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General Toeplitz Matrices Subject to Gaussian Perturbations

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Abstract. We study the spectra of general $N \times N$ Toeplitz matrices given by symbols in the Wiener Algebra perturbed by small complex Gaussian random matrices, in the regime $N \gg 1$. We prove an asymptotic formula for the number of eigenvalues of the perturbed matrix in smooth domains. We show that these eigenvalues follow a Weyl law with probability sub-exponentially close to 1, as $N \gg 1$, in particular that most eigenvalues of the perturbed Toeplitz matrix are close to the curve in the complex plane given by the symbol of the unperturbed Toeplitz matrix.

Contents

1. Introduction and main result

Let $a_\nu \in \mathbf{C}$, for $\nu \in \mathbf{Z}$ and assume that

$$|a_\nu| \leq \mathcal{O}(1)m(\nu), \tag{1.1}$$

where $m : \mathbf{Z} \to]0, +\infty[$ satisfies

$$(1 + |\nu|)m(\nu) \in \ell^1, \tag{1.2}$$

and

$$m(-\nu) = m(\nu), \ \forall \nu \in \mathbb{Z}. \tag{1.3}$$

Let

$$p(\tau) = \sum_{-\infty}^{+\infty} a_\nu \tau^\nu, \tag{1.4}$$

act on complex valued functions on $\mathbf{Z}$. Here τ denotes translation by 1 unit to the right: $\tau u(j) = u(j-1)$, $j \in \mathbf{Z}$. By (1.2) we know that $p(\tau) = \mathcal{O}(1) : \ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$. Indeed, for the corresponding operator norm, we have

$$\|p(\tau)\| \leq \sum |a_j| \|\tau^j\| = \|a\|_{\ell^1} \leq \mathcal{O}(1)\|m\|_{\ell^1}. \tag{1.5}$$

From the identity, $\tau(e^{ik\xi}) = e^{-i\xi}e^{ik\xi}$, we define the symbol of $p(\tau)$ by

$$p(e^{-i\xi}) = \sum_{-\infty}^{\infty} a_\nu e^{-i\nu\xi}. \tag{1.6}$$

It is an element of the Wiener algebra [4] and by (1.2) in $C^1(S^1)$.

We are interested in the *Toeplitz matrix*

$$P_N \stackrel{\text{def}}{=} 1_{[0,N[} p(\tau) 1_{[0,N[}, \tag{1.7}$$

acting on $\mathbf{C}^N \simeq \ell^2([0, N[)$, for $1 \ll N < \infty$. Furthermore, we frequently identify $\ell^2([0, N[)$ with the space $\ell^2_{[0,N[}(\mathbf{Z})$ of functions $u \in \ell^2(\mathbf{Z})$ with support in $[0, N[$.

The spectra of such Toeplitz matrices have been studied thoroughly, see [4] for an overview. Let P_∞ denote $p(\tau)$ as an operator $\ell^2(\mathbf{Z}) \to \ell^2(\mathbf{Z})$. It is a normal operator and by Fourier series expansions, we see that the spectrum of P_∞ is given by

$$\sigma(P_\infty) = p(S^1). \tag{1.8}$$

The restriction $P_{\mathbf{N}} = P_\infty|_{\ell^2(\mathbf{N})}$ of P_∞ to $\ell^2(\mathbf{N})$ is in general no longer normal, except for specific choices of the coefficients a_ν. The essential spectrum of the Toeplitz operator $P_{\mathbf{N}}$ is given by $p(S^1)$ and we have pointspectrum in all loops of $p(S^1)$ with nonzero winding number, i.e.,

$$\sigma(P_{\mathbf{N}}) = p(S^1) \cup \{z \in \mathbf{C}; \mathrm{ind}_{p(S^1)}(z) \neq 0\}. \tag{1.9}$$

By a result of Krein [4, Theorem 1.15], the winding number of $p(S^1)$ around the point $z \notin p(S^1)$ is related to the Fredholm index of $P_{\mathbf{N}} - z$: $\mathrm{Ind}(P_{\mathbf{N}} - z) = -\mathrm{ind}_{p(S^1)}(z)$.

The spectrum of the Toeplitz matrix P_N is contained in a small neighborhood of the spectrum of $P_{\mathbf{N}}$. More precisely, for every $\epsilon > 0$,

$$\sigma(P_N) \subset \sigma(P_{\mathbf{N}}) + D(0, \epsilon) \tag{1.10}$$

for $N > 0$ sufficiently large, where $D(z, r)$ denotes the open disc of radius r, centered at z. Moreover, the limit of $\sigma(P_N)$ as $N \to \infty$ is contained in a union of analytic arcs inside $\sigma(P_{\mathbf{N}})$, see [4, Theorem 5.28].

We show in Theorem 1.1 that after adding a small random perturbation to P_N, most of its eigenvalues will be close to the curve $p(S^1)$ with probability very close to 1. See Fig. 1 for a numerical illustration.

1.1. Small Gaussian perturbation

Consider the random matrix

$$Q_\omega \stackrel{\text{def}}{=} Q_\omega(N) \stackrel{\text{def}}{=} (q_{j,k}(\omega))_{1\le j,k\le N} \tag{1.11}$$

with complex Gaussian law

$$(Q_\omega)_*(d\mathbb{P}) = \pi^{-N^2} \mathrm{e}^{-\|Q\|^2_{\mathrm{HS}}} L(dQ),$$

where L denotes the Lebesgue measure on $\mathbf{C}^{N\times N}$. The entries $q_{j,k}$ of Q_ω are independent and identically distributed complex Gaussian random variables with expectation 0, and variance 1, i.e., $q_{j,k\sim\mathcal{N}_{\mathbf{C}}(0,1)}$.

We recall that the probability distribution of a complex Gaussian random variable $\alpha \sim \mathcal{N}_{\mathbf{C}}(0,1)$ is given by

$$\alpha_*(d\mathbb{P}) = \pi^{-1}\mathrm{e}^{-|\alpha|^2} L(d\alpha),$$

where $L(d\alpha)$ denotes the Lebesgue measure on $\mathbf{C}$. If $\mathbb{E}$ denotes the expectation with respect to the probability measure $\mathbb{P}$, then

$$\mathbb{E}[\alpha] = 0, \quad \mathbb{E}[|\alpha|^2] = 1.$$

We are interested in studying the spectrum of the random perturbations of the matrix $P_N^0 = P_N$:

$$P_N^\delta \stackrel{\text{def}}{=} P_N^0 + \delta Q_\omega, \quad 0 \le \delta \ll 1. \tag{1.12}$$

1.2. Eigenvalue asymptotics in smooth domains

Let $\Omega \Subset \mathbf{C}$ be an open simply connected set with smooth boundary $\partial\Omega$, which is independent of N, satisfying

(1) $\partial\Omega$ intersects $p(S^1)$ in at most finitely many points;
(2) $p(S^1)$ does not self-intersect at these points of intersection;
(3) these points of intersection are non-critical, i.e.,

$$dp \neq 0 \text{ on } p^{-1}(\partial\Omega \cap p(S^1));$$

(4) $\partial\Omega$ and $p(S^1)$ are transversal at every point of the intersection.

Theorem 1.1. *Let p be as in (1.6) and let P_N^δ be as in (1.12). Let Ω be as above, satisfying conditions (1)–(4), pick a $\delta_0 \in]0,1[$ and let $\delta_1 > 3$. If*

$$\mathrm{e}^{-N^{\delta_0}} \le \delta \ll N^{-\delta_1}, \tag{1.13}$$

then there exists $\varepsilon_N = o(1)$, as $N \to \infty$, such that

$$\left| \#(\sigma(P_N^\delta) \cap \Omega) - \frac{N}{2\pi} \int_{S^1 \cap p^{-1}(\Omega)} L_{S^1}(d\theta) \right| \le \varepsilon_N N, \tag{1.14}$$

with probability

$$\ge 1 - \mathrm{e}^{-N^{\delta_0}}. \tag{1.15}$$

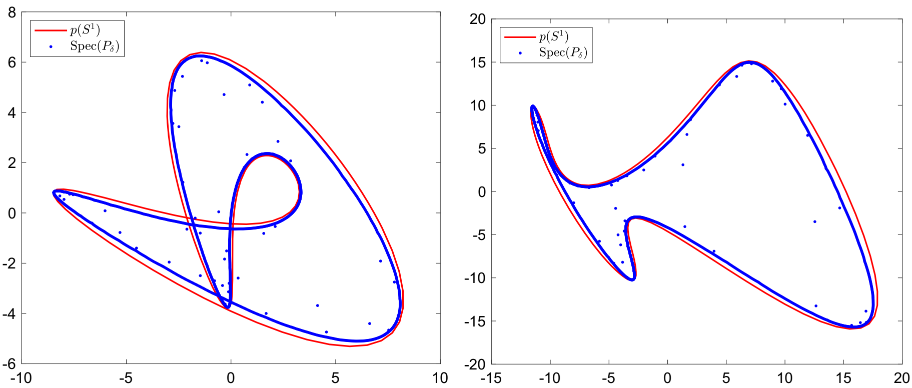

Figure 1. The left-hand side shows the spectrum of the perturbed Toeplitz matrix with symbol defined in (1.16), (1.17) and the right-hand side shows the spectrum of the perturbed Toeplitz matrix with symbol defined in (1.18), (1.17). The red line shows the symbol curve $p(S^1)$

In (1.14), we view p as a map from S^1 to $\mathbf{C}$. Theorem 1.1 shows that most eigenvalues of P_N^δ can be found close to the curve $p(S^1)$ with probability subexponentially close to 1. This is illustrated in Fig. 1 for two different symbols. The left-hand side of Fig. 1 shows the spectrum of a perturbed Toeplitz matrix with $N = 2000$ and $\delta = 10^{-14}$, given by the symbol $p = p_0 + p_1$ where

$$p_0(1/\zeta) = -\zeta^{-4} - (3+2i)\zeta^{-3} + i\zeta^{-2} + \zeta^{-1} + 10\zeta + (3+i)\zeta^2 + 4\zeta^3 + i\zeta^4 \quad (1.16)$$

and

$$p_1(1/\zeta) = \sum_{\nu\in\mathbf{Z}} a_\nu \zeta^\nu, \quad a_0 = 0,\ a_{-\nu} = 0.7|\nu|^{-5} + i|\nu|^{-9},\ a_\nu = -2i\nu^{-5} + 0.5\nu^{-9}\ \nu \in \mathbf{N}. \quad (1.17)$$

The red line shows the curve $p(S^1)$. The right-hand side of Fig. 1 similarly shows the spectrum of the perturbed Toeplitz matrix given by $p = p_0 + p_1$ where p_1 is as above and

$$p_0(1/\zeta) = -4\zeta^1 - 2i\zeta^2 + 2i\zeta^{-1} - \zeta^{-2} + 2\zeta^{-3}. \quad (1.18)$$

In our previous work [15], we studied Toeplitz matrices with a finite number of bands, given by symbols of the form

$$p(\tau) = \sum_{j=-N_-}^{N_+} a_j \tau^j, \quad a_{-N_-}, a_{-N_-+1}, \dots, a_{N_+} \in \mathbf{C},\ a_{\pm N_\pm} \neq 0. \quad (1.19)$$

In this case, the symbols are analytic functions on S^1 and we are able to provide in [15, Theorem 2.1] a version of Theorem 1.1 with a much sharper remainder estimate. See also [13,14], concerning the special cases of large Jordan block matrices $p(\tau) = \tau^{-1}$ and large bi-diagonal matrices $p(\tau) = a\tau + b\tau^{-1}$, $a, b \in \mathbf{C}$. However, Fig. 1 suggests that one could hope for a better remainder estimate in Theorem 1.1 as well.

Theorems 1.1 and 1.2 can be extended to allow for coupling constants with $\delta_1 > 1/2$. Furthermore, one can allow for much more general perturbations, for example perturbations given by random matrices whose entries are iid copies of a centered random variables with bounded fourth moment. However, both extensions require some extra work which we will present in a follow-up paper.

1.3. Convergence of the empirical measure and related results

An alternative way to study the limiting distribution of the eigenvalues of P_N^δ, up to errors of $o(N)$, is to study the *empirical measure* of eigenvalues, defined by

$$\xi_N \stackrel{\text{def}}{=} \frac{1}{N} \sum_{\lambda \in \mathrm{Spec}(P_N^\delta)} \delta_\lambda \tag{1.20}$$

where the eigenvalues are counted including multiplicity and δ_λ denotes the Dirac measure at $\lambda \in \mathbf{C}$. For any positive monotonically increasing function ϕ on the positive reals and random variable X, Markov's inequality states that $\mathbb{P}[|X| \geq \varepsilon] \leq \phi(\varepsilon)^{-1}\mathbb{E}[\phi(|X|)]$, assuming that the last quantity is finite. Using $\phi(x) = \mathrm{e}^{x/C}$, $x \geq 0$, with a sufficiently large $C > 0$, yields that for $C_1 > 0$ large enough

$$\mathbb{P}[\|Q_\omega\|_{\mathrm{HS}} \leq C_1 N] \geq 1 - \mathrm{e}^{-N^2}. \tag{1.21}$$

If $\delta \leq N^{-1}$, then (1.5) and the Borel–Cantelli Theorem shows that, almost surely, ξ_N has compact support for $N > 0$ sufficiently large.

We will show that, almost surely, ξ_N converges weakly to the push-forward of the uniform measure on S^1 by the symbol p.

Theorem 1.2. *Let $\delta_0 \in]0, 1[$, let $\delta_1 > 3$ and let p be as in (1.4). If (1.13) holds, i.e.,*

$$\mathrm{e}^{-N^{\delta_0}} \leq \delta \ll N^{-\delta_1}$$

then, almost surely,

$$\xi_N \rightharpoonup p_*\left(\frac{1}{2\pi} L_{S^1}\right), \quad N \to \infty, \tag{1.22}$$

weakly, where L_{S^1} denotes the Lebesgue measure on S^1.

This result generalizes [15, Corollary 2.2] from the case of Toeplitz matrices with a finite number of bands to the general case (1.4).

