




# Laguerre Ensemble: Correlators, Hurwitz Numbers and Hodge Integrals

Massimo Gisonni, Tamara Grava  and Giulio Ruzza

**Abstract.** We consider the Laguerre partition function and derive explicit generating functions for connected correlators with arbitrary integer powers of traces in terms of products of Hahn polynomials. It was recently proven in Cunden et al. (Ann. Inst. Henri Poincaré D, to appear) that correlators have a topological expansion in terms of weakly or strictly monotone Hurwitz numbers that can be explicitly computed from our formulae. As a second result, we identify the Laguerre partition function with only positive couplings and a special value of the parameter  $\alpha = -1/2$  with the modified GUE partition function, which has recently been introduced in Dubrovin et al. (Hodge-GUE correspondence and the discrete KdV equation. [arXiv:1612.02333](https://arxiv.org/abs/1612.02333)) as a generating function for Hodge integrals. This identification provides a direct and new link between monotone Hurwitz numbers and Hodge integrals.

## 1. Introduction and Results

### 1.1. Laguerre Unitary Ensemble (LUE) and Formulae for Correlators

The LUE is the statistical model on the cone  $H_N^+$  of positive definite Hermitian matrices of size  $N$  endowed with the probability measure

$$\frac{1}{Z_N(\alpha; \mathbf{0})} \det^\alpha X \exp \operatorname{tr}(-X) dX, \quad (1.1)$$

$dX$  being the restriction to  $H_N^+$  of the Lebesgue measure on the space  $H_N \simeq \mathbb{R}^{N^2}$  of Hermitian matrices  $X = X^\dagger$  of size  $N$ ;

$$dX := \prod_{1 \leq i \leq N} dX_{ii} \prod_{1 \leq i < j \leq N} d\operatorname{Re} X_{ij} d\operatorname{Im} X_{ij}. \quad (1.2)$$

The normalizing constant  $Z_N(\alpha; \mathbf{0})$  in (1.1) is computed explicitly as

$$Z_N(\alpha; \mathbf{0}) := \int_{\mathbb{H}_N^+} \det^\alpha X e^{-\text{tr} X} dX = \pi^{\frac{N(N-1)}{2}} \prod_{j=1}^N \Gamma(j + \alpha). \tag{1.3}$$

The parameter  $\alpha$  could be taken as an arbitrary complex number satisfying  $\text{Re} \alpha > -1$ . Writing  $\alpha = M - N$ , a random matrix  $X$  distributed according to the measure (1.1) is called complex *Wishart matrix* with parameter  $M$ ; in particular, when  $M$  is an integer, there is the equality in law  $X = \frac{1}{N} WW^\dagger$ , where  $W$  is an  $N \times M$  random matrix with independent identically distributed Gaussian entries [36].

Our first main result, Theorem 1.1, concerns explicit and effective formulae for *correlators* of the LUE

$$\langle \text{tr} X^{k_1} \cdots \text{tr} X^{k_r} \rangle := \frac{1}{Z_N(\alpha; \mathbf{0})} \int_{\mathbb{H}_N^+} \text{tr} X^{k_1} \cdots \text{tr} X^{k_r} \det^\alpha X \exp \text{tr} (-X) dX$$

for arbitrary nonzero integers  $k_1, \dots, k_r \in \mathbb{Z} \setminus \{0\}$ . Theorem 1.1 is best formulated in terms of *connected correlators*

$$\langle \text{tr} X^{k_1} \cdots \text{tr} X^{k_r} \rangle_c := \sum_{\mathcal{P} \text{ partition of } \{1, \dots, r\}} (-1)^{|\mathcal{P}|-1} (|\mathcal{P}| - 1)! \prod_{I \in \mathcal{P}} \left\langle \prod_{i \in I} \text{tr} X^{k_i} \right\rangle, \tag{1.4}$$

e.g.,

$$\langle \text{tr} X^{k_1} \rangle_c := \langle \text{tr} X^{k_1} \rangle, \quad \langle \text{tr} X^{k_1} \text{tr} X^{k_2} \rangle_c := \langle \text{tr} X^{k_1} \text{tr} X^{k_2} \rangle - \langle \text{tr} X^{k_1} \rangle \langle \text{tr} X^{k_2} \rangle.$$

The generating function for connected correlators

$$\left\langle \text{tr} \left( \frac{1}{x_1 - X} \right) \text{tr} \left( \frac{1}{x_2 - X} \right) \cdots \text{tr} \left( \frac{1}{x_r - X} \right) \right\rangle_c \tag{1.5}$$

can be expanded near  $x_j = \infty$  and/or  $x_j = 0$ , yielding the following generating functions up to some irrelevant terms; for  $r = 1$

$$C_{1,0}(x) := \sum_{k \geq 1} \frac{1}{x^{k+1}} \langle \text{tr} X^k \rangle, \quad C_{0,1}(x) := - \sum_{k \geq 1} x^{k-1} \langle \text{tr} X^{-k} \rangle, \tag{1.6}$$

for  $r = 2$

$$\begin{aligned} C_{2,0}(x_1, x_2) &:= \sum_{k_1, k_2 \geq 1} \frac{\langle \text{tr} X^{k_1} \text{tr} X^{k_2} \rangle_c}{x_1^{k_1+1} x_2^{k_2+1}}, \\ C_{1,1}(x_1, x_2) &:= - \sum_{k_1, k_2 \geq 1} \frac{x_2^{k_2-1}}{x_1^{k_1+1}} \langle \text{tr} X^{k_1} \text{tr} X^{-k_2} \rangle_c, \\ C_{0,2}(x_1, x_2) &:= \sum_{k_1, k_2 \geq 1} x_1^{k_1-1} x_2^{k_2-1} \langle \text{tr} X^{-k_1} \text{tr} X^{-k_2} \rangle_c, \end{aligned} \tag{1.7}$$

and, in general,

$$C_{r_+, r_-}(x_1, \dots, x_r) := (-1)^{r_-} \sum_{k_1, \dots, k_r \geq 1} \frac{\langle \text{tr} X^{\sigma_1 k_1} \cdots \text{tr} X^{\sigma_r k_r} \rangle_c}{x_1^{\sigma_1 k_1+1} \cdots x_r^{\sigma_r k_r+1}}, \tag{1.8}$$

where  $r = r_+ + r_-$  and we introduce the signs

$$\sigma_1 = \dots = \sigma_{r_+} = +, \quad \sigma_{r_++1} = \dots = \sigma_r = - . \tag{1.9}$$

We obtain the following explicit expression for these generating functions for correlators.

**Theorem 1.1.** *Introduce the matrix-valued formal series*

$$R_+(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \begin{pmatrix} \ell A_\ell(N, N+\alpha) & B_\ell(N+1, N+\alpha+1) \\ -N(N+\alpha)B_\ell(N, N+\alpha) & -\ell A_\ell(N, N+\alpha) \end{pmatrix}, \tag{1.10}$$

$$R_-(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell \geq 0} \frac{x^\ell}{(\alpha - \ell)_{2\ell+1}} \begin{pmatrix} (\ell+1)A_\ell(N, N+\alpha) & -B_\ell(N+1, N+\alpha+1) \\ N(N+\alpha)B_\ell(N, N+\alpha) & -(\ell+1)A_\ell(N, N+\alpha) \end{pmatrix}, \tag{1.11}$$

where, using a standard notation  $(p)_j := p(p+1)\dots(p+j-1)$  for the rising factorial,

$$A_\ell(N, M) := \begin{cases} N, & \ell = 0, \\ \frac{1}{\ell} \sum_{j=0}^{\ell-1} (-1)^j \frac{(N-j)_\ell (M-j)_\ell}{j!(\ell-1-j)!}, & \ell \geq 1, \end{cases} \tag{1.12}$$

$$B_\ell(N, M) := \sum_{j=0}^{\ell} (-1)^j \frac{(N-j)_\ell (M-j)_\ell}{j!(\ell-j)!}. \tag{1.13}$$

Then, the generating functions (1.8) can be expressed as

$$\begin{aligned} C_{1,0}(x) &= \frac{1}{x} \int_x^\infty [(R_+(y))_{11} - 1] dy, \\ C_{0,1}(x) &= \frac{1}{x} \int_0^x [1 - (R_-(y))_{11}] dy, \\ C_{2,0}(x_1, x_2) &= \frac{\text{tr}(R_+(x_1)R_+(x_2)) - 1}{(x_1 - x_2)^2}, \\ C_{1,1}(x_1, x_2) &= \frac{\text{tr}(R_+(x_1)R_-(x_2)) - 1}{(x_1 - x_2)^2}, \\ C_{0,2}(x_1, x_2) &= \frac{\text{tr}(R_-(x_1)R_-(x_2)) - 1}{(x_1 - x_2)^2}, \end{aligned} \tag{1.14}$$

and, in general,

$$C_{r_+, r_-}(x_1, \dots, x_r) = - \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \frac{\text{tr}(R_{\sigma_{i_1}}(x_{i_1}) \dots R_{\sigma_{i_r}}(x_{i_r})) - \delta_{r,2}}{(x_{i_1} - x_{i_2}) \dots (x_{i_{r-1}} - x_{i_r})(x_{i_r} - x_{i_1})}, \tag{1.15}$$

where  $r = r_+ + r_- \geq 2$ , the summation extends over the  $r$ -cycles  $(i_1, \dots, i_r)$  in the group of permutations of  $\{1, \dots, r\}$ , and we use the signs  $\sigma_1, \dots, \sigma_r$  defined in (1.9).

The proof is given in Sect. 2.3. Theorem 1.1 generalizes formulae for one-point correlators, since the formulae for the generating series  $C_{1,0}$  and  $C_{0,1}$  boil down to the following identities

$$\langle \text{tr } X^k \rangle = A_k(N, N + \alpha), \quad \langle \text{tr } X^{-k-1} \rangle = \frac{A_k(N, N + \alpha)}{(\alpha - k)_{2k+1}}, \quad k \geq 0, \quad (1.16)$$

which were already derived in the literature [23,44]. From Theorem 1.1, for example, one can deduce compact expressions for correlators of the form

$$\begin{aligned} \langle \text{tr } X^k \text{tr } X \rangle_c &= k A_k(N, N + \alpha), \\ \langle \text{tr } X^{-k} \text{tr } X^{-1} \rangle_c &= \frac{k A_k(N, N + \alpha)}{\alpha(\alpha - k)_{2k+1}}, \\ \langle \text{tr } X^k \text{tr } X^{-1} \rangle_c &= -\frac{k A_{k-1}(N, N + \alpha)}{\alpha}, \\ \langle \text{tr } X^{-k} \text{tr } X \rangle_c &= -\frac{k A_{k-1}(N, N + \alpha)}{(\alpha - k + 1)_{2k-1}}; \end{aligned} \quad (1.17)$$

for the derivation, see Example 2.14. For general positive moments, see also [48].

The entries  $A_\ell(N, M)$ , defined in (1.12), are known to satisfy a three-term recursion [24,42]. We deduce this recursion together with a similar three-term recursion for  $B_\ell(N, M)$ , in Lemma 2.11. It was pointed out in [23] that the entries  $A_\ell(N, M)$  are *hypergeometric orthogonal polynomials* (in particular suitably normalized *Hahn* and *dual Hahn polynomials* [23,50]), a fact which provides another interpretation of the same three-term recursion; this interpretation extends to the entries  $B_\ell(N, M)$ , see Remark 2.12. In Lemma 3.2, we provide an alternative expression for the entries  $A_\ell(N, M), B_\ell(N, M)$ , which makes clear that they are polynomials in  $N, M$  with integer coefficients.

Formulae of the same sort as (1.15) have been considered in [30] for the Gaussian Unitary Ensemble and already appeared in the Topological Recursion literature, see, e.g., [5,6,20,34,35]. Our approach is not directly based on the Matrix Resolvent method [30] or the Topological Recursion [20]; in particular, we provide a self-contained proof to Theorem 1.1 via orthogonal polynomials and their Riemann–Hilbert problem [46].

Insertion of *negative* powers of traces in the correlators and computation of *mixed* correlators are, to the best of our knowledge, novel aspects; as we shall see shortly, these general correlators have expansions with integer coefficients, a fact which generalizes results of, e.g., [22]. It would be interesting to implement this method to other invariant ensembles of random matrices [26,36]. With the aid of the formulae of Theorem 1.1, we have computed several LUE connected correlators which are reported in the tables of “Appendix A.” Moreover, we can make direct use of the formulae of Theorem 1.1 to prove (details in Sect. 3) the following result, concerning the formal structure as large  $N$  asymptotic series of *arbitrary* correlators of the LUE in the scaling

$$\alpha = (c - 1)N, \quad (1.18)$$

corresponding to  $M = cN$  in terms of the Wishart parameter  $M$ .

**Proposition 1.2.** *Arbitrary rescaled LUE correlators admit an asymptotic expansion for  $N \rightarrow \infty$  which is a series in  $N^{-2}$  with coefficients polynomial in  $c$  and  $(c-1)^{-1}$  with integer coefficients. More precisely, for all  $k_1, \dots, k_\ell \in \mathbb{Z} \setminus \{0\}$  we have*

$$N^{-[\ell \bmod 2] - \sum_{i=1}^{\ell} k_i} \langle \text{tr } X^{k_1} \dots \text{tr } X^{k_\ell} \rangle_c \sim \sum_{j \geq 0} \frac{f_j^{(k_1, \dots, k_\ell)}(c)}{N^{2j}}, \quad N \rightarrow \infty, \tag{1.19}$$

where  $f_j^{(k_1, \dots, k_\ell)}(c) \in \mathbb{Z}[c, (c-1)^{-1}]$  for all  $j \geq 0$ ; here, we also denote  $[\ell \bmod 2] \in \{0, 1\}$  the parity of  $\ell$ .

From this result, we infer that when  $c = 2$  (equivalently,  $\alpha = N$ ) the coefficients of this large  $N$  expansion are all integers.

From the tables in ‘‘Appendix A’’ one easily conjectures that actually a stronger version of this result holds true, namely that the asymptotic expansion for  $N^{\ell-2-\sum_{i=1}^{\ell} k_i} \langle \text{tr } X^{k_1} \dots \text{tr } X^{k_\ell} \rangle_c$ . (Note the different power of  $N$ ) as  $N \rightarrow \infty$  is a series in  $\mathbb{Z}[c, (c-1)^{-1}][[N^{-2}]]$ . Such stronger property holds true when all the  $k_j$ ’s have the same sign, see, e.g., [22, 24] and the following section.

**1.2. Topological Expansions and Hurwitz Numbers**

It has been shown in [14, 33] that for matrix models with convex potentials, as in our case, correlators, suitably rescaled by a power of  $N$ , as in (1.19), have a *topological expansion*, by which we mean an asymptotic expansion in non-negative powers of  $N^{-2}$ . As mentioned above, the topological expansion of the LUE correlators in the regime (1.18) was considered in [22, 24] where the connection with Hurwitz numbers was made explicit.

Hurwitz numbers are very important combinatorial quantities, counting factorizations in the symmetric group; they were first studied in the end of the nineteenth century by Hurwitz and there has been a recent renewal of interest in view of the connection with integrable systems and random matrices [45, 52]. The Hurwitz numbers related to this model [22] are a variant of *monotone* Hurwitz numbers [12, 13, 38–40] which can be defined as follows. For  $\mu, \nu$  partitions of the same integer  $d = |\mu| = |\nu|$ , define the *strictly* (resp. *weakly*) *monotone double Hurwitz numbers*  $h_g^>(\mu; \nu)$  (resp.  $h_g^{\geq}(\mu; \nu)$ ) as the number of tuples  $(\alpha, \tau_1, \dots, \tau_r, \beta)$  such that

- (i)  $r = \ell + s + 2g - 2$  where  $\ell$  is the length of  $\mu$  and  $s$  is the length of  $\nu$ ,
- (ii)  $\alpha, \beta$  are permutations of  $\{1, \dots, d\}$  of cycle type  $\mu, \nu$ , respectively, and  $\tau_1, \dots, \tau_r$  are transpositions such that  $\alpha\tau_1 \dots \tau_r = \beta$ ,
- (iii) the subgroup generated by  $\alpha, \tau_1, \dots, \tau_r$  acts transitively on  $\{1, \dots, d\}$ , and
- (iv) writing  $\tau_j = (a_j, b_j)$  with  $a_j < b_j$  we have  $b_1 < \dots < b_r$  (resp.  $b_1 \leq \dots \leq b_r$ ).

