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Hamiltonian Perturbations at the Second-Order Approximation

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Abstract. Integrability condition of Hamiltonian perturbations of integrable Hamiltonian PDEs of hydrodynamic type up to the second-order approximation is considered. Under a nondegeneracy assumption, we show that the Hamiltonian perturbation at the first-order approximation is integrable if and only if it is trivial, and that under a further assumption, the Hamiltonian perturbation at the second-order approximation is integrable if and only if it is quasi-trivial.

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1. Introduction and the Statements of the Results

Let M be an n-dimensional complex manifold. Consider the following system of Hamiltonian PDEs of hydrodynamic type:

$$\partial_t\big(v^\alpha\big) \;=\; \eta^{\alpha\beta}\partial_x\bigg(\frac{\delta H_0}{\delta v^\beta(x)}\bigg)\,, \qquad v=(v^1,\dots,v^n)\in M,\, x\in S^1,\, t\in\mathbb{R}\,, \quad (1.1)$$

where $(\eta^{\alpha\beta})$ is a given symmetric invertible constant matrix, $H_0 := \int_{S^1} h_0(v)\,\mathrm{d}x$ is a given local functional (called the Hamiltonian), and $\delta/\delta v^\beta(x)$ denotes the variational derivative. Here and below, free Greek indices take the integer values $1,\dots,n$, and the Einstein summation convention is assumed for repeated Greek indices with one-up and one-down; the matrix $(\eta^{\alpha\beta})$ and its inverse $(\eta_{\alpha\beta})$ are used to raise and lower Greek indices, e.g., $v_\alpha := \eta_{\alpha\beta}v^\beta$. The Hamiltonian density $h_0(v)$ is assumed to be a holomorphic function of v. More explicitly, Eq. (1.1) have the form:

$$\partial_t\big(v^\alpha\big) \;=\; A^\alpha_\gamma(v)\,v^\gamma_x\,, \quad \text{where } A^\alpha_\gamma(v) \;:=\; \eta^{\alpha\beta}\frac{\partial^2 h_0(v)}{\partial v^\beta\partial v^\gamma}\,.$$

Basic assumption: $(A^\alpha_\gamma(v))$ has pairwise distinct eigenvalues $\lambda_1(v),\dots,\lambda_n(v)$ on an open dense subset U of M.

Let us perform a change of variables $(v^1,\dots,v^n)\to(R_1,\dots,R_n)$ with non-degenerate Jacobian locally on U. We call $R_1,\dots,R_n$ a complete set of Riemann invariants, if evolutions along $R_1,\dots,R_n$ are all diagonal, namely,

$$\partial_t(R_i) \;=\; V_i(R)\,\partial_x(R_i)\,,\qquad i=1,\dots,n\,, \tag{1.2}$$

where V_i's are some functions of $R=(R_1,\dots,R_n)$. Below, free Latin indices take the integer values $1,\dots,n$ unless otherwise indicated. Clearly, Eq. (1.2) imply that the gradients of Riemann invariants are eigenvectors of (A^α_β), namely,

$$A^\alpha_\beta\,R_{i,\alpha} \;=\; \lambda_i\,R_{i,\beta}\,,\qquad V_i \;=\; \lambda_i \tag{1.3}$$

with $R_{i,\alpha}:=\partial_\alpha(R_i)$. Similar notations like $R_{i,j}:=\partial_j(R_i)$, $R_{i,jk}:=\partial_j\partial_k(R_i)$, $\dots$ will also be used. Here and below, $\partial_\alpha:=\partial_{v^\alpha}$, $\partial_i:=\partial_{R_i}$.

It was proven by Tsarev [23] that the integrability of Eq. (1.1) is equivalent to the existence of complete Riemann invariants. Here, "integrability" means existence of sufficiently many conservation laws/infinitesimal symmetries (See Definition 2.2). It was shown by B. Dubrovin [10,11] that existence of a complete set of Riemann invariants is equivalent to *vanishing* of the following Haantjes tensor:

$$H_{\alpha\beta\gamma} \;:=\; \big(A_{\alpha\rho\sigma}A_{\beta\phi}A_{\gamma\psi}+A_{\beta\rho\sigma}A_{\gamma\phi}A_{\alpha\psi}+A_{\gamma\rho\sigma}A_{\alpha\phi}A_{\beta\psi}\big)\,A^\rho_\nu\,\delta^{\sigma\nu\psi\phi}\,, \tag{1.4}$$

where $A_{\alpha\beta\gamma}:=\partial_\alpha\partial_\beta\partial_\gamma(h_0)$ and $\delta^{\alpha\beta\gamma\phi}:=\eta^{\alpha\gamma}\eta^{\beta\phi}-\eta^{\alpha\phi}\eta^{\beta\gamma}$. Note that $H_{\alpha\beta\gamma}$ automatically vanishes if the signature $\varepsilon(\alpha,\beta,\gamma)=0$; so for $n=1$ or for $n=2$, the system (1.1) is always integrable.

We proceed to the study of Hamiltonian perturbations [4,5,9–11,16,18] of (1.1)

$$\partial_t(v^\alpha) \;=\; \eta^{\alpha\beta}\partial_x\bigg(\frac{\delta H}{\delta v^\beta(x)}\bigg),\qquad x\in S^1\,,\ t\in\mathbb{R}\,,\ v=(v^1,\dots,v^n)\in M\,. \tag{1.5}$$

Here, $H:=\int_{S^1}h\,\mathrm{d}x=\sum_{j=0}^\infty\epsilon^jH_j$ with $H_j:=\int_{S^1}h_j(v,v_1,v_2,\dots,v_j)\,\mathrm{d}x$ is the Hamiltonian, and h_j are differential polynomials of v satisfying the following homogeneity condition:

$$\sum_{\ell=1}^{j}\ell\,v^\alpha_\ell\frac{\partial h_j}{\partial v^\alpha_\ell} \;=\; j\,h_j\,,\quad j\ge 0\,. \tag{1.6}$$

We recall that the variational derivative reads

$$\frac{\delta H}{\delta v^\beta(x)} \;=\; \sum_{\ell=0}^\infty(-\partial_x)^\ell\bigg(\frac{\partial h}{\partial v^\beta_\ell}\bigg).$$

In the above formulae, $v^\alpha_\ell:=\partial_x^\ell(v^\alpha)$, $\ell\ge0$, and we recall that a differential polynomial of v is a polynomial of $v_1,v_2,\dots$ whose coefficients are holomorphic functions of v. The ring of differential polynomials of v is denoted by $\mathcal{A}_v$. We remark that according to [4,14–16,18] the Hamiltonian system (1.5) that we are considering is general. Note that the Hamiltonian operator $\eta^{\alpha\beta}\partial_x$ defines a Poisson bracket $\{\,,\,\}$ on the space of local functionals $\mathcal{F} := \left\{\int_{S^1} f\,\mathrm{d}x \,|\, f \in \mathcal{A}_v[[\epsilon]]\right\}$, $\{\,,\,\} : \mathcal{F}\times\mathcal{F}\to\mathcal{F}$, by

$$\{F, G\} := \int_{S^1} \frac{\delta F}{\delta v^\alpha(x)}\eta^{\alpha\beta}\partial_x\left(\frac{\delta G}{\delta v^\beta(x)}\right)\mathrm{d}x\,, \qquad \forall\, F, G\in\mathcal{F}\,. \tag{1.7}$$

It is helpful to view $v^\alpha(x)$ as a "local functional" $v^\alpha(x) = \int_{S^1} v^\alpha(y)\,\delta(y-x)\,dy$, called the coordinate functional. Then, one can write Eq. (1.5) in the form

$$\partial_t(v^\alpha) \;=\; \left\{v^\alpha(x)\,,\,H\right\}.$$

Clearly, a system of Hamiltonian PDEs of hydrodynamic type (1.1) can be obtained from (1.5) simply by taking the dispersionless limit: $\epsilon\to 0$.

The perturbed system (1.5) is called *integrable* if its dispersionless limit is integrable and each conservation law of (1.1) can be extended to a conservation law of (1.5). In this paper, we start with a system of *integrable Hamiltonian PDEs of hydrodynamic type*, and study the conditions such that the perturbation (1.5) is integrable up to the second-order approximation.