Similar results to Theorem 1.2 have been proven in various settings. In [2, 3], the authors consider the special case of band Toeplitz matrices, i.e. P_N with p as in (1.19). In this case, they show that the convergence (1.22) holds weakly in probability for a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 1/2$. Furthermore, they prove a version of this theorem for Toeplitz matrices with non-constant coefficients in the bands, see [2, Theorem 1.3, Theorem 4.1]. They follow a different approach than we do: They compute directly the $\log|\det \mathcal{M}_N - z|$ by relating it to $\log|\det M_N(z)|$, where $M_N(z)$ is a truncation of $M_N - z$, where the smallest singular values of $M_N - z$ have been excluded. The level of truncation, however, depends on the strength of the coupling constant and it necessitates a very detailed analysis of the small singular values of $M_N - z$.

In the earlier work [9], the authors prove that the convergence (1.22) holds weakly in probability for the Jordan bloc matrix P_N with $p(\tau) = \tau^{-1}$ (1.4) and a perturbation given by a complex Gaussian random matrix whose entries are independent complex Gaussian random variables whose variances vanish (not necessarily at the same speed) polynomially fast, with minimal decay of order $N^{-1/2+}$. See also [6] for a related result.

In [20], using a replacement principle developed in [18], it was shown that the result of [9] holds for perturbations given by complex random matrices whose entries are independent and identically distributed random complex random variables with expectation 0 and variance 1 and a coupling constant $\delta = N^{-\gamma}$, with $\gamma > 2$.

1.4. Notation

We will frequently use the following notation: When we write $a \ll b$, we mean that $Ca \leq b$ for some sufficiently large constant $C > 0$. The notation $f = \mathcal{O}(N)$ means that there exists a constant $C > 0$ (independent of N) such that $|f| \leq CN$. When we want to emphasize that the constant $C > 0$ depends on some parameter k, then we write C_k, or with the above notation $\mathcal{O}_k(N)$.

2. The unperturbed operator

We are interested in the Toeplitz matrix

$$P_N = 1_{[0,N[}p(\tau)1_{[0,N[} : \ell^2([0,N[) \to \ell^2([0,N[) \tag{2.1}$$

for $1 \ll N < \infty$, see also (1.7). Here we identify $\ell^2([0,N[)$ with the space $\ell^2_{[0,N[}(\mathbf{Z})$ of functions $u \in \ell^2(\mathbf{Z})$ with support in $[0,N[$. Sometimes we write $P_N = P_{[0,N[}$ and identify P_N with $P_I = 1_I p(\tau) 1_I$ where $I = I_N$ is any interval in $\mathbf{Z}$ of "length" $|I| = \#I = N$.

Let $P_{\mathbf{N}} = P_{[0,+\infty[}$ and let $P_{\mathbf{Z}/\widetilde{N}\mathbf{Z}}$ denote $P = p(\tau)$, acting on $\ell^2(\mathbf{Z}/\widetilde{N}\mathbf{Z})$ which we identify with the space of $\widetilde{N}$-periodic functions on $\mathbf{Z}$. Here $\widetilde{N} \geq 1$. Using the discrete Fourier transform, we see that

$$\sigma(P_{\mathbf{Z}/\widetilde{N}\mathbf{Z}}) = p(S_{\widetilde{N}}), \tag{2.2}$$

where $S_{\widetilde{N}}$ is the dual of $\mathbf{Z}/\widetilde{N}\mathbf{Z}$ and given by

$$S_{\widetilde{N}} = \{e^{ik2\pi/\widetilde{N}}; 0 \leq k < \widetilde{N}\}.$$

Let

$$p_N(\tau) = \sum_{|\nu| \leq N} a_\nu \tau^\nu = \sum_{\nu \in \mathbf{Z}} a_\nu^N \tau^\nu, \quad a_\nu^N = 1_{[-N,N]}(\nu) a_\nu. \tag{2.3}$$

and notice that

$$P_N = 1_{[0,N[}\, p_N(\tau) 1_{[0,N[}. \tag{2.4}$$

We now consider $[0,N[$ as an interval I_N in $\mathbf{Z}/\widetilde{N}\mathbf{Z}$, $\widetilde{N}=N+M$, where $M\in\{1,2,..\}$ will be fixed and independent of N. The matrix of P_N, indexed over $I_N\times I_N$ is then given by

$$P_N(j,k)=a^N_{\widetilde{j}-\widetilde{k}},\ j,k\in I_N\subset \mathbf{Z}/\widetilde{N}Z, \tag{2.5}$$

where $\widetilde{j},\widetilde{k}\in\mathbf{Z}$ are the preimages of j,k under the projection $\mathbf{Z}\to\mathbf{Z}/\widetilde{N}\mathbf{Z}$ that belong to the interval $[0,N[\subset\mathbf{Z}$.

Let $\widetilde{P}_N$ be given by the formula (2.4), with the difference that we now view τ as a translation on $\ell^2(\mathbf{Z}/\widetilde{N}\mathbf{Z})$:

$$\widetilde{P}_N=1_{I_N}p_N(\tau)1_{I_N}. \tag{2.6}$$

The matrix of $\widetilde{P}_N$ is given by

$$\widetilde{P}_N(j,k)=\sum_{\substack{\nu\in\mathbf{Z},\\ \nu\equiv j-k \bmod \widetilde{N}\mathbf{Z}}} a^N_\nu,\quad j,k\in I_N. \tag{2.7}$$

Alternatively, if we let $\widetilde{j},\widetilde{k}$ be the preimages in $[0,N[$ of $j,k\in I_N$, then

$$\widetilde{P}_N(j,k)=\sum_{\widehat{j}\in\mathbf{Z};\ \widehat{j}\equiv\widetilde{j} \bmod \widetilde{N}\mathbf{Z}} a^N_{\widehat{j}-\widetilde{k}}. \tag{2.8}$$

Recall that the terms in (2.7), (2.8) with $|\nu|>N$ or $|\widehat{j}-\widetilde{k}|>N$ do vanish. This implies that with $\widetilde{j}$, $\widetilde{k}$ as in (2.8),

$$\widetilde{P}_N(j,k)-P_N(j,k)=a^N_{\widetilde{j}-\widetilde{N}-\widetilde{k}}+a^N_{\widetilde{j}+\widetilde{N}-\widetilde{k}}. \tag{2.9}$$

Here

$$\begin{aligned}\widetilde{j}-\widetilde{N}&\in[0,N[-\widetilde{N}=[-\widetilde{N},N-\widetilde{N}[=[-N-M,-M[,\\ \widetilde{j}+\widetilde{N}&\in[0,N[+\widetilde{N}=[\widetilde{N},N+\widetilde{N}[=[N+M,2N+M[.\end{aligned}$$

Since $\widetilde{k}\in[0,N[$, we have for the first term in (2.9) that $|\widetilde{j}-\widetilde{N}-\widetilde{k}|=\widetilde{k}+M+(N-\widetilde{j})$ with nonnegative terms in the last sum. Similarly for the second term in (2.9), we have $|\widetilde{j}+\widetilde{N}-\widetilde{k}|=\widetilde{j}+M+(N-\widetilde{k})$ where the terms in the last sum are all ≥ 0.

It follows that the trace class norm of $P_N-\widetilde{P}_N$ is bounded from above by

$$\begin{aligned}&\sum_{j<-M,\ k\geq 0}|a_{j-k}|+\sum_{j\geq N+M,\ k<N}|a_{j-k}|\\ &\quad=\sum_{k\geq 0,\ j\leq -M}|a_{j-k}|+\sum_{k\leq 0,\ j\geq M}|a_{j-k}|\\ &\quad\leq 2C\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}m(M+k+j)=2C\sum_{k=0}^{\infty}(k+1)m(M+k)\\ &\quad=2C\sum_{k=M}^{\infty}(k+1-M)m(k).\end{aligned}$$

By (1.2), it follows that

$$\|P_N - \widetilde{P}_N\|_{\mathrm{tr}} \leq 2C \sum_{k=M}^{+\infty} (k+1-M)m(k) \to 0, \ M \to \infty, \tag{2.10}$$

uniformly with respect to N. Here $\|A\|_{\mathrm{tr}} = \mathrm{tr}(A^*A)^{1/2}$ denotes the Schatten 1-norm for a trace class operator A.

Remark 2.1. To illustrate the difference between P_N and $\widetilde{P}_N$ let $N \gg 1$, $M > 0$ and consider the example of $p(\tau) = \tau^n$, so $a_n = 1$, for some fixed $n \in \mathbf{N}$, and $a_\nu = 0$ for $\nu \neq n$. Since $P_N(j,k) = a^N_{\widetilde{j}-\widetilde{k}}$, we see that

$$P_N(j,k) = \begin{cases} 1, & \widetilde{j} = n + \widetilde{k} \\ 0, & \text{else.} \end{cases}$$

In other words $P_N = (J^*)^n$ where J denotes the $N \times N$ Jordan block matrix. The matrix elements of $\widetilde{P}_N$ on the other hand are given by $\widetilde{P}_N(j,k) = a^N_{\widetilde{j}-\widetilde{N}-\widetilde{k}} + a^N_{\widetilde{j}-\widetilde{k}} + a^N_{\widetilde{j}+\widetilde{N}-\widetilde{k}}$, so

$$\widetilde{P}_N(j,k) = \begin{cases} 1, & \widetilde{j} = n + \widetilde{k} \\ 1, & \widetilde{j} = n + \widetilde{k} - (N+M) \\ 0, & \text{else.} \end{cases}$$

So $\widetilde{P}_N = P_N + J^{(N+M-n)}$, when $n \geq M$, otherwise $\widetilde{P}_N = P_N$.

3. A Grushin problem for $P_N - z$

Let $K \Subset \mathbf{C}$ be an open relatively compact set and let $z \in K$. Consider

$$J = [-M, 0[, \ I_N = [0, N[\tag{3.1}$$

as subsets of $\mathbf{Z}/(N+M)\mathbf{Z}$ so that

$$J \cup I_N = \mathbf{Z}/(N+M)\mathbf{Z} =: \mathbf{Z}_{N+M}$$

is a partition. Recall (2.3), (2.6) and consider

$$p_N(\tau) - z : \ell^2(\mathbf{Z}_{N+M}) \to \ell^2(\mathbf{Z}_{N+M})$$

and write this operator as a 2×2 matrix

$$p_N - z = \begin{pmatrix} \widetilde{P}_N - z & R_- \\ R_+ & R_{+-}(z) \end{pmatrix}, \tag{3.2}$$

induced by the orthogonal decomposition

$$\ell^2(\mathbf{Z}_{N+M}) = \ell^2(I_N) \oplus \ell^2(J). \tag{3.3}$$

The operator $p_N(\tau)$ is normal and we know by (2.2) that its spectrum is

$$\sigma(p_N(\tau)) = p_N(S_{N+M}). \tag{3.4}$$

Replacing $\widetilde{P}_N$ in (3.2) by P_N (2.4), we put

$$\mathcal{P}_N(z) = \begin{pmatrix} P_N - z & R_- \\ R_+ & R_{+-}(z) \end{pmatrix}. \tag{3.5}$$

Then, by (2.10),

$$\|\mathcal{P}_N(z) - (p_N - z)\|_{\mathrm{tr}} \le 2C \sum_{k=M}^{+\infty} (k+1-M)m(k) =: \epsilon(M). \tag{3.6}$$

If $\epsilon(M) < \operatorname{dist}(z, p_N(S_{N+M})) =: d_N(z)$, then $\mathcal{P}_N(z)$ is bijective and

$$\|\mathcal{P}_N(z)^{-1}\| \le \frac{1}{d_N(z) - \epsilon(M)}. \tag{3.7}$$

Write,

$$\begin{aligned}\mathcal{P}_N(z) &= p_N(\tau) - z + \mathcal{P}_N(z) - (p_N(\tau) - z)\\ &= (p_N(\tau) - z)\left(1 + (p_N(\tau) - z)^{-1}(\mathcal{P}_N(z) - (p_N(\tau) - z))\right).\end{aligned}$$

Here,

$$\begin{aligned}&\left|\det\left(1 + (p_N(\tau) - z)^{-1}(\mathcal{P}_N(z) - (p_N(\tau) - z))\right)\right|\\ &\qquad\le \exp \|(p_N(\tau) - z)^{-1}(\mathcal{P}_N(z) - (p_N(\tau) - z))\|_{\mathrm{tr}}\\ &\qquad\le \exp(\epsilon(M)/d_N(z)),\end{aligned}$$

so

$$|\det \mathcal{P}_N(z)| \le |\det(p_N(\tau) - z)|\, e^{\epsilon(M)/d_N(z)}. \tag{3.8}$$

Similarly from

$$\begin{aligned}p_N(\tau) - z &= \mathcal{P}_N(z) + p_N(\tau) - z - \mathcal{P}_N(z)\\ &= \mathcal{P}_N(z)\left(1 + \mathcal{P}_N(z)^{-1}(p_N(\tau) - z - \mathcal{P}_N(z)\right),\end{aligned}$$

we get

$$|\det(p_N(\tau) - z)| \le |\det \mathcal{P}_N(z)| e^{\frac{\epsilon(M)}{d_N(z) - \epsilon(M)}}. \tag{3.9}$$

In analogy with (3.5), we write

$$\mathcal{P}_N(z)^{-1} = \mathcal{E}_N(z) = \begin{pmatrix} E^N & E_+^N \\ E_-^N & E_{-+}^N \end{pmatrix} : \ \ell^2(I_N) \oplus \ell^2(J) \to \ell^2(I_N) \oplus \ell^2(J), \tag{3.10}$$

where J, I_N were defined in (3.1), still viewed as intervals in $\mathbf{Z}_{N+M}$. From (3.7), we get for the respective operator norms:

$$\|E^N\|, \|E_+^N\|, \|E_-^N\|, \|E_{-+}^N\| \le (d_N(z) - \epsilon(M))^{-1}. \tag{3.11}$$

4. Second Grushin problem

We begin with a result, which is a generalization of [16, Proposition 3.4] to the case where $R_{+-} \neq 0$.