**Theorem 1.3** ([22]). *The following asymptotic expansions as  $N \rightarrow \infty$  hold true:*

$$N^{\ell-|\mu|-2} \langle \text{tr } X^{\mu_1} \dots \text{tr } X^{\mu_\ell} \rangle_c = \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{s=1}^{1-2g+|\mu|-\ell} H_g^>(\mu; s)c^s, \tag{1.20}$$

for  $c > 1 - \frac{1}{N}$ , and

$$N^{\ell+|\mu|-2} \langle \text{tr } X^{-\mu_1} \dots \text{tr } X^{-\mu_\ell} \rangle_c = \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{s \geq 1} \frac{H_g^{\geq}(\mu; s)}{(c-1)^{2g-2+|\mu|+\ell+s}}, \tag{1.21}$$

for  $c > 1 + \frac{|\mu|}{N}$ . Here, we denote  $|\mu| := \mu_1 + \dots + \mu_\ell$ , and the coefficients can be expressed as

$$H_g^{\geq}(\mu; s) = \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{\geq}(\mu; \nu), \quad H_g^{\geq}(\mu; s) = \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{\geq}(\mu; \nu), \tag{1.22}$$

where  $z_\mu := \prod_{i \geq 1} (i^{m_i}) m_i!$ ,  $m_i$  being the multiplicity of  $i$  in the partition  $\mu$ .

From the structure of formula (1.21), it is clear that when  $c = 2$  (equivalently,  $\alpha = N$ ) the coefficients in this expansion are all positive integers.

*Remark 1.4.* The type of Hurwitz numbers appearing in expansions (1.20) and (1.21) can also be expressed in terms of the (connected) multiparametric weighted Hurwitz numbers  $\tilde{H}_G^d(\mu)$ , introduced and studied in [3, 10, 41, 45], which depend on a single partition  $\mu$  and are parameterized by a positive integer  $d$  and by a sequence  $g_1, g_2, \dots$  of complex numbers, the latter being encoded in the series  $G(z) = 1 + \sum_{i \geq 1} g_i z^i$ . To make the comparison precise, one has to identify

$$d = 2g - 2 - |\mu| - \ell(\mu) \tag{1.23}$$

and then we have

$$\sum_{s=1}^{1-2g+|\mu|-\ell} H_g^{\geq}(\mu; s) c^s = z_\mu c^{|\mu|-d} \tilde{H}_G^d(\mu), \quad G(z) = (1 + cz)(1 + z), \tag{1.24}$$

$$\sum_{s \geq 1} \frac{H_g^{\geq}(\mu; s)}{(c-1)^s} = \frac{z_\mu}{(c-1)^{|\mu|+d}} \tilde{H}_G^d(\mu), \quad G(z) = \frac{1 + (c-1)z}{1-z}, \tag{1.25}$$

where  $z_\mu := \prod_{i \geq 1} (i^{m_i}) m_i!$ ,  $m_i$  being the multiplicity of  $i$  in the partition  $\mu$ , as above.

### 1.3. Laguerre and Modified GUE Partition Functions and Hodge Integrals

Our arguments in the proof of Theorem 1.1 mainly revolve around the following generating function for correlators

$$Z_N(\alpha; \mathbf{t}_+, \mathbf{t}_-) = \int_{\mathbb{H}_N^+} \det^\alpha X \exp \text{tr} \left( -X + \sum_{k \neq 0} t_k X^k \right) dX, \tag{1.26}$$

which we call *LUE partition function*. Here,  $\mathbf{t}_+ = (t_1, t_2, \dots)$  and  $\mathbf{t}_- = (t_{-1}, t_{-2}, \dots)$  are two independent infinite vectors of times and  $\alpha$  is a complex parameter. For precise analytic details about the definition (1.26), we refer to the beginning of Sect. 2. Eventually, we are interested in the formal expansion as  $t_j \rightarrow 0$ ; more precisely, logarithmic derivatives of the LUE partition

function at  $\mathbf{t}_+ = \mathbf{t}_- = \mathbf{0}$  recover the connected correlators (1.4) as

$$\frac{\partial^\ell \log Z_N(\alpha; \mathbf{t}_+, \mathbf{t}_-)}{\partial t_{k_1} \cdots \partial t_{k_\ell}} \Big|_{\mathbf{t}_+ = \mathbf{t}_- = \mathbf{0}} = \langle \text{tr } X^{k_1} \cdots \text{tr } X^{k_\ell} \rangle_c. \tag{1.27}$$

It is known that  $Z_N(\alpha; \mathbf{t}_+, \mathbf{t}_-)$  is a *Toda lattice* tau function [2, 49] separately in the times  $\mathbf{t}_+$  and  $\mathbf{t}_-$ ; this point is briefly reviewed in Sect. 2.1.2. Our second main result is the identification (Theorem 1.5) of the LUE partition function (1.26) restricted to  $\mathbf{t}_- = \mathbf{0}$  with another type of tau function, the *modified Gaussian Unitary Ensemble (mGUE) partition function*, which has been introduced in [28] as a generating function for Hodge integrals (see below), within the context of the *Hodge-GUE correspondence* [28, 29, 31, 51, 54, 55].

The mGUE partition function  $\tilde{Z}_N(\mathbf{s})$  is defined in [28] starting from the *even GUE* partition function

$$Z_N^{\text{even}}(\mathbf{s}) := \int_{\mathbb{H}_N} \exp \text{tr} \left( -\frac{1}{2} X^2 + \sum_{k \geq 1} s_k X^{2k} \right) dX, \quad \mathbf{s} = (s_1, s_2, \dots), \tag{1.28}$$

which is the classical GUE partition function with couplings to odd powers set to zero. It is well known [46] that (1.28) is a tau function of the *discrete KdV* (also known as *Volterra lattice*) hierarchy, which is a reduction of the Toda lattice hierarchy (see Sect. 2.1.2 for a brief discussion of the Toda lattice hierarchy). As far as only formal dependence on  $N$  and on the times  $\mathbf{s}$  is concerned (see Sect. 4.2 for more details), it is then argued in [28] that the identity

$$\frac{Z_N^{\text{even}}(\mathbf{s})}{(2\pi)^N \text{Vol}(N)} = \tilde{Z}_{N-\frac{1}{2}}(\mathbf{s}) \tilde{Z}_{N+\frac{1}{2}}(\mathbf{s}), \quad \text{Vol}(N) := \frac{\pi^{\frac{N(N-1)}{2}}}{G(N+1)}, \tag{1.29}$$

uniquely defines a function  $\tilde{Z}_N(\mathbf{s})$ , termed mGUE partition function; in (1.29) and throughout this paper,  $G(z)$  is the Barnes  $G$ -function, with the particular evaluation

$$G(N+1) = 1!2! \cdots (N-1)! \tag{1.30}$$

for any integer  $N > 0$ . With respect to the normalizations in [28], we are setting  $\epsilon \equiv 1$  for simplicity; the dependence on  $\epsilon$  can be restored by the scaling  $N = x\epsilon$ . In [29], a new type of tau function for the discrete KdV hierarchy is introduced and the mGUE partition function is identified with a particular tau function of this kind.

We have the following interpretation for the mGUE partition function.

**Theorem 1.5.** *The modified GUE partition function  $\tilde{Z}_N(\mathbf{s})$  in (1.29) is identified with the Laguerre partition function  $Z_N(\alpha; \mathbf{t}_+, \mathbf{t}_-)$  in (1.26) by the relation*

$$\tilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) = C_N Z_N \left( \alpha = -\frac{1}{2}; \mathbf{t}_+, \mathbf{t}_- = \mathbf{0} \right), \tag{1.31}$$

where  $\mathbf{t}_+, \mathbf{s}$  are related by

$$t_k = 2^k s_k \tag{1.32}$$

and  $C_N$  is an explicit constant depending on  $N$  only:

$$C_N = \frac{2^{\frac{N}{2} - \frac{1}{8}} \pi^{\frac{N^2}{2}} \text{G}(\frac{1}{2})}{\text{G}(N + \frac{1}{2})}. \tag{1.33}$$

The proof is given in Sect. 4.3. Identity (1.31) can be recast as the following explicit relation:

$$\tilde{Z}_{2N - \frac{1}{2}}(\mathbf{s}) = \frac{2^{-N^2 + N - \frac{1}{8}} \pi^{\frac{N^2}{2}} \text{G}(\frac{1}{2})}{\text{G}(N + \frac{1}{2})} \int_{H_N^+} \exp \operatorname{tr} \left( -\frac{X}{2} + \sum_{k \geq 1} s_k X^k \right) \frac{dX}{\sqrt{\det X}}, \tag{1.34}$$

which is obtained from (1.31) by a change of variable  $X \mapsto \frac{X}{2}$  in the LUE partition function.

Theorem 1.5 provides a direct and new link (Corollary 1.6) between the monotone Hurwitz numbers in expansion (1.20) and *special cubic Hodge integrals*. To state this result, let us denote  $\overline{\mathcal{M}}_{g,n}$  the Deligne–Mumford moduli space of stable nodal Riemann surfaces,  $\psi_1, \dots, \psi_n \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$  and  $\kappa_j \in H^{2j}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$  ( $j = 1, 2, \dots$ ) the Mumford–Morita–Miller classes, and  $\Lambda(\xi) := 1 + \lambda_1 \xi + \dots + \lambda_g \xi^g$  the Chern polynomial of the Hodge bundle,  $\lambda_i \in H^{2i}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$ . For the definition of these objects, we refer to the literature, see, e.g., [56] and references therein.

**Corollary 1.6.** *For any partition  $\mu = (\mu_1, \dots, \mu_\ell)$  of length  $\ell$ , we have*

$$\sum_{g \geq 0} \epsilon^{2g-2} \mathcal{H}_{g,\mu} = 2^\ell \sum_{\gamma \geq 0} (2\epsilon)^{2\gamma-2} \sum_{s=1}^{1-2\gamma+|\mu|-\ell} \left( \lambda + \frac{\epsilon}{2} \right)^{2-2\gamma+|\mu|-\ell-s} \left( \lambda - \frac{\epsilon}{2} \right)^s H_\gamma^>(\mu; s), \tag{1.35}$$

where

$$\begin{aligned} \mathcal{H}_{g,\mu} := & 2^{g-1} \sum_{m \geq 0} \frac{(\lambda - 1)^m}{m!} \int_{\overline{\mathcal{M}}_{g,\ell+m}} \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \exp\left(-\sum_{d \geq 1} \frac{\kappa_d}{d}\right) \prod_{a=1}^{\ell} \frac{\mu_a \binom{2\mu_a}{\mu_a}}{1 - \mu_a \psi_a} \\ & + \frac{\delta_{g,0} \delta_{\ell,1}}{2} \left( \lambda - \frac{\mu_1}{\mu_1 + 1} \right) \binom{2\mu_1}{\mu_1} + \frac{\delta_{g,0} \delta_{\ell,2}}{2} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \binom{2\mu_1}{\mu_1} \binom{2\mu_2}{\mu_2}. \end{aligned} \tag{1.36}$$

The proof is given in Sect. 4.4. Note that  $\mathcal{H}_{g,\mu}$  in (1.36) is a well-defined formal power series in  $\mathbb{C}[[\lambda - 1]]$ , as for dimensional reasons each coefficient of  $(\lambda - 1)^m$  in (1.36) is a finite sum of intersection numbers of Mumford–Morita–Miller and Hodge classes on the moduli spaces of curves.

Matching coefficients in (1.35), we obtain the following *partial monotone ELSV-like formulae* valid for all partitions  $\mu = (\mu_1, \dots, \mu_\ell)$  of length  $\ell$ :

$$\begin{aligned} \sum_{s \geq 1} H_0^>(\mu; s) = & \frac{1}{2^{\ell-1}} \int_{\overline{\mathcal{M}}_{0,\ell}} \exp\left(-\sum_{d \geq 1} \frac{\kappa_d}{d}\right) \prod_{a=1}^{\ell} \frac{\mu_a \binom{2\mu_a}{\mu_a}}{1 - \mu_a \psi_a} \\ & + \delta_{\ell,1} \frac{1}{\mu_1 + 1} \binom{2\mu_1}{\mu_1} + \delta_{\ell,2} \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} \binom{2\mu_1}{\mu_1} \binom{2\mu_2}{\mu_2} \end{aligned} \tag{1.37}$$



in genus zero (see also Example 4.6) and

$$\sum_{\gamma=0}^g 2^{4\gamma} \sum_{s \geq 1} \left[ \sum_{p \geq 0} (-1)^p \binom{2 - 2\gamma + |\mu| - \ell - s}{p} \binom{s}{2g - 2\gamma - p} \right] H_\gamma^>(\mu; s) \\ = 2^{3g+1-\ell} \int_{\mathcal{M}_{g,\ell}} \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \exp\left(-\sum_{d \geq 1} \frac{\kappa_d}{d}\right) \prod_{a=1}^{\ell} \frac{\mu_a \binom{2\mu_a}{\mu_a}}{1 - \mu_a \psi_a} \quad (1.38)$$

in higher genus  $g \geq 1$ . Note that the left sides of (1.37) and (1.38) are finite sums.

The connection between Hurwitz numbers and Hodge integrals, the so-called ELSV formula, was introduced in [32], by T. Ekedahl, S. Lando, M. Shapiro and A. Vainshtein. Insertion of  $\kappa$  classes in ELSV-type formulae for *monotone* Hurwitz numbers has already been considered in the literature, e.g., in [4] for single monotone Hurwitz numbers and in [13] for *orbifold* monotone Hurwitz numbers.

The relation between Hodge integrals and Hurwitz numbers expressed by Corollary 1.6 is obtained from Theorem 1.5 by re-expanding the topological expansion (1.20). Indeed, fixing  $\alpha = -\frac{1}{2}$  implies that the parameter  $c$  in (1.20) is no longer independent of  $N$  (*soft-edge* limit) but actually scales as  $c = 1 - \frac{1}{2N}$  (*hard-edge* limit). This explains why we cannot derive from the Hodge-GUE correspondence an expression in terms of Hodge integrals for each Hurwitz number in (1.20), but only an expression for a combination of Hurwitz numbers in different genera.

In particular, to obtain the formulae of Corollary 1.6 one has to re-expand the topological expansion (1.20) in  $N$  after the substitution  $c = 1 - \frac{1}{2N}$ ; that the result of this re-expansion, namely the right side of (1.35), involves only even powers of  $\epsilon$  is a consequence of the invariance of positive LUE correlators under the involution  $(N, \alpha) \mapsto (N + \alpha, -\alpha)$ ; this symmetry is described in Lemma 4.2. More concretely, this symmetry implies the symmetry of the positive LUE correlators under the involution  $(N, c) \mapsto (Nc, c^{-1})$  which in view of (1.20) is equivalent to the identity

$$H_g^>(\mu; s) = H_g^>(\mu; 2 - 2g + |\mu| - \ell(\mu) - s). \quad (1.39)$$

The above identity implies that the small  $\epsilon$  expansion on the right-hand side of (1.35) contains only even powers of  $\epsilon$ . It is also possible to check the symmetry (1.39) by purely combinatorial arguments, see Remark 4.3.

*Remark 1.7.* It is known that special cubic Hodge integrals are related to a  $q$ -deformation of the representation theory of the symmetric group [53]; it would be interesting to directly provide a link to the monotone Hurwitz numbers under consideration here.

**Organization of the Paper**

In Sect. 2, we prove Theorem 1.1; a summary of the proof is given in the beginning of that section. In Sect. 3, we analyze the formulae of Theorem 1.1 to prove Proposition 1.2. In Sect. 4, we prove the identification of the mGUE

and LUE partition functions, namely Theorem 1.5; then, we recall the Hodge-GUE correspondence [28] and we deduce Corollary 1.6. Finally, in the tables of “Appendix A” we collect several connected correlators and weighted monotone double Hurwitz numbers, computed applying the formulae of Theorem 1.1.

## 2. Proof of Theorem 1.1

In this section, we prove our first main result, Theorem 1.1. The proof combines two main ingredients: on the one side the interpretation of the matrix integral (1.26) as an *isomonodromic tau function* [9] and on the other side some algebraic manipulations of residue formulae introduced in [7]. More in detail, we first introduce the relevant family of monic orthogonal polynomials and derive a compatible system of (*monodromy-preserving*) ODEs in the parameters  $\mathbf{t}$  (Proposition 2.1); throughout this section, in the interest of lighter notations, we set

$$\mathbf{t} := (\mathbf{t}_+, \mathbf{t}_-) = (\dots, t_{-2}, t_{-1}, t_1, t_2, \dots). \tag{2.1}$$

Such orthogonal polynomials reduce to monic Laguerre polynomials for  $\mathbf{t} = \mathbf{0}$ . With the aid of this system of deformations, we then compute arbitrary derivatives of the LUE partition function (1.26) in terms of formal residues of expressions that do not contain any derivative in  $\mathbf{t}$  (Propositions 2.4, 2.6 and 2.7). Finally, the formulae of Theorem 1.1 are found by evaluation of these residues at  $\mathbf{t} = \mathbf{0}$ ; the latter task is then to compute the asymptotic expansions of Cauchy transforms of Laguerre polynomials at zero and infinity (Propositions 2.9 and 2.13). It is worth stressing at this point that the two formal series  $R_{\pm}$  of (1.10)–(1.11) in Theorem 1.1 are actually asymptotic expansions of the *same* analytic function at two different points.

