x)\Phi'(t|u|)|u| \mathrm{d}x \\ - \int_{\mathbb{R}^N} f(x, tu)u \mathrm{d}x, \quad \forall\, t > 0.$$

On the other hand, the equality $J'(u)u = 0$ implies that

$$t^{c_\alpha - 1} \int_{\mathbb{R}^N} \Phi'(|\nabla u|)|\nabla u| \mathrm{d}x + t^{c_\alpha - 1} \int_{\mathbb{R}^N} V(x)\Phi'(|u|)|u| \mathrm{d}x = t^{c_\alpha - 1} \int_{\mathbb{R}^N} f(x, u)u \mathrm{d}x.$$

Therefore, we can write $h'(t)$ of the form

$$h'(t) = t^{c_\alpha - 1} \int_{\mathbb{R}^N} \left[\frac{\Phi'(t|\nabla u|)}{(t|\nabla u|)^{c_\alpha - 1}} - \frac{\Phi'(|\nabla u|)}{|\nabla u|^{c_\alpha - 1}} \right] |\nabla u|^{c_\alpha} \mathrm{d}x \\ + t^{c_\alpha - 1} \int_{\mathbb{R}^N} V(x) \left[\frac{\Phi'(t|u|)}{(t|u|)^{c_\alpha - 1}} - \frac{\Phi'(|u|)}{|u|^{c_\alpha - 1}} \right] |u|^{c_\alpha} \mathrm{d}x \\ + t^{c_\alpha - 1} \int_{\mathbb{R}^N} \left[\frac{f(x, |u|)}{|u|^{c_\alpha - 1}} - \frac{f(x, t|u|)}{(t|u|)^{c_\alpha - 1}} \right] |u|^{c_\alpha} \mathrm{d}x \quad \forall t > 0.$$

Using that $\Phi'(t)/t^{c_\alpha - 1}$ is non-increasing for $t > 0$, $f(x, t)/t^{c_\alpha - 1}$ is increasing for $t > 0$, and since $h'(1) = 0$, we conclude that $h'(t) > 0$ for $0 < t < 1$ and $h'(t) < 0$ for $t > 1$. Hence,

$$J(u) = \max_{t \geq 0} J(tu).$$

Now, defining $g : [0, 1] \to X$, $g(t) = tt_0u$, where t_0 is such that $J(t_0u) < 0$, we have $g \in \Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0 \;\text{ and }\; J(\gamma(1)) < 0\}$ and therefore

$$c \leq \max_{t \in [0,1]} J(g(t)) \leq \max_{t \geq 0} J(tu) = J(u).$$

Since $u \in S$ is arbitrary, $c \leq m$ and this shows that u_0 is a non-negative ground state. The positivity of u_0 was proved in Theorem 1.1.

8. Proof of Theorem 1.3

In this section, by considering the additional hypothesis (g_2) on $g(x, s)$ and the Ekeland variational principle, we show that problem (1.5) has a second solution. Firstly, we will need of the following lemma:

Lemma 8.1. *There exists $\varphi \in X$ such that $J_\lambda(t\varphi) < 0$ for $t > 0$ sufficiently small.*

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)\backslash\{0\}$ be such that $0 \leq \varphi(x) \leq t_1$ for all $x \in \mathbb{R}^N$. By condition (g_2) and Remark 2.2, for $0 < t < 1$, we have

$$\begin{aligned} J_\lambda(t\varphi) &\leq \xi_1(t)\int_{\mathbb{R}^N}[\Phi(|\nabla\varphi|)+V(x)\Phi(|\varphi|)]\mathrm{d}x - \lambda\int_{\mathbb{R}^N} G(x,t\varphi)\mathrm{d}x \\ &\leq t^{m_\alpha}\int_{\mathbb{R}^N}[\Phi(|\nabla\varphi|)+V(x)\Phi(|\varphi|)]\mathrm{d}x - \lambda\beta_1 t^{q_1}\int_{\mathrm{supp}(\varphi)}\varphi^{q_1}\mathrm{d}x \\ &= t^{m_\alpha}\left(\int_{\mathbb{R}^N}[\Phi(|\nabla\varphi|)+V(x)\Phi(|\varphi|)]\mathrm{d}x - \lambda\beta_1 t^{q_1-m_\alpha}\int_{\mathrm{supp}(\varphi)}\varphi^{q_1}\mathrm{d}x\right) < 0, \end{aligned}$$

for $t > 0$ sufficiently small, where we have used that $q_1 < m_\alpha$ and $0 \leq t\varphi(x) \leq t_1$ for all $0 < t < 1$ and $x \in \mathbb{R}^N$. □

Proposition 8.2. *Suppose that (g_2) is satisfied. For each $\lambda > 0$, problem (1.5) has a solution $v_\lambda \in X$ such that $J_\lambda(v_\lambda) < 0$.*

Proof. Fixed $\lambda > 0$, we can take $\rho > 0$ such that

$$\rho < \frac{K_{N,\alpha}^{1/\gamma}}{b^{1/\gamma}}.$$

By the Ekeland variational principle, there exists $(v_n) \subset B_\rho(0)$ satisfying $J_\lambda(v_n) \to m_\lambda$ and $J'_\lambda(v_n) \to 0$ in X', where

$$m_\lambda = \inf_{u\in\overline{B}_\rho(0)} J_\lambda(u),$$

which is negative by Lemma 8.1. Since (v_n) is bounded in X, there exists $v_\lambda \in X$ such that $v_n \rightharpoonup v_\lambda$ in X. Now, by using the same arguments as in the proof of Lemma 5.5, we can see that $v_n \to v_\lambda$ in X. Therefore,

$$J_\lambda(v_\lambda) = m_\lambda = \min_{u\in\overline{B}_\rho(0)} J_\lambda(u)$$

and consequently v_λ is a critical point of J_λ with negative energy. Analogously to the proof of Theorem 1.1, v_λ is also non-negative. □

The proof of Theorem 1.3 is now an immediate consequence of Proposition 8.2 and Theorem 1.1 since

$$J_\lambda(u_\lambda) = c_\lambda > 0 > J_\lambda(v_\lambda), \quad \forall\, \lambda \in (0, \lambda_*).$$

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