Proposition 4.1. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_\pm, \mathcal{S}_\pm$ be Banach spaces. If*

$$\mathcal{P} = \begin{pmatrix} P & R_- \\ R_+ & R_{+-} \end{pmatrix} : \mathcal{H}_1 \times \mathcal{H}_- \to \mathcal{H}_2 \times \mathcal{H}_+ \tag{4.1}$$

is bijective with bounded inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} : \mathcal{H}_2 \times \mathcal{H}_+ \to \mathcal{H}_1 \times \mathcal{H}_-,$$

and if

$$\mathcal{S} = \begin{pmatrix} E_{-+} & S_- \\ S_+ & 0 \end{pmatrix} : \mathcal{H}_+ \times \mathcal{S}_- \to \mathcal{H}_- \times \mathcal{S}_+ \tag{4.2}$$

is bijective with bounded inverse

$$\mathcal{F} = \begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix} : \mathcal{H}_- \times \mathcal{S}_+ \to \mathcal{H}_+ \times \mathcal{S}_-,$$

then

$$\mathcal{T} = \begin{pmatrix} P & R_- S_- \\ S_+ R_+ & S_+ R_{+-} S_- \end{pmatrix} =: \begin{pmatrix} P & T_- \\ T_+ & T_{+-} \end{pmatrix} : \mathcal{H}_1 \times \mathcal{S}_- \to \mathcal{H}_2 \times \mathcal{S}_+ \tag{4.3}$$

is bijective with bounded inverse

$$\mathcal{G} = \begin{pmatrix} G & G_+ \\ G_- & G_{-+} \end{pmatrix} = \begin{pmatrix} E - E_+ F E_- & E_+ F_+ \\ F_- E_- & -F_{-+} \end{pmatrix} : \mathcal{H}_2 \times \mathcal{S}_+ \to \mathcal{H}_1 \times \mathcal{S}_-. \tag{4.4}$$

Proof. We can essentially follow the proof of [16, Proposition 3.4]. We need to solve

$$\begin{cases} Pu + R_- S_- u_- = v \\ S_+ R_+ u + S_+ R_{+-} S_- u_- = v_+. \end{cases} \tag{4.5}$$

Putting $\widetilde{v}_+ = R_+ u + R_{+-} S_- u_-$, the first equation is equivalent to

$$\begin{cases} Pu + R_- S_- u_- = v \\ R_+ u + R_{+-} S_- u_- = \widetilde{v}_+, \end{cases} \quad \text{i.e.} \quad \mathcal{P} \begin{pmatrix} u \\ S_- u_- \end{pmatrix} = \begin{pmatrix} v \\ \widetilde{v}_+ \end{pmatrix},$$

and hence to

$$\begin{cases} u = Ev + E_+ \widetilde{v}_+ \\ S_- u_- = E_- v + E_{-+} \widetilde{v}_+. \end{cases} \tag{4.6}$$

Therefore, we can replace u by $\widetilde{v}_+$ and (4.5) is equivalent to

$$\begin{pmatrix} E_{-+} & S_- \\ S_+ & 0 \end{pmatrix} \begin{pmatrix} \widetilde{v}_+ \\ -u_- \end{pmatrix} = \begin{pmatrix} -E_- v \\ v_+ \end{pmatrix} \tag{4.7}$$

which can be solved by $\mathcal{F}$. Hence, (4.7) is equivalent to

$$\begin{cases} \widetilde{v}_+ = -F E_- v + F_+ v_+ \\ -u_- = -F_- E_- v + F_{-+} v_+, \end{cases}$$

and (4.6) gives the unique solution of (4.5)

$$\begin{cases} u = (E - E_+ F E_-) v + E_+ F_+ v_+ \\ u_- = F_- E_- v - F_{-+} v_+. \end{cases}$$

□

4.1. Grushin problem for $E_{-+}(z)$

We want to apply Proposition 4.1 to $\mathcal{P} = \mathcal{P}(z) = \mathcal{P}_N(z)$ in (3.5) with the inverse $\mathcal{E} = \mathcal{E}_N(z)$ in (3.10), where we sometimes drop the index N. We begin by constructing an invertible Grushin problem for E_{-+}:

Let $0 \le t_1 \le \cdots \le t_M$ denote the singular values of $E_{-+}(z)$. Let $e_1, \dots, e_M$ denote an orthonormal basis of eigenvectors of $E_{-+}^*E_{-+}$ associated to the eigenvalues $t_1^2 \le \cdots \le t_M^2$. Since E_{-+} is a square matrix, we have that $\dim \mathcal{N}(E_{-+}(z)) = \dim \mathcal{N}(E_{-+}^*(z))$[1]. Using the spectral decomposition $\ell^2(J) = \mathcal{N}(E_{-+}^*E_{-+}) \oplus_\perp \mathcal{R}(E_{-+}^*E_{-+})$ together with the fact that $\mathcal{N}(E_{-+}^*E_{-+}) = \mathcal{N}(E_{-+})$ and $\mathcal{R}(E_{-+}^*) = \mathcal{N}(E_{-+})^\perp$, it follows that $\mathcal{R}(E_{-+}^*) = \mathcal{R}(E_{-+}^*E_{-+})$. Similarly, we get that $\mathcal{R}(E_{-+}) = \mathcal{R}(E_{-+}E_{-+}^*)$. One then easily checks that $E_{-+} : \mathcal{R}(E_{-+}^*E_{-+}) \to \mathcal{R}(E_{-+}E_{-+}^*)$ is a bijection. Similarly, $E_{-+}^* : \mathcal{R}(E_{-+}E_{-+}^*) \to \mathcal{R}(E_{-+}^*E_{-+})$ is a bijection. Let $f_1, \dots, f_{M_0}$ denote an orthonormal basis of $\mathcal{N}(E_{-+}^*(z))$ and set

$$f_j = t_j^{-1} E_{-+} e_j, \quad j = M_0 + 1, \dots, M.$$

Then, $f_1, \dots, f_M$ is an orthonormal basis of $\ell^2(J)$ comprised of eigenfunctions of $E_{-+}E_{-+}^*$ associated with the eigenvalues $t_1^2 \le \cdots \le t_M^2$. In particular, $\sigma(E_{-+}E_{-+}^*) = \sigma(E_{-+}^*E_{-+})$ and

$$E_{-+}e_j = t_j f_j, \quad E_{-+}^* f_j = t_j e_j, \quad j = 1, \dots, M. \tag{4.8}$$

Let $0 \le t_1 \le ... \le t_k$ be the singular values of $E_{-+}(z)$ in the interval $[0, \tau]$ for $\tau > 0$ small. Let $\mathcal{S}_+, \mathcal{S}_- \subset \ell^2(J)$ be the corresponding (sums of) spectral subspaces for $E_{-+}^*E_{-+}$ and $E_{-+}E_{-+}^*$, respectively, corresponding to the eigenvalues $t_1^2 \le t_2^2 \le ... \le t_k^2$ in $[0, \tau^2]$. Using (4.8), we see that the restrictions (denoted by the same symbols)

$$E_{-+} : \mathcal{S}_+ \to \mathcal{S}_-, \ E_{-+}^* : \mathcal{S}_- \to \mathcal{S}_+,$$

have norms $\le \tau$. Also,

$$E_{-+} : \mathcal{S}_+^\perp \to \mathcal{S}_-^\perp, \ E_{-+}^* : \mathcal{S}_-^\perp \to \mathcal{S}_+^\perp \tag{4.9}$$

are bijective with inverses of norm $\le 1/\tau$.

Let S_+ be the orthogonal projection onto $\mathcal{S}_+$, viewed as an operator $\ell^2(J) \to \mathcal{S}_+$, whose adjoint is the inclusion map $\mathcal{S}_+ \to \ell^2(J)$. Let $S_- : \mathcal{S}_- \to \ell^2(J)$ be the inclusion map. Let $\mathcal{S}$ be the operator in (4.2) with $\mathcal{H}_\pm = \ell^2(J)$, corresponding to the problem

$$\begin{cases} E_{-+}g + S_- g_- = h \in \ell^2(J), \\ S_+ g = h_+ \in \mathcal{S}_+, \end{cases} \tag{4.10}$$

for the unknowns $g \in \ell^2(J)$, $g_- \in \mathcal{S}_-$. Using the orthogonal decompositions,

$$\ell^2(J) = \mathcal{S}_+^\perp \oplus \mathcal{S}_+, \ \ell^2(J) = \mathcal{S}_-^\perp \oplus \mathcal{S}_-,$$

[1]Here $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of a linear operator A.

we write $g = \sum_1^k g_j e_j + g^\perp$ and $h = \sum_1^k h_j f_j + h^\perp$. Then, (4.10) is equivalent to

$$\begin{cases} g^\perp = (E_{-+})^{-1} h^\perp \\ \begin{pmatrix} g_j \\ g_-^j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -t_j \end{pmatrix} \begin{pmatrix} h_j \\ h_+^j \end{pmatrix}, \ j = 1, \dots, M, \end{cases}$$

where we also used that $g_- = \sum_1^k g_-^j f_j$ and $h_+ = \sum_1^k h_+^j e_j$. It follows that

$$\begin{cases} g = (E_{-+})^{-1} h^\perp + \sum_1^k h_+^j e_j \\ g_- = \sum_1^k h^j f_j - \sum_1^k t_j h_+^j f_j. \end{cases} \tag{4.11}$$

Hence, the unique solution to (4.10) is given by

$$\begin{pmatrix} g \\ g_- \end{pmatrix} = \mathcal{F} \begin{pmatrix} h \\ h_+ \end{pmatrix} = \begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix} \begin{pmatrix} h \\ h_+ \end{pmatrix}, \tag{4.12}$$

where

$$\begin{aligned} F = E_{-+}^{-1} \Pi_{\mathcal{S}_-^\perp}, \quad F_+ = S_+^*, \\ F_- = S_-^*, \quad F_{-+} = -E_{-+|_{\mathcal{S}_+}} : \mathcal{S}_+ \to \mathcal{S}_-. \end{aligned} \tag{4.13}$$

Here Π_B denotes the orthogonal projection onto the subspace B of A, viewed as a self-adjoint operator $A \to A$. Notice that $F = \Pi_{\mathcal{S}_+^\perp} F$ and that

$$F_{-+} = -\sum_1^k t_j f_j \circ e_j^*, \quad \text{i.e. } F_{-+} u = -\sum_1^k t_j (u|e_j) f_j. \tag{4.14}$$

Using as well (4.9), we have

$$\|F\| \leq 1/\tau, \ \|F_+\|, \|F_-\| \leq 1, \|F_{-+}\| \leq \tau. \tag{4.15}$$

4.2. Composing the Grushin problems

From now on we assume that

$$0 < \alpha \ll 1, \quad \epsilon(M) \leq \alpha/2, \tag{4.16}$$

and the estimates below will be uniformly valid for $z \in K \setminus \gamma_\alpha$, $N \gg 1$, where K is some fixed relatively compact open set in $\mathbf{C}$ and

$$\gamma_\alpha = \{z \in \mathbf{C}; \operatorname{dist}(z, \gamma) \leq \alpha\}, \quad \gamma = p(S^1). \tag{4.17}$$

We apply Proposition 4.1 to $\mathcal{P}_N$ in (3.5) with the inverse $\mathcal{E}_N$ in (3.10), and to $\mathcal{S}$ defined in (4.10) with inverse in $\mathcal{F}$ in (4.12). Let $z \in K \backslash \gamma_\alpha$, then

$$\mathcal{T}_N = \begin{pmatrix} P_N - z & R_- S_- \\ S_+ R_+ & S_+ R_{+-} S_- \end{pmatrix} = \begin{pmatrix} P_N - z & T_- \\ T_+ & T_{+-} \end{pmatrix} : L^2(I_N) \times \mathcal{S}_- \to L^2(I_N) \times \mathcal{S}_+, \tag{4.18}$$

defined as in (4.3), is bijective with the bounded inverse

$$\mathcal{G}_N = \begin{pmatrix} G^N & G_+^N \\ G_-^N & G_{-+}^N \end{pmatrix} = \begin{pmatrix} E^N - E_+^N F E_-^N & E_+^N F_+ \\ F_- E_-^N & -F_{-+} \end{pmatrix}. \tag{4.19}$$

Since $S_\pm$ have norms ≤ 1, we get

$$\|T_\pm\| \leq \|R_\pm\| = \mathcal{O}(1), \tag{4.20}$$

uniformly in N, α and $z \in K$. Also, since the norms of E^N, E_+^N, E_-^N are $\leq 2/\alpha$ (uniformly as $N \to \infty$) by (3.11), we get from (4.4), (4.15), that

$$\|G^N\| \leq \frac{2}{\alpha} + \frac{4}{\tau\alpha^2},\ \|G_{-+}^N\| \leq \tau,\ \|G_\pm^N\| \leq \frac{2}{\alpha}. \tag{4.21}$$

Proposition 4.2. *Let $K \Subset \mathbf{C}$ be an open relatively compact set, let $z \in K\backslash\gamma_\alpha$, and let $\tau > 0$ be as in the definition of the Grushin problem (4.10). Then, for $\tau > 0$ small enough, depending only on K, we have that G_+^N is injective and G_-^N is surjective. Moreover, there exists a constant $C > 0$, depending only on K, such that for all $z \in K\backslash\gamma_\alpha$ the singular values s_j^+ of G_+^N, and s_j^- of $(G_-^N)^*$ satisfy*

$$\frac{1}{C} \leq s_j^\pm \leq \frac{2}{\alpha}, \quad 1 \leq j \leq k(z) = \operatorname{rank}(G_\pm^N). \tag{4.22}$$

Proof. To ease the notation we will omit the sub/superscript N. We begin with the injectivity of G_+. From

$$\begin{pmatrix} P-z & T_- \\ T_+ & T_{+-} \end{pmatrix} \begin{pmatrix} G & G_+ \\ G_- & G_{-+} \end{pmatrix} = 1, \tag{4.23}$$

we have $T_+G_+ + T_{+-}G_{-+} = 1$ which we write $T_+G_+ = 1 - T_{+-}G_{-+}$. Here

$$\|T_{+-}G_{-+}\| \leq \|R_{+-}\|\tau = \mathcal{O}(\tau),$$

where we used that $\|R_{+-}\| \leq \|p(\tau) - z\| = \mathcal{O}(1)\|m\|_{\ell^1}$, thus the error term above only depends on K. Choosing $\tau > 0$ small enough, depending on K but not on N, we get that $\|T_{+-}G_{+-}\| \leq 1/2$. Then, $1 - T_{+-}G_{-+}$ is bijective with $\|(1 - T_{+-}G_{-+})^{-1}\| \leq 2$ and G_+ has the left inverse

$$(1 - T_{+-}G_{-+})^{-1}T_+ \tag{4.24}$$

of norm $\leq 2\|R_+\| = \mathcal{O}(1)$, depending only on K.

