



Trigonometric Real Form of the Spin RS Model of Krichever and Zabrodin

M. Fairon, L. Fehér  and I. Marshall

Abstract. We investigate the trigonometric real form of the spin Ruijsenaars–Schneider system introduced, at the level of equations of motion, by Krichever and Zabrodin in 1995. This pioneering work and all earlier studies of the Hamiltonian interpretation of the system were performed in complex holomorphic settings; understanding the real forms is a non-trivial problem. We explain that the trigonometric real form emerges from Hamiltonian reduction of an obviously integrable ‘free’ system carried by a spin extension of the Heisenberg double of the $U(n)$ Poisson–Lie group. The Poisson structure on the unreduced real phase space $GL(n, \mathbb{C}) \times \mathbb{C}^{nd}$ is the direct product of that of the Heisenberg double and $d \geq 2$ copies of a $U(n)$ covariant Poisson structure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ found by Zakrzewski, also in 1995. We reduce by fixing a group valued moment map to a multiple of the identity and analyze the resulting reduced system in detail. In particular, we derive on the reduced phase space the Hamiltonian structure of the trigonometric spin Ruijsenaars–Schneider system and we prove its degenerate integrability.

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1. Introduction

The unbroken interest in integrable many-body systems of Calogero–Moser–Sutherland [8, 38, 54] and Ruijsenaars–Schneider (abbreviated RS) [50] types is due to their ubiquity in physical applications and rich web of connections to important areas of mathematics [3, 12, 40, 49, 55]. The same can be said about spin extensions of these models, which currently attract attention [5, 6, 10, 14–16, 28, 33, 42, 45–47, 51, 57].

Two kinds of spin many-body models are studied in the literature. Those that feature only ‘collective spin variables’ belonging to some group theoretic phase space such as a coadjoint orbit, and those that have ‘spin-vectors’ embodying internal degrees of freedom of the interacting particles. The former type of models arise rather naturally in harmonic analysis and its classical mechanical counterpart [12, 13, 21, 22, 34, 47]. The latter type of models, built on ‘individual spins,’ were introduced at the non-relativistic level by Gibbons and Hermsen [24], and their ‘relativistic’ generalization was later put forward by Krichever and Zabrodin [32].

In fact, in 1995 Krichever and Zabrodin introduced a family of spin RS models at the level of equations of motion and posed the question of their Hamiltonian structure and integrability. These models have rational, trigonometric/hyperbolic and elliptic versions and are usually studied in the holomorphic category. The elliptic model encodes the dynamics of the poles of elliptic solutions of the 2D non-Abelian Toda lattice [32], and a special hyperbolic degeneration is related to affine Toda solitons [7]. The existence of a Hamiltonian structure was established by Krichever [31] in the general case based on a universal construction that is hard to make explicit (see also [53]). The rational case was treated via Hamiltonian reduction by Arutyunov and Frolov [4] in 1997, utilizing a ‘spin extension’ of the holomorphic cotangent bundle

of $GL(n, \mathbb{C})$. More than twenty years later, there appeared two different treatments of the holomorphic trigonometric/hyperbolic models: by Chalykh and Fairon [10] based on double brackets and quasi-Hamiltonian structures, and by Arutyunov and Olivucci [5] based on Hamiltonian reduction of a spin extension of the Heisenberg double [52] of the standard factorizable Poisson–Lie group structure on $GL(n, \mathbb{C})$. In the present paper, we shall deal with the trigonometric real form of the models of [32] utilizing the Heisenberg double of the Poisson–Lie group $U(n)$, which is a natural generalization of $T^*U(n)$. The spin extension of the Heisenberg double that we consider is based on a $U(n)$ covariant Poisson structure on \mathbb{C}^n introduced by Zakrzewski [56].

Although the holomorphic systems are of great interest from several viewpoints, it is not easy to extract from them the features of the dynamics of the real forms, which should also be investigated. For motivation, it perhaps suffices to recall that all pioneering papers of the subject [8, 38, 50, 54] are devoted to point particles moving along the *real* line or circle.

The \mathbb{C} -valued dynamical variables of the Krichever–Zabrodin model are ‘particle positions’ x_i ($i = 1, \dots, n$) together with d -component row vectors c_i and column vectors a_i . The composite spin variables F_{ij} are built from these individual spins according to the rule

$$F_{ij} := c_i \cdot a_j := \sum_{\alpha=1}^d c_i^\alpha a_j^\alpha, \tag{1.1}$$

and the equations of motion can be written in first-order form as follows:

$$\dot{x}_i = F_{ii}, \quad \dot{a}_i^\alpha = \lambda_i a_i^\alpha + \sum_{k \neq i} V(x_{ik}) a_k^\alpha F_{ki}, \quad \dot{c}_j^\alpha = -\lambda_j c_j^\alpha - \sum_{k \neq j} V(x_{kj}) c_k^\alpha F_{jk}, \tag{1.2}$$

where $x_{ik} := x_i - x_k$. In the elliptic case, the ‘potential’ is given by $V(x) = \zeta(x) - \zeta(x + \gamma)$ with the Weierstrass zeta-function and an arbitrary complex ‘coupling constant’ $\gamma \neq 0$. The model admits hyperbolic/trigonometric degenerations for which one has $V^{\text{hyp}}(x) = \coth(x) - \coth(x + \gamma)$ and $V^{\text{rat}}(x) = x^{-1} - (x + \gamma)^{-1}$. The parameters λ_i in (1.2) are arbitrary. This is a hallmark of gauge invariance, and thus, it is natural to declare that the ‘physical observables’ are invariant with respect to arbitrary rescalings

$$a_i \mapsto \Lambda_i^{-1} a_i, \quad c_i \mapsto \Lambda_i c_i, \tag{1.3}$$

where the Λ_i may depend on the dynamical variables as well. One way to deal with this ambiguity is to impose a gauge fixing condition. Note also the interesting feature of the model that the spins a_i, c_i are not purely internal degrees of freedom, since they directly encode the velocities through the equations of motion $\dot{x}_i = F_{ii}$.

In the trigonometric real form of our interest, we put $x_j := \frac{1}{2}q_j$, where the q_j are real and are regarded as angles. In other words, we deal with particles located on the unit circle at the points $Q_j := \exp(iq_j)$. The spins c_i and a_i are

complex conjugates of each other, and we parametrize them as

$$c_i^\alpha = v(\alpha)_i, \quad a_i^\alpha = \bar{v}(\alpha)_i, \tag{1.4}$$

where $v(-)_i$ is regarded as a d -component row vector. For each α , $v(\alpha)$ is also viewed as an n -component column vector, and thus, $F = \sum_\alpha v(\alpha)v(\alpha)^\dagger$ is an n by n Hermitian matrix. The potential V is now chosen to be

$$V(x) := \cot(x) - \cot(x - i\gamma) \tag{1.5}$$

with a real, positive coupling constant γ . The gauge transformations are given by arbitrary $\Lambda_i \in U(1)$ and accordingly we have $\lambda_i \in i\mathbb{R}$. It can be checked that these reality constraints are consistent with the equations of motion (1.2). We remark in passing that they imply the second-order equation

$$\frac{1}{2}\ddot{q}_i = \sum_{j \neq i} F_{ij} F_{ji} \left[V\left(\frac{q_{ij}}{2}\right) - V\left(\frac{q_{ji}}{2}\right) \right] = \sum_{j \neq i} |F_{ij}|^2 \frac{2 \cot\left(\frac{q_{ij}}{2}\right)}{1 + \sinh^{-2}(\gamma) \sin^2\left(\frac{q_{ij}}{2}\right)}. \tag{1.6}$$

The equations of motion as given above are local in the sense that one does not know on what phase space their flow is complete, which is required for an integrable system. Neglecting this issue, let us assume that $\sum_\alpha v(\alpha)_i \neq 0$ for all i , which permits us to impose the conditions

$$\mathcal{U}_i := \sum_\alpha v(\alpha)_i > 0. \tag{1.7}$$

Note that (1.7) is a gauge fixing in disguise, since it amounts to setting the phase of \mathcal{U}_i to 0. Then, consistency with the requirement $\Im(\mathcal{U}_i) = 0$ can be used to uniquely determine the λ_i , and one finds the gauge fixed equations of motion

$$\frac{1}{2}\dot{q}_i = F_{ii}, \quad \dot{v}(\alpha)_j = i\eta_j v(\alpha)_j - \sum_{\ell \neq j} F_{j\ell} v(\alpha)_\ell V\left(\frac{q_{\ell j}}{2}\right), \tag{1.8}$$

with

$$i\eta_j = \frac{1}{2} \sum_{\ell \neq j} \frac{\mathcal{U}_\ell}{\mathcal{U}_j} \left[F_{j\ell} V\left(\frac{q_{\ell j}}{2}\right) + F_{\ell j} V\left(\frac{q_{j\ell}}{2}\right) \right]. \tag{1.9}$$

In this paper, we develop a Hamiltonian reduction approach to the real, trigonometric spin RS model specified above. In particular, this yields a phase space on which all flows of interest are complete. An open dense subset of the phase space will be associated with the gauge fixing condition (1.7), and on this submanifold we shall determine the explicit form of the Poisson brackets that generate the equations of motion (1.8) by means of the Hamiltonian

$$\mathcal{H} = \sum_i F_{ii}. \tag{1.10}$$

We shall also prove the degenerate integrability of the model by displaying $(2nd - n)$ independent, real-analytic integrals of motion that form a polynomial Poisson algebra whose n -dimensional Poisson center contains \mathcal{H} . These results will be derived by using the q_i and gauge fixed versions of the ‘dressed spins’

$v(\alpha)$ as coordinates on the reduced phase space obtained from Hamiltonian reduction. However, we will put forward another remarkable set of variables as well, which consists of canonical pairs q_i, p_i and ‘reduced primary spins’ w^α that decouple from q and p under the reduced Poisson bracket. On the overlap of their dense domains, the relation between the two sets of variables can be given explicitly, but the formula is very involved. The drawback of the variables q, p, w^α is that in terms of them \mathcal{H} and the equations of motion become complicated.

Notice that the Newton equations (1.6) imply the conservation of the sum of the velocities \dot{q}_i , which gives the Hamiltonian via (1.8) and (1.10), and \dot{q}_i is nonnegative by (1.8). The same features appear in the spinless chiral RS model [50] defined by the Hamiltonian

$$\mathcal{H}_{\text{RS}}^+ = \sum_i e^{2\theta_i} \prod_{j \neq i} \left[1 + \frac{\sinh^2 \gamma}{1 + \sin^2 \frac{q_i - q_j}{2}} \right]^{\frac{1}{2}}, \tag{1.11}$$

with Darboux coordinates¹ q_i, θ_j . The second-order equations of motion for q_i generated by this Hamiltonian reproduce the $d = 1$ special case of (1.6). To see this, note that $F_{ij}F_{ji} = F_{ii}F_{jj}$ if $d = 1$, and substitute $\dot{q}_i = 2F_{ii}$ from (1.8) into (1.6). In fact, the spinless RS model results from the $d = 1$ special case of our Hamiltonian reduction: in this case w^1 becomes gauge equivalent to a constant vector and one derives the model utilizing also a canonical transformation between q, p mentioned above and q, θ [19]. Thus, the spin RS systems of [32] are generalizations of the *chiral* RS model. We follow the general practice in dropping ‘chiral’ from their name.

Our result on the degenerate integrability of the system is not surprising, since the same property holds in the complex holomorphic case [5, 10] and it also holds generically for large families of related spin many-body models obtained by Hamiltonian reduction [44–46]. Despite these earlier results, the degenerate integrability of our specific real system cannot be obtained directly. Therefore, it requires a separate treatment, and we shall exhibit the desired integrals of motion in explicit form.

Here is an outline of this work and its main results. In Sect. 2, we present the master phase space \mathcal{M} which is an extension of the Heisenberg double of $U(n)$ by a space of primary spins. The latter space is formed of $d \geq 2$ copies of \mathbb{C}^n endowed with a $U(n)$ covariant Poisson bracket and a (Poisson–Lie) moment map, see Proposition 2.1. We also introduce the ‘free’ degenerate integrable system on \mathcal{M} that will be reduced. In Sect. 3, we define the Hamiltonian reduction and progress toward the description of the corresponding reduced phase space \mathcal{M}_{red} , which is a real-analytic symplectic manifold of dimension $2nd$. In particular, we exhibit two models of dense open subsets of \mathcal{M}_{red} ; the first one is used in the subsequent sections to derive the real form of the trigonometric spin RS system described above, while the second one allows

¹This form of the chiral RS Hamiltonian is the one found in [50]. Different Darboux variables, which avoid the appearance of square roots in the Hamiltonian, are also often used in the literature.

us to prove that \mathcal{M}_{red} is connected and it leads to a concise formula for the reduced symplectic form (see Theorem 3.14 and Corollary 3.15). The second model will also be used for recovering the Gibbons–Hermsen system through a scaling limit (see Remark 3.16). In Sect. 4, we characterize the projection of a family of free Hamiltonian vector fields of \mathcal{M} onto \mathcal{M}_{red} , and show in Corollary 4.3 that one of these projections reproduces the equations of motion (1.8). Then, in Sect. 5, we obtain the reduced Poisson bracket presented in Theorem 5.8. This offers an alternative way to derive the equations of motion (1.8), and we also provide a formula for the Poisson bracket of the Lax matrix that generates the commuting reduced Hamiltonians, see Proposition 5.10. In Sect. 6, we demonstrate the degenerate integrability of the real trigonometric spin RS system, with the final result formulated as Theorem 6.7. Section 7 concludes this work and gathers open questions. There are four appendices devoted to auxiliary results and proofs.

Note on conventions. The sign function sgn is such that $\text{sgn}(i - k)$ is $+1$ if $i > k$, -1 if $i < k$, and 0 for $i = k$. Similarly to Kronecker’s delta function, we define for any condition c the symbol δ_c which equals $+1$ if c is satisfied, and 0 otherwise. For example, $\delta_{(j < l \leq k)}$ equals $+1$ if $j < l$ and $l \leq k$, while it is 0 if one of those two conditions is not satisfied.

2. Heisenberg Double, Primary Spins and ‘Free’ Integrable System

Eventually, we shall obtain the real, trigonometric spin RS system by reduction of an ‘obviously integrable’ system on the phase space $\mathcal{M} := M \times \mathbb{C}^{n \times d}$, where M is the Heisenberg double of the Poisson–Lie group $U(n)$ and $\mathbb{C}^{n \times d}$ is the space of the so-called primary spin variables. In this section, we present a quick overview of these structures, to be used in the subsequent sections. More details can be found in the references [16, 19, 29, 30, 35, 52] and in “Appendix A.”

2.1. The Heisenberg Double and Its Models

Let us start with the real vector space direct sum

$$\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) + \mathfrak{b}(n), \quad (2.1)$$

where $\mathfrak{b}(n)$ denotes the Lie algebra of upper triangular complex matrices having real entries along the diagonal, and the unitary Lie algebra $\mathfrak{u}(n)$ consists of the skew-Hermitian matrices. These are isotropic subalgebras with respect to the non-degenerate, invariant bilinear form of $\mathfrak{gl}(n, \mathbb{C})$ given by

$$\langle X, Y \rangle := \Im \text{tr}(XY), \quad \forall X, Y \in \mathfrak{gl}(n, \mathbb{C}), \quad (2.2)$$

which means that we have a Manin triple at hand. Then define

$$R := \frac{1}{2} (P_{\mathfrak{u}(n)} - P_{\mathfrak{b}(n)}), \quad (2.3)$$

using the projection operators with ranges $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$, associated with the decomposition (2.1). For any $X \in \mathfrak{gl}(n, \mathbb{C})$, we may write

$$X = X_{\mathfrak{u}(n)} + X_{\mathfrak{b}(n)} \quad \text{with} \quad X_{\mathfrak{u}(n)} = P_{\mathfrak{u}(n)}(X), \quad X_{\mathfrak{b}(n)} = P_{\mathfrak{b}(n)}(X). \quad (2.4)$$

As a manifold, M is the real Lie group $GL(n, \mathbb{C})$, and for any smooth *real* function $f \in C^\infty(GL(n, \mathbb{C}))$ we introduce the $\mathfrak{gl}(n, \mathbb{C})$ -valued derivatives ∇f and $\nabla' f$ by

$$\langle \nabla f(K), X \rangle := \left. \frac{d}{dt} \right|_{t=0} f(e^{tX} K), \quad \langle \nabla' f(K), X \rangle := \left. \frac{d}{dt} \right|_{t=0} f(K e^{tX}), \quad \forall X \in \mathfrak{gl}(n, \mathbb{C}), \tag{2.5}$$

where K denotes the variable running over $GL(n, \mathbb{C})$. The commutative algebra of smooth real functions, $C^\infty(M)$, carries two natural Poisson brackets provided by

$$\{f, h\}_\pm := \langle \nabla f, R\nabla h \rangle \pm \langle \nabla' f, R\nabla' h \rangle. \tag{2.6}$$

The minus bracket makes $GL(n, \mathbb{C})$ into a real Poisson–Lie group, while the plus one corresponds to a symplectic structure on M . The former is called the Drinfeld double Poisson bracket and the latter the Heisenberg double Poisson bracket [52].

The real Poisson brackets can be extended to complex functions by requiring complex bilinearity. Then, the real Poisson brackets can be recovered if we know all Poisson brackets between the matrix elements of K and its complex conjugate \bar{K} . In the case of the Drinfeld double, we have

$$\{K_{ij}, K_{kl}\}_- = iK_{kj}K_{il} [\delta_{ik} + 2\delta_{(i>k)} - \delta_{lj} - 2\delta_{(l>j)}], \tag{2.7}$$

and

$$\{K_{ij}, \bar{K}_{kl}\}_- = iK_{ij}\bar{K}_{kl}[\delta_{ik} - \delta_{jl}] + 2i \left[\delta_{ik} \sum_{\beta>i} K_{\beta j} \bar{K}_{\beta l} - \delta_{jl} \sum_{\alpha<j} K_{i\alpha} \bar{K}_{k\alpha} \right]. \tag{2.8}$$

Consider the subgroup $B(n) < GL(n, \mathbb{C})$ of upper triangular matrices having positive entries along the diagonal, and the unitary subgroup $U(n) < GL(n, \mathbb{C})$. These subgroups correspond to the subalgebras in (2.1). It is well known that both $U(n)$ and $B(n)$ are Poisson submanifolds of the Drinfeld double $(GL(n, \mathbb{C}), \{ , \}_-)$. We denote their inherited Poisson structures by $\{ , \}_U$ and $\{ , \}_B$, which makes them Poisson–Lie groups.

The Poisson brackets on $C^\infty(U(n))$ and on $C^\infty(B(n))$ admit the following description. For any real function $\phi \in C^\infty(U(n))$ introduce the $\mathfrak{u}(n)$ -valued derivatives $D\phi$ and $D'\phi$ by

$$\langle D\phi(g), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \phi(e^{tX} g), \quad \langle D'\phi(g), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \phi(g e^{tX}), \quad \forall X \in \mathfrak{u}(n), \tag{2.9}$$

and for any $\chi \in C^\infty(B(n))$ similarly introduce the $\mathfrak{u}(n)$ -valued derivatives $D\chi$ and $D'\chi$. Then, we have

$$\{\phi_1, \phi_2\}_U(g) = -\langle D'\phi_1(g), g^{-1}(D\phi_2(g))g \rangle, \quad \forall g \in U(n), \tag{2.10}$$

where the conjugation takes place inside $GL(n, \mathbb{C})$. Similarly

$$\{\chi_1, \chi_2\}_B(b) = \langle D'\chi_1(b), b^{-1}(D\chi_2(b))b \rangle, \quad \forall b \in B(n). \tag{2.11}$$

The opposite signs in the last two formulae are due to our conventions.

By the Gram–Schmidt process, every element $K \in \text{GL}(n, \mathbb{C})$ admits the unique decompositions

$$K = b_L g_R^{-1} = g_L b_R^{-1} \quad \text{with } b_L, b_R \in \text{B}(n), g_L, g_R \in \text{U}(n), \quad (2.12)$$

and K can be recovered also from the pairs (g_L, b_L) and (g_R, b_R) , by utilizing the identity

$$b_L^{-1} g_L = g_R^{-1} b_R. \quad (2.13)$$

These decompositions give rise to the maps Λ_L, Λ_R into $\text{B}(n)$ and Ξ_L, Ξ_R into $\text{U}(n)$,

$$\Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R, \quad \Xi_L(K) := g_L, \quad \Xi_R(K) := g_R. \quad (2.14)$$

Then, we obtain the maps from $\text{GL}(n, \mathbb{C})$ onto $\text{U}(n) \times \text{B}(n)$,

$$(\Xi_L, \Lambda_R), \quad (\Xi_R, \Lambda_L), \quad (\Xi_L, \Lambda_L), \quad (\Xi_R, \Lambda_R), \quad (2.15)$$

which are all (real-analytic) diffeomorphisms. In particular, we shall use the diffeomorphism

$$m_1 := (\Xi_R, \Lambda_R) : \text{GL}(n, \mathbb{C}) \rightarrow \text{U}(n) \times \text{B}(n) \quad (2.16)$$

to transfer the Heisenberg double Poisson bracket to $C^\infty(\text{U}(n) \times \text{B}(n))$. The formula of the resulting Poisson bracket [16], called $\{ , \}_+^1$, can be written as follows:

$$\begin{aligned} \{\mathcal{F}, \mathcal{H}\}_+^1(g, b) &= \langle D'_2 \mathcal{F}, b^{-1}(D_2 \mathcal{H})b \rangle - \langle D'_1 \mathcal{F}, g^{-1}(D_1 \mathcal{H})g \rangle \\ &\quad + \langle D_1 \mathcal{F}, D_2 \mathcal{H} \rangle - \langle D_1 \mathcal{H}, D_2 \mathcal{F} \rangle, \end{aligned} \quad (2.17)$$

for any $\mathcal{F}, \mathcal{H} \in C^\infty(\text{U}(n) \times \text{B}(n))$. The derivatives on the right-hand side are taken at $(g, b) \in \text{U}(n) \times \text{B}(n)$; the subscripts 1 and 2 refer to derivatives with respect to the first and second arguments. As an application, one can determine the Poisson brackets between the matrix elements of $(g, b) := (g_R, b_R)$ on the Heisenberg double, which gives

$$\{g_{lm}, b_{jk}\}_+ = i\delta_{jl}g_{lm}b_{jk} + 2i\delta_{(j<l\leq k)}g_{jm}b_{lk}, \quad (2.18)$$

and

$$\{g_{lm}, \bar{b}_{jk}\}_+ = i\delta_{jl}g_{lm}\bar{b}_{jk} + 2i\delta_{jl} \sum_{j<\beta\leq k} g_{\beta m}\bar{b}_{\beta k}. \quad (2.19)$$

The same formulae are valid w.r.t. $\{ , \}_+^1$, and this was used for the computation.

Observe from the formula (2.17) that both Ξ_R and Λ_R are Poisson maps w.r.t. the Heisenberg double Poisson bracket and the Poisson brackets $\{ , \}_U$ and $\{ , \}_B$, respectively. The same is true regarding the maps Ξ_L and Λ_L . A further property that we use later is that

$$\{\Lambda_L^*(\chi_1), \Lambda_R^*(\chi_2)\}_+ = 0, \quad \forall \chi_1, \chi_2 \in C^\infty(\text{B}(n)). \quad (2.20)$$

Note, incidentally, that Ξ_L and Ξ_R enjoy the analogous identity.

The Heisenberg double admits another convenient model as well. This relies on the diffeomorphism between $\text{B}(n)$ and the manifold $\mathfrak{P}(n) = \exp(i\mathfrak{u}(n))$

of positive definite Hermitian matrices, defined by $b \mapsto L := bb^\dagger$. Then, we have the diffeomorphism

$$m_2 : \text{GL}(n, \mathbb{C}) \rightarrow \text{U}(n) \times \mathfrak{P}(n), \quad m_2 := (\Xi_R, \Lambda_R \Lambda_R^\dagger). \tag{2.21}$$

For $\psi \in C^\infty(\mathfrak{P}(n))$, define the $\mathfrak{u}(n)$ -valued derivative $d\psi$ by

$$\langle d\psi(L), X \rangle := \left. \frac{d}{dt} \right|_{t=0} \psi(L + tX), \quad \forall X \in \mathfrak{u}(n). \tag{2.22}$$

This definition makes sense since $(L + tX) \in \mathfrak{P}(n)$ for small t . By using m_2 , one can transfer the Poisson bracket $\{ , \}_+$ to $C^\infty(\text{U}(n) \times \mathfrak{P}(n))$. The resulting Poisson bracket is called $\{ , \}_+^2$ and is given [16] by the following explicit formula:

$$\begin{aligned} \{F, H\}_+^2(g, L) = & 4 \left\langle Ld_2F, (Ld_2H)_{\mathfrak{u}(n)} \right\rangle - \langle D_1'F, g^{-1}(D_1H)g \rangle \\ & + 2 \langle D_1F, Ld_2H \rangle - 2 \langle D_1H, Ld_2F \rangle, \end{aligned} \tag{2.23}$$

for any $F, H \in C^\infty(\text{U}(n) \times \mathfrak{P}(n))$, where the derivatives with respect to the first and second arguments are taken at $(g, L) \in \text{U}(n) \times \mathfrak{P}(n)$. The subscript $\mathfrak{u}(n)$ refers to the decomposition defined in (2.4).

We end this review of the Heisenberg double by recalling the symplectic form, denoted Ω_M , that corresponds to the non-degenerate Poisson structure $\{ , \}_+$. It can be displayed [1] as

$$\Omega_M = \frac{1}{2} \Im \text{tr}(d\Lambda_L \Lambda_L^{-1} \wedge d\Xi_L \Xi_L^{-1}) + \frac{1}{2} \Im \text{tr}(d\Lambda_R \Lambda_R^{-1} \wedge d\Xi_R \Xi_R^{-1}). \tag{2.24}$$

Here, $d\Lambda_L$ collects the exterior derivatives of the components of the matrix valued function Λ_L . To be clear about our conventions, we remark that the wedge does not contain $\frac{1}{2}$, and the Hamiltonian vector field X_h of h satisfies $dh = \Omega_M(, X_h)$ and $\{f, h\}_+ = df(X_h) = \Omega_M(X_h, X_f)$.

2.2. The Primary Spin Variables

We begin by recalling that the real, trigonometric spin Sutherland model of Gibbons–Hermsen [24] type can be derived via Hamiltonian reduction of $T^*\text{U}(n) \times \mathbb{C}^{n \times d}$, where $\mathbb{C}^{n \times d} \simeq \mathbb{R}^{2n \times d}$ carries its canonical Poisson structure. In particular, if the elements of $\mathbb{C}^{n \times d}$ are represented as a collection of \mathbb{C}^n column vectors

$$w^1, w^2, \dots, w^d, \tag{2.25}$$

then the d different copies pairwise Poisson commute. The symmetry group underlying the reduction is $\text{U}(n)$, which acts on \mathbb{C}^n in the obvious manner,

$$\mathcal{A}^{(n)} : \text{U}(n) \times \mathbb{C}^n \rightarrow \mathbb{C}^n \quad \text{given by} \quad \mathcal{A}^{(n)}(g, w) := gw. \tag{2.26}$$

For our generalization, it is natural to require this to be a Poisson action, i.e., $\mathcal{A}^{(n)}$ should be a Poisson map with respect to the Poisson structure (2.10) on $\text{U}(n)$ and a suitable Poisson structure on \mathbb{C}^n . A further requirement is that the $\text{U}(n)$ -action should be generated by a moment map.

