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Schrödinger Operators Generated by Locally Constant Functions on the Fibonacci Subshift

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Abstract. We investigate the spectral properties of discrete one-dimensional Schrödinger operators whose potentials are generated by sampling along the elements of the Fibonacci subshift with a locally constant function. The fundamental trace map formalism for this model is presented and related to its spectral features via an extension of a multitude of works on the classical model, where the sampling function only depends on a single entry of the sequence.

1. Introduction

The Fibonacci Hamiltonian is one of the most heavily studied Schrödinger operators, both in the physics literature and in the mathematics literature. It is the central model in the study of one-dimensional quasicrystals, and it exhibits interesting spectral phenomena, such as zero-measure Cantor spectrum and purely singular continuous spectral measures, in a persistent way. We refer the reader to the surveys [7–9] for background, known results, and discussion, as well as [16] for a recent study that essentially completes the spectral analysis of this model.

Let us briefly recall the definition of the Fibonacci Hamiltonian. In fact, there are two standard ways of generating the potential, either via a coding of a rotation or via the iteration of a primitive substitution.

In the first setting, we consider the potential

$$V_{\lambda,\theta}(n) = \lambda \chi_{[1-\alpha,1)}(n\alpha + \theta \bmod 1),$$

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where $\alpha = \frac{\sqrt{5}-1}{2}$ is the inverse of the golden ratio, $\lambda > 0$, and $\theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Thus, we consider the circle, written in additive notation, the transformation given by rotation by α, and sample along the orbit of an initial point θ under this transformation with a characteristic function of an interval, whose length happens to coincide with α. Whenever an iterate falls in this interval, we write down λ; otherwise, we write down 0. The two-sided sequence generated in this way serves as the potential of the discrete one-dimensional Schrödinger operator

$$[H_{\lambda,\theta}\psi](n) = \psi(n+1) + \psi(n-1) + V_{\lambda,\theta}(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$, which is called the *Fibonacci Hamiltonian.*

In the second setting, we start with the primitive substitution $S_F : a \mapsto ab$, $b \mapsto a$ on the alphabet $\mathcal{A} = \{a, b\}$. Iterating S_F on the symbol a, we obtain the words $s_k := S_F^k(a)$, $k \geq 0$. For example,

$$s_0 = a,\ s_1 = ab,\ s_2 = aba,\ s_3 = abaab,\ s_4 = abaababa, \ldots. \tag{1.1}$$

Obviously, we have

$$s_{k+1} = s_k s_{k-1}, \tag{1.2}$$

which follows quickly from the definition. Therefore, the "limit" $u_F = S_F^\infty(a)$, called the *Fibonacci sequence*, makes sense as the unique one-sided infinite sequence that has each s_k as a prefix. This then gives rise to the *Fibonacci subshift*

$$\Omega_F := \{\omega \in \mathcal{A}^{\mathbb{Z}} : \text{ each subword of } \omega \text{ is a subword of } u_F\}.$$

There is also a natural transformation on Ω_F, namely the shift $T : \Omega_F \to \Omega_F$ given by $(T\omega)(n) := \omega(n+1)$. We again can define potentials by sampling along the orbit of an initial point $\omega \in \Omega$ with a suitable sampling function, that is,

$$V_{f,\omega}(n) := f(T^n\omega),$$

where

$$f(\omega) = \begin{cases} \lambda & \omega_0 = a, \\ 0 & \omega_0 = b. \end{cases} \tag{1.3}$$

This gives rise to the discrete one-dimensional Schrödinger operator

$$[H_{f,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + V_{f,\omega}(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$.

It is by now well understood that for each fixed $\lambda > 0$, the families $\{H_{\lambda,\theta}\}_{\theta\in\mathbb{T}}$ and $\{H_{f,\omega}\}_{\omega\in\Omega}$ are essentially the same.[1] Which of the two representations of the potentials is used often depends on the specific aspect of them that needs to be highlighted: either the (generalized) quasi-periodicity or the self-similarity.

[1] The latter family contains one additional orbit, namely the operators with potentials $\lambda\chi_{(1-\alpha,1]}(n\alpha + m\alpha \bmod 1)$, $m \in \mathbb{Z}$.

Indeed, the self-similarity often turns out to be the more critical aspect, as it gives rise to the trace map, whose dynamical study underlies all the recent progress on this model; compare the papers [7–9, 16] mentioned above.

Given that the subshift perspective has turned out to be the critical one for the Fibonacci Hamiltonian, we can now explain our motivation for writing this paper. Once the Fibonacci subshift Ω_F, and hence the associated topological dynamical system (Ω_F, T), has been defined, the choice of sampling function given by (1.3) appears to be rather special. Note that $f(\omega)$ only depends on one entry of ω. It is therefore a locally constant function on Ω_F of a very special form. From the general perspective of dynamically defined Schrödinger operators (cf. [8, 12]), it would be natural to start with the general class of continuous sampling functions $f : \Omega_F \to \mathbb{R}$ and to only impose additional conditions as they become necessary for the proofs of the desired results.

Inspecting now the known results for the Fibonacci Hamiltonian, one observes that only the statement that the spectrum is a Cantor set of zero Lebesgue measure has been established for sampling functions more general than the ones in (1.3). Indeed, Damanik and Lenz showed in [21] that this spectral property holds for all potentials $V_\omega(n) = f(T^n\omega)$, $\omega \in \Omega$, provided that Ω is a subshift that satisfies the Boshernitzan condition (cf. [1, 2]) and $f : \Omega \to \mathbb{R}$ is locally constant. In [22], the same authors provide many examples of subshifts Ω that satisfy the Boshernitzan condition, and in particular the Fibonacci subshift Ω_F is among them. All of the other known results for the Fibonacci Hamiltonian have been proved only for sampling functions of the form (1.3).

Now that we have established that it would be natural to investigate whether those spectral results can be extended to more general sampling functions, in a setting for which we will fix assumptions and notation in Sect. 2, let us briefly comment on the immediate obstacle that needs to be overcome. The works [21, 22] rely only on the Boshernitzan condition and hence make no use of the self-similarity and the resulting trace map dynamical system. On the other hand, all the other results rely heavily on the trace map approach. If one considers a general locally constant $f : \Omega_F \to \mathbb{R}$, the very first objective is to clarify whether the resulting operator family can indeed be studied via the Fibonacci trace map.[2] This is discussed in Sect. 3.

From this point on, the analysis bifurcates. There are statements and proofs that rely merely on the trace map description of the spectrum and make use of it only in the sense of a recursion. This concerns most of the results obtained for the standard Fibonacci Hamiltonian prior to 2008. On the other hand, most of the developments around the standard Fibonacci Hamiltonian since 2008 make heavy use of properties of the Fibonacci trace map as a hyperbolic dynamical system and employ methods and tools from the theory of such maps.

[2] An alternative approach is based on realizing that our potentials are quasi-Sturmian and appealing to [20]. However, this approach cannot yield those results whose proofs actually employ the particular features of the Fibonacci trace map; compare Remark 2.1.

Some of the extensions will be rather straightforward, while others will require some work. Let us point out here that we found the following items to be crucial and non-trivial in the course of our investigation of this model:

- the matrix recursion and the resulting trace map formalism are no longer a direct consequence of the Fibonacci substitution rule,
- the property $x_{-1} = 1$ does not extend from the standard case to the general case,
- the proof of power-law bounds on transfer matrices for energies in the spectrum and *all* elements of the subshift requires considerations in both directions, and
- the curve of initial conditions is in general no longer a line and does not lie inside a single invariant surface of the trace map.

We will naturally give detailed arguments whenever the extension to the general case presents some obstacles or difficulties, but we will only sketch the proof of statements that largely follow the same arguments as the proof of the corresponding statement in the standard case.

In the discussion above, we have motivated our desire to extend the work on the Fibonacci Hamiltonian from sampling functions of the form (1.3) to general locally constant sampling functions intrinsically. Here we point out that there exists additional motivation. Namely, the paper [11] studied a certain class of quantum graphs, which may be reduced to suitable direct sums of generalized Sturm–Liouville operators. The combinatorial data of the quantum graphs studied in [11] generate the parameters of the resulting generalized Sturm–Liouville operators in a locally constant way, which, however, does not depend on a window of size one, but rather on a window of size two. This naturally prompts one to extend, in the Fibonacci case (which prominently features there, too), the known results for window size one to greater window sizes. Indeed, the extension from size one to size two already presents the same difficulties as the extension from size one to a general larger size, and hence, one should immediately work out the extension in the general case (which is exactly what we do in the present paper).

2. The Setting

In this short section, we fix the setting in which we work throughout the rest of the paper.

Given the Fibonacci subshift Ω_F as defined above, we generate potentials by sampling $T^n\omega$, $\omega \in \Omega_F$, $n \in \mathbb{Z}$, with an arbitrary non-constant locally constant sampling function $f : \Omega_F \to \mathbb{R}$.

Remark 2.1. It is not hard to see that the resulting potentials are quasi-Sturmian, and hence, we obtain a subclass of the operators studied in [20]. However, since the underlying subshift is Fibonacci, it is important to clarify the precise nature of the resulting trace map dynamics since much more is known for the Fibonacci trace map (especially the part of the theory that employs hyperbolic dynamics!) than for the trace map recursions used in the analysis of a general quasi-Sturmian operator. Concretely, while some of our results only recover what is known in the quasi-Sturmian case (and for these results, our proofs are quite succinct and omit details), others are not known in the general quasi-Sturmian case and their proofs require the more detailed analysis we carry out based on the Fibonacci trace map with a more general curve of initial conditions.

Since a locally constant function depends only on finitely many entries, we will assume (essentially without loss of generality) that there is an $N \in \mathbb{Z}_+$ such that $f(\omega)$ is completely determined by the N entries $\omega_0, \ldots, \omega_{N-1}$. It is well known that the Fibonacci sequence u_F is Sturmian, that is, it has precisely $N+1$ subwords of length N. This property extends to all elements of Ω_F, and hence, our function f is determined by $N+1$ real numbers $\lambda_1, \ldots, \lambda_{N+1}$, corresponding to the values of $f(\omega)$ as $\omega_0, \ldots, \omega_{N-1}$ runs through all the possible allowed choices. Fixing an order on this set of subwords of length N once and for all, the possible f's in question are then parametrized by $\lambda = (\lambda_1, \ldots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$, where the non-constancy of f corresponds to the assumption that not all λ_i are equal—such λ's will be called *non-degenerate*. The resulting potentials will then be denoted by $V_{\lambda,\omega}$.

Lemma 2.2. *If λ is non-degenerate, then $V_{\lambda,\omega}$ is non-periodic for every $\omega \in \Omega_F$.*

Before giving the proof of the lemma, we recall a useful fact: there are elements $\omega_F \in \Omega$ that coincide with the Fibonacci sequence u_F when restricted to the right half-line $\{0, 1, 2, \ldots\}$. In fact, there are precisely two such elements. We fix one of them for definiteness; the subsequent considerations will be independent of this choice.

Proof of Lemma 2.2. Suppose λ is such that $V_{\lambda,\omega}$ is periodic for some $\omega \in \Omega_F$. We have to show that λ is degenerate.

Note first that, by minimality, either all $V_{\lambda,\omega}$ are periodic or none of them are. In particular, our assumption implies that V_{λ,ω_F} is periodic.

Denote the period of V_{λ,ω_F} by $m \in \mathbb{Z}_+$. It is well known that the sequence $\{F_k \mod m\}_{k\geq 0}$ is periodic; see, for example, [34]. Since $F_0 = 1$, this implies that $F_k \equiv 1 \mod m$ for infinitely many values of $k \geq 0$.

Choose k large enough so that

(i) $F_k \equiv 1 \mod m$,
(ii) $F_{k-2} \geq m$, and
(iii) $F_{k-3} \geq N$.

Denote the values of V_{λ,ω_F} over one period, $(f(T^n\omega_F))_{n=0}^{m-1}$, by $(V_1, \cdots, V_m)$. By (1.2) and items (ii) and (iii) above, we have

$$\begin{aligned}(f(T^{F_k}\omega_F), f(T^{F_k+1}\omega_F), \cdots, f(T^{F_k+(m-1)}\omega_F)) &= (f(T^n\omega_F))_{n=0}^{m-1} \\ &= (V_1, \cdots, V_m).\end{aligned}$$

On the other hand, since $\{f(T^n\omega_F)\}_{n\in\mathbb{Z}}$ is m-periodic, item (i) implies

$$(f(T^{F_k}\omega_F), f(T^{F_k+1}\omega_F), \cdots, f(T^{F_k+(m-1)}\omega_F)) = (V_2, V_3, \cdots, V_m, V_1).$$

In conclusion, we have

$$(V_1, V_2, \cdots, V_{m-1}, V_m) = (V_2, V_3, \cdots, V_m, V_1),$$

and this implies $V_i = V_j$ for all $1 \le i, j \le m$. Thus, V_{λ,ω_F} is constant, which in turn implies that λ is degenerate. □

In the remainder of the paper, we will study the spectral properties of the operators $H_{\lambda,\omega}$ in $\ell^2(\mathbb{Z})$ given by

$$[H_{\lambda,\omega}\psi](n) = \psi(n+1) + \psi(n-1) + V_{\lambda,\omega}(n)\psi(n).$$

Let us discuss an example to clarify the setting and the notation.

Example 2.3. Let us consider the case $N = 3$. That is, the sampling function f depends on the three entries $\omega_0, \omega_1, \omega_2$ of its argument $\omega \in \Omega_F$. The first step is to determine the subwords of length 3 that appear in the Fibonacci sequence u_F and to fix an order. It is easy to check that these words are given by the $4 = N + 1$ words

$$aab,\ aba,\ baa,\ bab,$$

and we will order them in the way just specified.

Next, given a non-degenerate $\lambda \in \mathbb{R}^4$ and an $\omega \in \Omega$, we want to determine the values of the potential.

Suppose the ω in question looks like

$$\ldots abaababaabaababa|ababaabaababaabaab \ldots$$

around the origin, where the vertical bar denotes the position between the entries ω_{-1} and ω_0.

Suppose further that $\lambda = (1, 2, 3, 4)$. Then, for example,

$$\begin{aligned} V_{\lambda,\omega}(0) &= f(\omega) \\ &= \text{the entry of } \lambda \text{ associated with the word aba} \\ &= \text{the entry of } \lambda \text{ associated with the second word in the list} \\ &= \lambda_2 \\ &= 2. \end{aligned}$$

Continuing in this fashion, we find that $V_{\lambda,\omega}$ looks like

$$\ldots 2312423123124231|2423123124231231 \ldots$$

around the origin, where the vertical bar again sits between $V_{\lambda,\omega}(-1)$ and $V_{\lambda,\omega}(0)$.

3. The Self-similarity Relation and the Trace Map

In this section, we recall the transfer matrix formalism that is a standard tool in the spectral analysis of one-dimensional Schrödinger operators, and we exhibit self-similarity properties of the transfer matrices in the setting we consider, which then lead to the realization that the standard Fibonacci trace map can be associated with our more general model. This will then be the starting point for the work presented in the subsequent sections, which employ the trace map, either as a recursion or as a hyperbolic dynamical system, in an essential way.