Now we turn to the surjectivity of G_-. From

$$\begin{pmatrix} G & G_+ \\ G_- & G_{-+} \end{pmatrix} \begin{pmatrix} P-z & T_- \\ T_+ & T_{+-} \end{pmatrix} = 1,$$

we get

$$\begin{pmatrix} (P-z)^* & T_+^* \\ T_-^* & T_{+-}^* \end{pmatrix} \begin{pmatrix} G^* & G_-^* \\ G_+^* & G_{-+}^* \end{pmatrix} = 1,$$

and as above we then see that G_-^* has the left inverse $(1 - T_{+-}^*G_{-+}^*)^{-1}T_-^*$. Hence, G_- has the right inverse

$$T_-(1 - G_{-+}T_{+-})^{-1}, \tag{4.25}$$

of norm $\leq 2\|R_-\| = \mathcal{O}(1)$, depending only on K.

The lower bound on the singular values follows from the estimates on the left inverses of G_+ and G_-^*, and the upper bound follows from (4.21). ☐

5. Determinants

We continue working under the assumptions (4.16), (4.17). Additionally, we fix $\tau > 0$ sufficiently small (depending only on the fixed relatively compact set $K \Subset \mathbf{C}$) so that $\|T_{+-}G_{-+}\|$, $\|G_{-+}T_{+-}\|$ (both $= \mathcal{O}(\tau)$) are $\leq 1/2$, which implies that G_+ is injective and G_+ is surjective, see Proposition 4.2. Here, we sometimes drop the sub-/superscript N.

From now on, we will work with $z \in K\backslash\gamma_\alpha$. The constructions and estimates in Sect. 3 are then uniform in z for $N \gg 1$ and the same holds for those in Sect. 4.

Remark 5.1. To get the $o(N)$ error term in Theorem 1.1, we will take $\alpha > 0$ arbitrarily small, and $M > 1$ large enough (but fixed) so that $\varepsilon(M) \leq \alpha/2$, see (2.10) as well as $N > 1$ sufficiently large. In the following, the error terms will typically depend on α, although we will not always denote this explicitly, however, they will be uniform in $N > 1$ and in $z \in K\backslash\gamma_\alpha$.

5.1. The unperturbed operator

For $z \in K \setminus \gamma_\alpha$, we have $d_N(z) \geq \alpha$ and (3.8), (3.9) give

$$|\det \mathcal{P}_N(z)| \leq e^{\epsilon(M)/\alpha}|\det(p_N(\tau) - z)|, \tag{5.1}$$

$$|\det(p_N(\tau) - z)| \leq e^{2\epsilon(M)/\alpha}|\det \mathcal{P}_N(z)|, \tag{5.2}$$

where we also used that

$$\frac{\epsilon(M)}{d_N(z) - \epsilon(M)} \leq \frac{\epsilon(M)}{\alpha - \epsilon(M)} \leq \frac{2\epsilon(M)}{\alpha},$$

by the second inequality in (4.16). Recall here that $p_N(\tau)$ acts on $\ell^2(\mathbf{Z}/\widetilde{N}\mathbf{Z})$, $\widetilde{N} = N + M$.

By the Schur complement formula, we have

$$\begin{aligned}\det(P_N - z) &= \det \mathcal{P}_N(z) \det E_{-+}(z),\\ \det(P_N - z) &= \det \mathcal{T}_N(z) \det G_{-+}(z),\end{aligned} \tag{5.3}$$

so

$$\frac{\det \mathcal{T}_N}{\det \mathcal{P}_N} = \frac{\det E_{-+}}{\det G_{-+}}. \tag{5.4}$$

Recall from Sect. 4.1 that the singular values of E_{-+} are denoted by $0 \leq t_1 \leq t_2 \leq \cdots \leq t_M$ and that those of G_{-+} are $t_1, ..., t_k$, where $k = k(z, N)$ is determined by the condition $t_k \leq \tau < t_{k+1}$. Thus

$$\left|\frac{\det E_{-+}}{\det G_{-+}}\right| = \prod_{k+1}^{M} t_j$$

and we get (since $\tau \ll 1$)

$$\tau^M \leq \left|\frac{\det E_{-+}}{\det G_{-+}}\right| \leq \left(\frac{2}{\alpha}\right)^M.$$

Since $\tau > 0$ is small, but fixed depending only on K, we have uniformly for $z \in K \setminus \gamma_\alpha$, $N \gg 1$:

$$|\ln|\det E_{-+}| - \ln|\det G_{-+}|| \leq \mathcal{O}(1) \tag{5.5}$$

and by (5.4)

$$|\ln|\det \mathcal{T}_N| - \ln|\det \mathcal{P}_N|| \leq \mathcal{O}(1). \tag{5.6}$$

From (5.1), (5.2), we get

$$|\ln|\det \mathcal{P}_N| - \ln|\det(p_N(\tau) - z)|| \leq \mathcal{O}(1), \tag{5.7}$$

hence

$$|\ln|\det \mathcal{T}_N| - \ln|\det(p_N(\tau) - z)|| \leq \mathcal{O}(1). \tag{5.8}$$

5.2. The perturbed operator

We next extend the estimates to the case of a perturbed operator

$$P_N^\delta = P_N + \delta Q, \tag{5.9}$$

where $Q : \ell^2(I_N) \to \ell^2(I_N)$ satisfies

$$\delta\|Q\| \ll 1. \tag{5.10}$$

Proposition 5.2. *Let $K \Subset \mathbf{C}$ be an open relatively compact set and suppose that (4.16) hold. Recall (4.17) and (3.5), if $\delta\|Q\|\alpha^{-1} \ll 1$, then for all $z \in K\backslash\gamma_\alpha$*

$$\mathcal{P}_N^\delta = \begin{pmatrix} P_N^\delta - z & R_- \\ R_+ & R_{+-}(z) \end{pmatrix} = \mathcal{P} + \begin{pmatrix} \delta Q & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.11}$$

is bijective with bounded inverse

$$\mathcal{E}_N^\delta = \begin{pmatrix} E^\delta & E_+^\delta \\ E_-^\delta & E_{+-}^\delta \end{pmatrix}. \tag{5.12}$$

Recall (4.18), if $\delta\|Q\|\alpha^{-2} \ll 1$, then for all $z \in K\backslash\gamma_\alpha$

$$\mathcal{T}_N^\delta = \begin{pmatrix} P_N^\delta - z & T_- \\ T_+ & T_{+-} \end{pmatrix} = \mathcal{T}_N + \begin{pmatrix} \delta Q & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.13}$$

is bijective with bounded inverse

$$\mathcal{G}_N^\delta = \begin{pmatrix} G^\delta & G_+^\delta \\ G_-^\delta & G_{+-}^\delta \end{pmatrix}, \tag{5.14}$$

with

$$G_{-+}^\delta(z) = G_{-+} - G_- \delta Q(1 + G\delta Q)^{-1} G_+. \tag{5.15}$$

Moreover, $\|\mathcal{E}_N^\delta\| \leq 4/\alpha$, $\|\mathcal{G}_N^\delta\| \leq \mathcal{O}(\alpha^{-2})$, uniformly in $z \in K\backslash\gamma_\alpha$ and $N > 1$.

Proof. We sometimes drop the subscript N. By (3.10),

$$\mathcal{P}^\delta \mathcal{E} = 1 + \begin{pmatrix} \delta Q E & \delta Q E_+ \\ 0 & 0 \end{pmatrix}.$$

By (3.11), it follows that $\|E\| \leq 2/\alpha$, so if $\delta\|Q\|\alpha^{-1} \ll 1$, then by Neumann series argument, the above is invertible and

$$\mathcal{E}\left(1 + \begin{pmatrix} \delta Q E & \delta Q E_+ \\ 0 & 0 \end{pmatrix}\right)^{-1} \tag{5.16}$$

is a right inverse of $\mathcal{P}^\delta$, of norm $\leq 2\|\mathcal{E}\| \leq 4/\alpha$. Since $\mathcal{P}^\delta$ is Fredholm of index 0, this is also a left inverse.

The proof for $\mathcal{T}_N^\delta$ is similar, using that $\|G\| = \mathcal{O}(\alpha^{-2})$ by (4.21), since $\tau > 0$ is fixed. Finally, the expression (5.15) follows easily from expanding (5.16). □

We drop the subscript N until further notice. By (5.13), we have

$$\|\mathcal{T} - \mathcal{T}^\delta\|_{\mathrm{tr}} \leq \delta \|Q\|_{\mathrm{tr}}. \tag{5.17}$$

Recall from the text after (2.10) the definition of the Schatten norm $\|\cdot\|_{\mathrm{tr}}$. Write,

$$\mathcal{T}^\delta = \mathcal{T}(1 - \mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}^\delta)),$$

where

$$\|\mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}^\delta)\|_{\mathrm{tr}} \leq \mathcal{O}(\delta)\|Q\|_{\mathrm{tr}}. \tag{5.18}$$

Here, we used that $\|\mathcal{T}^{-1}\| = \|\mathcal{G}\| = \mathcal{O}(1)$, by (4.21) and the fact that $\tau > 0$ is fixed. We recall that the estimates here depend on α, yet are uniform in $z \in K\backslash\gamma_\alpha$ and $N > 1$. It follows that

$$|\det(1 - \mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}^\delta))| \leq \exp \|\mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}^\delta)\|_{\mathrm{tr}} \leq \exp(\mathcal{O}(\delta)\|Q\|_{\mathrm{tr}}),$$

and

$$\begin{aligned} |\det \mathcal{T}_\delta| &= |\det \mathcal{T}||\det(1 - \mathcal{T}^{-1}(\mathcal{T} - \mathcal{T}^\delta))| \\ &\leq \exp(\mathcal{O}(\delta)\|Q\|_{\mathrm{tr}})|\det \mathcal{T}|. \end{aligned} \tag{5.19}$$

Similarly from the identity

$$\mathcal{T} = \mathcal{T}^\delta(1 - \mathcal{T}_\delta^{-1}(\mathcal{T}^\delta - \mathcal{T})),$$

(putting δ as a subscript whenever convenient), we get

$$|\det \mathcal{T}| \leq \exp(\mathcal{O}(\delta)\|Q\|_{\mathrm{tr}})|\det \mathcal{T}^\delta|, \tag{5.20}$$

thus

$$|\ln|\det \mathcal{T}_\delta| - \ln|\mathcal{T}|| \leq \mathcal{O}(\delta)\|Q\|_{\mathrm{tr}}. \tag{5.21}$$

Assume that (uniformly in $N > 1$ and independently of α)

$$\delta\|Q\|_{\mathrm{tr}} \leq \mathcal{O}(1) \tag{5.22}$$

and recall (5.8). Then,

$$|\ln|\det \mathcal{T}_\delta| - \ln|\det(p_N(\tau) - z)|| \leq \mathcal{O}(1). \tag{5.23}$$

Notice that the error term depends on α. Using also the general identity (cf. (5.3)),

$$\det(P_N^\delta - z) = \det \mathcal{T}^\delta(z) \det G_{-+}^\delta(z), \tag{5.24}$$

we get

$$\ln|\det(P_N^\delta - z)| = \ln|\det(p_N(\tau) - z)| + \ln|\det G_{-+}^\delta| + \mathcal{O}(1), \tag{5.25}$$

uniformly for $z \in K \setminus \gamma_\alpha$, $N \gg 1$.

6. Lower bounds with probability close to 1

We now adapt the discussion in [15, Section 5] to $\mathcal{T}^\delta$. Let

$$P_N^\delta = P_N + \delta Q_\omega, \quad Q_\omega = (q_{j,k}(\omega))_{1\leq j,k\leq N}, \tag{6.1}$$

where $0 \leq \delta \ll 1$ and $q_{j,k}(\omega) \sim \mathcal{N}(0,1)$ are independent normalized complex Gaussian random variables. Recall from (1.21) that

$$\mathbf{P}[\|Q_\omega\|_{\mathrm{HS}} \leq C_1 N] \geq 1 - e^{-N^2}, \tag{6.2}$$

for some universal constant $C_1 > 0$. In the following, we restrict the attention to the case when

$$\|Q_\omega\|_{\mathrm{HS}} \leq C_1 N, \tag{6.3}$$

and (as before) $z \in K \setminus \gamma_\alpha$, $N \gg 1$. We assume that

$$\delta \ll N^{-3/2}. \tag{6.4}$$

Then,

$$\delta\|Q\|_{\mathrm{tr}} \leq \delta N^{1/2}\|Q\|_{\mathrm{HS}} \leq \delta C_1 N^{3/2} \ll 1,$$

and the estimates of the previous sections apply.