Specialized to $U(n)$ with the Poisson structure (2.10), the notion of moment map that we use can be summarized as follows.² Suppose that we have a Poisson manifold $(\mathcal{P}, \{ , \}_{\mathcal{P}})$ and a Poisson map $\Lambda : \mathcal{P} \rightarrow B(n)$, where $B(n)$ is endowed with the Poisson structure (2.11). Then, for any $X \in \mathfrak{u}(n)$, the following formula defines a vector field $X_{\mathcal{P}}$ on \mathcal{P} :

$$\mathcal{L}_{X_{\mathcal{P}}}(\mathcal{F}) \equiv X_{\mathcal{P}}[\mathcal{F}] := \langle X, \{ \mathcal{F}, \Lambda \}_{\mathcal{P}} \Lambda^{-1} \rangle, \quad \forall \mathcal{F} \in C^{\infty}(\mathcal{P}), \quad (2.27)$$

where the Poisson bracket is taken with every entry of the matrix Λ and $\mathcal{L}_{X_{\mathcal{P}}}$ denotes Lie derivative along $X_{\mathcal{P}}$. The map $X \mapsto X_{\mathcal{P}}$ is automatically a Lie algebra anti-homomorphism, representing an infinitesimal left action of $U(n)$. If it integrates to a global action of $U(n)$, then the resulting action is Poisson, i.e., the action map $\mathcal{A} : U(n) \times \mathcal{P} \rightarrow \mathcal{P}$ is Poisson. In the situation just outlined, Λ is called the (Poisson–Lie) moment map of the corresponding Poisson action.

In the next proposition, we collect the key properties of a Poisson structure on \mathbb{C}^n , which is a special case of the $U(n)$ covariant Poisson structures found by Zakrzewski [56].

Proposition 2.1. *The following formula defines a Poisson structure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$:*

$$\{w_i, w_l\} = i \operatorname{sgn}(i - l) w_i w_l, \quad \forall 1 \leq i, l \leq n, \quad (2.28)$$

$$\{w_i, \bar{w}_l\} = i \delta_{il} (2 + |w|^2) + i w_i \bar{w}_l + i \delta_{il} \sum_{r=1}^n \operatorname{sgn}(r - i) |w_r|^2. \quad (2.29)$$

These formulae imply

$$\{\bar{w}_i, \bar{w}_l\} = \overline{\{w_i, w_l\}}, \quad (2.30)$$

which means that the Poisson bracket of real functions is real. With respect to this Poisson bracket, the action (2.26) of $U(n)$ with (2.10) is Poisson and is generated by the moment map $\mathbf{b} : \mathbb{C}^n \rightarrow B(n)$ given by

$$\mathbf{b}_{jj}(w) = \sqrt{\mathcal{G}_j / \mathcal{G}_{j+1}}, \quad \mathbf{b}_{ij}(w) = \frac{w_i \bar{w}_j}{\sqrt{\mathcal{G}_j \mathcal{G}_{j+1}}}, \quad \forall 1 \leq i < j \leq n \quad (2.31)$$

with

$$\mathcal{G}_j = 1 + \sum_{k=j}^n |w_k|^2, \quad \mathcal{G}_{n+1} := 1, \quad (2.32)$$

The map \mathbf{b} satisfies the identity

$$\mathbf{b}(w) \mathbf{b}(w)^{\dagger} = \mathbf{1}_n + w w^{\dagger}. \quad (2.33)$$

The Poisson structure is non-degenerate, and the corresponding symplectic form is given by

²In full generality, the concept of Poisson–Lie moment map goes back to Lu [36]; our conventions are slightly different from hers.

$$\Omega_{\mathbb{C}^n} = \frac{i}{2} \sum_{k=1}^n \frac{1}{\mathcal{G}_k} dw_k \wedge d\bar{w}_k + \frac{i}{4} \sum_{k=1}^{n-1} \frac{1}{\mathcal{G}_k \mathcal{G}_{k+1}} d\mathcal{G}_{k+1} \wedge (\bar{w}_k dw_k - w_k d\bar{w}_k). \tag{2.34}$$

A variant of the factorization formula (2.33) (without connection to Zakrzewski’s Poisson bracket) was found earlier by Klimčík, as presented in an unpublished initial version of [19]. For convenience, we give a self-contained proof of the proposition in “Appendix A.”

Definition 2.2. The pairwise Poisson commuting w^1, \dots, w^d with each copy subject to the Poisson brackets (2.28), (2.29) are called *primary spin variables*. The Poisson space obtained in this manner is denoted $(\mathbb{C}^{n \times d}, \{ , \}_W)$, and we shall also use the notation

$$W := (w^1, \dots, w^d). \tag{2.35}$$

The corresponding symplectic form, Ω_W , is the sum of d -copies of $\Omega_{\mathbb{C}^n}$ (2.34), one for each variable w^α , $\alpha = 1, \dots, d$.

2.3. The Unreduced ‘Free’ Integrable System

Let \mathfrak{H} be an Abelian Poisson subalgebra of the Poisson algebra of (smooth, real-analytic, etc.) functions on a symplectic manifold \mathcal{M} of dimension $2N$, such that all elements of \mathfrak{H} generate complete Hamiltonian flows. Assume that the functional dimension of \mathfrak{H} is r ³ and that there exists also a Poisson subalgebra \mathfrak{C} of the functions on \mathcal{M} whose functional dimensions is $(2N - r)$ and its center contains \mathfrak{H} . Then \mathfrak{H} is called an *integrable system* with Hamiltonians \mathfrak{H} and algebra of constants of motion \mathfrak{C} . Liouville integrability is the $r = N$, $\mathfrak{C} = \mathfrak{H}$, special case. One calls the system *degenerate integrable* (or non-commutative integrable, or superintegrable) if $r < N$. In the degenerate case, similarly to Liouville integrability, the flows of the Hamiltonians belonging to \mathfrak{H} are linear in suitable coordinate systems on the joint level surfaces of \mathfrak{C} . For further details of this notion and its variants, and for the generalization of the Liouville–Arnold theorem, we refer to the papers [27, 37, 39, 45] and to Section 11.8 of the book [48].

Consider the Heisenberg double $(M, \{ , \}_+)$ and the space of primary spins $(\mathbb{C}^{n \times d}, \{ , \}_W)$ introduced in Sect. 2.2. Define

$$\mathcal{M} := M \times \mathbb{C}^{n \times d} \tag{2.36}$$

and equip it with the product Poisson structure, $\{ , \}_M$, which comes from the symplectic form

$$\Omega_M = \Omega_M + \Omega_W. \tag{2.37}$$

Let $C^\infty(\mathbb{B}(n))^{U(n)}$ denote those functions on $\mathbb{B}(n)$ that are invariant with respect to the dressing action of $U(n)$ on $\mathbb{B}(n)$, operating as

$$\text{Dress}_g(b) := \Lambda_L(gb), \quad \forall (g, b) \in U(n) \times \mathbb{B}(n). \tag{2.38}$$

³This means that the exterior derivatives of the elements of \mathfrak{H} span an r -dimensional subspace of the cotangent space for generic points of \mathcal{M} , which form a dense open submanifold.

It is well known that these invariant functions form the center of the Poisson bracket $\{ , \}_B$ (2.11). Extend all maps displayed in (2.14) to \mathcal{M} in the trivial manner, for example, by setting

$$\Lambda_R(K, W) := \Lambda_R(K) \quad \text{with} \quad (K, W) \in M \times \mathbb{C}^{n \times d}. \tag{2.39}$$

Then

$$\mathfrak{H} := \Lambda_R^*(C^\infty(\mathbb{B}(n)))^{U(n)} \tag{2.40}$$

is an Abelian Poisson subalgebra of $C^\infty(\mathcal{M})$. We call the elements of \mathfrak{H} ‘free Hamiltonians’ since their flows are easily written down explicitly. Indeed, for $H = \Lambda_R^*(h)$ the flow sends the initial value $(K(0), W(0))$ to

$$(K(t), W(t)) = (K(0) \exp(-tDh(b_R(0))), W(0)). \tag{2.41}$$

It follows that b_R and b_L are constants along the flow and we have the ‘free motion’ on $U(n)$ given by

$$g_R(t) = \exp(tDh(b_R(0))) g_R(0). \tag{2.42}$$

The functional dimension of \mathfrak{H} is n , and for independent generators one may take

$$H_k := \Lambda_R^*(h_k) \quad \text{with} \quad h_k(b) = \frac{1}{2k} \text{tr}(bb^\dagger)^k, \quad k = 1, \dots, n. \tag{2.43}$$

The invariance of these functions follows from the useful identity

$$(\text{Dress}_g(b))(\text{Dress}_g(b))^\dagger = g(bb^\dagger)g^{-1}. \tag{2.44}$$

The system is degenerate integrable, with \mathfrak{C} taken to be the algebra of all constants of motion, which are provided by arbitrary smooth functions depending on b_L, b_R and W . From the decomposition (2.12), we get

$$b_R b_R^\dagger = g_R (b_L^{-1} (b_L^{-1})^\dagger) g_R^{-1}, \tag{2.45}$$

and this entails n relations between the functions of b_L and b_R . Thus the functional dimension of \mathfrak{C} is $2N - n$, with $N = n^2 + nd$, as required. It is worth noting that the joint level surfaces of \mathfrak{C} are compact, since they can be viewed as closed subsets of $U(n)$.

There are several ways to enlarge \mathfrak{H} into an Abelian Poisson algebra of functional dimension N , i.e., to obtain Liouville integrability of the Hamiltonians in \mathfrak{H} . However, there is no canonical way to do so. Degenerate integrability is a stronger property than Liouville integrability, since it restricts the flows of the Hamiltonians to smaller level surfaces. For these reasons, we shall not pay attention to Liouville integrability in this paper.

3. Defining the Reduction and Solving the Moment Map Constraint

We first describe a Poisson action of $U(n)$ on \mathcal{M} and use it for defining the reduction of the free integrable system. Then we shall deal with two parametrizations of the ‘constraint surface,’ which is obtained by imposing the moment map constraint of Eq. (3.17) below.

3.1. Definition of the Reduction

Let us start by introducing the following Poisson map $\Lambda : \mathcal{M} \rightarrow \mathbb{B}(n)$,

$$\Lambda(K, W) = \Lambda_L(K)\Lambda_R(K)\mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d), \tag{3.1}$$

using the notations (2.14), (2.31). The Poisson property of Λ holds since all the factors are separately Poisson maps, and their matrix elements mutually Poisson commute. The infinitesimal action generated by Λ , via the formula (2.27), integrates to a global Poisson action of $U(n)$ on \mathcal{M} . This action turns out to have a nice form in terms of the new variables on \mathcal{M} given below.

Definition 3.1. For $\alpha = 1, \dots, d$, introduce

$$b_\alpha := \mathbf{b}(w^\alpha), \quad B_\alpha := b_R b_1 b_2 \cdots b_\alpha, \quad B_0 := b_R, \tag{3.2}$$

and define the *dressed spins* $v(\alpha)$ and the *half-dressed spins* v^α by the equalities

$$v(\alpha) := B_{\alpha-1}w^\alpha =: b_R v^\alpha. \tag{3.3}$$

Lemma 3.2. *The new variables on \mathcal{M} given by*

$$g_R, b_R, v(1), \dots, v(d) \tag{3.4}$$

are related by a diffeomorphism of $U(n) \times \mathbb{B}(n) \times \mathbb{C}^{n \times d}$ to the variables

$$g_R, b_R, w^1, \dots, w^d. \tag{3.5}$$

Proof. We have the relations,

$$w^1 = b_R^{-1}v(1), \quad w^2 = \mathbf{b}(w^1)^{-1}b_R^{-1}v(2), \dots, \quad w^d = \mathbf{b}(w^{d-1})^{-1} \cdots \mathbf{b}(w^1)^{-1}b_R^{-1}v(d), \tag{3.6}$$

which can be used to reconstruct the variables (3.5) from those in (3.4). \square

The statement analogous to Lemma 3.2 for the variables $(g_R, b_R, v^1, \dots, v^d)$ also holds. Later in the paper, we shall use the following identities enjoyed by the half-dressed spins and the dressed spins, which are direct consequences of (2.33). These identities and the subsequent proposition actually motivated the introduction of these variables.

Lemma 3.3. *With the above notations, one has the identities*

$$(b_1 b_2 \cdots b_d)(b_1 b_2 \cdots b_d)^\dagger = \mathbf{1}_n + \sum_{\alpha=1}^d v^\alpha (v^\alpha)^\dagger, \quad B_d B_d^\dagger = b_R b_R^\dagger + \sum_{\alpha=1}^d v(\alpha) v(\alpha)^\dagger. \tag{3.7}$$

Proposition 3.4. *The moment map $\Lambda : \mathcal{M} \rightarrow \mathbb{B}(n)$ given by (3.1) generates the action $\mathcal{A} : U(n) \times \mathcal{M} \rightarrow \mathcal{M}$ that operates as follows:*

$$\mathcal{A}_\eta : (g_R, b_R, v(1), \dots, v(d)) \mapsto (\tilde{\eta} g_R \tilde{\eta}^{-1}, \text{Dress}_{\tilde{\eta}}(b_R), \tilde{\eta} v(1), \dots, \tilde{\eta} v(d)), \quad \forall \eta \in U(n), \tag{3.8}$$

where $\tilde{\eta} = \Xi_R(\eta b_L)^{-1}$ with $b_L = \Lambda_L \circ m_1^{-1}(g_R, b_R)$ using (2.16). In other words, $b_L = \Lambda_L(K)$ with $K \in M$ parametrized by the pair (g_R, b_R) .

Proof. In order to avoid clumsy formulae and the introduction of further notations, in what follows we identify the variables $g_R, b_R, B_\alpha, w^\alpha$ and so on with the associated evaluation functions on M, \mathcal{M} and \mathbb{C}^n . We shall also use the infinitesimal dressing action corresponding to (2.38), which has the form

$$\text{dress}_X(b) = b(b^{-1}Xb)_{\mathfrak{b}(n)}, \quad \forall X \in \mathfrak{u}(n), b \in \mathbb{B}(n). \tag{3.9}$$

For any $X \in \mathfrak{u}(n)$, denote $X_{\mathcal{M}}, X_M, X_{\mathbb{C}^n}$ the vector fields associated with the moment maps $\Lambda : \mathcal{M} \rightarrow \mathbb{B}(n), \Lambda_L \Lambda_R : M \rightarrow \mathbb{B}(n)$ and $\mathfrak{b} : \mathbb{C}^n \rightarrow \mathbb{B}(n)$, respectively. The formula of X_M is known [19, 29] and $X_{\mathbb{C}^n}$ can be read off from Sect. 2.2. In fact, we have

$$\mathcal{L}_{X_M} g_R = [(b_L^{-1}Xb_L)_{\mathfrak{u}(n)}, g_R], \quad \mathcal{L}_{X_M} b_R = \text{dress}_{(b_L^{-1}Xb_L)_{\mathfrak{u}(n)}}(b_R) \tag{3.10}$$

and

$$\mathcal{L}_{X_{\mathbb{C}^n}} w = Xw, \quad \mathcal{L}_{X_{\mathbb{C}^n}} \mathfrak{b} = \text{dress}_X(\mathfrak{b}). \tag{3.11}$$

By using these, application of the definition (2.27) to the real and imaginary parts of the evaluation functions gives

$$\mathcal{L}_{X_{\mathcal{M}}} w^\alpha = ((b_L B_{\alpha-1})^{-1} X b_L B_{\alpha-1})_{\mathfrak{u}(n)} w^\alpha, \quad \mathcal{L}_{X_{\mathcal{M}}} B_\alpha = \text{dress}_{(b_L^{-1}Xb_L)_{\mathfrak{u}(n)}}(B_\alpha). \tag{3.12}$$

From the last two equalities, we obtain

$$\mathcal{L}_{X_{\mathcal{M}}} v(\alpha) = (b_L^{-1}Xb_L)_{\mathfrak{u}(n)} v(\alpha). \tag{3.13}$$

In conclusion, we see that the vector field $X_{\mathcal{M}}$ is encoded by the formula (3.13) together with

$$\mathcal{L}_{X_{\mathcal{M}}} g_R = [(b_L^{-1}Xb_L)_{\mathfrak{u}(n)}, g_R], \quad \mathcal{L}_{X_{\mathcal{M}}} b_R = \text{dress}_{(b_L^{-1}Xb_L)_{\mathfrak{u}(n)}}(b_R), \tag{3.14}$$

which follow from (3.10) and the structure of Λ (3.1). The completion of the proof now requires checking that the formula (3.8) indeed gives a left-action of $U(n)$ on \mathcal{M} , whose infinitesimal version reproduces the vector field \mathcal{M} found above. These last steps require some lines but are fully straightforward, and thus, we omit further details. \square

Remark 3.5. The action (3.8) on \mathcal{M} is called (extended) quasi-adjoint action, since if we forget the $v(\alpha)$, then it becomes the quasi-adjoint action on M that goes back to [29]. At any fixed (b_R, g_R) , the map $\eta \mapsto \tilde{\eta}$ that appears in (3.8) is a diffeomorphism on $U(n)$, and thus, the quasi-adjoint action and the so-called *obvious action* have the same orbits. The obvious action, denoted $A : U(n) \times \mathcal{M} \rightarrow \mathcal{M}$, operates as follows:

$$A_g(g_R, b_R, v) := (gg_Rg^{-1}, \text{Dress}_g(b_R), gv), \quad \forall g \in U(n), (g_R, b_R, v) \in \mathcal{M}, \tag{3.15}$$

where

$$v := (v(1), \dots, v(d)) \quad \text{and} \quad gv := (gv(1), \dots, gv(d)). \tag{3.16}$$

We are interested in the reduction of \mathcal{M} defined by imposing the moment map constraint

$$\Lambda = e^\gamma \mathbf{1}_n \quad \text{with a fixed constant } \gamma > 0. \tag{3.17}$$

The corresponding reduced phase space is

$$\mathcal{M}_{\text{red}} = \Lambda^{-1}(e^\gamma \mathbf{1}_n)/\text{U}(n). \tag{3.18}$$

According to Remark 3.5, it does not matter whether we use the quasi-adjoint or the obvious action for taking the quotient.

Denote $C^\infty(\mathcal{M})^{\text{U}(n)}$ the $\text{U}(n)$ invariant functions on \mathcal{M} . We may identify $C^\infty(\mathcal{M}_{\text{red}})$ as the restriction of $C^\infty(\mathcal{M})^{\text{U}(n)}$ to the ‘constraint surface’ $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$. Then, $C^\infty(\mathcal{M}_{\text{red}})$ is naturally a Poisson algebra, with bracket denoted $\{ \cdot, \cdot \}_{\text{red}}$. This is obtained by using that the Poisson bracket of any two invariant functions is again invariant, and its restriction to $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$ depends only on the restrictions of the two functions themselves. One sees this relying on the first class [25] character of the constraints that appear in (3.17).

Since the elements of \mathfrak{H} (2.40) are $\text{U}(n)$ invariant, they give rise to an Abelian Poisson subalgebra, $\mathfrak{H}_{\text{red}}$, of $C^\infty(\mathcal{M}_{\text{red}})$. The flows of the elements of $\mathfrak{H}_{\text{red}}$ on \mathcal{M}_{red} result by projection of the free flows (2.41), see Sect. 4.

Remark 3.6. By using that (3.17) can be written equivalently as $\Lambda\Lambda^\dagger = e^{2\gamma}\mathbf{1}_n$, it is not difficult to see that the triple (g_R, b_R, v) belongs to $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$ if and only if it satisfies

$$e^{2\gamma} g_R^{-1} (b_R b_R^\dagger) g_R - (b_R b_R^\dagger) = \sum_{\alpha=1}^d v(\alpha) v(\alpha)^\dagger. \tag{3.19}$$

We notice that the set of the triples $(g_R, b_R b_R^\dagger, v)$ subject to (3.19) is a subset of the set $\mathcal{M}_{n,d,q}^\times$ defined in [10], which contains the elements $(X, Z, \mathcal{A}_1, \dots, \mathcal{A}_d, \mathcal{B}_1, \dots, \mathcal{B}_d)$ satisfying

$$q^{-1} X Z X^{-1} - Z = \sum_{\alpha=1}^d \mathcal{A}_\alpha \mathcal{B}_\alpha \tag{3.20}$$

and the invertibility conditions

$$\left(Z + \sum_{\alpha=1}^k \mathcal{A}_\alpha \mathcal{B}_\alpha \right) \in \text{GL}(n, \mathbb{C}), \quad \forall k = 1, \dots, d, \tag{3.21}$$

where q is a nonzero complex constant, $X, Z \in \text{GL}(n, \mathbb{C})$, $\mathcal{A}_\alpha \in \mathbb{C}^{n \times 1}$ and $\mathcal{B}_\alpha \in \mathbb{C}^{1 \times n}$, for $\alpha = 1, \dots, d$. (These are equations (4.3) and (4.4) in [10].) It is known that if q is not a root of unity, then the action of $\text{GL}(n, \mathbb{C})$ on $\mathcal{M}_{n,d,q}^\times$, defined by

$$g \cdot (X, Z, \mathcal{A}_\alpha, \mathcal{B}_\alpha) := (g X g^{-1}, g Z g^{-1}, g \mathcal{A}_\alpha, \mathcal{B}_\alpha g^{-1}), \quad \forall g \in \text{GL}(n, \mathbb{C}), \tag{3.22}$$

is free. As explained in [9, 10], this goes back to results in representation theory [11]. Direct comparison of (3.19) and (3.20), and of the corresponding group actions, shows that the $\text{U}(n)$ action (3.15) on our ‘constraint surface’ $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$

is *free*. In our case the invertibility conditions (3.21) hold in consequence of the following identities that generalize (3.7):

$$b_R b_R^\dagger + \sum_{\alpha=1}^k v(\alpha) v(\alpha)^\dagger = (b_R \mathbf{b}(w^1) \cdots \mathbf{b}(w^k)) (b_R \mathbf{b}(w^1) \cdots \mathbf{b}(w^k))^\dagger. \tag{3.23}$$

Because $U(n)$ acts freely on it, $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$ is an embedded submanifold of \mathcal{M} , and \mathcal{M}_{red} (3.18) is a smooth symplectic manifold, whose Poisson algebra coincides with $(C^\infty(\mathcal{M}_{\text{red}}), \{ , \}_{\text{red}})$ presented in the preceding paragraph. Furthermore, since $(\mathcal{M}, \Omega_{\mathcal{M}})$ is actually a real-analytic symplectic manifold and the formulae of the $U(n)$ action and the moment map are all given by real-analytic functions, \mathcal{M}_{red} is also a *real-analytic* symplectic manifold. For the underlying general theory, the reader may consult [41] and also “Appendix D” in [18].

Remark 3.7. We will eventually prove the degenerate integrability of the reduced system by taking advantage of the following functions on \mathcal{M} :

$$I_{\alpha\beta}^k := \text{tr}(v(\alpha) v(\beta)^\dagger L^k) = v(\beta)^\dagger L^k v(\alpha), \quad 1 \leq \alpha, \beta \leq d, \quad k \geq 0, \tag{3.24}$$

where

$$L := b_R b_R^\dagger. \tag{3.25}$$

The identity (2.44) shows that L transforms by conjugation, and therefore, these integrals of motion are invariant under the $U(n)$ action (3.15) on \mathcal{M} . Their real and imaginary parts descend to *real-analytic* functions on the reduced phase space.

3.2. Solution of the Constraint in Terms of Q and Dressed Spins

Our fundamental task is to describe the set of $U(n)$ orbits in the ‘constraint surface’ $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$. For this purpose, it will be convenient to label the points of \mathcal{M} by g_R, L and $v = (v(1), \dots, v(d))$ using that L is given by (3.25). In the various arguments, we shall also employ alternative variables.

Since g_R can be diagonalized by conjugation, we see from the form of the $U(n)$ action (3.8) (or (3.15)) that every $U(n)$ orbit lying in the constraint surface intersects the set

$$\mathcal{M}_0 := \Lambda^{-1}(e^\gamma \mathbf{1}_n) \cap \Xi_R^{-1}(\mathbb{T}^n), \tag{3.26}$$

where \mathbb{T}^n is the subgroup of diagonal matrices in $U(n)$. Below,

$$Q := \text{diag}(Q_1, \dots, Q_n) \tag{3.27}$$

stands for an element of \mathbb{T}^n , and $\text{Ad}_{Q^{-1}}$ denotes conjugation by Q^{-1} . For any $\gamma \in \mathbb{R}^*$, $(e^{2\gamma} \text{Ad}_{Q^{-1}} - \text{id})$ is an invertible linear operator on $\mathfrak{gl}(n, \mathbb{C})$, which preserves the subspace of Hermitian matrices. After this preparation, we present a useful characterization of \mathcal{M}_0 .

Proposition 3.8. *If $(Q, L, v) \in \mathcal{M}_0$ (3.26), then L can be expressed in terms of Q and v as follows:*

$$L = (e^{2\gamma} \text{Ad}_{Q^{-1}} - id)^{-1} \left(\sum_{\alpha=1}^d v(\alpha)v(\alpha)^\dagger \right). \tag{3.28}$$

For the matrix elements of L , this gives

$$L_{ij} = \frac{F_{ij}}{e^{2\gamma} Q_j Q_i^{-1} - 1} \quad \text{with} \quad F := \sum_{\alpha=1}^d v(\alpha)v(\alpha)^\dagger. \tag{3.29}$$

Conversely, if the Hermitian matrix L given by the formula (3.28) is positive definite, then $(Q, L, v) \in \mathcal{M}_0$.

Proof. If $g_R = Q \in \mathbb{T}^n$, then we have

$$b_L Q^{-1} = Q^{-1} Q b_L Q^{-1} = Q^{-1} b_R^{-1}, \tag{3.30}$$

showing that $b_L = Q^{-1} b_R^{-1} Q$. Therefore, on \mathcal{M}_0 the moment map Λ (3.1) reads

$$\Lambda(Q, L, v) = Q^{-1} b_R^{-1} Q b_R \mathbf{b}(w^1) \mathbf{b}(w^2) \cdots \mathbf{b}(w^d), \tag{3.31}$$

where b_R and the w^α are viewed as functions of L and v , given by the invertible relations of Eq. (3.25) and Definition 3.1. We substitute this into the following equivalent form of the moment map constraint (3.17),

$$\Lambda(Q, L, v) \Lambda(Q, L, v)^\dagger = e^{2\gamma} \mathbf{1}_n, \tag{3.32}$$

and thus obtain the requirement

$$(b_R b_1 b_2 \cdots b_d) (b_R b_1 b_2 \cdots b_d)^\dagger = e^{2\gamma} Q^{-1} L Q \quad \text{with} \quad b_\alpha = \mathbf{b}(w^\alpha). \tag{3.33}$$

By using Lemma 3.3 and the definitions of L and F , this in turn is equivalent to

$$e^{2\gamma} Q^{-1} L Q - L = F. \tag{3.34}$$

It follows that if $(Q, L, v) \in \mathcal{M}_0$, then L is given by the formula (3.28).

To deal with the converse statement, notice that L as given by the formula (3.28) is Hermitian and automatically satisfies (3.34), but its positive definiteness is a non-trivial condition on the pair (Q, v) . Suppose that L (3.28) is positive definite. Then, there exists a unique $b_R \in B(n)$ for which $L = b_R b_R^\dagger$. Defining $v^\alpha := b_R^{-1} v(\alpha)$ (cf. (3.3)), we can convert (3.34) into

$$e^{2\gamma} b_R^{-1} Q^{-1} b_R Q (b_R^{-1} Q^{-1} b_R Q)^\dagger = \mathbf{1}_n + \sum_{\alpha=1}^d v^\alpha (v^\alpha)^\dagger. \tag{3.35}$$

Then, there exists unique w^1, \dots, w^d from \mathbb{C}^n for which (3.3) holds, and by (3.2) and (3.7) these variables satisfy

$$\mathbf{1}_n + \sum_{\alpha=1}^d v^\alpha (v^\alpha)^\dagger = (\mathbf{b}(w^1) \cdots \mathbf{b}(w^d)) (\mathbf{b}(w^1) \cdots \mathbf{b}(w^d))^\dagger. \tag{3.36}$$

Inserting into (3.35), and using (3.31), we see that (3.35) implies the constraint Eq. (3.32), whereby the proof is complete. \square

We have the following consequence of Proposition 3.8.