As a preliminary to the proof, let us comment on definition (1.26) of the LUE partition function. Even though a formal approach is sufficient to make sense of the LUE partition function as a generating function, we shall also regard it as genuine analytic function of the times  $\mathbf{t}$ . In this respect, let us point out that to make strict non-formal sense of (1.26) one can assume that the vector of times is finite, namely that

$$t_k \neq 0 \iff K_- \leq k \leq K_+, \tag{2.2}$$

and then, to ensure convergence of the matrix integral, that  $\text{Re } t_{K_-} < 0$  for  $K_- < 0$  and  $\text{Re } t_{K_+} < \delta_{K_+,1}$  for  $K_+ > 0$ .

Though we have to assume in our computations that we have chosen such an arbitrary truncation of the times, this is inconsequential in establishing the formulae of Theorem 1.1. More precisely, such truncation implies that (1.27) holds true only as long as  $K_+, K_-$  are large enough, and the formal generating functions  $C_{r_+, r_-}$  (as it follows from our arguments, see Sect. 2.3) are manifestly independent of  $K_{\pm}$  and are therefore obtained by a well-defined inductive limit  $K_+ \rightarrow \infty, K_- \rightarrow -\infty$ .

Moreover, in (1.26) the parameter  $\alpha$  has to satisfy  $\text{Re } \alpha > -1$ ; even worse, in (1.27) we have to assume that  $\text{Re } \alpha > -\sum_{i=1}^r k_i - 1$  to enforce convergence

of the matrix integral at  $X = 0$ . This restriction can be lifted, if  $\alpha$  is not an integer, by taking a suitable deformation of the contour of integration. This caveat is crucial to us, as we shall need the formal expansion of the matrix  $R(x)$  at all orders near  $x = 0$ , see (1.11); the coefficients of this expansion are in general ill-defined for integer  $\alpha$  (although truncated expansions are well defined if  $\alpha$  is confined to suitable right half-planes). It is clear how to overcome these issues by the aforementioned analytic continuation; hence, we do not dwell further on this point.

**2.1. Orthogonal Polynomials and Deformation Equations**

**2.1.1. Orthogonal Polynomials.** Let  $\pi_\ell^{(\alpha)}(x; \mathbf{t}) = x^\ell + \dots$  ( $\ell \geq 0$ ) be the family of *monic* orthogonal polynomials, uniquely defined by the property

$$\int_0^{+\infty} \pi_\ell^{(\alpha)}(x; \mathbf{t}) \pi_{\ell'}^{(\alpha)}(x; \mathbf{t}) e^{-V_\alpha(x; \mathbf{t})} dx = \delta_{\ell, \ell'} h_\ell(\mathbf{t}), \quad \ell, \ell' \geq 0, \tag{2.3}$$

where

$$V_\alpha(x; \mathbf{t}) := x - \alpha \log(x) - \sum_{k \neq 0} t_k x^k, \quad x > 0. \tag{2.4}$$

For  $\mathbf{t} = \mathbf{0}$ , they essentially reduce to the *generalized Laguerre polynomials*  $L_\ell^{(\alpha)}(x)$ ; more precisely, denoting  $\pi_\ell^{(\alpha)}(x) := \pi_\ell^{(\alpha)}(x; \mathbf{t} = \mathbf{0})$  we have the identity

$$\pi_\ell^{(\alpha)}(x) := (-1)^\ell \ell! L_\ell^{(\alpha)}(x) = \sum_{j=0}^{\ell} \frac{(-1)^{\ell-j} (\ell - j + 1)_j (j + 1 + \alpha)_{\ell-j}}{j!} x^j, \tag{2.5}$$

for all  $\ell \geq 0$ . Using *Rodrigues formula*

$$\pi_\ell^{(\alpha)}(x) = (-1)^\ell x^{-\alpha} e^x \left( \frac{d^\ell}{dx^\ell} (e^{-x} x^{\alpha+\ell}) \right) \tag{2.6}$$

and integration by parts, we obtain

$$\begin{aligned} \int_0^{+\infty} x^k \pi_\ell^{(\alpha)}(x) e^{-x} x^\alpha dx &= \int_0^{+\infty} \left( \frac{d^\ell}{dx^\ell} x^k \right) e^{-x} x^{\alpha+\ell} dx \\ &= \begin{cases} 0, & k < \ell, \\ \ell! \Gamma(\alpha + \ell + 1), & k = \ell. \end{cases} \end{aligned} \tag{2.7}$$

Hence, the orthogonality property (2.3) for  $\mathbf{t} = \mathbf{0}$  reads as

$$\int_0^{+\infty} \pi_\ell^{(\alpha)}(x) \pi_{\ell'}^{(\alpha)}(x) x^\alpha e^{-x} dx = h_\ell \delta_{\ell, \ell'}, \quad h_\ell = \ell! \Gamma(\alpha + \ell + 1), \tag{2.8}$$

where  $h_\ell = h_\ell(\mathbf{t} = \mathbf{0})$ . For general  $\mathbf{t}$  instead, the monic orthogonal polynomials  $\pi_0^{(\alpha)}(\mathbf{t}), \pi_1^{(\alpha)}(\mathbf{t}), \dots, \pi_{L-1}^{(\alpha)}(\mathbf{t})$  exist whenever the moment matrix

$$(m_{i+j})_{i,j=0}^{L-1}, \quad m_\ell := \int_0^{+\infty} x^\ell e^{-V_\alpha(x; \mathbf{t})} dx, \tag{2.9}$$

is non-degenerate. In the present case, their existence is ensured for real  $\mathbf{t}$  by the fact that the moment matrix  $(m_{i+j})_{i,j=0}^{L-1}$  is positive definite.

By standard computations, we have the following identity

$$Z_N(\alpha; \mathbf{t}) = \frac{\pi^{\frac{N(N-1)}{2}}}{G(N+1)} \prod_{\ell=0}^{N-1} h_\ell(\mathbf{t}), \tag{2.10}$$

where  $h_\ell(\mathbf{t})$  are defined by (2.3).

**2.1.2. Connection with Toda Lattice Hierarchy.** It is well known that the monic orthogonal polynomials  $\pi_\ell^{(\alpha)}(x; \mathbf{t})$  satisfy a three-term recurrence relation

$$x\pi_\ell^{(\alpha)}(x; \mathbf{t}) = \pi_{\ell+1}^{(\alpha)}(x; \mathbf{t}) + v_\ell^\alpha(\mathbf{t})\pi_\ell^{(\alpha)}(x; \mathbf{t}) + w_\ell^\alpha(\mathbf{t})\pi_{\ell-1}^{(\alpha)}(x; \mathbf{t}). \tag{2.11}$$

That is, the orthogonal polynomials are eigenvectors of the second order difference operator

$$(L\psi)_\ell = \psi_{\ell+1} + v_\ell^\alpha\psi_\ell + w_\ell^\alpha\psi_{\ell-1}. \tag{2.12}$$

The corresponding half-infinite tri-diagonal matrix, also denoted  $L = (L_{ij})$ ,  $i, j \geq 0$ , takes the form

$$L = \begin{pmatrix} v_0^\alpha & 1 & 0 & 0 & \cdots \\ w_1^\alpha & v_1^\alpha & 1 & 0 & \cdots \\ 0 & w_2^\alpha & v_2^\alpha & 1 & \cdots \\ 0 & 0 & w_3^\alpha & v_3^\alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{2.13}$$

It is a standard fact that  $L$  and therefore the coefficients  $v_n^\alpha(\mathbf{t})$  and  $w_n^\alpha(\mathbf{t})$  evolve with respect to positive times  $\mathbf{t}_+ = (t_1, t_2, \dots)$ , for any fixed  $\mathbf{t}_- = (t_{-1}, t_{-2}, \dots)$ , according to the *Toda lattice hierarchy* [2, 19, 27, 30, 49]

$$\frac{\partial L}{\partial t_k} = \left[ (L^k)_+, L \right], \tag{2.14}$$

where for any matrix  $P$ ,  $P_+$  denotes the lower triangular part of  $P$ , i.e., the matrix with entries

$$(P_+)_{ij} := \begin{cases} P_{ij} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}, \tag{2.15}$$

where  $P_{ij}$  are the entries of  $P$ . Setting  $\mathbf{t}_- = \mathbf{0}$ , we can also write the initial data of the Toda hierarchy as

$$v_\ell^\alpha(\mathbf{t}_+ = \mathbf{t}_- = \mathbf{0}) = 2\ell + 1 + \alpha, \quad w_\ell^\alpha(\mathbf{t}_+ = \mathbf{t}_- = \mathbf{0}) = \ell(\ell + \alpha) \tag{2.16}$$

that are the recurrence coefficients for the monic generalized Laguerre polynomials (2.5). Moreover, it is well known, see loc. cit., that  $Z_N(\alpha; \mathbf{t}_+, \mathbf{t}_- = \mathbf{0})$  is the Toda lattice tau function corresponding to this solution.

It can be observed that the evolution with respect to the negative times  $\mathbf{t}_- = (t_{-1}, t_{-2}, \dots)$  is also described by a Toda lattice hierarchy and a simple shift in  $\alpha$ . More precisely, we claim that  $Z_N(\alpha - 2N, \mathbf{t}_+ = \mathbf{0}, \mathbf{t}_-)$  is also a Toda lattice tau function, with a *different* initial datum; namely, it is associated with the tri-diagonal matrix  $\tilde{L}$  satisfying the Toda hierarchy

$$\frac{\partial \tilde{L}}{\partial t_{-k}} = \left[ (\tilde{L}^k)_+, \tilde{L} \right] \tag{2.17}$$

constructed as above from the three-term recurrence of monic orthogonal polynomials, this time with respect to the measure

$$\exp\left(-\frac{1}{x} + \sum_{k>0} t_{-k}x^k\right) \frac{dx}{x^\alpha} \tag{2.18}$$

on  $(0, +\infty)$ . To see it, let us rewrite

$$\begin{aligned} Z_N(\alpha; \mathbf{t}_+ = \mathbf{0}, \mathbf{t}_-) &= \int_{\mathbb{H}_N^+} \det^\alpha X \exp \operatorname{tr} \left(-X + \sum_{k<0} t_k X^k\right) dX \\ &= \int_{\mathbb{H}_N^+} \det^{-\alpha} \tilde{X} \exp \operatorname{tr} \left(-\tilde{X}^{-1} + \sum_{k>0} t_{-k} \tilde{X}^k\right) d\left(\tilde{X}^{-1}\right), \end{aligned} \tag{2.19}$$

where we perform the change of variable  $\tilde{X} = X^{-1}$ , which is a diffeomorphism of  $\mathbb{H}_N^+$ . The Lebesgue measure (1.2) can be rewritten (on the full-measure set of semisimple matrices) as

$$dX = dU \prod_{i<j} (x_i - x_j)^2 dx_1 \cdots dx_N, \tag{2.20}$$

where  $dU$  is a suitably normalized Haar measure on  $U(N)/(U(1))^N$  and  $x_1, \dots, x_N$  are the eigenvalues of  $X$ . Therefore, the measure transforms as

$$\begin{aligned} d\tilde{X} &= dU \prod_{i<j} \left(\frac{1}{x_i} - \frac{1}{x_j}\right)^2 d\left(\frac{1}{x_1}\right) \cdots d\left(\frac{1}{x_N}\right) \\ &= \frac{dU}{(x_1 \cdots x_N)^{2N}} \prod_{i<j} (x_i - x_j)^2 dx_1 \cdots dx_N = \frac{dX}{\det^{2N} X}, \end{aligned} \tag{2.21}$$

yielding

$$dX = \frac{d\tilde{X}}{\det^{2N} \tilde{X}}. \tag{2.22}$$

Summarizing, we have

$$Z_N(\alpha; \mathbf{t}_+ = \mathbf{0}, \mathbf{t}_-) = \int_{\mathbb{H}_N^+} \frac{\exp \operatorname{tr} \left(-\tilde{X}^{-1} + \sum_{k>0} t_{-k} \tilde{X}^k\right)}{\det^{\alpha+2N} \tilde{X}} d\tilde{X} \tag{2.23}$$

and the standard arguments of loc. cit. now apply to the matrix integral  $Z_N(\alpha - 2N; \mathbf{t}_+ = \mathbf{0}, \mathbf{t}_-)$  to show that it is indeed the Toda lattice tau function associated with the solution  $\tilde{L}$ .

For our purposes, we need to describe the simultaneous dependence on  $\mathbf{t}_+$  and  $\mathbf{t}_-$ ; this is achieved by the zero-curvature condition (2.34) of the system of compatible ODEs (2.30) which we now turn our attention to.

**2.1.3. Cauchy Transform and Deformation Equations.** Let us denote by

$$\widehat{\pi}_\ell^{(\alpha)}(x; \mathbf{t}) := \frac{1}{2\pi i} \int_0^{+\infty} \pi_\ell^{(\alpha)}(\xi; \mathbf{t}) e^{-V_\alpha(\xi; \mathbf{t})} \frac{d\xi}{\xi - x} \tag{2.24}$$

the *Cauchy transforms* of the orthogonal polynomials  $\pi_\ell^{(\alpha)}(x; \mathbf{t})$ . Then, for fixed  $N$  introduce the following  $2 \times 2$  matrix

$$Y(x; \mathbf{t}) := \begin{pmatrix} \pi_N^{(\alpha)}(x; \mathbf{t}) & \widehat{\pi}_N^{(\alpha)}(x; \mathbf{t}) \\ -\frac{2\pi i}{h_{N-1}(\mathbf{t})} \pi_{N-1}^{(\alpha)}(x; \mathbf{t}) & -\frac{2\pi i}{h_{N-1}(\mathbf{t})} \widehat{\pi}_{N-1}^{(\alpha)}(x; \mathbf{t}) \end{pmatrix}, \tag{2.25}$$

where, for the interest of clarity, we drop the dependence on  $N, \alpha$ . The matrix  $Y(x; \mathbf{t})$  was introduced in the seminal paper [46] to study the general connection between orthogonal polynomials and random matrix models. The rest of this section follows from [46]. Matrix (2.25) solves the following Riemann–Hilbert problem for orthogonal polynomials; it is analytic for  $x \in \mathbb{C} \setminus [0, \infty)$  and continuous up to the boundary  $(0, \infty)$  where it satisfies the *jump condition*

$$Y_+(x; \mathbf{t}) = Y_-(x; \mathbf{t}) \begin{pmatrix} 1 & e^{-V_\alpha(x; \mathbf{t})} \\ 0 & 1 \end{pmatrix}, \quad x \in (0, \infty), \tag{2.26}$$

where  $Y_\pm(x; \mathbf{t}) = \lim_{\epsilon \rightarrow 0_\pm} Y(x \pm i\epsilon; \mathbf{t})$ . Moreover, at the endpoints  $x = \infty, 0$  we have

$$Y(x; \mathbf{t}) \sim (\mathbf{1} + \mathcal{O}(x^{-1})) x^{N\sigma_3}, \quad x \rightarrow \infty, \tag{2.27}$$

$$Y(x; \mathbf{t}) \sim G_0(\mathbf{t}) (\mathbf{1} + \mathcal{O}(x)), \quad x \rightarrow 0, \tag{2.28}$$

within the sector  $0 < \arg x < 2\pi$ ; the matrix  $G_0(\mathbf{t})$  in (2.28) is independent of  $x$  and it is invertible. (Actually, it has unit determinant, as we now explain.)

The jump matrix in (2.26) has unit determinant; hence,  $\det Y(x; \mathbf{t})$  is analytic for all complex  $x$  but possibly for isolated singularities at  $x = 0, \infty$ ; however,  $\det Y(x; \mathbf{t}) \sim 1$  when  $x \rightarrow \infty$ , see (2.27), and is bounded as  $x \rightarrow 0$ , see (2.28); therefore, we conclude by the Liouville theorem that  $\det Y(x; \mathbf{t}) \equiv 1$  identically.

Introduce the  $2 \times 2$  matrix

$$\Psi(x; \mathbf{t}) := Y(x; \mathbf{t}) \exp\left(-V_\alpha(x; \mathbf{t}) \frac{\sigma_3}{2}\right). \tag{2.29}$$

Here, we choose the branch of the logarithm appearing in  $V_\alpha(x; \mathbf{t})$ , see (2.4), analytic for  $x \in \mathbb{C} \setminus [0, \infty)$  satisfying  $\lim_{\epsilon \rightarrow 0_\pm} \log(x + i\epsilon) \in \mathbb{R}$ ; to be consistent with (2.4), we shall identify  $V_\alpha(x; \mathbf{t})$ , without further mention, with  $V_{\alpha,+}(x; \mathbf{t}) = \lim_{\epsilon \rightarrow 0_\pm} V_\alpha(x + i\epsilon; \mathbf{t})$  whenever  $x > 0$ .