Let $\mathcal{Q}_{C_1N}$ be the set of matrices satisfying (6.3). As in [15, Section 5.3], we study the map (5.15), i.e.,

$$\begin{aligned} \mathcal{Q}_{C_1N} \ni Q \mapsto G^\delta_{-+}(z) &= G_{-+} - G_-\delta Q(1+G\delta Q)^{-1}G_+ \\ &= G_{-+} - \delta G_-(Q + T(z,Q))G_+, \end{aligned} \tag{6.5}$$

where

$$T(z,Q) = \sum_1^\infty (-\delta)^n Q(GQ)^n, \tag{6.6}$$

and notice first that by (4.21)

$$\|T\|_{\mathrm{HS}} \leq \mathcal{O}(\delta\alpha^{-2}N^2). \tag{6.7}$$

We strengthen the assumption (6.4) to

$$\delta \ll N^{-2}\alpha^2. \tag{6.8}$$

At the end of Sect. 4, we have established the uniform injectivity and surjectivity respectively for G_+ and G_-. This means that the singular values $s_j^\pm$ of $G_\pm$ for $1 \leq j \leq k(z) = \mathrm{rank}\,(G_-) = \mathrm{rank}\,(G_+)$ satisfy

$$\frac{1}{C} \leq s_j^\pm \leq \frac{2}{\alpha} \tag{6.9}$$

This corresponds to [15, (5.27)] and the subsequent discussion there carries over to the present situation with the obvious modifications. Similarly to [15, (5.42)], we strengthen the assumption on δ to

$$\delta \ll N^{-3}\alpha^2 \tag{6.10}$$

Notice that assumption (6.10) is stronger than the assumptions on δ in Proposition 5.2. The same reasoning as in [15, Section 5.3] leads to the following adaptation of Proposition 5.3 in [15]:

Proposition 6.1. *Let $K \subset \mathbf{C}$ be compact, $0 < \alpha \ll 1$ and choose M so that $\epsilon(M) \leq \alpha/2$. Let δ satisfy (6.10). Then, the second Grushin problem with matrix $\mathcal{T}^\delta$ is well posed with a bounded inverse $\mathcal{G}^\delta$ introduced in Proposition 5.2. The following holds uniformly for $z \in K \setminus \gamma_\alpha$, $N \gg 1$:*

There exist positive constants C_0, C_2 such that

$$\mathbf{P}\left(\ln|\det G^\delta_{-+}(z)|^2 \geq -t \text{ and } \|Q\|_{\mathrm{HS}} \leq C_1 N\right) \geq 1 - e^{-N^2} - C_2\delta^{-M}e^{-t/2},$$

when

$$t \geq C_0 - 2M\ln\delta, \quad 0 < \delta \ll N^{-3}\alpha^2.$$

7. Counting eigenvalues in smooth domains

In this section, we will prove Theorem 1.1. We will begin with a brief outline of the key steps:

We wish to count the zeros of the holomorphic function $u(z) = \det(P_N^\delta - z)$, which depends on the large parameter $N > 0$, in smooth domains $\Omega \Subset \mathbf{C}$ as in Theorem 1.1.

1. We work in some sufficiently large but fixed compact set $K \Subset \mathbf{C}$ containing Ω. In Sect. 7.1, we begin by showing that $u(z)$ satisfies with probability close to 1 an upper bound of the form

$$\ln|u(z)| \leq N(\phi(z) + \varepsilon), \tag{7.1}$$

for $z \in K$. Here, $0 < \varepsilon \ll 1$ and $\phi(z)$ is some suitable continuous subharmonic function. Next, we will show that $u(z)$ satisfies for any fixed point z_0 in $K\backslash\Gamma_\alpha$ a lower bound of the form

$$\ln|u(z_0)| \geq N(\phi(z_0) - \varepsilon) \tag{7.2}$$

with probability close to 1. Here, Γ_α denotes the set γ_α suitably enlarged to be a compact set with smooth boundary, see Fig. 2 for an illustration. The function ϕ will be constructed in the following way : Outside Γ_α we set $\phi(z)$ to be $\ln|\det(p_N(\tau) - z)|$, which in view of (5.25) and Proposition 6.1 yields the estimates (7.1), (7.2) outside Γ_α. Inside Γ_α, we set ϕ to be the solution to the Dirichlet problem for the Laplace operator on Γ_α with boundary conditions $\phi\restriction_{\partial\Gamma_\alpha} = \ln|\det(p_N(\tau) - z)|\restriction_{\partial\Gamma_\alpha}$. Since $\ln|u(z)|$ is subharmonic we have that the bound (7.1) holds in all of K.

2. In Sect. 7.2, we will use (7.1), (7.2) and [12, Theorem 1.1] (see also [13, Chapter 12]) to estimate the number of zeros of u in Ω and thus the number of eigenvalues of P_N^δ in Ω, i.e.,

$$\#(\sigma(P_N^\delta) \cap \Omega) = \#(u^{-1}(0) \cap \Omega) \sim \frac{N}{2\pi}\int_\Omega \Delta\phi L(dz), \tag{7.3}$$

see (7.22).

3. In Sect. 7.3, we study the measure $\Delta\phi$ by analyzing the Poisson and Green kernel of Γ_α. We will use this analysis to give precise error estimates on

the asymptotics (7.3) and we will show that $\frac{N}{2\pi}\Delta\phi$ integrated over Ω is, up to a small error, given by the number of eigenvalues λ_j of $p_N(\tau)$ (3.4) in Ω, i.e.,

$$\frac{N}{2\pi}\int_\Omega \Delta\phi L(dz) = \#\{\lambda_j \in \Omega\} + \mathcal{O}(\alpha N),$$

see (7.53). This, in combination with (7.3), see (7.22), will let us conclude Theorem 1.1.

7.1. Estimates on the log-determinant

We work under the assumptions of Proposition 6.1 and from now on we assume that δ satisfies (1.13), i.e.,

$$\mathrm{e}^{-N^{\delta_0}} \leq \delta \ll N^{-\delta_1}, \tag{7.4}$$

for some fixed $\delta_0 \in]0,1[$ and $\delta_1 > 3$. Notice that (6.10) holds for $N > 1$ sufficiently large (depending on α). Then with probability $\geq 1 - e^{-N^2}$, we have $G^\delta_{-+}(z) = \mathcal{O}(1)$ for every $z \in K \setminus \gamma_\alpha$, hence by (5.25)

$$\ln|\det(P_N^\delta - z)| \leq \ln|\det(p_N(\tau) - z)| + \mathcal{O}(1). \tag{7.5}$$

On the other hand, by (5.25) and Proposition 6.1, we have for every $z \in K\setminus\gamma_\alpha$ that

$$\ln|\det(P_N^\delta - z)| \geq \ln|\det(p_N(\tau) - z)| - \frac{t}{2} - \mathcal{O}(1) \tag{7.6}$$

with probability

$$\geq 1 - e^{-N^2} - C_2\delta^{-M}e^{-t/2}, \tag{7.7}$$

when

$$t \geq C_0 - 2M\ln\delta. \tag{7.8}$$

Next we enlarge γ_α to Γ_α, away from a neighborhood of the region $\partial\Omega\cap\gamma$, so that Γ_α has a smooth boundary. More precisely, let $g \in C^\infty(\mathbf{C};\mathbf{R})$ be a boundary defining function of Ω, so that $g(z) < 0$ for $z \in \Omega$ and $dg \neq 0$ on $\partial\Omega$. Then, for $C > 0$ sufficiently large and $\alpha > 0$ sufficiently small, we define

$$\Gamma_\alpha^0 \stackrel{\text{def}}{=} \gamma_\alpha \cup \{z \in \mathbf{C}; g(z) < -1/C\} \cup \{z \in \mathbf{C}; g(z) > 1/C \text{ and } |z| \leq C\}, \tag{7.9}$$

Notice that due to the assumption that the intersection of $\partial\Omega$ with γ is transversal, the boundary of Γ_α^0 may be only Lipschitz near the intersection points

$$\{z_0,\dots,z_q\} = \partial\gamma_\alpha \cap \partial G, \quad \text{where } G \stackrel{\text{def}}{=} \{z \in \mathbf{C}; |g(z)| \leq 1/C\}.$$

By the assumptions on Ω, we have that $q < \infty$. Away from these points, we have that $\partial\Gamma_\alpha^0$ is smooth. To remedy this lack of regularity, we will slightly deform Γ_α^0 in an α-neighborhood of these points.

Pick $z_0 \in \partial\gamma_\alpha\cap\partial G$. Since $\partial\gamma_\alpha\cap D(z_0,\alpha)$ and $\partial G\cap D(z_0,\alpha)$ are transversal to each other, it follows that there exists new affine coordinates $\widetilde{z} = U(z - z_0)$, $\mathbf{R}^2 \simeq \mathbf{C} \ni z = (z^1, z^2)$ being the old coordinates, where U is orthogonal, and smooth functions f_1, f_2 independent of α, such that $\gamma_\alpha \cap D(z_0,\alpha)$ takes the form

$$A = \{z \in D(z_0,\alpha); \widetilde{z}^2 \leq f_2(\widetilde{z}^1),\ |\widetilde{z}^1| < \alpha,\ \|\widetilde{z}\| < \alpha\},$$

and that $(\mathbf{C}\backslash\mathring{G}) \cap D(z_0,\alpha)$ takes the form

$$B = \{z \in D(z_0,\alpha); \widetilde{z}^2 \leq f_1(\widetilde{z}^1),\ |\widetilde{z}^1| < \alpha,\ \|\widetilde{z}\| < \alpha\}.$$

Here, f_1, respectively f_2, is (after translation and rotation) a smooth local parametrization of ∂G, resp. $\partial\gamma_\alpha$, near z_0. Moreover, $f_2(0) = f_1(0)$ and the transversality assumption yields that $\widetilde{z}^1 = 0$ is the only point in the interval $]-\alpha,\alpha[$ where $f_2(\widetilde{z}^1) = f_1(\widetilde{z}^1)$.

Then, $\Gamma^0_\alpha \cap D(z_0,\alpha)$ takes the form

$$A \cup B = \{z \in D(z_0,\alpha); \widetilde{z}^2 \leq \max\{f_1(\widetilde{z}^1), f_2(\widetilde{z}^1)\},\ |\widetilde{z}^1| < \alpha,\ \|\widetilde{z}\| < \alpha\}.$$

Continuing, let $\chi \in C^\infty_c(\mathbf{R};[0,1])$ so that $\chi = 1$ on $[-1/4,1/4]$ and $\chi = 0$ outside $]-1/2,1/2[$, and let $C > 0$ be sufficiently large. Set

$$f(t) = \left(1 - \chi\left(\frac{t}{\alpha}\right)\right)\max\{f_1(t), f_2(t)\} + \chi\left(\frac{t}{\alpha}\right)\frac{\alpha}{C}, \quad t \in]-\alpha,\alpha[,$$

which is a smooth function. Then, let Γ^1_α be equal to Γ^0_α outside $D(z_0,\alpha)$, and equal to

$$\{z \in D(z_0,\alpha); \widetilde{z}^2 \leq f(\widetilde{z}^1),\ |\widetilde{z}^1| < \alpha,\ \|\widetilde{z}\| < \alpha\},$$

inside $D(z_0,\alpha)$. Summing up, we have that the boundary of Γ^1_α is smooth at z_0 and $\Gamma^0_\alpha \subset \Gamma^1_\alpha$.

Next, we perform the same procedure for Γ^1_α at the point z_1 and obtain Γ^2_α whose boundary is smooth at z_0 and z_1 and which contains Γ^1_α. Continuing in this way until z_q, and defining

$$\Gamma_\alpha \stackrel{\text{def}}{=} \Gamma^q_\alpha, \tag{7.10}$$

we have that Γ_α has a smooth boundary and it contains Γ^0_α (7.9), and thus γ_α. Figure 2 presents an illustration of this "fattening" of γ_α.

Remark 7.1. Notice that the deformation of the boundary of Γ^0_α (7.9) has been done in such a way that the rescaled domain $\frac{1}{\alpha}\Gamma_\alpha$ has a smooth boundary which can be locally parametrized by a smooth function f with $\partial^\beta f = \mathcal{O}(1)$, $\beta \in \mathbf{N}$, uniformly in α.

Continuing, we define $\phi(z) = \phi_N(z)$ by requiring that

$$N\phi(z) = \ln|\det(p_N(\tau) - z)| \text{ on } K \setminus \Gamma_\alpha, \tag{7.11}$$

and

$$\phi(z) \text{ is continuous in } K \text{ and harmonic in } \mathring{\Gamma}_\alpha \tag{7.12}$$

Here we assume that K is large enough to contain a neighborhood of Γ_α. Choose

$$t = N^{\epsilon_0}, \tag{7.13}$$

for some fixed $\epsilon_0 \in]0,1[$ with $\delta_0 < \varepsilon_0$, see (7.4), (1.13). Then,

$$C_2\delta^{-M}e^{-t/2} = \exp\left(\ln C_2 - M\ln\delta - N^{\epsilon_0}/2\right),$$

and we require from δ that

$$\ln C_2 - M\ln\delta - N^{\epsilon_0}/2 \leq -N^{\epsilon_0}/4,$$

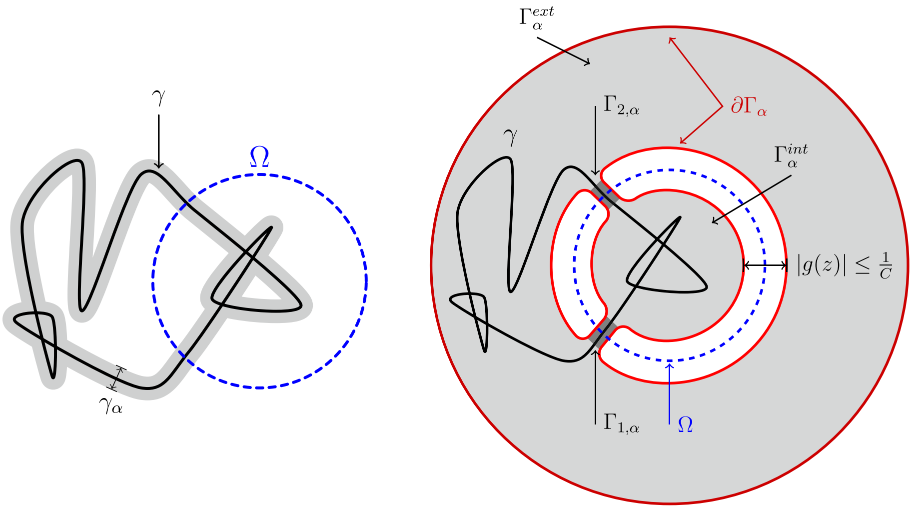

FIGURE 2. Left-hand side shows the curve γ surrounded by the tube γ_α and the domain Ω (dashed line) where we are counting the eigenvalues of P_N^δ. The right-hand side shows the same picture with γ_α enlarged to $\Gamma_\alpha = \Gamma_\alpha^{ext} \cup \Gamma_\alpha^{int} \cup \Gamma_{1,\alpha} \cup \Gamma_{2,\alpha}$, i.e., the whole gray area. The decomposition into an "exterior" part, an "interior" part and into the thin tubes $\Gamma_{j,\alpha}$ connecting exterior and interior will play a role in the proof of Lemma 7.3

i.e.,

$$\ln \delta \geq \frac{\ln C_2}{M} - \frac{N^{\epsilon_0}}{4M}.$$

This is fulfilled if $N \gg 1$ and

$$\ln \delta \geq -\frac{N^{\epsilon_0}}{5M},$$

i.e.,

$$\delta \geq \exp\left(-\frac{1}{5M} N^{\epsilon_0}\right) \tag{7.14}$$

and (7.13), (7.14) imply (7.8) when $N \gg 1$. Notice that (7.4) implies (7.14) for $N \gg 1$.

