



Remarks on Scattering Matrices for Schrödinger Operators with Critically Long-Range Perturbations

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Abstract. We consider scattering matrix for Schrödinger-type operators on \mathbb{R}^d with perturbation $V(x) = O(\langle x \rangle^{-1})$ as $|x| \rightarrow \infty$. We show that the scattering matrix (with time-independent modifiers) is a pseudodifferential operator and analyze its spectrum. We present examples of which the spectrum of the scattering matrices has dense point spectrum, and absolutely continuous spectrum, respectively. These give a partial answer to an open question posed by Yafaev (Scattering theory: some old and new problems. Springer Lecture Notes in Mathematical, vol 1735, 2000).

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1. Introduction

In this note, we consider the scattering matrices for Schrödinger-type operators

$$H = H_0 + V \quad \text{on } \mathcal{H} = L^2(\mathbb{R}^d),$$

where $H_0 = p_0(D_x)$ is a Fourier multiplier, and $V = V^W(x, D_x)$ is a long-range perturbation of H_0 . We will explain the general setup in the next section, and here we present our main results for the standard Schrödinger operators with potential perturbations, i.e., $H_0 = -\frac{1}{2}\Delta$, and $V = V(x)$. We say the potential

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$V(x)$ is a long-range perturbation, if $V(x)$ is a real-valued smooth function, and there is $\mu \in (0, 1]$ such that for any multi-index $\alpha \in \mathbb{Z}_+^d$,

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

with some $C_\alpha > 0$, where $\langle x \rangle = (1 + |x|^2)^{1/2}$. We consider the case $\mu \in (0, 1)$ in another paper [10], and we concentrate on the case $\mu = 1$ in this paper. Namely, we suppose

Assumption A. $V(x) \in C^\infty(\mathbb{R}^d; \mathbb{R})$, and for any $\alpha \in \mathbb{Z}^d$, there is $C_\alpha > 0$ such that

$$|\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-1-|\alpha|}, \quad x \in \mathbb{R}^d.$$

At first, we show the scattering matrix is a pseudodifferential operator and compute the principal symbol.

Theorem 1.1. *Under Assumption A, for any $\lambda > 0$, the scattering matrix $S(\lambda) \in \mathcal{B}(L^2(S^{d-1}))$ is a pseudodifferential operator on S^{d-1} , and the principal symbol is given by*

$$s_0(\lambda, x, \xi) = \exp\left(-i \int_{-\infty}^{\infty} (V(x + t\sqrt{2\lambda}\xi) - V(t\sqrt{2\lambda}\xi))dt\right),$$

for $\xi \in S^{d-1}$, $x \in T_\xi^*S^{d-1} \simeq \xi^\perp$. More precisely, if we write the symbol of $S(\lambda)$ by $s(\lambda, x, \xi)$, then $s(\lambda, \cdot, \cdot) \in S_{1,0}^\delta(T^*S^{d-1})$, and $s(\lambda, \cdot, \cdot) - s_0(\lambda, \cdot, \cdot) \in S_{1,0}^{-1+\delta}(T^*S^{d-1})$ with any $\delta > 0$.

Remark 1.1. This is essentially a refined version of a result by Yafaev [13] for the case $\mu = 1$, and our proof for generalized model follows the argument of Nakamura [8] for short-range perturbations. This argument works for $\mu > 1/2$, as in the paper [13], though we have more precise results if we employ Fourier integral operator formulation as in [10], unless $\mu = 1$. Thus one of the purposes of this note is to fill a gap left in [10].

Remark 1.2. By a simple change of integration variable, we have

$$s_0(\lambda, x, \xi) = \exp\left(-i(2\lambda)^{-1/2} \int_{-\infty}^{\infty} (V(x + t\xi) - V(t\xi))dt\right),$$

though the expression in Theorem 1.1 might be more natural since $\sqrt{2\lambda}\xi$ is the velocity corresponding to $\xi \in S^{d-1}$ at the energy λ . If we write

$$\psi(x, \xi) = \int_{-\infty}^{\infty} (V(x + t\xi) - V(t\xi))dt, \quad \xi \in S^{d-1}, x \in T_\xi^*S^{d-1} \simeq \xi^\perp,$$

then it is easy to see that ψ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta \psi(x, \xi)| \leq \begin{cases} C_{\alpha\beta} \langle \log \langle x \rangle \rangle, & \text{if } \alpha = 0, \\ C_{\alpha\beta} \langle x \rangle^{-|\alpha|}, & \text{if } \alpha \neq 0, \end{cases}$$

for any $\alpha, \beta \in \mathbb{Z}_+^{d-1}$ in a local coordinate. Thus we learn

$$s_0(\lambda, x, \xi) = \exp(-i(2\lambda)^{-1/2}\psi(x, \xi)) \in S_{1,0}^\delta(T^*S^{d-1})$$

with any $\delta > 0$.

Next, we consider the spectral properties of $S(\lambda)$ using the above representation.

Theorem 1.2. *Suppose Assumption A, and suppose V is rotation symmetric and*

$$|x \cdot \partial_x V(x)| \geq c|x|^{-1} \quad \text{for } |x| \geq R,$$

with some $c, R > 0$. Then for any $\lambda > 0$ the scattering matrix has dense pure point spectrum on the whole unit circle.

This result is due to Yafaev [14], §9.7. For the moment, we need the rotation symmetry to show the pure point spectrum, but we can show the absence of absolutely continuous spectrum under weaker assumptions (Theorem 3.3). We discuss these in Sect. 3.

Theorem 1.3. *Suppose $d = 2$, and let*

$$V(x) = a \frac{x_1}{\langle x \rangle^2}, \quad x = (x_1, x_2) \in \mathbb{R}^2,$$

with $a \neq 0$. Then, $\sigma_{\text{ess}}(S(\lambda)) = \{e^{i\theta} \mid |\theta| \leq |a|\pi(2\lambda)^{-1/2}\}$, and $S(\lambda)$ has absolutely continuous spectrum on $\sigma_{\text{ess}}(S(\lambda)) \setminus \{e^{\pm ia\pi(2\lambda)^{-1/2}}\}$, except for possible eigenvalues of finite multiplicities. The eigenvalues may accumulate only at $e^{\pm ia\pi(2\lambda)^{-1/2}}$.

The absolutely continuous spectrum is relatively stable under small perturbations, and we have the same properties if we add lower-order perturbations.

There is extensive literature concerning the two-body long-range scattering. We refer textbooks, Reed-Simon Volume 3 [11] §X1-9, Yafaev [14] Part 2, [15] Chapter 10, Dereziński-Gérard [1], and references therein. About the scattering matrix for long-range scattering, there are detailed analysis by Yafaev, especially [13]. Our approach is closely related to his result, though our formulation is more general and the proof is substantially different. Actually it is a direct extension of a previous paper by the author [8]. In particular, this argument is easily generalized to discrete Schrödinger operators with long-range perturbations [7, 12]. Our example of scattering matrix with pure point spectrum is discussed in §9.7 in Yafaev [14], though in a different manner, and we also discuss generalizations. Thus the author feels it would be useful to include an independent proof. Our results give a partial answer to an open question by Yafaev [13], Problem 9.12.

Theorems 1.2 and 1.3 are proved in Sects. 3, 4, respectively. In Sect. 4, we use functional calculus of unitary pseudodifferential operators, and for the completeness we give a proof of the functional calculus in “Appendix A”. A construction of approximate logarithm of unitary pseudodifferential operators is discussed in “Appendix B”, and a simple result of trace-class scattering theory for unitary operators is discussed in “Appendix C”.

In the following, we use the Weyl quantization of a symbol $a \in C^\infty(\mathbb{R}^{2d})$:

$$\text{Op}(a)\varphi(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi\right)\varphi(y)dyd\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

We denote the Kohn-Nirenberg symbol class in ξ -space by $S_{\rho,\delta}^m$, i.e., $a \in S_{\rho,\delta}^m$ if $a \in C^\infty(\mathbb{R}^{2d})$ and for any $\alpha, \beta \in \mathbb{Z}_+^d$ there is $C_{\alpha\beta}$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad x, \xi \in \mathbb{R}^d.$$

We also use the Hörmander $S(m, g)$ symbol class notation [4], but we will use it for specific metrics g and \tilde{g} , and we explain later. For a symbol class Σ , we denote the corresponding operator set by $\text{Op}\Sigma = \{\text{Op}(a) \mid a \in \Sigma\}$. We refer Hörmander [4], Dimassi-Sjöstrand [2] and Zworski [16] for the pseudodifferential operator calculus.

2. Representation Formula of the Scattering Matrix

Here we define long-range wave operators and scattering operators using time-independent modifiers originally due to Isozaki and Kitada [5,6]. We follow the formulation of Nakamura [8] and sketch the proof of Theorem 1.1 in a generalized setting.