Accordingly,  $\Psi(x; \mathbf{t})$  is analytic for  $x \in \mathbb{C} \setminus [0, \infty)$ .

**Proposition 2.1.** *The matrix  $\Psi$  in (2.29) satisfies a compatible system of linear  $2 \times 2$  matrix ODEs with rational coefficients:*

$$\frac{\partial \Psi(x; \mathbf{t})}{\partial x} = \mathcal{A}(x; \mathbf{t}) \Psi(x; \mathbf{t}), \quad \frac{\partial \Psi(x; \mathbf{t})}{\partial t_k} = \Omega_k(x; \mathbf{t}) \Psi(x; \mathbf{t}), \quad k \neq 0. \tag{2.30}$$

In particular, for  $k > 0$ , the matrices  $\Omega_k(x; \mathbf{t})$  are polynomials in  $x$  of degree  $k$ , whilst for  $k < 0$  they are polynomials in  $x^{-1}$  of degree  $|k|$  without constant term; more precisely, they admit the representations

$$\Omega_k(x; \mathbf{t}) = \operatorname{res}_\xi \left( Y(\xi; \mathbf{t}) \frac{\sigma_3}{2} Y^{-1}(\xi; \mathbf{t}) \frac{\xi^k d\xi}{x - \xi} \right), \tag{2.31}$$

where  $\operatorname{res}_\xi$  denotes  $\operatorname{res}_{\xi=\infty}$  when  $k > 0$  and  $\operatorname{res}_{\xi=0}$  when  $k < 0$ . On the other hand,  $\mathcal{A}(x; \mathbf{t})$  is a Laurent polynomial in  $x$ , provided times are truncated according to (2.2).

*Proof.* We note that (2.26) implies the following jump condition for the matrix  $\Psi$ , with a constant jump matrix:

$$\Psi_+(x; \mathbf{t}) = \Psi_-(x; \mathbf{t}) \begin{pmatrix} e^{-i\pi\alpha} & e^{-i\pi\alpha} \\ 0 & e^{i\pi\alpha} \end{pmatrix}, \quad x \in (0, \infty). \tag{2.32}$$

Here,  $\Psi_\pm(x; \mathbf{t}) = \lim_{\epsilon \rightarrow 0_\pm} \Psi(x \pm i\epsilon; \mathbf{t})$ ; to prove this relation, we observe that the branch of the logarithm we are using satisfies  $\log_+(x) = \log_-(x) - 2\pi i$  for  $x \in (0, \infty)$  and so  $V_{\alpha,+}(x; \mathbf{t}) = V_{\alpha,-}(x; \mathbf{t}) + 2i\pi\alpha$ , with a similar notation for the  $\pm$ -boundary values along  $(0, \infty)$ . Hence, all derivatives of  $\Psi$  satisfy the same jump condition, with the same jump matrix. It follows that the ratios  $\mathcal{A} := \frac{\partial \Psi}{\partial x} \Psi^{-1}$  and  $\Omega_k := \frac{\partial \Psi}{\partial t_k} \Psi^{-1}$  (for all  $k \neq 0$ ) are regular along the positive real axis; however, they may have isolated singularities at  $x = 0$  and at  $x = \infty$ . Let us start from  $\Omega_k$  for  $k > 0$ . In such case, it follows from (2.27) and (2.28) that  $\Omega_k$  has a polynomial growth at  $x = \infty$  and it is regular at  $x = 0$ :

$$\begin{aligned} \Omega_k &= \frac{\partial Y(x; \mathbf{t})}{\partial t_k} Y^{-1}(x; \mathbf{t}) + Y(x; \mathbf{t}) \frac{\sigma_3}{2} Y^{-1}(x; \mathbf{t}) x^k \\ &\sim \begin{cases} \frac{\sigma_3}{2} x^k + \mathcal{O}(x^{k-1}), & x \rightarrow \infty \\ \mathcal{O}(1), & x \rightarrow 0. \end{cases} \end{aligned} \tag{2.33}$$

From the Liouville theorem, we conclude that  $\Omega_k$  for  $k > 0$  is a polynomial, which therefore equals the polynomial part of its expansion at  $x = \infty$ , which is computed as in (2.31), since at  $x = \infty$  the term  $\frac{\partial Y}{\partial t_k} Y^{-1} = \mathcal{O}(x^{-1})$  does not contribute to the polynomial part of the expansion. The statement for  $\Omega_k$  for  $k < 0$  follows along similar lines. Likewise,  $\mathcal{A}(x; \mathbf{t})$  in (2.30) has a polynomial growth at  $x = \infty$  and a pole at  $x = 0$  and therefore it is a Laurent polynomial.  $\square$

The compatibility of (2.30) is ensured by the existence of the solution  $\Psi(x; \mathbf{t})$ . In particular, this implies the *zero curvature equations*

$$\frac{\partial \mathcal{A}}{\partial t_k} - \frac{\partial \Omega_k}{\partial x} = [\Omega_k, \mathcal{A}], \quad k \neq 0. \tag{2.34}$$

*Remark 2.2.* Since the determinants of  $Y(x; \mathbf{t})$  and  $\Psi(x; \mathbf{t})$  are identically equal to 1, it follows that  $\Omega_k(x; \mathbf{t})$  and  $\mathcal{A}(x; \mathbf{t})$ , introduced in (2.30), are traceless.

We end this paragraph by considering the restriction  $\mathbf{t} = \mathbf{0}$ . The matrix  $\Psi(x) := \Psi(x; \mathbf{t} = \mathbf{0})$  is obtained from the Laguerre polynomials (2.5). The

matrix  $\mathcal{A}(x) = \frac{\partial \Psi(x)}{\partial x} \Psi(x)^{-1}$  takes the form

$$\mathcal{A}(x) := \mathcal{A}(x; \mathbf{t} = \mathbf{0}) = -\frac{1}{2}\sigma_3 + \frac{1}{x} \begin{pmatrix} N + \frac{\alpha}{2} & -\frac{h_N}{2\pi i} \\ \frac{2\pi i}{h_{N-1}} & -N - \frac{\alpha}{2} \end{pmatrix} \tag{2.35}$$

which has a *Fuchsian* singularity at  $x = 0$  and an *irregular* singularity of *Poincaré rank 1* at  $x = \infty$ .

*Remark 2.3.* The *Frobenius indices* of (2.35) at  $x = 0$  are  $\pm \frac{\alpha}{2}$ , and so the Fuchsian singularity  $x = 0$  is *non-resonant* if and only if  $\alpha$  is not an integer. It is worth pointing out that the monodromy matrix  $\frac{\alpha}{2}\sigma_3$  at  $x = 0$  is preserved under the  $\mathbf{t}$ -deformation (2.30).

**2.2. Residue Formulae for Correlators**

**2.2.1. One-Point Correlators.** The general type of formulae of Proposition 2.4 first appeared in [9], where the authors consider a very general case. Such formulae identify the LUE partition function with the *isomonodromic tau function* [47] of the *monodromy-preserving* deformation system (2.30). The starting point for the following considerations is the representation (2.10) for the LUE partition function (1.26).

**Proposition 2.4.** *Logarithmic derivatives of the LUE partition function admit the following expression in terms of formal residues:*

$$\frac{\partial \log Z_N(\alpha; \mathbf{t})}{\partial t_k} = -\operatorname{res}_x \operatorname{tr} \left( Y^{-1}(x; \mathbf{t}) \frac{\partial Y(x; \mathbf{t})}{\partial x} \frac{\sigma_3}{2} \right) x^k dx, \tag{2.36}$$

where the symbol  $\operatorname{res}_x$  denotes  $\operatorname{res}_{x=\infty}$  when  $k > 0$  and  $\operatorname{res}_{x=0}$  when  $k < 0$ .

*Proof.* For the proof, we follow the lines of [21]. Omitting the dependence on  $x, \mathbf{t}$  for the sake of brevity, we first differentiate the orthogonality relation (2.3)

$$\frac{\partial h_\ell}{\partial t_k} = \int_0^{+\infty} \left( \pi_\ell^{(\alpha)} \right)^2 x^k e^{-V_\alpha} dx \tag{2.37}$$

and recall the *confluent Christoffel–Darboux* formula for orthogonal polynomials

$$\begin{aligned} \sum_{\ell=0}^{N-1} \frac{\left( \pi_\ell^{(\alpha)} \right)^2}{h_\ell} &= \frac{1}{h_{N-1}} \left( \pi_{N-1}^{(\alpha)} \frac{\partial \pi_N^{(\alpha)}}{\partial x} - \frac{\partial \pi_{N-1}^{(\alpha)}}{\partial x} \pi_N^{(\alpha)} \right) \\ &= \frac{1}{2\pi i} \left( Y^{-1} \frac{\partial Y}{\partial x} \right)_{21}, \end{aligned} \tag{2.38}$$

where in the last step one uses  $\det Y(x; \mathbf{t}) \equiv 1$ . It can be checked that the jump relation (2.26) implies

$$\operatorname{tr} \left( Y_+^{-1} \frac{\partial Y_+}{\partial x} \frac{\sigma_3}{2} \right) = \operatorname{tr} \left( Y_-^{-1} \frac{\partial Y_-}{\partial x} \frac{\sigma_3}{2} \right) - \left( Y^{-1} \frac{\partial Y}{\partial x} \right)_{21} e^{-V_\alpha}. \tag{2.39}$$

Therefore, starting from (2.10), we compute

$$\frac{\partial \log Z_N(\alpha; \mathbf{t})}{\partial t_k} = \sum_{\ell=0}^{N-1} \frac{1}{h_\ell} \frac{\partial h_\ell}{\partial t_k} = \sum_{\ell=0}^{N-1} \int_0^{+\infty} \frac{\left( \pi_\ell^{(\alpha)} \right)^2}{h_\ell} x^k e^{-V_\alpha} dx$$



$$= \frac{1}{2\pi i} \int_0^{+\infty} \text{tr} \left[ \left( Y_-^{-1} \frac{\partial Y_-}{\partial x} - Y_+^{-1} \frac{\partial Y_+}{\partial x} \right) \frac{\sigma_3}{2} \right] x^k dx. \tag{2.40}$$

Such an integral of a jump can be performed by a residue computation. First of all, note that although  $x^k \text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \frac{\sigma_3}{2} \right)$  is not analytic at  $x = \infty$ , it has a large  $x$  asymptotic expansion given by

$$x^k \text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \frac{\sigma_3}{2} \right) = \sum_{j=-1}^{k-2} c_j x^j + \mathcal{O} \left( \frac{1}{x^2} \right) \tag{2.41}$$

for any  $k \in \mathbb{N}$ , where  $-c_{-1}$  is, by definition, the *formal* residue at infinity of  $x^k \text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \frac{\sigma_3}{2} \right)$ . Then, recalling our choice for the branch of the logarithm and using contour deformation, we can express (2.40) as

$$\frac{\partial \log Z_N(\alpha; \mathbf{t})}{\partial t_k} = - \text{res}_{x=0} \text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \frac{\sigma_3}{2} \right) x^k dx - \text{res}_{x=\infty} \text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \frac{\sigma_3}{2} \right) x^k dx, \tag{2.42}$$

the residues being intended in the formal sense explained above. Finally, the proof is complete by noting that for  $k > 0$  (resp.  $k < 0$ ) the formal residue at  $x = 0$  (resp.  $x = \infty$ ) vanishes.  $\square$

For later convenience, let us slightly rewrite the result of the above proposition. To this end, introduce the matrix

$$R(x; \mathbf{t}) := Y(x; \mathbf{t}) E_{11} Y^{-1}(x; \mathbf{t}), \tag{2.43}$$

denoting  $E_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$  from now on.

**Corollary 2.5.** *We have*

$$\frac{\partial \log Z_N(\alpha; \mathbf{t})}{\partial t_k} = - \text{res}_x \left( \text{tr} (\mathcal{A}(x; \mathbf{t}) R(x; \mathbf{t})) + \frac{1}{2} \frac{\partial}{\partial x} V_\alpha(x; \mathbf{t}) \right) x^k dx, \tag{2.44}$$

where  $R(x; \mathbf{t})$  is introduced in (2.43) and again  $\text{res}_x$  denotes  $\text{res}_{x=\infty}$  when  $k > 0$  and  $\text{res}_{x=0}$  when  $k < 0$ .

*Proof.* We have from (2.29) and (2.30)

$$\frac{\partial}{\partial x} Y = \mathcal{A}Y + Y \frac{\sigma_3}{2} \frac{\partial}{\partial x} V_\alpha \tag{2.45}$$

(omitting again the dependence on  $x, \mathbf{t}$  for the sake of clarity) so that

$$\text{tr} \left( Y^{-1} \frac{\partial Y}{\partial x} \frac{\sigma_3}{2} \right) = \text{tr} \left( Y^{-1} \mathcal{A}Y \frac{\sigma_3}{2} \right) + \frac{1}{2} \frac{\partial}{\partial x} V_\alpha = \text{tr} (\mathcal{A}R) + \frac{1}{2} \frac{\partial}{\partial x} V_\alpha, \tag{2.46}$$

where in the last step we have used that

$$\text{tr} \left( Y^{-1} \mathcal{A}Y \frac{\sigma_3}{2} \right) = \text{tr} (Y^{-1} \mathcal{A}Y E_{11}) = \text{tr} (\mathcal{A}R), \tag{2.47}$$

where the first equality follows from  $\text{tr} \mathcal{A} = 0$  and the second one from the cyclic property of the trace and definition (2.43).  $\square$

**2.2.2. Multipoint Connected Correlators.** We first consider two-point connected correlators.

**Proposition 2.6.** *For every nonzero integers  $k_1, k_2$ , we have*

$$\frac{\partial^2 \log Z_N(\alpha; \mathbf{t})}{\partial t_{k_2} \partial t_{k_1}} = \operatorname{res}_{x_1} \operatorname{res}_{x_2} \frac{\operatorname{tr}(R(x_1; \mathbf{t})R(x_2; \mathbf{t})) - 1}{(x_1 - x_2)^2} x_1^{k_1} x_2^{k_2} dx_1 dx_2, \tag{2.48}$$

where the symbol  $\operatorname{res}_{x_i}$  denotes  $\operatorname{res}_{x_i=\infty}$  (resp.  $\operatorname{res}_{x_i=0}$ ) if  $k_i > 0$  (resp.  $k_i < 0$ ).