Combining (7.6), (7.11), (7.13) and (7.14), we get for each $z \in K \setminus \Gamma_\alpha$ that

$$\ln|\det(P_N^\delta - z)| \geq N(\phi(z) - \epsilon_1), \tag{7.15}$$

with probability

$$\geq 1 - e^{-N^2} - e^{-N^{\epsilon_0}/4} \tag{7.16}$$

where

$$\epsilon_1 = N^{\epsilon_0 - 1}. \tag{7.17}$$

Here and in the following, we assume that $N \geq N(\alpha, K)$ sufficiently large.

On the other hand, with probability $\geq 1 - e^{-N^2}$, we have by (7.5)

$$\ln|\det(P_N^\delta - z)| \leq N(\phi(z) + \epsilon_1) \tag{7.18}$$

for all $z \in K \setminus \Gamma_\alpha$. Then, since the left- hand side in (7.18) is subharmonic and the right-hand side is harmonic in Γ_α, we see that (7.18) remains valid also in Γ_α and hence in all of K.

7.2. Counting zeros of holomorphic functions with exponential growth

Let $\Omega \Subset \mathbf{C}$ be as in Theorem 1.1, so that $\partial\Omega$ intersects γ at finitely many points $\widetilde{z}_1, ..., \widetilde{z}_{k_0}$ which are not critical values of p and where the intersection is transversal. Choose $z_1, ..., z_L \in \partial\Omega \setminus \Gamma_\alpha$ such that with $r_0 = C_0\alpha$, $C_0 \gg 1$, we have

$$\frac{r_0}{4} \leq |z_{j+1} - z_j| \leq \frac{r_0}{2} \tag{7.19}$$

where the z_j are distributed along the boundary in the positively oriented sense and with the cyclic convention that $z_{L+1} = z_1$. Notice that $L = \mathcal{O}(1/\alpha)$. Then,

$$\partial\Omega \subset \bigcup_1^L D(z_j, r_0/2)$$

and we can arrange so that $z_j \notin \Gamma_\alpha$ and even so that

$$\mathrm{dist}\,(z_j, \Gamma_\alpha) \geq \alpha, \tag{7.20}$$

for $\alpha > 0$ sufficiently small.

Choose K above so that $\overline{\Omega} \Subset K$. Combining (7.18) and (7.15), we have that $\det(P_N^\delta - z)$ satisfies the upper bound (7.18) for all $z \in K$ and the lower bound (7.15) for $z = z_1, \ldots, z_L$ with probability

$$\geq 1 - \mathcal{O}(\alpha^{-1})(e^{-N^2} + e^{-N^{\epsilon_0}/4}). \tag{7.21}$$

Since ϕ is continuous and subharmonic, we can apply [12, Theorem 1.1] (see also [13, Chapter 12]) to the holomorphic function $\det(P_N^\delta - z)$ and get

$$\begin{aligned}\left|\#(\sigma(P_N^\delta) \cap \Omega) - \frac{N}{2\pi}\int_\Omega \Delta\phi L(dz)\right| &\leq \mathcal{O}(N)\\ &\times \left(L\epsilon_1 + \int_{\partial\Omega + D(0,r_0)} \Delta\phi L(dz) + \sum_1^L \int_{D(z_j,r_0)} \Delta\phi(z)\left|\ln\frac{|z - z_j|}{r_0}\right| L(dz)\right)\end{aligned} \tag{7.22}$$

with probability (7.21).

Recall that $L = \mathcal{O}(1/\alpha)$ (hence $\mathcal{O}(1)$ for every fixed α). $\Delta\phi$ is supported in Γ_α and the number of discs $D(z_j, r_0)$ that intersect Γ_α is $\leq \mathcal{O}(1)$ uniformly with respect to α. Also $\ln(|z - z_j|/r_0) = \mathcal{O}(1)$ on the intersection of each such disc with Γ_α. Since $\epsilon_1 = N^{\epsilon_0 - 1}$, we get from (7.22):

$$\left|\#(\sigma(P_N^\delta)\cap\Omega)-\frac{N}{2\pi}\int_\Omega \Delta\phi L(dz)\right|$$
$$\leq \mathcal{O}(N)\left(\mathcal{O}_\alpha(N^{\epsilon_0-1})+\int_{(\gamma\cap\partial\Omega)+D(0,2r_0)}\Delta\phi(z)L(dz)\right). \tag{7.23}$$

7.3. Analysis of the measure $\Delta\phi$

By (3.4), we have that

$$\ln|\det(p_N(\tau)-z)|=\sum_1^{N+M}\ln|z-\lambda_j|, \tag{7.24}$$

where

$$\lambda_j=p\left(\exp\frac{2\pi ij}{N+M}\right),\ 1\leq j\leq N+M,$$

and this expression is equal to $N\phi(z)$ in $K\setminus\Gamma_\alpha$.

Define

$$\psi(z)=\phi(z)-\frac{1}{N}\sum_1^{N+M}\ln|z-\lambda_j|, \tag{7.25}$$

so that ψ is continuous away from the $\lambda_j\in\gamma$,

$$\psi(z)=0 \text{ in } \mathbf{C}\setminus\Gamma_\alpha, \tag{7.26}$$
$$\psi\restriction_{\partial\Gamma_\alpha}=0, \tag{7.27}$$
$$\Delta\psi=-\frac{2\pi}{N}\sum_1^{N+M}\delta_{\lambda_j} \text{ in } \overset{\circ}{\Gamma}_\alpha\ . \tag{7.28}$$

It follows that in Γ_α:

$$\psi(z)=-\frac{2\pi}{N}\sum_1^{N+M}G_{\Gamma_\alpha}(z,\lambda_j), \tag{7.29}$$

where G_{Γ_α} is the Green kernel for Γ_α.

ϕ is harmonic away from $\partial\Gamma_\alpha$, so for ϕ as a distribution on $\mathbf{C}$, we have $\operatorname{supp}\Delta\phi\subset\partial\Gamma_\alpha$. Now $\psi-\phi$ is harmonic near $\partial\Gamma_\alpha$, so $\Delta\psi=\Delta\phi$ near $\partial\Gamma_\alpha$. In the interior of Γ_α we have (7.28) and in order to compute $\Delta\psi$ globally, we let $v\in C_0^\infty(\mathbf{C})$ and apply Green's formula to get

$$\langle\Delta\psi,v\rangle=\langle\psi,\Delta v\rangle=\int_{\Gamma_\alpha}\psi\Delta vL(dz)$$
$$=\int_{\Gamma_\alpha}\Delta\psi vL(dz)+\int_{\partial\Gamma_\alpha}\psi\partial_\nu v|dz|-\int_{\partial\Gamma_\alpha}\partial_\nu\psi v|dz|.$$

Here ν is the exterior unit normal and in the last term, it is understood that we apply ∂_ν to the restriction of ψ to $\overset{\circ}{\Gamma}_\alpha$ then take the boundary limit. (7.27), (7.28) and (7.29) imply that in the sense of distributions on $\mathbf{C}$,

$$\Delta\psi=-\frac{2\pi}{N}\sum_1^{N+M}\delta_{\lambda_j}+\frac{2\pi}{N}\partial_\nu\left(\sum_1^{N+M}G_{\Gamma_\alpha}(\cdot,\lambda_j)\right)L_{\partial\Gamma_\alpha}(dz) \tag{7.30}$$

where $L_{\partial\Gamma_\alpha}$ denotes the (Lebesgue) arc length measure supported on $\partial\Gamma_\alpha$.

By the preceding discussion, we conclude that

$$\Delta\phi = \frac{2\pi}{N}\left(\sum_1^{N+M} \partial_\nu G_{\gamma_\alpha}(\cdot,\lambda_j) L_{\partial\Gamma_\alpha}(dz)\right). \tag{7.31}$$

Each term in the sum is a nonnegative measure of mass 1:

$$\int \partial_\nu G(z,\lambda_j) L_{\partial\Gamma_\alpha}(dz) = 1. \tag{7.32}$$

Before continuing, we will present two technical lemmas.

Lemma 7.2. *Let $X \Subset \mathbf{C}$ be an open relatively compact, simply connected domain with smooth boundary. Let $u \in C^\infty(\overline{X})$ with $u\restriction_{\partial X}= 0$. Let $z_0 \in \partial X$ and let $\widetilde{W} \Subset W \Subset \mathbf{C}$ be two open relatively compact small complex neighborhoods of z_0, so that the closure of $\widetilde{W}$ is contained in W. If u is harmonic in $X \cap W$, then for any $s \in \mathbf{N}$*

$$\|u\|_{H^s(X\cap\widetilde{W})} \leq \mathcal{O}_{s,\widetilde{W}}(1)\|u\|_{H^0(X\cap W)}. \tag{7.33}$$

Here H^s are the standard Sobolev spaces.

Proof. The proof is standard, and we present it here for the reader's convenience.

1. Let $W_1 \Subset W \Subset \mathbf{C}$ be two open relatively compact small complex neighborhoods of z_0, so that the closure of W_1 is contained in W. Let $\chi \in C_c^\infty(\mathbf{C};[0,1])$ be so that $\chi = 1$ on W_1 and $\mathrm{supp}\chi \subset W$. Integration by parts then yields that

$$\begin{aligned}\int_{X\cap W} |\chi\nabla u|^2 dx &= \int_{X\cap W} \chi\nabla u\cdot(\nabla(\chi\overline{u}) - \overline{u}\,\nabla\chi)dx \\ &= -\int_{X\cap W} \chi\overline{u}\,\nabla(\chi\nabla u) + \chi\overline{u}\,\nabla u\cdot\nabla\chi)dx \\ &= -2\int_{X\cap W} \chi\overline{u}\,\nabla u\cdot\nabla\chi dx.\end{aligned}$$

In the last equality, we used as well that u is harmonic in $X \cap W$. By the Cauchy–Schwarz inequality

$$\|\chi\nabla u\|^2_{L^2(X\cap W)} \leq \mathcal{O}(1)\|\chi\nabla u\|_{L^2(X\cap W)}\|u\|_{L^2(X\cap W)},$$

which implies that

$$\|\chi\nabla u\|_{L^2(X\cap W)} \leq \mathcal{O}(1)\|u\|_{L^2(X\cap W)}.$$

Hence,

$$\|u\|_{H^1(X\cap W_1)} \leq \mathcal{O}(1)\|u\|_{L^2(X\cap W)}. \tag{7.34}$$

2. Since W is small, we may pass to new local coordinates y, and we can suppose that $z_0 = 0$ and that locally $\partial X = \{y_2 = 0\}$. If ϕ is a local diffeomorphism realizing this change of variables, then the Laplacian can be formally written in the new coordinates as

$$L \stackrel{\mathrm{def}}{=} {}^t((\phi')^{-1}\nabla_y)\cdot((\phi')^{-1}\nabla_y), \quad \text{with } \Delta_x = (\phi^{-1})^* \circ \Delta \circ \phi^*. \tag{7.35}$$

Here, L is an elliptic second-order differential operator, and ϕ' is the Jacobian map associated with the diffeomorphism ϕ.

Working from now on in these new coordinates, we proceed by an induction argument: suppose that

$$\|u\|_{H^{s+1}(X\cap W_1)} \leq \mathcal{O}(1)\|u\|_{H^s X\cap W)}. \tag{7.36}$$

holds for some $s \in \mathbf{N}$. Here we write as well W, W_1 for the respective sets in the new coordinates to ease notation. We want to show that we then also have

$$\|u\|_{H^{s+2}(X\cap W_2)} \leq \mathcal{O}(1)\|u\|_{H^{s+1} X\cap W_1)}. \tag{7.37}$$

where $W_2 \Subset W_1$ is a slightly smaller neighborhood of $z_0 = 0$, whose closure is contained inside W_1.