Assumption B. Let $p_0(\xi) \in C^\infty(\mathbb{R}^d; \mathbb{R})$ and elliptic in the following sense: There is $\nu > 0$ such that $p_0 \in S^\nu$, i.e., $\partial_\xi^\alpha p_0(\xi) = O(\langle \xi \rangle^{\nu-|\alpha|})$ for any $\alpha \in \mathbb{Z}_+^d$, and

$$p_0(\xi) \geq c_0 \langle \xi \rangle^\nu - c_1, \quad \xi \in \mathbb{R}^d,$$

with some $c_0, c_1 > 0$. Let $I \Subset \mathbb{R}$ be a compact interval. We suppose there is $c_0 > 0$ such that

$$|\partial_\xi p_0(\xi)| \geq c_0 \quad \text{for } \xi \in p_0^{-1}(I).$$

We set

$$H_0 = p_0(D_x) = \mathcal{F}^* p_0(\cdot) \mathcal{F},$$

where \mathcal{F} is the Fourier transform, and we also write the free velocity by

$$v(\xi) = \partial_\xi p_0(\xi), \quad \xi \in \mathbb{R}^d.$$

We suppose the perturbation V is a symmetric pseudodifferential operator with the real-valued Weyl symbol $V(x, \xi)$, i.e.,

$$V\varphi(x) = (2\pi)^{-d} \iint e^{i(x-y)\cdot\xi} V\left(\frac{x+y}{2}, \xi\right) \varphi(y) dy d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

We denote the metric $g = dx^2/\langle x \rangle^2 + d\xi^2$, and the symbol class $S(m, g)$ is defined as follows: $a \in S(m, g)$ if and only if $a \in C^\infty(\mathbb{R}^{2d})$ and

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \langle x \rangle^{-|\alpha|}, \quad x, \xi \in \mathbb{R}^d$$

for any $\alpha, \beta \in \mathbb{Z}_+^d$, with some $C_{\alpha\beta} > 0$.

Assumption C. $V(x, \xi)$ is real valued and $V \in S(\langle x \rangle^{-1} \langle \xi \rangle^\nu, g)$.

We write

$$H = H_0 + V = p_0(D_x) + V^W(x, D_x)$$

be our Hamiltonian, and we suppose:

Assumption D. H is essentially self-adjoint on $H^\nu(\mathbb{R}^d)$.

We write the symbol of H by

$$p(x, \xi) = p_0(\xi) + V(x, \xi).$$

Remark 2.1. It might be natural to assume the ellipticity:

$$|p(x, \xi)| \geq c_0 \langle \xi \rangle^\nu - c_1, \quad \text{for } x, \xi \in \mathbb{R}^d.$$

It implies the self-adjointness on $H^\nu(\mathbb{R}^d)$, but it is not essential in the following argument.

For $\varepsilon > 0$, we denote

$$\Omega_\pm^\varepsilon = \{(x, \xi) \in \mathbb{R}^{2d} \mid \pm \cos(x, v(\xi)) > -1 + \varepsilon, |x| \geq 1, p_0(\xi) \in I\}.$$

As well as in [8] Section 3, we can construct symbols $a^\pm \in S(1, g)$ such that

$$H\text{Op}(a^\pm) - \text{Op}(a^\pm)H_0 \sim 0$$

in the formal symbol sense as $|x| \rightarrow \infty$ in Ω_\pm^ε . a_\pm have the form:

$$a^\pm(x, \xi) \sim e^{i\psi_\pm(x, \xi)} (1 + a_1^\pm(x, \xi) + a_2^\pm(x, \xi) + \dots)$$

where

$$\psi_\pm(x, \xi) = \int_0^{\pm\infty} (V(x + tv(\xi), \xi) - V(tv(\xi), \xi)) dt.$$

We note $\psi_\pm(x, \xi) \notin S(1, g)$ (on Ω_\pm^ε) in general, but for any $\alpha, \beta \in \mathbb{Z}_+^d$,

$$|\partial_\xi^\beta \psi_\pm(x, \xi)| \leq C_\beta \langle \log \langle x \rangle \rangle,$$

and if $\alpha \neq 0$,

$$|\partial_x^\alpha \partial_\xi^\beta \psi_\pm(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|}$$

on Ω_\pm^ε . We note ψ_\pm satisfies

$$v(\xi) \cdot \partial_x \psi_\pm(x, \xi) + V(x, \xi) = 0$$

as well as in the short-range case (see [8] Section 3).

We introduce a new metric \tilde{g} by

$$\tilde{g} = \langle x \rangle^{-2} dx^2 + \langle \log \langle x \rangle \rangle^2 d\xi^2 \quad \text{on } \mathbb{R}^{2d}.$$

Then the corresponding symbol class $S(m, \tilde{g})$ is defined as follows: $a \in S(m, \tilde{g})$ if and only if, for any $\alpha, \beta \in \mathbb{Z}_+^d$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \langle x \rangle^{-|\alpha|} \langle \log \langle x \rangle \rangle^{|\beta|}$$

with some $C_{\alpha\beta} > 0$. We note, hence, for any $\delta > 0$, $S(m, \tilde{g}) \subset S(m \langle x \rangle^\delta, g)$.

By the same construction of a_j^\pm as in [8], Section 3, and direct computations, we can easily show $a_j^\pm \in S(\langle x \rangle^{-j} \langle \log \langle x \rangle \rangle^j, \tilde{g})$ on Ω_\pm^ε . Hence, a^\pm , which is an asymptotic sum of $\{a_j^\pm\}$, is an element of $S(1, \tilde{g}) \subset S(\langle x \rangle^\delta, g)$, with any $\delta > 0$ on Ω_\pm^ε . We also note $a_\pm - e^{i\psi_\pm} \in S(\langle x \rangle^{-1} \langle \log \langle x \rangle \rangle, \tilde{g}) \subset S(\langle x \rangle^{-1+\delta}, g)$ on Ω_\pm^ε .

We choose smooth cut-off functions χ, ζ and η such that: $\chi \in C_0^\infty(I)$ with $\chi(\lambda) = 1$ on $I' \Subset I$; $\zeta(x) = 0$ in a neighborhood of 0 and $\text{supp}[1 - \zeta] \subset \{|x| \leq 2\}$; and $\eta(\sigma) = 1$ if $\sigma > -1 + 2\varepsilon$ and $\eta(\sigma) = 0$ if $\sigma \leq -1 + \varepsilon$ with sufficiently small $\varepsilon > 0$. With these cut-off functions, we set

$$\tilde{a}^\pm(x, \xi) = \chi(p_0(\xi))\zeta(|x|)\eta(\pm \cos(x, v(\xi)))a^\pm(x, \xi).$$

Then we have symbols $\tilde{a}^\pm \in S(1, \tilde{g})$. We set

$$J_\pm = \text{Op}(\tilde{a}^\pm).$$

We note the principal symbols of $J_\pm^* J_\pm$ are $|\chi(p_0(\xi))\zeta(|x|)\eta(\pm \cos(x, v(\xi)))|^2$, and the remainder terms are in $S(\langle x \rangle^{-1+\delta}, g)$. Hence J_\pm are bounded in L^2 , and we can utilize standard pseudodifferential operator calculus as if they are in $S(1, g)$. We call J_\pm the *time-independent modifiers*, or the *Isozaki-Kitada modifiers* [5, 6]. By the construction,

$$\text{EssSupp}[a^\pm] \subset \{p_0(\xi) \in I \setminus I'\} \cup \{\pm \cos(x, v(\xi)) \in [-1 + \varepsilon, -1 + 2\varepsilon]\} \cup \{|x| \leq 2\},$$

where $\text{EssSupp}[\cdot]$ denotes the essential support of the symbol. Using this fact and the standard non-stationary phase argument, we can show the existence of modified wave operators:

$$W_\pm E_{I'}(H_0) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} J_\pm e^{-itH_0} E_{I'}(H_0)$$

where $E_I(A)$ denotes the spectral projection. We recall W_\pm has the intertwining property:

$$HW_\pm E_{I'}(H_0) = W_\pm E_{I'}(H_0)H_0.$$

We set the (modified) scattering operator S by

$$SE_{I'}(H_0) = (W_+ E_{I'})^* W_- E_{I'}(H_0),$$

and then $SE_{I'}(H_0)$ is a unitary operator on $E_{I'}(H_0)\mathcal{H}$. By the above intertwining property, S commutes with H_0 .