Corollary 3.9. *Let $L(Q, v)$ be given by (3.28) and define the set*

$$\mathcal{P}_0 := \{(Q, v) \in \mathbb{T}^n \times \mathbb{C}^{n \times d} \mid L(Q, v) \text{ is positive definite}\}. \quad (3.37)$$

The formula (3.28) establishes a bijection between \mathcal{M}_0 (3.26) and \mathcal{P}_0 , which is an open subset of $\mathbb{T}^n \times \mathbb{C}^{n \times d}$.

Let us call Q regular if it belongs to

$$\mathbb{T}_{\text{reg}}^n := \{Q = \text{diag}(Q_1, \dots, Q_n) \mid Q_i \in \text{U}(1), Q_i \neq Q_j \ \forall i \neq j\}. \quad (3.38)$$

Define

$$\mathcal{M}_0^{\text{reg}} := \{(Q, L(Q, v), v) \in \mathcal{M}_0 \mid Q \in \mathbb{T}_{\text{reg}}^n\}, \quad (3.39)$$

where $L(Q, v)$ is specified by (3.28). Notice that any $g \in \text{U}(n)$ for which A_g (3.15) maps an element of $\mathcal{M}_0^{\text{reg}}$ to $\mathcal{M}_0^{\text{reg}}$ must belong to the normalizer $\mathcal{N}(n)$ of \mathbb{T}^n inside $\text{U}(n)$. Therefore, the quotient of the regular part of the constraint surface by $\text{U}(n)$, denoted $\mathcal{M}_{\text{red}}^{\text{reg}}$, can be identified as

$$\mathcal{M}_{\text{red}}^{\text{reg}} = \mathcal{M}_0^{\text{reg}} / \mathcal{N}(n), \quad (3.40)$$

where the quotient refers to the restriction of the obvious $\text{U}(n)$ action (3.15) to the subgroup $\mathcal{N}(n) < \text{U}(n)$.

Let us call $Q \in \mathbb{T}^n$ admissible if $Q \in \Xi_R^{-1}(\Lambda^{-1}(e^\gamma \mathbf{1}_n))$. In the next section, we present an alternative procedure for solving the moment map constraint (3.17), which will show that all elements of $\mathbb{T}_{\text{reg}}^n$ are admissible.

Remark 3.10. Equation (3.34) implies $(e^{2\gamma} - 1)\text{tr}(L) = \text{tr}(F)$, and this can hold for a nonzero real γ only if $\gamma > 0$ and $\text{tr}(F) > 0$, since L must be positive definite and $\text{tr}(F) \geq 0$ by the definition (3.29). This is why we assumed that $\gamma > 0$. It would be desirable to describe the elements of the set \mathcal{P}_0 (3.37) explicitly. For $d = 1$ the solution of this problem can be read off from [19]. On account of the next two observations, we expect that the structure of \mathcal{P}_0 is very different for $d < n$ and for $d \geq n$. First, let us notice that $Q = \mathbf{1}_n$ is not admissible if $d < n$, since in this case the rank of L given by the formula (3.28) is at most d , while the rank of any positive definite L is n . Second, note that if $d \geq n$, then we can arrange to have $F = \mathbf{1}_n$ by suitable choice of v . Let v_0 be such a choice. Then, $L(\mathbf{1}_n, v_0)$ is a positive multiple of $\mathbf{1}_n$, and therefore, there is an open neighborhood of $(\mathbf{1}_n, v_0)$ in $\mathbb{T}^n \times \mathbb{C}^{n \times d}$ that belongs to \mathcal{P}_0 .

3.3. Solution of the Constraint in Terms of Q, p and Primary Spins

Now we return to using the variables g_R, b_R and W (2.35) for labeling the points of \mathcal{M} . We can uniquely decompose every element $b \in \text{B}(n)$ as the product of a diagonal matrix, b_0 , and an upper triangular matrix, b_+ , with unit diagonal. Applying this to $b = b_R$, we write

$$b_R = b_0 b_+, \quad (3.41)$$

and introduce also

$$S(W) := \mathbf{b}(w^1)\mathbf{b}(w^2) \cdots \mathbf{b}(w^d) =: S_0(W)S_+(W). \tag{3.42}$$

The moment map constraint on \mathcal{M}_0 (3.26) reads

$$\Lambda(Q, b_R, W) = Q^{-1}b_R^{-1}Qb_RS(W) = Q^{-1}b_+^{-1}Qb_+S_0(W)S_+(W) = e^\gamma \mathbf{1}_n. \tag{3.43}$$

Since b_0 drops out from the formula of Λ , it is left arbitrary, and we parametrize it as

$$b_0 = e^p \quad \text{with} \quad p = \text{diag}(p_1, \dots, p_n), \quad p_i \in \mathbb{R}. \tag{3.44}$$

A crucial observation is that (3.43) can be separated according to the diagonal and strictly upper-triangular parts, since it is equivalent to the two requirements

$$S_0(W) = e^\gamma \mathbf{1}_n \tag{3.45}$$

and

$$Q^{-1}b_+^{-1}Qb_+S_+(W) = \mathbf{1}_n. \tag{3.46}$$

The constraint (3.45) is responsible for a reduction of the primary spin variables. Next we make a little detour and present a general analysis of such reductions.

Let us introduce the map $\phi : \mathbb{C}^{n \times d} \rightarrow \mathfrak{b}(n)_0$ by writing

$$S_0(W) := \exp(\phi(W)), \tag{3.47}$$

and notice from Remark A.5 that ϕ is the moment map for the ordinary Hamiltonian action of \mathbb{T}^n on the symplectic manifold $(\mathbb{C}^{n \times d}, \Omega_{\mathcal{W}})$ of the primary spins. Here, the dual of the Lie algebra of the torus $\mathbb{T}^n < U(n)$ is identified with the space $\mathfrak{b}(n)_0$ of real diagonal matrices. The torus action in question is given by

$$\tau \cdot (w^1, \dots, w^d) = (\tau w^1, \dots, \tau w^d), \quad \forall \tau \in \mathbb{T}^n. \tag{3.48}$$

Taking any moment map value from the range of ϕ ,

$$\Gamma := \text{diag}(\gamma_1, \dots, \gamma_n), \tag{3.49}$$

we define the reduced space of primary spins:

$$\mathbb{C}_{\text{red}}^{n \times d}(\Gamma) := \phi^{-1}(\Gamma)/\mathbb{T}^n. \tag{3.50}$$

Proposition 3.11. *The moment map $\phi : \mathbb{C}^{n \times d} \rightarrow \mathfrak{b}(n)_0$ defined by (3.47) with (3.42) is proper, i.e., the inverse image of any compact set is compact. Fixing any moment map value Γ for which $\gamma_j > 0$ for all j , the reduced spin-space (3.50) is a smooth, compact and connected symplectic manifold of dimension $2n(d - 1)$.*

Proof. We first prove that the map ϕ is proper. Since the compact sets of Euclidean spaces are the bounded and closed sets, and since ϕ is continuous, it is enough to show that the inverse image of any bounded subset of $\mathfrak{b}(n)_0 \simeq \mathbb{R}^n$ is a bounded subset of $\mathbb{C}^{n \times d}$. Due to the definition of ϕ and Eq. (2.31), the formula of $\phi = \text{diag}(\phi_1, \dots, \phi_n)$ is determined by the equality

$$\exp(2\phi_j(W)) = \prod_{\alpha=1}^d \frac{\mathcal{G}_j(w^\alpha)}{\mathcal{G}_{j+1}(w^\alpha)} = \prod_{\alpha=1}^d \left[1 + \frac{|w_j^\alpha|^2}{\mathcal{G}_{j+1}(w^\alpha)} \right]. \tag{3.51}$$

The second equality shows that $\phi_j(W) \geq 0$. On the other hand, using the first equality and that $\mathcal{G}_{n+1} = 1$, we see that

$$W \in \phi^{-1}(\text{diag}(\gamma_1, \dots, \gamma_n)) \tag{3.52}$$

if and only if

$$\prod_{\alpha=1}^d \mathcal{G}_j(w^\alpha) = \exp\left(2 \sum_{k=j}^n \gamma_k\right), \quad \forall j = 1, \dots, n. \tag{3.53}$$

Now, if $\text{diag}(\gamma_1, \dots, \gamma_n)$ is from a bounded set, then $\sum_{k=1}^n \gamma_k \leq C$ with some constant C . By using this and the $j = 1$ special case of (3.53), we obtain

$$1 + \sum_{\alpha=1}^d |w^\alpha|^2 \leq \prod_{\alpha=1}^d (1 + |w^\alpha|^2) = \prod_{\alpha=1}^d \mathcal{G}_1(w^\alpha) \leq e^{2C}, \tag{3.54}$$

which implies that the inverse image of any bounded set is bounded.

If $\gamma_j > 0$ for all j , then we see from the formula (3.51) that for any $W \in \Phi^{-1}(\Gamma)$ and for each $1 \leq j \leq n$ there must exist an index $1 \leq \alpha(j) \leq d$ such that

$$w_j^{\alpha(j)} \neq 0. \tag{3.55}$$

This implies immediately that the action (3.48) of \mathbb{T}^n is free on $\phi^{-1}(\Gamma)$, and therefore $\mathbb{C}_{\text{red}}^{n \times d}(\Gamma)$ (3.50) is a smooth symplectic manifold of dimension $2n(d-1)$. Since the moment map ϕ is proper, all its fibers $\phi^{-1}(\Gamma)$ are compact, and by Theorem 4.1 in [26] they are also connected. Hence, $\mathbb{C}_{\text{red}}^{n \times d}(\Gamma)$ is also compact and connected. \square

Remark 3.12. If $\gamma_j > 0$ for all j , then $\mathbb{C}_{\text{red}}^{n \times d}(\Gamma)$ is actually a *real-analytic* symplectic manifold. To cover it with charts, for any map $\mu : \{1, \dots, n\} \rightarrow \{1, \dots, d\}$ we introduce the set

$$X(\mu) := \{W \in \phi^{-1}(\Gamma) \mid w_j^{\mu(j)} \neq 0, \forall j\}. \tag{3.56}$$

Then, the reduced spin-space is the union of the open subsets $Y(\mu) := X(\mu)/\mathbb{T}^n$, and a model of $Y(\mu)$ is provided by

$$Z(\mu) := \{W \in \phi^{-1}(\Gamma) \mid w_j^{\mu(j)} > 0, \forall j\}. \tag{3.57}$$

One can specify coordinates on $Z(\mu)$ by solving the constraints (3.53) for the $w_j^{\mu(j)}$ in terms of the remaining free variables, the w_j^α with $\alpha \neq \mu(j)$, which

take their values in a certain open subset of $\mathbb{C}^{n(d-1)}$. It is an interesting exercise to fill out the details and to also write down the reduced symplectic form by using these charts.

If $d = 1$, then the reduced spin-space consists of a single point. This is also true in the trivial case for which $\gamma_j = 0$ for all j . If some of the γ_j are zero and the others are positive, then the moment map constraint $\phi(W) = \Gamma$ leads to a stratified symplectic space. Finally, note that for the case corresponding to Eq. (3.45) $\gamma_j = \gamma > 0$ for all j .

Now returning to our main problem, it is useful to recast (3.46) in the form

$$b_+ S_+(W) = Q^{-1} b_+ Q. \tag{3.58}$$

By using the principal gradation of $n \times n$ matrices, this equation can be solved recursively for b_+ if $S_+(W)$ and Q are given, with Q regular. In fact, the following lemma is obtained by a word-by-word application of the arguments of Section 5 in [15]; hence, we omit the proof.

Lemma 3.13. *Suppose that $S_+ = S_+(W)$ and Q are given, with $Q \in \mathbb{T}_{\text{reg}}^n$. Then, Eq. (3.58) admits a unique solution for b_+ , denoted $b_+(Q, W)$. Using the notation*

$$\mathcal{I}^{a,a+j} = \frac{1}{Q_{a+j} Q_a^{-1} - 1}, \quad a = 1, \dots, n-1, \tag{3.59}$$

and placing the matrix indices in the upstairs position, we have

$$b_+^{a,a+1} = \mathcal{I}^{a,a+1} S_+^{a,a+1}, \tag{3.60}$$

and for $k = 2, \dots, n-a$ we have

$$b_+^{a,a+k} = \mathcal{I}^{a,a+k} S_+^{a,a+k} + \sum_{\substack{m=2,\dots,k \\ (i_1,\dots,i_m) \in \mathbb{N}^m \\ i_1+\dots+i_m=k}} \prod_{\alpha=1}^m \mathcal{I}^{a,a+i_1+\dots+i_\alpha} S_+^{a+i_1+\dots+i_{\alpha-1},a+i_1+\dots+i_\alpha}. \tag{3.61}$$

Now we restrict ourselves to the regular part of \mathcal{M}_0 , stressing that it is defined without reference to any particular parametrization:

$$\mathcal{M}_0^{\text{reg}} \equiv \Lambda^{-1}(e^\gamma \mathbf{1}_n) \cap \Xi_R^{-1}(\mathbb{T}_{\text{reg}}^n). \tag{3.62}$$

Any gauge transformation that maps an element of $\mathcal{M}_0^{\text{reg}}$ to $\mathcal{M}_0^{\text{reg}}$ is given by the obvious action (3.15) of the normalizer $\mathcal{N}(n)$ of \mathbb{T}^n inside $U(n)$. The normalizer has the normal subgroup \mathbb{T}^n , and the corresponding factor group is the permutation group

$$S_n = \mathcal{N}(n)/\mathbb{T}^n. \tag{3.63}$$

Consequently, we have

$$\mathcal{M}_{\text{red}}^{\text{reg}} = \mathcal{M}_0^{\text{reg}}/\mathcal{N}(n) = (\mathcal{M}_0^{\text{reg}}/\mathbb{T}^n)/S_n. \tag{3.64}$$

It is plain that $\mathcal{M}_{\text{red}}^{\text{reg}}$ is a dense, open subset of the reduced phase space, and the above consecutive quotients show that $\mathcal{M}_0^{\text{reg}}/\mathbb{T}^n$ is an S_n covering space⁴ of this dense open subset.

Theorem 3.14. *By solving the moment map constraint for b_R in the form $b_R = e^p b_+(Q, W)$ as explained above, the manifold $\mathcal{M}_0^{\text{reg}}$ (3.62) can be identified with the model space*

$$\tilde{\mathcal{P}}_0^{\text{reg}} := \mathbb{T}_{\text{reg}}^n \times \mathfrak{b}(n)_0 \times \phi^{-1}(\Gamma) = \{(Q, p, W) \mid Q \in \mathbb{T}_{\text{reg}}^n, p \in \mathfrak{b}(n)_0, W \in \phi^{-1}(\Gamma)\}, \tag{3.65}$$

where $\Gamma = \gamma \mathbf{1}_n$. Utilizing this model, the covering space $\mathcal{M}_0^{\text{reg}}/\mathbb{T}^n$ of the regular part of the reduced phase space becomes identified with the symplectic manifold

$$T^*\mathbb{T}_{\text{reg}}^n \times \mathbb{C}_{\text{red}}^{n \times d}(\Gamma) \tag{3.66}$$

equipped with its natural product symplectic structure.

Proof. The restriction of the action (3.15) to \mathbb{T}^n translates into the action

$$A_\tau(Q, p, W) = (Q, p, \tau \cdot W), \quad \tau \in \mathbb{T}^n, \tag{3.67}$$

on the model space $\tilde{\mathcal{P}}_0^{\text{reg}}$, from which we obtain the identification $\mathcal{M}_0^{\text{reg}}/\mathbb{T}^n \simeq T^*\mathbb{T}_{\text{reg}}^n \times \mathbb{C}_{\text{red}}^{n \times d}(\Gamma)$ at the level of manifolds. Let $\xi_1 : \tilde{\mathcal{P}}_0^{\text{reg}} \rightarrow \mathcal{M}$ and $\xi_2 : \phi^{-1}(\Gamma) \rightarrow \mathbb{C}^{n \times d}$ denote the natural inclusions, and write $Q_j = e^{iq_j}$. Then, a simple calculation gives

$$\xi_1^*(\Omega_{\mathcal{M}}) = \sum_{j=1}^n dp_j \wedge dq_j + \xi_2^*(\Omega_W), \tag{3.68}$$

which proves the claimed identification at the level of *symplectic* manifolds. \square

Corollary 3.15. *The dense open submanifold $\mathcal{M}_{\text{red}}^{\text{reg}} \subseteq \mathcal{M}_{\text{red}}$ is connected, and consequently, \mathcal{M}_{red} is also connected.*

Proof. Since $\phi^{-1}(\Gamma)$ is connected by Proposition 3.11, the connected components of $\tilde{\mathcal{P}}_0^{\text{reg}}$ correspond to the connected components of $\mathbb{T}_{\text{reg}}^n$. It is well known (see, e.g., ‘‘Appendix A’’) that any two connected components of $\mathbb{T}_{\text{reg}}^n$ are related by permutations. Thus, $\mathcal{M}_{\text{red}}^{\text{reg}}$ is the continuous image of a single connected component of $\tilde{\mathcal{P}}_0^{\text{reg}}$, implying its connectedness. The proof is finished by recalling that if a dense open subset of a topological space is connected, then the space itself is connected. \square

Remark 3.16. In this long remark, we explain how the (trigonometric real form of the) spin Sutherland model of Gibbons and Hermsen [24] can be obtained from our construction via a scaling limit. For this, we introduce a positive parameter ϵ and replace the variables p by ϵp and W by $\epsilon^{\frac{1}{2}}W$, while keeping Q unchanged. The formulae (2.31) imply

$$\mathbf{b}(\epsilon^{\frac{1}{2}}w)_{kk} = 1 + \frac{1}{2}\epsilon|w_k|^2 + o(\epsilon), \quad \forall k \quad \text{and} \quad \mathbf{b}(\epsilon^{\frac{1}{2}}w)_{ij} = \epsilon w_i \bar{w}_j + o(\epsilon), \quad \forall i < j.$$

⁴More precisely, $\mathcal{M}_0^{\text{reg}}/\mathbb{T}^n$ is a principal fiber bundle with structure group S_n over the base $\mathcal{M}_{\text{red}}^{\text{reg}}$.

$$(3.69)$$

By using this, we see that the matrix $b = b_+$ in (3.41), given in explicit form by Lemma 3.13, has the expansion

$$b(Q, p, \epsilon^{\frac{1}{2}}W)_{ij} = \epsilon(Q_j Q_i^{-1} - 1)^{-1} \sum_{\alpha=1}^d w_i^\alpha \bar{w}_j^\alpha + o(\epsilon), \quad \forall i < j. \quad (3.70)$$

Then, for $L = b_R b_R^\dagger$ with $b_R = \exp(\epsilon p)b(Q, \epsilon p, \epsilon^{\frac{1}{2}}W)$, we find

$$\text{tr}(L^{\pm 1}) = n \pm 2\epsilon \text{tr}(p) + 2\epsilon^2 \text{tr}(p^2) + \epsilon^2 \sum_{i < j} \frac{|(w_i^\bullet, w_j^\bullet)|^2}{|Q_j Q_i^{-1} - 1|^2} + o(\epsilon^2), \quad (3.71)$$

where $w_i^\bullet \in \mathbb{C}^d$ with components w_i^α , and $(w_i^\bullet, w_j^\bullet) := \sum_{\alpha=1}^d w_i^\alpha \bar{w}_j^\alpha$. Therefore, we obtain

$$\lim_{\epsilon \rightarrow 0} \frac{1}{8\epsilon^2} (\text{tr}(L) + \text{tr}(L^{-1}) - 2n) = \frac{1}{2} \text{tr}(p^2) + \frac{1}{32} \sum_{i \neq j} \frac{|(w_i^\bullet, w_j^\bullet)|^2}{\sin^2 \frac{q_i - q_j}{2}}, \quad (3.72)$$

which is just the standard Hamiltonian of the (real, trigonometric) Gibbons–Hershen model.

Replacing γ by $-\epsilon\gamma$ and taking the limit, the residual constraint (3.45) gives $(w_j^\bullet, w_j^\bullet) = 2\gamma$. Then, rescaling not only the variables but also the symplectic form (3.68), one gets

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} (\xi^* \Omega_{\mathcal{M}}) = \sum_{j=1}^n dp_j \wedge dq_j + \frac{i}{2} \sum_{j=1}^n \sum_{\alpha=1}^d dw_j^\alpha \wedge d\bar{w}_j^\alpha, \quad (3.73)$$

which reproduces the symplectic form of the Gibbons–Hershen model.

It is known [19] that the standard spinless RS Hamiltonian [50] can be derived as the reduction of $\text{tr}(L) + \text{tr}(L^{-1})$ in the $d = 1$ case. For $d \geq 2$, we shall show (see Corollary 4.3 and Corollary 5.9) that the Hamiltonian of the (real, trigonometric) spin RS model of Krichever of Zabrodin [32] is the reduction of $\text{tr}(L)$. As was already discussed in Introduction, the term *chiral* spin RS model could have been a more fitting name for the model of [32], but we follow the literature in dropping ‘chiral’ in this context.

In this subsection, we have derived an almost complete description of the reduced system. We have established that $T^* \mathbb{T}_{\text{reg}}^n \times \mathbb{C}_{\text{red}}^{n \times d}(\Gamma)$ is an S_n covering space of a dense, open subset of the reduced phase space, and we can write down the Hamiltonians $\text{tr}(L^k)$ by using the explicit formula $b_R = e^p b_+(Q, W)$. Why is the paper not finished at this stage? Well, one reason is that although Q, p and W are very nice variables for presenting the reduced symplectic form, they are not the ones that feature in the Krichever–Zabrodin equations of motion, which we wish to reproduce in our setting. In fact, the usage of the dressed spins $v(\alpha)$ (3.3) will turn out indispensable for this purpose. (Notationwise, we took this into account already in Eq. (1.8).) Another, closely related, reason is

that the action of permutations is practically intractable in terms of the primary spins.⁵ More precisely, the action on the components of Q is the obvious one, but on p and W it is known only in an implicit manner, via the realization of these variables as functions of $L = b_R b_R^\dagger$ and the dressed spins, on which the full $U(n)$ action, and thus also the permutation action, is governed by the simple formula (3.15). In short, both the formula $b_+(Q, W)$ and the change of variables from Q, p, W to Q and dressed spins $v(\alpha)$ are complicated, and for some purposes the latter will prove to be more convenient variables.

4. The Reduced Equations of Motion

Below, we first present a characterization of the projection of the Hamiltonian vector field of arbitrary elements of \mathfrak{H} (2.40) to the dense open submanifold $\mathcal{M}_{\text{red}}^{\text{reg}}$ (3.40) of the reduced phase space \mathcal{M}_{red} (3.18). Then, we reproduce the trigonometric real form of the equations of motion (1.2) of [32] as the simplest special case of the reduced dynamics.

As in Sect. 3.2, we parametrize the points of \mathcal{M} by g_R, v and $L = b_R b_R^\dagger$. For any $h \in C^\infty(\mathbb{B}(n))^{\text{U}(n)}$ we put

$$\mathcal{V}(L) := Dh(b_R). \quad (4.1)$$

Denoting the Hamiltonian vector field of $H = \Lambda_R^*(h)$ by X_H and viewing $g_R, v(\alpha)$ and L as evaluation functions, in correspondence to (2.41), we have

$$X_H[g_R] = \mathcal{V}(L)g_R, \quad X_H[v(\alpha)] = 0, \quad X_H[L] = 0. \quad (4.2)$$

It is clear that X_H admits a well-defined projection on \mathcal{M}_{red} , which encodes the reduced dynamics. Of course, one may add any infinitesimal gauge transformation to the vector field X_H without modifying its projection on \mathcal{M}_{red} , i.e., instead of X_H one may equally well consider any Y_H of the form

$$\begin{aligned} Y_H[g_R] &= \mathcal{V}(L)g_R + [Z(g_R, L, v), g_R], \\ Y_H[v(\alpha)] &= Z(g_R, L, v)v(\alpha), \\ Y_H[L] &= [Z(g_R, L, v), L], \end{aligned} \quad (4.3)$$

with arbitrary $Z(g_R, L, v) \in \mathfrak{u}(n)$. It is also clear that one may use the restriction of Y_H to \mathcal{M}_0 (3.26) for determining the projection, and Z can be chosen in such a manner to guarantee the tangency of the restricted vector field to \mathcal{M}_0 .

Let us consider the vector space decomposition

$$\mathfrak{u}(n) = \mathfrak{u}(n)_0 + \mathfrak{u}(n)_\perp, \quad (4.4)$$

where $\mathfrak{u}(n)_0$ and $\mathfrak{u}(n)_\perp$ consist of diagonal and off-diagonal matrices, respectively. Accordingly, for any $T \in \mathfrak{u}(n)$ we have

$$T = T_0 + T_\perp, \quad T = \mathfrak{u}(n)_0, \quad T_\perp \in \mathfrak{u}(n)_\perp. \quad (4.5)$$

⁵This difficulty evaporates in the scaling limit discussed in Remark 3.16.

Using $\text{Ad}_Q(T) = QTQ^{-1}$, the restriction of the operator $(\text{Ad}_Q - \text{id})$ to $\mathfrak{u}(n)_\perp$ is invertible, and we define

$$K(Q, L) := ((\text{Ad}_Q - \text{id})|_{\mathfrak{u}(n)_\perp})^{-1} \mathcal{V}(L)_\perp. \tag{4.6}$$

More explicitly, setting $Q = \exp(iq)$, we have $K_{kk} = 0$ and

$$K(Q, L)_{kl} = -\frac{1}{2}\mathcal{V}(L)_{kl} - \frac{i}{2}\mathcal{V}(L)_{kl} \cot\left(\frac{q_k - q_l}{2}\right), \quad \forall k \neq l. \tag{4.7}$$

Proposition 4.1. *For any $H \in \mathfrak{H}$ (2.40), applying the previous notations, the following formulae yield a vector field Y_H^0 on $\mathcal{M}_0^{\text{reg}}$ (3.39),*

$$\begin{aligned} Y_H^0[Q] &= \mathcal{V}(L)_0 Q, \\ Y_H^0[v(\alpha)] &= (K(Q, L) + \mathcal{Z}(Q, L, v))v(\alpha), \\ Y_H^0[L] &= [K(Q, L) + \mathcal{Z}(Q, L, v), L], \end{aligned} \tag{4.8}$$

for any $\mathcal{Z}(Q, L, v) \in \mathfrak{u}(n)_0$ (4.4), with $L = L(Q, v)$ given by (3.28). The vector field Y_H^0 admits a well-defined projection on $\mathcal{M}_{\text{red}}^{\text{reg}} \subset \mathcal{M}_{\text{red}}$ (3.40), which coincides with the corresponding restriction of the projection of the Hamiltonian vector field X_H (4.2) to \mathcal{M}_{red} .