Let $\chi \in C_c^\infty(\mathbf{C};[0,1])$ be so that $\chi = 1$ on W_2 and $\mathrm{supp}\chi \subset W_1$. Let $\partial_{t,j}u(y) := t^{-1}(u(y+te_j) - u(y))$, where $x \in \mathbf{C} \simeq \mathbf{R}^2$ and e_1, e_2 is the standard orthonormal basis of $\mathbf{R}^2$. Then, by the hypothesis (7.36) applied to $\partial_{t,j}\chi u$, for $|t| \ll 1$, we get

$$\begin{aligned}\|\partial_{t,1}\chi u\|_{H^{s+1}(X\cap W_1)} &\leq \mathcal{O}(1)\|\partial_{t,1}\chi u\|_{H^s(X\cap W)}\\ &\leq \mathcal{O}(1)\|\chi\partial_{t,1}u\|_{H^s(X\cap W)} + \mathcal{O}(1)\|[\partial_{t,1},\chi]u\|_{H^s(X\cap W)}\\ &\leq \mathcal{O}(1)\|u\|_{H^{s+1}(X\cap W_1)} + \mathcal{O}(1)\|u\|_{H^s(X\cap W_1)},\end{aligned}$$

uniformly in $|t| \ll 1$. In the last inequality, we used as well that $\chi\partial_{t,1}u$ and $[\partial_{t,1},\chi]u = (\partial_{t,1}\chi)u(\cdot + te_1)$ are supported in W_1 for $|t| \ll 1$. Performing the limit $t \to 0$, we get

$$\|\partial_{y_1}\chi u\|_{H^{s+1}(X\cap W_1)} \leq \mathcal{O}(1)\|u\|_{H^{s+1} X\cap W_1)}. \tag{7.38}$$

Thus, for $j = 1, 2$, we have that

$$\|\partial_{y_1}\partial_{y_j}\chi u\|_{H^s(X\cap W_1)} \leq \mathcal{O}(1)\|\partial_{y_1}u\|_{H^{s+1}(X\cap W_1)} \leq \mathcal{O}(1)\|u\|_{H^{s+1}(X\cap W_1)}. \tag{7.39}$$

By (7.35), it follows that there exists some smooth function $a \neq 0$, such that

$$\partial_{y_2}^2\chi u = \frac{1}{a}L\chi u - \widetilde{L}\chi u, \tag{7.40}$$

where $\widetilde{L}$ is a second-order differential operator with smooth coefficients and which does not contain the derivative $\partial_{y_2}^2$. Since u is harmonic in $X \cap W$, it follows that $L\chi u = [L,\chi]u$. Since $[L,\chi]$ is a differential operator of order 1, it follows from (7.40) and (7.39) that

$$\|\partial_{y_2}\chi u\|_{H^{s+1}(X\cap W_1)} \leq \mathcal{O}(1)\sum_1^2 \|\partial_{y_j}\partial_{y_2}\chi u\|_{H^s(X\cap W_1)} \leq \mathcal{O}(1)\|u\|_{H^{s+1}(X\cap W_1)}. \tag{7.41}$$

In combination with (7.38), this yields

$$\|u\|_{H^{s+2}(X\cap W_2)} \leq \|\chi u\|_{H^{s+2}(X\cap W_1)} \leq \mathcal{O}(1)\|u\|_{H^{s+1} X\cap W_1)}. \tag{7.42}$$

Thus, by choosing a decreasing sequence of nested compact neighborhoods of z_0, say $\widetilde{W} = W_{s+1} \Subset W_s \cdots \Subset W_0 = W$, we may iterate the estimate (7.36), which then in combination with (7.34) yields (7.33). □

Lemma 7.3. *There exists a $C>0$ independent of $\alpha>0$, such that for any $1\leq j\leq N+M$*

$$|\partial_\nu G_{\Gamma_\alpha}(z,\lambda_j)|\leq \frac{1}{\alpha}e^{-\frac{|z-\lambda_j|}{C\alpha}}, \tag{7.43}$$

for $z\in\partial\Gamma_\alpha\cap\mathrm{neigh}\,(\gamma\cap\partial\Omega)$, $\lambda_j\in\Gamma_\alpha$, $|z-\lambda_j|\geq\alpha/C$. (7.43) also holds when $z\in\partial\Gamma_\alpha$, $\lambda_j\in\Gamma_\alpha$, $|z-\lambda_j|\geq\alpha/C$ and $(z,\lambda_j)\in(\Omega\times(\mathbf{C}\setminus\Omega))\cup((\mathbf{C}\setminus\Omega)\times\Omega)$.

Proof. 1. By scaling of the harmonic function $G_{\Gamma_\alpha}(\cdot,\lambda_j)$ by a factor $1/\alpha$, it suffices to show that

$$|G_{\Gamma_\alpha}(z,\lambda_j)|\leq e^{-\frac{|z-\lambda_j|}{C\alpha}}, \tag{7.44}$$

for (z,λ_j) as after (7.43) with the difference that z now varies in Γ_α instead of $\partial\Gamma_\alpha$.

To see this, recall from the construction of Γ_α after (7.8) that $\mathrm{dist}(\partial\Gamma_\alpha,\lambda_j)\geq\alpha$ and fix a point $z_0\in\partial\Gamma_\alpha$, let $C_1>0$ be sufficiently large so that for any $z\in D(z_0,\alpha/C_1)\cap\Gamma_\alpha$ we have that (z,λ_j) satisfies the conditions after (7.43) with z varying in $D(z_0,\alpha/C_1)\cap\Gamma_\alpha$ instead of $\partial\Gamma_\alpha$.

Let $u(z):=G_{\gamma_\alpha}(\alpha z,\lambda_j)$, $z\in\frac{1}{\alpha}\Gamma_\alpha$, be the scaled function, and recall Remark 7.1. Let $\chi\in C_c^\infty(\mathbf{C};[0,1])$ be so that $\chi=1$ on $D(z_0/\alpha,1/(4C_1))$, $\mathrm{supp}\chi\subset D(z_0/\alpha,1/2C_1)=:W'$ and $\partial^\beta=\mathcal{O}(1)$, uniformly in α for any $\beta\in\mathbf{N}^2$. Moreover, put $W=D(z_0/\alpha,1/C_1)$.

Then, $\chi u\in H^s(\Gamma_\alpha\cap W')$ for any $s>0$. We can find an extension $v\in H^s(\mathbf{R}^2)$ of χu so that $\|v\|_{H^s}\leq\mathcal{O}(1)\|\chi u\|_{H^s(\Gamma_\alpha\cap W')}$. Using the Fourier transform, we see that for $s>2$ and for $z\in D(z_0/\alpha,1/(4C_1))$

$$|\nabla v(z)|\leq\mathcal{O}(1)\||\xi|\widehat{v}\|_{L^2}\leq\mathcal{O}(1)\||\xi|\langle\xi\rangle^{-s}\|_{L^2}\|v\|_{H^s}\leq\mathcal{O}(1)\|\chi u\|_{H^s(\Gamma_\alpha\cap W')}. \tag{7.45}$$

By Lemma 7.2 and (7.44), we see that

$$|\partial_\nu v(z)|\leq\mathcal{O}(1)\|u\|_{L^\infty(\Gamma_\alpha\cap W)}\leq\mathcal{O}(1)\,e^{-\frac{|z-\lambda_j/\alpha|}{C}}, \tag{7.46}$$

and

$$|\alpha(\partial_\nu G_{\Gamma_\alpha})(\alpha z,\lambda_j)|\leq\mathcal{O}(1)\,e^{-\frac{|z-\lambda_j/\alpha|}{C}}, \tag{7.47}$$

which implies (7.44) after rescaling and potentially slightly increasing the constant $C>0$.

2. We decompose Γ_α as $\Gamma^{int}\cup\Gamma^{ext}\cup\Gamma_{1,\alpha}\cup...\cup\Gamma_{T,\alpha}$, where Γ^{int} and Γ^{ext} are the enlarged parts of Γ_α with $\Gamma^{int}\subset\Omega$, $\Gamma^{ext}\subset\mathbf{C}\setminus\Omega$ and $\Gamma_{1,\alpha},...,\Gamma_{T,\alpha}$ are the regular parts of width 2α, corresponding to the segments of γ, that intersect $\partial\Omega$ transversally, see Fig. 2 for an illustration. Here, T is the number of intersections of γ with $\partial\Omega$, notice that T is finite and independent of N,α.

For simplicity, we assume that Γ^{int} and Γ^{ext} are connected and that each segment $\Gamma_{k,\alpha}$ links Γ^{int} to Γ^{ext} and crosses $\partial\Omega$ once. We may think of Γ_α as a graph with the vertices Γ^{int}, Γ^{ext} and with $\Gamma_{k,\alpha}$ as the edges.

Let first λ_j belong to Γ^{int}. We apply the first estimate in Proposition 2.2 in [12] or equivalently Proposition 12.2.2 in [13] and see that $-G_{\Gamma_\alpha}(z,\lambda_j)\leq\mathcal{O}(1)$ for $z\in\Gamma_\alpha$, $|z-\lambda_j|\geq 1/\mathcal{O}(1)$. Here and in the following the constants $\mathcal{O}(1)$ are independent of j and α. Furthermore, the notation $1/\mathcal{O}(1)$ means $1/C$ for some sufficiently large constant $C>0$.

Possibly, after cutting away a piece of $\Gamma_{k,\alpha}$ and adding it to Γ^{int}, we may assume that $-G_{\Gamma_\alpha}(z,\lambda_j) \le \mathcal{O}(1)$ in $\Gamma_{k,\alpha}$. Consider one of the $\Gamma_{k,\alpha}$ as a finite band with the two ends given by the closure of the set of $z \in \partial\Gamma_{k,\alpha}$ with $\mathrm{dist}\,(z, \partial\Gamma_\alpha) < \alpha$. Let $G_{\Gamma_{k,\alpha}}$ denote the Green kernel of $\Gamma_{k,\alpha}$. Then, the second estimate in the quoted proposition applies and we find

$$-G_{\Gamma_{k,\alpha}}(x,y) \le \mathcal{O}(1)e^{-|x-y|/(\alpha\,\mathcal{O}(1))}, \text{ when } x, y \in \Gamma_{k,\alpha},\ |x-y| \ge \alpha/\mathcal{O}(1).$$

Let

$$u = \chi G_{\Gamma_\alpha}(\cdot, \lambda_j)\restriction_{\Gamma_{k,\alpha}},$$

where $\chi \in C^\infty(\Gamma_{k,\alpha}; [0,1])$ vanishes near the ends of $\Gamma_{k,\alpha}$, is equal to 1 away from an α-neighborhood of these end points and with the property that $\nabla\chi = \mathcal{O}(1/\alpha)$, $\nabla^2\chi = \mathcal{O}(1/\alpha^2)$. Then, $u_{|\partial\Gamma_{k,\alpha}} = 0$ and $\Delta u = \mathcal{O}(\alpha^{-2})$ is supported in an α-neighborhood of the union of the two ends and hence of uniformly bounded L^1-norm. Now we apply the second estimate in the quoted proposition to $u = \int G_{\Gamma_{k,\alpha}}(\cdot, y)\Delta u(y)L(dy)$ and we see that

$$G_{\Gamma_\alpha}(\cdot, \lambda_j) = \mathcal{O}(e^{-1/(\alpha\,\mathcal{O}(1))}). \tag{7.48}$$

in $\{x \in \Gamma_{k,\alpha};\ \mathrm{dist}\,(x, \partial\Omega \cap \Gamma_{k,\alpha}) \le 1/\mathcal{O}(1)\}$. Here, we also recall that $\lambda_j \in \gamma \Subset \mathring{\Gamma}_\alpha$. Varying k, we get (7.48) in $\{x \in \Gamma_\alpha;\ \mathrm{dist}\,(x, \partial\Omega \cap \Gamma) \le 1/\mathcal{O}(1)\}$. Applying the maximum principle to the harmonic function $G_{\Gamma_\alpha}(\cdot, \lambda_j)\restriction_{(\mathbf{C}\setminus\Omega)\cap\Gamma_\alpha}$, we see that (7.48) holds uniformly in $(\mathbf{C}\setminus\Omega)\cap\Gamma_\alpha$.

Similarly, we have (7.48) uniformly in

$$\{x \in \Gamma_\alpha;\ \mathrm{dist}\,(x, \partial\Omega \cap \gamma) \le 1/\mathcal{O}(1)\} \cup (\Omega \cap \Gamma_\alpha),$$

when $\lambda_j \in \Gamma^{ext}$ and we have shown (7.44), (7.43) when $\lambda_j \in \Gamma^{int} \cup \Gamma^{ext}$. Similarly, we have (7.43) when $\lambda_j \in \gamma_{k,\alpha}$ is close to one of the ends.

It remains to treat the case when $\lambda_j \in \gamma_{k,\alpha}$ is at distance $\ge 1/\mathcal{O}(1)$ from the ends of $\gamma_{k,\alpha}$. Defining $u = \chi G_{\Gamma_\alpha}(\cdot, \lambda_j)\restriction_{\gamma_{k,\alpha}}$ as before we now have

$$\Delta u = [\Delta, \chi]G_{\Gamma_\alpha(\cdot,\lambda_j)} + \delta_{\lambda_j},$$

where the first term in the right-hand side has its support in an α-neighborhood of the union of the ends and is $\mathcal{O}(1)$ in L^1. By the second part of the quoted proposition, we have

$$u(x) = \mathcal{O}(1)\exp\left(-\frac{1}{\mathcal{O}(1)\alpha}\min\left(\mathrm{dist}\,(x, \mathrm{ends}\,(\gamma_{k,\alpha})), |x-\lambda_j|\right)\right), \tag{7.49}$$

away from an α-neighborhood of $\mathrm{ends}\,(\gamma_{k,\alpha}) \cup \{\lambda_j\}$. Here $\mathrm{ends}\,(\gamma_{k,\alpha})$ denotes the union of the two ends of $\gamma_{k,\alpha}$. Since u is harmonic away from λ_j and from α-neighborhoods of the ends, we get from (7.49) that

$$\nabla u(x) = \mathcal{O}\left(\frac{1}{\alpha}\right)\exp\left(-\frac{1}{\mathcal{O}(1)\alpha}\min\left(\mathrm{dist}\,(x, \mathrm{ends}\,(\gamma_{k,\alpha})), |x-\lambda_j|\right)\right), \quad (7.50)$$

which gives (7.43) near $\partial\Omega \cap \gamma$. By using the maximum principle as before, we can extend the validity of (7.43) to all of $\partial\Gamma_\alpha \setminus D(\lambda_j, \alpha/\mathcal{O}(1))$. □

Continuing, notice that by (3.4), (7.24)

$$\#\{\sigma(P_{S_{\widetilde{N}}}) \cap \eta\} = \#\{\widehat{S}_{\widetilde{N}} \cap p_N^{-1}(\eta)\}, \quad \widetilde{N} = N + M, \quad (7.51)$$

for $\eta \subset \gamma$. Since two consecutive points of $\widehat{S}_{\widetilde{N}}$ differ by an angle of $2\pi/\widetilde{N}$ and by the assumptions (1)-(4) prior to Theorem 1.1, we get that

$$\#\{\lambda_j;\, \mathrm{dist}\,(\lambda_j, \partial\Omega\cap\gamma) < 4r_0\} = \mathcal{O}(\alpha N)$$

and also

$$\#\{\lambda_j;\, \mathrm{dist}\,(\lambda_j, \partial\Omega\cap\gamma) \in [2^k r_0, 2^{k+1} r_0[\} = \mathcal{O}(\alpha 2^k N),\ \ k = 2, 3, \ldots$$