Given a potential $V_{\lambda,\omega}$ generated by the Fibonacci subshift and a locally constant sampling function f, where $\lambda = (\lambda_1, \dots, \lambda_{N+1}) \in \mathbb{R}^{N+1}$ and $\omega \in \Omega_F$, as described at the end of the previous section, we consider the associated difference equation

$$u(n+1) + u(n-1) + V_{\lambda,\omega}(n)u(n) = Eu(n), \tag{3.1}$$

where $n \in \mathbb{Z}$ and $E \in \mathbb{R}$. It is well known that the properties of the solutions u to the difference equation (3.1) are intimately connected to the spectral properties of the Schrödinger operator $H_{\lambda,\omega}$ in $\ell^2(\mathbb{Z})$ with potential $V_{\lambda,\omega}$; compare, for example, [8,12].

The solutions to (3.1) can be expressed via transfer matrices,

$$\begin{pmatrix} u(n) \\ u(n-1) \end{pmatrix} = A^n_{\lambda,E}(\omega) \begin{pmatrix} u(0) \\ u(-1) \end{pmatrix},$$

which are defined by $(T, A_{\lambda,E})^n = (T^n, A^n_{\lambda,E})$, where

$$(T, A_{\lambda,E}) : \Omega_F \times \mathbb{R}^2 \to \Omega_F \times \mathbb{R}^2, \quad (\omega, v) \mapsto (T\omega, A_{\lambda,E}(\omega))$$

and

$$A_{\lambda,E}(\omega) = \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}.$$

(Recall that we are implicitly using the association $f \leftrightarrow \lambda$ explained earlier.)

It is an essential feature of the standard Fibonacci Hamiltonian that a certain sequence of transfer matrices of this type satisfies a recursion. It is our immediate goal to extend this feature to the general locally constant case we consider.

In the standard Fibonacci case, this sequence of matrices is easy to define: it is simply the sequence of (energy-dependent) matrices that map across the words $s_k = S_F^k(a)$. In the general case at hand, we cannot define the matrices in this way because they are defined using a "look-ahead" that takes one outside the word s_k.

Definition. Let $\omega_F \in \Omega$ be an element that coincides with the Fibonacci sequence u_F when restricted to the right half-line $\{0, 1, 2, \dots\}$. With the Fibonacci numbers $\{F_k\}$ given by $F_0 = 1$, $F_1 = 2$, and $F_{k+1} = F_k + F_{k-1}$, $k \geq 1$, we set

$$M_k := M_k(E, \lambda) := A^{F_k}_{\lambda,E}(\omega_F). \tag{3.2}$$

Since the look-ahead points to the right, the definition of the M_k is independent of the choice of ω_F (recall there are two possible choices).[3]

The following lemma addresses the first issue one encounters when passing from window size one to the general case.

[3]This is the primary reason why we have normalized the window inspected by f in the way we did.

Lemma 3.1. *Let*

$$k_0 := \min\{k : F_{k-2} \geq N-1\}. \tag{3.3}$$

Then, we have

$$M_{k+1} = M_{k-1}M_k \tag{3.4}$$

for every $\lambda \in \mathbb{R}^{N+1}$*,* $E \in \mathbb{R}$*, and* $k \geq k_0$*.*

Proof. Applying (1.2) several times, we find

$$\begin{aligned} s_{k+2} &= s_{k+1}s_k \\ &= s_k s_{k-1} s_{k-1} s_{k-2} \\ &= s_k s_{k-1} s_{k-2} s_{k-3} s_{k-3} s_{k-4} \\ &= s_k s_{k-1} s_{k-2} s_{k-3} s_{k-4} s_{k-5} s_{k-4} \\ &= s_k s_k s_{k-2} s_{k-5} s_{k-4}. \end{aligned}$$

Thus, since s_{k-2} is a prefix of s_k, which in turn is a prefix of u_F, (3.4) follows from (3.2) and (3.3) since the length F_{k-2} of s_{k-2} exceeds the look-ahead $N-1$ that is necessary to define the matrices. □

Remark 3.2. If we want the recursion (3.4) to hold also for $k < k_0$, we can force this by iteratively *redefining* the matrices

$$M_k := M_{k+2}M_{k+1}^{-1}$$

for $k = k_0 - 2, k_0 - 3, \cdots$. Of course this will come at the price of the matrices M_k, $k \leq k_0 - 2$ not corresponding to actual transfer matrices. Thus, one needs to be careful in all arguments that use this interpretation of the matrices to ensure that k is large enough.

It is well known that the matrix recursion (3.4) gives rise to the recursion

$$x_{k+1} = 2x_k x_{k-1} - x_{k-2} \tag{3.5}$$

for the variables

$$x_k := x_k(E,\lambda) := \frac{1}{2}\mathrm{Tr}M_k. \tag{3.6}$$

To explain the terminology, note that (3.5) can be rewritten as

$$(x_{k+1}, x_k, x_{k-1}) = T(x_k, x_{k-1}, x_{k-2})$$

with the so-called *Fibonacci trace map*

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x,y,z) \mapsto (2xy - z, x, y). \tag{3.7}$$

The specific properties of T will not play a role in the present section and Sect. 4, as all arguments in this part of the paper are based on a study of (3.5) as a recursion. However, for our work in Sect. 5 on the extension of the results for the classical Fibonacci Hamiltonian whose proofs involve (partially) hyperbolic dynamics, the specific properties of the map T, and especially the fact that, once it is restricted to one of its invariant surfaces, the non-wandering set is hyperbolic, will be absolutely crucial.

The recursion (3.5) will hold for $k \geq k_0 + 1$ if we use the actual transfer matrices, and it will hold for all $k \in \mathbb{Z}$ if we use the redefined matrices; compare Remark 3.2.

Remark 3.3. In the remainder of the paper, we will consider the variables $\{x_k\}_{k\in\mathbb{Z}}$ only in the redefined sense, that is, when we talk about x_k for some $k < k_0 - 1$, we are referring to the value that is obtained by solving the recursion (3.5) backwards, starting from $k_0 + 1$.

It is also well known that any solution of the recursion (3.5) must obey an invariance relation. Namely, the quantity

$$I := x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 \tag{3.8}$$

is independent of $k \in \mathbb{Z}$. With our variables, we note that I depends on both λ and E. We will write $I(E)$ or $I(E, \lambda)$ whenever we want to make this dependence explicit. Also, foreshadowing Sect. 5, let us rephrase (3.8) in terms of the map T defined in (3.7): with the so-called *Fricke–Vogt invariant*

$$I(x, y, z) := x^2 + y^2 + z^2 - 2xyz - 1, \tag{3.9}$$

we have

$$I \circ T = I. \tag{3.10}$$

Even in the standard case, it is helpful to solve the recursion (3.5) backwards and compute the values at least up to index $k = -1$. One then realizes two things:

(i) The variable x_{-1} takes the value 1, independently of E and λ.
(ii) The variables x_0 and x_1 take the values $\frac{E}{2}$ and $\frac{E-\lambda}{2}$, respectively.

Item (i) is crucial in the formulation and proof of Sütő's central Lemma 2 in [33]. Item (ii) (combined with item (i)) means that the map $E \mapsto (x_1, x_0, x_{-1})$ is affine and the image is a line. This so-called *line of initial conditions* is a central object in the part of the analysis of the standard Fibonacci Hamiltonian that employs hyperbolic dynamics.

The next lemma will address the absence of item (i) in the general case. We will deal with the absence of item (ii) in the general case in Sect. 5.

Lemma 3.4. *Let the sequence $\{x_k\}_{k\in\mathbb{Z}}$ be defined as above and set*

$$b := \max\{1, |x_{-1}|\}. \tag{3.11}$$

Then, $\{x_k\}_{k\geq 0}$ is unbounded if and only if there exists $K \geq 0$ such that

$$|x_{K-1}| \leq b, \; |x_K| > b, \; |x_{K+1}| > b. \tag{3.12}$$

If such a K exists, it is unique, and we have $|x_{k+2}| > |x_{k+1}x_k| > b$ for every $k \geq K$, and there exists $d > 1$ such that

$$|x_k| > d^{F_{k-K}}$$

for every $k \geq K$.

On the other hand, if no such K exists, then

$$|x_k| \leq b^2 + \sqrt{(1 - b^2)^2 + I}. \tag{3.13}$$

for every $k \geq 0$.

Proof. Suppose that (3.12) holds for some $K \geq 0$. Then, by (3.5), the triangle inequality, and the assumptions,

$$\begin{aligned} |x_{K+2}| &= |2x_{K+1}x_K - x_{K-1}| \\ &\geq |x_{K+1}x_K| + (|x_{K+1}x_K| - |x_{K-1}|) \\ &> |x_{K+1}x_K| \\ &> b. \end{aligned}$$

By induction we get $|x_{k+2}| > |x_{k+1}x_k|$ for every $k \geq K$. Since $\log|x_{k+2}| > \log|x_{k+1}| + \log|x_k|$ shows that $\log|x_k|$ increases faster than the Fibonacci sequence, we find $|x_k| > d^{F_{k-K}}$ for some $d > 1$ and all $k \geq K$. Note that

$$|x_{k-1}| \leq b < |x_k|, |x_{k+1}| < |x_{k+2}| < |x_{k+3}| < \cdots$$

if $k = K$. It is obvious that the above inequalities cannot simultaneously hold for any other value of k, and hence, K is unique.

Suppose now that (3.12) fails for every $K \geq 0$. If $|x_k| > b$ for some $k \geq 0$, we must have $|x_{k-1}| \leq b$ and $|x_{k+1}| \leq b$. Otherwise, (3.12) holds for some $K \leq k$ since $|x_{-1}| \leq b$.

Observe that

$$\begin{aligned} (|x_k| - |x_{k-1}x_{k+1}|)^2 &\leq (x_k - x_{k-1}x_{k+1})^2 \\ &= x_k^2 - 2x_kx_{k-1}x_{k+1} + x_{k-1}^2x_{k+1}^2 \\ &= (1 - x_{k-1}^2)(1 - x_{k+1}^2) + x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 \\ &= (1 - x_{k-1}^2)(1 - x_{k+1}^2) + I; \end{aligned}$$

recall definition (3.8) of I.

By combining the inequalities above, we find

$$|x_k| \leq |x_{k-1}x_{k+1}| + \sqrt{(1 - x_{k-1}^2)(1 - x_{k+1}^2) + I} \tag{3.14}$$

We claim that the maximum of the right-hand side of (3.14), subject to $|x_{k-1}|, |x_{k+1}| \leq b$, is attained at $|x_{k-1}|, |x_{k+1}| = b$. This yields

$$|x_k| \leq b^2 + \sqrt{(1 - b^2)^2 + I},$$

as claimed in (3.13).

Let us show this claim. If $b = 1$, the assertion holds by [33].

Consider the case $1 < b \leq \sqrt{2}$. By [33], the right-hand side of (3.14) does not attain its maximum at $|x_{k-1}|, |x_{k+1}| < 1$. If it attains its maximum at $1 \leq |x_{k-1}|, |x_{k+1}| < b$. Then, we have

$$\sqrt{(x_{k-1}^2 - 1)(x_{k+1}^2 - 1) + I} \leq \sqrt{(b^2 - 1)(b^2 - 1) + I} = \sqrt{(1 - b^2)(1 - b^2) + I},$$

which is a contradiction. If it attains its maximum at $1 \leq |x_{k+1}| < b$ and $|x_{k-1}| < 1$, then

$$\sqrt{(1 - x_{k-1}^2)(1 - x_{k+1}^2) + I} \leq \sqrt{I} \leq \sqrt{(1 - b^2)(1 - b^2) + I},$$

which again is a contradiction.

Finally, consider the case $b \geq \sqrt{2}$. If $1 - x_{k-1}^2 \geq 0$ and $1 - x_{k+1}^2 \geq 0$, then

$$\sqrt{(1 - x_{k-1}^2)(1 - x_{k+1}^2) + I} \leq \sqrt{1 + I} \leq \sqrt{(1 - b^2)(1 - b^2) + I}.$$

If $1 - x_{k-1}^2 < 0$ and $1 - x_{k+1}^2 < 0$, then

$$\sqrt{(x_{k-1}^2 - 1)(x_{k+1}^2 - 1) + I} \leq \sqrt{(b^2 - 1)(b^2 - 1) + I} = \sqrt{(1 - b^2)(1 - b^2) + I}.$$

If $(1 - x_{k-1}^2)(1 - x_{k+1}^2) \leq 0$, then

$$\sqrt{(1 - x_{k-1}^2)(1 - x_{k+1}^2) + I} \leq \sqrt{I} \leq \sqrt{(1 - b^2)(1 - b^2) + I}.$$

This covers all cases and hence proves the claim. □

4. Extension of Classical Results Not Relying on Hyperbolicity

In this section, we discuss those results for our generalized Fibonacci Hamiltonian whose proofs do not make use of the hyperbolicity of the trace map. This corresponds roughly to those results obtained for the standard Fibonacci Hamiltonian that were obtained prior to 2008.

4.1. Uniformity of the Lyapunov Exponent and Zero-Measure Spectrum

The first result is stated merely for completeness, as it is already known. As was discussed in introduction, the zero-measure property of the spectrum is actually the only result that was already known for the generalized Fibonacci Hamiltonian, generated by a general locally constant sampling function over the Fibonacci subshift, and its proof can be given without trace map considerations. It proceeds by showing that the associated Schrödinger cocycles are uniform via [21,22] and then appealing to results by Johnson [26] and/or Lenz [28] and Kotani [27].

Theorem 4.1. *Fix a non-degenerate* $\lambda \in \mathbb{R}^{N+1}$. *Then, for every* $E \in \mathbb{R}$, *there is* $L_\lambda(E) \geq 0$, *called the Lyapunov exponent, such that*

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_{\lambda,E}^n(\omega)\| = L_\lambda(E) \quad \text{uniformly in } \omega \in \Omega_F. \tag{4.1}$$

Moreover, there is a Cantor set Σ_λ *of zero Lebesgue measure such that*

$$\sigma(H_{\lambda,\omega}) = \Sigma_\lambda \quad \text{for every } \omega \in \Omega_F. \tag{4.2}$$

In fact, with $\mathcal{Z}_\lambda := \{E \in \mathbb{R} : L_\lambda(E) = 0\}$, *we have*

$$\Sigma_\lambda = \mathcal{Z}_\lambda. \tag{4.3}$$

Proof. The ω-independence of the spectrum (4.2) is an immediate consequence of the minimality of (Ω_F, T) and the continuity of the sampling function. The uniform convergence statement (4.1) follows from the general theory developed in [21,22], where it is derived from three ingredients: aperiodicity, the Boshernitzan condition for the subshift, and the local constancy of the sampling function. Once the uniform convergence statement (4.1) has been established, (4.3) follows from [26,28]. Moreover, by [27] and Lemma 2.2, the set $\mathcal{Z}_\lambda$ has zero Lebesgue measure. Finally, Σ_λ is compact and has no isolated points by general principles. □

It should be noted that the approach from [21,22] does not give any additional information about the fractal properties of the Cantor set in question, while the trace map approach does. Thus, even when the results from [21,22] cover a given model, it is still worthwhile to explore the trace map approach to the given model, as one can obtain additional information in this way.