Proof. As a consequence of $(\text{Ad}_Q - \text{id})K(Q, L) = \mathcal{V}(L)_\perp$, we have the identity

$$\mathcal{V}(L)Q + [K(Q, L), Q] = \mathcal{V}(L)_0 Q. \tag{4.9}$$

This shows that Y_H^0 is obtained by restricting Y_H (4.3) to $\mathcal{M}_0^{\text{reg}}$, where

$$Z(Q, L, v) = K(Q, L) + \mathcal{Z}(Q, L, v). \tag{4.10}$$

This choice of Z guarantees that the restricted vector field is tangent to $\mathcal{M}_0^{\text{reg}}$. The fact that \mathcal{Z} is left undetermined reflects the residual $\mathcal{N}(n)$ gauge transformations acting on $\mathcal{M}_0^{\text{reg}}$. \square

Remark 4.2. Only the first two relations in (4.8) are essential, since the third one follows from them via the formula (3.28) of $L = L(Q, v)$. Now take an initial value $(Q^0, L^0, v^0) \in \mathcal{M}_0^{\text{reg}}$ and $\epsilon > 0$ ($\epsilon = \infty$ is allowed) such that

$$g_R(t) = \exp(t\mathcal{V}(L^0))Q^0 \in \text{U}(n)_{\text{reg}} \quad \text{for} \quad -\epsilon < t < \epsilon, \tag{4.11}$$

where the elements of $\text{U}(n)_{\text{reg}}$ have n distinct eigenvalues. Notice from (2.42) that $g_R(t)$ describes the unreduced solution curve and that a small enough ϵ will certainly do. Then, for $-\epsilon < t < \epsilon$ there exists a unique smooth curve $\eta(t) \in \text{U}(n)$ for which

$$Q(t) := \eta(t)g_R(t)\eta(t)^{-1} \in \mathbb{T}_{\text{reg}}^n \quad \text{and} \quad \eta(0) = \mathbf{1}_n, \quad (\dot{\eta}(t)\eta(t)^{-1})_0 = 0. \tag{4.12}$$

It is easy to see that $(Q(t), L(t), v(\alpha)(t))$ given by the above $Q(t)$ and

$$L(t) = \eta(t)L^0\eta(t)^{-1}, \quad v(\alpha)(t) = \eta(t)v(\alpha)^0 \tag{4.13}$$

yields the integral curve of the vector field (4.8) with $\mathcal{Z} = 0$. We here used the property $\mathcal{V}(gLg^{-1}) = g\mathcal{V}(L)g^{-1}$ ($\forall g \in \text{U}(n), L \in \mathfrak{P}(n)$), which follows from the definition (4.1). The auxiliary conditions imposed in (4.12) fix the ambiguity of the ‘diagonalizer’ $\eta(t)$ of $g_R(t)$. The reduction approach leads

to this solution algorithm naturally, but we should stress that an analogous algorithm was found long ago by Ragnisco and Suris [43] using a direct method.

Corollary 4.3. *Consider $H \in \mathfrak{H}$ defined by*

$$H = \Lambda_R^*(h) \quad \text{with} \quad h(b) := (e^{2\gamma} - 1)\text{tr}(bb^\dagger). \tag{4.14}$$

Then, the evolution equation on $\mathcal{M}_0^{\text{reg}}$ corresponding to the vector field Y_H^0 (4.8) with $\mathcal{Z} = 0$ can be written explicitly as follows:

$$\frac{1}{2}\dot{q}_j := \frac{1}{2i}Y_H^0[Q_j]Q_j^{-1} = F_{jj}, \tag{4.15}$$

$$\dot{v}(\alpha)_i := Y_H^0[v(\alpha)_i] = - \sum_{j \neq i} F_{ij}v(\alpha)_j V \left(\frac{q_j - q_i}{2} \right), \tag{4.16}$$

where $F = \sum_\alpha v(\alpha)v(\alpha)^\dagger$ and the ‘potential function’ V reads

$$V(x) = \cot x - \cot(x - i\gamma). \tag{4.17}$$

These formulae reproduce the spin RS equations of motion (1.2) by setting $x_i = q_i/2$ and imposing the additional reality condition (1.4).

Proof. In this case the definition (4.1) gives

$$\mathcal{V}(L) = 2i(e^{2\gamma} - 1)L. \tag{4.18}$$

Equation (4.15) follows immediately from (4.8) since $\mathcal{V}(L)_{jj} = 2iF_{jj}$ by (3.28) and we have $Q_j = \exp(iq_j)$. Taking advantage of Hermite’s cotangent identity,

$$\cot(z - a_1) \cot(z - a_2) = -1 + \cot(a_1 - a_2) \cot(z - a_1) + \cot(a_2 - a_1) \cot(z - a_2), \tag{4.19}$$

it is not difficult to re-cast the off-diagonal matrix function $K(Q, L(Q, v))$ (4.7) in the form

$$K_{kl} = F_{kl} \left[\cot \left(\frac{q_k - q_l}{2} \right) - \cot \left(\frac{q_k - q_l}{2} + i\gamma \right) \right], \tag{4.20}$$

for all $k \neq l$. This gives (4.16) with (4.17). The validity of the last sentence of the corollary can also be checked directly. □

Remark 4.4. The restriction of the Hamiltonian H (4.14) to \mathcal{M}_0 gives

$$H(Q, v) = (e^{2\gamma} - 1)\text{tr}(L(Q, v)) = \sum_{j=1}^n F_{jj}. \tag{4.21}$$

We shall confirm in Sect. 5.2 that the ensuing reduced Hamiltonian generates the equations of motion (1.8) via the reduced Poisson structure described in coordinates using the gauge fixing condition (1.7).

5. The Reduced Poisson Structure

The main purpose of this section is to present the explicit form of the reduced Poisson structure in terms of the variables that feature in the equations of motion (1.8). The first subsection contains a couple of auxiliary lemmata, in which we provide explicit formulae for the Poisson brackets of the half-dressed and dressed spins, and the matrix entries of g_R and L . These permit us to establish that the $U(n)$ invariant integrals of motion (3.24) form a closed polynomial Poisson algebra on the unreduced phase space, which automatically descends to the reduced phase space. This interesting algebra is given by Proposition 5.5. In the second subsection, we utilize the Poisson brackets of another set of $U(n)$ invariant functions in order to characterize the reduced Poisson structure. We shall rely on the fact that the restriction of the Poisson brackets of $U(n)$ invariant functions to a gauge slice in the ‘constraint surface’ must coincide with the Poisson brackets of the restricted functions calculated from the reduced Poisson structure.

All calculations required by this section are straightforward, but they are quite voluminous and not enlightening. We strive to give just enough details to provide the gist of these calculations, and so that an interested reader may reproduce them. Some of these details are relegated to “Appendix B and C.”

5.1. Some Poisson Brackets Before Reduction

Using the results from Sect. 2, the Poisson structure $\{ , \}_{\mathcal{M}}$ on \mathcal{M} can be described in terms of the (complex-valued) functions returning the entries of the matrices $(g_R, b_R, w^1, \dots, w^d)$ and their complex conjugates. Namely, we can use (2.7)–(2.8) with $K = g_R$ or $K = b_R$, then (2.18)–(2.19) to characterize the Poisson structure restricted to functions on the Heisenberg double; for fixed $\alpha = 1, \dots, d$, the Poisson brackets involving w^α are given by (2.28)–(2.29). The Poisson brackets between functions of w^α and functions of g_R and b_R vanish. Our aim is to translate these relations to the matrices $(g_R, L = b_R b_R^\dagger, v(1) = b_R v^1, \dots, v(d) = b_R v^d)$, which are more convenient to understand the reduced phase space \mathcal{M}_{red} , see Sect. 3. As a first step, we express the Poisson structure on the half-dressed spins $v^\alpha = b_1 \cdots b_{\alpha-1} w^\alpha$ defined in (3.3). We let $\{ , \} := \{ , \}_{\mathcal{M}}$ for the rest of the section.

Lemma 5.1. *The Poisson brackets of the half-dressed spins are given by the following formulae*

$$\{v_i^\alpha, v_k^\beta\} = -i \operatorname{sgn}(k - i) v_k^\alpha v_i^\beta + i \operatorname{sgn}(\beta - \alpha) v_k^\alpha v_i^\beta, \tag{5.1}$$

$$\{v_i^\alpha, \bar{v}_k^\beta\} = i \delta_{ik} v_i^\alpha \bar{v}_k^\beta + 2i \delta_{ik} \sum_{r>k} v_r^\alpha \bar{v}_r^\beta + i \delta_{\alpha\beta} v_i^\alpha \bar{v}_k^\beta + 2i \delta_{\alpha\beta} \sum_{\mu<\alpha} v_i^\mu \bar{v}_k^\mu + 2i \delta_{ik} \delta_{\alpha\beta}, \tag{5.2}$$

where $1 \leq i, k \leq n$ and $1 \leq \alpha, \beta \leq d$. In particular, this defines a Poisson structure on $\mathbb{C}^{nd} \simeq \mathbb{R}^{2nd}$.

This result is proved in Appendix B. From the reality of the Poisson bracket, we have

$$\{\bar{v}_i^\alpha, \bar{v}_k^\beta\} = +i \operatorname{sgn}(k - i) \bar{v}_k^\alpha \bar{v}_i^\beta - i \operatorname{sgn}(\beta - \alpha) \bar{v}_k^\alpha \bar{v}_i^\beta. \tag{5.3}$$

Remark 5.2. If we complexify the formulae of Lemma 5.1 by introducing $a_{i\alpha} := (v_i^\alpha)^\mathbb{C}$ and $b_{\alpha i} := (\bar{v}_i^\alpha)^\mathbb{C}$, we get a complex holomorphic Poisson structure on \mathbb{C}^{2nd} given by

$$\{a_{i\alpha}, a_{k\beta}\} = -i \operatorname{sgn}(k - i) a_{k\alpha} a_{i\beta} + i \operatorname{sgn}(\beta - \alpha) a_{k\alpha} a_{i\beta}, \tag{5.4}$$

$$\{b_{\alpha i}, b_{\beta k}\} = +i \operatorname{sgn}(k - i) b_{\alpha k} b_{\beta i} - i \operatorname{sgn}(\beta - \alpha) b_{\alpha k} b_{\beta i}, \tag{5.5}$$

$$\begin{aligned} \{a_{i\alpha}, b_{\beta k}\} &= i\delta_{ik} a_{i\alpha} b_{\beta k} + 2i\delta_{ik} \sum_{r>k} a_{r\alpha} b_{\beta r} + i\delta_{\alpha\beta} a_{i\alpha} b_{\beta k} \\ &\quad + 2i\delta_{\alpha\beta} \sum_{\mu<\alpha} a_{i\mu} b_{\mu k} + 2i\delta_{ik} \delta_{\alpha\beta}. \end{aligned} \tag{5.6}$$

After appropriate rescaling, this reproduces the *minus* Poisson bracket introduced by Arutyunov and Olivucci in their treatment of the complex holomorphic spin RS system by Hamiltonian reduction [5]. Considering the analogous construction with the variables $v_{+,i}^\alpha := v_i^{d-\alpha+1}$ instead, we obtain the *plus* Poisson bracket introduced in [5].

From now on, we let $b = b_R, g = g_R$. Using Lemma 5.1 and the Poisson structure of the Heisenberg double, we can easily write the Poisson brackets involving the entries $v(\alpha)_i$ of the dressed spins $v(\alpha) = b_R v^\alpha$.

Lemma 5.3. *The Poisson brackets of the dressed spins are given by the following formulae*

$$\{v(\alpha)_i, v(\beta)_k\} = -i \operatorname{sgn}(k - i) v(\alpha)_k v(\beta)_i + i \operatorname{sgn}(\beta - \alpha) v(\alpha)_k v(\beta)_i, \tag{5.7}$$

$$\begin{aligned} \{v(\alpha)_i, \bar{v}(\beta)_k\} &= i\delta_{ik} v(\alpha)_i \bar{v}(\beta)_k + 2i\delta_{ik} \sum_{r>k} v(\alpha)_r \bar{v}(\beta)_r + i\delta_{\alpha\beta} v(\alpha)_i \bar{v}(\beta)_k \\ &\quad + 2i\delta_{\alpha\beta} \sum_{\mu<\alpha} v(\mu)_i \bar{v}(\mu)_k + 2i\delta_{\alpha\beta} (bb^\dagger)_{ik}. \end{aligned} \tag{5.8}$$

The Poisson brackets of the dressed spins and the matrices b, g are given by the following formulae

$$\{v(\alpha)_i, g_{kl}\} = -i\delta_{ik} v(\alpha)_i g_{kl} - 2i\delta_{(i<k)} v(\alpha)_k g_{il}, \tag{5.9}$$

$$\{\bar{v}(\beta)_i, g_{kl}\} = -i\delta_{ik} \bar{v}(\beta)_i g_{kl} - 2i\delta_{ik} \sum_{r>i} \bar{v}(\beta)_r g_{rl}, \tag{5.10}$$

$$\{v(\alpha)_i, b_{kl}\} = 2i\delta_{(k<i)} v(\alpha)_k b_{il} + i\delta_{ik} v(\alpha)_k b_{il} - 2i \sum_{s<l} b_{ks} v_s^\alpha b_{il} - i b_{kl} v_l^\alpha b_{il}, \tag{5.11}$$

$$\{\bar{v}(\beta)_i, b_{kl}\} = -i\delta_{ik}\bar{v}(\beta)_i b_{kl} - 2i\delta_{ik} \sum_{r>k} \bar{v}(\beta)_r b_{rl} + i\bar{b}_{il}\bar{v}_i^\beta b_{kl} + 2i \sum_{s<l} \bar{b}_{is}\bar{v}_i^\beta b_{ks}. \tag{5.12}$$

Finally, we can express the Poisson structure in terms of the elements $(g, L, v(1), \dots, v(d))$ where $L = bb^\dagger$. This is a direct application of Lemma 5.3 and the relations (2.7)–(2.8), (2.18)–(2.19) of the Heisenberg double by using that $L_{kl} = \sum_r b_{ks}\bar{b}_{ls}$. Alternatively, one may use the formula (2.23) to derive equations (5.14)–(5.15).

Lemma 5.4. *The Poisson brackets involving L are given by the following formulae*

$$\{v(\alpha)_i, L_{kl}\} = -i(2\delta_{(k>i)} + \delta_{ik})v(\alpha)_k L_{il} + i\delta_{il}v(\alpha)_i L_{kl} + 2i\delta_{il} \sum_{r>l} v(\alpha)_r L_{kr}, \tag{5.13}$$

$$\{g_{ij}, L_{kl}\} = i(\delta_{ik} + \delta_{il})g_{ij} L_{kl} + 2i\delta_{(k<i)}g_{kj} L_{il} + 2i\delta_{il} \sum_{r>i} L_{kr}g_{rj}, \tag{5.14}$$

$$\begin{aligned} \{L_{ij}, L_{kl}\} &= i[2\delta_{(i>k)} + \delta_{ik} - 2\delta_{(j>l)} - \delta_{ij}]L_{il}L_{kj} \\ &\quad + i(\delta_{il} - \delta_{jk})L_{ij}L_{kl} + 2i\delta_{il} \sum_{r>i} L_{kr}L_{rj} - 2i\delta_{jk} \sum_{r>k} L_{ir}L_{rl}. \end{aligned} \tag{5.15}$$

Now we present an interesting application of the above auxiliary results. Recall that our ‘free Hamiltonians’ (2.40) Poisson commute with the functions $I_{\alpha\beta}^k$ defined in (3.24), and hence, they Poisson commute with the elements of the polynomial algebra

$$\mathcal{I}_L = \mathbb{R}[\text{tr}L^k, \mathfrak{R}(I_{\alpha\beta}^k), \mathfrak{S}(I_{\alpha\beta}^k) \mid 1 \leq \alpha, \beta \leq d, k \geq 0]. \tag{5.16}$$

The algebra \mathcal{I}_L is finitely generated as a consequence of the Cayley–Hamilton theorem for L . We also note that for an arbitrary non-commutative polynomial P obtained as a linear combination of products of the matrices L and $v(\alpha)v(\beta)^\dagger$, $1 \leq \alpha, \beta \leq d$, we have that $\text{tr}(P) \in \mathcal{I}_L$ in view of the identity

$$\text{tr}(L^{a_0}v(\alpha_0)v(\beta_1)^\dagger L^{a_1}v(\alpha_1)v(\beta_2)^\dagger \dots L^{a_l}v(\alpha_l)v(\beta_0)^\dagger L^{a_{l+1}}) = I_{\alpha_0\beta_0}^{a_0+a_{l+1}} I_{\alpha_1\beta_1}^{a_1} \dots I_{\alpha_l\beta_l}^{a_l}. \tag{5.17}$$

A key property of \mathcal{I}_L is that it is a Poisson subalgebra of $C^\infty(\mathcal{M})$. This follows from the next result, which can be proved by direct calculation.

Proposition 5.5. *For any $M, N \geq 0$ and $1 \leq \alpha, \beta, \gamma, \epsilon \leq d$,*

$$\begin{aligned} \{I_{\alpha\beta}^M, I_{\gamma\epsilon}^N\} &= 2i\delta_{\alpha\epsilon}I_{\gamma\beta}^{M+N+1} - 2i\delta_{\gamma\beta}I_{\alpha\epsilon}^{M+N+1} \\ &\quad + i(\delta_{\alpha\epsilon} - \delta_{\gamma\beta})I_{\alpha\beta}^M I_{\gamma\epsilon}^N + 2i\delta_{\alpha\epsilon} \sum_{\mu<\alpha} I_{\gamma\mu}^N I_{\mu\beta}^M - 2i\delta_{\gamma\beta} \sum_{\lambda<\beta} I_{\alpha\lambda}^M I_{\lambda\epsilon}^N \\ &\quad + i\text{sgn}(\gamma - \alpha)I_{\gamma\beta}^M I_{\alpha\epsilon}^N - i\text{sgn}(\epsilon - \beta)I_{\gamma\beta}^N I_{\alpha\epsilon}^M \\ &\quad + i \left(\sum_{b=0}^{M-1} + \sum_{b=0}^{N-1} \right) \left(I_{\gamma\beta}^b I_{\alpha\epsilon}^{M+N-b} - I_{\gamma\beta}^{M+N-b} I_{\alpha\epsilon}^b \right). \end{aligned} \tag{5.18}$$

Using that $\overline{I_{\alpha\beta}^M} = I_{\beta\alpha}^M$, one can verify that the complex Poisson brackets in (5.18) enjoy the property

$$\{\overline{I_{\alpha\beta}^M}, \overline{I_{\gamma\epsilon}^N}\} = \overline{\{I_{\alpha\beta}^M, I_{\gamma\epsilon}^N\}}, \tag{5.19}$$

which together with (5.18) implies that \mathcal{I}_L (5.16) is indeed a real Poisson algebra. Since the elements of \mathcal{I}_L are invariant with respect to the $U(n)$ action on \mathcal{M} , they descend to the reduced phase space. We shall further inspect these integrals of motion in Sect. 6.

5.2. The Reduced Poisson Bracket in Local Coordinates

In this subsection, we shall derive explicit formulae for the reduced Poisson structure, restricting ourselves to an open dense subset $\check{\mathcal{M}}_{\text{red}}^{\text{reg}}$ of the reduced phase space. More precisely, it will be more convenient to work on a covering space of $\check{\mathcal{M}}_{\text{red}}^{\text{reg}}$ that supports residual S_n gauge transformations.

We start by introducing the open dense subset $\check{\mathcal{M}}_0^{\text{reg}} \subset \mathcal{M}_0^{\text{reg}}$ (3.62), which is defined as

$$\check{\mathcal{M}}_0^{\text{reg}} := \{(Q, L(Q, v), v) \in \mathcal{M}_0^{\text{reg}} \mid \sum_{1 \leq \alpha \leq d} v(\alpha)_i \neq 0 \text{ for } i = 1, \dots, n\}. \tag{5.20}$$

The corresponding open dense subset of the reduced phase space is

$$\check{\mathcal{M}}_{\text{red}}^{\text{reg}} := \check{\mathcal{M}}_0^{\text{reg}} / \mathcal{N}(n). \tag{5.21}$$

Using that $S_n = \mathcal{N}(n) / \mathbb{T}^n$, we can take the quotient in two steps. Thus, similarly to (3.64), we have

$$\check{\mathcal{M}}_0^{\text{reg}} / \mathcal{N}(n) = (\check{\mathcal{M}}_0^{\text{reg}} / \mathbb{T}^n) / S_n. \tag{5.22}$$

For our purpose, we choose to identify $\check{\mathcal{M}}_0^{\text{reg}} / \mathbb{T}^n$ with the following subset of $\check{\mathcal{M}}_0^{\text{reg}}$

$$\check{\mathcal{M}}_{0,+}^{\text{reg}} := \{(Q, L(Q, v), v) \in \mathcal{M}_0^{\text{reg}} \mid \sum_{1 \leq \alpha \leq d} v(\alpha)_i > 0 \text{ for } i = 1, \dots, n\}. \tag{5.23}$$

Indeed, if $(Q, L, v) \in \check{\mathcal{M}}_0^{\text{reg}}$ we can write the vector $\sum_{1 \leq \alpha \leq d} v(\alpha)$ as $(U_1 e^{i\phi_1}, \dots, U_n e^{i\phi_n})^T$ with $U_i > 0$, $\phi_i \in \mathbb{R}$ for all i . Acting on (Q, L, v) by $\tau = \text{diag}(e^{-i\phi_1}, \dots, e^{-i\phi_n}) \in \mathbb{T}^n$ yields an element of $\check{\mathcal{M}}_{0,+}^{\text{reg}}$, and it is clear that τ is the unique element of \mathbb{T}^n with this property. The upshot is the identification

$$\check{\mathcal{M}}_{\text{red}}^{\text{reg}} \equiv \check{\mathcal{M}}_{0,+}^{\text{reg}} / S_n. \tag{5.24}$$

The main reason for introducing the particular gauge slice $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ for the free \mathbb{T}^n action on $\check{\mathcal{M}}_0^{\text{reg}}$ is that S_n still acts on it in the obvious manner, by permuting the n entries of Q and the components of each column vector $v(\alpha) \in \mathbb{C}^n$. Similar ‘democratic gauge fixing’ was employed in the previous papers dealing with holomorphic systems [4, 5, 10]. The relation between the spaces just defined and those given in Sect. 3 is summarized in Fig. 1.

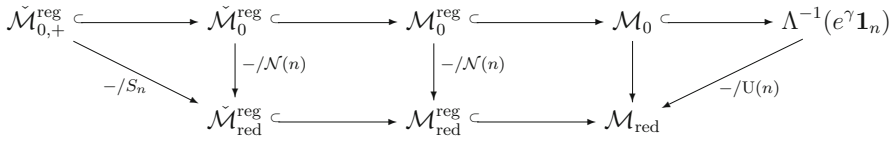


FIGURE 1. (From right to left.) \mathcal{M}_0 is the subspace (3.26) of the constraint surface $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$ where each point (Q, b_R, v) satisfies that $Q \in \mathbb{T}^n$. $\mathcal{M}_0^{\text{reg}} \subset \mathcal{M}_0$ is the subspace (3.39) where $Q \in \mathbb{T}_{\text{reg}}^n$, $\tilde{\mathcal{M}}_0^{\text{reg}} \subset \mathcal{M}_0^{\text{reg}}$ is the subspace (5.20) where the vector $\sum_{\alpha=1}^d v(\alpha)$ has only nonzero entries, while $\tilde{\mathcal{M}}_{0,+}^{\text{reg}} \subset \tilde{\mathcal{M}}_0^{\text{reg}}$ is the subspace (5.23) obtained by imposing to the vector $\sum_{\alpha=1}^d v(\alpha)$ to have positive entries. The spaces appearing on the second line are the sets corresponding to the $U(n)$ -orbits inside $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$

Let

$$\xi : \tilde{\mathcal{M}}_{0,+}^{\text{reg}} \rightarrow \mathcal{M} \tag{5.25}$$

be the tautological inclusion. General principles of reduction theory [25, 41] ensure that the pull-back $\xi^* \Omega_{\mathcal{M}}$ is symplectic and satisfies $\xi^* \Omega_{\mathcal{M}} = \tilde{\pi}^*(\Omega_{\text{red}})$, where $\tilde{\pi} : \tilde{\mathcal{M}}_{0,+}^{\text{reg}} \rightarrow \mathcal{M}_{\text{red}}$ is the canonical projection and Ω_{red} is the reduced symplectic form. We let $\{ , \}_{\text{red}}$ denote the Poisson bracket on $C^\infty(\tilde{\mathcal{M}}_{0,+}^{\text{reg}})$ that corresponds to $\xi^* \Omega_{\mathcal{M}}$ (2.37), and note that it possesses the key property

$$\xi^* \{F_1, F_2\} = \{\xi^* F_1, \xi^* F_2\}_{\text{red}}, \quad \forall F_1, F_2 \in C^\infty(\mathcal{M})^{U(n)}, \tag{5.26}$$

where $\{F_1, F_2\} := \{F_1, F_2\}_{\mathcal{M}}$ is the Poisson bracket associated with $\Omega_{\mathcal{M}}$ (2.37). We shall determine the form of this reduced Poisson bracket by applying the identity (5.26) to a judiciously chosen set of invariant functions.

Remark 5.6. The bracket $\{ , \}_{\text{red}}$ is also known as the Dirac bracket [25] associated with the gauge slice $\tilde{\mathcal{M}}_{0,+}^{\text{reg}}$. To avoid any potential confusion, we stress that our notation $\{ , \}_{\text{red}}$ involves a slight abuse of terminology, since not all elements of $C^\infty(\tilde{\mathcal{M}}_{0,+}^{\text{reg}})$ arise as restrictions of elements of $C^\infty(\mathcal{M})^{U(n)}$ (which carries the reduced Poisson algebra in the strict sense). For example, all those restricted $U(n)$ invariants are S_n invariant function on $\tilde{\mathcal{M}}_{0,+}^{\text{reg}}$. However, the Poisson algebra $(C^\infty(\tilde{\mathcal{M}}_{0,+}^{\text{reg}}), \{ , \}_{\text{red}})$ encodes all information about $(C^\infty(\mathcal{M})^{U(n)}, \{ , \})$, since $\tilde{\mathcal{M}}_{0,+}^{\text{reg}}$ projects onto a dense open subset of \mathcal{M}_{red} . We shall see shortly that it underlies the Hamiltonian interpretation of the spin RS equations of motion given by (1.8).