We now define the scattering matrix $S(\lambda)$ for $\lambda \in I'$. We denote the energy surface with the energy $\lambda \in I$ by

$$\Sigma_\lambda = \{\xi \in \mathbb{R}^d \mid p_0(\xi) = \lambda\} = p_0^{-1}(\{\lambda\}).$$

We note Σ_λ is a smooth hypersurface by the above assumption. Let

$$m_\lambda = |p_0(\xi)|^{-1} dS(\xi)$$

be a measure on Σ_λ , where $dS(\xi)$ is the surface measure on Σ_λ , so that

$$\int \varphi d\xi = \int_I \left(\int_{\Sigma_\lambda} \varphi|_{\Sigma_\lambda} dm_\lambda \right) d\lambda$$

for $\varphi \in C_0^\infty(p_0^{-1}(I))$. Hence we have the integral decomposition

$$L^2(p_0^{-1}(I), d\xi) \simeq \int_I^\oplus L^2(\Sigma_\lambda, m_\lambda) d\lambda.$$

Since S commutes with H_0 , the operator $\mathcal{F}SE_{I'}(H_0)\mathcal{F}^*$ commutes with $p_0(\xi)\cdot$, and hence it is decomposed to operators on $L^2(\Sigma_\lambda, m_\lambda)$:

$$\mathcal{F}SE_{I'}(H_0)\mathcal{F}^* \simeq \int_{I'}^\oplus S(\lambda)d\lambda \quad \text{on} \quad \int_{I'}^\oplus L^2(\Sigma_\lambda, m_\lambda)d\lambda.$$

The family of operators $\{S(\lambda)\}_{\lambda \in I'}$ is called the scattering matrix.

Given the above construction, we can prove the following theorem in exactly the same argument as in [8] (see also [10]). We note the microlocal resolvent estimate, which is crucial in the proof, is proved in [9] under our setting. We describe the microlocal resolvent estimate briefly: If A_\pm are microlocal cut-off to out-going/in-coming subspaces, then

$$\langle x \rangle^N A_\mp (H - \lambda \mp i0)^{-1} A_\pm^* \langle x \rangle^N \in B(L^2(\mathbb{R}^d)), \quad \lambda > 0,$$

for any $N > 0$. These imply scattering from out-going subspace to in-coming subspace is very weak. For the detail, we refer [9] and references therein.

Theorem 2.1. *Let $\lambda \in I' \setminus \sigma_p(H)$. Then $S(\lambda)$ is a pseudodifferential operator on Σ_λ . If we denote the symbol by $s(\lambda, x, \xi)$, then it satisfies for any $\alpha, \beta \in \mathbb{Z}_+^{d-1}$,*

$$|\partial_x^\alpha \partial_\xi^\beta s(\lambda, x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \log \langle x \rangle \rangle^{|\beta|}$$

for $\xi \in \Sigma_\lambda, x \in T_\xi^* \Sigma_\lambda$. Moreover, the principal symbol is given by

$$s_0(\lambda, x, \xi) = \exp\left(-i \int_{-\infty}^\infty (V(x + tv(\xi), \xi) - V(tv(\xi), \xi))dt\right),$$

i.e., $s(\lambda, \cdot, \cdot) - s_0(\lambda, \cdot, \cdot) \in S(\langle x \rangle^{-1+\delta}, g)$ with any $\delta > 0$.

3. Scattering Matrix with Pure Point Spectrum

We first note that, if $H_0 = -\frac{1}{2}\Delta$, and if the perturbation is rotation symmetric, then the scattering matrix is also rotation symmetric. Then we can easily show that such operator has pure point spectrum. This model is also discussed in [14] §9.7.

Lemma 3.1. *Suppose U is a rotation symmetric bounded pseudodifferential operator on S^{d-1} , then the spectrum is pure point.*

Proof. In the geodesic local coordinate with the center at ξ_0 , the symbol of the operator U has the form $u(\xi_0, |x|^2)$ by virtue of the symmetry (with respect the rotation around ξ_0). Then, again by the symmetry, the symbol is independent of ξ_0 , i.e., the symbol has the form $u(\xi, |x|^2) = g(|x|^2)$ in the geodesic local coordinate. This implies $U = g(-\Delta)$, where Δ is the Laplace-Beltrami operator on S^{d-1} . Since the spectrum of $-\Delta$ is pure point, the spectrum of $U = g(-\Delta)$ is also pure point.

We now observe the spectrum of the scattering matrix tends to cover the whole unit circle.

Lemma 3.2. *Suppose $V = V(x)$ is a rotationally symmetric potential and satisfies Assumption A. Suppose, moreover, V satisfies*

$$|x \cdot \partial_x V(x)| \geq c|x|^{-1}, \quad |x| \geq R,$$

with some $c > 0$ and $R > 0$. Then for any $\lambda > 0$, $\sigma(S(\lambda)) = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Proof. We suppose $x \cdot \partial_x V(x) \geq c_0|x|^{-1}$ for large x . Let $\theta_0 \in [0, 2\pi]$ be fixed, and we show $e^{-i\theta_0} \in \sigma(S(\lambda))$. We write $V(x) = g(|x|)$.

We write, for $\xi \in S^{d-1}$, $x \perp \xi$ and $|x| \geq R$,

$$\begin{aligned} \psi(x, \xi) &= \int_{-\infty}^{\infty} (V(x + t\xi) - V(t\xi))dt, \\ &= \int_{-\infty}^{\infty} \left(\int_0^1 x \cdot \partial_x V(sx + t\xi)ds \right) dt. \end{aligned} \tag{3.1}$$

We note, since $V(x)$ is rotationally symmetric, we have

$$x \cdot \partial_x V(x) = |x|g'(|x|) \geq c_0|x|^{-1},$$

and hence

$$\begin{aligned} x \cdot \partial_x V(sx + t\xi) &= x \cdot \frac{sx + t\xi}{|sx + t\xi|} g'(|sx + t\xi|) \\ &= \frac{s|x|^2}{|sx + t\xi|} g'(|sx + t\xi|) \geq \frac{c_0s|x|^2}{\langle sx + t\xi \rangle^3}. \end{aligned}$$

Thus we have

$$\begin{aligned} \psi(x, \xi) &\geq \int_{-\infty}^{\infty} \left(\int_0^1 \frac{c_0s|x|^2}{\langle sx + t\xi \rangle^3} ds \right) dt \\ &= \int_0^1 \left(\int_{-\infty}^{\infty} \frac{c_0s|x|^2}{(s^2|x|^2 + t^2 + 1)^{3/2}} dt \right) ds \\ &= 2c_0 \int_0^1 \frac{s|x|^2}{s^2|x|^2 + 1} ds = 2c_0 \int_0^{|x|} \frac{sds}{s^2 + 1} = 2c_0 \log \langle x \rangle. \end{aligned}$$

Here we have used the formula: $\int_0^\infty (a^2 + t^2)^{-3/2} dt = a^{-2}$, $a > 0$. In particular $\psi(x, \xi) \rightarrow \infty$ as $|x| \rightarrow \infty$, and hence, for any $N > 0$ we can find (x_N, ξ_N) such that $|x_N| \geq N$ and $\psi(x_N, \xi_N) \equiv \sqrt{2\lambda}\theta_0 \pmod{(2\pi\mathbb{Z})}$. We set

$$\varphi_N(\xi) = c_N \exp(ix_N \cdot (\xi - \xi_N) - |\xi - \xi_N|^2/|x_N|)$$

in a neighborhood inside a local coordinate of ξ_N , where c_N is chosen so that $\|\varphi_N\| = 1$. Then φ_N is supported essentially in

$$\{(x, \xi) \mid |x - x_N| = O(\langle x_N \rangle^{1/2}), |\xi - \xi_N| = O(\langle x_N \rangle^{-1/2})\}.$$

We also recall $e^{-i(2\lambda)^{-1/2}\psi(x, \xi)}$ is the principal symbol of $S(\lambda)$, and $\partial_x \psi(x, \xi) = O(|x|^{-1})$, $\partial_\xi \psi(x, \xi) = O(\log \langle x \rangle)$ as $|\xi| \rightarrow \infty$. These imply

$$\langle \varphi_N, S(\lambda)\varphi_N \rangle - e^{-i\theta_0} \|\varphi_N\|^2 = O(\langle x_N \rangle^{-1/2} \log \langle x_N \rangle) \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

and we may assume $\{\varphi_N\}$ are asymptotically orthogonal (since they have essentially disjoint supports in the phase space). Then by the Weyl's criterion

([11] Theorem VII.12), we conclude $e^{i\theta_0} \in \sigma_{\text{ess}}(S(\lambda))$. The proof for the case $x \cdot \partial_x V(x) \leq -c_0|x|^{-1}$ ($|x| \geq R$) is essentially the same.

Theorem 1.2 follows immediately from the above two lemmas.

We now consider slightly more general potentials. We write

$$\partial_r f(x) = \hat{x} \cdot \partial_x f(x), \quad \hat{x} = \frac{x}{|x|},$$

and

$$\partial_r^\perp f(x) = \partial_x f(x) - \partial_r f(x)\hat{x} = (E - \hat{x} \otimes \hat{x})\partial_x f(x),$$

for $f \in C^1(\mathbb{R}^d)$.

Theorem 3.3. *Suppose V satisfies Assumption A, and there are constants $c_1, c_2, R > 0$ such that $c_1 > c_2$ and*

$$|\partial_r V(x)| \geq \frac{c_1}{|x|^2}, \quad |\partial_r^\perp V(x)| \leq \frac{c_2}{|x|^2}, \quad \text{if } |x| \geq R. \tag{3.2}$$

Then $\sigma(S(\lambda)) = S^1$, and $S(\lambda)$ has no absolutely continuous spectrum for $\lambda > 0$.