From (7.43) and (7.31), we get

$$\begin{aligned}\frac{N}{2\pi}\int_{(\partial\Omega\cap\gamma)+D(0,2r_0)} \Delta\phi L(dz) &= \sum_j \int_{((\partial\Omega\cap\gamma)+D(0,2r_0))\cap\partial\Gamma_\alpha} \partial_\nu G_{\Gamma_\alpha}(z,\lambda_j)L(dz)\\ &= \mathcal{O}(\alpha N) + \sum_{k=2}^{\infty} \sum_{\substack{\lambda_j;\\ \mathrm{dist}\,(\lambda_j,\partial\Omega\cap\gamma)\in[2^k r_0, 2^{k+1}r_0[}} e^{-2^k/\mathcal{O}(1)}\\ &= \mathcal{O}(1)\left(\alpha N + \sum_{k=2}^{\infty} e^{-2^k/\mathcal{O}(1)}\alpha 2^k N\right)\\ &= \mathcal{O}(\alpha N) + \mathcal{O}(1)N\alpha\int_0^\infty e^{-t/\mathcal{O}(1)}dt\\ &= \mathcal{O}(\alpha N).\end{aligned} \quad (7.52)$$

Combining (7.32) and (7.43), we get when $\mathrm{dist}\,(\lambda_j, \partial\Omega\cap\gamma) \geq 2r_0$:

$$\int_{\partial\Gamma_\alpha\cap\Omega} \partial_\nu G_{\gamma_\alpha}(z,\lambda_j) L_{\partial\Gamma_\alpha}(dz) = \begin{cases} 1 + \mathcal{O}(1)e^{-\mathrm{dist}\,(\lambda_j,\partial\Omega\cap\gamma)/\mathcal{O}(\alpha)}, & \text{when } \lambda_j \in \Omega,\\ \mathcal{O}(1)e^{-\mathrm{dist}\,(\lambda_j,\partial\Omega\cap\gamma)/\mathcal{O}(\alpha)}, & \text{when } \lambda_j \notin \Omega.\end{cases}$$

We now get

$$
\begin{aligned}
\frac{N}{2\pi}\int_{\Omega}\Delta\phi L(dz) =& \sum_{j;\, \mathrm{dist}\,(\lambda_j,\gamma\cap\partial\Omega)\le 4r_0}\int_{\partial\Gamma_\alpha\cap\Omega}\partial_\nu G_{\Gamma_\alpha}(z,\lambda_j)L_{\partial\Gamma_\alpha}(dz)\\
&+\sum_{k=2}^{\infty}\sum_{\substack{\lambda_j\in\Omega,\\ \mathrm{dist}\,(\lambda_j,\gamma\cap\partial\Omega)\in[2^k r_0,2^{k+1}r_0[}}\int_{\partial\Gamma_\alpha\cap\Omega}\partial_\nu G_{\Gamma_\alpha}(z,\lambda_j)L_{\partial\gamma_\alpha}(dz)\\
&+\sum_{k=2}^{\infty}\sum_{\substack{\lambda_j\in\mathbf{C}\setminus\Omega,\\ \mathrm{dist}\,(\lambda_j,\gamma\cap\partial\Omega)\in[2^k r_0,2^{k+1}r_0[}}\int_{\partial\Gamma_\alpha\cap\Omega}\partial_\nu G_{\Gamma_\alpha}(z,\lambda_j)L_{\partial\gamma_\alpha}(dz)\\
=&\mathcal{O}(\alpha N)+\sum_{k=2}^{\infty}\sum_{\substack{\lambda_j\in\Omega,\\ \mathrm{dist}\,(\lambda_j,\gamma\cap\partial\Omega)\in[2^k r_0,2^{k+1}r_0[}}(1+\mathcal{O}(1)e^{-2^k/\mathcal{O}(1)})\\
&+\sum_{k=2}^{\infty}\sum_{\substack{\lambda_j\in\mathbf{C}\setminus\Omega,\\ \mathrm{dist}\,(\lambda_j,\gamma\cap\partial\Omega)\in[2^k r_0,2^{k+1}r_0[}}\mathcal{O}(1)e^{-2^k/\mathcal{O}(1)}\\
&=\#\{\lambda_j\in\Omega\}+\mathcal{O}(\alpha N).
\end{aligned}
\tag{7.53}
$$

Thus, (7.23) gives

$$
\begin{aligned}
\#(\sigma(P_N^\delta)\cap\Omega) &= \#(\{\lambda_j\}\cap\Omega)+\mathcal{O}(\alpha N)+\mathcal{O}_\alpha(N^{\epsilon_0})\\
&=\frac{N}{2\pi}\left(\int_{S^1\cap p^{-1}(\Omega)}L_{S^1}(d\theta)\right)+\mathcal{O}(\alpha N)+\mathcal{O}_\alpha(N^{\epsilon_0})+o(N),
\end{aligned}
\tag{7.54}
$$

with a probability as in (7.21) which is bounded from below by the probability (1.15) for $N>1$ sufficiently large. Here and in the next formula, we view p_N and p as maps from S^1 to $\mathbf{C}$. In the second equality, we used that by (7.51)

$$
\begin{aligned}
\#(\{\lambda_j\}\cap\Omega) &= \frac{\widetilde{N}}{2\pi}\int_{S^1\cap p_N^{-1}(\Omega)}L_{S^1}(d\theta)+\mathcal{O}(1)\\
&=\frac{N}{2\pi}\int_{S^1\cap p_N^{-1}(\Omega)}L_{S^1}(d\theta)+\mathcal{O}(M)\\
&=\frac{N}{2\pi}\int_{S^1\cap p^{-1}(\Omega)}L_{S^1}(d\theta)+o(N),
\end{aligned}
\tag{7.55}
$$

where we used that $p_N\to p$ uniformly on S^1 and where the measure $L_{S^1}(d\theta)$ in the integral denotes the Lebesgue measure on S^1.

Theorem 1.1 follows by taking $\alpha>0$ in (7.54) arbitrarily small and $N>1$ sufficiently large.

8. Convergence of the empirical measure

In this section, we present a proof of Theorem 1.2 following the strategy of [15, Section 7.3]. An alternative, and perhaps more direct way, to conclude the weak convergence of the empirical measure from a counting theorem as Theorem 1.2, is presented in [15, Section 7.1].

Recall the definition of the empirical measure ξ_N (1.20). By (1.21), (1.5) combined with a Borel Cantelli argument, it follows that almost surely

$$\mathrm{supp}\xi_N \subset \overline{D(0, \|p\|_{L^\infty(S^1)} + 1)} \stackrel{\mathrm{def}}{=} K \subset D(0, \|p\|_{L^\infty(S^1)} + 2) \stackrel{\mathrm{def}}{=} K' \tag{8.1}$$

for N sufficiently large. For p as in (1.4), put

$$\xi = p_* \left(\frac{1}{2\pi} L_{S^1} \right) \tag{8.2}$$

which has compact support,

$$\mathrm{supp}\xi = p(S^1) \subset K. \tag{8.3}$$

Here, $\frac{1}{2\pi} L_{S^1}$ denotes the normalized Lebesgue measure on S^1.

We recall [15, Theorem 7.1]:

Theorem 8.1. *Let $K, K' \Subset \mathbf{C}$ be open relatively compact sets with $\overline{K} \subset K'$, and let $\{\mu_n\}_{n\in\mathbf{N}} \in \mathcal{P}(\mathbf{C})$ be as sequence of random measures so that almost surely*

$$\mathrm{supp}\mu_n \subset K \text{ for } n \text{ sufficiently large.}$$

Suppose that for a.e. $z \in K'$ almost surely

$$U_{\mu_n}(z) \to U_\mu(z), \quad n \to \infty,$$

where $\mu \in \mathcal{P}(\mathbf{C})$ is some probability measure with $\mathrm{supp}\mu \subset K$. Then, almost surely,

$$\mu_n \rightharpoonup \mu, \quad n \to \infty, \quad \text{weakly.}$$

This theorem is a modification of a classical result which allows to deduce the weak convergence of measures from the point-wise convergence of the associated Logarithmic potentials, see for instance [17, Theorem 2.8.3] or [1].

In view of Theorem 8.1, it remains to show that for almost every $z \in K'$ we have that $U_{\xi_N}(z) \to U_\xi(z)$ almost surely, where

$$U_{\xi_N}(z) = -\int \log|z - x|\xi_N(dx), \quad U_\xi(z) = -\int \log|z - x|\xi(dx).$$

For $z \notin \sigma(P_N^\delta)$

$$U_{\xi_N}(z) = -\frac{1}{N} \log|\det(P_N^\delta - z)|. \tag{8.4}$$

For any $z \in \mathbf{C}$ the set $\Sigma_z = \{Q \in \mathbf{C}^{N\times N}; \det(P_N + \delta Q - z) = 0\}$ has Lebesgue measure 0, since $\mathbf{C}^{N\times N} \ni Q \mapsto \det(P_N^\delta - z)$ is analytic and not constantly 0. Thus, $\mu_N(\Sigma_z) = 0$, where μ_N is the Gaussian measure given in after (1.11), and for every $z \in \mathbf{C}$ (8.4) holds almost surely.

Let δ satisfy (1.13) for some fixed $\delta_0 \in]0, 1[$ and $\delta_1 > 3$. Pick a $\varepsilon_0 \in]\delta_0, 1[$. Let $z \in K' \backslash p(S^1)$. Recall (4.17). For $\alpha > 0$ sufficiently small, we have that $z \in K' \backslash \gamma_\alpha$.

Put $t = N^{\varepsilon_0}$ as in (7.13), which together with (7.14) implies (7.8) when $N \gg 1$. Since (1.13) implies (7.14), it follows by combining (7.14), (7.5), (7.6) and (7.7) that

$$\left| \frac{1}{N} \log |\det(P_N^\delta - z)| - \phi(z) \right| \leq \mathcal{O}(N^{\varepsilon_0 - 1}). \tag{8.5}$$

with probability $\geq 1 - \mathrm{e}^{-N^2} - \mathrm{e}^{-N^{\varepsilon_0/4}}$. Here, $\phi(z) := N^{-1} \ln |\det(p_N(\tau) - z)|$, since $z \notin \gamma_\alpha$.

Using a Riemann sum argument and the fact that $p_N \to p$ uniformly on S^1, we have that

$$|\phi(z) + U_\xi(z)| \longrightarrow 0, \quad \text{as } N \to \infty. \tag{8.6}$$

Thus, by (8.5), (8.6), we have for any $z \in K' \backslash p(S^1)$ that

$$|U_{\xi_N}(z) - U_\xi(z)| = o(1) \tag{8.7}$$

with probability $\geq 1 - \mathrm{e}^{-N^2} - \mathrm{e}^{-N^{\varepsilon_0/4}}$. By the Borel–Cantelli theorem, if follows that for every $z \in K' \backslash p(S^1)$

$$U_{\xi_N}(z) \longrightarrow U_\xi(z), \quad \text{as } N \to \infty, \text{ almost surely}, \tag{8.8}$$

which by Theorem 8.1 concludes the proof of Theorem 1.2.

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References

[1] Bordenave, C., Chafaï, D.: Lecture notes on the circular law. In: Vu, V.H. (ed.) Modern Aspects of Random Matrix Theory, vol. 72, pp. 1–34. American Mathematical Society, London (2013)

[2] Basak, A., Paquette, E., Zeitouni, O.: Regularization of non-normal matrices by gaussian noise-the banded toeplitz and twisted toeplitz cases. Forum Math. Sigma **7**, e3 (2019)

[3] Basak, A., Paquette, E., Zeitouni, O.: Spectrum of random perturbations of toeplitz matrices with finite symbols. Trans. Am. Math. Soc. **373**(7), 4999–5023 (2020)

[4] Böttcher, A., Silbermann, B.: Introduction to Large Truncated Toeplitz Matrices. Springer, Berlin (1999)

[5] Davies, E.B.: Non-self-adjoint operators and pseudospectra, volume 76 of Proceedings of Symposia in Pure Mathematics, AMS (2007)

[6] Davies, E.B., Hager, M.: Perturbations of Jordan matrices. J. Approx. Theory **156**(1), 82–94 (2009)

[7] Dimassi, M., Sjöstrand, J.: Spectral Asymptotics in the Semi-Classical Limit. Cambridge University Press, Cambridge (1999)

[8] Embree, M., Trefethen, L.N.: Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators. Princeton University Press, Princeton (2005)

[9] Guionnet, A., Matchett Wood, P., Zeitouni, O.: Convergence of the spectral measure of non-normal matrices. Proc. AMS **142**(2), 667–679 (2014)

[10] Hager, M., Sjöstrand, J.: Eigenvalue asymptotics for randomly perturbed nonselfadjoint operators. Math. Annal. **342**, 177–243 (2008)

[11] Kallenberg, O.: Foundations of Modern Probability. Probability and its Applications. Springer, Berlin (1997)

[12] Sjöstrand, J.: Counting zeros of holomorphic functions of exponential growth. J. Pseudodiffer. Oper. Appl. **1**(1), 75–100 (2010)

[13] Sjöstrand, J.: Non-Self-Adjoint Differential Operators, Spectral Asymptotics and Random Perturbations Pseudo-Differential Operators Theory and Applications, vol. 14. Birkhäuser, Basel (2019)

[14] Sjöstrand, J., Vogel, M.: Large bi-diagonal matrices and random perturbations. J. Spectral Theory **6**(4), 977–1020 (2016)

[15] Sjöstrand, J., Vogel, M.: Toeplitz band matrices with small random perturbations. Indagationes Mathematicae (2020). https://doi.org/10.1016/j.indag.2020.09.001

[16] Sjöstrand, J., Zworski, M.: Elementary linear algebra for advanced spectral problems. Ann. l'Inst. Fourier **57**, 2095–2141 (2007)

[17] Tao, T.: Topics in Random Matrix Theory. Graduate Studies in Mathematics, vol. 132. American Mathematical Society, London (2012)

[18] Tao, T., Vu, V., Krishnapur, M.: Random matrices: universality of esds and the circular law. Ann. Probab. **38**(5), 2023–2065 (2010)

[19] Vogel, M.: The precise shape of the eigenvalue intensity for a class of Non-Self-Adjoint operators under random perturbations. Ann. Henri Poincaré **18**, 435–517 (2017)

[20] Wood, P.M.: Universality of the esd for a fixed matrix plus small random noise: a stability approach. Ann. l'Inst. Henri Poinc. Probab. Stat. **52**(4), 1877–1896 (2016)

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