Theorem 1.1. *Assume that the matrix* (A^α_β) *associated with* (1.1) *has distinct eigenvalues* $\lambda_1,\ldots,\lambda_n$ *on an open dense subset* $U\subset M$. *Assume that* (1.1) *is integrable and denote by* $R = \left(R_1,\ldots,R_n\right)$ *the associated complete Riemann invariants. A Hamiltonian perturbation of* (1.1) *of the form* $H = H_0 + \epsilon\, H_1 + \mathcal{O}(\epsilon^2)$ *with* $H_0 = \int_{S^1} h(v)\,\mathrm{d}x$, $H_1 = \int_{S^1}\sum_{i=1}^n p_i(R)\,R_{ix}\,\mathrm{d}x$ *is integrable at the first-order approximation iff either of the following is true:*

(i) *it is trivial;*
(ii) *the following equations hold true for* p_i*:*

$$\omega_{ij,k} - \omega_{ik,j} \;=\; a_{ij}\,\omega_{ik} \;+\; a_{ji}\,\omega_{jk} \;-\; a_{ik}\,\omega_{ij} \;-\; a_{ki}\,\omega_{kj}\,, \quad \forall\,\varepsilon(i,j,k) = \pm 1\,. \tag{1.8}$$

Here, a_{ij} *and* ω_{ij} *are defined by*

$$a_{ij} \;:=\; \frac{\lambda_{i,j}}{\lambda_i - \lambda_j}\,, \quad \omega_{ij} \;:=\; \frac{p_{i,j} - p_{j,i}}{\lambda_i-\lambda_j}\,, \qquad \forall\; i\neq j\,. \tag{1.9}$$

In the above statement, we recall that a Hamiltonian perturbation is called trivial if it is Miura equivalent to its dispersionless limit; for more details about triviality, see Sect. 2. Due to Theorem 1.1, to study the integrable Hamiltonian perturbation (1.5) of an integrable PDE of hydrodynamic type (1.1) up to the second-order approximation, it suffices to consider the case with vanishing H_1. Here, it should also be noted that the basic assumption proposed in the beginning of the paper has been assumed as it is written again in the statement.

Theorem 1.2. *Assume that the matrix* (A^α_β) *associated with (1.1) has distinct eigenvalues* $\lambda_1,\ldots,\lambda_n$ *on an open dense subset* $U\subset M$ *and that* $\lambda_{i,i}(v)\neq 0$ *for* $v\in U$. *Assume that (1.1) is integrable and denote by* $R = \left(R_1,\ldots,R_n\right)$ *the associated complete Riemann invariants. A Hamiltonian perturbation of (1.1) of the form*

$$H \;=\; H_0 \;+\; \epsilon^2\, H_2 \;+\; \mathcal{O}(\epsilon^3) \tag{1.10}$$

with $H_0 = \int_{S^1} h_0(u)\,\mathrm{d}x$, $H_2 = \int_{S^1} \sum_{i,j=1}^n d_{ij}(R)\, R_{ix} R_{j_x}\,\mathrm{d}x$ $(d_{ij} = d_{ji})$ *is* $\mathcal{O}(\epsilon^2)$*-integrable iff either of the followings is true:*

(i) *it is quasi-trivial;*

(ii) *there exist functions* $C_i(R_i)$, $i = 1, \ldots, n$ *such that*

$$d_{ii} \;=\; -\,C_i(R_i)\,\lambda_{i,i}\,, \tag{1.11}$$

$$\left(\frac{d_{ij}}{\lambda_i-\lambda_j}\right)_{,k} + \left(\frac{d_{jk}}{\lambda_j-\lambda_k}\right)_{,i} + \left(\frac{d_{ki}}{\lambda_k-\lambda_i}\right)_{,j} \;=\; 0\,, \quad \forall\, \varepsilon(i,j,k) = \pm 1\,. \tag{1.12}$$

For the meaning of quasi-triviality, see Sects. 2 and 3 . Note that an equivalent description of (1.11)–(1.12) is that the density h_2 can be written in the form

$$h_2 \;=\; -\sum_{i=1}^n C_i(R_i)\,\lambda_{i,i}\,R_{ix}^2 \;+\; \frac{1}{2}\sum_{i\neq j} (\lambda_i-\lambda_j)\, s_{ij}\, R_{ix}\, R_{j_x}\,, \tag{1.13}$$

where $s_{ij} := \phi_{i,j} - \phi_{j,i}$ for some functions $\phi_i(R)$.

For the cases $n = 1, 2$, Theorems 1.1 and 1.2 agree with the results of [20] and [9].

The paper is organized as follows. In Sect. 2, we review some terminologies about Hamiltonian PDEs. In Sect. 3, we study integrability of (1.5) up to the second-order approximation. An example of non-integrable perturbation is given in Sect. 4.

2. Preliminaries

In this section, we will recall several terminologies in the theory of Hamiltonian perturbations; more terminologies can be found in, e.g., [6–8,10,12,16,22,23].

Definition 2.1. A local functional $F_0 = \int_{S^1} f_0(v)\,\mathrm{d}x$ is called a *conserved quantity* of (1.1) if

$$\frac{dF_0}{dt} \;=\; 0\,. \tag{2.1}$$

Here, the density $f_0(v)$ is a given holomorphic function of v.

We also often call a conserved quantity a conservation law. Note that for simplicity we will exclude the degenerate ones with $f_0(v) \equiv \text{const}$ from conservation laws.

Since (1.1) is a Hamiltonian system, Eq. (2.1) can be written equivalently as

$$\big\{H_0, F_0\big\} \;=\; 0\,, \tag{2.2}$$

where $\{\,,\}$ denotes the Poisson bracket defined in (1.7). (This is straightforward to verify.) According to Noether's theorem, (2.1) is also equivalent to the statement that the following Hamiltonian flow generated by F_0

$$v^\alpha_s \;:=\; \{v^\alpha(x), F_0\}$$

commutes with (1.1). Let $(M_{\alpha\beta})$ denote the Hessian of f, i.e., $M_{\alpha\beta} := \partial_\alpha \partial_\beta(f)$. Equation (2.1) then reads

$$A^\alpha_\gamma\, M^\gamma_\beta \;=\; M^\alpha_\gamma\, A^\gamma_\beta. \tag{2.3}$$

Definition 2.2. The PDE system (1.1) is called integrable if it possesses an infinite family of conserved quantities parametrized by n arbitrary functions of one variable.

A necessary and sufficient condition for integrability of (1.1) is the vanishing of the Haantjes tensor $H_{\alpha\beta\gamma}$ (1.4) as recalled already in the introduction. We will assume that (1.1) is integrable and study its perturbations. Recall that vanishing of the Haantjes tensor ensures the existence of a complete set of Riemann invariants $\{R_1, \ldots, R_n\}$. We have

$$A^\alpha_\beta\, R_{i,\alpha} \;=\; \lambda_i\, R_{i,\beta}\,, \tag{2.4}$$

$$M^\alpha_\beta\, R_{i,\alpha} \;=\; \mu_i\, R_{i,\beta}\,. \tag{2.5}$$

Here, μ_i are eigenvalues of (M^α_β). For a generic conserved quantity F_0, the eigenvalues $\mu_1, \ldots, \mu_n$ on the U are also pairwise distinct. In terms of λ_i, μ_i, the flow commutativity is equivalent to

$$a_{ij} \;=\; b_{ij}\,, \qquad \forall\, i \neq j\,, \tag{2.6}$$

where

$$a_{ij} \;:=\; \frac{\lambda_{i,j}}{\lambda_i - \lambda_j}\,, \qquad b_{ij} \;:=\; \frac{\mu_{i,j}}{\mu_i - \mu_j}\,. \tag{2.7}$$

The compatibility condition

$$\mu_{i,jk} \;=\; \mu_{i,kj}\,, \qquad \forall\, \varepsilon(i,j,k) = \pm 1$$

for Eq. (2.6) reads as follows

$$(\mu_i - \mu_k)(a_{ij,k} - a_{ik,j}) - (\mu_j - \mu_k)(a_{ij,k} + a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik}) \;=\; 0\,. \tag{2.8}$$

Definition 2.2 requires that equation (2.8) is true for infinitely many F_0 parametrized by n arbitrary functions of one variable. So the coefficients of $\mu_i - \mu_k$ and of $\mu_j - \mu_k$ must vanish:

$$a_{ij,k} \,-\, a_{ik,j} \;=\; 0\,, \qquad \forall\, \varepsilon(i,j,k) = \pm 1\,, \tag{2.9}$$

$$a_{ij,k} \,+\, a_{ij}a_{jk} \,+\, a_{ik}a_{kj} \,-\, a_{ij}a_{ik} \;=\; 0\,, \qquad \forall\, \varepsilon(i,j,k) = \pm 1\,. \tag{2.10}$$

Note that (2.10) is implied by Eqs. (2.9) and (2.7).