4.2. The Description of the Spectrum in Terms of the Trace Map

Recall that the x_k are E-dependent and that $b = b(E)$ is defined by $b = \max\{1, |x_{-1}|\}$. Let, for $k \geq 0$,

$$\rho_k := \{E \in \mathbb{R} : |x_k(E)| > b(E)\}, \quad \sigma_k := \{E \in \mathbb{R} : |x_k(E)| \leq b(E)\},$$

and

$$B_\infty := \{E \in \mathbb{R} : \{x_k\}_{k \geq 0} \text{ is bounded}\}.$$

Proposition 4.2. *The set B_∞^c is open, and*

$$B_\infty^c = \bigcup_{k=k_0}^{\infty} (\rho_k \cap \rho_{k+1}) \tag{4.4}$$

for every $k_0 \geq 0$. Moreover,

$$\rho_k \cap \rho_{k+1} = \bigcap_{k'=k}^{\infty} \rho_{k'} \tag{4.5}$$

for every $k \geq 0$.

Proof. By Lemma 3.4,

$$B_\infty^c = (\rho_0 \cap \rho_1) \cup \bigcup_{k=1}^{\infty} (\sigma_{k-1} \cap \rho_k \cap \rho_{k+1})$$

is a disjoint decomposition of B_∞^c. Obviously,

$$\rho_k \cap \rho_{k+1} = (\rho_0 \cap \rho_1) \cup \bigcup_{k_1=1}^{k} (\sigma_{k_1-1} \cap \rho_{k_1} \cap \rho_{k_1+1})$$

for all $k \geq 0$, and we therefore have

$$\rho_k \cap \rho_{k+1} \subseteq \rho_{k+1} \cap \rho_{k+2}. \tag{4.6}$$

This implies (4.4) and also that B_∞^c is open, since ρ_k is open for every k. Finally, $\rho_k \cap \rho_{k+1} \subseteq \bigcap_{k'=k}^{\infty} \rho_{k'}$ is a consequence of Lemma 3.4 and the other inclusion is trivial. □

The next step is to identify the set σ_k as the spectrum of a suitable periodic Schrödinger operator.

Definition. Let $k \geq k_0$ with k_0 as defined in (3.3). Consider the periodic sequence

$$\omega^{(k)} = \dots s_k | s_k s_k \dots \in \{a, b\}^{\mathbb{Z}},$$

where as usual the bar is located between positions -1 and 0. Define the F_k-periodic potential $V^{(k)}$ by

$$V^{(k)}(n) = f(T^n \omega^{(k)}).$$

and consider the associated periodic Schrödinger operator

$$[H^{(k)} \psi](n) = \psi(n+1) + \psi(n-1) + V^{(k)}(n)\psi(n)$$

in $\ell^2(\mathbb{Z})$.

Lemma 4.3. *For every $k \geq k_0$, we have*

$$\sigma(H^{(k)}) = \{E \in \mathbb{R} : |x_k(E)| \leq 1\} \subseteq \sigma_k. \tag{4.7}$$

Moreover, for the common spectrum Σ_λ of the operators $\{H_{\lambda,\omega}\}_{\omega \in \Omega_F}$, we have

$$\Sigma_\lambda = \bigcap_{k \geq 0} (\sigma_k \cup \sigma_{k+1}) = B_\infty. \tag{4.8}$$

Proof. The identity $\sigma(H^{(k)}) = \{E \in \mathbb{R} : |x_k(E)| \leq 1\}$ in (4.7) follows from standard Floquet theory, and the inclusion $\{E \in \mathbb{R} : |x_k(E)| \leq 1\} \subseteq \sigma_k$ follows from the definition of σ_k.

One inclusion underlying the identity (4.8) is essentially a consequence of strong convergence. There is a subtlety, however. It is not true that the sequence $\{H^{(k)}\}_{k \geq 0}$ converges strongly because to the left of the bar, the two rightmost values do (in general) not stabilize. Namely, the last two symbols of the words s_k (for $k \geq 1$) alternate between ab and ba, compare (1.1) (and the statement follows quickly inductively from (1.2)). This can be dealt with by considering suitable translates of the operators $H^{(k)}$, which of course have the same spectrum.

In order to choose suitable translates that ensure strong convergence of the associated sequence of operators, recall that by (1.2), we have $s_k = s_{k-1} s_{k-2}$. Thus, if we arrange the period relative to the origin as follows,

$$\dots s_k | s_k \dots \quad \text{for } k \geq k_0 + 1 \text{ even, and}$$
$$\dots s_{k-1} | s_{k-2} s_{k-1} s_{k-2} \dots \quad \text{for } k \geq k_0 + 1 \text{ odd,}$$

and then, as we move from level k to $k+1$, the two sequences coincide on at least F_{k-1} positions to the left of the bar and on at least F_{k-2} positions to the right of the bar. This shows that for the associated operators ($H^{(k)}$ defined as before for k even, and shifted to the left by F_{k-1} positions for k odd) converges strongly to H_{ω_F}. Thus,

$$\begin{aligned} \Sigma_\lambda &= \sigma(H_{\omega_F}) \\ &\subseteq \bigcap_{k \geq k_0+1} \overline{\bigcup_{k' \geq k} \sigma(H^{(k')})} \end{aligned}$$

$$\begin{aligned}
&\subseteq \bigcap_{k\ge k_0+1} \overline{\bigcup_{k'\ge k} \sigma_{k'}} \\
&\subseteq \bigcap_{k\ge k_0+1} \overline{\sigma_k \cup \sigma_{k+1}} \\
&\subseteq \bigcap_{k\ge k_0+1} \sigma_k \cup \sigma_{k+1} \\
&= B_\infty,
\end{aligned}$$

where we used (4.2) in the first line, a standard strong operator convergence result in the second line, (4.7) in the third line, Proposition 4.2 in the fourth line and the sixth line, and the fifth line is obvious.

Conversely, the inclusion $B_\infty \subseteq \sigma(H_{\omega_F}) = \Sigma_\lambda$ follows as in the standard case using the two-block Gordon argument along the sequence of the lengths $\{F_k\}$. Namely, due to the presence of squares to the right of the bar and the boundedness of the associated transfer matrix trace for each $E \in B_\infty$, the Gordon lemma shows that no solution is square-summable at infinity (this will be discussed in more detail in the next subsection), which in turn implies that the energy in question must belong to the spectrum. □

4.3. Existence and Uniqueness of *k*-Partitions

In this subsection, we recall the partitions of elements of the Fibonacci subshift introduced by Damanik and Lenz in [18], which are an important tool used to establish spectral results that hold uniformly in $\omega \in \Omega_F$, and extend them to the setting of this paper.

The starting point is given by the construction of the subshift Ω_F. Recall that each element looks locally like the Fibonacci sequence u_F, which in turn is given as the limit of the words $\{s_k\}$, which obey the recursion (1.2). From there, it is not too difficult to show that each $\omega \in \Omega_F$ can be partitioned, for every $k \ge 1$, into subwords of type s_k or s_{k-1}. Moreover, blocks of type s_{k-1} are isolated (i.e., surrounded by blocks of type s_k), whereas blocks of type s_k occur with multiplicity either one or two. This partition of a given $\omega \in \Omega_F$ is unique and called the k-partition of ω.

In the classical case, this induces directly a k-partition of the associated potentials since we simply replace the symbols a, b by real numbers $\lambda, 0$, and do so in a bijective way.

In the case at hand, a small amount of care has to be exercised. To consistently apply the standard Gordon two-block argument, we need the following:

(i) k needs to be large enough so that the look-ahead only inspects one additional block,
(ii) the resulting potential value has to be the same, regardless of which type of block, s_k or s_{k-1}, occurs to the right of the current one,

To ensure item (i), we simply need $|s_{k-1}| = F_{k-1} \ge N - 1$, that is, $k \ge k_0 - 1$; compare (3.3). Item (ii) holds for these values of k as well since s_{k-1} is a prefix of s_k.

4.4. Singular Continuous Spectrum

In this very short subsection, we note that the results above determine the spectral type completely for all parameters:

Theorem 4.4. *Fix a non-degenerate $\lambda \in \mathbb{R}^{N+1}$. Then, for every $\omega \in \Omega_F$, the operator $H_{\lambda,\omega}$ has purely singular continuous spectrum.*

Proof. The absence of absolutely continuous spectrum follows from Theorem 4.1 since a set of zero Lebesgue measure cannot support an absolutely continuous measure.

The absence of point spectrum follows from the existence and uniqueness of the k-partitions of the potentials, as discussed in the previous subsection, along with the boundedness of the transfer matrix traces across the blocks of the partition for every energy in the spectrum (cf. (4.8)), together with the combinatorial case-by-case analysis from [18]. Indeed, these ingredients allow one to apply the Gordon two-block lemma [6, Lemma 3.1] and deduce the absence of solutions that decay near $+\infty$, which of course implies the absence of square-summable solutions, and hence the absence of eigenfunctions. □

4.5. Quantitative Spectral Continuity

Once it is known that all spectral measures are purely singular continuous, it is of interest to study dimensional properties of these measures. Upper bounds can, for example, be established by proving upper bounds for the dimension of the spectrum, since the latter set supports all spectral measures. We will discuss this approach later. Lower bounds for the dimensional properties of spectral measures, on the other hand, need statements to the effect that these measures give no weight to sets that are too small in a suitable dimensional sense. The latter issue will be discussed in the present subsection from the perspective of Hausdorff dimension. Results in this direction for the standard Fibonacci Hamiltonian appeared in [5,17,25]. The overall structure of the proofs is largely similar in the general case, and hence, parts of the presentation will be somewhat brief. However, some parts will be quite detailed as the extension is not straightforward and requires us to delve quite deeply into the inner workings of the arguments.

To clarify the goal of this section, let us state the main theorem we wish to prove.

Theorem 4.5. *Fix a non-degenerate $\lambda \in \mathbb{R}^{N+1}$. Then, there is $\alpha_\lambda > 0$ such that for every $\omega \in \Omega_F$, all spectral measures of $H_{\lambda,\omega}$ are α_λ-continuous, that is, they give zero weight to sets $S \subseteq \mathbb{R}$ with $h^{\alpha_\lambda}(S) = 0$.*

Here, h^α denotes the α-dimensional Hausdorff measure on $\mathbb{R}$. Since $\alpha_\lambda > 0$ and any countable set S obeys $h^\alpha(S) = 0$ for every $\alpha > 0$, Theorem 4.5 is a strengthening of the absence of point spectrum part of Theorem 4.4.

Remark 4.6. It is not necessary to assume in Theorem 4.5 that λ is non-degenerate, and the assumption will not be used in the proof. However, the statement is obviously true when λ is degenerate, and hence, the assumption is made to clarify that our focus is on the non-trivial case, where a proof is necessary. The same remark applies to the other results presented in this subsection.

It was shown in [17] that α-continuity of spectral measures associated with Schrödinger operators in $\ell^2(\mathbb{Z})$ can be established by proving power-law upper and lower bounds for solutions on one of the two half lines. Given that correspondence, Theorem 4.5 follows from the following:

Theorem 4.7. *Fix a non-degenerate* $\lambda \in \mathbb{R}^{N+1}$*. Then, there are* $C_1, C_2, \gamma_1, \gamma_2 \in (0, \infty)$ *such that for every* $E \in \Sigma_\lambda$ *and every* $\omega \in \Omega_F$*, every solution* u *of*

$$u(n+1) + u(n-1) + V_{\lambda,\omega}(n)u(n) = Eu(n)$$

that is normalized in the sense that $|u(0)|^2 + |u(1)|^2 = 1$ *obeys*

$$C_1 L^{\gamma_1} \leq \|u\|_L \leq C_2 L^{\gamma_2} \tag{4.9}$$

for every $L \geq 1$.

Remark 4.8. (a) The local ℓ^2 norm $\|\cdot\|_L$ is defined by

$$\|u\|_L^2 := \sum_{n=0}^{L} |u(n)|^2 + (L - \lfloor L \rfloor)|u(\lfloor L \rfloor + 1)|^2.$$

(b) While it is clear from the way Theorem 4.7 is formulated, we emphasize that the constants $C_1, C_2, \gamma_1, \gamma_2$ depend on λ (but are uniform in E and ω). The value of α_λ for which the statement in Theorem 4.5 can then by derived via [17] is given by

$$\alpha_\lambda = \frac{2\gamma_1}{\gamma_1 + \gamma_2}.$$

The proof of the lower bound in (4.9) follows largely the same arguments as in the standard case, given the results already established about boundedness of traces for energies in the spectrum and the existence of unique k-partitions. Thus, this part of the proof of Theorem 4.7 will only be sketched below. On the other hand, the proof of the upper bound in (4.9) turns out to be significantly more difficult than in the standard case and we will in fact have to work out an adapted version of the results from [24]. This part of the analysis will be presented with full details.

We introduce the following notation. For $E \in \Sigma_\lambda$, let

$$c := \sup_{k \geq 0} |x_k|. \tag{4.10}$$

Note that c is E-dependent. By (3.13), with b defined in (3.11) we have $c \leq b^2 + \sqrt{(1-b^2)^2 + I}$ for every $E \in \Sigma_\lambda$, which leads to the uniform estimate

$$\sup_{E \in \Sigma_\lambda} c \leq \sup_{E \in \Sigma_\lambda} b^2 + \sqrt{(1-b^2)^2 + I}. \tag{4.11}$$

Let us recall an observation from the proof of [24, Lemma 4] and adjust it to the present setting.