Remark 3.1. Suppose $V(x) = -f(\theta)/r$, $x = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2$ for $|x| \geq R$, $f(\theta) > 0$. Then the condition (3.2) is equivalent to

$$\inf_{\theta} f(\theta) = c_1 > c_2 = \sup_{\theta} |f'(\theta)|.$$

Lemma 3.4. *Suppose V satisfies (3.2), then there is $c_3 > 0$ such that*

$$\psi(x, \xi) \geq 2(c_1 - c_2) \log |x| - c_3, \quad \xi \in S^{d-1}, x \perp \xi.$$

Proof. Here we suppose $\partial_r V(x) \geq c_1/|x|^2$. The other case is considered similarly. We may suppose $|x| \geq R$ without loss of generality. We recall (3.1). We write $y = sx + t\xi$, and compute

$$x \cdot \partial_x V(y) = \partial_r V(y)(x \cdot \hat{y}) + x \cdot \partial_r^\perp V(y).$$

At first, we note

$$x \cdot \hat{y} = \frac{x \cdot (sx + t\xi)}{|sx + t\xi|} = \frac{s|x|^2}{(s^2|x|^2 + t^2)^{1/2}}.$$

We also note

$$\begin{aligned} (E - \hat{y} \otimes \hat{y})x &= x - (x \cdot \hat{y})\hat{y} = x - \frac{s|x|^2(sx + t\xi)}{s^2|x|^2 + t^2} \\ &= \frac{(s^2|x|^2 + t^2) - s^2|x|^2}{s^2|x|^2 + t^2}x - \frac{s|x|^2 t}{s^2|x|^2 + t^2}\xi \\ &= \frac{t^2x - st|x|^2\xi}{s^2|x|^2 + t^2}, \end{aligned}$$

and thus

$$|(E - \hat{y} \otimes \hat{y})x| = \frac{(t^4|x|^2 + s^2t^2|x|^4)^{1/2}}{s^2|x|^2 + t^2} = \frac{|t||x|}{(s^2|x|^2 + t^2)^{1/2}}.$$

Hence we learn

$$\begin{aligned} \int_{-\infty}^{\infty} \partial_r V(y)(x \cdot \hat{y}) dt &\geq \int_{-\infty}^{\infty} \frac{c_1}{|sx + t\xi|^2} \cdot \frac{s|x|^2}{(s^2|x|^2 + t^2)^{1/2}} dt \\ &= \int_{-\infty}^{\infty} \frac{c_1 s|x|^2 dt}{(s^2|x|^2 + t^2)^{3/2}} = \frac{2c_1 s|x|^2}{s^2|x|^2} = \frac{2c_1}{s}, \end{aligned}$$

provided $s|x| \geq R$. Similarly, we learn

$$\begin{aligned} \int_{-\infty}^{\infty} |x \cdot \partial_r^\perp V(y)| dt &\leq \int_{-\infty}^{\infty} \frac{c_2}{|sx + t\xi|^2} \cdot \frac{|t||x|}{(s^2|x|^2 + t^2)^{1/2}} dt \\ &= \int_{-\infty}^{\infty} \frac{c_2|x||t| dt}{(s^2|x|^2 + t^2)^{3/2}} = \frac{2c_2|x|}{s|x|} = \frac{2c_2}{s}, \end{aligned}$$

if $s|x| \geq R$. Here we have used the formula: $\int_0^\infty t(a^2 + t^2)^{-3/2} dt = a^{-1}$. Thus we have

$$\begin{aligned} \int_{R/|x|}^1 \left(\int_{-\infty}^{\infty} x \cdot \partial_x V(sx + t\xi) dt \right) ds &\geq \int_{R/|x|}^1 \frac{2(c_1 - c_2)}{s} ds \\ &= 2(c_1 - c_2) \log(|x|/R) = 2(c_1 - c_2) \log|x| - 2(c_1 - c_2) \log R. \end{aligned}$$

On the other hand, if $s|x| \leq R$, we use

$$|x \cdot \partial_x V(sx + t\xi)| \leq C|x|\langle t\xi \rangle^{-2} = C|x|\langle t \rangle^{-2},$$

with some $C > 0$, which follows directly from Assumption A. Hence, we learn

$$\int_0^{R/|x|} \left(\int_{-\infty}^{\infty} |x \cdot \partial_x V(sx + t\xi)| dt \right) ds \leq C|x| \cdot \frac{R}{|x|} \int_{-\infty}^{\infty} \langle t \rangle^{-2} dt = C\pi R.$$

Combining these, we obtain

$$\int_0^1 \left(\int_{-\infty}^{\infty} x \cdot \partial_x V(sx + t\xi) dt \right) ds \geq 2(c_1 - c_2) \log|x| - c_3,$$

where $c_3 = 2(c_1 - c_2) \log R + C\pi R$.

Proof of Theorem 3.3. The claim $\sigma(S(\lambda)) = S^1$ is proved exactly as in the proof of Lemma 3.2 using Lemma 3.4.

By Theorem B.1 in ‘‘Appendix B’’, we learn there is a real-valued symbol $\Psi \in S(\langle \log \langle x \rangle \rangle, g)$ such that $S(\lambda) \equiv \exp(-i(2\lambda)^{-1/2} \text{Op}(\Psi))$ modulo $S(\langle x \rangle^{-\infty}, g)$, where $g = dx^2/\langle x \rangle^2 + d\xi^2$. Moreover, the principal symbol of Ψ is ψ computed above, i.e., $\Psi - \psi \in S(\langle x \rangle^{-1+\delta}, g)$ with any $\delta > 0$. In particular, by Lemma 3.4, $\Psi(x, \xi) \rightarrow +\infty$ as $|x| \rightarrow \infty$. This implies $\text{Op}(\Psi)$ has a compact resolvent, and its spectrum is discrete. Hence $\exp(-i(2\lambda)^{-1/2} \text{Op}(\Psi))$ has pure point spectrum. Now we note $K = S(\lambda) - \exp(-i(2\lambda)^{-1/2} \text{Op}(\Psi)) \in \text{Op}S(\langle x \rangle^{-\infty}, g)$ is a trace class operator, and we can apply the scattering theory for trace class perturbation (see ‘‘Appendix C’’) to conclude $\sigma_{\text{ac}}(S(\lambda)) = \sigma_{\text{ac}}(\exp(-i(2\lambda)^{-1/2} \text{Op}(\Psi))) = \emptyset$.

4. Scattering Matrix with Absolutely Continuous Spectrum

Here we suppose $d = 2$ and consider the potential

$$V(x) = a \frac{x_1}{\langle x \rangle^2}, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

At first we compute the principal part of $\psi(x, \xi) = \int_{-\infty}^{\infty} (V(x + t\xi) - V(t\xi))dt$ for $|\xi| = 1, x \perp \xi$. We use the standard coordinate for S^1 : We denote a point $\xi \in S^1$ by $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that

$$\xi = (\cos \theta, \sin \theta), \quad \theta \in [0, 2\pi) \simeq \mathbb{T}.$$

The cotangent space at θ is identified with the orthogonal space at θ , i.e.,

$$x = (-\omega \sin \theta, \omega \cos \theta), \quad \omega \in \mathbb{R}.$$

We use $(\theta, \omega) \in \mathbb{T} \times \mathbb{R}$ as the coordinate system of T^*S^1 . As in the last section, we write

$$\psi(x, \xi) = \int_{-\infty}^{\infty} (V(x + t\xi) - V(t\xi))dt$$

so that $\exp(-i(2\lambda)^{-1/2}\psi(x, \xi))$ is the principal symbol of $S(\lambda)$.

Lemma 4.1. *Let V and the coordinate of T^*S^1 as above. Then*

$$\psi(x, \xi) = -a\pi \sin \theta \frac{\omega}{\langle \omega \rangle}, \quad (\theta, \omega) \in T^*S^1.$$

Proof. We again recall (3.1) and we compute

$$\partial_x V(x) = \left(\frac{a}{\langle x \rangle^2}, 0 \right) + a \left(\frac{-2x_1^2}{\langle x \rangle^4}, \frac{-2x_1x_2}{\langle x \rangle^4} \right) = \left(\frac{a}{\langle x \rangle^2}, 0 \right) - \frac{2ax_1}{\langle x \rangle^4} x.$$

Then we have

$$\begin{aligned} x \cdot \partial_x V(sx + t\xi) &= \frac{ax_1}{\langle sx + t\xi \rangle^2} - 2a \frac{sx_1 + t\xi_1}{\langle sx + t\xi \rangle^4} x \cdot (sx + t\xi) \\ &= \frac{ax_1(s^2|x|^2 + t^2 + 1) - 2as^2x_1|x|^2}{(s^2|x|^2 + t^2 + 1)^2} - \frac{2as|x|^2\xi_1 t}{(s^2|x|^2 + t^2 + 1)^2} \\ &= ax_1 \frac{t^2 - s^2|x|^2 + 1}{(s^2|x|^2 + t^2 + 1)^2} - \frac{2as|x|^2\xi_1 t}{(s^2|x|^2 + t^2 + 1)^2}. \end{aligned}$$