*Proof.* From (2.44), we have

$$\frac{\partial \log Z_N(\alpha; \mathbf{t})}{\partial t_{k_1}} = - \operatorname{res}_{x_1} \left( \operatorname{tr}(\mathcal{A}(x_1; \mathbf{t})R(x_1; \mathbf{t})) + \frac{1}{2} \frac{\partial}{\partial x_1} V_\alpha(x_1; \mathbf{t}) \right) x_1^k dx_1. \tag{2.49}$$

Let us take one more time derivative

$$\begin{aligned} & \frac{\partial^2 \log Z_N(\alpha; \mathbf{t})}{\partial t_{k_2} \partial t_{k_1}} \\ &= - \operatorname{res}_{x_1} \left( \operatorname{tr} \left( \frac{\partial \mathcal{A}(x_1; \mathbf{t})}{\partial t_{k_2}} R(x_1; \mathbf{t}) + \frac{\partial R(x_1; \mathbf{t})}{\partial t_{k_2}} \mathcal{A}(x_1; \mathbf{t}) \right) - \frac{1}{2} k_2 x_1^{k_2-1} \right) x_1^{k_1} dx_1, \end{aligned} \tag{2.50}$$

and note that using (2.34) and  $\partial_{t_k} R(x; \mathbf{t}) = [\Omega_k(x; \mathbf{t}), R(x; \mathbf{t})]$

$$\operatorname{tr} \left( \frac{\partial \mathcal{A}(x_1; \mathbf{t})}{\partial t_{k_2}} R(x_1; \mathbf{t}) + \frac{\partial R(x_1; \mathbf{t})}{\partial t_{k_2}} \mathcal{A}(x_1; \mathbf{t}) \right) = \operatorname{tr} \left( \frac{\partial \Omega_{k_2}(x_1; \mathbf{t})}{\partial x_1} R(x_1; \mathbf{t}) \right). \tag{2.51}$$

Now let us write  $\Omega_{k_2}$  from (2.31) as

$$\begin{aligned} \Omega_{k_2}(x_1; \mathbf{t}) &= \operatorname{res}_{x_2} \left( Y(x_2; \mathbf{t}) \frac{\sigma_3}{2} Y^{-1}(x_2; \mathbf{t}) \frac{x_2^{k_2}}{x_1 - x_2} \right) dx_2 \\ &= \operatorname{res}_{x_2} R(x_2; \mathbf{t}) \frac{x_2^{k_2} dx_2}{x_1 - x_2} - \frac{1}{2} \operatorname{res}_{x_2} \frac{x_2^{k_2} dx_2}{x_1 - x_2}, \end{aligned} \tag{2.52}$$

yielding

$$\frac{\partial \Omega_{k_2}(x_1; \mathbf{t})}{\partial x_1} = - \operatorname{res}_{x_2} R(x_2; \mathbf{t}) \frac{x_2^{k_2} dx_2}{(x_1 - x_2)^2} + \frac{1}{2} \operatorname{res}_{x_2} \frac{x_2^{k_2} dx_2}{(x_1 - x_2)^2}. \tag{2.53}$$

Finally, the identity

$$-k_2 x_1^{k_2-1} = \operatorname{res}_{x_2} \frac{x_2^{k_2}}{(x_1 - x_2)^2} dx_2 \tag{2.54}$$

holds true irrespectively of the sign of  $k_2$ , and the proof is completed by inserting (2.51), (2.53) and (2.54) in (2.50), along with  $\operatorname{tr} R(x; \mathbf{t}) \equiv 1$ .  $\square$

To compute higher-order logarithmic derivatives of the LUE partition function, let us introduce the functions

$$S_r(x_1, \dots, x_r; \mathbf{t}) := - \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \frac{\operatorname{tr}(R(x_{i_1}; \mathbf{t}) \cdots R(x_{i_r}; \mathbf{t})) - \delta_{r,2}}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-1}} - x_{i_r})(x_{i_r} - x_{i_1})}, \tag{2.55}$$

where, as explained in the statement of Theorem 1.1, the sum extends over cyclic permutations of  $\{1, \dots, r\}$ . Due to the cyclic invariance of the trace and of the polynomial  $(x_1 - x_2) \cdots (x_r - x_1)$ , it follows that  $S_r(x_1, \dots, x_r; \mathbf{t})$  is symmetric in  $x_1, \dots, x_r$ .

The following proof is reported for the sake of completeness; it has appeared in the literature several times, e.g., see [7, 8, 11, 30]. The only slight difference here is that we consider two different set of times and correspondingly the residues are taken at two different points.

**Proposition 2.7.** *For every  $r \geq 2$ , we have*

$$\frac{\partial^r \log Z_N(\alpha; \mathbf{t})}{\partial t_{k_r} \cdots \partial t_{k_1}} = (-1)^r \operatorname{res}_{x_1} \cdots \operatorname{res}_{x_r} S_r(x_1, \dots, x_r; \mathbf{t}) x_1^{k_1} \cdots x_r^{k_r} dx_1 \cdots dx_r, \tag{2.56}$$

where, as above, the symbol  $\operatorname{res}_{x_i}$  denotes  $\operatorname{res}_{x_i=\infty}$  (resp.  $\operatorname{res}_{x_i=0}$ ) if  $k_i > 0$  (resp.  $k_i < 0$ ).

*Proof.* We have

$$\frac{\partial}{\partial t_k} R(x; \mathbf{t}) = [\Omega_k(x; \mathbf{t}), R(x; \mathbf{t})] = \operatorname{res}_{\xi} \frac{[R(\xi; \mathbf{t}), R(x; \mathbf{t})]}{x - \xi} \xi^k d\xi, \tag{2.57}$$

where we have used (2.31) and  $\operatorname{res}_{\xi}$  denotes the formal residue at  $\xi = \infty$  if  $k > 0$  or the formal residue at  $\xi = 0$  if  $k < 0$ . Hence, we compute

$$\begin{aligned} & \frac{\partial S_r(x_1, \dots, x_r; \mathbf{t})}{\partial t_k} \\ &= - \operatorname{res}_{\xi} \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \sum_{j=1}^r \frac{\operatorname{tr} (R(x_{i_1}; \mathbf{t}) \cdots [R(\xi; \mathbf{t}), R(x_{i_j}; \mathbf{t})] \cdots R(x_{i_r}; \mathbf{t}))}{(x_{i_1} - x_{i_2}) \cdots (x_{i_r} - x_{i_1})(x_{i_j} - \xi)} \xi^k d\xi. \end{aligned} \tag{2.58}$$

Expanding  $[R(\xi; \mathbf{t}), R(x_{i_j}; \mathbf{t})] = R(\xi; \mathbf{t})R(x_{i_j}; \mathbf{t}) - R(x_{i_j}; \mathbf{t})R(\xi; \mathbf{t})$ , we note that each term involving the expression

$$\operatorname{tr} (R(x_{i_1}; \mathbf{t}) \cdots R(\xi; \mathbf{t})R(x_{i_j}; \mathbf{t}) \cdots R(x_{i_r}; \mathbf{t})) \tag{2.59}$$

appears twice, but with different denominators; collecting these terms gives

$$\begin{aligned} & - \operatorname{res}_{\xi} \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \sum_{j=1}^r \frac{\operatorname{tr} (R(x_{i_1}; \mathbf{t}) \cdots R(\xi; \mathbf{t})R(x_{i_j}; \mathbf{t}) \cdots R(x_{i_r}; \mathbf{t}))}{(x_{i_1} - x_{i_2}) \cdots (x_{i_r} - x_{i_1})} \\ & \times \left( \frac{1}{x_{i_j} - \xi} - \frac{1}{x_{i_{j-1}} - \xi} \right) \xi^k d\xi \end{aligned}$$

$$\begin{aligned}
 &= \operatorname{res}_\xi \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \sum_{j=1}^r \frac{\operatorname{tr} (R(x_{i_1}; \mathbf{t}) \cdots R(\xi; \mathbf{t}) R(x_{i_j}; \mathbf{t}) \cdots R(x_{i_r}; \mathbf{t}))}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{j-1}} - \xi)(\xi - x_{i_j}) \cdots (x_{i_r} - x_{i_1})} \xi^k d\xi \\
 &= - \operatorname{res}_\xi S_{r+1}(x_1, \dots, x_r, \xi) \xi^k d\xi,
 \end{aligned} \tag{2.60}$$

where the index  $j$  in the internal summation is taken mod  $r$ , namely  $i_0 := i_r$ . Summarizing, we have shown that for all  $r \geq 2$

$$\frac{\partial S_r(x_1, \dots, x_r; \mathbf{t})}{\partial t_k} = - \operatorname{res}_\xi S_{r+1}(x_1, \dots, x_r, \xi) \xi^k d\xi \tag{2.61}$$

and the proof now follows by induction on  $r \geq 2$ , the base  $r = 2$  being established in Proposition 2.6.  $\square$

*Remark 2.8.* The functions  $S_r(x_1, \dots, x_r)$  are regular along the diagonals  $x_i = x_j$ . In the case  $r = 2$ , this can be seen from the fact that

$$\operatorname{tr} (R^2(x; \mathbf{t})) \equiv 1, \tag{2.62}$$

and hence, the function  $\operatorname{tr} (R(x_1; \mathbf{t}) R(x_2; \mathbf{t})) - 1$  is symmetric in  $x_1$  and  $x_2$  and vanishes for  $x_1 = x_2$ . Therefore, the zero on the diagonal  $x_1 = x_2$  is of order at least 2 and so  $S_2(x_1, x_2)$  is regular at  $x_1 = x_2$ . For  $r \geq 3$ , instead we can reason as follows: since  $S_r$  is symmetric, we can focus on the case  $x_{r-1} = x_r$ , and the only addends in  $S_r$  which are singular at  $x_{r-1} = x_r$  are those coming from the  $r$ -cycles  $(i_1, \dots, i_{r-2}, r-1, r)$  and  $(i_1, \dots, i_{r-2}, r, r-1)$ , namely the terms

$$\begin{aligned}
 &\sum_{(i_1, \dots, i_{r-2}, r-1, r)} \frac{\operatorname{tr} (R(x_{i_1}; \mathbf{t}) \cdots R(x_{i_{r-2}}; \mathbf{t}) R(x_{r-1}; \mathbf{t}) R(x_r; \mathbf{t}))}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-2}} - x_{r-1})(x_{r-1} - x_r)(x_r - x_{i_1})} \\
 &+ \sum_{(i_1, \dots, i_{r-2}, r, r-1)} \frac{\operatorname{tr} (R(x_{i_1}; \mathbf{t}) \cdots R(x_{i_{r-2}}; \mathbf{t}) R(x_r; \mathbf{t}) R(x_{r-1}; \mathbf{t}))}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-2}} - x_r)(x_r - x_{r-1})(x_{r-1} - x_{i_1})},
 \end{aligned} \tag{2.63}$$

and this expression is manifestly regular at  $x_{r-1} = x_r$ .

In particular, the order in which residues are carried out in (2.56) is immaterial.

Finally, we remark that it would be interesting to extend the above formulation to other matrix ensembles like the GOE, see, e.g., [26].

### 2.3. Asymptotic Expansions and Proof of Theorem 1.1

To compute LUE correlators, we have to set  $\mathbf{t} = \mathbf{0}$  in the residue formulae of Corollary 2.5 and of Propositions 2.6 and 2.7. To this end, we now consider

$$R(x) := R(x; \mathbf{t} = \mathbf{0}), \tag{2.64}$$

where  $R(x; \mathbf{t})$  is introduced in (2.43) and compute explicitly series expansions as  $x \rightarrow \infty, 0$ . We start with the expansion as  $x \rightarrow \infty$ .

**Proposition 2.9.** *The matrix  $R(x)$  admits the asymptotic expansion*

$$TR(x)T^{-1} \sim R_+(x), \quad x \rightarrow \infty, \tag{2.65}$$

uniformly within the sector  $0 < \arg x < 2\pi$ . Here,  $R_+$  is the formal series introduced in the beginning of this paper, see (1.10), and  $T$  is defined as

$$T := \begin{pmatrix} 1 & 0 \\ 0 & \frac{h_N}{2\pi i} \end{pmatrix} \tag{2.66}$$

where  $h_N = N!\Gamma(N + \alpha + 1)$  as in (2.8).

*Remark 2.10.* The matrix  $T$  is independent of  $x$  and is introduced for convenience as it simplifies the coefficients in the expansions. This simplification does not affect the residue formulae of the previous paragraph, as it involves a constant conjugation of  $R(x)$ .

*Proof.* First off, we recall that

$$Y(x) := Y(x; \mathbf{t} = \mathbf{0}) = \begin{pmatrix} \pi_N^{(\alpha)}(x) & \widehat{\pi}_N^{(\alpha)}(x) \\ -\frac{2\pi i}{h_{N-1}} \pi_{N-1}^{(\alpha)}(x) & -\frac{2\pi i}{h_{N-1}} \widehat{\pi}_{N-1}^{(\alpha)}(x) \end{pmatrix}, \tag{2.67}$$

where the polynomials  $\pi_\ell^{(\alpha)}(x)$  and their Cauchy transforms  $\widehat{\pi}_\ell^{(\alpha)}(x)$  have been given in (2.5) and (2.24), respectively, while  $h_\ell$  is in (2.8). We can expand  $\widehat{\pi}_\ell^{(\alpha)}$  as  $x \rightarrow \infty$  as

$$\begin{aligned} \widehat{\pi}_\ell^{(\alpha)}(x) &= \frac{1}{2\pi i} \int_0^{+\infty} \pi_\ell^{(\alpha)}(\xi) \xi^\alpha e^{-\xi} \frac{d\xi}{\xi - x} \\ &\sim -\frac{1}{2\pi i} \sum_{j \geq 0} \frac{1}{x^{j+1}} \int_0^{+\infty} \pi_\ell^{(\alpha)}(\xi) \xi^{\alpha+j} e^{-\xi} d\xi \\ &= -\frac{1}{2\pi i} \sum_{j \geq 0} \frac{1}{x^{j+\ell+1}} \int_0^{+\infty} \pi_\ell^{(\alpha)}(\xi) \xi^{\alpha+j+\ell} e^{-\xi} d\xi \\ &= -\frac{1}{2\pi i} \sum_{j \geq 0} \frac{1}{x^{j+\ell+1}} \int_0^{+\infty} (-1)^\ell \left( \frac{d^\ell}{d\xi^\ell} (e^{-\xi} \xi^{\alpha+\ell}) \right) \xi^{j+\ell} d\xi \\ &= -\frac{1}{2\pi i} \sum_{j \geq 0} \frac{1}{x^{j+\ell+1}} \int_0^{+\infty} \left( \frac{d^\ell}{d\xi^\ell} \xi^{j+\ell} \right) \xi^{\alpha+\ell} e^{-\xi} d\xi \\ &= -\frac{1}{2\pi i} \sum_{j \geq 0} \frac{(j+1)_\ell \Gamma(j+\ell+1+\alpha)}{x^{\ell+j+1}}, \end{aligned} \tag{2.68}$$

where we have used the orthogonality property to shift the sum in the first place, then Rodrigues formula (2.6) and integration by parts. The expansion (2.68) is formal; however, it has an analytic meaning of asymptotic expansion as  $x \rightarrow \infty$ . Indeed, for any  $J \geq 0$  the difference between the Cauchy transform and its truncated formal expansion is

$$\begin{aligned} \widehat{\pi}_\ell^{(\alpha)}(x) + \frac{1}{2\pi i} \sum_{j=0}^{J-1} \frac{1}{x^{j+1}} \int_0^{+\infty} \pi_\ell^{(\alpha)}(\xi) \xi^{\alpha+j} e^{-\xi} d\xi \\ = \frac{1}{2\pi i x^J} \int_0^{+\infty} \pi_\ell^{(\alpha)}(\xi) \xi^{\alpha+J} e^{-\xi} \frac{d\xi}{\xi - x} = \mathcal{O}\left(\frac{1}{x^{J+1}}\right), \end{aligned} \tag{2.69}$$

where the last step holds as  $x \rightarrow \infty$ , uniformly in  $\mathbb{C} \setminus [0, +\infty)$ . Hence, using (2.5) and (2.68),

$$Y(x) \sim \sum_{j \geq 0} \frac{1}{j! x^j} \begin{pmatrix} (-1)^j (N-j+1+\alpha)_j (N-j+1)_j & -\frac{h_{N-1}}{2\pi i x} (N+\alpha)_{j+1} (N)_{j+1} \\ -\frac{2\pi i}{h_{N-1} x} (-1)^j (N-j+\alpha)_j (N-j)_j & (N+\alpha)_j (N)_j \end{pmatrix} x^{N\sigma_3} \tag{2.70}$$

as  $x \rightarrow \infty$  within the sector  $0 < \arg x < 2\pi$ . Since  $\det Y(x) \equiv 1$ , we have

$$\begin{aligned} TR(x)T^{-1} &= TY(x)E_{11}Y^{-1}(x)T^{-1} \\ &= \begin{pmatrix} 1 + Y_{21}(x)Y_{12}(x) - \frac{2\pi i}{h_N} Y_{11}(x)Y_{12}(x) \\ \frac{h_N}{2\pi i} Y_{21}(x)Y_{22}(x) & -Y_{21}(x)Y_{12}(x) \end{pmatrix}, \end{aligned} \tag{2.71}$$

from which the expansion at  $x = \infty$  can be computed as follows. For the  $(1, 1)$ -entry we have

$$Y_{21}(x)Y_{12}(x) \sim \sum_{\ell \geq 0} \frac{1}{x^{\ell+2}} \sum_{j=0}^{\ell} \frac{(-1)^j (N+\alpha)_{\ell-j+1} (N)_{\ell-j+1} (N-j+\alpha)_j (N-j)_j}{j!(\ell-j)!}, \tag{2.72}$$

and noting a trivial simplification of rising factorials

$$(N+\alpha)_{\ell-j+1} (N-j+\alpha)_j = (N-j+\alpha)_{\ell+1}, \quad (N)_{\ell-j+1} (N-j)_j = (N-j)_{\ell+1}, \tag{2.73}$$

it follows that as  $x \rightarrow \infty$

$$\begin{aligned} (TR(x)T^{-1})_{11} &\sim 1 + \sum_{\ell \geq 0} \frac{1}{x^{\ell+2}} (\ell+1) A_{\ell+1}(N, N+\alpha) \\ &= 1 + \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \ell A_\ell(N, N+\alpha) = (R_+)_{11}(x), \end{aligned} \tag{2.74}$$

with  $A_\ell(N, M)$  as in (1.12). In a similar way, we compute the  $(1, 2)$ -entry

$$\begin{aligned} -\frac{2\pi i}{h_N} Y_{11}(x)Y_{12}(x) &\sim \frac{1}{N(N+\alpha)} \\ &\times \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \sum_{j=0}^{\ell} (-1)^j \frac{(N-j+1+\alpha)_j (N-j+1)_j (N+\alpha)_{\ell-j+1} (N)_{\ell-j+1}}{j!(\ell-j)!} \\ &= \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \sum_{j=0}^{\ell} (-1)^j \frac{(N-j+1+\alpha)_\ell (N-j+1)_\ell}{j!(\ell-j)!}, \end{aligned} \tag{2.75}$$

where in the second relation we use a similar version of (2.73), and therefore, from the above relation and (2.71) we conclude that

$$(TR(x)T^{-1})_{12} \sim \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} B_\ell(N+1, N+1+\alpha) = (R_+)_{12}(x), \tag{2.76}$$

with  $B_\ell(N, M)$  as in (1.13). Finally, the  $(2, 1)$ -entry of the expansion of  $TR(x)T^{-1}$  is computed in a similar way as

$$\begin{aligned} & \frac{2\pi i}{h_N} Y_{21}(x) Y_{22}(x) \\ & \sim -N(N + \alpha) \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \sum_{j=0}^{\ell} (-1)^j \frac{(N - j + \alpha)_j (N - j)_j (N + \alpha)_{\ell-j} (N)_{\ell-j}}{j!(\ell - j)!} \\ & = -N(N + \alpha) \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \sum_{j=0}^{\ell} (-1)^j \frac{(N - j + \alpha)_\ell (N - j)_\ell}{j!(\ell - j)!} \\ & = -N(N + \alpha) \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} B_\ell(N, N + \alpha), \end{aligned} \tag{2.77}$$

and the proof is complete. □

Let us note a recurrence property of the coefficients  $A_\ell(N, M)$  and  $B_\ell(N, M)$ .