In order to implement the above ideas, now we introduce the following $U(n)$ invariant elements of $C^\infty(\mathcal{M})$

$$f_m^{\alpha\beta} := \text{tr}(v(\alpha)v(\beta)^\dagger g_R^m) = v(\beta)^\dagger g_R^m v(\alpha), \quad f_m := \text{tr}(g_R^m), \quad m \in \mathbb{N}, \quad 1 \leq \alpha, \beta \leq d. \tag{5.27}$$

Lemma 5.7. *For any $M, N \in \mathbb{N}$ and $1 \leq \alpha, \beta \leq d$ we have the following Poisson bracket relations in $C^\infty(\mathcal{M})^{U(n)}$:*

$$\{f_M, f_N\} = 0, \quad \{f_M, \bar{f}_N\} = 0, \quad \{\bar{f}_M, \bar{f}_N\} = 0, \tag{5.28}$$

$$\{f_M^{\alpha\beta}, f_N\} = -2iN f_{M+N}^{\alpha\beta}. \tag{5.29}$$

Furthermore, letting $\phi^{\mu\nu}(a, c) := \text{tr}[v(\mu)v(\nu)^\dagger g_R^a Lg_R^c]$ for $a, b \in \mathbb{N}$, we have

$$\begin{aligned} \{f_M^{\alpha\beta}, f_N^{\gamma\epsilon}\} = & 2i \left(\sum_{a=1}^M - \sum_{a=1}^N \right) f_a^{\alpha\epsilon} f_{M+N-a}^{\gamma\beta} - i f_M^{\alpha\epsilon} f_N^{\gamma\beta} + i f_N^{\alpha\epsilon} f_M^{\gamma\beta} \\ & + \text{isgn}(\gamma - \alpha) f_N^{\alpha\epsilon} f_M^{\gamma\beta} - \text{isgn}(\epsilon - \beta) f_M^{\alpha\epsilon} f_N^{\gamma\beta} + i(\delta_{\alpha\epsilon} - \delta_{\gamma\beta}) f_M^{\alpha\beta} f_N^{\gamma\epsilon} \\ & + 2i\delta_{\alpha\epsilon} \sum_{\mu < \alpha} f_N^{\gamma\mu} f_M^{\mu\beta} - 2i\delta_{\gamma\beta} \sum_{\lambda < \beta} f_M^{\alpha\lambda} f_N^{\lambda\epsilon} \\ & + 2i\delta_{\alpha\epsilon} \phi^{\gamma\beta}(M, N) - 2i\delta_{\gamma\beta} \phi^{\alpha\epsilon}(N, M). \end{aligned} \tag{5.30}$$

Proof. The identities (5.28) are well known. To establish (5.29), we use the decomposition (here $g = g_R$)

$$\begin{aligned} \{f_M^{\alpha\beta}, f_N\} = & N \sum_{ijkl} \{v(\alpha)_i, g_{kl}\} \bar{v}(\beta)_j g_{ji}^M g_{lk}^{N-1} + N \sum_{ijkl} \{\bar{v}(\beta)_j, g_{kl}\} g_{ji}^M v(\alpha)_i g_{lk}^{N-1} \\ & + N \sum_{ijkl} \sum_{b=0}^{M-1} \{g_{ij}, g_{kl}\} (g^{M-b-1} v(\alpha) v(\beta)^\dagger g^b)_{ji} g_{lk}^{N-1}, \end{aligned} \tag{5.31}$$

then use for these three terms (5.9), (5.10) and (2.7), respectively. Some obvious cancellations yield (5.29). Finally, (5.30) only requires some of the Poisson brackets gathered in §5.1 and it can be proved in a way similar to (5.29). \square

Convenient variables on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ are provided by the evaluation functions $Q_j = e^{iq_j} \in U(1)$ and the real and imaginary parts of the $v(\alpha)_j \in \mathbb{C}$. The latter are not all independent, since they obey the gauge fixing conditions

$$\mathcal{U}_j = \Re(\mathcal{U}_j) > 0, \quad \text{with} \quad \mathcal{U}_j := \sum_{1 \leq \alpha \leq d} v(\alpha)_j, \quad 1 \leq j \leq n. \tag{5.32}$$

It is clear that all these functions belong to $C^\infty(\check{\mathcal{M}}_{0,+}^{\text{reg}})$ and their mutual Poisson brackets completely determine $\{, \}_{\text{red}}$.

The pull-backs of the functions (5.27) can be written in the local variables on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ as

$$\xi^* f_m^{\alpha\beta} = \sum_{i=1}^n v(\alpha)_i Q_i^m \bar{v}(\beta)_i, \quad \xi^* f_m = \sum_{i=1}^n Q_i^m, \tag{5.33}$$

and we note that

$$\sum_{\beta} \xi^* f_m^{\alpha\beta} = \sum_{i=1}^n \mathcal{U}_i v(\alpha)_i Q_i^m, \quad \sum_{\alpha} \xi^* f_m^{\alpha\beta} = \sum_{i=1}^n \mathcal{U}_i \bar{v}(\beta)_i Q_i^m, \quad \sum_{\alpha, \beta} \xi^* f_m^{\alpha\beta} = \sum_{i=1}^n \mathcal{U}_i^2 Q_i^m. \tag{5.34}$$

In conjunction with Lemma 5.7 and Eq. (5.26), these expressions can be used to determine the reduced Poisson brackets of the variables Q_j and $v(\alpha)$. To

state the result, we introduce the $n \times n$ matrix-valued functions S^0 and R^α , $1 \leq \alpha \leq d$, whose entries are given by

$$S_{ij}^0 = \frac{1}{4} \sum_{\mu, \nu} \operatorname{sgn}(\nu - \mu) v(\nu)_i v(\mu)_j - \frac{1}{4} \sum_{\mu} v(\mu)_i \bar{v}(\mu)_j - \frac{1}{2} \sum_{\nu} \sum_{\mu < \nu} v(\mu)_i \bar{v}(\mu)_j - \frac{d}{2} L_{ij}, \tag{5.35}$$

$$R_{ij}^\alpha = L_{ij} - \frac{1}{2} \sum_{\kappa} \operatorname{sgn}(\kappa - \alpha) v(\kappa)_i v(\alpha)_j + \frac{1}{2} v(\alpha)_i \bar{v}(\alpha)_j + \sum_{\kappa < \alpha} v(\kappa)_i \bar{v}(\kappa)_j. \tag{5.36}$$

We also define the matrix S with entries $S_{ij} = S_{ij}^0 - \bar{S}_{ij}^0$.

Theorem 5.8. *In terms of the functions $(Q_j = e^{iq_j}, v(\alpha)_j)$ defined on $\check{\mathcal{M}}_{0,+}^{\operatorname{reg}}$, and using the formulae (3.29) for L and (5.32) for \mathcal{U}_j , we can write the reduced Poisson bracket as*

$$\begin{aligned} \{q_i, q_j\}_{\operatorname{red}} &= 0, \quad \{v(\alpha)_i, q_j\}_{\operatorname{red}} = -\delta_{ij} v(\alpha)_i, \tag{5.37} \\ \{v(\alpha)_i, v(\gamma)_j\}_{\operatorname{red}} &= i \operatorname{sgn}(\gamma - \alpha) v(\alpha)_j v(\gamma)_i + i \frac{v(\alpha)_i}{u_i} \frac{v(\gamma)_j}{u_j} S_{ij} + i \frac{v(\gamma)_j}{u_j} R_{ij}^\alpha - i \frac{v(\alpha)_i}{u_i} R_{ji}^\gamma \\ &+ \frac{1}{2} i \delta_{(i \neq j)} \frac{Q_i + Q_j}{Q_i - Q_j} \left[2v(\alpha)_j v(\gamma)_i + v(\alpha)_i v(\gamma)_j - \frac{u_i}{u_j} v(\alpha)_j v(\gamma)_j - \frac{u_j}{u_i} v(\alpha)_i v(\gamma)_i \right], \tag{5.38} \end{aligned}$$

$$\begin{aligned} \{v(\alpha)_i, \bar{v}(\epsilon)_j\}_{\operatorname{red}} &= i \delta_{\alpha \epsilon} (v(\alpha)_i \bar{v}(\epsilon)_j + 2 \sum_{\kappa < \alpha} v(\kappa)_i \bar{v}(\kappa)_j + 2L_{ij}) \\ &+ \frac{1}{2} i \delta_{(i \neq j)} \frac{Q_i + Q_j}{Q_i - Q_j} \left[-v(\alpha)_i \bar{v}(\epsilon)_j + \frac{u_i}{u_j} v(\alpha)_j \bar{v}(\epsilon)_j + \frac{u_j}{u_i} v(\alpha)_i \bar{v}(\epsilon)_i \right] \tag{5.39} \\ &- i \frac{v(\alpha)_i}{u_i} \frac{\bar{v}(\epsilon)_j}{u_j} S_{ij} - i \frac{\bar{v}(\epsilon)_j}{u_j} R_{ij}^\alpha - i \frac{v(\alpha)_i}{u_i} \bar{R}_{ji}^\epsilon. \end{aligned}$$

The bracket $\{-, -\}_{\operatorname{red}}$ is invariant under simultaneous permutations of the n components of the variables q and $v(\alpha)$ for $\alpha = 1, \dots, d$.

The proof of this result is the subject of ‘‘Appendix C.’’ Let us already mention that the reader can check the reality condition $\{\bar{v}(\alpha)_i, v(\epsilon)_j\}_{\operatorname{red}} = \overline{\{v(\alpha)_i, \bar{v}(\epsilon)_j\}_{\operatorname{red}}}$.

We know from Corollary 4.3 that the projection of the Hamiltonian vector field of $H = (e^{2\gamma} - 1) \operatorname{tr}(L)$ onto the gauge slice $\check{\mathcal{M}}_{0,+}^{\operatorname{reg}}$ leads to the equations of motion (1.8). Of course, the corresponding reduced Hamiltonian must generate the same evolution equations via the reduced Poisson bracket. The reduced Hamiltonian is encoded by the pull-back $\mathcal{H} := \xi^* H$ on $\check{\mathcal{M}}_{0,+}^{\operatorname{reg}}$. Thus, the next result shows the consistency of the computations performed in Sects. 4 and 5.

Corollary 5.9. *Consider the reduced Hamiltonian*

$$\mathcal{H}(Q, v) = (e^{2\gamma} - 1) \operatorname{tr}(L(Q, v)) = \sum_{k=1}^n F_{kk}, \quad F_{kk} = \sum_{\alpha=1}^d v(\alpha)_k \bar{v}(\alpha)_k, \tag{5.40}$$

on the gauge slice $\check{\mathcal{M}}_{0,+}^{\text{reg}}$. Then, the Hamiltonian vector field generated by \mathcal{H} via the reduced Poisson bracket of Theorem 5.8 reproduces the equations of motion (1.8)–(1.9).

Proof. We get from (5.37) that

$$\dot{q}_j := \{q_j, \mathcal{H}\}_{\text{red}} = \sum_k \{q_j, F_{kk}\}_{\text{red}} = 2F_{jj}, \tag{5.41}$$

which is just (1.8). To compute $\dot{v}(\alpha)_i$, we use that $F_{kl} = \sum_{\alpha=1}^d v(\alpha)_k \bar{v}(\alpha)_l$ together with Theorem 5.8 in order to obtain

$$\begin{aligned} \{v(\alpha)_i, F_{kk}\}_{\text{red}} &= i\delta_{(i \neq k)} v(\alpha)_k \left[F_{ik} + \frac{Q_i + Q_k}{Q_i - Q_k} F_{ik} + 2L_{ik} \right] \\ &\quad - \frac{1}{2} i\delta_{(i \neq k)} v(\alpha)_i \frac{\mathcal{U}_k}{\mathcal{U}_i} \left[\left(F_{ik} + \frac{Q_i + Q_k}{Q_i - Q_k} F_{ik} + 2L_{ik} \right) + \left(F_{ki} + \frac{Q_k + Q_i}{Q_k - Q_i} F_{ki} + 2L_{ki} \right) \right]. \end{aligned} \tag{5.42}$$

Noticing the identity

$$i \left(F_{ik} + \frac{Q_i + Q_k}{Q_i - Q_k} F_{ik} + 2L_{ik} \right) = -F_{ik} V \left(\frac{q_k - q_i}{2} \right), \tag{5.43}$$

where $V(x)$ is the potential (1.5), this allows us to write

$$\begin{aligned} \{v(\alpha)_i, F_{kk}\}_{\text{red}} &= -\delta_{(i \neq k)} v(\alpha)_k F_{ik} V \left(\frac{q_k - q_i}{2} \right) \\ &\quad + \frac{1}{2} \delta_{(i \neq k)} v(\alpha)_i \frac{\mathcal{U}_k}{\mathcal{U}_i} \left[F_{ik} V \left(\frac{q_k - q_i}{2} \right) + F_{ki} V \left(\frac{q_i - q_k}{2} \right) \right]. \end{aligned} \tag{5.44}$$

Summing over k precisely gives $\dot{v}(\alpha)_i$ in (1.8) with (1.9). □

As a second consequence of Theorem 5.8, we can write down the reduced Poisson brackets of the ‘collective spins’ (F_{ij}) , which can be found in ‘Appendix D.’ By using Eq. (3.29), then we can obtain the formula for the Poisson brackets of the entries of the Lax matrix on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$, which implies that the symmetric functions of L are in involution. This is in agreement with the fact that \mathfrak{h} given in (2.40) is an Abelian Poisson subalgebra of $C^\infty(\mathcal{M})$. To present the desired formula, we use the matrix S defined before Theorem 5.8. We also define

$$\begin{aligned} r_{12} &:= \sum_{a \neq b} \frac{iQ_b}{Q_a - Q_b} E_{aa} \otimes \left(E_{bb} - \frac{\mathcal{U}_b}{\mathcal{U}_a} E_{ba} \right) - \sum_{a \neq b} \frac{iQ_a}{Q_a - Q_b} E_{ab} \otimes \left(\frac{\mathcal{U}_a}{\mathcal{U}_b} E_{bb} - 2E_{ba} \right) \\ &\quad + \sum_{a,b} i \frac{S_{ab}}{\mathcal{U}_a \mathcal{U}_b} E_{aa} \otimes E_{bb} + i \sum_a E_{aa} \otimes E_{aa}, \end{aligned} \tag{5.45}$$

and

$$s_{12} := \sum_{a \neq b} \frac{iQ_a}{Q_a - Q_b} E_{aa} \otimes \left(\frac{\mathcal{U}_b}{\mathcal{U}_a} E_{ab} - E_{bb} \right) + \sum_{a \neq b} \frac{iQ_a}{Q_a - Q_b} \frac{\mathcal{U}_a}{\mathcal{U}_b} E_{ab} \otimes E_{bb}$$

$$-\sum_{a,b} i \frac{S_{ab}}{\mathcal{U}_a \mathcal{U}_b} E_{aa} \otimes E_{bb} + \frac{1}{2} i \sum_a E_{aa} \otimes E_{aa}, \tag{5.46}$$

where E_{ab} is the $n \times n$ elementary matrix with only nonzero entry equal to $+1$ in position (a, b) .

Proposition 5.10. *On the gauge slice $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ (5.23), the entries of the Lax matrix L (3.29) satisfy*

$$\{L_1, L_2\}_{\text{red}} = r_{12} L_1 L_2 + L_1 L_2 t_{12} - L_1 s_{21} L_2 + L_2 s_{12} L_1, \tag{5.47}$$

where $t_{12} = -s_{12} + s_{21} - r_{12}$. This relation implies that the functions $\text{tr}(L^k)$ are in involution.

In (5.47), we used the standard notations $L_1 = L \otimes \mathbf{1}_n$, $L_2 = \mathbf{1}_n \otimes L$, and $\{L_1, L_2\}_{\text{red}} = \sum_{ijkl} \{L_{ij}, L_{kl}\}_{\text{red}} E_{ij} \otimes E_{kl}$, where the entries of L are seen as evaluation functions on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$.

Remark 5.11. The formulae of Theorem 5.8 exhibit an interesting two-body structure in the sense that the Poisson brackets of the basic variables with particle labels i and j close on this subset of the variables. This is consistent with the fact that the Hamiltonian (1.10) is the sum of one-body terms, while the equations of motion (1.8)–(1.9) reflect two-body interactions. It should be stressed that this interpretation is based on viewing q_i and the dressed spin $v(-)_i$ as degrees of freedom belonging to particle i . The same features hold in the complex holomorphic spin RS models as well [4, 5, 10]. It is also worth noting that the formulae of Theorem 5.8 enjoy a nice homogeneity property. Namely, let us define a \mathbb{Z}^n -valued weight $\text{wt}[-]$ by setting

$$\text{wt}[1] = \text{wt}[q_j] = \text{wt}[Q_j] = 0, \quad \text{wt}[\bar{v}(\alpha)_j] = \text{wt}[v(\alpha)_j] = \mathbf{e}_j, \quad \text{for } 1 \leq j \leq n, \tag{5.48}$$

where $\mathbf{e}_j \in \mathbb{Z}^n$ is $+1$ in its j -th entry and zero everywhere else. Extending this weight by $\text{wt}[fg] = \text{wt}[f] + \text{wt}[g]$ for homogeneous elements f, g , we easily get that

$$\text{wt}[\mathcal{U}_j^{\pm 1}] = \pm \mathbf{e}_j, \quad \text{wt}[F_{ij}] = \mathbf{e}_i + \mathbf{e}_j, \quad \text{wt}[L_{ij}] = \mathbf{e}_i + \mathbf{e}_j. \tag{5.49}$$

We can then observe from (5.37)–(5.39) that the reduced Poisson bracket preserves this weight.

6. Degenerate Integrability of the Reduced System

We discussed the degenerate integrability of the unreduced free system in Sect. 2.3 and now wish to show that this property is inherited by the reduced system. This is expected to hold not only in view of the earlier results on holomorphic spin RS systems [5, 10] and related models [44–46], but also on account of a general result in reduction theory. In fact, it is known (Theorem 2.16 in [58], see also [27]) that the integrability of invariant Hamiltonians on a manifold descends generically to the reduced space of Poisson reduction. However, the pertinent spaces of group orbits are typically not smooth manifolds.

The existing results provide strong motivation, but do not help us directly to establish integrability in our concrete case.

Our goal is to prove the degenerate integrability of the reduced system in the real-analytic category by explicitly displaying the required integrals of motion. Specifically, we wish to show that the n reduced Hamiltonians arising from the functions

$$\text{tr}(L^k), \quad k = 1, \dots, n, \tag{6.1}$$

are functionally independent and that one can complement them to $(2nd - n)$ independent functions using suitable reduced integrals of motion that arise from the real and imaginary parts of the $U(n)$ invariant functions

$$I_{\alpha\beta}^k = v(\beta)^\dagger L^k v(\alpha). \tag{6.2}$$

These integrals of motion appeared before in Proposition 5.5. As throughout the paper, we assume that $d \geq 2$.

The independence of functions means linear independence of their exterior derivatives at generic points, and this can be translated into the non-vanishing of a suitable Jacobian determinant. For real-analytic functions, the determinant at issue is also real-analytic, and hence, it is generically nonzero if it is nonzero at a single point. Thus, by patching together analytic charts, one sees that on a connected real-analytic manifold independence of real analytic functions follows from the linear independence of their derivatives at a single point. We can use this observation since we know (see Remark 3.6 and Corollary 3.15) that \mathcal{M}_{red} is a connected real-analytic manifold.

6.1. Construction of Local Coordinates

Our first goal below is to construct local coordinates around certain points of the reduced phase space in which the formulae of the integrals of motion become simple. The coordinates will involve the eigenvalues of L , whereby the Hamiltonians $\text{tr}(L^k)$ acquire a trivial form. We start by noting that the moment map constraint admits solutions for which only a single one of the vectors $v(\alpha)$ is nonzero. Concerning those elements of $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$, the following useful result can be obtained from (the proof of) Lemma 5.2 of [19].

Lemma 6.1. *Consider any $y \in \mathbb{R}^n$ whose components y_1, \dots, y_n satisfy the inequalities*

$$y_i > e^{2\gamma} y_{i+1} \quad \forall i = 1, \dots, n \quad \text{with} \quad y_{n+1} := 0. \tag{6.3}$$

Then, there exists $(g^0, L^0, v^0) \in \Lambda^{-1}(e^\gamma \mathbf{1}_n)$ such that $L^0 = \text{diag}(y_1, \dots, y_n)$ and $v(\alpha)^0 = 0$ for each $1 \leq \alpha < d$ (where $d \geq 2$ and $\gamma > 0$). For such elements, all components of the vector $v(d)^0$ are nonzero.

Proof. Given $L^0 = \text{diag}(y_1, \dots, y_n)$ and $v(1)^0 = \dots = v(d-1)^0 = 0$, we have to find $g^0 \in U(n)$ and $v(d)^0 \in \mathbb{C}^n$ such that the moment map constraint (3.17) holds. Using (3.19), this means that

$$e^{2\gamma}(g^0)^{-1} L^0 g^0 = L^0 + v(d)^0 (v(d)^0)^\dagger. \tag{6.4}$$

This is equivalent to the requirement that there exists $v(d)^0 \in \mathbb{C}^n$ such that $L^0 + v(d)^0(v(d)^0)^\dagger$ and $e^{2\gamma}L^0$ have the same spectrum. But this holds if and only if we have the equality of polynomials in λ

$$\det(L^0 + v(d)^0(v(d)^0)^\dagger - \lambda \mathbf{1}_n) = \det(e^{2\gamma}L^0 - \lambda \mathbf{1}_n) = \prod_{k=1}^n (e^{2\gamma}y_k - \lambda). \tag{6.5}$$

We can expand the left-hand side as follows :

$$\begin{aligned} \det(L^0 + v(d)^0(v(d)^0)^\dagger - \lambda \mathbf{1}_n) &= \det(L^0 - \lambda \mathbf{1}_n)[1 + (v(d)^0)^\dagger(L^0 - \lambda \mathbf{1}_n)^{-1}v(d)^0] \\ &= \prod_{k=1}^n (y_k - \lambda) + \sum_{j=1}^n |v(d)_j^0|^2 \prod_{k \neq j} (y_k - \lambda). \end{aligned} \tag{6.6}$$

Thus, we seek $v(d)^0 \in \mathbb{C}^n$ such that

$$\prod_{k=1}^n (e^{2\gamma}y_k - \lambda) - \prod_{k=1}^n (y_k - \lambda) = \sum_{j=1}^n |v(d)_j^0|^2 \prod_{k \neq j} (y_k - \lambda). \tag{6.7}$$

Evaluating this identity at $\lambda = y_l$ yields

$$|v(d)_l^0|^2 = (e^{2\gamma} - 1)y_l \prod_{k \neq l} \frac{e^{2\gamma}y_k - y_l}{y_k - y_l}, \tag{6.8}$$

which is *positive due to (6.3)*. It now suffices to pick $v(d)^0$ whose components have moduli given by (6.8), while we pick for g^0 any unitary matrix diagonalizing $L^0 + v(d)^0(v(d)^0)^\dagger$ into $e^{2\gamma}L^0$. \square

Remark 6.2. Notice that a completely gauge fixed normal form of the elements appearing in Lemma 6.1 can be obtained by requiring all components of the vector $v(d)^0$ to be positive. We also note in passing that in the $d = 1$ case the set of possible (ordered) eigenvalues of L in $(g_R, L, v) \in \Lambda^{-1}(e^\gamma \mathbf{1}_n)$ is given [19] by the polyhedron in \mathbb{R}^n specified by the conditions $y_i \geq e^{2\gamma}y_{i+1}$ for all $i = 1, \dots, n - 1$ and $y_n > 0$.

Now we introduce two subsets of the inverse image of the ‘constraint surface.’

Definition 6.3. Denote

$$\mathcal{S} = \{(g_R, L, v) \in \Lambda^{-1}(e^\gamma \mathbf{1}_n) \mid L = \text{diag}(y_1, \dots, y_n), y_i > y_{i+1}, v(1)_i > 0 \forall i\}. \tag{6.9}$$

The open subset $\mathcal{S}_1 \subset \mathcal{S}$ is defined by imposing the further condition that the matrix

$$L_1 := L + \sum_{\alpha=1}^{d-1} v(\alpha)v(\alpha)^\dagger \tag{6.10}$$

is conjugate to $\text{diag}(\mu_1, \dots, \mu_n)$, where the μ_i satisfy the inequalities

$$e^{2\gamma}y_i > \mu_i > e^{2\gamma}y_{i+1} \quad \forall i = 1, \dots, n - 1 \quad \text{and} \quad e^{2\gamma}y_n > \mu_n. \tag{6.11}$$

Note that \mathcal{S} is non-empty since we can apply the analogue of Lemma 6.1 to obtain elements of $\Lambda^{-1}(e^\gamma \mathbf{1}_n)$ for which only $v(1)$ is nonzero, and \mathcal{S}_1 is non-empty since for those elements $L_1 = L$. It is clear that \mathcal{S} can serve as a model of an open dense subset of the reduced phase space. Below, we provide a full characterization of the elements of \mathcal{S}_1 .

Taking y and μ subject to the inequalities in (6.11), define $\mathbf{V}(y, \mu) \in \mathbb{R}^n$ by

$$V_l(y, \mu) := \left[(e^{2\gamma} y_l - \mu_l) \prod_{k \neq l} \frac{e^{2\gamma} y_k - \mu_l}{\mu_k - \mu_l} \right]^{\frac{1}{2}} \quad \forall l = 1, \dots, n. \tag{6.12}$$

Observe that the function under the square root is positive, and the positive root is taken.

Lemma 6.4. *For any $(g_R, L, v) \in \mathcal{S}_1$ pick a matrix $g_1 \in U(n)$ for which*

$$g_1 L_1 g_1^{-1} = \text{diag}(\mu_1, \dots, \mu_n) \tag{6.13}$$

with μ satisfying (6.11). Then, $v(d)$ is of the form

$$v(d) = g_1^{-1} \text{diag}(\tau_1, \dots, \tau_n) \mathbf{V}(y, \mu) \quad \text{with some } \tau \in \mathbb{T}^n. \tag{6.14}$$

Furthermore, g_R is of the form

$$g_R = \text{diag}(\Gamma_1, \dots, \Gamma_n) g_R^0 \quad \text{with some } \Gamma \in \mathbb{T}^n, \tag{6.15}$$

where $g_R^0 \in U(n)$ is a fixed solution of the constraint equation

$$g_R^{-1} e^{2\gamma} L g_R = L_1 + v(d)v(d)^\dagger. \tag{6.16}$$

Conversely, take any positive definite $L = \text{diag}(y_1, \dots, y_n)$ and \mathbb{C}^n vectors $v(1), \dots, v(d-1)$ such that L and L_1 given by (6.10) satisfy the spectral conditions (6.11), and all components of $v(1)$ are positive. Choose a diagonalizer g_1 according to (6.13) and define $v(d) \in \mathbb{C}^n$ by the formula (6.14) using an arbitrary $\tau \in \mathbb{T}^n$. Then, Eq. (6.16) admits solutions for g_R , the general solution has the form (6.15) with arbitrary $\Gamma \in \mathbb{T}^n$, and all so obtained triples (g_R, L, v) belong to \mathcal{S}_1 .

Proof. By using the definitions (3.25) of L and (6.10) of L_1 , we can always recast the moment map constraint (3.19) in the form (6.16), which implies the equality of characteristic polynomials

$$\det(e^{2\gamma} L - \lambda \mathbf{1}_n) = \det(L_1 + v(d)v(d)^\dagger - \lambda \mathbf{1}_n). \tag{6.17}$$

Since L is diagonal for $(g, L, v) \in \mathcal{S}_1$, we have

$$\det(e^{2\gamma} L - \lambda \mathbf{1}_n) = \prod_j (e^{2\gamma} y_j - \lambda). \tag{6.18}$$

By using (6.13) and introducing

$$\tilde{u} := g_1 v(d), \tag{6.19}$$

we can write the polynomial on the right-hand side of (6.17) as

$$\begin{aligned} \det(L_1 - \lambda \mathbf{1}_n)[1 + v(d)^\dagger(L_1 - \lambda \mathbf{1}_n)^{-1}v(d)] &= \prod_{k=1}^n (\mu_k - \lambda) \left[1 + \sum_{j=1}^n \tilde{u}_j^\dagger \frac{1}{\mu_j - \lambda} \tilde{u}_j \right] \\ &= \prod_{k=1}^n (\mu_k - \lambda) + \sum_{j=1}^n |\tilde{u}_j|^2 \prod_{k \neq j} (\mu_k - \lambda). \end{aligned} \tag{6.20}$$

Thus, (6.17) evaluated at $\lambda = \mu_l$ yields

$$|\tilde{u}_l|^2 = (e^{2\gamma} y_l - \mu_l) \prod_{k \neq l} \frac{e^{2\gamma} y_k - \mu_l}{\mu_k - \mu_l} = \mathbf{V}_l(y, \mu)^2, \tag{6.21}$$

which is positive due to (6.11). We conclude from this and Eq. (6.19) that $v(d)$ has the form (6.14). The claim (6.15) about the form of g_R follows from (6.16) since L is diagonal and has distinct eigenvalues.

The converse statement is proved by utilizing that the equality of the polynomials in λ (6.17) is *equivalent* to the existence of a unitary matrix g_R that solves the constraint equation (6.16). Then, we simply turn the above arguments backward. The crux is that the spectral assumption (6.11) ensures the positivity of the expression in (6.12), whence $v(d)$ can be constructed starting from the vector $\tilde{u} = \text{diag}(\tau_1, \dots, \tau_n)\mathbf{V}(y, \mu)$. \square

From now on, we write

$$v(\alpha)_j = v(\alpha)_j^{\Re} + iv(\alpha)_j^{\Im} \quad \text{for } \alpha = 2, \dots, d-1, \tag{6.22}$$

with real-valued $v(\alpha)_j^{\Re}, v(\alpha)_j^{\Im}$. In the next statement, we summarize how Lemma 6.4 gives us coordinates on \mathcal{S}_1 .