Now we note

$$\int_{-\infty}^{\infty} \frac{2s|x|^2\xi_1 t}{(s^2|x|^2 + t^2 + 1)^2} dt = 0$$

since the integrand is odd. We also note, since

$$\frac{d}{dt} \left(\frac{t}{b^2 + t^2} \right) = \frac{b^2 - t^2}{(b^2 + t^2)^2}, \quad b > 0,$$

we have

$$\int_{-\infty}^{\infty} \frac{b^2 - t^2}{(b^2 + t^2)^2} dt = \lim_{T \rightarrow \infty} \left[\frac{t}{b^2 + t^2} \right]_{-T}^T = 0.$$

Using this, we learn

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{t^2 - s^2|x|^2 + 1}{(s^2|x|^2 + t^2 + 1)^2} dt &= \int_{-\infty}^{\infty} \left(\frac{t^2 - s^2|x|^2 - 1}{(s^2|x|^2 + t^2 + 1)^2} + \frac{2}{(s^2|x|^2 + t^2 + 1)^2} \right) dt \\ &= \int_{-\infty}^{\infty} \frac{2}{(s^2|x|^2 + t^2 + 1)^2} dt = \pi(s^2|x|^2 + 1)^{-3/2}. \end{aligned}$$

Here we have used the well-known formula: $\int_{-\infty}^{\infty} (b^2 + t^2)^{-2} dt = \pi/(2b^3)$. Combining these, we learn

$$\psi(\theta, \omega) = a\pi \int_0^1 \frac{x_1}{\langle s x \rangle^3} ds = a\pi \frac{x_1}{|x|} \int_0^{|x|} \frac{du}{\langle u \rangle^3} = a\pi \frac{x_1}{|x|} \cdot \frac{|x|}{\langle x \rangle} = a\pi \frac{x_1}{\langle x \rangle}.$$

We then substitute $x_1 = -\omega \sin \theta$ and $|x| = |\omega|$ to conclude the assertion.

Then the essential spectrum of $S(\lambda)$ is easy to locate using the Weyl theorem.

Lemma 4.2. *For the above Hamiltonian, we have*

$$\sigma_{\text{ess}}(S(\lambda)) = \{e^{i\tau} \mid |\tau| \leq |a|\pi(2\lambda)^{-1/2}\}, \quad \lambda > 0.$$

In particular, if $|a| \geq \sqrt{2\lambda}$ then the essential spectrum is the whole circle.

Now we construct a simple scattering theory to show that the essential spectrum is absolutely continuous. We set

$$q(\theta, \omega) = \text{sgn}(a) \cos \theta \langle \omega \rangle, \quad (\theta, \omega) \in T^*S^1,$$

and we define an operator Q on $L^2(S^1)$ by

$$Q = \text{Op}(q) \equiv \text{sgn}(a) \cos \theta \langle -D_\theta \rangle \quad \text{mod } \text{Op}(S_{1,0}^0).$$

We note, since we are working in θ -space, it is convenient to quantize function $a(x, \xi)$ as $a(-D_\theta, \theta)$. We may assume Q is formally self-adjoint, since we may quantize it, for example, by

$$Qf(\theta) = \frac{1}{2\pi} \iint e^{-i(\theta-\tau)\omega} \eta(\theta - \tau) q\left(\frac{\theta+\tau}{2}, \omega\right) f(\tau) d\tau d\omega,$$

where $\eta \in C^\infty(\mathbb{T})$ such that $\eta(\tau) = 1$ if $|\tau| \leq 1/8$; $= 0$ if $|\tau| \geq 1/4$, and $f \in C^\infty(\mathbb{T})$, and this Q is formally self-adjoint.

Lemma 4.3. *Q is essentially self-adjoint on $H^1(\mathbb{T})$.*

Proof. We set $N = \langle D_\theta \rangle$ on $L^2(\mathbb{T})$. Then it is easy to see N is self-adjoint with $\mathcal{D}(N) = H^1(\mathbb{T})$ and $N \geq 1$. Moreover, by symbol calculus, it is easy to see Q and $[N, Q]$ are bounded from $H^{1/2}(\mathbb{T})$ to $H^{-1/2}(\mathbb{T})$, since the symbols of Q and $[N, Q]$ are in $S_{1,0}^1$. Hence, by the commutator theorem ([11] Theorem X.36), Q is essentially self-adjoint on $H^1(\mathbb{T})$.

Now we note, $[Q, S(\lambda)], [Q, [Q, S(\lambda)]]$, etc., are bounded in $L^2(\mathbb{T})$ since symbols of these operators are in $S_{1,0}^0$. Namely, $S(\lambda)$ is Q -smooth in the sense of the Mourre theory.

Lemma 4.4. *Suppose $I \subset S^1$ be a compact interval such that $I \cap \{e^{\pm ia\pi(2\lambda)^{-1/2}}\} = \emptyset$. Then there is $c > 0$ and a compact operator $K(\lambda)$ such that*

$$E_I(S(\lambda))S(\lambda)^*[Q, S(\lambda)]E_I(S(\lambda)) \geq cE_I(S(\lambda)) + K(\lambda), \quad \lambda > 0,$$

where $E_I(S)$ denotes the spectral projection for a unitary operator S .

Proof. For simplicity, we suppose $a > 0$. The other case is similar.

Let $f \in C_0^\infty(S^1)$. Then using the functional calculus of unitary pseudodifferential operators, Theorem A.4, we learn the principal symbol of $f(S(\lambda))S(\lambda)^*[Q, S(\lambda)]f(S(\lambda))$ is given by

$$i(f \circ s_0(\lambda; \cdot))^2 s_0(\lambda; \cdot)^* \{q, s_0(\lambda; \cdot)\} = -(f \circ s_0(\lambda; \cdot))^2 \{q, a\pi(2\lambda)^{-1/2} \sin \theta(\omega/\langle \omega \rangle)\},$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket. By direct computations, we have

$$\begin{aligned} -\{\cos \theta \langle \omega \rangle, \sin \theta(\omega/\langle \omega \rangle)\} &= \sin \theta \langle \omega \rangle \cdot \sin \theta \langle \omega \rangle^{-3} + \cos \theta \omega \langle \omega \rangle^{-1} \cdot \cos \theta \omega \langle \omega \rangle^{-1} \\ &= \frac{\sin^2 \theta}{\langle \omega \rangle^2} + \cos^2 \theta \frac{\omega^2}{\langle \omega \rangle^2} \geq \cos^2 \theta \frac{\omega^2}{\langle \omega \rangle^2}, \end{aligned}$$

and hence

$$-\{q, a\pi(2\lambda)^{-1/2} \sin \theta(\omega/\langle \omega \rangle)\} \geq a\pi(2\lambda)^{-1/2} \cos^2 \theta \frac{\omega^2}{\langle \omega \rangle^2}.$$

Now we choose $I' \Subset S^1$ so that $I \Subset I'$ and $I' \cap \{e^{\pm ia\pi(2\lambda)^{-1/2}}\} = \emptyset$, and then choose $f \in C^\infty(\mathbb{T}; \mathbb{R})$ such that $f = 1$ on I and $\text{supp}[f] \subset I'$. Then, by this condition, $a\pi(2\lambda)^{-1/2} \sin \theta \neq \pm a\pi(2\lambda)^{-1/2}$ on the support of $f \circ s_0$, and hence $|\sin \theta| \leq (1 - \varepsilon^2)^{1/2}$ with some $\varepsilon > 0$, i.e., $\cos^2 \theta \geq \varepsilon^2$. Thus we learn

$$i(f \circ s_0(\lambda; \cdot))^2 s_0(\lambda; \cdot)^* \{q, s_0(\lambda; \cdot)\} \geq \varepsilon^2 (f \circ s_0(\lambda; \cdot))^2 \frac{\omega^2}{\langle \omega \rangle^2},$$

and this implies

$$f(S(\lambda))S(\lambda)^*[Q, S(\lambda)]f(S(\lambda)) \geq \varepsilon^2 f(S(\lambda))^2 + K_1(\lambda)$$

with some compact operator $K_1(\lambda)$ on $L^2(S^1)$. Then, multiplying $E_I(S(\lambda))$ from the both sides, we arrive at the assertion.

Then, by the Mourre theory for unitary operators (see, e.g., Fernández-Richard-Tiedra [3]), we have the following result.