Lemma 4.9. *For $E \in \Sigma_\lambda$, let*

$$a := \max\{2c, 2, \|M_1\|, \|M_2\|, \|M_3\|\}. \tag{4.12}$$

Then, we have

$$\|M_k\| \le a^k \quad \textit{for every } k \ge 1. \tag{4.13}$$

Proof. By (3.4) and the Cayley-Hamilton Theorem, we have

$$M_k = M_{k-2}M_{k-1} = M_{k-2}(2x_{k-1}I - M_{k-1}^{-1}) = 2x_{k-1}M_{k-2} - M_{k-3}^{-1}. \tag{4.14}$$

Moreover, since $M_{k-3} \in \mathrm{SL}(2,\mathbb{R})$, we have

$$\|M_{k-3}^{-1}\| = \|M_{k-3}\|. \tag{4.15}$$

The bound (4.13) follows by induction from (4.10) and (4.14)–(4.15). □

Let us establish a few useful recursive relations that follow from (3.4) and the Cayley–Hamilton Theorem:

Lemma 4.10 (Lemma 6 of [24]).

$$M_kM_{k+2} = 2x_kM_{k+2} - M_{k+1}, \tag{4.16}$$

$$M_kM_{k+3} = 2x_{k+2}M_{k+2} - I. \tag{4.17}$$

Lemma 4.11. *We have*

$$M_kM_{k-1} = 2x_{k-1}M_k - M_{k-2}, \tag{4.18}$$

$$M_kM_{k-2} = 2x_{k-2}M_k - 2x_{k-3}M_{k-2} + M_{k-4}, \tag{4.19}$$

$$M_kM_{k-3} = 4x_{k-1}x_{k-3}M_{k-2} - 2x_{k-1}M_{k-4} - I. \tag{4.20}$$

Proof. We note that

$$\begin{aligned} M_kM_{k-1} &= M_{k-2}M_{k-1}^2 \\ &= M_{k-2}(2x_{k-1}M_{k-1} - I) \\ &= 2x_{k-1}M_k - M_{k-2}, \\ M_kM_{k-2} &= M_{k-2}M_{k-1}M_{k-2} \\ &= M_{k-2}M_{k-3}M_{k-2}^2 \\ &= M_{k-2}M_{k-3}(2x_{k-2}M_{k-2} - I) \\ &= 2x_{k-2}M_k - M_{k-2}M_{k-3} \\ &= 2x_{k-2}M_k - 2x_{k-3}M_{k-2} + M_{k-4}, \end{aligned}$$

where we used (4.18) in the last step, and

$$\begin{aligned} M_kM_{k-3} &= M_{k-2}M_{k-1}M_{k-3} \\ &= M_{k-2}M_{k-3}M_{k-2}M_{k-3} \\ &= (M_{k-2}M_{k-3})^2 \\ &= \mathrm{Tr}(M_{k-2}M_{k-3})M_{k-2}M_{k-3} - I \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Tr}(M_{k-3}M_{k-2})M_{k-2}M_{k-3} - I \\
&= \mathrm{Tr}(M_{k-1})M_{k-2}M_{k-3} - I \\
&= 2x_{k-1}M_{k-2}M_{k-3} - I \\
&= 2x_{k-1}(2x_{k-3}M_{k-2} - M_{k-4}) - I \\
&= 4x_{k-1}x_{k-3}M_{k-2} - 2x_{k-1}M_{k-4} - I,
\end{aligned}
$$

where we again used (4.18) in the penultimate step. This establishes (4.18)–(4.20). □

Recall that all these quantities depend on λ and E, and that this dependence is often left implicit. However, in the following lemmas, the statements need to be interpreted in such a way that λ is fixed and E is the independent variable. That is, when we state that a certain quantity is a polynomial, what we mean is that it is a λ-dependent polynomial in the variable E.

Lemma 4.12. *For $i = 1, 2, \cdots, 6$, let $P^{(i)}$ be a polynomial of the variables $(x_{k-1}, x_k, \cdots, x_{k+m})$ for some $m \geq 1$.*

Then, we have

$$
\begin{aligned}
&\Big(P^{(1)}M_{k+2} + P^{(2)}M_{k+1} + P^{(3)}M_k + P^{(4)}M_{k-1} + P^{(5)}M_{k-2} + P^{(6)}I\Big)M_{k-1} \\
&= Q^{(1)}M_{k+1} + Q^{(2)}M_k + Q^{(3)}M_{k-1} + Q^{(4)}M_{k-2} + Q^{(5)}M_{k-3} + Q^{(6)}I,
\end{aligned}
$$

where $Q^{(i)}$ is a polynomial of the variables $(x_{k-2}, x_{k-1}, \cdots, x_{k+m})$.

Moreover, for every $E \in \Sigma_\lambda$, we have

$$
\sum_{i=1}^{6} |Q^{(i)}| \leq (4c^2 + 2c + 1) \sum_{i=1}^{6} |P^{(i)}|, \tag{4.21}
$$

$$
\sum_{i=1}^{6} |Q^{(i)}| + |Q^{(4)}| \leq (4c^2 + 4c + 1) \sum_{i=1}^{6} |P^{(i)}| \tag{4.22}
$$

and

$$
\sum_{i=1}^{6} |Q^{(i)}| + |Q^{(5)}| \leq (4c^2 + 2c + 1) \sum_{i=1}^{6} |P^{(i)}|. \tag{4.23}
$$

Proof. Using Lemmas 4.10 and 4.11, we have

$$
\begin{aligned}
&\Big(P^{(1)}M_{k+2} + P^{(2)}M_{k+1} + P^{(3)}M_k + P^{(4)}M_{k-1} + P^{(5)}M_{k-2} + P^{(6)}I\Big)M_{k-1} \\
&\quad = M_{k+1}\Big[2P^{(2)}x_{k-1}\Big] \\
&\qquad + M_k\Big[4P^{(1)}x_{k+1}x_{k-1} + 2P^{(3)}x_{k-1} + P^{(5)}\Big] \\
&\qquad + M_{k-1}\Big[-2P^{(2)}x_{k-2} + 2P^{(4)}x_{k-1} + P^{(6)}\Big] \\
&\qquad + M_{k-2}\Big[-2P^{(1)}x_{k+1} - P^{(3)}\Big] \\
&\qquad + M_{k-3}\Big[P^{(2)}\Big]
\end{aligned}
$$

$$+\,\mathbb{I}\Big[-P^{(1)}-P^{(4)}\Big].$$

Set $Q^{(i)}$ as follows:

$$\begin{aligned}
Q^{(1)} &= 2P^{(2)}x_{k-1},\\
Q^{(2)} &= 4P^{(1)}x_{k+1}x_{k-1}+2P^{(3)}x_{k-1}+P^{(5)},\\
Q^{(3)} &= -2P^{(2)}x_{k-2}+2P^{(4)}x_{k-1}+P^{(6)},\\
Q^{(4)} &= -2P^{(1)}x_{k+1}-P^{(3)},\\
Q^{(5)} &= P^{(2)},\\
Q^{(6)} &= -P^{(1)}-P^{(4)}.
\end{aligned}$$

All properties are easy to check. □

Lemma 4.13. *Let $k,\ell\in\mathbb{N}$ with $k\geq 2$ and $k-\ell\geq 2$. Then, we have*

$$M_kM_{k-\ell}=Q^{(1)}M_{k-\ell+2}+Q^{(2)}M_{k-\ell+1}+Q^{(3)}M_{k-\ell} \tag{4.24}$$

$$+\,Q^{(4)}M_{k-\ell-1}+Q^{(5)}M_{k-\ell-2}+Q^{(6)}I, \tag{4.25}$$

where $Q^{(i)}$ is a polynomial of variables $(x_{k-\ell-2},x_{k-\ell-1},\cdots,x_{k-3},x_{k-2})$ and the dependence of all quantities on E and λ is left implicit.

Moreover, for every $E\in\Sigma_\lambda$, we have

$$\sum_{i=1}^{6}|Q^{(i)}|\leq(4c^2+2c+1)^\ell, \tag{4.26}$$

$$\sum_{i=1}^{6}|Q^{(i)}|+|Q^{(4)}|\leq(4c^2+4c+1)^\ell \tag{4.27}$$

and

$$\sum_{i=1}^{6}|Q^{(i)}|+|Q^{(5)}|\leq(4c^2+2c+1)^\ell. \tag{4.28}$$

Proof. Note that we may write

$$M_kM_{k-\ell}=M_{k-2}M_{k-3}\cdots M_{k-\ell}M_{k-\ell+1}M_{k-\ell}.$$

Observe that, by (4.18),

$$M_{k-2}M_{k-3}=2x_{k-3}M_{k-2}-M_{k-4}. \tag{4.29}$$

In particular, we may write

$$M_{k-2}M_{k-3}=P_1^{(1)}M_{k-1}+P_1^{(2)}M_{k-2}+P_1^{(3)}M_{k-3}+P_1^{(4)}M_{k-4}+P_1^{(5)}M_{k-5}+P_1^{(6)}I,$$

where $P_1^{(i)}$ is a polynomial of the variables $(x_{k-4},x_{k-3},x_{k-2})$.

Thus, by applying Lemma 4.12, we have

$$\begin{aligned}
M_{k-2}M_{k-3}M_{k-4} \;&= P_2^{(1)}M_{k-2}+P_2^{(2)}M_{k-3}+P_2^{(3)}M_{k-4}+P_2^{(4)}M_{k-5}\\
&\quad+P_2^{(5)}M_{k-6}+P_2^{(6)}I,
\end{aligned}$$

where $P_2^{(i)}$ is a polynomial of the variables $(x_{k-5}, x_{k-4}, x_{k-3}, x_{k-2})$.

Again, by applying Lemma 4.12, we have

$$\begin{aligned} M_{k-2}M_{k-3}M_{k-4}M_{k-5} &= P_3^{(1)}M_{k-3} + P_3^{(2)}M_{k-4} + P_3^{(3)}M_{k-5} + P_3^{(4)}M_{k-6} \\ &\quad + P_3^{(5)}M_{k-7} + P_3^{(6)}I, \end{aligned}$$

where $P_3^{(i)}$ is a polynomial of the variables $(x_{k-6}, x_{k-5}, x_{k-4}, x_{k-3}, x_{k-2})$.

By applying lemma 4.12 repeatedly, we have

$$\begin{aligned} &M_{k-2}M_{k-3}\cdots M_{k-\ell+1}M_{k-\ell} \\ &\quad = P_{\ell-2}^{(1)}M_{k-\ell+2} + P_{\ell-2}^{(2)}M_{k-\ell+1} + P_{\ell-2}^{(3)}M_{k-\ell} \\ &\quad\quad + P_{\ell-2}^{(4)}M_{k-\ell-1} + P_{\ell-2}^{(5)}M_{k-\ell-2} + P_{\ell-2}^{(6)}I, \end{aligned}$$

where $P_{\ell-2}^{(i)}$ is a polynomial of the variables $(x_{k-\ell-1}, x_{k-\ell}, \cdots, x_{k-3}, x_{k-2})$.

Let $E \in \Sigma_\lambda$ and let $c := \sup_k\{x_k\}$. By observing (4.29), we have

$$\sum_{i=1}^{6} |P_1^{(i)}| \leq 2c + 1 < 4c^2 + 2c + 1.$$

Thus, by using (4.21), (4.22) and (4.23) recursively, we have

$$\sum_{i=1}^{6} |P_{\ell-2}^{(i)}| < (4c^2 + 2c + 1)^{\ell-2},$$

$$\sum_{i=1}^{6} |P_{\ell-2}^{(i)}| + |P_{\ell-2}^{(4)}| < (4c^2 + 4c + 1)^{\ell-2},$$

and

$$\sum_{i=1}^{6} |P_{\ell-2}^{(i)}| + |P_{\ell-2}^{(5)}| < (4c^2 + 2c + 1)^{\ell-2}.$$

A direct calculation using Lemma 4.10 and Lemma 4.11 shows

$$\begin{aligned} &\Big[M_{k-2}M_{k-3}\cdots M_{k-\ell+1}M_{k-\ell}\Big]M_{k-\ell+1} \\ &\quad = \Big[P_{\ell-2}^{(1)}M_{k-\ell+2} + P_{\ell-2}^{(2)}M_{k-\ell+1} + P_{\ell-2}^{(3)}M_{k-\ell} \\ &\quad\quad + P_{\ell-2}^{(4)}M_{k-\ell-1} + P_{\ell-2}^{(5)}M_{k-\ell-2} + P_{\ell-2}^{(6)}I\Big]M_{k-\ell+1} \\ &\quad = M_{k-\ell+2}\Big[P_{\ell-2}^{(3)} + P_{\ell-2}^{(1)}2x_{k-\ell+1}\Big] \\ &\quad\quad + M_{k-\ell+1}\Big[P_{\ell-2}^{(4)}2x_{k-\ell-1} + P_{\ell-2}^{(2)}2x_{k-\ell+1} + P_{\ell-2}^{(6)}\Big] \\ &\quad\quad + M_{k-\ell}\Big[P_{\ell-2}^{(5)}2x_{k-\ell} - P_{\ell-2}^{(4)} - P_{\ell-2}^{(1)}\Big] \\ &\quad\quad + \mathbb{I}\Big[-P_{\ell-2}^{(5)} - P_{\ell-2}^{(2)}\Big]. \end{aligned}$$

In particular, we may write

$$\begin{aligned}&M_{k-2}M_{k-3}\cdots M_{k-\ell+1}M_{k-\ell}M_{k-\ell+1}\\&\quad= P_{\ell-1}^{(1)}M_{k-\ell+2}+P_{\ell-1}^{(2)}M_{k-\ell+1}+P_{\ell-1}^{(3)}M_{k-\ell}+P_{\ell-1}^{(4)}M_{k-\ell-1}\\&\qquad+P_{\ell-1}^{(5)}M_{k-\ell-2}+P_{\ell-1}^{(6)}I,\end{aligned}$$

where $P_{\ell-1}^{(i)}$ is a polynomial of the variables $(x_{k-\ell-1},x_{k-\ell},\cdots,x_{k-3},x_{k-2})$.

Also, it is easy to see that

$$\sum_{i=1}^{6}|P_{\ell-1}^{(i)}|\le(4c^2+2c+1)\cdot\sum_{i=1}^{6}|P_{\ell-2}^{(i)}|<(4c^2+2c+1)^{\ell-2},$$

$$\sum_{i=1}^{6}|P_{\ell-1}^{(i)}|+|P_{\ell-1}^{(4)}|\le(4c^2+4c+1)\cdot\sum_{i=1}^{6}|P_{\ell-2}^{(i)}|<(4c^2+4c+1)^{\ell-1},$$

and

$$\sum_{i=1}^{6}|P_{\ell-1}^{(i)}|+|P_{\ell-1}^{(5)}|\le(4c^2+2c+1)\cdot\sum_{i=1}^{6}|P_{\ell-2}^{(i)}|<(4c^2+4c+1)^{\ell-1}.$$

Again, a direct calculation using Lemma 4.10 and 4.11 shows

$$\begin{aligned}&\Big[M_{k-2}M_{k-3}\cdots M_{k-\ell}M_{k-\ell+1}\Big]M_{k-\ell}\\&\quad=\Big[P_{\ell-1}^{(1)}M_{k-\ell+2}+P_{\ell-1}^{(2)}M_{k-\ell+1}+P_{\ell-1}^{(3)}M_{k-\ell}\\&\qquad+P_{\ell-1}^{(4)}M_{k-\ell-1}+P_{\ell-1}^{(5)}M_{k-\ell-2}+P_{\ell-1}^{(6)}I\Big]M_{k-\ell}\\&\quad=M_{k-\ell+2}\Big[P_{\ell-1}^{(1)}2x_{k-\ell}\Big]+M_{k-\ell+1}\Big[P_{\ell-1}^{(2)}2x_{k-\ell}+P_{\ell-1}^{(4)}\Big]\\&\qquad+M_{k-\ell}\Big[-P_{\ell-1}^{(1)}2x_{k-\ell-1}+P_{\ell-1}^{(3)}2x_{k-\ell}+P_{\ell-1}^{(5)}2x_{k-\ell-2}+P_{\ell-1}^{(6)}\Big]\\&\qquad+M_{k-\ell-1}\Big[-P_{\ell-1}^{(2)}-P_{\ell-1}^{(5)}\Big]+M_{k-\ell-2}\Big[P_{\ell-1}^{(1)}\Big]+\mathbb{I}\Big[P_{\ell-1}^{(3)}\Big].\end{aligned}$$

In particular, we may write

$$\begin{aligned}&M_{k-2}M_{k-3}\cdots M_{k-\ell+1}M_{k-\ell}M_{k-\ell+1}M_{k-\ell}\\&\quad=P_{\ell}^{(1)}M_{k-\ell+2}+P_{\ell}^{(2)}M_{k-\ell+1}+P_{\ell}^{(3)}M_{k-\ell}+P_{\ell}^{(4)}M_{k-\ell-1}\\&\qquad+P_{\ell}^{(5)}M_{k-\ell-2}+P_{\ell}^{(6)}I,\end{aligned}$$

where $P_{\ell}^{(i)}$ is a polynomial of the variables $(x_{k-\ell-2},x_{k-\ell-1},\cdots,x_{k-3},x_{k-2})$.