Definition 2.3. A local functional $F := \sum_{j=0}^\infty \epsilon^j F_j$ is called a conserved quantity of (1.5), if

$$\frac{dF}{dt} \;=\; 0\,. \tag{2.11}$$

Here, $F_j = \int_{S^1} f_j(v, v_1, \ldots, v_j)\, \mathrm{d}x$, $j \geq 0$ with f_j being differential polynomials of v homogeneous of degree j.

Conserved quantities (or say conservation laws) considered in this paper are always of the form as in Definition 2.3.

Equation (2.11) can be equivalently written as

$$\{H,F\} \;=\; 0\,,$$

which is recast into an infinite sequence of equations

$$\begin{aligned}&\{H_0,F_0\} \;=\; 0\,,\\ &\{H_0,F_1\} \;+\; \{H_1,F_0\} \;=\; 0\,,\\ &\{H_0,F_2\} \;+\; \{H_1,F_1\} \;+\; \{H_2,F_0\} \;=\; 0\,,\\ &\text{etc.}\end{aligned}$$

Definition 2.4. A Hamiltonian perturbation (1.5) is called integrable if its dispersionless limit (1.1) is integrable and generic conservation laws of (1.1) can be extended to those of (1.5). For $N \ge 1$, (1.5) is called $\mathcal{O}(\epsilon^N)$-integrable if its dispersionless limit (1.1) is integrable and every generic conservation law F_0 of (1.1) can be extended to a local functional F, s.t.

$$\{H,F\} \;=\; \mathcal{O}(\epsilon^{N+1}). \tag{2.12}$$

One important tool of studying Hamiltonian perturbations is to use Miura-type and quasi-Miura transformations [16]. Recall that a Miura-type transformation near identity is given by an invertible map of the form

$$v \mapsto w\,, \qquad w^\alpha \;:=\; \sum_{j=0}^{\infty} \epsilon^j\, W_j^\alpha(v, v_1, \dots, v_\ell)\,,\; W_0^\alpha = v^\alpha\,, \tag{2.13}$$

where W_j^α, $j \ge 0$ are differential polynomials of v homogeneous of degree j with respect to the degree assignments $\deg v_\ell^\alpha \;=\; \ell$, $\ell \ge 1$. A Miura-type transformation is called *canonical* if there exists a local functional K, such that

$$w^\alpha \;=\; v^\alpha \;+\; \epsilon\,\bigl\{v^\alpha(x), K\bigr\} \;+\; \frac{\epsilon^2}{2!}\,\bigl\{\bigl\{v^\alpha(x), K\bigr\}, K\bigr\} \;+\; \cdots \tag{2.14}$$

where $K = \sum_{j=0}^{\infty} \epsilon^j K_j$. Two Hamiltonian perturbations of the same form (1.5) are called *equivalent* if they are related via a canonical Miura-type transformation. A Hamiltonian perturbation (1.5) is called *trivial* if it is equivalent to (1.1).

A map of the form (2.13) is called a *quasi-Miura* transformation, if W_ℓ^α, $\ell \ge 1$ are allowed to have rational and logarithmic dependence in v_x. The Hamiltonian perturbation (1.5) is called *quasi-trivial* or possessing *quasi-triviality*, if it is related via a canonical quasi-Miura transformation to (1.1). We recall that many interesting nonlinear PDE systems possess quasi-triviality; for example, it was shown in [12] that if (1.5) is *bihamiltonian* then it is quasi-trivial. The precise definition used in this paper for quasi-Miura transformation will be given in the next section.

3. Proofs of Theorems 1.1 and 1.2

In this section, we study integrability of the Hamiltonian system (1.5) up to the second-order approximation, and prove Theorems 1.1 and 1.2 .

Assume that (1.1) is integrable.

We start with the first-order approximation. Let us first look at the integrability condition of the $\mathcal{O}(\epsilon^1)$-approximation. Denote

$$H \;=\; H_0 \;+\; \epsilon\, H_1 \;+\; \mathcal{O}(\epsilon^2) \tag{3.1}$$

with $H_1 = \int_{S^1} \tilde{p}_\alpha(u)\, u_x^\alpha \,\mathrm{d}x = \sum_{i=1}^n \int_{S^1} p_i(R)\, R_{ix}\,\mathrm{d}x$. Here, the functions p_α and p_i are assumed to satisfy $\tilde{p}_\alpha = \sum_{i=1}^n p_i R_{i,\alpha}$.

Proof of Theorem 1.1. Denote by $\tilde{\theta}_{\alpha\beta}$ the exterior differential of the 1-form $\tilde{p}_\alpha du^\alpha$

$$\tilde{\theta}_{\alpha\beta} \;=\; \tilde{p}_{\alpha,\beta} \;-\; \tilde{p}_{\beta,\alpha}\,. \tag{3.2}$$

In the coordinate chart of the Riemann invariants $R_1,\ldots,R_n$, we have

$$\theta_{ij} \;=\; \partial_i u^\alpha\, \tilde{\theta}_{\alpha\beta}\, \partial_j u^\beta \;=\; p_{i,j} \;-\; p_{j,i}\,.$$

The $\mathcal{O}(\epsilon^1)$-integrability says any local functional $F_0 = \int_{S^1} f(u)\,\mathrm{d}x$ satisfying

$$\{H_0, F_0\} \;=\; 0$$

can be extended to a local functional

$$F \;=\; F_0 \;+\; \epsilon\, F_1 \;+\; \mathcal{O}(\epsilon^2)\,,$$

such that

$$\{H, F\} \;=\; \mathcal{O}(\epsilon^2). \tag{3.3}$$

Here, the local function F_1 is of the form

$$F_1 \;=\; \int_{S^1} \tilde{q}_\alpha(u)\, u_x^\alpha\,\mathrm{d}x \;=\; \sum_{i=1}^n \int_{S^1} q_i(R)\, R_{ix}\,\mathrm{d}x\,. \tag{3.4}$$

Eq. (3.3) reads as follows

$$\{H_0, F_1\} \;+\; \{H_1, F_0\} \;=\; 0\,,$$

which is equivalent to

$$\tilde{\theta}_{\alpha\gamma} M^\gamma_\beta \;+\; \tilde{\theta}_{\beta\gamma} M^\gamma_\alpha \;=\; \tilde{\Theta}_{\alpha\gamma} A^\gamma_\beta \;+\; \tilde{\Theta}_{\beta\gamma} A^\gamma_\alpha \tag{3.5}$$

or, in the coordinate system of the Riemann invariants, to

$$\frac{\theta_{ij}}{\lambda_i - \lambda_j} \;=\; \frac{\Theta_{ij}}{\mu_i - \mu_j}\,, \qquad \forall\, i \neq j. \tag{3.6}$$

Here, $\tilde{\Theta}_{\alpha\beta} := \tilde{q}_{\alpha,\beta} - \tilde{q}_{\beta,\alpha}$, $\Theta_{ij} := q_{i,j} - q_{j,i}$. The compatibility condition of (3.6) is given by

$$\Theta_{ij,k} \;+\; \Theta_{jk,i} \;+\; \Theta_{ki,j} \;=\; 0\,, \quad \forall\, \varepsilon(i,j,k) = \pm 1. \tag{3.7}$$

Introduce the notations

$$\omega_{ij} \;=\; \frac{\theta_{ij}}{\lambda_i - \lambda_j}\,, \qquad i \neq j\,. \tag{3.8}$$