Theorem 4.5. *Let H and $S(\lambda)$ be as above, and let $\lambda > 0$. Let Γ be the set of eigenvalues of $S(\lambda)$. Then Γ can accumulate only at $\{e^{\pm ia\pi(2\lambda)^{-1/2}}\}$. For $\xi \in S^1 \setminus \{\Gamma \cup \{e^{\pm ia\pi(2\lambda)^{-1/2}}\}\}$, the limits*

$$\lim_{\varepsilon \downarrow 0} \langle Q \rangle^{-1} (S(\lambda) - (1 \pm \varepsilon)\xi)^{-1} \langle Q \rangle^{-1} = \langle Q \rangle^{-1} (S(\lambda) - (1 \pm 0)\xi)^{-1} \langle Q \rangle^{-1}$$

exist, locally uniformly in $S^1 \setminus \{\Gamma \cup \{e^{\pm ia\pi(2\lambda)^{-1/2}}\}\}$. Hence, in particular, $\sigma_{\text{sc}}(S(\lambda)) = \emptyset$ and $S(\lambda)$ has no singular spectrum on $S^1 \setminus \Gamma$.

Theorem 1.3 follows immediately from the above theorem and Lemma 4.2. □

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Appendix A: Functional Calculus of Unitary Pseudodifferential Operators

In Appendices A and B, we consider pseudodifferential operators on \mathbb{R}^d , but it can be generalized easily to pseudodifferential operators on manifolds. We restrict ourselves to the \mathbb{R}^d case mostly to simplify notations related to Beal’s characterization of pseudodifferential operators.

Let $\delta \in [0, 1)$, and we consider a unitary operator U on L^2 with the symbol $u \in \bigcap_{\delta > 0} S_{1,0}^\delta$. We consider operators on \mathbb{R}^d , or in a local coordinate in a d -dimensional manifold. We show that $f(U)$, the function of U , is a pseudodifferential operator and compute the principal symbol. At first, we note

Lemma A.1. *Suppose $a \in S_{1,0}^1$, and the symbol is bounded. Then $\text{Op}(a)$ is bounded in L^2 .*

Proof. The proof is essentially the same as the Gårding inequality. Without loss of generality, we may suppose a is real valued, and we write $A = \text{Op}(a)$. Let $M > \sup |a|$. We set $b(x, \xi) = (M^2 - a(x, \xi)^2)^{1/2} \in S_{1,0}^1$, and $B = \text{Op}(b)$. Then by the symbol calculus, we learn

$$R = A^*A + B^*B - M^2 \in \text{Op}(S_{1,0}^0).$$

Hence

$$\|Au\|^2 \leq \|Au\|^2 + \|Bu\|^2 \leq M^2\|u\|^2 + \|Ru\|\|u\| \leq C\|u\|^2$$

since R is bounded in L^2 .

Lemma A.2. *Suppose $U = \text{Op}(u)$ is unitary with $u \in S_{1,0}^\delta$, $\delta \in [0, 1)$. Then for any $s \in \mathbb{R}$,*

$$\|U^k\|_{H^s \rightarrow H^s} \leq C_s \langle k \rangle^{|s|/(1-\delta)}, \quad k \in \mathbb{Z}.$$

Proof. We let $\nu = 1 - \delta \in (0, 1]$, $s = N\nu$ and show

$$\|U^k\|_{H^{N\nu} \rightarrow H^{N\nu}} \leq C \langle k \rangle^N, \quad k \in \mathbb{Z}.$$

We first suppose $k > 0$. We consider the commutator:

$$[\langle D_x \rangle^\nu, U^k] = \sum_{j=1}^{d-1} U^j [\langle D_x \rangle^\nu, U] U^{k-1-j}.$$

Since the symbol of the operator $[\langle D_x \rangle^\nu, U]$ is in $S_{1,0}^0$, it is bounded in L^2 , and hence $\|[\langle D_x \rangle^\nu, U^k]\| \leq C \langle k \rangle$. This implies $\|U^k\|_{H^\nu \rightarrow H^\nu} \leq C \langle k \rangle$.

More generally, we compute

$$\begin{aligned} \langle D_x \rangle^{N\nu}, U^k &= \sum_{j=1}^{k-1} U^j \langle D_x \rangle^{N\nu}, U U^{k-1-j} \\ &= \sum_{j=1}^{k-1} \sum_{\ell=0}^{N-1} U^j \langle D_x \rangle^{\ell\nu} [\langle D_x \rangle^\nu, U] \langle D_x \rangle^{(N-1-\ell)\nu} U^{k-1-j}. \end{aligned}$$

Now we use the induction in N . Suppose the claim holds for $N \leq N_0$. Then we have

$$\begin{aligned} & [\langle D_x \rangle^{N_0\nu}, U^k] \langle D_x \rangle^{-N_0\nu} \\ &= \sum_{j=1}^{k-1} \sum_{\ell=0}^{N_0-1} U^j \langle D_x \rangle^{\ell\nu} [\langle D_x \rangle^\nu, U] \langle D_x \rangle^{(N_0-1-\ell)\nu} U^{k-1-j} \langle D_x \rangle^{-N_0\nu} \\ &= \sum_{j=1}^{k-1} \sum_{\ell=0}^{N_0-1} U^j (\langle D_x \rangle^{\ell\nu} [\langle D_x \rangle^\nu, U] \langle D_x \rangle^{-\ell\nu}) \\ &\quad \times (\langle D_x \rangle^{(N_0-1)\nu} U^{k-1-j} \langle D_x \rangle^{-(N_0-1)\nu}) \langle D_x \rangle^{-1}. \end{aligned}$$

By the induction hypothesis and the fact $[\langle D_x \rangle^\nu, U]$ is bounded in $H^{\ell\nu}$, each term in the sum is bounded in L^2 , and the norm is $O(\langle k \rangle^{(N_0-1)\nu})$. By summing up these norms, we arrive at the claim with $N = N_0$. For $k < 0$, we use the same argument for $U^{-1} = U^*$. Then the assertion for general $s \in \mathbb{R}$ follows by the interpolation and the duality argument.

Now we consider functional calculus of a unitary operator U . For $f \in C^\infty(S^1)$, we write the Fourier series expansion by $\hat{f}[k]$, i.e.,

$$\hat{f}[k] = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} f(e^{i\theta}) d\theta, \quad k \in \mathbb{Z},$$

and hence

$$f(e^{i\theta}) = \sum_{k \in \mathbb{Z}} \hat{f}[k] e^{ik\theta}, \quad \theta \in [0, 2\pi).$$

We recall $\hat{f}[n]$ is rapidly decreasing in n . Then we write

$$f(U) = \sum_{k \in \mathbb{Z}} \hat{f}[k] U^k \in \mathcal{B}(L^2).$$

It is well-known that $f(U)$ is the same function of U defined in terms of the spectral decomposition. We show $f(U)$ is a pseudodifferential operator using the Beals characterization of pseudodifferential operators.

For an operator A , we write

$$K_j A = i[D_{x_j}, A], \quad L_j A = -i[x_j, A], \quad j = 1, \dots, d,$$

and multiple commutators by $L^\alpha A, K^\beta A$, etc., for $\alpha, \beta \in \mathbb{Z}_+^d$. We recall $A = \text{Op}(a)$ with $a \in S_{1,0}^\delta$ if and only if $K^\alpha L^\beta A$ is bounded from L^2 to $H^{-\delta+|\beta|}$ for any $\alpha, \beta \in \mathbb{Z}_+^d$ (cf. Dimassi-Sjöstrand [2], Zworski [16]). We compute

$$\begin{aligned} K^\alpha L^\beta (U^k) &= \sum_{\substack{\alpha^1 + \dots + \alpha^N = \alpha, \\ \beta^1 + \dots + \beta^N = \beta, \\ \alpha^j + \beta^j \neq 0, \\ k_1 + \dots + k_{N+1} = k}} U^{k_1} (K^{\alpha^1} L^{\beta^1} U) U^{k_2} (K^{\alpha^2} L^{\beta^2} U) \\ &\quad \times \dots \times U^{k_N} (K^{\alpha^N} L^{\beta^N} U) U^{k_{N+1}}. \end{aligned}$$

Since $K^{\alpha_j} L^{\beta_j} U$ is bounded from H^s to $H^{-\delta+|\beta^j|}$, we have, using Lemma A.2,

$$\|K^{\alpha} L^{\beta}(U^k)\|_{L^2 \rightarrow H^{-N_0\delta+|\beta|}} \leq C \langle k \rangle^{N_1},$$

where $N_0 = |\alpha + \beta|$, $N_1 = (N_0\delta + |\beta|)/(1 - \delta) + N_0$. Thus we learn

$$K^{\alpha} L^{\beta}(f(U)) \in \mathcal{B}(L^2, H^{-|\alpha+\beta|\delta+|\beta|}),$$

and we have the following lemma: We write

$$S_{1,0}^{+0} = \bigcap_{\delta>0} S_{1,0}^{\delta}.$$

Lemma A.3. *Suppose $U = \text{Op}(u)$ is unitary with $u \in S_{1,0}^{+0}$. Then $f(U)$ is a pseudodifferential operator with the symbol in $S_{1,0}^{+0}$.*