Corollary 6.5. *Via the formulae of Lemma 6.4 for $v(d)$ and g_R , the elements of \mathcal{S}_1 are uniquely parametrized by the $2n(d-1)$ variables*

$$y_j, v(1)_j, v(\alpha)_j^{\Re}, v(\alpha)_j^{\Im}, \quad j = 1, \dots, n, \quad \alpha = 2, \dots, d-1, \tag{6.23}$$

together with the $2n$ variables

$$\tau_j \in \mathbf{U}(1), \quad \Gamma_j \in \mathbf{U}(1), \quad j = 1, \dots, n. \tag{6.24}$$

The variables (6.23) take values in an open subset of $\mathbb{R}^{2n(d-1)}$. The matrix elements of g_1 (6.13) can be chosen to be real-analytic functions of the $2n(d-1)$ variables (6.23), and then, the components of $v(d)$ (6.14) are also real-analytic functions of these variables and the τ_j . Likewise, the matrix elements of g_R^0 can be chosen to be real-analytic functions of the variables (6.23) and the τ_j . Consequently, the variables (6.23) together with t_j and γ_j in $\tau_j = e^{it_j}$ and $\Gamma_j = e^{i\gamma_j}$ define a coordinate system on the open submanifold of the reduced phase space corresponding to \mathcal{S}_1 .

Proof. The variables (6.23) run over an open set simply because the eigenvalues of L_1 depend continuously on them. This dependence is actually analytic since those eigenvalues are all distinct. Regarding the dependence of g_1 and g_R^0 on the variables, we use the well-known fact that the eigenvectors of regular Hermitian

matrices can be chosen as analytic functions of the independent parameters of the matrix elements. □

6.2. Degenerate Integrability

The reduced integrals of motion arising from (6.1) and (6.2) take a simple form in terms of our coordinates on \mathcal{S}_1 . Relying on this, we shall inspect the following $2n(d - 1)$ reduced integrals of motion:

$$\begin{aligned} \text{tr}(L^k) &= \sum_j y_j^k, \quad I_{1,1}^k = \sum_j v(1)_j^2 y_j^k, \\ \Re[I_{\alpha,1}^k] &= \sum_j v(1)_j y_j^k v(\alpha)_j^{\Re}, \quad \Im[I_{\alpha,1}^k] = \sum_j v(1)_j y_j^k v(\alpha)_j^{\Im}, \end{aligned} \tag{6.25}$$

where $k = 1, \dots, n$ and $\alpha = 2, \dots, d - 1$, and the additional $2n$ integrals of motion supplied by the real and imaginary parts of

$$I_{d,1}^k = \sum_j v(1)_j y_j^k v(d)_j \quad \text{with} \quad v(d) = g_1^{-1} \text{diag}(\tau_1, \dots, \tau_n) \mathbf{V}. \tag{6.26}$$

Proposition 6.6. *The $2n(d - 1)$ reduced integrals of motion (6.25), which include the n reduced Hamiltonians $\text{tr}(L^k)$, are functionally independent on \mathcal{S}_1 . On each connected component of \mathcal{S}_1 , n further integrals of motion may be selected from the real and imaginary parts of the functions (6.26) in such a way that together with (6.25) they provide a set of $2nd - n$ independent functions.*

Proof. We are going to prove functional independence by inspection of Jacobian determinants using the coordinates on \mathcal{S}_1 exhibited in Corollary 6.5. Let us first consider the functions given by (6.25). If we order the $2n(d - 1)$ functions as written in (6.25) and also order the $2n(d - 1)$ coordinates as written in (6.23), then the corresponding Jacobian matrix J takes a block lower-triangular form, with $n \times n$ blocks. The first diagonal block, $(\partial \text{tr} L^k / \partial y_j)$, is given by $Y \in \text{Mat}_{n \times n}(\mathbb{R})$ with $Y_{kj} = ky_j^{k-1}$, while all other diagonal blocks are given by XD_1 with $X_{kj} = y_j^k$ and $D_1 = \text{diag}(v(1)_1, \dots, v(1)_n)$, except the second one, $(\partial I_{1,1}^k / \partial v(1)_j)$, which equals $2XD_1$. By the definition of \mathcal{S}_1 , the coordinates y_j are positive and distinct, while the $v(1)_j$ are positive, so that X , Y and D_1 are invertible. Hence, J has rank $2n(d - 1)$.

To continue, consider the $2n$ functions

$$\Re(I_{d,1}^k), \Im(I_{d,1}^k) \quad \text{with} \quad 1 \leq k \leq n. \tag{6.27}$$

It is clear that any function G taken from (6.25) satisfies $\partial G / \partial t_j = 0$. So our claim will follow if there exists a subset of n functions F_1, \dots, F_n from those in (6.27) for which the Jacobian matrix $\left(\frac{\partial F_k}{\partial t_l} \right)_{kl}$ is invertible.

Note from Lemma 6.4 and Corollary 6.5 that

$$\frac{\partial v(d)_j}{\partial t_l} = i(g_1^{-1})_{jl} e^{it_l} \mathbf{V}_l, \tag{6.28}$$

since \mathbf{V} and g_1 depend only on the variables (6.23). In particular, the matrix

$$\frac{\partial v(d)}{\partial t} := \left(\frac{\partial v(d)_j}{\partial t_l} \right)_{1 \leq j, l \leq n} \tag{6.29}$$

is invertible, because so are $g_1, \text{diag}(e^{it_1}, \dots, e^{it_n})$ and $\text{diag}(\mathbf{V}_1, \dots, \mathbf{V}_n)$.

If the $2n \times n$ real matrix

$$\left(\frac{\partial \left(\Re(I_{d,1}^k), \Im(I_{d,1}^k) \right)}{\partial t_l} \right)_{1 \leq k, l \leq n} \tag{6.30}$$

has rank n , then we are done. Assume by contradiction that this matrix has rank less than n . From (6.26), we have

$$\frac{\partial I_{d,1}^k}{\partial t_l} = \sum_j v(1)_j y_j^k \frac{\partial v(d)_j}{\partial t_l}, \tag{6.31}$$

and therefore we can write the following equality of complex matrices

$$\frac{\partial I_{d,1}}{\partial t} := \left(\frac{\partial I_{d,1}^k}{\partial t_l} \right)_{1 \leq k, l \leq n} = X \text{diag}(v(1)_1, \dots, v(1)_n) \frac{\partial v(d)}{\partial t}, \tag{6.32}$$

where X is given by $X_{kj} = y_j^k$ as before. We have already established that all three factors in the above product of matrices are invertible. Thus, $\partial I_{d,1}/\partial t$ is invertible and hence has rank n .

To finish the proof, it suffices to remark that the complex matrix $\partial I_{d,1}/\partial t$ is a complex linear combination of the rows of the matrix given in (6.30). If the latter matrix has rank strictly less than n , then so does $\partial I_{d,1}/\partial t$, which gives a contradiction. \square

Let us recall from Proposition 5.5 that the unreduced phase space supports the polynomial Poisson algebra \mathcal{I}_L (5.16), whose Poisson center contains the polynomial algebra

$$\mathfrak{H}_{\text{tr}} := \mathbb{R}[\text{tr} L^k, k \geq 0]. \tag{6.33}$$

Since these Poisson algebras consist of $U(n)$ invariant functions, they engender corresponding Poisson algebras $\mathcal{I}_L^{\text{red}}$ and $\mathfrak{H}_{\text{tr}}^{\text{red}}$ over the reduced phase space \mathcal{M}_{red} . Our final result is a direct consequence of Proposition 6.6.

Theorem 6.7. *The reduced polynomial algebras of functions $\mathfrak{H}_{\text{tr}}^{\text{red}}$ and $\mathcal{I}_L^{\text{red}}$ inherited from \mathfrak{H}_{tr} (6.33) and \mathcal{I}_L (5.16) have functional dimension n and $2nd - n$, respectively. In particular, on the phase space \mathcal{M}_{red} of dimension $2nd$, the Abelian Poisson algebra $\mathfrak{H}_{\text{tr}}^{\text{red}}$ yields a real-analytic, degenerate integrable system with integrals of motion $\mathcal{I}_L^{\text{red}}$.*

Proof. Let us consider $\mathcal{I}_L^{\text{red}}$ and its Poisson center $\mathcal{Z}(\mathcal{I}_L^{\text{red}})$. Denote r and r_0 the functional dimensions of these polynomial algebras of functions. Observe from Proposition 6.6 that

$$r \geq (2nd - n) \quad \text{and} \quad r_0 \geq n. \tag{6.34}$$

The second inequality holds since $\mathfrak{H}_{\text{tr}}^{\text{red}}$ is contained in $\mathcal{Z}(\mathcal{I}_L^{\text{red}})$, and Proposition 6.6 implies that the functional dimension of $\mathfrak{H}_{\text{tr}}^{\text{red}}$ is n .

In a neighborhood U_0 of a generic point of \mathcal{M}_{red} , we can choose a system of coordinates given by $2nd$ functions F_1, \dots, F_{2nd} such that the first r

functions belong to $\mathcal{I}_L^{\text{red}}$, of which the first r_0 belong to $\mathcal{Z}(\mathcal{I}_L^{\text{red}})$. In terms of such coordinates, the Poisson matrix $P = (\{F_i, F_j\})_{i,j}$ can be decomposed into blocks as

$$P = \begin{pmatrix} 0_{r_0 \times r_0} & 0_{r_0 \times (r-r_0)} & B \\ 0_{(r-r_0) \times r_0} & A & * \\ -B^T & * & * \end{pmatrix}. \tag{6.35}$$

This matrix must be non-degenerate since the reduced phase space is a symplectic manifold. In particular, this implies that the r_0 rows of B must be independent. Then the number of independent columns of B must be also r_0 , which cannot be bigger than the number of columns. This gives $r_0 \leq (2nd - r)$, or equivalently

$$r_0 + r \leq 2nd. \tag{6.36}$$

By combining (6.34) with (6.36), we obtain that $r_0 = n$ and $r = (2nd - n)$. □

We see from the above proof that $\mathcal{Z}(\mathcal{I}_L^{\text{red}})$ and $\mathfrak{H}_{\text{tr}}^{\text{red}}$ have the same functional dimension. Since $\mathfrak{H}_{\text{tr}}^{\text{red}} \subseteq \mathcal{Z}(\mathcal{I}_L^{\text{red}})$, we expect that these polynomial algebras of functions actually coincide.

Remark 6.8. Let us explain that our coordinates on \mathcal{S}_1 are very close to action-angle variables. To start, we recall that the joint level surfaces of the integrals of motion of the unreduced free system are compact, because (with the help of the variables (g_R, L, W)) they can be identified with closed subsets of $U(n)$. This compactness property is inherited by the reduced system. If we restrict ourselves to the open subset of the reduced phase space parametrized by \mathcal{S}_1 , then the connected components of the joint level surfaces of the elements of $\mathcal{I}_L^{\text{red}}$ (5.16) are the n -dimensional ‘ Γ -tori’ obtained by fixing all variables in (6.23) and (6.24) except the Γ_j . Both the gauge slice \mathcal{S} (6.9) and its subset \mathcal{S}_1 are invariant under the flow (2.42) of the Hamiltonian $H_k := \frac{1}{2k} \text{tr}(L^k)$, for every $k = 1, \dots, n$, which gives the following linear flow on the Γ -torus:

$$\Gamma_j(t) = \exp(iy_j^k t) \Gamma_j^0, \quad \forall j = 1, \dots, n, \tag{6.37}$$

where Γ_j^0 refers to the initial value. This statement holds since for $H_k(L) \equiv h_k(b_R)$ one has $Dh_k(b_R) = iL^k$. The flow (6.37) entails that on \mathcal{S}_1 the variables $\hat{p}_j := \frac{1}{2} \log y_j$ are canonical conjugates to the angles γ_j in $\Gamma_j = e^{i\gamma_j}$, i.e., they satisfy $\{\gamma_j, \hat{p}_l\}_{\text{red}} = \delta_{jl}$.

7. Conclusion

In this paper, we investigated a trigonometric real form of the spin RS system (1.2) introduced originally by Krichever and Zabrodin [32] and studied subsequently in [5, 10] in the complex holomorphic setting. We have shown that this real form arises from Hamiltonian reduction of a free system on a spin extended Heisenberg double of the $U(n)$ Poisson–Lie group and exploited the reduction approach for obtaining a detailed characterization of its main features. In particular, we presented two models of dense open subsets of the reduced phase

space where the system lives. The model developed in Sect. 3.3 led to an elegant description of the reduced symplectic form (Theorem 3.14), while the equations of motion and the corresponding Hamiltonian are complicated in the pertinent variables based on the ‘primary spins.’ On the other hand, the model studied in Sect. 3.2 and in Sects. 4 and 5 allowed us to recover the spin RS equations of motion (1.8) from the projection of a free flow (Corollary 4.3), but the reduced Poisson brackets (Theorem 5.8) take a relatively complicated form in the underlying ‘dressed spin’ variables. In our framework, the solvability of the evolution equations by linear algebraic manipulations emerges naturally (Remark 4.2), and we also proved their degenerate integrability by explicitly exhibiting the required number of constants of motion (Theorem 6.7).

A basic ingredient of the unreduced phase space that we started with was a $U(n)$ covariant Poisson structure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ that goes back to Zakrzewski [56], for which we found the corresponding moment map (Proposition A.3) and symplectic form (Proposition A.6).

We finish by highlighting a few open problems related to our current research. As always in the reduction treatment of an integrable Hamiltonian system, one should gain as complete an understanding of the global structure of the reduced phase space as possible. The basic point is that the projections of free flows are automatically complete, but only on the full reduced phase space. In the present case, one should actually construct two global models of the reduced phase space: one fitted to the system that we have studied and another one that should be associated with its action-angle dual. Without going into details, we refer to the literature [20, 23, 44, 49] where it is explained that the integrable many-body systems usually come in dual pairs, and the same holds for their several spin extensions. In our case, the commuting Hamiltonians of the dual system are expected to arise from the reduction of the Abelian Poisson algebra $\hat{\mathfrak{h}} = \Xi_R^*(C^\infty(U(n)))$, which is in some sense dual to \mathfrak{h} (2.40) on which our system was built.

It could be interesting to explore generalizations of the construction employed in our study. For example, one may obtain new variants of the trigonometric spin RS model by replacing some or all of the primary spins w^α by z^α subject to the Poisson bracket described at the end of Appendix A (Remark A.7). Generalization of our reduction in which the Heisenberg double is replaced by a quasi-Hamiltonian double of the form $U(n) \times U(n)$ [2], and the primary spins are also modified suitably, should lead to compactified spin RS systems. It should be possible to uncover a reduction picture behind the hyperbolic real form of the spinless and spin RS models, too. All these issues, as well as the questions of quantization and the reduction approach to elliptic spin RS models, pose challenging problems for future work.

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A. Properties of the primary spin variables

In this appendix, we first elaborate the properties of the primary spin variables that were summarized in Proposition 2.1. As was already mentioned, the pertinent Poisson structure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ is a special case of the $U(n)$ covariant Poisson structures due to Zakrzewski [56]. Nevertheless, to make our text self-contained, we shall also verify its Jacobi identity and covariance property. Then, we present the corresponding moment map and symplectic form, which have not been considered in previous work.

For any real function $F \in C^\infty(\mathbb{C}^n)$, we define its \mathbb{C}^n -valued ‘gradient’ ∇F by the equality⁶

$$\Im((\nabla F(w))^\dagger V) := \left. \frac{d}{dt} \right|_{t=0} F(w + tV), \quad \forall w, V \in \mathbb{C}^n, \quad (\text{A.1})$$

where the elements of \mathbb{C}^n are viewed as column vectors. We note that any real linear function on the real vector space \mathbb{C}^n is of the form

$$F_\xi(w) := \Im(\xi^\dagger w), \quad (\text{A.2})$$

for some $\xi \in \mathbb{C}^n$, and for such function $\nabla F_\xi = \xi$. Next we give a convenient presentation of Zakrzewski's Poisson bracket.

Proposition A.1. *For real functions $F, H \in C^\infty(\mathbb{C}^n)$, let $\xi(w) := \nabla F(w)$ and $\eta(w) := \nabla H(w)$. Then, the following formula*

$$\{F, H\}(w) = \Im(\xi(w)^\dagger (\eta(w)^\dagger)_{u(n)} w - \frac{1}{2} \xi(w)^\dagger w \eta(w)^\dagger w - \frac{1}{2} \xi(w)^\dagger w w^\dagger \eta(w))$$

⁶This is a *symplectic* gradient associated with the standard symplectic form, $\omega(\xi, \eta) = \Im(\xi^\dagger \eta)$, on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

$$-\xi(w)^\dagger \eta(w), \tag{A.3}$$

where the notation (2.4) is used, defines a Poisson bracket on $C^\infty(\mathbb{C}^n)$. Equivalently to the formula (A.3), the Hamiltonian vector field V_H associated with $H \in C^\infty(\mathbb{C}^n)$ is given by

$$V_H(w) = (w\eta(w)^\dagger)_{\mathfrak{u}(n)} w - \eta(w) - \frac{1}{2}(\eta(w)^\dagger w + w^\dagger \eta(w))w, \quad \eta(w) = \nabla H(w). \tag{A.4}$$

Extending the real Poisson bracket to complex functions by complex bilinearity, the Poisson brackets of the component functions $w \mapsto w_i$ satisfy the explicit formulae (2.28) and (2.29).

Proof. The anti-symmetry of the last two terms of (A.3) is obvious, while the anti-symmetry of the sum of the first and second terms is seen from the identity

$$\Im (\xi^\dagger (w\eta^\dagger)_{\mathfrak{u}} w - \frac{1}{2}\xi^\dagger w\eta^\dagger w) = \frac{1}{2}\Im \operatorname{tr} ((w\eta^\dagger)_{\mathfrak{u}} (w\xi^\dagger)_{\mathfrak{b}} - (w\xi^\dagger)_{\mathfrak{u}} (w\eta^\dagger)_{\mathfrak{b}}), \tag{A.5}$$

where we used constant ξ and η for simplicity. Here and below, the subscripts \mathfrak{u} and \mathfrak{b} stand for $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$.

Regarding the Jacobi identity, it is enough to verify it for linear functions F_ξ, F_η and F_ζ for arbitrary $\xi, \eta, \zeta \in \mathbb{C}^n$. In this verification, we may use the formula (A.4), since this expresses the identity $\{F, H\}(w) = \Im(\xi(w)^\dagger V_H(w))$ and does not rely on the Jacobi identity.

We start by calculating the gradient of $\{F_\xi, F_\eta\}$ from (A.3) and find

$$(\nabla\{F_\xi, F_\eta\}(w))^\dagger = \xi^\dagger (w\eta^\dagger)_{\mathfrak{u}} - \eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} + \frac{1}{2}\eta^\dagger w\xi^\dagger - \frac{1}{2}\xi^\dagger w\eta^\dagger - \frac{1}{2}(w^\dagger \eta)\xi^\dagger + \frac{1}{2}(w^\dagger \xi)\eta^\dagger. \tag{A.6}$$

Combining this with $V_{F_\zeta}(w)$ from (A.4), we have to inspect

$$\begin{aligned} \mathcal{J}(w) &:= \{\{F_\xi, F_\eta\}, F_\zeta\}(w) + \text{cycl. perm.} \\ &= \Im \left[(\nabla\{F_\xi, F_\eta\}(w))^\dagger ((w\xi^\dagger)_{\mathfrak{u}} w - \zeta - \frac{1}{2}(\zeta^\dagger w)w - \frac{1}{2}(w^\dagger \zeta)w) \right] + \text{c.p.} \end{aligned} \tag{A.7}$$

By spelling this out, we obtain

$$\begin{aligned} \mathcal{J}(w) &= \Im \left[\eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} \zeta - \xi^\dagger (w\eta^\dagger)_{\mathfrak{u}} \zeta + \frac{1}{2}(\xi^\dagger w)\eta^\dagger \zeta \right. \\ &\quad \left. - \frac{1}{2}(\eta^\dagger w)\xi^\dagger \zeta + \frac{1}{2}(w^\dagger \eta)\xi^\dagger \zeta - \frac{1}{2}(w^\dagger \xi)\eta^\dagger \zeta + \text{c.p.} \right] \\ &+ \Im \left[\xi^\dagger (w\eta^\dagger)_{\mathfrak{u}} (w\xi^\dagger)_{\mathfrak{u}} w - \eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} (w\xi^\dagger)_{\mathfrak{u}} w + \frac{1}{2}\eta^\dagger w\xi^\dagger (w\xi^\dagger)_{\mathfrak{u}} w - \frac{1}{2}\xi^\dagger w\eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} w \right. \\ &\quad \left. + \frac{1}{2}(w^\dagger \xi)\eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} w - \frac{1}{2}(w^\dagger \eta)\xi^\dagger (w\xi^\dagger)_{\mathfrak{u}} w + \text{c.p.} \right. \\ &\quad \left. - \frac{1}{2}(\zeta^\dagger w)\xi^\dagger (w\eta^\dagger)_{\mathfrak{u}} w + \frac{1}{2}(\zeta^\dagger w)\eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} w - \frac{1}{4}(\eta^\dagger w)(\xi^\dagger w)(\zeta^\dagger w) + \frac{1}{4}(\xi^\dagger w)(\eta^\dagger w)(\zeta^\dagger w) \right. \\ &\quad \left. + \frac{1}{4}(w^\dagger \eta)(\xi^\dagger w)(\zeta^\dagger w) - \frac{1}{4}(w^\dagger \xi)(\eta^\dagger w)(\zeta^\dagger w) + \text{c.p.} \right. \\ &\quad \left. - \frac{1}{2}(w^\dagger \zeta)\xi^\dagger (w\eta^\dagger)_{\mathfrak{u}} w + \frac{1}{2}(w^\dagger \zeta)\eta^\dagger (w\xi^\dagger)_{\mathfrak{u}} w - \frac{1}{4}(w^\dagger \zeta)(\eta^\dagger w)(\xi^\dagger w) + \frac{1}{4}(w^\dagger \zeta)(\xi^\dagger w)(\eta^\dagger w) \right. \\ &\quad \left. + \frac{1}{4}(w^\dagger \eta)(w^\dagger \zeta)(\xi^\dagger w) - \frac{1}{4}(w^\dagger \xi)(w^\dagger \zeta)(\eta^\dagger w) + \text{c.p.} \right] \end{aligned}$$

After making several self-evident cancellations, and using cyclic permutations to reorganize terms in a convenient way, we get

$$\begin{aligned} \mathcal{J}(w) &= \Im \left(\zeta^\dagger (w\eta^\dagger)_u \xi - \xi^\dagger (w\eta^\dagger)_u \zeta + \frac{1}{2} (\xi^\dagger w - w^\dagger \xi) \eta^\dagger \zeta - \frac{1}{2} (\eta^\dagger w - w^\dagger \eta) \xi^\dagger \zeta + \text{c.p.} \right) \\ &\quad + \Im \text{tr} \left(w\xi^\dagger [(w\eta^\dagger)_u, (w\zeta^\dagger)_u] + (\eta^\dagger w) w\xi^\dagger (w\zeta^\dagger)_u - (\xi^\dagger w) w\eta^\dagger (w\zeta^\dagger)_u + \text{c.p.} \right). \end{aligned}$$

It is not difficult to see that the first line gives zero. Rearranging the second line, we have

$$\begin{aligned} \mathcal{J}(w) &= \Im \text{tr} \left(w\xi^\dagger [(w\eta^\dagger)_u, (w\zeta^\dagger)_u] + w\eta^\dagger w\xi^\dagger (w\zeta^\dagger)_u - w\xi^\dagger w\eta^\dagger (w\zeta^\dagger)_u + \text{c.p.} \right) \\ &= \Im \text{tr} \left(-w\xi^\dagger [(w\eta^\dagger)_b, (w\zeta^\dagger)_u] + \text{c.p.} \right) \\ &= -\Im \text{tr} \left((w\xi^\dagger)_u ([(w\eta^\dagger)_b, (w\zeta^\dagger)_u] + [w\eta^\dagger, (w\zeta^\dagger)_b]) \right. \\ &\quad \left. + (w\xi^\dagger)_b ([(w\eta^\dagger)_b, (w\zeta^\dagger)_u] + [(w\eta^\dagger)_u, w\zeta^\dagger]) \right) \\ &= -\Im \text{tr} \left((w\xi^\dagger)_u ([(w\eta^\dagger)_b, (w\zeta^\dagger)_u] + [(w\eta^\dagger)_u, (w\zeta^\dagger)_b] + [(w\eta^\dagger)_b, (w\zeta^\dagger)_b]) \right. \\ &\quad \left. + (w\xi^\dagger)_b ([(w\eta^\dagger)_b, (w\zeta^\dagger)_u] + [(w\eta^\dagger)_u, (w\zeta^\dagger)_u] + [(w\eta^\dagger)_u, (w\zeta^\dagger)_b]) \right) \\ &= \Im \text{tr} (w\xi^\dagger [w\eta^\dagger, w\zeta^\dagger]) + \text{cycl. perm.} = 0. \end{aligned}$$

Having verified the Jacobi identity, it remains to calculate the Poisson brackets of the components of w and their complex conjugates. Let e_k ($k = 1, \dots, n$) denote the canonical basis of \mathbb{C}^n . One obtains by tedious calculation that the Hamiltonian vector fields of the linear functions given by the real and imaginary parts of the components w_k have the following form:

$$V_{\Re w_k}(w) = \begin{cases} i\Re(w_k)w_k e_k + i \sum_{r>k} (w_k w_r e_r + |w_r|^2 e_k) + i e_k - \frac{1}{2} i (w_k - \bar{w}_k) w & k < n, \\ i\Re(w_n)w_n e_n + i e_n - \frac{1}{2} i (w_n - \bar{w}_n) w & k = n. \end{cases}$$

and

$$V_{\Im w_k}(w) = \begin{cases} i\Im(w_k)w_k e_k + \sum_{r>k} (w_k w_r e_r - |w_r|^2 e_k) - e_k - \frac{1}{2} (w_k + \bar{w}_k) w & k < n, \\ i\Im(w_n)w_n e_n - e_n - \frac{1}{2} (w_n + \bar{w}_n) w & k = n. \end{cases}$$

By using these, one can check that the formulae (2.28) and (2.29) follow. If desired, the reader can supply the details. \square

The bracket (A.3) has the nice property that the natural action of $U(n)$ on \mathbb{C}^n is Poisson [56], and this can also be checked using linear functions F_ξ . To this end, for any $g \in U(n)$ and $w \in \mathbb{C}^n$ we define the functions $F_\xi(g \cdot) \in C^\infty(\mathbb{C}^n)$ and $F_\xi(\cdot w) \in C^\infty(U(n))$ by

$$F_\xi(g \cdot)(w) = F_\xi(gw) = F_\xi(\cdot w)(g). \tag{A.8}$$

Then, an easy calculation gives that

$$\{F_\xi, F_\eta\}(gw) - \{F_\xi(g \cdot), F_\eta(g \cdot)\}(w) \tag{A.9}$$

is equal to

$$\Im \text{tr}(gw\xi^\dagger(gw\eta^\dagger)_{\mathfrak{u}(n)} - w\xi^\dagger g(w\eta^\dagger g)_{\mathfrak{u}(n)}), \tag{A.10}$$

which in turn is equal to the value at g of the Poisson bracket (2.10) of the functions $F_\xi(\cdot w)$ and $F_\eta(\cdot w)$ on $\mathfrak{U}(n)$. The last equality follows using $DF_\xi(\cdot w)(g) = (gw\xi^\dagger)_{\mathfrak{b}(n)}$ and elementary manipulations. Thus, we have

$$\{F_\xi, F_\eta\}(gw) = \{F_\xi(g\cdot), F_\eta(g\cdot)\}(w) + \{F_\xi(\cdot w), F_\eta(\cdot w)\}_U(g), \tag{A.11}$$

which means that the map $\mathfrak{U}(n) \times \mathbb{C}^n \ni (g, w) \mapsto gw \in \mathbb{C}^n$ is indeed a Poisson map.