**Lemma 2.11.** *The entries  $A_\ell(N, M)$  and  $B_\ell(N, M)$  ( $\ell \geq 0$ ), defined in (1.12) and (1.13), satisfy the following three-term recursions*

$$\begin{aligned} (\ell + 2)A_{\ell+1}(N, M) &= (2\ell + 1)(N + M)A_\ell(N, M) \\ &\quad + (\ell - 1)(\ell^2 - (M - N)^2)A_{\ell-1}(N, M), \\ (\ell + 1)B_{\ell+1}(N, M) &= (2\ell + 1)(N + M - 1)B_\ell(N, M) \\ &\quad + \ell(\ell^2 - (M - N)^2)B_{\ell-1}(N, M), \end{aligned} \tag{2.78}$$

for  $\ell \geq 1$ , with initial data given as

$$A_0(N, M) = N, \quad A_1(N, M) = NM, \quad B_0(N, M) = 1, \quad B_1(N, M) = N + M - 1. \tag{2.79}$$

*Proof.* Introduce the matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2.80}$$

and write

$$TR(x)T^{-1} = \frac{1}{2}\mathbf{1} + r_3\sigma_3 + r_+\sigma_+ + r_-\sigma_-, \tag{2.81}$$

where we use that  $\text{tr } R \equiv 1$ ; hereafter, we omit the dependence on  $x$  for brevity. Recalling the first equation in (2.30), we infer that

$$\frac{\partial}{\partial x} R(x) = [\mathcal{A}(x), R(x)] \Rightarrow \frac{\partial}{\partial x} (TR(x)T^{-1}) = [T\mathcal{A}(x)T^{-1}, TR(x)T^{-1}], \tag{2.82}$$

and writing

$$T\mathcal{A}(x)T^{-1} = -\frac{1}{2}\sigma_3 + \frac{1}{x} \begin{pmatrix} N + \frac{\alpha}{2} & -1 \\ N(N + \alpha) & -N - \frac{\alpha}{2} \end{pmatrix} = a_3\sigma_3 + a_+\sigma_+ + a_-\sigma_- \tag{2.83}$$

using (2.35), we deduce from (2.82) the system of linear ODEs

$$\partial_x r_3 = a_+ r_- - a_- r_+, \quad \partial_x r_+ = 2(a_3 r_+ - a_+ r_3), \quad \partial_x r_- = 2(a_- r_3 - a_3 r_-), \tag{2.84}$$

which in turn implies the following *decoupled* third-order equations for  $\partial_x r_3$ ,  $r_+$ , and  $r_-$ ,

$$3(2N + \alpha - x)\partial_x r_3 + (4 - \alpha^2 + 2(2N + \alpha)x - x^2)\partial_x^2 r_3 + 5x\partial_x^3 r_3 + x^2\partial_x^4 r_3 = 0, \tag{2.85}$$

$$(2N + \alpha \pm 1 - x)r_{\pm} + (1 - \alpha^2 + 2(2N + \alpha \pm 1)x - x^2)\partial_x r_{\pm} + 3x\partial_x^2 r_{\pm} + x^2\partial_x^3 r_{\pm} = 0. \tag{2.86}$$

Finally, using the Wishart parameter  $M = N + \alpha$ , we substitute the expansion at  $x = \infty$  given by (1.10) into the ODEs (2.85) and (2.86) to obtain the claimed recursion relations. □

*Remark 2.12.* Let us remark that the recursion for  $A_\ell(N, M)$  in Lemma 2.11 is also deduced, by different means, in [42]. In [23], it is pointed out that such three term recursion is a manifestation of the fact that  $A_\ell(N, M)$  is expressible in terms of *hypergeometric orthogonal polynomials*; this property extends to the entries  $B_\ell(N, M)$ , as we now show. Introducing the *generalized hypergeometric function*  ${}_3F_2$

$${}_3F_2\left(\begin{matrix} p_1, p_2, p_3 \\ q_1, q_2 \end{matrix} \middle| \zeta\right) := \sum_{j \geq 0} \frac{(p_1)_j (p_2)_j (p_3)_j \zeta^j}{(q_1)_j (q_2)_j j!}, \tag{2.87}$$

we can rewrite the coefficients  $A_\ell(N, M)$  and  $B_\ell(N, M)$  in the form

$$A_\ell(N, M) := \frac{(N)_\ell (M)_\ell}{\ell!} {}_3F_2\left(\begin{matrix} 1 - N, 1 - M, 1 - \ell \\ 1 - N - \ell, 1 - M - \ell \end{matrix} \middle| 1\right), \tag{2.88}$$

$$B_\ell(N, M) := \frac{(N)_\ell (M)_\ell}{\ell!} {}_3F_2\left(\begin{matrix} 1 - N, 1 - M, -\ell \\ 1 - N - \ell, 1 - M - \ell \end{matrix} \middle| 1\right). \tag{2.89}$$

Alternatively, introducing the *Hahn* and *dual Hahn polynomials* [23, 50]

$$Q_j(x; \mu, \nu, k) := {}_3F_2\left(\begin{matrix} -x, j + \mu + \nu + 1, -j \\ -k, \mu + 1 \end{matrix} \middle| 1\right), \tag{2.90}$$

$$R_j(\lambda(x); \gamma, \delta, k) := {}_3F_2\left(\begin{matrix} -j, x + \gamma + \delta + 1, -x \\ -k, \gamma + 1 \end{matrix} \middle| 1\right), \tag{2.91}$$

$\lambda(x) = x(x + \gamma + \delta + 1)$

the coefficients  $A_\ell(N, M)$  and  $B_\ell(N, M)$  can be rewritten in the form

$$\begin{aligned} \frac{\ell!}{(N)_\ell (M)_\ell} A_\ell(N, M) &= Q_{\ell-1}(N - 1; -M - \ell, 1, N + \ell - 1) \\ &= R_{N-1}(\ell - 1; -M - \ell, 1, N + \ell - 1), \end{aligned} \tag{2.92}$$

$$\begin{aligned} \frac{\ell!}{(N)_\ell (M)_\ell} B_\ell(N, M) &= Q_\ell(N - 1; -M - \ell, 0, N + \ell - 1) \\ &= R_{N-1}(\ell; -M - \ell, 0, N + \ell - 1). \end{aligned} \tag{2.93}$$



Let us note that the first differential equation in (2.84) implies, at the level of the coefficients of the power series  $r_3, r_-, r_+$ , the following relation

$$\ell(\ell + 1)A_\ell(N, M) = NM (B_\ell(N + 1, M + 1) - B_\ell(N, M)), \tag{2.94}$$

which is used in Example 2.14 to prove formulae (1.17).

Let us now consider the asymptotic expansion as  $x \rightarrow 0$ .

**Proposition 2.13.** *The matrix  $R(x)$  admits the asymptotic expansion*

$$TR(x)T^{-1} \sim R_-(x), \quad x \rightarrow 0, \tag{2.95}$$

uniformly in  $\mathbb{C} \setminus [0, +\infty)$ . Here,  $R_-$  is the formal series introduced in the beginning of this paper, see (1.11), and  $T$  is defined in (2.66).

*Proof.* First, we observe that by arguments which are entirely analogous to those employed in the proof of Proposition 2.9, the matrices  $Y(x)$  and (consequently)  $R(x)$  possess asymptotic expansions in integer powers of  $x$  as  $x \rightarrow 0$ , which are uniform in  $\mathbb{C} \setminus [0, +\infty)$ . The first coefficients of these expansions at  $x = 0$  can be computed from

$$\pi_\ell^{(\alpha)} = (-1)^\ell ((\alpha + 1)_\ell - \ell(\alpha + 2)_{\ell-1}x + \mathcal{O}(x^2)), \tag{2.96}$$

$$\widehat{\pi}_\ell^{(\alpha)} \sim \frac{(-1)^\ell}{2\pi i} (\ell! \Gamma(\alpha) + (\ell + 1)! \Gamma(\alpha - 1)x + \mathcal{O}(x^2)), \tag{2.97}$$

where the former is found directly from (2.5) and the latter by a computation analogous to (2.68); hence, recalling the definition (2.25) we have

$$\begin{aligned} Y(x) \sim & (-1)^N \left( \begin{array}{cc} (\alpha + 1)_N & \frac{N! \Gamma(\alpha)}{2\pi i} \\ \frac{2\pi i}{h_{N-1}} (\alpha + 1)_{N-1} & \frac{(N-1)! \Gamma(\alpha)}{h_{N-1}} \end{array} \right) \\ & + (-1)^N \left( \begin{array}{cc} -N(\alpha + 2)_{N-1} & \frac{(N+1)! \Gamma(\alpha-1)}{2\pi i} \\ -\frac{2\pi i}{h_{N-1}} (N-1)(\alpha + 2)_{N-2} & \frac{N!}{h_{N-1}} \Gamma(\alpha - 1) \end{array} \right) x + \mathcal{O}(x^2) \end{aligned} \tag{2.98}$$

as  $x \rightarrow 0$  within  $0 < \arg x < 2\pi$ ; this implies that in the same regime we have

$$\begin{aligned} TR(x)T^{-1} \sim & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\alpha} \begin{pmatrix} N & -1 \\ N(N + \alpha) & -N \end{pmatrix} \\ & + \begin{pmatrix} 2N(N + \alpha) & -2N - \alpha - 1 \\ N(N + \alpha)(2N + \alpha - 1) & -2N(N + \alpha) \end{pmatrix} \frac{x}{(\alpha - 1)\alpha(\alpha + 1)} \\ & + \mathcal{O}(x^2). \end{aligned} \tag{2.99}$$

Therefore, our goal is just to show that the coefficients of the latter expansion are related to those of the expansion at  $x = \infty$  as stated in the formulae (1.10) and (1.11). To this end let us write, in terms of the decomposition (2.81),

$$\begin{aligned} r_3(x) \sim & \frac{1}{2} + \sum_{\ell \geq 0} (\ell + 1) \widetilde{A}_\ell(N, N + \alpha) \frac{x^\ell}{(\alpha - \ell)_{2\ell+1}}, \\ r_\pm(x) \sim & \sum_{\ell \geq 0} \widetilde{B}_\ell^\pm(N, N + \alpha) \frac{x^\ell}{(\alpha - \ell)_{2\ell+1}}, \end{aligned} \tag{2.100}$$

for some, yet undetermined coefficients  $\tilde{A}_\ell(N, M), \tilde{B}_\ell^\pm(N, M)$ . From (2.99), we read the first coefficients  $\tilde{A}_\ell(N, M), \tilde{B}_\ell^\pm(N, M)$  in (2.100) as

$$\begin{aligned} \tilde{A}_0(N, M) &= N = A_0(N, M), \\ \tilde{B}_0^+(N, M) &= -1 = -B_0(N + 1, M + 1), \\ \tilde{B}_0^-(N, M) &= NM = NMB_0(N, M) \\ \tilde{A}_1(N, M) &= NM = A_1(N, M), \\ \tilde{B}_1^+(N, M) &= -N - M - 1 = -B_1(N + 1, M + 1), \\ \tilde{B}_1^-(N, M) &= NM(N + M - 1) = NMB_1(N, M). \end{aligned} \tag{2.101}$$

Finally, it can be checked that substituting (2.100) in (2.85) and (2.86), we obtain, again using  $M = N + \alpha$ , the recursions

$$\begin{aligned} (\ell + 2)\tilde{A}_{\ell+1}(N, M) &= (2\ell + 1)(N + M)\tilde{A}_\ell(N, M) \\ &\quad + (\ell - 1)(\ell^2 - (M - N)^2)\tilde{A}_{\ell-1}(N, M), \\ (\ell + 1)\tilde{B}_{\ell+1}^\pm(N, M) &= (2\ell + 1)(N + M \pm 1)\tilde{B}_\ell^\pm(N, M) \\ &\quad + \ell(\ell^2 - (M - N)^2)\tilde{B}_{\ell-1}^\pm(N, M), \end{aligned} \tag{2.102}$$

for  $\ell \geq 1$ . In view of Lemma 2.11, the linear recursions (2.102) with initial data (2.101) are uniquely solved as

$$\begin{aligned} \tilde{A}_\ell(N, M) &= A_\ell(N, M), \\ \tilde{B}_\ell^+(N, M) &= -B_\ell(N + 1, M + 1), \quad \tilde{B}_\ell^-(N, M) = NMB_\ell(N, M). \end{aligned} \tag{2.103}$$

Therefore, from (2.81), (2.100) and the above relation we obtain

$$\begin{aligned} TR(x)T^{-1} &\sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell \geq 0} \frac{x^\ell}{(\alpha - \ell)_{2\ell+1}} \\ &\quad \times \begin{pmatrix} (\ell + 1)A_\ell(N, N + \alpha) & -B_\ell(N + 1, N + 1 + \alpha) \\ N(N + \alpha)B_\ell(N, N + \alpha) & -(\ell + 1)A_\ell(N, N + \alpha) \end{pmatrix} \end{aligned} \tag{2.104}$$

with  $\alpha = M - N$  and  $A_\ell(N, M)$  and  $B_\ell(N, M)$  as in (1.12) and (1.13). The proof is complete  $\square$

We are finally ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Let us first consider the one-point generating functions  $C_{1,0}(x)$  and  $C_{0,1}(x)$ . It is convenient to introduce the scalar function

$$S_1(x) := \text{tr}(\mathcal{A}(x)R(x)). \tag{2.105}$$

Indeed, from (2.44) we see that for all  $k \neq 0$  we have

$$\begin{aligned} \left. \frac{\partial Z_N(\alpha; \mathbf{t})}{\partial t_k} \right|_{\mathbf{t}=\mathbf{0}} &= -\text{res}_x \text{tr}(\mathcal{A}(x)R(x))x^k dx + \text{res}_x \left( \frac{\alpha}{2x} - \frac{1}{2} \right) x^k dx \\ &= -\text{res}_x (xS_1(x))x^{k-1} dx - \frac{1}{2}\delta_{k,-1} \end{aligned}$$

$$\begin{aligned}
 &= \operatorname{res}_x \frac{\partial(xS_1(x))}{\partial x} \frac{x^k}{k} dx - \frac{1}{2} \delta_{k,-1} \\
 &= \operatorname{res}_x \left( \frac{\partial(xS_1(x))}{\partial x} + \frac{1}{2} \right) \frac{x^k}{k} dx.
 \end{aligned}
 \tag{2.106}$$

We now claim that

$$\frac{\partial}{\partial x}(xS_1(x)) = \frac{1}{2} - R_{11}(x).
 \tag{2.107}$$

Indeed, we have

$$\partial_x S_1(x) = \operatorname{tr}((\partial_x \mathcal{A}(x))R(x)) + \operatorname{tr}(\mathcal{A}(x)(\partial_x R(x))),
 \tag{2.108}$$

and noting the following identities

$$\partial_x \mathcal{A}(x) = -\frac{1}{x} \left( \mathcal{A}(x) + \frac{1}{2} \sigma_3 \right), \quad \partial_x R(x) = [\mathcal{A}(x), R(x)],
 \tag{2.109}$$

we can rewrite (2.108) as

$$\partial_x S_1(x) = -\frac{1}{x} \operatorname{tr}(\mathcal{A}(x)R(x)) - \frac{1}{2x} \operatorname{tr}(\sigma_3 R(x)) + \operatorname{tr}(\mathcal{A}(x)[\mathcal{A}(x), R(x)])
 \tag{2.110}$$

and (2.107) follows noting  $\operatorname{tr}(\mathcal{A}(x)[\mathcal{A}(x), R(x)]) = \operatorname{tr}([\mathcal{A}(x), \mathcal{A}(x)R(x)]) = 0$  and

$$\frac{1}{2} \operatorname{tr}(\sigma_3 R(x)) = \operatorname{tr}(E_{11}R(x)) - \frac{1}{2} \operatorname{tr} R(x) = R_{11}(x) - \frac{1}{2}
 \tag{2.111}$$

as  $\operatorname{tr} R(x) \equiv 1$ . Hence, substituting (2.107) into (2.106) we obtain, irrespectively of the sign of  $k$ ,

$$\left\langle \operatorname{tr} X^k \right\rangle \stackrel{(1.27)}{=} \left. \frac{\partial Z_N(\alpha; \mathbf{t})}{\partial t_k} \right|_{\mathbf{t}=\mathbf{0}} = -\frac{1}{k} \operatorname{res}_x (R_{11}(x) - 1) x^k dx.
 \tag{2.112}$$

At the level of generating functions, for  $C_{1,0}(x)$  we have

$$\begin{aligned}
 \partial_x(xC_{1,0}(x)) &\stackrel{(1.6)}{=} -\sum_{k \geq 1} \frac{k \langle \operatorname{tr} X^k \rangle}{x^{k+1}} \stackrel{(2.112)}{=} \sum_{k \geq 1} \frac{1}{x^{k+1}} \operatorname{res}_{\xi=\infty} (R_{11}(\xi) - 1) \xi^k d\xi \\
 &= -(R_+(x))_{11} + 1,
 \end{aligned}
 \tag{2.113}$$

which, after integration, is the formula in the statement of Theorem 1.1; in the last step of the last chain of equalities, we have to observe that  $(R_+)_{11} = 1 + \mathcal{O}(x^{-2})$  as  $x \rightarrow \infty$ , see (2.71).