We then compute the principal symbol of $f(U)$. If $U = \text{Op}(u)$ is unitary with $u \in S_{1,0}^{\delta}$, then the symbol of $1 = U^*U$ is $1 = |u(x, \xi)|^2$ modulo $S_{1,0}^{\delta-1}$. Thus, we may assume u_0 , the principal symbol of U modulo $S_{1,0}^{\delta-1}$, has modulus 1. This implies, in particular, $u_0^j \in S_{1,\delta}^0$ for any $j \geq 0$. We show $f(U)$ has the principal symbol $f \circ u_0$. We note

$$\begin{aligned} U^k - \text{Op}(u_0^k) &= \sum_{j=0}^{k-1} (U^{j+1} \text{Op}(u_0^{k-j-1}) - U^j \text{Op}(u_0^{k-j})) \\ &= \sum_{j=0}^{k-1} U^j (U - \text{Op}(u_0)) \text{Op}(u_0^{k-j-1}) \\ &\quad - \sum_{j=0}^{k-1} U^j (\text{Op}(u_0^{k-j}) - u_0 \# (u_0^{k-j-1})), \end{aligned}$$

where $a \# b$ denotes the operator composition: $\text{Op}(a \# b) = \text{Op}(a)\text{Op}(b)$. By the symbol calculus, we learn $u_0^{k-j} - u_0 \# (u_0^{k-j-1}) \in S_{1,\delta}^{\delta-1}$, and each seminorm of it is bounded by $C \langle k \rangle^M$ with some $M > 0$. Thus, after direct computations, we learn that $U^k - \text{Op}(u_0^k) \in S_{1,\delta}^{\delta-1}$ and its seminorm is bounded by $C \langle k \rangle^M$ with some M . Hence we have the following claim: We note $\bigcap_{\delta>0} S_{1,0}^{\delta-1} = \bigcap_{\delta>0} S_{1,\delta}^{\delta-1}$.

Theorem A.4. *Suppose $U = \text{Op}(u)$ is unitary with $u \in S_{1,0}^{+0}$, and let u_0 be a principal symbol such that $|u_0(x, \xi)| = 1$. Let $f \in C^\infty(S^1)$. Then $f(U)$ is a pseudodifferential operator with its symbol in $S_{1,0}^{+0}$ and the principal symbol is given by $f \circ u_0$ modulo $S_{1,0}^{\delta-1}$ with any $\delta > 0$.*

Remark A.1. We can actually compute the asymptotic expansion of $f(U)$ in terms of derivatives of $f \circ u$ and derivatives of u . Thus, in particular, the support of these terms is contained in the support of $f \circ u$, and hence the essential support of the symbol of $f(U)$ is contained in the support of $f \circ u$.

Remark A.2. In our application, we consider the case $u \in S(1, \tilde{g})$, i.e., for any $\alpha, \beta \in \mathbb{Z}_+ d$,

$$|\partial_x^\alpha \partial_\xi^\beta u(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{-|\beta|} \langle \log \langle \xi \rangle \rangle^{|\alpha|}.$$

Then we can apply Theorem A.4 to learn $f(U)$ is a pseudodifferential operator with the symbol in $S_{1,0}^{+0}$. Moreover, since the principal symbol is $f \circ u \in S(1, \tilde{g})$, and the remainder is in $S_{1,0}^{-1+\delta}$ for any $\delta > 0$, we actually learn the symbol is in $S(1, \tilde{g})$.

Appendix B: Logarithm of Unitary Pseudodifferential Operators

For notational convenience, we write $\ell(\xi) = \langle \log \langle \xi \rangle \rangle$ for $\xi \in \mathbb{R}^d$. We use the following metrics on $T^*\mathbb{R}^d$:

$$g = dx^2 + \frac{d\xi^2}{\langle \xi \rangle^2}, \quad \tilde{g} = \ell(\xi)^2 dx^2 + \frac{d\xi^2}{\langle \xi \rangle^2}.$$

We recall, $a \in S(m, g)$ if and only if, for any $\alpha, \beta \in \mathbb{Z}^d$, $\exists C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \langle \xi \rangle^{-|\beta|}, \quad x, \xi \in \mathbb{R}^d,$$

and $a \in S(m, \tilde{g})$ if and only if, for any $\alpha, \beta \in \mathbb{Z}^d$, $\exists C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \ell(\xi)^{|\alpha|} \langle \xi \rangle^{-|\beta|}, \quad x, \xi \in \mathbb{R}^d.$$

Assumption E. Let $\psi_0 \in S(\ell(\xi), g)$, real-valued, and $\partial_\xi \psi_0 \in S(\langle \xi \rangle^{-1}, g)$. Let U be a unitary pseudodifferential operator on $L^2(\mathbb{R}^d)$ such that the principal symbol is given by $e^{i\psi_0}$, i.e., $U \in \text{Op}S(1, \tilde{g})$ and $U - \text{Op}(e^{i\psi_0}) \in \text{Op}S(\ell(\xi)/\langle \xi \rangle, \tilde{g})$.

We note $e^{i\psi_0} \in S(1, \tilde{g})$, and natural remainder terms are in the symbol class $S(\ell(\xi)/\langle \xi \rangle, \tilde{g})$.

Theorem B.1. Suppose ψ_0 and U as in Assumption E. Then there is $\psi \in S(\ell(\xi), g)$ such that $U - \exp(i\text{Op}(\psi)) \in \text{Op}S(\langle \xi \rangle^{-\infty}, g)$, and $\psi - \psi_0 \in S(\ell(\xi)/\langle \xi \rangle, \tilde{g})$.

Lemma B.2. Let $\varphi \in S(\ell(\xi), g)$, real-valued, and $\partial_\xi \varphi \in S(\langle \xi \rangle^{-1}, g)$. Then $\text{Op}(\varphi)$ is essentially self-adjoint and $\exp(it\text{Op}(\varphi)) \in \text{Op}S(1, \tilde{g})$, $t \in \mathbb{R}$. Moreover,

$$e^{it\text{Op}(\varphi)} - \text{Op}(e^{it\varphi}) \in \text{Op}S(\ell(\xi)/\langle \xi \rangle, \tilde{g}),$$

and is uniformly bounded for $t \in [0, 1]$.

Proof. The essential self-adjointness of $\text{Op}(\varphi)$ follows by the commutator theorem with an auxiliary operator $N = \langle D_x \rangle$.

In order to show $e^{it\text{Op}(\varphi)} \in \text{Op}S(1, \tilde{g})$, we use Beal’s characterization. Let K_j and L_j ($j = 1, \dots, d$) as in “Appendix A”. We note, by a simple commutator argument as in Appendix A, we can show, for any $k, \ell \in \mathbb{Z}$, $T > 0$,

$$\sup_{|t| \leq T} \left\| \langle D_x \rangle^k \ell \langle D_x \rangle^\ell e^{it\text{Op}(\varphi)} \ell \langle D_x \rangle^{-\ell} \langle D_x \rangle^{-k} \right\|_{L^2 \rightarrow L^2} < \infty.$$

We compute, for example,

$$L_j [e^{it\text{Op}(\varphi)}] = i \int_0^t e^{is\text{Op}(\varphi)} L_j [\text{Op}(\varphi)] e^{i(t-s)\text{Op}(\varphi)} ds.$$

Since $L_j[\text{Op}(\varphi)] = \text{Op}(\partial_{\xi_j}\varphi) \in \text{Op}S(\langle \xi \rangle^{-1}, g)$, we learn $\langle D_x \rangle L_j[e^{it\text{Op}(\varphi)}]$ is bounded in H^s with any $s \in \mathbb{R}$. Similarly, since $K_j[\text{Op}(\varphi)] = \text{Op}(\partial_{x_j}\varphi) \in \text{Op}S(\ell(\xi), g)$, we learn $\ell(D_x)^{-1}K_j[e^{it\text{Op}(\varphi)}]$ is bounded in $H^s, \forall s \in \mathbb{R}$. Iterating this procedure, we learn, for any $\alpha, \beta \in \mathbb{Z}_+^d$,

$$\ell(D_x)^{-|\alpha|}\langle D_x \rangle^{|\beta|}(K^\alpha L^\beta[e^{it\text{Op}(\varphi)}]) : H^s \rightarrow H^s, \text{ bounded,}$$

with any $s \in \mathbb{R}$. By Beal’s characterization, this implies $e^{it\text{Op}(\varphi)} \in \text{Op}S(1, \tilde{g})$, and bounded locally uniformly in t .

Then we show the principal symbol of $e^{it\text{Op}(\varphi)}$ is $e^{it\varphi}$. We have

$$\begin{aligned} e^{it\text{Op}(\varphi)} - \text{Op}(e^{it\varphi}) &= \int_0^t \frac{d}{ds} (e^{is\text{Op}(\varphi)}\text{Op}(e^{i(t-s)\varphi})) ds \\ &= i \int_0^t e^{is\text{Op}(\varphi)} (\text{Op}(\varphi)\text{Op}(e^{i(t-s)\varphi}) - \text{Op}(\varphi e^{i(t-s)\varphi})) ds \\ &\in \text{Op}S(\ell(\xi)/\langle \xi \rangle, \tilde{g}) \end{aligned}$$

by the asymptotic expansion.