Also, it is easy to see that

$$\sum_{i=1}^{6}|P_{\ell}^{(i)}|\le(4c^2+2c+1)\cdot\sum_{i=1}^{6}|P_{\ell-1}^{(i)}|<(4c^2+2c+1)^{\ell},$$

$$\sum_{i=1}^{6}|P_{\ell}^{(i)}|+|P_{\ell}^{(4)}|\le(4c^2+4c+1)\cdot\sum_{i=1}^{6}|P_{\ell-1}^{(i)}|<(4c^2+4c+1)^{\ell},$$

and

$$\sum_{i=1}^{6} |P_\ell^{(i)}| + |P_\ell^{(5)}| \leq (4c^2 + 2c + 1) \cdot \sum_{i=1}^{6} |P_{\ell-1}^{(i)}| < (4c^2 + 2c + 1)^\ell.$$

This completes the proof by setting $Q^{(i)} = P_\ell^{(i)}$. □

Lemma 4.14. *There exist constants $\bar{C} > 0$ and $\tau > 0$ such that for $E \in \Sigma_\lambda$ and $k_0 < k_1 < \cdots < k_t$, we have*

$$\|M_{k_t} \cdots M_{k_1} M_{k_0}\| \leq \bar{C} n^\tau,$$

where

$$n = \sum_{i=0}^{t} F_{k_i}.$$

Proof. Throughout this proof we denote $M(k) := M_k$ for the sake of better legibility since we use subindices.

Let us prove first that

$$\|M(k_t) \cdots M(k_1) M(k_0)\| \leq d^{k_t - k_0 + 2(t-1)} \cdot a^{k_0+2} \tag{4.30}$$

with $d = 4c^2 + 4c + 1$, where a is the constant from (4.12).

We begin with the case $t = 1$. By Lemma 4.13, we have for $E \in \Sigma_\lambda$,

$$\begin{aligned} \|M(k_1)M(k_0)\| &\leq |P^{(1)}| \cdot \|M(k_0+2)\| + |P^{(2)}| \cdot \|M(k_0+1)\| \\ &\quad + |P^{(3)}| \cdot \|M(k_0)\| + |P^{(4)}| \cdot \|M(k_0-1)\| \\ &\quad + |P^{(5)}| \cdot \|M(k_0-2)\| + |P^{(6)}| \\ &\leq \sum_{i=1}^{6} |P^{(i)}| \cdot a^{k_0+2} \\ &\leq (4c^2+2c+1)^{k_1-k_0} \cdot a^{k_0+2} \\ &\leq d^{k_1-k_0} \cdot a^{k_0+2}. \end{aligned}$$

Assume now that (4.30) holds for $t \in \{1, \cdots, \ell\}$. We claim that (4.30) holds for $t = \ell + 1$. We have three cases.

Case 1: $k_\ell - k_{\ell-1} > 2$.

By Lemma 4.13, we have

$$\begin{aligned} &\|M(k_{\ell+1})M(k_\ell) \cdots M(k_1)M(k_0)\| \\ &= \|\,[M(k_{\ell+1})M(k_\ell)]\, M(k_{\ell-1}) \cdots M(k_1)M(k_0)\| \\ &\leq |P^{(1)}| \cdot \|M(k_\ell+2)M(k_{\ell-1}) \cdots M(k_0)\| \\ &\quad + |P^{(2)}| \cdot \|M(k_\ell+1)M(k_{\ell-1}) \cdots M(k_0)\| \\ &\quad + |P^{(3)}| \cdot \|M(k_\ell)M(k_{\ell-1}) \cdots M(k_0)\| \\ &\quad + |P^{(4)}| \cdot \|M(k_\ell-1)M(k_{\ell-1}) \cdots M(k_0)\| \\ &\quad + |P^{(5)}| \cdot \|M(k_\ell-2)M(k_{\ell-1}) \cdots M(k_0)\| \end{aligned}$$

$$+ |P^{(6)}| \cdot \|M(k_{\ell-1}) \cdots M(k_0)\|$$

By inductive hypothesis and Lemma 4.13, we have

$$\begin{aligned}
RHS &\leq |P^{(1)}| \cdot d^{k_\ell+2-k_0+2(\ell-1)} a^{k_0+2} + |P^{(2)}| \cdot d^{k_\ell+1-k_0+2(\ell-1)} a^{k_0+2} \\
&\quad + |P^{(3)}| \cdot d^{k_\ell-k_0+2(\ell-1)} a^{k_0+2} + |P^{(4)}| \cdot d^{k_\ell-1-k_0+2(\ell-1)} a^{k_0+2} \\
&\quad + |P^{(5)}| \cdot d^{k_\ell-2-k_0+2(\ell-1)} a^{k_0+2} + |P^{(6)}| \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} a^{k_0+2} \\
&\leq d^{k_\ell-k_0+2\ell} \cdot a^{k_0+2} \cdot \sum_{i=1}^{6} |P^{(i)}| \\
&\leq d^{k_\ell-k_0+2\ell} \cdot a^{k_0+2} \cdot d^{k_{\ell+1}-k_\ell} \text{ (by (4.26))} \\
&\leq d^{k_{\ell+1}-k_0+2[(\ell+1)-1]} \cdot a^{k_0+2},
\end{aligned}$$

and hence, (4.30) holds for $t = \ell + 1$ as well.

Case 2: $k_\ell - k_{\ell-1} = 2$.

Note that in this case, we have

$$M(k_\ell - 2)M(k_{\ell-1}) = M(k_{\ell-1})M(k_{\ell-1}) = 2x_{k_{\ell-1}} M(k_{\ell-1}) - \mathbb{I}. \quad (4.31)$$

By Lemma 4.13 and (4.31), we have

$$\begin{aligned}
&\|M(k_{\ell+1})M(k_\ell) \cdots M(k_1)M(k_0)\| \\
&= \| [M(k_{\ell+1})M(k_\ell)]\, M(k_{\ell-1}) \cdots M(k_1)M(k_0)\| \\
&\leq |P^{(1)}| \cdot \|M(k_\ell + 2)M(k_{\ell-1}) \cdots M(k_0)\| \\
&\quad + |P^{(2)}| \cdot \|M(k_\ell + 1)M(k_{\ell-1}) \cdots M(k_0)\| \\
&\quad + |P^{(3)}| \cdot \|M(k_\ell)M(k_{\ell-1}) \cdots M(k_0)\| \\
&\quad + |P^{(4)}| \cdot \|M(k_\ell - 1)M(k_{\ell-1}) \cdots M(k_0)\| \\
&\quad + |P^{(5)}| \cdot |2c| \cdot \|M(k_{\ell-1}) \cdots M(k_0)\| + |P^{(5)}| \cdot \|M(k_{\ell-2}) \cdots M(k_0)\| \text{ (by (4.31))} \\
&\quad + |P^{(6)}| \cdot \|M(k_{\ell-1}) \cdots M(k_0)\|
\end{aligned}$$

By the inductive hypothesis and Lemma 4.13, we have

$$\begin{aligned}
RHS &\leq |P^{(1)}| \cdot d^{k_\ell+2-k_0+2(\ell-1)} a^{k_0+2} + |P^{(2)}| \cdot d^{k_\ell+1-k_0+2(\ell-1)} a^{k_0+2} \\
&\quad + |P^{(3)}| \cdot d^{k_\ell-k_0+2(\ell-1)} a^{k_0+2} + |P^{(4)}| \cdot d^{k_\ell-1-k_0+2(\ell-1)} a^{k_0+2} \\
&\quad + |P^{(5)}| \cdot |2c| \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} a^{k_0+2} + |P^{(5)}| \cdot d^{k_{\ell-2}-k_0+2(\ell-3)} a^{k_0+2} \\
&\quad + |P^{(6)}| \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} a^{k_0+2} \\
&\leq d^{k_\ell-k_0+2\ell} \cdot a^{k_0+2} \cdot \left(\sum_{i=1}^{6} |P^{(i)}| \right) + |P^{(5)}| \cdot |2c| \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} \cdot a^{k_0+2} \\
&\leq d^{k_\ell-k_0+2\ell} \cdot a^{k_0+2} \cdot \left(\sum_{i=1}^{6} |P^{(i)}| \right) + |P^{(5)}| \cdot d \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} \cdot a^{k_0+2}
\end{aligned}$$

$$\le d^{k_\ell - k_0 + 2\ell} \cdot a^{k_0+2} \cdot \Big(\sum_{i=1}^{6} |P^{(i)}| + |P^{(5)}| \Big)$$
$$\le d^{k_\ell - k_0 + 2\ell} \cdot a^{k_0+2} \cdot d^{k_{\ell+1} - k_\ell} \text{ (by (4.28))}$$
$$\le d^{k_{\ell+1} - k_0 + 2[(\ell+1)-1]} \cdot a^{k_0+2},$$

and hence, (4.30) holds for $t = \ell + 1$ as well.

Case 3: $k_\ell - k_{\ell-1} = 1$.

Note that in this case, we have

$$M(k_\ell - 2)M(k_{\ell-1}) = M(k_{\ell-1} - 1)M(k_{\ell-1}) = M(k_{\ell-1} + 1) \tag{4.32}$$

and

$$M(k_\ell - 1)M(k_{\ell-1}) = M(k_{\ell-1})^2 = 2x_{k_{\ell-1}} M(k_{\ell-1}) - \mathbb{I}. \tag{4.33}$$

By Lemma 4.13, we have

$$\begin{aligned}
&\|M(k_{\ell+1})M(k_\ell)\cdots M(k_1)M(k_0)\| \\
&= \| [M(k_{\ell+1})M(k_\ell)]\, M(k_{\ell-1})\cdots M(k_1)M(k_0)\| \\
&\le |P^{(1)}| \cdot \|M(k_\ell + 2)M(k_{\ell-1})\cdots M(k_0)\| \\
&\quad + |P^{(2)}| \cdot \|M(k_\ell + 1)M(k_{\ell-1})\cdots M(k_0)\| \\
&\quad + |P^{(3)}| \cdot \|M(k_\ell)M(k_{\ell-1})\cdots M(k_0)\| \\
&\quad + |P^{(4)}| \cdot \|M(k_\ell - 1)M(k_{\ell-1})\cdots M(k_0)\| \\
&\quad + |P^{(5)}| \cdot \|M(k_\ell - 2)M(k_{\ell-1})\cdots M(k_0)\| \\
&\quad + |P^{(6)}| \cdot \|M(k_{\ell-1})\cdots M(k_0)\|
\end{aligned}$$

By (4.32) and (4.33),

$$\begin{aligned}
RHS \le\ & |P^{(1)}| \cdot \|M(k_\ell + 2)M(k_{\ell-1})\cdots M(k_0)\| \\
&+ |P^{(2)}| \cdot \|M(k_\ell + 1)M(k_{\ell-1})\cdots M(k_0)\| \\
&+ |P^{(3)}| \cdot \|M(k_\ell)M(k_{\ell-1})\cdots M(k_0)\| \\
&+ |P^{(4)}| \cdot |2c| \cdot \|M(k_{\ell-1})M(k_{\ell-2})\cdots M(k_0)\| \\
&+ |P^{(4)}| \cdot \|M(k_{\ell-2})M(k_{\ell-3})\cdots M(k_0)\| \text{ (by (4.33))} \\
&+ |P^{(5)}| \cdot \|M(k_{\ell-1} + 1)M(k_{\ell-2})\cdots M(k_0)\| \text{ (by (4.32))} \\
&+ |P^{(6)}| \cdot \|M(k_{\ell-1})\cdots M(k_0)\|
\end{aligned}$$

By inductive hypothesis and Lemma 4.13,

$$\begin{aligned}
RHS \le\ & |P^{(1)}| \cdot d^{k_\ell + 2 - k_0 + 2(\ell-1)} a^{k_0+2} + |P^{(2)}| \cdot d^{k_\ell + 1 - k_0 + 2(\ell-1)} a^{k_0+2} \\
&+ |P^{(3)}| \cdot d^{k_\ell - k_0 + 2(\ell-1)} a^{k_0+2} \\
&+ |P^{(4)}| \cdot |2c| \cdot d^{k_{\ell-1} - k_0 + 2(\ell-2)} a^{k_0+2} + |P^{(4)}| \cdot d^{k_{\ell-2} - k_0 + 2(\ell-3)} a^{k_0+2}
\end{aligned}$$

$$
\begin{aligned}
&+ |P^{(5)}| \cdot d^{k_{\ell-1}+1-k_0+2(\ell-2)} a^{k_0+2} \\
&+ |P^{(6)}| \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} a^{k_0+2} \\
&\le d^{k_\ell - k_0 + 2\ell} \cdot a^{k_0+2} \cdot \Big(\sum_{i=1}^{6} |P^{(i)}| \Big) + |P^{(4)}| \cdot |2c| \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} \cdot a^{k_0+2} \\
&\le d^{k_\ell - k_0 + 2\ell} \cdot a^{k_0+2} \cdot \Big(\sum_{i=1}^{6} |P^{(i)}| \Big) + |P^{(4)}| \cdot d \cdot d^{k_{\ell-1}-k_0+2(\ell-2)} \cdot a^{k_0+2} \\
&\le d^{k_\ell - k_0 + 2\ell} \cdot a^{k_0+2} \cdot \Big(\sum_{i=1}^{6} |P^{(i)}| + |P^{(4)}| \Big) \\
&\le d^{k_\ell - k_0 + 2\ell} \cdot a^{k_0+2} \cdot d^{k_{\ell+1}-k_\ell} \text{ (by (4.27))} \\
&\le d^{k_{\ell+1} - k_0 + 2[(\ell+1)-1]} \cdot a^{k_0+2}.
\end{aligned}
$$

and hence, (4.30) holds for $t = \ell + 1$ as well. This completes the proof of (4.30) for all cases.