Let us recall the diffeomorphism

$$b \mapsto bb^\dagger \tag{A.12}$$

from the group $\mathfrak{B}(n)$ to the space $\mathfrak{P}(n)$ of positive definite Hermitian matrices. By this diffeomorphism, the Poisson structure (2.11) on $\mathfrak{B}(n)$ is converted into a Poisson structure on $\mathfrak{P}(n)$, which is given by the first term of (2.23), i.e. ,

$$\{f, h\}_{\mathfrak{P}}(L) = 4\langle Ldf(L), (Ldh(L))_{\mathfrak{u}(n)} \rangle \tag{A.13}$$

for all $f, h \in C^\infty(\mathfrak{P}(n))$. Here, the $\mathfrak{u}(n)$ -valued derivatives df and dh are defined by (2.22).

Proposition A.2. *With respect to the brackets (A.3) and (A.13), the map*

$$\Phi : w \mapsto \mathbf{1}_n + ww^\dagger \tag{A.14}$$

from \mathbb{C}^n to $\mathfrak{P}(n)$ is Poisson.

Proof. Let $X, Y \in \mathfrak{u}(n)$ and consider the pull-backs $\Phi^*(f_X)$ and $\Phi^*(f_Y)$ of the functions $f_X(L) := \langle X, L \rangle$ and $f_Y(L) := \langle Y, L \rangle$. We have

$$\Phi^*(f_X)(w) = \Im(w^\dagger Xw) + \Im \text{tr}(X) \quad \text{and} \quad \Phi^*(f_Y)(w) = \Im(w^\dagger Yw) + \Im \text{tr}(Y). \tag{A.15}$$

Using the formula (A.3) with $(\nabla \Phi^*(f_X))(w) = -2Xw$ and similar for f_Y , we can compute

$$\begin{aligned} \{\Phi^*(f_X), \Phi^*(f_Y)\}(w) &= 4\Im(w^\dagger X(ww^\dagger Y)_{\mathfrak{u}}w + w^\dagger XYw \\ &\quad - \frac{1}{2}w^\dagger Xww^\dagger Yw + \frac{1}{2}w^\dagger Xww^\dagger Yw) \\ &= 4\Im \text{tr}((\mathbf{1}_n + ww^\dagger)X((\mathbf{1}_n + ww^\dagger)Y)_{\mathfrak{u}(n)}) = \{f_X, f_Y\}_{\mathfrak{P}}(\Phi(w)). \end{aligned} \tag{A.16}$$

Here, we have taken into account that, for example, $\Im \text{tr}(XY) = 0$ for $X, Y \in \mathfrak{u}(n)$. The statement follows since the linear functions of the form f_X can serve as coordinates on $\mathfrak{P}(n)$. \square

Let $\mathbf{b} : \mathbb{C}^n \rightarrow \mathfrak{B}(n)$ be the map determined by the condition

$$\Phi = \mathbf{b}\mathbf{b}^\dagger. \tag{A.17}$$

It follows from Proposition A.2 that this is a Poisson map with respect to the Poisson brackets (A.3) on \mathbb{C}^n and (2.11) on $\mathfrak{B}(n)$.

Proposition A.3. *The map \mathbf{b} defined by (A.17) with (A.14) is the moment map for the Poisson action (2.26) of $U(n)$ on \mathbb{C}^n . According to (2.27), this means that we have*

$$\Im \left((\nabla F(w))^\dagger X w \right) = \Im \text{tr} \left(X \{F, \mathbf{b}\}(w) \mathbf{b}(w)^{-1} \right), \quad \forall X \in \mathfrak{u}(n), w \in \mathbb{C}^n, F \in C^\infty(\mathbb{C}^n). \tag{A.18}$$

Proof. For ease of notation, we verify the relation for linear functions F_ξ on \mathbb{C}^n , which is sufficient. For this, we have to calculate the $\mathfrak{b}(n)$ -valued function

$$\beta_F := \{ \mathbf{b}, F \} \mathbf{b}^{-1}, \quad F := F_\xi. \tag{A.19}$$

Since (A.12) is a diffeomorphism, β_F is uniquely determined by

$$\{ \Phi, F \} = \beta_F \Phi + \Phi \beta_F^\dagger, \tag{A.20}$$

and this can be calculated as follows. First, we rearrange the expression (A.4) of the Hamiltonian vector field in the form

$$V_F(w) = \frac{1}{2} (\xi^\dagger w - w^\dagger \xi) w - \xi - (w \xi^\dagger)_{\mathfrak{b}(n)} w. \tag{A.21}$$

Then, as $(\xi^\dagger w - w^\dagger \xi) \in i\mathbb{R}$, we obtain

$$\begin{aligned} \{ \Phi, F \}(w) &= V_F(w) w^\dagger + w (V_F(w))^\dagger \\ &= -(w \xi^\dagger)_{\mathfrak{b}(n)} w w^\dagger - w w^\dagger (w \xi^\dagger)_{\mathfrak{b}(n)}^\dagger - \xi w^\dagger - w \xi^\dagger \\ &= -(w \xi^\dagger)_{\mathfrak{b}(n)} \Phi(w) - \Phi(w) (w \xi^\dagger)_{\mathfrak{b}(n)}^\dagger + ((w \xi^\dagger)_{\mathfrak{b}(n)} - w \xi^\dagger) \\ &\quad + ((w \xi^\dagger)_{\mathfrak{b}(n)}^\dagger - \xi w^\dagger). \end{aligned} \tag{A.22}$$

But the last two terms cancel, and hence we see that

$$\beta_F(w) = - (w \xi^\dagger)_{\mathfrak{b}(n)}. \tag{A.23}$$

By using this, the right-hand-side of (A.18) becomes

$$- \Im \text{tr} (X \beta_F(w)) = \Im \text{tr} (X w \xi^\dagger) = \Im (\xi^\dagger X w), \tag{A.24}$$

whereby the proof is complete. □

Remark A.4. We had no need for the explicit formula of $\mathbf{b}(w)$ in the above, but in some other calculations it is needed. The reader can verify directly that it obeys equation (2.31).

Remark A.5. The maximal torus $\mathbb{T}^n < U(n)$ is a Poisson subgroup with vanishing Poisson bracket, and therefore, the restriction of the $U(n)$ action to \mathbb{T}^n gives an ordinary Hamiltonian action. One can identify the dual Poisson–Lie group of \mathbb{T}^n with $B(n)_0$, the group of positive diagonal matrices, with zero Poisson bracket. Then, the corresponding group valued moment map is provided by $w \mapsto \mathbf{b}(w)_0$, which is the diagonal part of $\mathbf{b}(w)$. Writing

$$\mathbf{b}(w)_0 = \exp(\phi(w)), \tag{A.25}$$

we get the ordinary moment map $w \mapsto \phi(w) \in \mathfrak{b}(n)_0$, where $\mathfrak{b}(n)_0$ (the space of real diagonal matrices) is identified with the linear dual of $\mathfrak{u}(n)_0$.

The following proposition represents one of the side results of the paper.

Proposition A.6. *The Poisson bracket (A.3) is symplectic and, with $\mathcal{G}_j = 1 + \sum_{k=j}^n |w_j|^2$, the corresponding symplectic form on \mathbb{C}^n is given by*

$$\Omega_{\mathbb{C}^n} = \frac{i}{2} \sum_{k=1}^n \frac{1}{\mathcal{G}_k} dw_k \wedge d\bar{w}_k + \frac{i}{4} \sum_{k=1}^{n-1} \frac{1}{\mathcal{G}_k \mathcal{G}_{k+1}} d\mathcal{G}_{k+1} \wedge (\bar{w}_k dw_k - w_k d\bar{w}_k). \tag{A.26}$$

Proof. We start from the coordinate form of the Poisson bracket, copied here for convenience:

$$\begin{aligned} \{w_i, w_k\} &= i \operatorname{sgn}(i - k) w_i w_k \\ \{w_i, \bar{w}_l\} &= i\delta_{il}(2 + |w|^2) + iw_i \bar{w}_l + i\delta_{il} \sum_{r=1}^n \operatorname{sgn}(r - l) |w_r|^2. \end{aligned} \tag{A.27}$$

We shall first invert the Poisson tensor on the dense open submanifold on which all $|w_j| > 0$, where we can use the parametrization $w_j = e^{i\varphi_j} |w_j|$.

Let us consider

$$\begin{aligned} \{w_i, |w_k|^2\} &= i \operatorname{sgn}(i - k) |w_k|^2 w_i + i |w_k|^2 w_i + i\delta_{ik}(2 + |w|^2) w_i \\ &\quad + i\delta_{ik} w_i \sum_{r=1}^n \operatorname{sgn}(r - k) |w_r|^2, \end{aligned} \tag{A.28}$$

from which we easily obtain

$$\{|w_i|^2, |w_k|^2\} = 0. \tag{A.29}$$

Using this, and restricting now to our submanifold, the relation (A.28) implies

$$\{e^{i\varphi_i}, |w_k|^2\} = \left\{ \frac{w_i}{|w_i|}, |w_k|^2 \right\} = i[1 - \delta_{ik} + \operatorname{sgn}(i - k)] |w_k|^2 e^{i\varphi_i} + 2i\delta_{ik} \mathcal{G}_k e^{i\varphi_i}. \tag{A.30}$$

Plainly, we have the identity

$$\{w_j, w_k\} + e^{2i\varphi_j} e^{2i\varphi_k} \{\bar{w}_j, \bar{w}_k\} = 2|w_j w_k| \{e^{i\varphi_j}, e^{i\varphi_k}\}. \tag{A.31}$$

The left-hand side can be checked to vanish, and thus, we get

$$\{e^{i\varphi_j}, e^{i\varphi_k}\} = 0. \tag{A.32}$$

It is convenient to change variables, noting that the map $(|w_1|^2, \dots, |w_n|^2) \mapsto (\mathcal{G}_1, \dots, \mathcal{G}_n)$ is invertible. Then, it is elementary to derive from (A.30) the relation

$$\{e^{i\varphi_i}, \mathcal{G}_k\} = \begin{cases} 2i\mathcal{G}_k e^{i\varphi_i}, & k \leq i \\ 0, & k > i \end{cases} \tag{A.33}$$

that can be also written as

$$\{\varphi_i, \ln \mathcal{G}_k\} = \begin{cases} 2, & k \leq i \\ 0, & k > i \end{cases} \tag{A.34}$$

This means that the matrix of Poisson brackets, in the variables $\varphi_i, \ln \mathcal{G}_k$ has the form

$$P = 2 \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} \tag{A.35}$$

with

$$A = \mathbf{1}_n + B + B^2 + \dots + B^{n-1}, \tag{A.36}$$

where B is the nilpotent matrix having the entries $B_{ik} = \delta_{i,k+1}$. Both A and P are invertible, and their inverses are

$$A^{-1} = \mathbf{1}_n - B \quad \text{and} \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -(A^{-1})^T \\ A^{-1} & 0 \end{pmatrix}. \tag{A.37}$$

Consequently, we obtain the symplectic form⁷ (x^α represent the local variables φ_i and $\ln \mathcal{G}_k$)

$$\begin{aligned} \Omega &= \frac{1}{2} \sum_{\alpha, \beta=1}^{2n} (P^{-1})_{\alpha\beta} dx^\alpha \wedge dx^\beta \\ &= \frac{1}{2} \sum_{k=1}^{n-1} [d \ln \mathcal{G}_k - d \ln \mathcal{G}_{k+1}] \wedge d\varphi_k + \frac{1}{2} d \ln \mathcal{G}_n \wedge d\varphi_n. \end{aligned} \tag{A.38}$$

If we now substitute the identities

$$d \ln \mathcal{G}_k - d \ln \mathcal{G}_{k+1} = \frac{\mathcal{G}_{k+1} d|w_k|^2 - |w_k|^2 d\mathcal{G}_{k+1}}{\mathcal{G}_k \mathcal{G}_{k+1}} \tag{A.39}$$

and

$$d\varphi_k = (2i|w_k|^2)^{-1} (\bar{w}_k dw_k - w_k d\bar{w}_k), \tag{A.40}$$

then Ω (A.38) takes the form

$$\Omega = \frac{i}{2} \sum_{k=1}^n \frac{1}{\mathcal{G}_k} dw_k \wedge d\bar{w}_k + \frac{i}{4} \sum_{k=1}^{n-1} \frac{1}{\mathcal{G}_k \mathcal{G}_{k+1}} d\mathcal{G}_{k+1} \wedge (\bar{w}_k dw_k - w_k d\bar{w}_k). \tag{A.41}$$

It is clear that both the original Poisson tensor corresponding to (A.27) and Ω (A.41) are regular over the whole of \mathbb{C}^n . As a result, their inverse relationship extends from the dense open submanifold (where $|w_j| > 0$ for all j) to the full phase space. \square

Remark A.7. The image of the map $w \mapsto \mathbf{1}_n + ww^\dagger$ is the union of the $U(n)$ orbits in $\mathfrak{P}(n)$ passing through the degenerate diagonal matrices

$$\text{diag}(1 + R^2, 1, \dots, 1), \quad R \geq 0. \tag{A.42}$$

⁷In our convention, the wedge does not contain $\frac{1}{2}$ and $dH = \Omega(\cdot, V_H)$ with the Hamiltonian vector field V_H .

For any fixed $R > 0$, the orbit is a symplectic leaf in $\mathfrak{P}(n)$ of dimension $2(n - 1)$; $R = 0$ corresponds to a trivial symplectic leaf. The union of the orbits consisting of the conjugates of the matrices

$$\text{diag}(1 - r^2, 1, \dots, 1), \quad 0 \leq r < 1 \tag{A.43}$$

is the image of the map

$$z \mapsto \mathbf{1}_n - zz^\dagger \tag{A.44}$$

from

$$\mathcal{B}(1) := \{z \in \mathbb{C}^n \mid |z|^2 < 1\} \tag{A.45}$$

to $\mathfrak{P}(n)$. In fact, the open ball $\mathcal{B}(1)$, identified as a subset of \mathbb{R}^{2n} , can be equipped with the Poisson bracket

$$\begin{aligned} \{z_i, z_k\} &= i \operatorname{sgn}(i - k) z_i z_k \\ \{z_i, \bar{z}_l\} &= i(|z|^2 - 2)\delta_{il} + iz_i \bar{z}_l + i\delta_{il} \sum_{r=1}^n \operatorname{sgn}(r - l) |z_r|^2 \end{aligned} \tag{A.46}$$

with respect to which the map (A.44) is Poisson. This is also a special case of the Poisson structures found in [56]. The analogue of Proposition A.3 holds for the map $\mathbf{b}_- : \mathcal{B}(1) \rightarrow \mathcal{B}(n)$ defined by

$$\mathbf{1}_n - zz^\dagger = \mathbf{b}_-(z)\mathbf{b}_-(z)^\dagger. \tag{A.47}$$

The Poisson map \mathbf{b}_- can be used to introduce variants of our reduction. Concretely, one may replace one or more of the \mathbf{b} factors in (3.1) by \mathbf{b}_- and study the reduced system. The restriction $\gamma > 0$ in the moment map constraint (3.17) then might not be necessary. Let us also note that one obtains a Poisson pencil on \mathbb{C}^n if one replaces the last term of (A.3) by $-\lambda \Im(\xi(w)^\dagger \eta(w))$ for any real parameter λ , and the formula (A.46) corresponds to $\lambda = -1$.

B. Proof of Lemma 5.1

In this section, we work over $(\mathbb{C}^{n \times d}, \{ , \}_{\mathcal{W}})$ with the primary spins (w^α) , see Sect. 2.2. We set $\{ , \} := \{ , \}_{\mathcal{W}}$ to simplify notations.

As noted in Sect. 3.1, the half-dressed spins v^α can be defined in $\mathbb{C}^{n \times d}$ in terms of the primary spins. It is convenient to introduce the matrices $b_\alpha = \mathbf{b}(w^\alpha)$ and $B^\alpha = b_1 \cdots b_\alpha$, so that

$$v^\alpha = B^{\alpha-1} w^\alpha. \tag{B.1}$$

Remark that B^α is related to the matrix B_α introduced in (3.2) by $B_\alpha = b_R B^\alpha$. We also note the following lemma, which follows from Proposition 2.1 by straightforward computations.

Lemma B.1. *For any $1 \leq \alpha \leq d$, $1 \leq i, j, l \leq n$,*

$$\{w_i^\alpha, (b_\alpha)_{jl}\} = i[\delta_{ij} + 2\delta_{(i>j)}] w_j^\alpha (b_\alpha)_{il}, \tag{B.2}$$

$$\{\bar{w}_i^\alpha, (b_\alpha)_{jl}\} = -i\delta_{ij} \bar{w}_i^\alpha (b_\alpha)_{il} - 2i\delta_{ij} \sum_{k=j+1}^l \bar{w}_k^\alpha (b_\alpha)_{kl}. \tag{B.3}$$

Furthermore, the Poisson bracket evaluated on $((b_\alpha)_{ij}, (\bar{b}_\alpha)_{ij})$ is given by (2.7)–(2.8).

Next, we need to describe the Poisson brackets between the matrix entries of (B^α, w^α) , which appear in the decomposition (B.1). To write them down, we introduce the matrices

$$B^{\alpha;\gamma} = b_\alpha \cdots b_\gamma, \quad 1 \leq \alpha \leq \gamma \leq d, \tag{B.4}$$

which are such that $B^{1;\alpha} = B^\alpha$ and $B^{\alpha;\alpha} = b_\alpha$. We also set $B^{\alpha+1;\alpha} := \mathbf{1}_n$ and $B^0 := \mathbf{1}_n$.

Lemma B.2. *For any $1 \leq \alpha, \beta \leq d, 1 \leq i, k, l \leq n$,*

$$\{w_i^\alpha, B_{kl}^\beta\} = -i\delta_{(\alpha \leq \beta)} B_{ki}^{\alpha-1} w_i^\alpha B_{il}^{\alpha;\beta} + 2i\delta_{(\alpha \leq \beta)} \sum_{k' \leq i} B_{kk'}^{\alpha-1} w_{k'}^\alpha B_{il}^{\alpha;\beta}, \tag{B.5}$$

$$\{\bar{w}_i^\alpha, B_{kl}^\beta\} = +i\delta_{(\alpha \leq \beta)} B_{ki}^{\alpha-1} \bar{w}_i^\alpha B_{il}^{\alpha;\beta} - 2i\delta_{(\alpha \leq \beta)} B_{ki}^{\alpha-1} \sum_{i \leq u} \bar{w}_u^\alpha B_{ul}^{\alpha;\beta}. \tag{B.6}$$

Proof. By construction, for $\beta \neq \alpha$ we have $\{w_i^\alpha, w_k^\beta\} = 0$, hence $\{w_i^\alpha, (b_\beta)_{kl}\} = 0$. We get that

$$\{w_i^\alpha, B_{kl}^\beta\} = 0, \quad \alpha < \beta; \quad \{w_i^\alpha, B_{kl}^\beta\} = \sum_{k \leq l' \leq l} \{w_i^\alpha, B_{kl'}^\alpha\} B_{l'l}^{\alpha+1;\beta}, \quad \beta > \alpha \tag{B.7}$$

When $\beta = \alpha$, $\{w_i^\alpha, (b_\alpha)_{kl}\}$ is given by (B.2) and we get

$$\{w_i^\alpha, B_{kl}^\alpha\} = -iB_{ki}^{\alpha-1} w_i^\alpha (b_\alpha)_{il} + 2i \sum_{k' \leq i} B_{kk'}^{\alpha-1} w_{k'}^\alpha (b_\alpha)_{il}, \tag{B.8}$$

from which the first identity can be obtained. The second case is proved in the same way. □

Lemma B.3. *For any $1 \leq \alpha, \beta \leq d, 1 \leq i, j, k, l \leq n$,*

$$\begin{aligned} \{B_{ij}^\alpha, B_{kl}^\beta\} &\stackrel{\alpha \leq \beta}{=} -2iB_{kj}^\alpha \sum_{r>j} B_{ir}^\alpha B_{rl}^{\alpha+1;\beta} - iB_{kj}^\alpha B_{ij}^\alpha B_{jl}^{\alpha+1;\beta} + 2i\delta_{(i>k)} B_{kj}^\alpha B_{il}^\beta + i\delta_{ik} B_{kj}^\alpha B_{il}^\beta, \\ \{B_{ij}^\alpha, \bar{B}_{kl}^\beta\} &\stackrel{\alpha \leq \beta}{=} -iB_{ij}^\alpha \bar{B}_{kj}^\alpha \bar{B}_{jl}^{\alpha+1;\beta} - 2i \sum_{s<j} B_{is}^\alpha \bar{B}_{ks}^\alpha \bar{B}_{jl}^{\alpha+1;\beta} + i\delta_{ik} B_{ij}^\alpha \bar{B}_{kl}^\beta + 2i\delta_{ik} \sum_{r>k} B_{rj}^\alpha \bar{B}_{rl}^\beta, \\ \{B_{ij}^\alpha, \bar{B}_{kl}^\beta\} &\stackrel{\alpha \geq \beta}{=} -iB_{il}^\beta \bar{B}_{kl}^\beta B_{lj}^{\beta+1;\alpha} - 2i \sum_{s<l} B_{is}^\beta \bar{B}_{ks}^\beta B_{lj}^{\beta+1;\alpha} + i\delta_{ik} B_{ij}^\alpha \bar{B}_{kl}^\beta + 2i\delta_{ik} \sum_{r>k} B_{rj}^\alpha \bar{B}_{rl}^\beta. \end{aligned}$$

Proof. For the first equality, we have for $\alpha \leq \beta$ that

$$\{B_{ij}^\alpha, B_{kl}^\beta\} = \sum_{1 \leq \gamma \leq \alpha} \sum_{i', j', k', l'} (B^{\gamma-1})_{ii'} (B^{\gamma-1})_{kk'} \{(b_\gamma)_{i'j'}, (b_\gamma)_{k'l'}\} B_{j'j}^{\gamma+1;\alpha} B_{l'l}^{\gamma+1;\beta}. \tag{B.9}$$

A similar expansion holds for $\{B_{ij}^\alpha, \bar{B}_{kl}^\beta\}$. It then suffices to use Lemma B.1. □

Note that in the case $\beta = \alpha$, the Poisson brackets from Lemma B.3 take the usual form (2.7)–(2.8) on $B(n)$. We can also see that we can use $\beta = 0$ in Lemma B.2 and $\alpha, \beta = 0$ in Lemma B.3, since in such cases the Poisson bracket vanishes on $B^0 = \mathbf{1}_n$.

We can now prove Lemma 5.1 using Lemmae B.2 and B.3. We will use the definition of the half-dressed spins given by (B.1). To show (5.1) we need to write

$$\begin{aligned} \{v_i^\alpha, v_k^\beta\} &= \sum_{j,l} \{B_{ij}^{\alpha-1} w_j^\alpha, B_{kl}^{\beta-1} w_l^\beta\} \\ &= \sum_{j,l} \{B_{ij}^{\alpha-1}, B_{kl}^{\beta-1}\} w_j^\alpha w_l^\beta + \sum_{j,l} \{B_{ij}^{\alpha-1}, w_l^\beta\} w_j^\alpha B_{kl}^{\beta-1} \\ &\quad + \sum_{j,l} \{w_j^\alpha, B_{kl}^{\beta-1}\} B_{ij}^{\alpha-1} w_l^\beta + \sum_{j,l} \{w_j^\alpha, w_l^\beta\} B_{ij}^{\alpha-1} B_{kl}^{\beta-1} \end{aligned} \tag{B.10}$$

where we assume $\alpha \leq \beta$ without loss of generality. We can then use Lemmae B.2 and B.3 to show that

$$\{v_i^\alpha, v_k^\alpha\} = -i \operatorname{sgn}(k - i) v_k^\alpha v_i^\alpha; \quad \{v_i^\alpha, v_k^\beta\} = -i \operatorname{sgn}(k - i) v_k^\alpha v_i^\beta + i v_k^\alpha v_i^\beta, \quad \alpha < \beta. \tag{B.11}$$

By anti-symmetry, (B.11) implies that (5.1) holds.

The Poisson bracket (5.2) is computed in the same way and requires to remark in the case $\alpha = \beta$ that

$$\sum_s B_{is}^\gamma \overline{B}_{ks}^\gamma = \sum_{\mu=1}^\gamma v_i^\mu \overline{v}_k^\mu + \delta_{ik}. \tag{B.12}$$

This identity is equivalent to $B^\gamma (B^\gamma)^\dagger = \sum_{\mu=1}^\gamma v^\mu (\overline{v}^\mu)^\dagger + \mathbf{1}_n$, which is obtained by induction on γ using (2.33); it becomes (3.7) when $\gamma = d$.

C. Proof of Theorem 5.8

Recall that we work over the gauge slice $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ (5.23) and wish to compute the reduced Poisson brackets $\{ , \}_{\text{red}}$ of the basic evaluation functions $Q_j = e^{i q_j} \in U(1)$ and $v(\alpha)_j \in \mathbb{C}$, where the latter obey the relations (5.32). Our fundamental tool will be the identity (5.26), which concerns $U(n)$ invariant functions on \mathcal{M} and their pull-backs on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$. Knowing the left-hand side of (5.26), we will be able to determine the reduced Poisson brackets. In the particular case at hand, we consider the invariant functions $f_m, f_m^{\alpha\beta} \in C^\infty(\mathcal{M})$ defined by (5.27). Their Poisson brackets on \mathcal{M} are given by Lemma 5.7, and their restrictions (pull-backs) to $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ are displayed in (5.33). The point is that the right-hand side of (5.26) can be also expressed through the reduced Poisson brackets of the basic variables on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$, which enables us to derive the explicit formulae of Theorem 5.8.

We begin by giving an auxiliary lemma, which will be used below.

Lemma C.1. *The $n \times n$ matrices $\mathcal{E}, \tilde{\mathcal{E}}$ given by*

$$\mathcal{E}_{kl} = Q_l^k \quad \text{and} \quad \tilde{\mathcal{E}}_{kl} = Q_l^k \mathcal{U}_l \tag{C.1}$$

are invertible on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$.