In particular, we have

$$Ue^{-i\text{Op}(\psi_0)} - 1 \in \text{Op}S(\ell(\xi)/\langle \xi \rangle, \tilde{g}),$$

and hence there is a real-valued symbol $\psi_1 \in S(\ell(\xi)/\langle \xi \rangle, \tilde{g})$ such that

$$Ue^{-i\text{Op}(\psi_0)} - \text{Op}(e^{i\psi_1}) \in \text{Op}S(\ell(\xi)^2/\langle \xi \rangle^2, \tilde{g}).$$

This implies,

$$Ue^{-i\text{Op}(\psi_0)}e^{-i\text{Op}(\psi_1)} - 1 \in \text{Op}S(\ell(\xi)^2/\langle \xi \rangle^2, \tilde{g}). \tag{4.1}$$

We use the next lemma to rewrite $e^{-i\text{Op}(\psi_0)}e^{-i\text{Op}(\psi_1)}$.

Lemma B.3. *Let $\varphi \in S(\ell(\xi), g)$, real-valued, and $\partial_\xi\varphi \in S(\langle \xi \rangle^{-1}, g)$. Let $\eta \in S(\ell(\xi)^k/\langle \xi \rangle^k, \tilde{g})$, real-valued, with $k \geq 1$. Then*

$$e^{i\text{Op}(\eta)}e^{i\text{Op}(\varphi)} - e^{i\text{Op}(\varphi+\eta)} \in \text{Op}S(\ell(\xi)^{k+1}/\langle \xi \rangle^{k+1}, \tilde{g}).$$

Proof. We have, for any self-adjoint operators A and B , at least formally,

$$\begin{aligned} e^{i(A+B)}e^{-iA}e^{-iB} - 1 &= \int_0^1 \frac{d}{dt} (e^{it(A+B)}e^{-itA}e^{-itB}) dt \\ &= i \int_0^1 (e^{it(A+B)}(A+B-A)e^{-itA}e^{-itB} - e^{-t(A+B)}e^{-itA}Be^{-itB}) dt \\ &= i \int_0^1 e^{it(A+B)}[B, e^{-itA}]e^{-itB} dt \\ &= - \int_0^1 \left(\int_0^t e^{it(A+B)}e^{i(t-s)A}[A, B]e^{-isA}e^{-itB} ds \right) dt. \end{aligned}$$

This computation is easily justified when $A = \text{Op}(\varphi)$ and $B = \text{Op}(\eta)$, and since $[\text{Op}(\varphi), \text{Op}(\eta)] \in \text{Op}S(\ell(\xi)^{k+1}/\langle \xi \rangle^{k+1}, \tilde{g})$, $e^{i\text{Op}(\varphi)} \in \text{Op}S(1, \tilde{g})$, etc., we have

$$e^{i\text{Op}(\varphi+\eta)}e^{-i\text{Op}(\varphi)}e^{-i\text{Op}(\eta)} - 1 \in \text{Op}S(\ell(\xi)^{k+1}/\langle \xi \rangle^{k+1}, \tilde{g}),$$

and this implies the assertion.

Proof of Theorem B.1. Combining (4.1) with lemma B.3, we have

$$Ue^{-i\text{Op}(\psi_0+\psi_1)} - 1 \in \text{Op}S(\ell(\xi)^2/\langle \xi \rangle^2, \tilde{g}).$$

We note $\psi_0 + \psi_1 \in S(\ell(\xi), g) + S(\ell(\xi)^2/\langle \xi \rangle, \tilde{g}) \subset S(1, g)$. Iterating this procedure, we construct $\psi_k \in S(\ell(\xi)^k/\langle \xi \rangle^k, \tilde{g})$, real-valued, such that

$$Ue^{-i\text{Op}(\psi_0+\dots+\psi_k)} - 1 \in \text{Op}S(\ell(\xi)^{k+1}/\langle \xi \rangle^{k+1}, \tilde{g}).$$

for $k = 2, 3, \dots$. Then we choose an asymptotic sum: $\psi \sim \sum_{k=0}^\infty \psi_k$, i.e., $\psi \in S(\ell(\xi), g)$ and

$$\psi - \sum_{k=0}^N \psi_k \in S(\ell(\xi)^{N+1}/\langle \xi \rangle^{N+1}, \tilde{g})$$

for any $N > 0$. Then we have

$$Ue^{-i\text{Op}(\psi)} - 1 \in \text{Op}S(\langle \xi \rangle^{-\infty}, \tilde{g}) = \text{Op}S(\langle \xi \rangle^{-\infty}, g),$$

and we complete the proof of Theorem B.1.

Appendix C: Trace Class Scattering for Unitary Operators

The next theorem, the unitary version of the Kuroda-Birman theorem, seems well-known, but the author could not find an appropriate reference. Here we give a proof for the completeness.

Theorem C.1. *Let U_1 and U_2 be unitary operators on a separable Hilbert space, and suppose $U_1 - U_2$ is a trace class operator. Then $\sigma_{\text{ac}}(U_1) = \sigma_{\text{ac}}(U_2)$.*

Proof. Since the eigenvalues of U_1 and U_2 are at most countable, we can find $\theta \in \mathbb{R}$ such that $e^{-i\theta}$ is not an eigenvalue of both U_1 and U_2 . Then, by replacing U_1 and U_2 by $e^{i\theta}U_1$ and $e^{i\theta}U_2$, respectively, we may suppose 1 is not an eigenvalue of both U_1 and U_2 . Then we can define the Cayley transform of U_1 and U_2 by

$$H_j = i(U_j + 1)(U_j - 1)^{-1}, \quad j = 1, 2.$$

By the definition, we have

$$U_j = (H_j + i)(H_j - i)^{-1} = 1 + 2i(H_j - i)^{-1}, \quad j = 1, 2,$$

and hence

$$(H_1 + i)^{-1} - (H_2 + i)^{-1} = \frac{1}{2i}(U_1 - U_2),$$

is in the trace class. Thus we can apply the Kuroda-Birman theorem ([11], Theorem XI.9) to learn $\sigma_{\text{ac}}(H_1) = \sigma_{\text{ac}}(H_2)$. This implies the assertion since

$$\sigma_{\text{ac}}(U_j) = \{(s - i)(s + i)^{-1} \mid s \in \sigma_{\text{ac}}(H_j)\}, \quad j = 1, 2,$$

by the spectral decomposition theorem.

References

- [1] Dereziński, J., Gérard, C.: *Scattering Theory of Classical and Quantum N-Particle Systems*. Texts and Monographs in Physics. Springer, Berlin (1997)
- [2] Dimassi, M., Sjöstrand, J.: *Spectral asymptotics in the semi-classical limit*. London Mathematical Society Lecture Note Series, **268**. Cambridge Univ. Press, Cambridge, (1999)
- [3] Fernández, C., Richard, S., de Tiedra Aldecoa, R.: Commutator methods for unitary operators. *J. Spectr. Theory* **3**, 271–292 (2013)
- [4] Hörmander, L.: *The Analysis of Linear Partial Differential Operators*. I–IV, Springer, New York, pp. 1983–1985
- [5] Isozaki, H., Kitada, H.: Modified wave operators with time-independent modifiers. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **32**(1), 77–104 (1985)
- [6] Isozaki, H., Kitada, H.: Scattering matrices for two-body Schrödinger operators. *Sci. Papers College Arts Sci. Univ. Tokyo* **35**(2), 81–107 (1985)
- [7] Nakamura, S.: Modified wave operators for discrete Schrödinger operators with long-range perturbations. *J. Math. Phys.* **55** (2014), 112101 (8 pages)
- [8] Nakamura, S.: Microlocal properties of scattering matrices. *Commun. Partial Differ. Equ.* **41**, 894–912 (2016)
- [9] Nakamura, S.: Microlocal resolvent estimates, revisited. *J. Math. Sci. Univ. Tokyo* **24**, 239–257 (2017)
- [10] Nakamura, S.: Long-range scattering matrix for Schrödinger-type operators. Preprint, 2018 <https://arxiv.org/abs/1804.05488>
- [11] Reed, M., Simon, B.: *The Methods of Modern Mathematical Physics*, Volumes I–IV, Academic Press, 1972–1979
- [12] Tadano, Y.: Long-range scattering for discrete Schrödinger operators. *Ann. Henri Poincaré* **20**, 1439–1469 (2019)
- [13] Yafaev, D.R.: The Scattering amplitude for the Schrödinger equation with a long-range potential. *Commun. Math. Phys.* **191**, 183–218 (1998)
- [14] Yafaev, D. R.: *Scattering Theory: Some Old and New Problems*. Springer Lecture Notes in Math. **1735**, 2000
- [15] Yafaev, D. R.: *Mathematical Scattering Theory. Analytic Theory*. Mathematical Surveys and Monographs, 158. American Mathematical Society, Providence, RI, (2010)
- [16] Zworski, M.: *Semiclassical analysis*. Graduate Studies in Mathematics, **138**. American Math. Soc., Providence, RI, (2012)

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