Then, Eq. (3.7) imply

$$\partial_k \left[\omega_{ij} \left(\mu_i - \mu_j \right) \right] + \partial_i \left[\omega_{jk} \left(\mu_j - \mu_k \right) \right] + \partial_j \left[\omega_{ki} \left(\mu_k - \mu_i \right) \right] \;=\; 0\,, \quad \forall\, \varepsilon(i,j,k) = \pm 1\,,$$

i.e.,

$$\omega_{ij,k} \left(\mu_i - \mu_j \right) \;+\; \omega_{ij} \left(\mu_{i,k} - \mu_{j,k} \right) \;+\; \text{cyclic} \;=\; 0\,, \qquad \forall\, \varepsilon(i,j,k) = \pm 1\,. \tag{3.9}$$

Substituting Eqs. (2.6), (2.7) in Eq. (3.9), we obtain

$$\omega_{ij,k} \left(\mu_i - \mu_j \right) \;+\; \omega_{ij} \left(a_{ik}(\mu_i - \mu_k) - a_{jk}(\mu_j - \mu_k) \right) \;+\; \text{cyclic} \;=\; 0\,, \tag{3.10}$$

from which we obtain that for any pairwise distinct i, j, k,

$$\begin{aligned} &(\mu_i - \mu_k)\big(\omega_{ij,k} + \omega_{ij}\, a_{ik} - \omega_{jk}\, a_{ji} + \omega_{jk}\, a_{ki} - \omega_{ki,j} - \omega_{ki}\, a_{ij}\big) \\ &\quad +\; (\mu_j - \mu_k)\big(-\omega_{ij,k} - \omega_{ij}\, a_{jk} + \omega_{jk,i} + \omega_{jk}\, a_{ji} - \omega_{ki}\, a_{kj} + \omega_{ki}\, a_{ij}\big) \;=\; 0\,. \end{aligned} \tag{3.11}$$

As a result, we conclude that

$$\omega_{ij,k} \;+\; \omega_{ij}\, a_{ik} \;-\; \omega_{jk}\, a_{ji} \;+\; \omega_{jk}\, a_{ki} - \omega_{ki,j} \;-\; \omega_{ki}\, a_{ij} \;=\; 0\,, \quad \forall\, \varepsilon(i,j,k) = \pm 1\,, \tag{3.12}$$

$$-\,\omega_{ij,k} \;-\; \omega_{ij}\, a_{jk} \;+\; \omega_{jk,i} \;+\; \omega_{jk}\, a_{ji} - \omega_{ki}\, a_{kj} \;+\; \omega_{ki}\, a_{ij} \;=\; 0\,, \quad \forall\, \varepsilon(i,j,k) = \pm 1\,. \tag{3.13}$$

This arguments above can be reversed to get (3.7). We therefore conclude that integrability at the first order of approximation is equivalent to (1.8). □

Let us now consider the condition of (quasi-)triviality at the first order of approximation. The Hamiltonian perturbation (3.1) is *quasi-trivial* at the first-order approximation, if there exists a local functional

$$K_0 \;=\; \int_{S^1} k_0(v)\, \mathrm{d}x$$

such that

$$\{H_0, K_0\} \;=\; H_1\,. \tag{3.14}$$

Clearly, quasi-triviality at the first-order approximation is the same as triviality at the first-order approximation. Equation (3.14) is equivalent to the existence of a function ψ satisfying

$$\tilde{p}_\alpha \;=\; \frac{\partial k_0}{\partial u^\gamma} A^\gamma_\alpha \;+\; \frac{\partial \psi}{\partial u^\alpha}\,. \tag{3.15}$$

Eliminating ψ in the above equation we find the following equivalent equation to (3.14):

$$\tilde{\theta}_{\alpha\beta} \;=\; \frac{\partial^2 k_0}{\partial u^\beta \partial u^\gamma} A^\gamma_\alpha \;-\; \frac{\partial^2 k_0}{\partial u^\alpha \partial u^\gamma} A^\gamma_\beta\,. \tag{3.16}$$

In the coordinate chart of Riemann invariants, Eqs. (3.15) and (3.16) become

$$p_i \;=\; \lambda_i\, k_{0,i} + \psi_{,i}\,, \tag{3.17}$$

$$\frac{\theta_{ij}}{\lambda_i - \lambda_j} \;=\; k_{0,ij} \;+\; a_{ij}\, k_{0,i} \;+\; a_{ji}\, k_{0,j}\,, \qquad i \neq j\,. \tag{3.18}$$

The compatibility condition of Eq. (3.18) is given by

$$\partial_k k_{0,ij} \;=\; \partial_j k_{0,ik}\,, \qquad \forall\, \varepsilon(i,j,k) = \pm 1,$$

which yields

$$\partial_k\left(\frac{\theta_{ij}}{\lambda_i-\lambda_j} - a_{ij}\,k_{0,i} - a_{ji}\,k_{0,j}\right) \;=\; \partial_j\left(\frac{\theta_{ik}}{\lambda_i-\lambda_k} - a_{ik}\,k_{0,i} - a_{ki}\,k_{0,k}\right). \quad (3.19)$$

Substituting Eq. (3.18) into (3.19), we find

$$\begin{aligned}
&\omega_{ij,k} \;-\; a_{ij}\,\omega_{ik} \;-\; a_{ji}\,\omega_{jk} \;-\; k_{0,i}\,a_{ij,k} \;+\; k_{0,j}(a_{ji}a_{jk} - a_{ji,k})\\
&\quad +\; k_{0,k}(a_{ki}a_{ij} + a_{kj}a_{ji})\\
&= \;\omega_{ik,j} \;-\; a_{ik}\,\omega_{ij} \;-\; a_{ki}\,\omega_{kj} \;-\; k_{0,i}\,a_{ik,j} \;+\; k_{0,k}(a_{ki}a_{kj} - a_{ki,j})\\
&\quad +\; k_{0,j}(a_{ji}a_{ik} + a_{jk}a_{ki})\,.
\end{aligned} \quad (3.20)$$

Finally substituting Eqs. (2.9) and (2.10) into (3.20), we have

$$\omega_{ij,k} \;-\; a_{ij}\,\omega_{ik} \;-\; a_{ji}\,\omega_{jk} \;=\; \omega_{ik,j} \;-\; a_{ik}\,\omega_{ij} \;-\; a_{ki}\,\omega_{kj}\,, \quad \forall\,\varepsilon(i,j,k) = \pm 1\,. \quad (3.21)$$

The procedure can again be reversed. So we proved the equivalence between (1.8) and triviality at the first-order approximation. The theorem is proved. □

We proceed with the second-order approximation. Let

$$H \;=\; H_0 \;+\; \epsilon\, H_1 \;+\; \epsilon^2\, H_2 \;+\; \mathcal{O}(\epsilon^3) \quad (3.22)$$

be a Hamiltonian perturbation of (1.1) with

$$H_2 \;=\; \int_{S^1} \tilde{d}_{\alpha\beta}(v)\, v^\alpha_x\, v^\beta_x \,\mathrm{d}x \;=\; \int_{S^1} \sum_{i,j=1}^{n} d_{ij} R_{i\,x} R_{j\,x}\, dx \quad (3.23)$$

and

$$\tilde{d}_{\alpha\beta} \;=\; \tilde{d}_{\beta\alpha}\,, \qquad d_{ij} \;=\; d_{ji} \;:=\; \tilde{d}_{\alpha\beta}\, v^\alpha_{,i}\, v^\beta_{,j}\,. \quad (3.24)$$

Assume as always that (1.1) is integrable, and assume that (3.22) is $\mathcal{O}(\epsilon^1)$-integrable. According to Theorem 1.1, there exists a canonical Miura-type transformation reducing H_1 to the zero functional. So the assumption that $H_1 = 0$ used in (1.10) in the statement of Theorem 1.1 does not lose generality as we already pointed it out in the Introduction.

Proof of Theorem 1.2. The proof will be given with the following order: firstly, we show that $\mathcal{O}(\epsilon^2)$-integrability implies (1.11)–(1.12); secondly, we show that (1.11)–(1.12) is equivalent to quasi-triviality at the second-order approximation; thirdly, we show that quasi-triviality implies $\mathcal{O}(\epsilon^2)$-integrability.