Similarly, for  $C_{0,1}(x)$  we have

$$\begin{aligned}
 \partial_x(xC_{0,1}(x)) &\stackrel{(1.6)}{=} -\sum_{k \geq 1} kx^{k-1} \langle \operatorname{tr} X^{-k} \rangle \\
 &\stackrel{(2.112)}{=} -\sum_{k \geq 1} x^{k-1} \operatorname{res}_{\xi=0} (R_{11}(\xi) - 1) \xi^{-k} d\xi = -(R_-(x))_{11} + 1,
 \end{aligned}
 \tag{2.114}$$

which, after integration, is the formula in the statement of Theorem 1.1. Here, we have noted that  $(R(x))_{11} = (TR(x)T^{-1})_{11}$  since  $T$  is diagonal, see (2.66).

The formulae for  $r \geq 2$  are proven instead by the following computation;

$$\begin{aligned}
 & C_{r_+, r_-}(x_1, \dots, x_r) \\
 & \stackrel{(1.8)}{=} \sum_{k_1, \dots, k_r \geq 1} \frac{(-1)^{r-}}{x_1^{\sigma_1 k_1 + 1} \dots x_r^{\sigma_r k_r + 1}} \langle \text{tr } X^{\sigma_1 k_1} \dots \text{tr } X^{\sigma_r k_r} \rangle_{\mathbf{c}} \\
 & \stackrel{(1.27)}{=} \sum_{k_1, \dots, k_r \geq 1} \frac{(-1)^{r-}}{x_1^{\sigma_1 k_1 + 1} \dots x_r^{\sigma_r k_r + 1}} \left. \frac{\partial^r \log Z_N(\alpha; \mathbf{t})}{\partial t_{\sigma_1 k_1} \dots \partial t_{\sigma_r k_r}} \right|_{\mathbf{t}=\mathbf{0}} \\
 & \stackrel{(2.56)}{=} \sum_{k_1, \dots, k_r \geq 1} \frac{(-1)^{r+} \text{res}_{\xi_1} \dots \text{res}_{\xi_r} S_r(\xi_1, \dots, \xi_r; \mathbf{t} = \mathbf{0}) \xi_1^{\sigma_1 k_1} \dots \xi_r^{\sigma_r k_r} d\xi_1 \dots d\xi_r}{x_1^{\sigma_1 k_1 + 1} \dots x_r^{\sigma_r k_r + 1}} \\
 & \stackrel{(2.55), (2.65), (2.95)}{=} - \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \frac{\text{tr} (R_{\sigma_{i_1}}(x_{i_1}) \dots R_{\sigma_{i_r}}(x_{i_r})) - \delta_{r,2}}{(x_{i_1} - x_{i_2}) \dots (x_{i_r} - x_{i_1})}, \quad (2.115)
 \end{aligned}$$

where we have noted that the transformation  $R \mapsto TRT^{-1}$  leaves the expression  $S_r$  invariant, and therefore, we are free to use the expansions  $R_{\pm}$  of Propositions 2.9 and 2.13; the signs  $\sigma_i$  are those defined in (1.9). The proof is complete.  $\square$

*Example 2.14.* As an application of Theorem 1.1, let us show how to prove formulae (1.17). Combining (1.7) and (1.14) gives

$$\langle \text{tr } X^k \text{tr } X \rangle_{\mathbf{c}} = \text{res}_{x_1=\infty} \text{res}_{x_2=\infty} \frac{\text{tr} (R_+(x_1)R_+(x_2)) - 1}{(x_1 - x_2)^2} x_1^k x_2 \, dx_1 dx_2. \quad (2.116)$$

Let us write the matrix  $R_+(x)$  as

$$R_+(x) = E_{11} + \sum_{\ell \geq 0} \frac{R_{\ell}^+}{x^{\ell+1}}, \quad R_{\ell}^+ = \begin{pmatrix} \ell A_{\ell}(N, M) & B_{\ell}(N + 1, M + 1) \\ -NMB_{\ell}(N, M) & -\ell A_{\ell}(N, M) \end{pmatrix} \quad (2.117)$$

and expand the denominator in  $1/(x_1 - x_2)^2$  as a geometric series (the order we carry out the expansions in  $x_1, x_2$  is irrelevant, as explained in Remark 2.8) to rewrite the right side of (2.116) as

$$\frac{1}{x_2^2} \sum_{h_1, h_2 \geq 0} \frac{x_1^{h_1+h_2}}{x_2^{h_1+h_2}} \left( \sum_{\ell_1 \geq 0} \frac{\text{tr} (E_{11}R_{\ell_1}^+)}{x_1^{\ell_1+1}} + \sum_{\ell_2 \geq 0} \frac{\text{tr} (E_{11}R_{\ell_2}^+)}{x_2^{\ell_2+1}} + \sum_{\ell_1, \ell_2 \geq 0} \frac{\text{tr} (R_{\ell_1}^+R_{\ell_2}^+)}{x_1^{\ell_1+1}x_2^{\ell_2+1}} \right). \quad (2.118)$$

Finally, the residues extract the coefficient in front of  $x_1^{-k-1}x_2^{-2}$ , yielding

$$\langle \text{tr } X^k \text{tr } X \rangle_{\mathbf{c}} = \text{tr} (E_{11}R_k^+) = kA_k(N, M). \quad (2.119)$$

In a similar way, from the relation

$$\langle \text{tr } X^{-k} \text{tr } X^{-1} \rangle_{\mathbf{c}} = \text{res}_{x_1=0} \text{res}_{x_2=0} \frac{\text{tr} (R_-(x_1)R_-(x_2)) - 1}{(x_1 - x_2)^2} x_1^{-k} x_2^{-1} dx_1 dx_2 \quad (2.120)$$

and

$$\begin{aligned}
 R_-(x) &= \frac{1}{\alpha} \begin{pmatrix} M & -1 \\ NM & -N \end{pmatrix} + \sum_{\ell \geq 1} \frac{x^\ell}{(\alpha - \ell)_{2\ell+1}} R_\ell^-, \\
 R_\ell^- &= \begin{pmatrix} (\ell + 1)A_\ell(N, M) & -B_\ell(N + 1, M + 1) \\ NMB_\ell(N, M) & -(\ell + 1)A_\ell(N, M) \end{pmatrix}
 \end{aligned} \tag{2.121}$$

we obtain

$$\begin{aligned}
 &\left\langle \text{tr } X^{-k} \text{tr } X^{-1} \right\rangle_c \\
 &= \frac{1}{\alpha} \text{tr} \left[ \begin{pmatrix} M & -1 \\ NM & -N \end{pmatrix} R_{k+1}^- \right] \\
 &= \frac{(k + 2)(N + M)A_{k+1}(N, M) - NM(B_{k+1}(N + 1, M + 1) + B_{k+1}(N, M))}{\alpha(\alpha - k - 1)_{2k+3}} \\
 &= \frac{kA_k(N, M)}{\alpha(\alpha - k)_{2k+1}}.
 \end{aligned} \tag{2.122}$$

The last equality follows from the recursion relations (2.78) and formula (2.94). The computations of  $\langle \text{tr } X^k \text{tr } X^{-1} \rangle_c$  and  $\langle \text{tr } X^{-k} \text{tr } X^1 \rangle_c$  follow in a similar way.

### 3. Proof of Proposition 1.2

In this section, we prove Proposition 1.2 by means of the explicit formulae for the matrices  $R_\pm(x)$  of Theorem 1.1. The proof follows from two main lemmas; the first one explains how rescaled correlators can be written as series in even powers of  $N$  only. We recall that we are working in the regime  $\alpha = (c - 1)N$ , i.e.,  $M = cN$ , with  $c$  independent of  $N$ . From (1.8), we can write generating functions for the rescaled correlators appearing in (1.19) as

$$\sum_{k_1, \dots, k_r \geq 1} \frac{N^{-\sum_{i=1}^r \sigma_i k_i} (-1)^{r-}}{x_1^{\sigma_1 k_1 + 1} \dots x_r^{\sigma_r k_r + 1}} \langle \text{tr } X^{\sigma_1 k_1} \dots \text{tr } X^{\sigma_r k_r} \rangle_c = N^r C_{r_+, r_-}(Nx_1, \dots, Nx_r), \tag{3.1}$$

where we use the signs in (1.9).

Let us preliminarily observe two properties of formula (1.15), which are crucial to our proof of Proposition 1.2. First, such formula is invariant under replacing the matrices  $R_\pm(x)$  with  $GR_\pm(x)G^{-1}$  for some *constant* non-degenerate matrix  $G$ , and second, it is invariant (up to a simple modification for the two-point function) under replacing  $R_\pm(x)$  with  $R_\pm(x) + \gamma \mathbf{1}$  for any constant  $\gamma \in \mathbb{C}$ . While the first property is trivial, the second one requires few lines of explanation. When  $r = 2$ , one can exploit the fact that  $\text{tr } R_\pm(x) \equiv 1$  to write

$$\frac{\text{tr} (R_\pm(x_1)R_\pm(x_2)) - 1}{(x_1 - x_2)^2} = \frac{\text{tr} ((R_\pm(x_1) + \gamma \mathbf{1})(R_\pm(x_2) + \gamma \mathbf{1})) - (1 + 2\gamma + 2\gamma^2))}{(x_1 - x_2)^2}. \tag{3.2}$$

When  $r \geq 3$ , instead we reason as follows. Let us write every  $r$ -cycle  $(i_1, \dots, i_r)$  with  $i_r = r$ , namely

$$\begin{aligned} & \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \frac{\text{tr}(R_{i_1} \cdots R_{i_r})}{(x_{i_1} - x_{i_2}) \cdots (x_{i_r} - x_{i_1})} \\ &= \sum_{(i_1, \dots, i_{r-1}, r) \in \mathcal{C}_r} \frac{\text{tr}(R_{i_1} \cdots R_{i_{r-1}} R_r)}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-1}} - x_r)(x_r - x_{i_1})}, \end{aligned} \tag{3.3}$$

where for the purpose of this explanation we adopt a short notation  $R_i := R_{\sigma(i)}(x_i)$ ; we point out that the role of the “fixed” matrix  $R_r$  is completely arbitrary, as function (1.15) is symmetric. Let us now show that this expression is invariant under the transformation  $R_r \mapsto R_r + \gamma \mathbf{1}$ ; indeed the difference between the two expressions is computed from the last formula to be proportional to

$$\begin{aligned} & \sum_{(i_1, \dots, i_{r-1}, r) \in \mathcal{C}_r} \frac{\text{tr}(R_{i_1} \cdots R_{i_{r-1}})}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-1}} - x_r)(x_r - x_{i_1})} \\ &= \sum_{(i_1, \dots, i_{r-1}, r) \in \mathcal{C}_r} \frac{\text{tr}(R_{i_1} \cdots R_{i_{r-1}})}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-1}} - x_{i_1})} \left( \frac{1}{x_{i_{r-1}} - x_r} - \frac{1}{x_{i_1} - x_r} \right) = 0. \end{aligned} \tag{3.4}$$

It follows that in (1.15) one may inductively substitute all  $R_i$ ’s by  $R_i + \gamma \mathbf{1}$  (in principle, even with different  $\gamma$ ’s for each  $R_i$ , but we do not need such freedom) without affecting formula (1.15).

**Lemma 3.1.**  $C_{r_+, r_-}(Nx_1, \dots, Nx_r)$  is an even function of  $N$  for every  $r_+, r_-$ .

*Proof.* Using formula (1.15) in Theorem 1.1, we have

$$C_{r_+, r_-}(Nx_1, \dots, Nx_r) = -\frac{1}{N^r} \sum_{(i_1, \dots, i_r) \in \mathcal{C}_r} \frac{\text{tr}(R_{\sigma_{i_1}}(Nx_{i_1}) \cdots R_{\sigma_{i_r}}(Nx_{i_r})) - \delta_{r,2}}{(x_{i_1} - x_{i_2}) \cdots (x_{i_{r-1}} - x_{i_r})(x_{i_r} - x_{i_1})}. \tag{3.5}$$

After the considerations exposed just before this lemma, it is clear that we are done if we find a matrix  $G$  such that the matrices  $GR_{\pm}(Nx)G^{-1} - \frac{1}{2}\mathbf{1}$  are both odd in  $N$ . We claim that the matrix

$$G = \begin{pmatrix} \sqrt{c} & N^{-1} \\ -\sqrt{c}N & 1 \end{pmatrix} \tag{3.6}$$

serves this purpose. Indeed, we have

$$\begin{aligned} GR_+(Nx)G^{-1} &= \frac{1}{2} \begin{pmatrix} 1 & -N^{-1} \\ -N & 1 \end{pmatrix} \\ &+ \frac{1}{2} \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \begin{pmatrix} D_{\ell}(c, N) & N^{-2}(\ell E_{\ell}(c, N) + F_{\ell}(c, N)) \\ \ell E_{\ell}(c, N) - F_{\ell}(c, N) & -D_{\ell}(c, N) \end{pmatrix}, \end{aligned} \tag{3.7}$$

where

$$D_{\ell}(c, N) := \frac{\sqrt{c}}{N^{\ell}} (B_{\ell}(N + 1, cN + 1) - B_{\ell}(N, cN)), \tag{3.8}$$

$$E_\ell(c, N) := -\frac{2}{N^\ell} A_\ell(N, cN), \tag{3.9}$$

$$F_\ell(c, N) := \frac{\sqrt{c}}{N^{\ell-1}} (B_\ell(N + 1, cN + 1) + B_\ell(N, cN)). \tag{3.10}$$

Therefore, our claim is equivalent to the statement that  $D_\ell, E_\ell, F_\ell$  are odd functions of  $N$ . This is easily seen from the linear recursions of Lemma 2.11. For the coefficients  $E_\ell$ , the initial datum of the recursion is

$$E_0(c, N) = -2N, \quad E_1(c, N) = -2cN, \tag{3.11}$$

and the recursion reads

$$\begin{aligned} N^2(\ell + 2)E_{\ell+1}(c, N) &= N^2(2\ell + 1)(c + 1)E_\ell(c, N) \\ &\quad + (\ell - 1)(\ell^2 - (c - 1)^2 N^2)E_{\ell-1}(c, N) \end{aligned} \tag{3.12}$$

and the claim follows by induction, as the initial datum is odd and the recursion is even in  $N$ . Similarly, for the coefficients  $D_\ell, F_\ell$ , the initial datum of the recursion is odd in  $N$

$$D_0(c, N) = 0, \quad D_1(c, N) = \frac{2\sqrt{c}}{N}, \quad F_0(c, N) = 2\sqrt{c}N, \quad F_1(c, N) = 2N\sqrt{c}(c + 1), \tag{3.13}$$

and the recursion is even in  $N$

$$\begin{aligned} N^2(\ell + 1)D_{\ell+1}(c, N) &= N^2(c + 1)(2\ell + 1)D_\ell(c, N) \\ &\quad + (2\ell + 1)F_\ell + \ell(\ell^2 - N^2(c - 1)^2)D_{\ell-1}(c, N), \end{aligned} \tag{3.14}$$

$$\begin{aligned} N^2(\ell + 1)F_{\ell+1}(c, N) &= N^2(c + 1)(2\ell + 1)F_\ell(c, N) \\ &\quad + N^2(2\ell + 1)D_\ell + \ell(\ell^2 - N^2(c - 1)^2)F_{\ell-1}(c, N). \end{aligned} \tag{3.15}$$

The same claim for  $R_-(Nx)$  is proven exactly in the same way, as we have

$$\begin{aligned} GR_-(Nx)G^{-1} &= \frac{1}{2} \begin{pmatrix} 1 & -N^{-1} \\ -N & 1 \end{pmatrix} + \frac{1}{2} \sum_{\ell \geq 0} \frac{N^{2\ell+1} x^\ell}{(\alpha - \ell)_{2\ell+1}} \\ &\quad \times \begin{pmatrix} -D_\ell(c, N) & N^{-2}((\ell+1)E_\ell(c, N) - F_\ell(c, N)) \\ (\ell+1)E_\ell(c, N) + F_\ell(c, N) & D_\ell(c, N) \end{pmatrix} \end{aligned} \tag{3.16}$$

and, since  $\alpha = (c - 1)N$ ,

$$\frac{N^{2\ell+1}}{(\alpha - \ell)_{2\ell+1}} = \frac{N^{2\ell}}{(c - 1)} \prod_{j=1}^{\ell} \frac{1}{N^2(c - 1)^2 - j^2} \tag{3.17}$$

which is even in  $N$ . □

The second lemma regards integrality of the coefficients.