Since $k_{j+1} - k_j \ge 1$ for $j \in \{0, 1, \cdots, t-1\}$, we have $k_t - k_0 \ge t$. Setting $g = d^3 a^3$, (4.30) gives

$$
\begin{aligned}
\|M(k_t) \cdots M(k_1) M(k_0)\| &\le d^{k_t - k_0 + 2(t-1)} \cdot a^{k_0+2} \\
&\le d^{k_t + 2t} \cdot a^{k_0+2} \\
&\le d^{k_t + 2k_t} \cdot a^{k_t + 2k_t} \\
&= (d^3 a^3)^{k_t} \\
&= g^{k_t}.
\end{aligned}
$$

Moreover,

$$
F_k = \frac{1}{\sqrt{5}} [\alpha^{-k} + (-1)^{k+1} \alpha^k], \text{ and } \lim_{k \to \infty} |k - \frac{\log(\sqrt{5} F_k)}{\log \alpha^{-1}}| = 0
$$

Therefore, for large k,

$$
\begin{aligned}
\|M(k_t) \cdots M(k_1) M(k_0)\| &\le g^{[\log(\sqrt{5} F_{k_t})]/\log \alpha^{-1}} \\
&\le (g^{(\log \sqrt{5})/\log \alpha^{-1}})^{\log n} \\
&\le \bar{C} n^{\tau}
\end{aligned}
$$

with $\tau = (\log \sqrt{5} \log g)/\log \alpha^{-1}$ and for some constant $\bar{C}$. □

Proposition 4.15. *Fix a non-degenerate* $\lambda \in \mathbb{R}^{N+1}$. *Then, there exist* $C > 0$ *and* $\gamma > 0$ *such that for every* $E \in \Sigma_\lambda$, *every* $\omega \in \Omega_F$, *and every* $n \ge 1$, *we have* $\|A^n_{\lambda,E}(\omega)\| \le C n^{\gamma}$

Proof. The proof will consist of two steps. In the first step we prove the assertion for $\omega = \omega_F$. Thus, we extend a result from [24] from the standard Fibonacci Hamiltonian to the generalized Fibonacci Hamiltonian. In the second step, we consider the case of a general $\omega \in \Omega_F$, but use the result from the first step in the proof. This extends a result from [19] from the standard Fibonacci Hamiltonian to the generalized Fibonacci Hamiltonian.

Step 1: Given $n \geq 1$ we may choose a unique index set of positive integers $\{k_j\}_{j=-S}^{J}$ such that

(1) $k_{j+1} - k_j \geq 2$,
(2) $F_{k_j} < N$ if $-S \leq j < 0$ and $F_{k_j} \geq N$ if $0 \leq j \leq J$,
(3) $n = \sum_{j=-S}^{K} F_{k_j}$.

Consider the case $J \geq 0$. Then, we may write $A_{\lambda,E}^{n}(\omega_F)$ as

$$\begin{aligned} A_{\lambda,E}^{n}(\omega_F) &= A_{\lambda,E}^{\sum_{j=-S}^{J-1} F_{k_j} + F_{k_J}}(\omega_F) \cdots A_{\lambda,E}^{1+F_{k_J}} A_{\lambda,E}^{F_{k_J}}(\omega_F) \cdots A_{\lambda,E}^{1}(\omega_F) \\ &= A_{\lambda,E}^{\sum_{j=-S}^{J-1} F_{k_j}}(\omega_F) \cdots A_{\lambda,E}^{1} M_{k_J}(E) \\ &\vdots \\ &= A_{\lambda,E}^{\sum_{j=-S}^{-1} F_{k_j}}(\omega_F) \cdots A_{\lambda,E}^{1} M_{k_0}(E) \cdots M_{k_J}(E). \end{aligned}$$

By using a similar argument as in the proof of [24, Theorem 1], we have

$$\|M_{k_0}(E) \cdots M_{k_J}(E)\| \leq h^{2J} (ah)^{k_J},$$

where $h := 4c+1$. By item (1), we have $k_J - k_0 \geq 2J$, and hence, $\|M_{k_0}(E) \cdots M_{k_J}(E)\| \leq f^{n_K}$, $f := ah^2$.

Therefore, for all large enough J,

$$\|M_{k_0}(E) \cdots M_{k_J}(E)\| \leq f^{\frac{\log(\sqrt{5} F_{k_J})}{\log \alpha^{-1}}} \leq \left(f^{\frac{\log \sqrt{5}}{\log \alpha^{-1}}} \right)^{\log n^*},$$

where $n^* := \sum_{j=0}^{J} F_{k_j} \leq n$.

Let $A := A_{\lambda,E}^{\sum_{j=-S}^{-1} F_{k_j}}(\omega_F) \cdots A_{\lambda,E}^{1}$. Define

$$\overline{C} = \overline{C}(E) := \max\{\|A_{\lambda,E}^{m}(\omega)\| : 0 \leq m < N,\, \omega \in \Omega_F\}.$$

In conclusion, we have

$$\|A_{\lambda,E}^{n}(\omega_F)\| \leq \|A\| \cdot \|M_{k_0}(E) \cdots M_{k_J}(E)\| \leq \overline{C} (n^*)^{\mu} \leq \overline{C} n^{\mu}$$

with $\mu = \frac{\log \sqrt{5} \log d}{\log \alpha^{-1}}$.

Step 2: Let now $\omega \in \Omega_F$ be arbitrary and consider $A_{\lambda,E}^{n}(\omega)$. Viewing the underlying matrix product relative to a k-partition (for k sufficiently large) as in [19], we can partition it relative to the (at most) two consecutive blocks in which we fall. The norm of $A_{\lambda,E}^{n}(\omega)$ is then bounded from above by the product of the norms of the two pieces.

The right piece is covered by Step 1 since it is aligned at the left endpoint of a block, which is also the starting point of the matrix products associated with ω_F.

The left piece is estimated using an inversion of the order of the pieces $M_{k_j}(E)$ we break the long product into. The difference is that now the index decreases from left to right (while it increases in Step 1). This is the reason why we had to prove Lemma 4.14. Using this lemma, we obtain a power-law estimate also for the left piece.

Putting the two estimates together, the desired result follows. □

Proof of Theorem 4.7. The proof of the lower bound in (4.9) follows by the same arguments as in the standard case: using the k-partitions of the potentials, one can identify sufficiently many length scales so that, using the Gordon two-block argument and the trace bounds for energies on the spectrum, the mass-reproduction technique developed in [5] can inductively prove the desired lower bound for every $\omega \in \Omega_F$; see [17] for details.

The upper bound in (4.9) follows readily from the power-law upper bound for the transfer matrices corresponding to energies in the spectrum, as established in Proposition 4.15. □

5. Extension of Results Whose Proofs are Based on Hyperbolicity

In this section we discuss those results for our generalized Fibonacci Hamiltonian whose proofs do make use of the hyperbolicity of the trace map. This corresponds roughly to those results obtained for the standard Fibonacci Hamiltonian that have been obtained since 2008, starting with [10].

5.1. General Setup

The result on the zero measure property of Σ_λ naturally leads one to ask about the fractal dimension of this set. There are several ways to measure the fractal dimension of a nowhere dense subset of the real line, for example the Hausdorff dimension or the box counting dimension. Given $S \subset \mathbb{R}$, we denote by $\dim_{\mathrm{H}}(S)$ and $\dim_{\mathrm{B}}(S)$ the Hausdorff dimension and the box counting dimension of S, respectively. The local Hausdorff dimension and box counting dimension of S at $s \in S$ are given by

$$\dim_{\mathrm{H}}^{\mathrm{loc}}(S, s) := \lim_{\epsilon \to 0} \dim_{\mathrm{H}}(S \cap (s - \epsilon, s + \epsilon)),$$

and

$$\dim_{\mathrm{B}}^{\mathrm{loc}}(S, s) := \lim_{\epsilon \to 0} \dim_{\mathrm{B}}(S \cap (s - \epsilon, s + \epsilon)).$$

Recall the Fibonacci trace map

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, y, z) \mapsto (2xy - z, x, y)$$

and the Fricke–Vogt invariant,

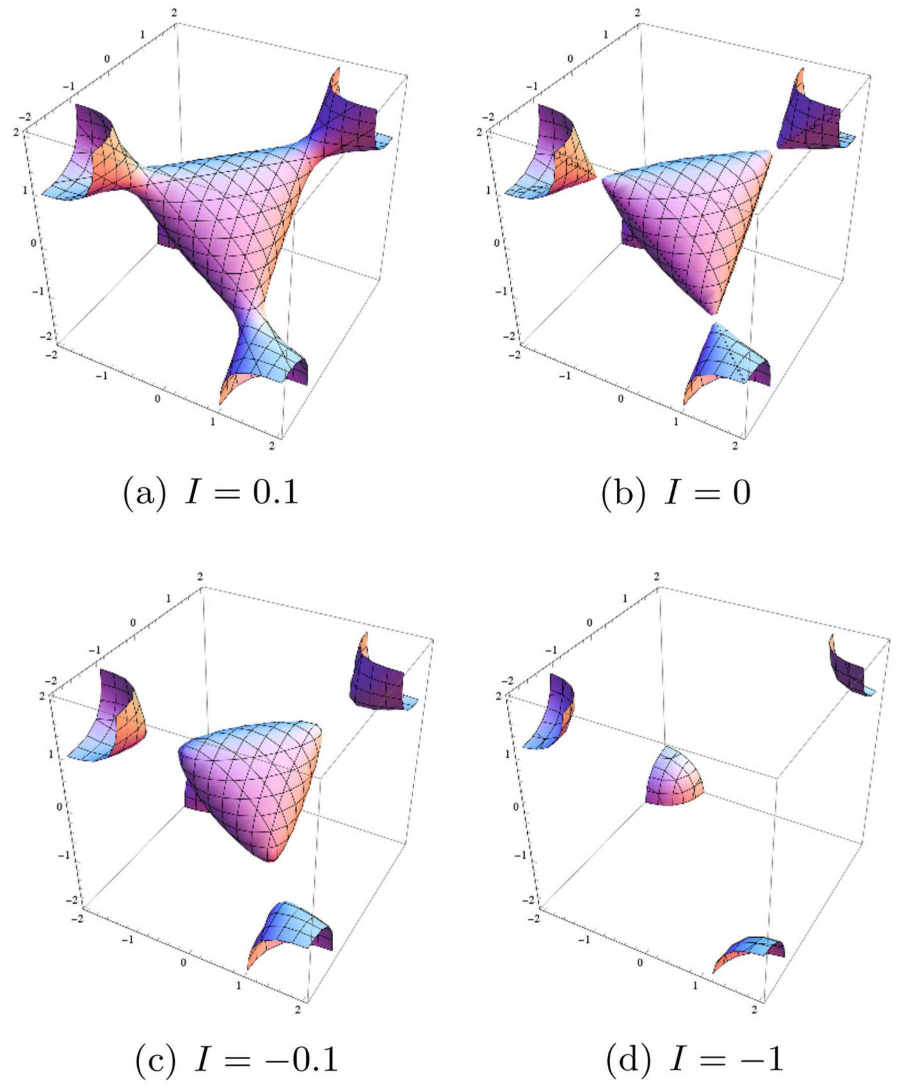
$$I(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1,$$

for which we have

$$I \circ T = I.$$

Define

$$S_I := \left\{(x, y, z) : x^2 + y^2 + z^2 - 2xyz - 1 = I\right\}.$$



(a) $I = 0.1$ (b) $I = 0$

(c) $I = -0.1$ (d) $I = -1$

Figure 1. Invariant surfaces S_I for four values of I

It is well known that for $I > 0$, S_I is a smooth, connected, non-compact two-dimensional submanifold of $\mathbb{R}^3$, homeomorphic to the four-punctured sphere. When $I = 0$, S_I develops four conic singularities, away from which it is smooth. When $-1 < I < 0$, S_I contains five smooth connected components: four non-compact components, each homeomorphic to the two-disc, and one compact component, homeomorphic to the two-sphere. When $I = -1$, S_I consists of the four smooth noncompact discs and a point at the origin. When $I < -1$, S_I consists only of the four noncompact two-discs (Fig. 1).

To investigate the dynamics of the trace map, we consider the following initial conditions

$$\gamma_\lambda(E) = (x_1(E), x_0(E), x_{-1}(E))$$

and ask how points on it behave under iteration of the map T.

Recall that the quantity

$$\begin{aligned} I &= x_{k+1}^2 + x_k^2 + x_{k-1}^2 - 2x_{k+1}x_kx_{k-1} - 1 \\ &= x_1^2 + x_0^2 + x_{-1}^2 - 2x_1x_0x_{-1} - 1 \end{aligned}$$

is independent of $k \in \mathbb{Z}$, and we note that I depends on both λ and E. We will write $I(E)$ or $I(E, \lambda)$ whenever we want to make this dependence explicit.

Given a point $p \in S_I$, the forward semi-orbit of p under T is given by

$$\mathcal{O}_T^+(p) := \{p, T(p), T^2(p), \cdots\}.$$

We say that a point p satisfies property **B** if p has a bounded forward semi-orbit. Note that (4.8) shows for non-degenerate $\lambda \in \mathbb{R}^{N+1}$,

$$\Sigma_\lambda = \{E \in \mathbb{R} : \gamma_\lambda(E) \text{ is a type-}\mathbf{B}\text{ point}\},$$

and this set is a Cantor set of zero Lebesgue measure. Moreover, for every $E \in \Sigma_\lambda$, $I(E) \geq 0$ since if there exists $E \in \Sigma_\lambda$ such that $I(E) < 0$, by the assumption of $I(E) < 0$ and the continuity of I, we could find an open neighborhood of E that belongs to Σ_λ, contradicting the fact that Σ_λ is a Cantor set. Therefore, we mainly focus on the cases $I \geq 0$ in the remainder of this paper.

For $I = 0$, we define the surface

$$\mathbb{S} := S_0 \cap \left\{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, |y| \leq 1, |z| \leq 1\right\}, \tag{5.1}$$

which is smooth everywhere except at the four points $P_1 = (1, 1, 1)$, $P_2 = (-1, -1, 1)$, $P_3 = (1, -1, -1)$, $P_4 = (-1, -1, 1)$. By invariance of $\mathbb{S}$ under T it follows that all points of $\mathbb{S}$ are of type **B**. Beside these points, there exist type **B** points in $S_0 \backslash \mathbb{S}$, these points form a disjoint union of four smooth injectively immersed connected one-dimensional submanifolds of $S_0 \backslash \mathbb{S}$, $\mathcal{W}_1, \cdots, \mathcal{W}_4$; see [23, Lemma 2.2].