Assume that (3.22) with $H_1 = 0$ is $\mathcal{O}(\epsilon^2)$-integrable. This means that, for a generic conservation law F_0 of (1.1), there exists a local functional of the form

$$F_2 \;=\; \int_{S^1} \tilde{D}_{\alpha\beta}(u)\, u^\alpha_x\, u^\beta_x \,\mathrm{d}x \;=\; \sum_{i,j=1}^{n} \int_{S^1} D_{ij}(R) R_{i\,x} R_{j\,x}\,\mathrm{d}x \quad (3.25)$$

such that

$$\{H_0, F_2\} \;+\; \{H_2, F_0\} \;=\; 0\,. \quad (3.26)$$

Note that equation (3.26) implies

$$M^\rho_\sigma\,\tilde d_{\rho\beta} \;-\; M^\rho_\beta\,\tilde d_{\rho\sigma} \;=\; A^\rho_\sigma\,\tilde D_{\rho\beta} \;-\; A^\rho_\beta\,\tilde D_{\rho\sigma}\,, \tag{3.27}$$

$$\begin{aligned} &M^\rho_\gamma\,\tilde d_{\rho\sigma,\beta} \;+\; M^\rho_\sigma\,\tilde d_{\rho\beta,\gamma} \;+\; M^\rho_\beta\,\tilde d_{\rho\gamma,\sigma} \;-\; M^\rho_{\sigma\gamma}\,\tilde d_{\rho\beta} \;-\; M^\rho_{\sigma\beta}\,\tilde d_{\rho_\gamma} \;-\; M^\rho_{\beta\gamma}\,\tilde d_{\rho\sigma} \\ &\quad -\; M^\rho_\sigma\,\tilde d_{\beta\gamma,\rho} \;-\; M^\rho_\beta\,\tilde d_{\sigma\gamma,\rho} \;-\; M^\rho_\gamma\,\tilde d_{\sigma\beta,\rho} \\ &\quad =\; (M \leftrightarrow A, d \leftrightarrow D)\,. \end{aligned} \tag{3.28}$$

In the coordinate system of the complete Riemann invariants, (3.27) and (3.28) become

$$\frac{D_{ij}}{\mu_i-\mu_j} \;=\; \frac{d_{ij}}{\lambda_i-\lambda_j}\,, \quad \forall\; i\neq j, \tag{3.29}$$

$$\begin{aligned} &\lambda_{i,l}D_{ij} \;+\; \lambda_{j,i}D_{jl} \;+\; \lambda_{i,j}D_{il} \;+\; (\lambda_i-\lambda_l)D_{lj,i} \\ &\quad +\; (\lambda_j-\lambda_l)D_{li,j} \;+\; (\lambda_l-\lambda_j)D_{ij,l} \\ &\quad =\; \mu_{i,l}\,d_{ij} \;+\; \mu_{j,i}\,d_{jl} \;+\; \mu_{i,j}\,d_{il} \;+\; (\mu_i-\mu_l)\,d_{lj,i} \;+\; (\mu_j-\mu_l)\,d_{li,j} \\ &\quad +\; (\mu_l-\mu_j)\,d_{ij,l}\,, \qquad \forall\; i,j,l\,. \end{aligned} \tag{3.30}$$

Here, in the derivation of (3.30), we have used (3.29).

Taking $j=l=i$ in (3.30), we obtain

$$\lambda_{i,i}\,D_{ii} \;=\; \mu_{i,i}\,d_{ii}\,. \tag{3.31}$$

By assumption, in the subset U of M, λ_i satisfy $\lambda_{i,i}\neq 0$. Thus, there exist functions $C_i(R)$ such that

$$D_{ii} \;=\; -\,C_i(R)\,\mu_{i,i}\,, \qquad d_{ii} \;=\; -\,C_i(R)\,\lambda_{i,i}\,. \tag{3.32}$$

Taking $l=j$ and $i\neq j$ in (3.30), we find

$$\lambda_{j,i}D_{jj} \;+\; (\lambda_i-\lambda_j)D_{jj,i} \;=\; \mu_{j,i}\,d_{jj} \;+\; (\mu_j-\mu_i)\,d_{jj,i}\,, \qquad \forall\; j\neq i. \tag{3.33}$$

Substituting (3.32) into (3.33) and using (2.9) we obtain

$$C_{j,i}\,\big((\lambda_i-\lambda_j)\mu_{j,j}-(\mu_i-\mu_j)\lambda_{j,j}\big) \;=\; 0\,, \qquad \forall\, j\neq i\,, \tag{3.34}$$

which implies

$$C_{j,i} \;=\; 0\,, \qquad \forall\, j\neq i\,,$$

i.e.,

$$C_j(R) \;=\; C_j(R_j)\,.$$

Taking $l=i$ and $j\neq i$ in (3.30) and using (3.31),(3.33), we find

$$\lambda_{i,i}D_{ij} \;+\; (\lambda_i-\lambda_j)D_{ij,i} \;=\; \mu_{i,i}\,d_{ij} \;+\; (\mu_i-\mu_j)\,d_{ij,i}\,. \tag{3.35}$$

Taking $j=i$ and $l\neq i$ in (3.30) and using (3.33), we find

$$\lambda_{i,i}D_{li} \;+\; (\lambda_i-\lambda_l)D_{li,i} \;=\; \mu_{i,i}\,d_{li} \;+\; (\mu_i-\mu_l)\,d_{li,i}\,, \tag{3.36}$$

which coincides with (3.35). It is straightforward to check that (3.29) and (2.9) imply (3.35). So (3.35) does not give new constraints to d_{ij}, $i\neq j$.

Now we use (3.30) with $\varepsilon(i,j,l)=\pm 1$. First, by (3.29) it is convenient to write

$$D_{ij} \;=\; s_{ij}(\mu_i-\mu_j)\,, \quad d_{ij} \;=\; s_{ij}(\lambda_i-\lambda_j)\,, \qquad i\neq j\,, \tag{3.37}$$

where s_{ij} are some anti-symmetric fields. Substituting (3.37) in (3.30) and using (2.9), we obtain

$$(s_{lj,i}+s_{ji,l}+s_{il,j})\,((\lambda_i-\lambda_l)(\mu_j-\mu_l)-(\lambda_j-\lambda_l)(\mu_i-\mu_l)) \;=\; 0\,, \quad \forall\,\varepsilon(i,j,l)=\pm 1\,. \tag{3.38}$$

Hence,

$$s_{lj,i} \;+\; s_{ji,l} \;+\; s_{il,j} \;=\; 0\,, \qquad \forall\,\varepsilon(i,j,l)=\pm 1\,. \tag{3.39}$$

This proves (1.11)–(1.12).

We now consider the condition of quasi-triviality for (3.22) with $H_1=0$. Such a perturbation is called *quasi-trivial* if there exists a local functional K of the form

$$K \;=\; \epsilon\, K_1 \;+\; \mathcal{O}(\epsilon^2), \quad K_1=\int_{S^1} k_1(u;u_x)\,\mathrm{d}x, \tag{3.40}$$

such that

$$H_0 \;+\; \epsilon\,\{H_0,K\} \;=\; H\,. \tag{3.41}$$

Here, k_1 is also required to satisfy the following homogeneity condition:

$$\sum_{r\geq 1} r\,u^\alpha_r\,\frac{\partial}{\partial u^\alpha_r}\left(\frac{\partial k_1}{\partial u^\beta}-\partial_x\left(\frac{\partial k_1}{\partial u^\beta_x}\right)\right) \;=\; \frac{\partial k_1}{\partial u^\beta}-\partial_x\left(\frac{\partial k_1}{\partial u^\beta_x}\right). \tag{3.42}$$

(The above (3.40)–(3.42) is the precise definition used in this paper for quasi-triviality at the second-order approximation.)