Proof. We can write that $\mathcal{E} = VQ$ with $Q = \text{diag}(Q_1, \dots, Q_n)$ and $V = (V_{kl})$, $V_{kl} = Q_k^{l-1}$, which is a Vandermonde matrix. Since $Q \in \mathbb{T}_{\text{reg}}^n$ on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$, both V and Q are invertible. We also have that $\tilde{\mathcal{E}} = \mathcal{E}D$ where $D = \text{diag}(\mathcal{U}_1, \dots, \mathcal{U}_n)$. As $\mathcal{U}_j > 0$ on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$, $\tilde{\mathcal{E}}$ is also invertible. \square

Deriving (5.37)

Lemma C.2. *For any $i, j = 1, \dots, n$, $\{q_i, q_j\}_{\text{red}} = 0$.*

Proof. From (5.28) and (5.33), we get for any $M, N \in \mathbb{N}$,

$$0 = \xi^* \{f_M, f_N\} = \{\xi^* f_M, \xi^* f_N\}_{\text{red}} = -MN \sum_{i,j=1}^n e^{iMq_i} e^{iNq_j} \{q_i, q_j\}_{\text{red}} .$$

Considering this equality for $M, N = 1, \dots, n$, this is equivalent to

$$\mathcal{E} \hat{U}^{(0)} \mathcal{E}^T = 0_{n \times n} ,$$

where $\hat{U}^{(0)} \in \text{Mat}_{n \times n}(\mathbb{C})$ is given by $\hat{U}_{kl}^{(0)} = \{q_k, q_l\}_{\text{red}}$. By Lemma C.1, \mathcal{E} is invertible on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$ so that $\hat{U}^{(0)}$ is the zero matrix. \square

Lemma C.3. *For any $i, j = 1, \dots, n$,*

$$\{\mathcal{U}_i, q_j\}_{\text{red}} = -\delta_{ij} \mathcal{U}_i, \quad \{v(\alpha)_i, q_j\}_{\text{red}} = -\delta_{ij} v(\alpha)_i, \quad \{\bar{v}(\alpha)_i, q_j\}_{\text{red}} = -\delta_{ij} \bar{v}(\alpha)_i . \tag{C.2}$$

Proof. From (5.29), after summing over all α, β we get for any $M, N \in \mathbb{N}$

$$\begin{aligned} \sum_{i,j} \{\mathcal{U}_i^2 e^{iMq_i}, e^{iNq_j}\}_{\text{red}} &= \sum_{\alpha,\beta} \{\xi^* f_M^{\alpha\beta}, \xi^* f_N\}_{\text{red}} = -2iN \sum_{\alpha,\beta} \xi^* f_{M+N}^{\alpha\beta} \\ &= -2iN \sum_i \mathcal{U}_i^2 e^{i(M+N)q_i} . \end{aligned}$$

Using Lemma C.2, we obtain

$$\sum_{i,j} e^{iMq_i} \mathcal{U}_i e^{iNq_j} \{\mathcal{U}_i, q_j\}_{\text{red}} = - \sum_i \mathcal{U}_i^2 e^{i(M+N)q_i} .$$

We can rewrite this for $N, M = 1, \dots, n$ as

$$\tilde{\mathcal{E}} \hat{U}^{(1)} \mathcal{E}^T = \tilde{\mathcal{E}} U^{(1)} \mathcal{E}^T ,$$

where the $n \times n$ matrices are given by $\hat{U}_{kl}^{(1)} = \{\mathcal{U}_k, q_l\}_{\text{red}}$, $U_{kl}^{(1)} = -\delta_{kl} \mathcal{U}_k$. By Lemma C.1, both \mathcal{E} and $\tilde{\mathcal{E}}$ are invertible. Hence, $\hat{U}^{(1)} = U^{(1)}$.

For the second identity, we use (5.29) with summation over all β , and we get for any $M, N \in \mathbb{N}$

$$\sum_{i,j} \{\mathcal{U}_i v(\alpha)_i e^{iMq_i}, e^{iNq_j}\}_{\text{red}} = -2iN \sum_i v(\alpha)_i \mathcal{U}_i e^{i(M+N)q_i} .$$

Now that the first identity is proved, we can use it to get

$$\sum_{i,j} e^{iMq_i} \mathcal{U}_i e^{iNq_j} \{v(\alpha)_i, q_j\}_{\text{red}} = - \sum_i v(\alpha)_i \mathcal{U}_i e^{i(M+N)q_i}.$$

As before, we write this for $N, M = 1, \dots, n$ as

$$\tilde{\mathcal{E}} \hat{U}^{(2)} \mathcal{E}^T = \tilde{\mathcal{E}} U^{(2)} \mathcal{E}^T,$$

where the $n \times n$ matrices are given by $\hat{U}_{kl}^{(2)} = \{v(\alpha)_k, q_l\}_{\text{red}}, U_{kl}^{(2)} = -\delta_{kl} v(\alpha)_k$. Again by invertibility of \mathcal{E} and $\tilde{\mathcal{E}}$, we get $\hat{U}^{(2)} = U^{(2)}$.

The last identity follows from the second one by complex conjugation. \square

From now on, we do not provide complete proofs of the different results that are stated. They can be successively obtained by direct computations in the same way as we got Lemmas C.2 and C.3.

Deriving (5.38). We first need two preliminary lemmatae.

Lemma C.4. For any $i, j = 1, \dots, n$,

$$\begin{aligned} \{\mathcal{U}_i, \mathcal{U}_j\}_{\text{red}} &= \frac{1}{2} i \delta_{(i \neq j)} \frac{Q_i + Q_j}{Q_i - Q_j} \mathcal{U}_i \mathcal{U}_j + \frac{1}{4} i \sum_{\mu, \nu} \text{sgn}(\nu - \mu) [v(\nu)_i v(\mu)_j - \bar{v}(\nu)_i \bar{v}(\mu)_j] \\ &+ \frac{1}{4} i \sum_{\mu} [v(\mu)_i \bar{v}(\mu)_j - v(\mu)_j \bar{v}(\mu)_i] + \frac{d}{2} i (L_{ij} - L_{ji}) \\ &+ \frac{1}{2} i \sum_{\nu} \sum_{\mu < \nu} [v(\mu)_i \bar{v}(\mu)_j - v(\mu)_j \bar{v}(\mu)_i]. \end{aligned} \tag{C.3}$$

Proof. It suffices to use (5.30) where we sum over all $\alpha, \beta, \gamma, \epsilon$. After elementary manipulations, we arrive at

$$\sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j \{ \mathcal{U}_i, \mathcal{U}_j \}_{\text{red}} = \sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j U_{ij}^{(3)}, \tag{C.4}$$

where $U_{ij}^{(3)}$ is the right-hand side of (C.3). We can then write the equalities with $N, M = 1, \dots, n$ as

$$\tilde{\mathcal{E}} \hat{U}^{(3)} \tilde{\mathcal{E}}^T = \tilde{\mathcal{E}} U^{(3)} \tilde{\mathcal{E}}^T, \tag{C.5}$$

where the $n \times n$ matrix $\hat{U}^{(3)}$ is given by $\hat{U}_{kl}^{(3)} = \{\mathcal{U}_k, \mathcal{U}_l\}_{\text{red}}$. By invertibility of $\tilde{\mathcal{E}}$, this proves the claim (C.3). \square

Lemma C.5. For any $i, j = 1, \dots, n$,

$$\begin{aligned} \{v(\alpha)_i, \mathcal{U}_j\}_{\text{red}} &= \frac{1}{2} i \delta_{(i \neq j)} \frac{Q_i + Q_j}{Q_i - Q_j} v(\alpha)_j \mathcal{U}_i + \frac{1}{2} i \sum_{\kappa} \text{sgn}(\kappa - \alpha) v(\alpha)_j v(\kappa)_i \\ &\quad - \frac{1}{4} i \frac{v(\alpha)_i}{\mathcal{U}_i} \sum_{\mu, \nu} \text{sgn}(\nu - \mu) [v(\nu)_i v(\mu)_j + \bar{v}(\nu)_i \bar{v}(\mu)_j] \\ &\quad + \frac{1}{2} i v(\alpha)_i \bar{v}(\alpha)_j - \frac{1}{4} i \frac{v(\alpha)_i}{\mathcal{U}_i} \sum_{\mu} [v(\mu)_i \bar{v}(\mu)_j + v(\mu)_j \bar{v}(\mu)_i] \quad (\text{C.6}) \\ &\quad + i \sum_{\kappa < \alpha} v(\kappa)_i \bar{v}(\kappa)_j - \frac{1}{2} i \frac{v(\alpha)_i}{\mathcal{U}_i} \sum_{\nu} \sum_{\mu < \nu} [v(\mu)_i \bar{v}(\mu)_j + v(\mu)_j \bar{v}(\mu)_i] \\ &\quad + \frac{1}{2} i \left[2L_{ij} - d \frac{v(\alpha)_i}{\mathcal{U}_i} (L_{ij} + L_{ji}) \right]. \end{aligned}$$

Proof. It suffices to use (5.30) after summing over β, γ, ϵ . We arrive at

$$\sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j \{v(\alpha)_i, \mathcal{U}_j\}_{\text{red}} = \sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j U_{ij}^{(4)}, \quad (\text{C.7})$$

where $U_{ij}^{(4)}$ is the right-hand side of (C.6). We can then write the equalities with $N, M = 1, \dots, n$ as

$$\tilde{\mathcal{E}} \hat{U}^{(4)} \tilde{\mathcal{E}}^T = \tilde{\mathcal{E}} U^{(4)} \tilde{\mathcal{E}}^T,$$

where the $n \times n$ matrix $\hat{U}^{(4)}$ is given by $\hat{U}_{kl}^{(4)} = \{v(\alpha)_k, \mathcal{U}_l\}_{\text{red}}$. By invertibility of $\tilde{\mathcal{E}}$, we obtain the equality (C.5). \square

Summing over β, ϵ in (5.30) and using the previous results, we can get

$$\sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j \{v(\alpha)_i, v(\gamma)_j\}_{\text{red}} = \sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j U_{ij}^{(5)}, \quad (\text{C.8})$$

where $U_{ij}^{(5)}$ is the right-hand side of (5.38). We can then write the equalities (C.8) with $N, M = 1, \dots, n$ as

$$\tilde{\mathcal{E}} \hat{U}^{(5)} \tilde{\mathcal{E}}^T = \tilde{\mathcal{E}} U^{(5)} \tilde{\mathcal{E}}^T,$$

where the $n \times n$ matrix $\hat{U}^{(5)}$ is given by $\hat{U}_{kl}^{(5)} = \{v(\alpha)_k, v(\gamma)_l\}_{\text{red}}$. By invertibility of $\tilde{\mathcal{E}}$, this implies that (5.38) holds.

Deriving (5.39). By anti-symmetry and complex conjugation, we get $\{\mathcal{U}_i, \bar{v}(\epsilon)_j\}_{\text{red}}$ from Lemma C.5. We can then use the previous results as well as (5.30) after summing over β, γ in order to get

$$\sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j \{v(\alpha)_i, \bar{v}(\epsilon)_j\}_{\text{red}} = \sum_{i,j} Q_i^M \mathcal{U}_i Q_j^N \mathcal{U}_j U_{ij}^{(6)}, \quad (\text{C.9})$$

where $U_{ij}^{(6)}$ is the right-hand side of (5.39). We can then write the equalities (C.9) with $N, M = 1, \dots, n$ as

$$\tilde{\mathcal{E}} \hat{U}^{(6)} \tilde{\mathcal{E}}^T = \tilde{\mathcal{E}} U^{(6)} \tilde{\mathcal{E}}^T,$$

where the $n \times n$ matrix $\hat{U}^{(6)}$ is given by $\hat{U}_{kl}^{(6)} = \{v(\alpha)_k, \bar{v}(\epsilon)_l\}_{\text{red}}$. By invertibility of $\tilde{\mathcal{E}}$, we can conclude that (5.39) holds.

D. Poisson Brackets of Collective Spins

Recall the matrix (S_{ij}) defined before Theorem 5.8. The reduced Poisson brackets of the so-called collective spins F (3.29) can be computed in the following form.

Lemma D.1. Denoting $q_{ab} := q_a - q_b$, the following identity holds on $\check{\mathcal{M}}_{0,+}^{\text{reg}}$

$$\begin{aligned} \{F_{ij}, F_{kl}\}_{\text{red}} &= i \left(\frac{S_{ik}}{\mathcal{U}_i \mathcal{U}_k} - \frac{S_{lj}}{\mathcal{U}_l \mathcal{U}_j} + \frac{S_{kj}}{\mathcal{U}_k \mathcal{U}_j} - \frac{S_{il}}{\mathcal{U}_i \mathcal{U}_l} \right) F_{ij} F_{kl} \\ &+ \frac{1}{2} \left[\delta_{(i \neq k)} \cot\left(\frac{q_{ik}}{2}\right) + \delta_{(j \neq l)} \cot\left(\frac{q_{jl}}{2}\right) + \delta_{(k \neq j)} \cot\left(\frac{q_{kj}}{2}\right) + \delta_{(i \neq l)} \cot\left(\frac{q_{li}}{2}\right) \right] F_{ij} F_{kl} \\ &+ \left[\delta_{(i \neq k)} \cot\left(\frac{q_{ik}}{2}\right) + \delta_{(j \neq l)} \cot\left(\frac{q_{jl}}{2}\right) - \cot\left(\frac{q_{jk}}{2} - i\gamma\right) + \cot\left(\frac{q_{li}}{2} - i\gamma\right) \right] F_{il} F_{kj} \\ &+ \frac{1}{2} \left[\delta_{(k \neq i)} \cot\left(\frac{q_{ki}}{2}\right) - \cot\left(\frac{q_{li}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_k}{\mathcal{U}_i} F_{ij} F_{il} \\ &+ \frac{1}{2} \left[\delta_{(j \neq k)} \cot\left(\frac{q_{jk}}{2}\right) + \cot\left(\frac{q_{lj}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_k}{\mathcal{U}_j} F_{ij} F_{jl} \\ &+ \frac{1}{2} \left[\delta_{(i \neq k)} \cot\left(\frac{q_{ki}}{2}\right) + \cot\left(\frac{q_{jk}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_i}{\mathcal{U}_k} F_{kj} F_{kl} \\ &+ \frac{1}{2} \left[\delta_{(i \neq l)} \cot\left(\frac{q_{il}}{2}\right) - \cot\left(\frac{q_{jl}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_i}{\mathcal{U}_l} F_{lj} F_{kl} \\ &+ \frac{1}{2} \left[\delta_{(i \neq l)} \cot\left(\frac{q_{il}}{2}\right) - \cot\left(\frac{q_{ik}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_l}{\mathcal{U}_i} F_{ij} F_{ki} \\ &+ \frac{1}{2} \left[\delta_{(l \neq j)} \cot\left(\frac{q_{lj}}{2}\right) + \cot\left(\frac{q_{jk}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_l}{\mathcal{U}_j} F_{ij} F_{kj} \\ &+ \frac{1}{2} \left[\delta_{(j \neq k)} \cot\left(\frac{q_{jk}}{2}\right) + \cot\left(\frac{q_{ki}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_j}{\mathcal{U}_k} F_{ik} F_{kl} \\ &+ \frac{1}{2} \left[\delta_{(j \neq l)} \cot\left(\frac{q_{lj}}{2}\right) - \cot\left(\frac{q_{li}}{2} - i\gamma\right) \right] \frac{\mathcal{U}_j}{\mathcal{U}_l} F_{il} F_{kl} \end{aligned}$$

This follows from Theorem 5.8 by direct calculation. The reader can easily check the reality condition $\{F_{ji}, F_{lk}\}_{\text{red}} = \{\overline{F}_{ij}, \overline{F}_{kl}\}_{\text{red}} = \overline{\{F_{ij}, F_{kl}\}_{\text{red}}}$. Taking $i = j$ and $k = l$ in Lemma D.1, everything cancels out except for the third line, which can be rewritten as follows:

$$\{F_{jj}, F_{kk}\}_{\text{red}} = F_{jk} F_{kj} \frac{2 \cot\left(\frac{q_{jk}}{2}\right)}{1 + \sinh^{-2}(\gamma) \sin^2\left(\frac{q_{jk}}{2}\right)}, \quad \text{for } j \neq k. \quad (\text{D.1})$$

Let us now assume that $d = 1$, so that $F_{jk} F_{kj} = F_{jj} F_{kk}$. Note that the formula of L (3.29) shows that $F_{jj} > 0$. Motivated by the form of the equations of motion (1.8) and the spinless Hamiltonian (1.11), we make the change of variables

$$F_{jj} = e^{2\theta_j} \prod_{i \neq j} \left[1 + \frac{\sinh^2 \gamma}{1 + \sin^2 \frac{q_i - q_j}{2}} \right]^{\frac{1}{2}}. \quad (\text{D.2})$$

Using (5.37) and (D.1), it turns out that (q_j, θ_j) are Darboux variables, and we recover the standard chiral RS Hamiltonian (1.11) for $\mathcal{H} = \sum_j F_{jj}$.

References

- [1] Alekseev, A.Y., Malkin, A.Z.: Symplectic structures associated to Lie-Poisson groups. *Commun. Math. Phys.* **162**, 147–173 (1994). [arXiv:hep-th/9303038](#)
- [2] Alekseev, A., Malkin, A., Meinrenken, E.: Lie group valued moment maps. *J. Differ. Geom.* **48**, 445–495 (1998). [arXiv:dg-ga/9707021](#)
- [3] Arutyunov, G.: *Elements of Classical and Quantum Integrable Systems*. Springer, Berlin (2019)
- [4] Arutyunov, G.E., Frolov, S.A.: On the Hamiltonian structure of the spin Ruijsenaars–Schneider model. *J. Phys. A* **31**, 4203–4216 (1998). [arXiv:hep-th/9703119](#)
- [5] Arutyunov, G., Olivucci, E.: Hyperbolic spin Ruijsenaars–Schneider model from Poisson reduction. *Proc. Steklov Inst. Math.* **309**, 31–45 (2020). [arXiv:1906.02619](#)
- [6] Arthamonov, S., Reshetikhin, N.: Superintegrable systems on moduli spaces of flat connections. [arXiv:1909.08682](#)
- [7] Braden, H.W., Hone, N.W.: Affine Toda solitons and systems of Calogero–Moser type. *Phys. Lett. B* **380**, 296–302 (1996). [arXiv:hep-th/9603178](#)
- [8] Calogero, F.: Solution of the one-dimensional N-body problem with quadratic and/or inversely quadratic pair potentials. *J. Math. Phys.* **12**, 419–436 (1971)
- [9] Chalykh, O., Fairon, M.: Multiplicative quiver varieties and generalised Ruijsenaars–Schneider models. *J. Geom. Phys.* **121**, 413–437 (2017). [arXiv:1704.05814](#)
- [10] Chalykh, O., Fairon, M.: On the Hamiltonian formulation of the trigonometric spin Ruijsenaars–Schneider system. *Lett. Math. Phys.* **110**, 2893–2940 (2020). [arXiv:1811.08727](#)
- [11] Crawley-Boevey, W., Shaw, P.: Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem. *Adv. Math.* **201**, 180–208 (2006). [arXiv:math/0404186](#)
- [12] Etingof, P.: *Calogero–Moser Systems and Representation Theory*. European Mathematical Society, Zurich (2007)
- [13] Etingof, P.I., Frenkel, I.B., Kirillov Jr., A.A.: Spherical functions on affine Lie groups. *Duke Math. J.* **80**, 59–90 (1995). [arXiv:hep-th/9407047](#)
- [14] Fairon, M.: Spin versions of the complex trigonometric Ruijsenaars–Schneider model from cyclic quivers. *J. Integrable Syst.* **4**, xyz008 (2019). [arXiv:1811.08717](#)
- [15] Fehér, L.: Poisson–Lie analogues of spin Sutherland models. *Nucl. Phys. B* **949**, 114807 (2019). [arXiv:1809.01529](#)
- [16] Fehér, L.: Reduction of a bi-Hamiltonian hierarchy on $T^*U(n)$ to spin Ruijsenaars–Sutherland models. *Lett. Math. Phys.* **110**, 1057–1079 (2020). [arXiv:1908.02467](#)
- [17] Fehér, L., Ayadi, V.: Trigonometric Sutherland systems and their Ruijsenaars duals from symplectic reduction. *J. Math. Phys.* **51**, 103511 (2010). [arXiv:1005.4531](#)
- [18] Fehér, L., Görbe, T.: On a Poisson–Lie deformation of the BC_n Sutherland system. *Nucl. Phys. B* **901**, 85–114 (2015). [arXiv:1508.04991](#)
- [19] Fehér, L., Klimčík, C.: Poisson–Lie interpretation of trigonometric Ruijsenaars duality. *Commun. Math. Phys.* **301**, 55–104 (2011). [arXiv:0906.4198](#)

- [20] Fehér, L., Marshall, I.: Global description of action-angle duality for a Poisson-Lie deformation of the trigonometric BC_n Sutherland system. *Annales Henri Poincaré* **20**, 1217–1262 (2019). [arXiv:1710.08760](#)
- [21] Fehér, L., Pusztai, B.G.: A class of Calogero type reductions of free motion on a simple Lie group. *Lett. Math. Phys.* **79**, 263–277 (2007). [arXiv:math-ph/0609085](#)
- [22] Fehér, L., Pusztai, B.G.: Hamiltonian reductions of free particles under polar actions of compact Lie groups. *Theor. Math. Phys.* **155**, 646–658 (2008). [arXiv:0705.1998](#)
- [23] Fock, V., Gorsky, A., Nekrasov, N., Rubtsov, V.: Duality in integrable systems and gauge theories. *JHEP* **07**, 028 (2000). [arXiv:hep-th/9906235](#)
- [24] Gibbons, J., Hermsen, T.: A generalisation of the Calogero–Moser system. *Physica D* **11**, 337–348 (1984)
- [25] Henneaux, M., Teitelboim, C.: *Quantization of Gauge Systems*. Princeton University Press, Princeton (1992)
- [26] Hilgert, J., Neeb, K.-H., Plank, W.: Symplectic convexity theorems and coadjoint orbits. *Compositio Mathematica* **94**, 129–180 (1994)
- [27] Jovanovic, B.: Symmetries and integrability. *Publ. Institut Math.* **49**, 1–36 (2008). [arXiv:0812.4398](#)
- [28] Kharchev, S., Levin, A., Olshanetsky, M., Zotov, A.: Quasi-compact Higgs bundles and Calogero–Sutherland systems with two types spins. *J. Math. Phys.* **59**, 103509 (2018). [arXiv:1712.08851](#)
- [29] Klimčík, C.: On moment maps associated to a twisted Heisenberg double. *Rev. Math. Phys.* **18**, 781–821 (2006). [arXiv:math-ph/0602048](#)
- [30] Korogodski, L.I., Soibelman, Y.S.: *Algebras of Functions on Quantum Groups: Part I*. American Mathematical Society, Providence (1998)
- [31] Krichever, I.: Elliptic solutions to difference nonlinear equations and nested Bethe Ansatz equations. In: van Diejen, J.F., Vinet, L. (eds.), *Calogero–Moser–Sutherland Models*, CRM Series in Mathematical Physics, Springer, New York, pp. 249–271 (2000). [arXiv:solv-int/9804016](#)
- [32] Krichever, I., Zabrodin, A.: Spin generalization of the Ruijsenaars–Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra. *Russ. Math. Surv.* **50**, 1101–1150 (1995). [arXiv:hep-th/9505039](#)
- [33] Li, L.-C.: Poisson involutions, spin Calogero–Moser systems associated with symmetric Lie subalgebras and the symmetric space spin Ruijsenaars–Schneider models. *Commun. Math. Phys.* **265**, 333–372 (2006). [arXiv:math-ph/0506025](#)
- [34] Li, L.-C., Xu, P.: A class of integrable spin Calogero–Moser systems. *Commun. Math. Phys.* **231**, 257–286 (2002). [arXiv:math/0105162](#)
- [35] Lu, J.-H.: *Multiplicative and affine Poisson structures on Lie groups*, Ph.D. Thesis, University of California, Berkeley, 74 pp (1990)
- [36] Lu, J.-H.: Momentum mappings and reduction of Poisson actions. In: *Symplectic Geometry, Groupoids, and Integrable Systems*, Springer, pp. 209–226 (1991)
- [37] Mischenko, A.S., Fomenko, A.T.: Generalized Liouville method for integrating Hamiltonian systems. *Funct. Anal. Appl.* **12**, 113–125 (1978)
- [38] Moser, J.: Three integrable Hamiltonian systems connected with isospectral deformations. *Adv. Math.* **16**, 197–220 (1975)

- [39] Nekhoroshev, N.N.: Action-angle variables and their generalizations. *Trans. Mosc. Math. Soc.* **26**, 180–197 (1972)
- [40] Nekrasov, N.: Infinite-dimensional algebras, many-body systems and gauge theories. In: *Moscow Seminar in Mathematical Physics*, AMS Translations Series 2, vol. 191, American Mathematical Society, , pp. 263–299 (1999)
- [41] Ortega, J.-P., Ratiu, T.: *Momentum Maps and Hamiltonian Reduction*. Birkhäuser, Basel (2004)
- [42] Penciak, M.: Spectral description of the spin Ruijsenaars–Schneider system. [arXiv:1909.08107](https://arxiv.org/abs/1909.08107)
- [43] Ragnisco, O., Suris, Yu.B.: Integrable discretizations of the spin Ruijsenaars–Schneider models. *J. Math. Phys.* **38**, 4680–4691 (1997). [arXiv:solv-int/9605001](https://arxiv.org/abs/solv-int/9605001)
- [44] Reshetikhin, N.: Degenerate integrability of spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems. *Lett. Math. Phys.* **63**, 55–71 (2003). [arXiv:math/0202245](https://arxiv.org/abs/math/0202245)
- [45] Reshetikhin, N.: Degenerately integrable systems. *J. Math. Sci.* **213**, 769–785 (2016). [arXiv:1509.00730](https://arxiv.org/abs/1509.00730)
- [46] Reshetikhin, N.: Spin Calogero–Moser models on symmetric spaces. [arXiv:1903.03685](https://arxiv.org/abs/1903.03685)
- [47] Reshetikhin, N., Stokman, J.: N-point spherical functions and asymptotic boundary KZB equations. [arXiv:2002.02251](https://arxiv.org/abs/2002.02251)
- [48] Rudolph, G., Schmidt, M.: *Differential Geometry and Mathematical Physics. Part I. Manifolds, Lie Groups and Hamiltonian Systems*. Springer, Berlin (2013)
- [49] Ruijsenaars, S.N.M.: Systems of Calogero–Moser type. In: *Proceedings of the 1994 CRM–Banff Summer School: Particles and Fields*, Springer, , pp. 251–352 (1999)
- [50] Ruijsenaars, S.N.M., Schneider, H.: A new class of integrable systems and its relation to solitons. *Ann. Phys.* **170**, 370–405 (1986)
- [51] Schomerus, V., Sobko, E.: From spinning conformal blocks to matrix Calogero–Sutherland models. *JHEP* **04**, 052 (2018). [arXiv:1711.02022](https://arxiv.org/abs/1711.02022)
- [52] Semenov-Tian-Shansky, M.A.: Dressing transformations and Poisson group actions. *Publ. RIMS* **21**, 1237–1260 (1985)
- [53] Soloviev, F.L.: On the Hamiltonian form of the equations of the elliptic spin Ruijsenaars–Schneider model. *Russian Math. Surv.* **64**, 1142–1144 (2009). [arXiv:0808.3875](https://arxiv.org/abs/0808.3875)
- [54] Sutherland, B.: Exact results for a quantum many-body problem in one dimension. *Phys. Rev. A* **4**, 2019–2021 (1971)
- [55] van Diejen, J.F., Vinet, L. (eds.): *Calogero–Moser–Sutherland Models*. Springer, Berlin (2000)
- [56] Zakrzewski, S.: Phase spaces related to standard classical r -matrices. *J. Phys. A Math. Gen.* **29**, 1841–1857 (1996). [arXiv:q-alg/9511002](https://arxiv.org/abs/q-alg/9511002)
- [57] Zotov, A.V.: Relativistic interacting integrable elliptic tops. *Teoret. Mat. Fiz.* **201**, 175–192 (2019). [arXiv:1910.08246](https://arxiv.org/abs/1910.08246)
- [58] Zung, N.T.: Torus actions and integrable systems. In: *Topological Methods in the Theory of Integrable Systems*, Cambridge Scientific Publishers Ltd, pp. 289–328 (2006). [arXiv:math/0407455](https://arxiv.org/abs/math/0407455)

M. Fairon
School of Mathematics and Statistics
University of Glasgow
University Place
Glasgow G12 8QQ
UK
e-mail: maxime.fairon@glasgow.ac.uk

L. Fehér
Department of Theoretical Physics
University of Szeged
Tisza Lajos krt 84-86
Szeged 6720
Hungary
e-mail: lfehér@physx.u-szeged.hu

and

Institute for Particle and Nuclear Physics
Wigner Research Centre for Physics
Budapest 1525 P.O.B. 49
Hungary

I. Marshall
Faculty of Mathematics, Higher School of Economics
National Research University
Usacheva 6
Moscow
Russia
e-mail: imarshall@hse.ru

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