For fixed $I > 0$, the set of all bounded two-sided orbits of T in S_I coincides with the nonwandering set Λ_I, and the set Λ_I is a compact locally maximal T-invariant hyperbolic subset of S_I; see [3,4,13]. Therefore, a point p is a type-**B** point in S_I if and only if there exists $q \in \Lambda_I$, such that $p \in W^s(q)$, the stable manifold at q. We define

$$\Lambda := \bigcup_{I>0} \Lambda_I$$

and define a smooth three-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^3$ by

$$\mathcal{M} := \bigcup_{I>0} S_I.$$

There exists a family, denoted by $\mathcal{W}^s$, of smooth 2-dimensional connected injectively immersed submanifolds of $\mathcal{M}$, whose members we denote by W^{cs}

and call center-stable manifolds, such that for every $p \in \Lambda$, there exist a unique $W^{cs} \in \mathcal{W}^s$ containing p and the type-**B** points of $\mathcal{M}$ are precisely $\bigcup_{W^{cs}\in\mathcal{W}^s} W^{cs}$; see [23, Theorem 2.6].

5.2. Fractal Dimensions in the Case of General N

Denote by γ_λ the curve of the initial conditions

$$\gamma_\lambda = \left\{(x_1(E), x_0(E), x_{-1}(E)) : E \in \mathbb{R}\right\}.$$

Define

$$B_\infty(\gamma_\lambda) := \{E \in \mathbb{R} : \gamma_\lambda(E) \text{ is a type-}\mathbf{B}\text{ point}\}.$$

Theorem 5.1. *There exists a non-empty set $\mathfrak{N} \subset \mathbb{R}^{N+1}$ of Lebesgue measure zero, such that for non-degenerate $\lambda \in \mathbb{R}^{N+1}$,*

(a) *if γ_λ lies entirely in some $S_{I\circ\gamma_\lambda(E)}$, that is, $\frac{\partial I\circ\gamma_\lambda}{\partial E} \equiv 0$, then $I\circ\gamma_\lambda(E) \equiv c > 0$ and for every $E \in B_\infty(\gamma_\lambda)$, we have*

$$0 < \dim_{\mathrm{H}}^{\mathrm{loc}}\left(B_\infty(\gamma_\lambda), E\right) < 1, \tag{5.2}$$

and

$$\begin{aligned}\dim_{\mathrm{H}}^{\mathrm{loc}}\left(B_\infty(\gamma_\lambda), E\right) &= \dim_{\mathrm{B}}^{\mathrm{loc}}\left(B_\infty(\gamma_\lambda), E\right)\\ &= \dim_{\mathrm{H}}(B_\infty(\gamma_\lambda))\\ &= \dim_{\mathrm{B}}(B_\infty(\gamma_\lambda)).\end{aligned} \tag{5.3}$$

(b) *if γ_λ does not lie entirely in some $S_{I\circ\gamma_\lambda(E)}$, that is, $\frac{\partial I\circ\gamma_\lambda}{\partial E} \not\equiv 0$, we have*
  - (b.1) *$B_\infty(\gamma_\lambda) \ni E \mapsto \dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E)$ is continuous;*
  - (b.2) *there exists a finite set $F \subseteq B_\infty(\gamma_\lambda)$ such that for $E \in B_\infty(\gamma_\lambda)\backslash F$, $\dim_{\mathrm{B}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E)$ exists and is equal to $\dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E)$.*

(c)
  - (c.1) *for all $\lambda \notin \mathfrak{N}$ and for all $E \in B_\infty(\gamma_\lambda)$, we have $0 < \dim_{\mathrm{H}}^{\mathrm{loc}} (B_\infty(\gamma_\lambda), E) < 1$;*
  - (c.2) *for all $\lambda \in \mathfrak{N}$, $0 < \dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E) < 1$ for all $E \in B_\infty(\gamma_\lambda)$ away from the lower and upper boundary points of the spectrum, and $\dim_{\mathrm{H}}(B_\infty(\gamma_\lambda))) = 1$.*

Proof. (a) By assumption we have $I\circ\gamma_\lambda(E) \equiv c$. We have already argued above that c must be non-negative, as the spectrum is non-empty and for E's in the spectrum, the invariant cannot take negative values. By a similar argument it follows that c also cannot be zero. Note that the curve of initial conditions γ_λ does not intersect $\mathbb{S}$ (see (5.1)), that is, $\mathbb{S} \cap \gamma_\lambda = \emptyset$. Indeed, by invariance of $\mathbb{S}$ under T it follows that all points of $\mathbb{S}$ are of type-**B**, and if there exists $E = E(\lambda)$ such that $\gamma_\lambda(E) \in \mathbb{S}$, then $E \in B_\infty(\gamma_\lambda)$. On the other hand, by the continuity of the curve, we could find an open neighborhood of E that belongs to $B_\infty(\gamma_\lambda)$, contradicting the fact that Σ_λ is a Cantor set. Therefore, any type-**B** point $\gamma_\lambda(E)$ must lie in one of the four curves $\mathcal{W}_1, \cdots, \mathcal{W}_4$. But this is again at odds with the fact that for non-degenerate λ, the spectrum is a Cantor set.

Thus, we know that $I\circ\gamma_\lambda(E) \equiv c > 0$. The curve of the initial conditions γ_λ intersects $W^s(\Lambda_c)$ transversally; see [16, Theorem 1.5]. As a consequence of this, the box counting dimension of the spectrum Σ_λ exists and coincides with the Hausdorff dimension and we have (5.3); see [16, Theorem 1.1].

If $I \circ \gamma_\lambda(E) \equiv c > 0$ and $\gamma_\lambda(E)$ is a type-**B** point, by using the proof of Theorem 2.1–iii in [36], we have

$$\dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E) = \frac{1}{2}\dim_{\mathrm{H}}(\Lambda_c). \tag{5.4}$$

Combining (5.4) with the fact (see [14] and [15]) that

$$\text{for all } c > 0,\ 0 < \dim_{\mathrm{H}}(\Lambda_c) < 2,$$

this implies (5.2), and hence, we complete the proof of (a).

(b) Since $I \circ \gamma_\lambda(E)$ is E dependent, we can follow the arguments Yessen used in the proof of [35, Theorem 2.3].

Assume $c > 0$, and let $\gamma_\lambda(E) \in \gamma_\lambda \cap S_c$ be a point of transversal intersection with the center-stable manifold, we have

$$\dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E) = \frac{1}{2}\dim_{\mathrm{H}}(\Lambda_c).$$

Since $c \to \dim_{\mathrm{H}}(\Lambda_c)$ is continuous, this proves the continuity result (b.1); see the proof of [35, Theorem 2.3–(i)] for more details.

Since the curve of initial conditions γ_λ is analytic and it is contained in no single invariant surface, it may have only isolated tangencies with invariant surfaces; hence, we have the result (b.2); see [23, Theorem 3.2].

As for the proof of (c), let $E_0 : \mathbb{R}^{N+1} \to \mathbb{R}$ be such that $I \circ \gamma_\lambda \circ E_0(\lambda) = 0$. Define

$$\mathcal{C} \overset{\text{def}}{=} \{(x, y, z) : I(x, y, z) = 0 \text{ and } |x|, |y|, |z| \leq 1\}^{\mathrm{c}}.$$

Then, $\mathcal{C}$ is a smooth two-dimensional submanifold of $\mathbb{R}^3$ with four connected components, and there exist four smooth curves in $\mathcal{C}$, $\mathcal{W}_1, \cdots, \mathcal{W}_4$ whose union we denote by τ, such that for all $x \in \mathcal{C}$, $\mathcal{O}_T^+(x)$ is bounded if and only if $x \in \tau$. We define the continuous map $F : \mathbb{R}^{N+1} \to \mathcal{C}$ by

$$F(\lambda) = \gamma_\lambda \circ E_0(\lambda).$$

Let

$$\mathfrak{N} = F^{-1}(\tau).$$

Clearly, $\mathfrak{N}$ has zero Lebesgue measure. We also claim that $\mathfrak{N}$ is non-empty. Let $P_1 = (1, 1, 1)$. One of the four curves $\mathcal{W}_1, \cdots, \mathcal{W}_4$ is a branch of the strong stable manifold at P_1, which we denoted by W^{ss}. The tangent space $T_{P_1}W^{ss}$ is transversal to the plane $\{(x, y, z) : x, y \in \mathbb{R}, z = 1\}$. Hence, $W^{ss} \cap \{z \approx 1\} \neq \emptyset$. Let us assume that $x_{-1} = x_{-1}(E, \lambda) \equiv c \approx 1$, for any $p = (x_1, x_0, x_{-1}) \in \{z = c\}$, we can always find $\lambda \in \mathbb{R}^{N+1}$ such that $p \in \gamma_\lambda$. Thus, $\mathfrak{N} \neq \emptyset$; see the proof of [35, Theorem 2.3–(iib)] for more details.

If $\lambda \notin \mathfrak{N}$, the intersection of the corresponding γ_λ with the center-stable manifolds is away from S_0. Hence, for all $E \in B_\infty(\gamma_\lambda)$, we have

$$0 < \dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E) < 1.$$

If $\lambda \in \mathfrak{N}$, pick $\gamma_\lambda(E) \in \gamma_\lambda \cap S_0$. Then, E is one of the two extreme boundary points of the spectrum, and away from it, we have $\dim_{\mathrm{H}}^{\mathrm{loc}}(B_\infty(\gamma_\lambda), E) \in (0,1)$. On the other hand, we have $\dim_{\mathrm{H}}(B_\infty(\gamma_\lambda)) = 1$ due to $\lim_{c\to 0^+} \dim_{\mathrm{H}}(\Lambda_c) = 2$. □

Remark 5.2. As we have seen that the value of the local fractal dimension at a point in the spectrum is determined by the value of the invariant at that point, it is worth pointing out that the former has explicitly known asymptotics in the regime of small [15] and large [10] values of the latter.

5.3. Fractal Dimension of the Spectrum for the case $N = 2$

In this subsection, we illustrate the results from the previous subsection in the special case $N = 2$, where explicit calculations are easy to carry out and the resulting expressions may be readily analyzed.

We notice that in this case, the locally constant function $f(\omega)$ depends on the window $(\ldots\omega_0, \omega_1\ldots)$ of size *two*. For $g : \{a,b\}^2 \to \mathbb{R}$, we define $g(a,b) = \lambda_1$, $g(b,a) = \lambda_2$ and $g(a,a) = \lambda_3$. We define the locally constant function $f(\omega)$ as the following

$$f(\omega) := \begin{cases} g(ab) & \omega_0\omega_1 = ab, \\ g(ba) & \omega_0\omega_1 = ba, \\ g(aa) & \omega_0\omega_1 = aa, \end{cases}$$

that is,

$$f(\omega) := \begin{cases} \lambda_1 & \omega_0\omega_1 = ab, \\ \lambda_2 & \omega_0\omega_1 = ba, \\ \lambda_3 & \omega_0\omega_1 = aa. \end{cases}$$

As the subshift Ω is minimal, we consider $\omega = \omega_F$, where $\omega_F \in \Omega$ is such that its restriction to the right half-line $\{0,1,2,\ldots\}$ coincides with the Fibonacci sequence u_F. That is, ω looks like

$$\ldots ababaabaab|abaababaabaab\ldots$$

around the origin, where the vertical bar denotes the position between the entries ω_{-1} and ω_0.

When the size of window $N = 2$, we recall from (3.3) that

$$\begin{aligned} k_0 &:= \min\{k : F_{k-2} \geq N-1\} \\ &= \min\{k : F_{k-2} \geq 1\} \\ &= 2, \end{aligned}$$

where $\{F_k\}_{k\geq 0}$ is the sequence of Fibonacci numbers given by $F_0 = 1$, $F_1 = 2$, and $F_{k+1} = F_k + F_{k-1}$, $k \geq 1$. Then, Lemma 3.1 implies that for any $k \geq 2$,

$$M_{k+1} = M_{k-1}M_k.$$

The exact expression for M_1 and M_2 is

$$M_1 = \begin{pmatrix} E - f(T\omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} E - g(ba) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - g(ab) & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} E - \lambda_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda_1 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 + (E-\lambda_1)(E-\lambda_2) & -E + \lambda_2 \\ E - \lambda_1 & 0 \end{pmatrix},$$
$$M_2 = \begin{pmatrix} E - f(T^2\omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - f(T\omega) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - f(\omega) & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} E - g(aa) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - g(ba) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - g(ab) & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} E - \lambda_3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda_2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E - \lambda_1 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -E + (E-\lambda_1)(-1 + (E-\lambda_2)(E-\lambda_3)) + \lambda_3 & 1 - (E-\lambda_2)(E-\lambda_3) \\ -1 + (E-\lambda_1)(E-\lambda_2) & -E + \lambda_2 \end{pmatrix}.$$

We define

$$\begin{aligned} M_0 :&= M_2 M_1^{-1} \\ &= \begin{pmatrix} E - \lambda_3 & -1 \\ 1 & 0 \end{pmatrix}, \\ M_{-1} :&= M_1 M_0^{-1} \\ &= \begin{pmatrix} E - \lambda_2 & -1 - (E-\lambda_2)(\lambda_1 - \lambda_3) \\ 1 & -\lambda_1 + \lambda_3 \end{pmatrix}, \end{aligned}$$

and then define

$$\begin{aligned} x_1 = x_1(E,\lambda) &:= \frac{1}{2}\mathrm{Tr} M_1 = \frac{-2 + (E-\lambda_1)(E-\lambda_2)}{2}, \\ x_0 = x_0(E,\lambda) &:= \frac{1}{2}\mathrm{Tr} M_0 = \frac{E - \lambda_3}{2}, \\ x_{-1} = x_{-1}(E,\lambda) &:= \frac{1}{2}\mathrm{Tr} M_{-1} = \frac{E - \lambda_1 - \lambda_2 + \lambda_3}{2}; \end{aligned}$$

therefore, the curve of initial conditions is

$$\begin{aligned} \gamma_\lambda &= \{(x_1(E), x_0(E), x_{-1}(E)) : E \in \mathbb{R}\} \\ &= \left\{ \left(\frac{-2 + (E-\lambda_1)(E-\lambda_2)}{2}, \frac{E-\lambda_3}{2}, \frac{E - \lambda_1 - \lambda_2 + \lambda_3}{2} \right), E \in \mathbb{R} \right\}. \end{aligned}$$

The Fricke–Vogt invariant is

$$\begin{aligned} I \circ \gamma_\lambda(E) &= x_{-1}^2(E) + x_0^2(E) + x_1^2(E) - 2x_{-1}^2(E) x_0^2(E) x_1^2(E) - 1 \\ &= -1 + \frac{1}{4}\left(-2 + E^2 + \lambda_1\lambda_2 - E(\lambda_1 + \lambda_2)\right)^2 + \frac{1}{4}(E - \lambda_3)^3 \\ &\quad - \frac{1}{4}\left(-2 + E^2 + \lambda_1\lambda_2 - E(\lambda_1+\lambda_2)\right)(E-\lambda_3)(E - \lambda_1 - \lambda_2 + \lambda_3) \\ &\quad + \frac{1}{4}(E - \lambda_1 - \lambda_2 + \lambda_3), \end{aligned}$$

which obeys

$$\frac{\partial I\circ\gamma_\lambda}{\partial E}=\frac{1}{4}(2E-\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3).$$