Equation (3.42) is equivalent to the following linear PDE system:

$$u^\alpha_x\,k_{1,u^\alpha_x u^\beta_x u^\gamma_x} \;+\; k_{1,u^\beta_x u^\gamma_x} \;=\; 0\,, \tag{3.43}$$

$$u^\alpha_x\,k_{1,u^\alpha_x u^\beta} \;-\; u^\alpha_x u^\gamma_x\,k_{1,u^\gamma u^\alpha_x u^\beta_x} \;-\; k_{1,u^\beta} \;=\; 0\,. \tag{3.44}$$

From Eq. (3.41), we obtain

$$\{H_0,K_1\} \;=\; H_2\,,$$

which is equivalent to

$$\frac{\delta}{\delta u^\rho(x)}\left(H_2+\int_{S^1}\frac{\delta K_1}{\delta u^\alpha(x)}A^\alpha_\gamma u^\gamma_x\,\mathrm{d}x\right) \;=\; 0\,. \tag{3.45}$$

Eq. (3.45) read more explicitly as follows:

$$\sum_{j=0}^{2}(-1)^j\,\partial_x^j\,\frac{\partial}{\partial u^\rho_j}\left[\tilde{d}_{\alpha\beta}\,u^\alpha_x\,u^\beta_x+A^\alpha_\gamma\,u^\gamma_x\left(\frac{\partial k_1}{\partial u^\alpha}-\partial_x\Big(\frac{\partial k_1}{\partial u^\alpha_x}\Big)\right)\right] \;=\; 0\,. \tag{3.46}$$

Comparing the coefficients of u^σ_{xxx} of both sides of Eq. (3.46) gives

$$A^\alpha_\rho\,k_{1,u^\alpha_x u^\sigma_x} \;=\; A^\alpha_\sigma\,k_{1,u^\alpha_x u^\rho_x}\,. \tag{3.47}$$

In terms of the Riemann invariants, Eq. (3.47) read

$$\sum_{i\neq j} k_{1,R_{i\,x}R_{j\,x}}\,R_{i,\sigma}\,R_{j,\rho}\,(\lambda_j-\lambda_i) \;=\; 0\,,$$

which imply

$$k_{1,R_{i\,x}R_{j\,x}} \;=\; 0\,, \qquad \forall\; i\neq j\,. \tag{3.48}$$

Lemma 3.1. *Up to a total x-derivative, k_1 must have the form*

$$k_1 = \sum_{i=1}^{n} C_i(R_1,\ldots,R_n)R_{ix}\log R_{ix} - C_i(R_1,...,R_n)R_{ix} + \phi_i(R_1,\ldots,R_n)R_{ix} \tag{3.49}$$

for some C_i, ϕ_i. Moreover, if k_1 has the form (3.49) then it satisfies (3.43), (3.44), (3.47).

Proof. Eq. (3.48) imply that k_1 must have the variable separation form

$$k_1 = \sum_{i=1}^{n} B_i(R_1,\ldots,R_n;R_{ix})\,. \tag{3.50}$$

Noting that

$$\begin{aligned} k_{1,u_x^\alpha} &= \sum_{i=1}^{n} k_{1,R_{ix}} R_{i,\alpha}\,, \\ k_{1,u_x^\alpha u_x^\beta} &= \sum_{i,j=1}^{n} k_{1,R_{ix}R_{jx}} R_{i,\alpha}R_{j,\beta}\,, \\ k_{1,u_x^\alpha u_x^\beta u_x^\gamma} &= \sum_{i,j,k=1}^{n} k_{1,R_{ix}R_{jx}R_{kx}} R_{i,\alpha}R_{j,\beta}R_{k,\gamma} \end{aligned}$$

and substituting Eq. (3.50) into Eq. (3.43), we obtain

$$R_{ix}\,B_{i,R_{ix}R_{ix}R_{ix}} + 2B_{i,R_{ix}R_{ix}} = 0\,.$$

If follows that

$$B_i = E_i(R) + \phi_i(R)R_{ix} + C_i(R)R_{ix}\log R_{ix} - C_i(R)R_{ix} \tag{3.51}$$

for some functions C_i, ϕ_i, E_i. Finally, noticing that

$$\begin{aligned} k_{1,u^\beta} &= \sum_{i=1}^{n}\left(k_{1,R_i}R_{i,\beta} + k_{1,R_{ix}}R_{i,\beta\sigma}u_x^\sigma\right), \\ k_{1,u_x^\alpha u^\beta} &= \sum_{i,j=1}^{n}\left(k_{1,R_{ix}R_j}R_{j,\beta} + k_{1,R_{ix}R_{jx}}R_{j,\beta\sigma}u_x^\sigma\right)R_{i,\alpha} + \sum_{i=1}^{n} k_{1,R_{ix}}R_{i,\alpha\beta}\,, \\ k_{1,u_x^\alpha u_x^\beta u^\gamma} &= \sum_{i,j,k=1}^{n}\left(k_{1,R_{ix}R_{jx}R_k}R_{k,\gamma} + k_{1,R_{ix}R_{jx}R_{kx}}R_{k,\gamma\sigma}u_x^\sigma\right)R_{i,\alpha}R_{j,\beta} \\ &\quad + \sum_{i,j=1}^{n} k_{1,R_{ix}R_{jx}}\left(R_{i,\alpha\gamma}R_{j,\beta} + R_{i,\alpha}R_{j,\beta\gamma}\right), \end{aligned}$$

and substituting (3.50), (3.51) into (3.44), we obtain

$$\partial_\beta\left(\sum_{i=1}^{n} E_i(R)\right) = 0\,, \tag{3.52}$$

which finishes the proof. □

Now collect the terms of (3.46) containing $u^{\beta}_{xx}u^{\sigma}_{xx}$:

$$u^{\beta}_{xx}\,u^{\sigma}_{xx}\left(A^{\alpha}_{\rho}\frac{\partial^3 k_1}{\partial u^{\alpha}_x\partial u^{\beta}_x\partial u^{\sigma}_x}+A^{\alpha}_{\beta}\frac{\partial^3 k_1}{\partial u^{\alpha}_x\partial u^{\rho}_x\partial u^{\sigma}_x}-2A^{\alpha}_{\sigma}\frac{\partial^3 k_1}{\partial u^{\alpha}_x\partial u^{\beta}_x\partial u^{\rho}_x}\right)\;=\;0\,. \tag{3.53}$$

Lemma 3.2. *If k_1 satisfies (3.48), then it automatically satisfies (3.53).*

Proof. We have

$$\begin{aligned}
&\text{LHS of (3.53)}\\
&= u^{\beta}_{xx}\,u^{\sigma}_{xx}\sum_{i,j,l=1}^{n}k_{1,R_{i\,x}R_{j\,x}R_{l\,x}}\,R_{l,\alpha}\left(A^{\alpha}_{\rho}R_{i,\beta}R_{j,\sigma}+A^{\alpha}_{\beta}R_{i,\rho}R_{j,\sigma}-2A^{\alpha}_{\sigma}R_{i,\beta}R_{j,\rho}\right)\\
&= u^{\beta}_{xx}\,u^{\sigma}_{xx}\sum_{i=1}^{n}k_{1,R_{i\,x}R_{i\,x}R_{i\,x}}\,R_{i,\alpha}\left(A^{\alpha}_{\rho}R_{i,\beta}R_{i,\sigma}+A^{\alpha}_{\beta}R_{i,\rho}R_{i,\sigma}-2A^{\alpha}_{\sigma}R_{i,\beta}R_{i,\rho}\right)\\
&= u^{\beta}_{xx}\,u^{\sigma}_{xx}\sum_{i=1}^{n}k_{1,R_{i\,x}R_{i\,x}R_{i\,x}}\,\lambda_i\left(R_{i,\rho}R_{i,\beta}R_{i,\sigma}+R_{i,\beta}R_{i,\rho}R_{i,\sigma}-2R_{i,\sigma}R_{i,\beta}R_{i,\rho}\right)\\
&= 0\,.
\end{aligned}$$

The lemma is proved. □

Comparing the coefficients of u^{β}_{xx} of the both sides of (3.46) yields

$$\begin{aligned}
&2\,A^{\alpha}_{\rho}\,k_{1,u^{\alpha}_x u^{\beta}_x u^{\gamma}}u^{\gamma}_x\;-\;A^{\alpha}_{\beta}\,k_{1,u^{\alpha}_x u^{\rho}_x u^{\gamma}}u^{\gamma}_x\;-\;3\,A^{\alpha}_{\beta\gamma}k_{1,u^{\alpha}_x u^{\rho}_x}u^{\gamma}_x\;-\;A^{\alpha}_{\gamma\epsilon}\,k_{1,u^{\alpha}_x u^{\rho}_x u^{\beta}_x}u^{\epsilon}_x u^{\gamma}_x\\
&\quad+\;A^{\alpha}_{\beta}\left(k_{1,u^{\alpha}_x u^{\rho}}-k_{1,u^{\alpha}u^{\rho}_x}\right)\;+\;A^{\alpha}_{\rho}\left(k_{1,u^{\alpha}_x u^{\beta}}-k_{1,u^{\alpha}u^{\beta}_x}\right)\;-\;2\,\tilde{d}_{\rho\beta}\;=\;0\,.
\end{aligned} \tag{3.54}$$

Substituting (3.49) into (3.54), we obtain the following lemma.

Lemma 3.3. *The functions C_i must satisfy*

$$C_{i,j}\;=\;0\,,\quad\forall\;i\neq j. \tag{3.55}$$

Proof. Noting that

$$\begin{aligned}
k_{1,R_{i\,x}} &= C_i\log R_{ix}\;+\;\phi_i\,,\\
k_{1,R_{i\,x}R_j} &= C_{i,j}\log R_{ix}\;+\;\phi_{i,j}\,,\\
k_{1,R_{i\,x}R_{j\,x}} &= C_i\,\delta_{ij}\,R_{ix}^{-1}\,,
\end{aligned}$$

we find that the only possible terms containing $\log R_{ix}$ in Eq. (3.54) are

$$A^{\alpha}_{\rho}\left(k_{1,u^{\alpha}_x u^{\beta}}-k_{1,u^{\alpha}u^{\beta}_x}\right),\qquad A^{\alpha}_{\beta}\left(k_{1,u^{\alpha}_x u^{\rho}}-k_{1,u^{\alpha}u^{\rho}_x}\right).$$

If follows that $\sum_{i,j=1}^{n}C_{i,j}(\lambda_i-\lambda_j)\big(R_{i,\beta}R_{j,\rho}+R_{i,\rho}R_{j,\beta}\big)\,\log R_{ix}\;=\;0$, which yields

$$\sum_{j\neq i}C_{i,j}(\lambda_i-\lambda_j)\big(R_{i,\beta}R_{j,\rho}+R_{i,\rho}R_{j,\beta}\big)\;=\;0\,.$$

This gives (3.55). The lemma is proved. □

Lemma 3.4. *The $\tilde{d}_{\alpha\beta}$ must have the form*

$$\tilde{d}_{\alpha\beta} = -\frac{1}{2}\sum_{i=1}^{n} C_i(R_i)\left(\lambda_{i,\alpha} R_{i,\beta} + \lambda_{i,\beta} R_{i,\alpha}\right) + \frac{1}{2}\sum_{i\neq j} s_{ij}\left(\lambda_i - \lambda_j\right) R_{i,\alpha} R_{j,\beta}, \tag{3.56}$$

where $s_{ij} = \phi_{i,j} - \phi_{j,i}$ for some functions ϕ_i.

Proof. Using Eq. (3.54), we obtain

$$2\,\tilde{d}_{\alpha\beta}\, u_x^\alpha\, u_x^\beta = -2\sum_{i=1}^{n} C_i(R_i)\,\lambda_{ix}\, R_{ix} + \sum_{i,j=1}^{n} s_{ij}\left(\lambda_i - \lambda_j\right) R_{ix}\, R_{jx}. \tag{3.57}$$

The lemma is proved. □

Let us further show that the expression (3.56) is equivalent to the expression (1.13) (therefore is also equivalent to (1.11)–(1.12)). Indeed, in the coordinate chart of the complete Riemann invariants, (3.56) becomes

$$d_{ij} = -\frac{1}{2}\Big(C_i(R_i)\lambda_{i,j} + C_j(R_j)\lambda_{j,i}\Big) + \frac{1}{2}\sum_{i\neq j}^{n} s_{ij}\left(\lambda_i - \lambda_j\right), \tag{3.58}$$

where $s_{ij} = \phi_{i,j} - \phi_{j,i}$ for some functions ϕ_i. It then suffices to show that $-\frac{1}{2}\big(C_i(R_i)\lambda_{i,j} + C_j(R_j)\lambda_{j,i}\big)$, $\forall i \neq j$ can be absorbed into the term $\frac{1}{2}\sum_{i\neq j}^{n} s_{ij}\left(\lambda_i - \lambda_j\right)$. This is true because

$$\partial_k\left(\frac{C_i(R_i)\lambda_{i,j} + C_j(R_j)\lambda_{j,i}}{\lambda_i - \lambda_j}\right) + \partial_i\left(\frac{C_j(R_j)\lambda_{j,k} + C_k(R_k)\lambda_{k,j}}{\lambda_j - \lambda_k}\right) + \partial_j\left(\frac{C_k(R_k)\lambda_{k,i} + C_i(R_i)\lambda_{i,k}}{\lambda_k - \lambda_i}\right) = 0, \quad \forall\, \varepsilon(i,j,k) = \pm 1. \tag{3.59}$$

Finally, let us check that equalities (3.46) hold true if $\tilde{d}_{\alpha\beta}$ and k_1 are given by (3.56) and (3.49). Collecting the rest terms of both sides of (3.46), we find that it suffices to show

$$\begin{aligned} &-\left(\tilde{d}_{\alpha\beta,\rho} u_x^\beta u_x^\alpha - 2\,\tilde{d}_{\rho\beta,\gamma} u_x^\gamma u_x^\beta\right) \\ &\quad = A^\alpha_\gamma u_x^\gamma\left(k_{1,u^\alpha u^\rho} - u_x^\sigma\, k_{1,u^\sigma u^\alpha u_x^\rho}\right) - A^\alpha_\rho u_x^\gamma\left(k_{1,u^\gamma u^\alpha} - u_x^\sigma\, k_{1,u^\sigma u^\gamma u_x^\alpha}\right) \\ &\quad\quad - A^\alpha_{\gamma\beta\epsilon}\, u_x^\epsilon u_x^\beta u_x^\gamma\, k_{1,u_x^\alpha u_x^\rho} + A^\alpha_{\gamma\sigma} u_x^\sigma u_x^\gamma\left(k_{1,u_x^\alpha u^\rho} - k_{1,u^\alpha u_x^\rho} - u_x^\beta\, k_{1,u^\beta u_x^\alpha u_x^\rho}\right), \end{aligned} \tag{3.60}$$

where $A^\alpha_{\gamma\beta\epsilon} := \eta^{\alpha\delta}\,\partial_\delta\partial_\gamma\partial_\beta\partial_\epsilon(h)$. Indeed, the contribution of ϕ_i-terms is just a result of canonical Miura-type transformation and note that Eq. (3.46) depend on k_1 *linearly*, so we can assume $\phi_i = 0$, $i = 1, \ldots, n$. Then, by straightforward calculations, we find that the both sides of Eq. (3.60) are equal to $-\sum_{i=1}^{n} C_i(R_i)\big(\lambda_{i,\beta\delta} R_{i,\rho} + \lambda_{i,\rho} R_{i,\beta\delta}\big)\, u_x^\beta u_x^\delta$.

Hence, we have proved that the Hamiltonian perturbation (3.22) is quasi-trivial at the second-order approximation iff $\tilde{d}_{\alpha\beta}$ has the form (1.13).