We consider the following two cases:

Case 1: $\lambda_1=\lambda_3$ or $\lambda_2=\lambda_3$. $\frac{\partial I\circ\gamma_\lambda}{\partial E}\equiv 0$; hence, the Fricke–Vogt invariant is

$$I\circ\gamma_\lambda(E)=I=\frac{1}{4}(\lambda_1-\lambda_2)^2>0$$

(if $\lambda_1=\lambda_2$, then $\lambda_1=\lambda_2=\lambda_3$, that means $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}^3$ is degenerate); thus, the curve of the initial conditions lies in single invariant surface S_I, and it intersects $W^s(\Lambda_I)$ transversally. Hence, the Hausdorff dimension of Σ_λ is strictly between zero and one and for every $E\in\Sigma_\lambda$ and every $\varepsilon>0$, we have

$$\begin{aligned}\dim_{\mathrm{H}}\left((E-\varepsilon,E+\varepsilon)\cap\Sigma_\lambda\right)&=\dim_{\mathrm{B}}\left((E-\varepsilon,E+\varepsilon)\cap\Sigma_\lambda\right)\\&=\dim_{\mathrm{H}}\Sigma_\lambda=\dim_{\mathrm{B}}\Sigma_\lambda.\end{aligned}$$

Case 2: $\lambda_1\neq\lambda_3$ and $\lambda_2\neq\lambda_3$.

$$\frac{\partial I\circ\gamma_\lambda}{\partial E}=\frac{1}{4}(2E-\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\not\equiv 0.$$

Case 2.1: $\lambda_1>\lambda_3$, $\lambda_2>\lambda_3$. $I\circ\gamma_\lambda(E)$ decreases monotonically on the interval $(-\infty,\frac{\lambda_1+\lambda_2}{2})$, increases monotonically on the interval $(\frac{\lambda_1+\lambda_2}{2},+\infty)$, so $I\circ\gamma_\lambda(E)$ takes its minimum at $\frac{\lambda_1+\lambda_2}{2}$. In particular, we have

$$I\circ\gamma_\lambda\left(\frac{\lambda_1+\lambda_2}{2}\right)=-\frac{1}{16}\left(\lambda_1-\lambda_2)^2(\lambda_1(\lambda_2-\lambda_3)-\lambda_2\lambda_3+\lambda_3^2-4\right).$$

Case 2.1.1: $I\circ\gamma_\lambda(\frac{\lambda_1+\lambda_2}{2})>0$. γ_λ intersects the invariant surfaces $\{S_I\}_{I>0}$ transversally; therefore, for all $E\in\Sigma_\lambda$, we have $0<\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda,E)<1$.

Case 2.1.2: $I\circ\gamma_\lambda(\frac{\lambda_1+\lambda_2}{2})=0$. If $E=\frac{\lambda_1+\lambda_2}{2}$ is such that

$$\gamma_\lambda\circ E\in\tau,$$

where τ is the union of four smooth curves $\mathcal{W}_1,\cdots,\mathcal{W}_4$ in $\mathcal{C}$ (see the proof of (c) in Theorem 5.1), then $E=\frac{\lambda_1+\lambda_2}{2}\in\Sigma_\lambda$. Therefore, for this spectral point E, $\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda,E)=1$, and for other $E\in\Sigma_\lambda$, we have $0<\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda,E)<1$.

Case 2.1.3: $I\circ\gamma_\lambda(\frac{\lambda_1+\lambda_2}{2})<0$. We take $E_1=E_1(\lambda_1,\lambda_2,\lambda_3)$ and $E_2=E_2(\lambda_1,\lambda_2,\lambda_3)$ such that $I\circ\gamma_\lambda(E_1)=I\circ\gamma_\lambda(E_2)=0$, then the spectrum $\Sigma_\lambda\subset(-\infty,E_1]\cup[E_2,+\infty)$. If E_1 and E_2 are such that

$$\gamma_\lambda\circ E_1\in\tau,\quad\gamma_\lambda\circ E_1\in\tau,$$

then $E_1\in\Sigma_\lambda$ and $E_2\in\Sigma_\lambda$. Therefore, for $E\in\{E_1,E_2\}$, $\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda,E)=1$, and for $E\in\Sigma_\lambda\backslash\{E_1,E_2\}$, $0<\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda,E)<1$ (Fig. 2).

Case 2.2: $\lambda_1>\lambda_3$, $\lambda_2<\lambda_3$. $I\circ\gamma_\lambda(E)$ increases monotonically on the interval $(-\infty,\frac{\lambda_1+\lambda_2}{2})$, decreases monotonically on the interval $(\frac{\lambda_1+\lambda_2}{2},+\infty)$, $I\circ\gamma_\lambda(E)$ takes its maximum at $\frac{\lambda_1+\lambda_2}{2}$.

Case 2.2.1: $I\circ\gamma_\lambda(\frac{\lambda_1+\lambda_2}{2})<0$. Since $I\circ\gamma_\lambda(E)\geq 0$ for every $E\in\Sigma_\lambda$, this case cannot happen.

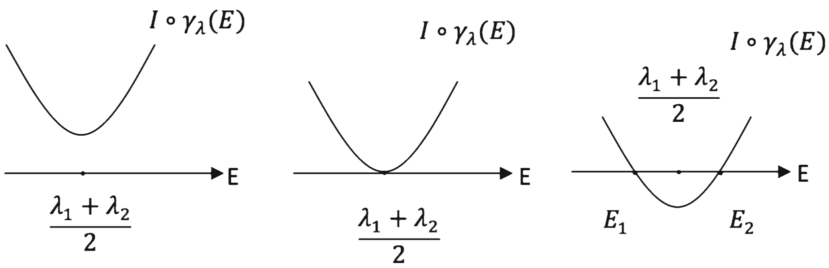


FIGURE 2. *Case 2.1.1–Case 2.1.3*

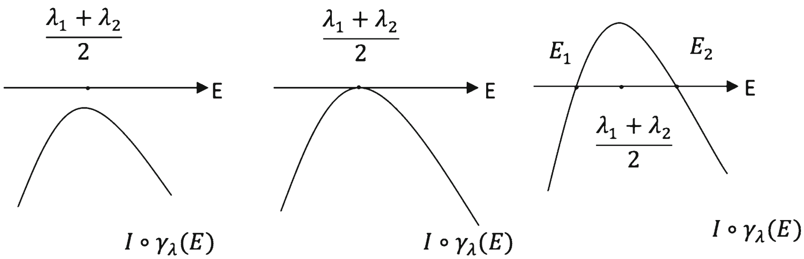


FIGURE 3. *Case 2.2.1–Case 2.2.3*

Case 2.2.2: $I \circ \gamma_\lambda(\frac{\lambda_1+\lambda_2}{2}) = 0$. The spectrum Σ_λ at most consists of one single point, that is, $\frac{\lambda_1+\lambda_2}{2}$, it contradicts the Cantor spectrum Σ_λ. This case cannot happen.

Case 2.2.3: $I \circ \gamma_\lambda(\frac{\lambda_1+\lambda_2}{2}) > 0$. We take $E_1 = E_1(\lambda_1, \lambda_2, \lambda_3)$ and $E_2 = E_2(\lambda_1, \lambda_2, \lambda_3)$ such that $I \circ \gamma_\lambda(E_1) = I \circ \gamma_\lambda(E_2) = 0$, then the spectrum $\Sigma_\lambda \subset [E_1, E_2]$. And if E_1 and E_2 are such that

$$\gamma_\lambda \circ E_1 \in \tau, \quad \gamma_\lambda \circ E_1 \in \tau,$$

then $E_1 \in \Sigma_\lambda$ and $E_2 \in \Sigma_\lambda$. Therefore, for $E \in \{E_1, E_2\}$, $\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda, E) = 1$, and for $E \in \Sigma_\lambda \backslash \{E_1, E_2\}$, we have $0 < \dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda, E) < 1$ (Fig. 3).

Case 2.3: $\lambda_1 < \lambda_3$, $\lambda_2 > \lambda_3$. $I \circ \gamma_\lambda(E)$ increases monotonically on the interval $(-\infty, \frac{\lambda_1+\lambda_2}{2})$, decreases monotonically on the interval $(\frac{\lambda_1+\lambda_2}{2}, +\infty)$, so $I \circ \gamma_\lambda(E)$ takes its maximum at $\frac{\lambda_1+\lambda_2}{2}$. This is a situation similar to *Case 2.2*, so we omit it here.

Case 2.4: $\lambda_1 < \lambda_3$, $\lambda_2 < \lambda_3$. $I \circ \gamma_\lambda(E)$ decreases monotonically on the interval $(-\infty, \frac{\lambda_1+\lambda_2}{2})$, increases monotonically on the interval $(\frac{\lambda_1+\lambda_2}{2}, +\infty)$, $I \circ \gamma_\lambda(E)$ takes its minimum at $\frac{\lambda_1+\lambda_2}{2}$. This is a situation similar to *Case 2.1*, so we omit it here.

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Appendix A. Notions and Results From Hyperbolic Dynamics

An invariant closed set Λ of a diffeomorphism $f : M \to M$ is *hyperbolic* if there exists a splitting $T_xM = E^s_x \oplus E^u_x$ of the tangent space at every point $x \in \Lambda$ that is invariant under Df, and Df exponentially contracts vectors from the stable subspaces $\{E^s_x\}$ and exponentially expands vectors from the unstable subspaces $\{E^u_x\}$.

Let us recall that an invariant set Λ of a diffeomorphism $f : M \to M$ is *locally maximal* if there exists a neighborhood $U(\Lambda)$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U).$$

The set Λ is called *transitive* if it contains a dense orbit. It is not hard to prove that the splitting $E^s_x \oplus E^u_x$ depends continuously on $x \in \Lambda$; hence, $\dim(E^{s,u}_x)$ is locally constant. If Λ is transitive, then $\dim(E^{s,u}_x)$ is constant on Λ.

Consider a locally maximal invariant transitive hyperbolic set $\Lambda \subset M$, $\dim M = 2$, of a diffeomorphism $f \in \mathrm{Diff}^r(M)$, $r \geq 1$. We have $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(U(\Lambda))$ for some neighborhood $U(\Lambda)$. Assume that $\dim E^s = \dim E^u = 1$. Then, the following properties hold.

A.1. Stability. There is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of the map f such that for every $g \in \mathcal{U}$, the set

$$\Lambda_g = \bigcap_{n \in \mathbb{Z}} g(U(\Lambda))$$

is a locally maximal invariant hyperbolic set of g. Moreover, there is a homeomorphism $h : \Lambda \to \Lambda_g$ that conjugates $f|_\Lambda$ and $g|_{\Lambda_g}$, that is, the following diagram commutes:

$$\begin{array}{ccc} \Lambda & \xrightarrow[f|_\Lambda]{} & \Lambda \\ \downarrow h & & h \downarrow \\ \Lambda_g & \xrightarrow{g|_{\Lambda_g}} & \Lambda_g \end{array}$$

Also h can be taken arbitrarily close to the identity by taking $\mathcal{U}$ sufficiently small.

A.2. Invariant Manifolds. For $x \in \Lambda$ and small $\varepsilon > 0$, consider the local stable and unstable sets

$$W^s_\varepsilon(x) = \{w \in M : d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \geq 0\},$$

$$W^u_\varepsilon(x) = \{w \in M : d(f^n(x), f^n(w)) \leq \varepsilon \text{ for all } n \leq 0\}.$$

If ε is small enough, these sets are embedded C^r-disks with $T_x W^s_\varepsilon(x) = E^s_x$ and $T_x W^u_\varepsilon(x) = E^u_x$. Define the global stable and unstable sets by

$$W^s(x) = \bigcup_{n\in\mathbb{N}} f^{-n}\left(W^s_\varepsilon(x)\right), \quad W^u(x) = \bigcup_{n\in\mathbb{N}} f^{n}\left(W^u_\varepsilon(x)\right).$$

Define also

$$W^s(\Lambda) = \bigcup_{x\in\Lambda} W^s(x), \quad W^u(\Lambda) = \bigcup_{x\in\Lambda} W^u(x).$$

A.3. Invariant Foliations. A stable foliation for Λ is a foliation $\mathcal{F}^s$ of a neighborhood of Λ such that

(a) for each $x \in \lambda$, $\mathcal{F}(x)$, the leaf containing x, is tangent to E^s_x,
(b) for each x sufficiently close to Λ, $f(\mathcal{F}^s(x)) \subset \mathcal{F}^s(f(x))$.

An unstable foliation $\mathcal{F}^u$ can be defined in a similar way.

For a locally maximal hyperbolic set $\Lambda \subset M$ for $f \in \mathrm{Diff}^1(M)$, $\dim(M) = 2$, stable and unstable C^0 foliations with C^1 leaves can be constructed, see [29]; in case $f \in \mathrm{Diff}^2(M)$, C^1 invariant foliations exist, see [32].

A.4. Local Hausdorff Dimension and Box Counting Dimension. Consider, for $x \in \Lambda$ and small $\varepsilon > 0$, the set $W^s_\varepsilon(x) \cap \Lambda$. Its Hausdorff dimension does not depend on $x \in \Lambda$ and $\varepsilon > 0$, and coincides with its box counting dimension

$$\dim_{\mathrm{H}} W^s_\varepsilon(x) \cap \Lambda = \dim_{\mathrm{B}} W^s_\varepsilon(x) \cap \Lambda.$$

In a similar way,

$$\dim_{\mathrm{H}} W^u_\varepsilon(x) \cap \Lambda = \dim_{\mathrm{B}} W^u_\varepsilon(x) \cap \Lambda.$$

Denote $h^s = \dim_{\mathrm{H}} W^s_\varepsilon(x) \cap \Lambda$ and $h^u = \dim_{\mathrm{H}} W^u_\varepsilon(x) \cap \Lambda$. We will say that h^s and h^u are the *local stable and unstable Hausdorff dimension* of Λ.

For properly chosen small $\varepsilon > 0$, the sets $W^s_\varepsilon \cap \Lambda$ and $W^u_\varepsilon \cap \Lambda$ are dynamically defined Cantor set, and this implies that

$$h^s < 1 \ \text{ and } \ h^u < 1.$$

A.5. Global Hausdorff dimension. The Hausdorff dimension of Λ is equal to its box counting dimension, and

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = h^s + h^u,$$

see [30,31] for more details.

A.6. Continuity of the Hausdorff Dimension. The local Hausdorff dimensions $h^s(\Lambda)$ and $h^u(\Lambda)$ depend continuously on $f : M \to M$ in the C^1-topology; see [30,31]. Therefore, $\dim_{\mathrm{H}} \Lambda_f = \dim_{\mathrm{B}} \Lambda_f = h^s(\Lambda_f) + h^u(\Lambda_f)$ also depends continuously on f in the C^1-topology. Moreover, for a C^r diffeomorphism $f : M \to M$, $r \geq 2$, the Hausdorff dimension of a hyperbolic set Λ_f is a C^{r-1} function of f, see [29].

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