We proceed with proving that quasi-triviality at the second-order approximation implies $\mathcal{O}(\epsilon^2)$-integrability. We have shown that there exist functions

$C_i(R_i)$ and $\phi_i(R)$ such that Eqs. (3.56) hold true. And the quasi-triviality is generated by $\epsilon\, K_1 + \mathcal{O}(\epsilon^2)$:

$$K_1 \;=\; \int_{S^1} \sum_{i=1}^{n} C_i(R_i)\, R_{ix} \log R_{ix} - C_i(R_i) R_{ix} + \phi_i(R_1, \ldots, R_n) R_{ix}\, \mathrm{d}x. \quad (3.61)$$

For a generic conservation law $F_0 = \int_{S^1} f_0(v)\, \mathrm{d}x$ of (1.1), denote by $\mu_1, \ldots, \mu_n$ the distinct eigenvalues of the Hessian (M^α_β) of f_0. The calculations above can be applied to F_0, which give

$$F_2 \;:=\; \{F_0, K_1\} \;=\; \int_{S^1} \left(-\sum_{i=1}^{n} C_i(R_i)\, \mu_{ix} R_{ix} + \frac{1}{2} \sum_{i\neq j} (\mu_i - \mu_j)\, s_{ij}\, R_{ix} R_{jx} \right) \mathrm{d}x\,. \quad (3.62)$$

Then, using the Jacobi identity, we obtain $\{H_0, F_2\} + \{H_2, F_0\} = 0$. Hence, we have proved the $\mathcal{O}(\epsilon^2)$-integrability.

The theorem is proved. □

4. Example

The two component irrotational water wave equations in 1 + 1 dimensions [1,25] are given by

$$\int_{-\infty}^{\infty} e^{-ikx} \mathrm{d}x \left\{ i\, \eta_t \cosh\left[k\, \epsilon\, (1 + \mu\, \eta)\right] - \frac{q_x}{\epsilon} \sinh\left[k\, \epsilon\, (1 + \mu\, \eta)\right] \right\} \;=\; 0\,, \quad (4.1)$$

$$q_t \;+\; \eta \;+\; \frac{\mu}{2} q_x^2 \;=\; \frac{\mu \epsilon^2}{2} \frac{(\eta + \mu\, q_x \eta_x)^2}{1 + \mu^2 \epsilon^2 \eta_x^2} \;+\; \frac{\sigma \epsilon^2 \eta_{xx}}{(1 + \mu^2 \epsilon^2 \eta_x^2)^{3/2}}\,. \quad (4.2)$$

Here, μ and σ are constants. For simplicity, we will only consider the case $\sigma \equiv 0$. Denote $r = 1 + \mu\, \eta,\ v = \mu\, q_x$. Then, we can rewrite (4.1)–(4.2) as the perturbation of a system of Hamiltonian PDEs of hydrodynamic type:

$$r_t = (1 + Q)^{-1} \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^{2j-2}}{(2j-1)!} \partial_x^{2j-1} (r^{2j-1} v), \quad (4.3)$$

$$v_t = -r_x - v v_x + \frac{\epsilon^2}{2} \partial_x \left(\frac{v\, r_x + (1 + Q)^{-1} \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^{2j-2}}{(2j-1)!} \partial_x^{2j-1} (r^{2j-1} v)}{1 + \epsilon^2 r_x^2} \right), \quad (4.4)$$

where Q is an operator defined by $Q := \sum_{j=1}^{\infty} \frac{(-1)^j \epsilon^{2j}}{(2j)!} \partial_x^{2j} \circ r^{2j}$. The dispersionless limit of (4.3)–(4.4) was studied by Whitham [24] and is integrable. Now we look at the second-order approximation of (4.3)–(4.4):

$$r_t \;=\; -\,(rv)_x \;+\; \epsilon^2 \left(-r^2 r_x v_x - \frac{1}{3} r^3 v_{xx} \right)_x \;+\; \mathcal{O}(\epsilon^4)\,, \quad (4.5)$$

$$v_t \;=\; -\,r_x \;-\; v v_x \;+\; \epsilon^2 \left(\frac{1}{2} r^2 v_x^2 \right)_x \;+\; \mathcal{O}(\epsilon^4)\,. \quad (4.6)$$

This approximation has the Hamiltonian structure:

$$(r_t, v_t)^T = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \left(\frac{\delta H}{\delta r(x)}, \frac{\delta H}{\delta v(x)} \right)^T, \tag{4.7}$$

$$H = H_0 + \epsilon^2 H_2 + \mathcal{O}(\epsilon^3), \tag{4.8}$$

$$H_0 = -\int_{S^1} \frac{1}{2} r v^2 + \frac{r^2}{2} \, \mathrm{d}x \,, \qquad H_2 = \int_{S^1} \frac{1}{6} r^3 v_x^2 \, \mathrm{d}x \,. \tag{4.9}$$

Proposition 4.1. *The system* (4.3)–(4.4) *is not integrable in the sense of Definition* 2.2.

Proof. The Riemann invariants are $R_1 = v/2 + \sqrt{r}$, $R_2 = v/2 - \sqrt{r}$. And the eigenvalues are

$$\lambda_1 = -v - \sqrt{r} = -\frac{3}{2} R_1 - \frac{1}{2} R_2, \quad \lambda_2 = -v + \sqrt{r} = -\frac{1}{2} R_1 - \frac{3}{2} R_2. \tag{4.10}$$

This gives $\lambda_{1,1} = \lambda_{2,2} = -3/2$. According to Theorem 1.2, the perturbation (4.8) is quasi-trivial at the second-order approximation iff the following equation has a solution:

$$- \big((R_1 - R_2)(\phi_{1,2} - \phi_{2,1}) \big) R_{1x} R_{2x} + \frac{3}{2} C_1(R_1) \, R_{1x}^{\,2} + \frac{3}{2} C_2(R_2) \, R_{2x}^{\,2} = \frac{(R_1 - R_2)^6}{384} (R_{1x} + R_{2x})^2.$$

However, the solution set to this equation is empty as the coefficients of $R_{1x}^{\,2}$ on the both sides already produce a contradiction. The proposition is proved. □

Let us provide additional but more straightforward evidence supporting the already proved statement of Proposition 4.1. It is easy to verify that up to the second-order approximations, system (4.3)–(4.4) has four linearly independent conservation laws:

$$\int_{S^1} r \, \mathrm{d}x, \ \int_{S^1} v \, \mathrm{d}x, \ \int_{S^1} r v \, dx, \ -H.$$

We will show these form all possible conservation laws in all-order for (4.3)–(4.4). (They are actually indeed conservation laws all-order, but we do not prove this in the present paper; instead we refer to [1,3,25].) The precise statement that we will now prove is that only the following *four* conservation laws of the dispersionless limit of (4.3)–(4.4)

$$\int_{S^1} r \, \mathrm{d}x, \ \int_{S^1} v \, \mathrm{d}x, \ \int_{S^1} r v \, \mathrm{d}x, \ \int_{S^1} \frac{1}{2} r v^2 + \frac{r^2}{2} \, \mathrm{d}x \tag{4.11}$$

can be extended to conservation laws at the second-order approximation for (4.3)–(4.4). To see this, denote $u^1 = r$, $u^2 = v$, and let

$$F = F_0 + \epsilon^2 F_2 + \mathcal{O}(\epsilon^3) = \int_{S^1} f(u) \, \mathrm{d}x + \epsilon^2 \int_{S^1} D_{\alpha\beta}(u) \, u_x^\alpha u_x^\beta + \mathcal{O}(\epsilon^3)$$

be a conserved quantity of (4.3)–(4.4) at the second-order approximation. Then, we have

$$f_{vv} = r\, f_{rr}, \tag{4.12}$$

$$\mu_1 = f_{rv} - \sqrt{r}\, f_{rr}, \quad \mu_2 = f_{rv} + \sqrt{r}\, f_{rr}, \tag{4.13}$$

$$d_{11} = d_{22} = \frac{1}{384}(r_1 - r_2)^2, \tag{4.14}$$

$$D_{11} = -\frac{\partial_{R_1}(\mu_1)}{576}(r_1 - r_2)^6, \quad D_{22} = -\frac{\partial_{R_2}(\mu_2)}{576}(r_1 - r_2)^6. \tag{4.15}$$

Substituting these equations in (3.33) and using (4.12), we find $f_{rrv} = 0$. It yields five solutions:

$$f = r, \quad f = v, \quad f = rv, \quad f = \frac{1}{2}rv^2 + \frac{1}{2}r^2, \quad f = \frac{v^2}{2} + r\log r. \tag{4.16}$$

However, through one by one verifications, only the first four can be (and are indeed) extended to the second-order approximation.

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