**Lemma 3.2.** *The functions  $A_\ell(N, M)$  and  $B_\ell(N, M)$  in (1.12) and (1.13) admit the alternative expressions*

$$A_\ell(N, M) = \sum_{\substack{a, b \geq 0 \\ a+b \leq \ell-1}} \frac{\ell!(\ell - 1)!(N - a)_{a+1}(M - b)_{b+1}}{(a + 1)!(b + 1)!a!b!(\ell - 1 - a - b)!}, \quad \ell \geq 1, \tag{3.18}$$

$$B_\ell(N, M) = \sum_{\substack{a, b \geq 0 \\ a+b \leq \ell}} \frac{\ell!(\ell-1)!(N-a)_a(M-b)_b}{a!^2 b!^2 (\ell-a-b)!}, \quad \ell \geq 0. \tag{3.19}$$

*Proof.* Using the identity

$$(\beta)_k = \beta(\beta+1) \cdots (\beta+k-1) = \frac{\partial^k}{\partial x^k} x^{\beta+k-1} \Big|_{x=1}, \tag{3.20}$$

we rewrite (1.12), for  $\ell \geq 1$ , as

$$\begin{aligned} A_\ell(N, M) &= \frac{1}{\ell!} \sum_{k=0}^{\ell-1} (-1)^k \binom{\ell-1}{k} (N-k)_\ell (M-k)_\ell \\ &= \frac{\partial^\ell}{\partial x^\ell} \frac{\partial^\ell}{\partial y^\ell} \frac{x^N y^M (xy-1)^{\ell-1}}{\ell!} \Big|_{x=1, y=1}, \end{aligned} \tag{3.21}$$

and then, we rewrite this expression, by a change of variable  $1+\xi = x, 1+\eta = y$ , as

$$\begin{aligned} &\frac{\partial^\ell}{\partial \xi^\ell} \frac{\partial^\ell}{\partial \eta^\ell} \frac{(1+\xi)^N (1+\eta)^M (\xi\eta + \xi + \eta)^{\ell-1}}{\ell!} \Big|_{\xi=0, \eta=0} \\ &= \sum_{\substack{a, b \geq 0 \\ a+b \leq \ell-1}} \frac{\ell!(\ell-1)!(N-a)_{a+1}(M-b)_{b+1}}{(a+1)!(b+1)!a!b!(\ell-1-a-b)!}. \end{aligned} \tag{3.22}$$

Similarly, for all  $\ell \geq 0$  we have

$$\begin{aligned} B_\ell(N, M) &= \frac{1}{\ell!} \sum_{k=0}^{\ell} (-1)^k \binom{\ell}{j} (N-k)_\ell (M-k)_\ell \\ &= \frac{\partial^\ell}{\partial x^\ell} \frac{\partial^\ell}{\partial y^\ell} \frac{x^{N-1} y^{M-1} (xy-1)^\ell}{\ell!} \Big|_{x=1, y=1} \\ &= \frac{\partial^\ell}{\partial \xi^\ell} \frac{\partial^\ell}{\partial \eta^\ell} \frac{(1+\xi)^{N-1} (1+\eta)^{M-1} (\xi\eta + \xi + \eta)^\ell}{\ell!} \Big|_{\xi=0, \eta=0} \\ &= \sum_{\substack{a, b \geq 0 \\ a+b \leq \ell-1}} \frac{\ell!(\ell-1)!(N-a)_a(M-b)_b}{a!^2 b!^2 (\ell-a-b)!}, \end{aligned} \tag{3.23}$$

and the proof is complete. □

Expression (3.18) is also derived, in a different way, in [42].

It can be checked that the coefficients  $\frac{\ell!(\ell-1)!}{(a+1)!(b+1)!a!b!(\ell-1-a-b)!}$  in (3.18) are integers within the range of summation  $a, b \geq 0, a + b \leq \ell - 1$ ; indeed, if  $a + b \leq \ell - 2$ , one can write such coefficient as

$$\frac{\ell!(\ell-1)!}{(a+1)!(b+1)!a!b!(\ell-1-a-b)!} = \binom{\ell}{a+1} \binom{\ell-1}{b} \binom{\ell-b-1}{a} (b+2)_{\ell-a-b-2} \tag{3.24}$$



which is manifestly integer, while if  $a + b = \ell - 1$  the same coefficient is written as

$$\frac{\ell!(\ell - 1)!}{(a + 1)!(b + 1)!a!b!(\ell - 1 - a - b)!} = \frac{1}{\ell} \binom{\ell}{a} \binom{\ell}{a + 1}, \tag{3.25}$$

which is also manifestly integer since  $a \leq \ell - 1$ . Similarly, the coefficients  $\frac{\ell!(\ell-1)!}{a!2b!2(\ell-a-b)!}$  in (3.19) are integers within the range of summation  $a, b \geq 0, a + b \leq \ell$ .

*Proof of Proposition 1.2.* Lemma 3.2 implies that  $A_\ell(N, cN)$  and  $B_\ell(N, cN)$  are polynomials in  $N$  and  $c$  with integer coefficients. Then, the dependence on  $N^2$  follows from Lemma 3.1 and the expansion of (3.17) as series in  $N$  and  $(c - 1)$  with integer coefficients as

$$\frac{1}{N(c - 1)} \prod_{j=1}^{\ell} \frac{1}{N^2(c - 1)^2 - j^2} = \frac{1}{(N(c - 1))^{2\ell+1}} \sum_{k_1, \dots, k_\ell \geq 0} \frac{1^{2k_1} 2^{2k_2} \dots \ell^{2k_\ell}}{(N(c - 1))^{2k_1 + \dots + 2k_\ell}}, \tag{3.26}$$

provided  $\ell < N(c - 1)$ .

Finally, we note, e.g., from the recursions, that  $D_\ell \sim \mathcal{O}(N^{-1}), E_\ell, F_\ell = \mathcal{O}(N)$ , as  $N \rightarrow \infty$ ; hence, from (3.7) and (3.16) we see that  $GR_\pm(Nx)G^{-1} = \text{diag}(1, N)\mathcal{O}(1)\text{diag}(1, N^{-1})$ , where  $\mathcal{O}(1)$  refers to the behavior as  $N \rightarrow \infty$ . We conclude that (3.1) is  $\mathcal{O}(1)$  as  $N \rightarrow \infty$ , and has the same parity in  $N \mapsto -N$  as  $r$  (Lemma 3.1), completing the proof.  $\square$

*Example 3.3.* Here, we obtain the formulae of Theorem 1.1 in genus zero for one- and two-point correlators. In these cases, equivalent formulae have already appeared in the literature [25, 37, 54]. In the regime  $\alpha = N(c - 1)$  with  $N \rightarrow \infty$ , we have

$$\lim_{N \rightarrow \infty} \frac{A_\ell(N, cN)}{N^{\ell+1}} = \frac{1}{\ell} \sum_{b=0}^{\ell-1} \binom{\ell}{b+1} \binom{\ell}{b} c^{b+1}, \tag{3.27}$$

$$\lim_{N \rightarrow \infty} \frac{B_\ell(N, cN)}{N^\ell} = \sum_{b=0}^{\ell} \binom{\ell}{b}^2 c^b. \tag{3.28}$$

The above relations follow from Lemma 3.2 and the trivial estimate  $\binom{N}{k} \sim \frac{N^k}{k!}$ . In particular, due to (1.16), in the regime  $N \rightarrow \infty$  with  $\alpha = N(c - 1)$  we have

$$\lim_{N \rightarrow \infty} \frac{\langle \text{tr } X^\ell \rangle}{N^{\ell+1}} = \sum_{s=1}^{\ell} \mathcal{N}_{\ell, s} c^s, \tag{3.29}$$

where

$$\mathcal{N}_{\ell, s} := \frac{1}{\ell} \binom{\ell}{s} \binom{\ell}{s-1}, \quad \ell \geq 1, \quad s = 1, \dots, \ell \tag{3.30}$$

are the *Narayana numbers*. Formula (3.29) agrees with Wigner’s computation of positive moments of the Laguerre equilibrium measure

$$\rho(x) = \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi cx} 1_{x \in (x_-, x_+)}, \tag{3.31}$$

where  $x_{\pm} := (1 \pm \sqrt{c})^2$ , see [36]. From the one-point function, we obtain the weighted strictly monotone and weakly monotone double Hurwitz numbers of genus zero with partition  $\mu = (k)$  and  $\nu$  of length  $s$  as

$$H_{g=0}^{>}((k); s) = \frac{1}{(k-1)!} \sum_{\nu \text{ of length } s} h_{g=0}^{>}((k); \nu) = \mathcal{N}_{k,s} = \frac{1}{k} \binom{k}{s-1} \binom{k}{s}, \tag{3.32}$$

$$H_{g=0}^{\geq}((k); s) = \frac{1}{(k-1)!} \sum_{\nu \text{ of length } s} h_{g=0}^{\geq}((k); \nu) = \binom{k-1}{k-s} \frac{(s+1)_{k-2}}{(k-1)!}. \tag{3.33}$$

Similarly, for all two-point generating functions, we obtain the *planar limit*  $g = 0$  as

$$\begin{aligned} \lim_{N \rightarrow \infty} N^2 C_{2,0}(Nx_1, Nx_2) &= \lim_{N \rightarrow \infty} N^2 C_{0,2}(Nx_1, Nx_2) \\ &= \frac{\phi(x_1, x_2) - \sqrt{\phi(x_1, x_1)\phi(x_2, x_2)}}{2\sqrt{\phi(x_1, x_1)\phi(x_2, x_2)}(x_1 - x_2)^2} \end{aligned} \tag{3.34}$$

$$\lim_{N \rightarrow \infty} N^2 C_{1,1}(Nx_1, Nx_2) = -\frac{\phi(x_1, x_2) + \sqrt{\phi(x_1, x_1)\phi(x_2, x_2)}}{2\sqrt{\phi(x_1, x_1)\phi(x_2, x_2)}(x_1 - x_2)^2}, \tag{3.35}$$

where

$$\phi(x_1, x_2) := c^2 - c(2 + x_1 + x_2) + (x_1 - 1)(x_2 - 1). \tag{3.36}$$

The two-point planar limit is strictly related [35] to the so-called canonical symmetric bi-differential (called also Bergman kernel) associated with the spectral curve  $x^2y^2 = (x - x_+)(x - x_-) = c^2 - 2c(x + 1) + (x - 1)^2$ .

### 4. Hodge-LUE Correspondence

#### 4.1. Factorization of Matrix Models with Even Potential

For the purposes of the present section, let us introduce two sequences of monic orthogonal polynomials:  $p_n^{\text{even}}(x) = x^n + \dots$  satisfying

$$\int_{-\infty}^{+\infty} p_n^{\text{even}}(x)p_m^{\text{even}}(x)e^{-V(x^2)}dx = h_n^{\text{even}}\delta_{n,m} \tag{4.1}$$

and, for  $\text{Re } \alpha > -1$ ,  $p_n^{(\alpha)}(x) = x^n + \dots$  satisfying

$$\int_0^{+\infty} p_n^{(\alpha)}(x)p_m^{(\alpha)}(x)x^\alpha e^{-V(x)}dx = h_n^{(\alpha)}\delta_{n,m}, \tag{4.2}$$

where  $V(x)$  is an arbitrary potential for which the polynomials are well defined. The following lemma is elementary and the proof can be found in [21].

**Lemma 4.1.** *For all  $n \geq 0$ , we have*

$$p_{2n}^{\text{even}}(x) = p_n^{(-\frac{1}{2})}(x^2), \quad p_{2n+1}^{\text{even}}(x) = xp_n^{(\frac{1}{2})}(x^2) \tag{4.3}$$

and

$$h_{2n}^{\text{even}} = h_n^{(-\frac{1}{2})}, \quad h_{2n+1}^{\text{even}} = h_n^{(\frac{1}{2})}. \tag{4.4}$$

Next, we recall the relation between matrix integrals and the norming constants of the above orthogonal polynomials

$$\frac{1}{\text{Vol}(N)} \int_{\mathbb{H}_N} \exp \text{tr} (-V(X^2)) dX = h_0^{\text{even}} h_1^{\text{even}} \dots h_{N-1}^{\text{even}}, \tag{4.5}$$

$$\frac{1}{\text{Vol}(N)} \int_{\mathbb{H}_N^+} \det^\alpha X \exp \text{tr} (-V(X)) dX = h_0^{(\alpha)} h_1^{(\alpha)} \dots h_{N-1}^{(\alpha)}, \tag{4.6}$$

where  $\text{Vol}(N)$  is defined in (1.29).

Using the above relations and (4.4) in the case  $V(x) = \frac{x}{2} - \sum_{k \geq 1} s_k x^k$ , we obtain the following identity between the GUE partition function  $Z_{2N}^{\text{even}}(\mathbf{s})$  in (1.28) and the Laguerre partition function  $Z_N(\pm \frac{1}{2}; \mathbf{t}_+)$  in (1.26) with  $\mathbf{t}_- = \mathbf{0}$

$$\frac{Z_{2N}^{\text{even}}(\mathbf{s})}{Z_{2N}^{\text{even}}(\mathbf{0})} = \frac{Z_N(-\frac{1}{2}; \mathbf{t}_+)}{Z_N(-\frac{1}{2}; \mathbf{0})} \frac{Z_N(\frac{1}{2}; \mathbf{t}_+)}{Z_N(\frac{1}{2}; \mathbf{0})}, \quad t_k := 2^k s_k, \tag{4.7}$$

where  $Z_N^{\text{even}}(\mathbf{0})$  is given in (4.10) and  $Z_N(\pm \frac{1}{2}; \mathbf{0})$  in (1.3). There is a similar, slightly more involved, factorization for the matrix model  $Z_{2N+1}^{\text{even}}$ , but we do not need its formulation for our present purposes.

**4.2. Formal Matrix Models and mGUE Partition Function**

In this section, we review the definition of mGUE partition function. First, the logarithm of the even GUE partition function can be considered as a formal Taylor expansion for small  $s_k$  as

$$\log Z_N^{\text{even}}(\mathbf{s}) := \log Z_N^{\text{even}}(\mathbf{0}) + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{s_{k_1} \dots s_{k_r}}{r!} \langle \text{tr} X^{k_1} \dots \text{tr} X^{k_r} \rangle_c^{\text{even}}, \tag{4.8}$$

where the connected even GUE correlators are introduced as in (1.27)

$$\langle \text{tr} X^{k_1} \dots \text{tr} X^{k_r} \rangle_c^{\text{even}} := \left. \frac{\partial^r \log Z_N^{\text{even}}(\mathbf{s})}{\partial s_{k_1} \dots \partial s_{k_r}} \right|_{\mathbf{s}=\mathbf{0}} \tag{4.9}$$

and the normalizing constant  $Z_N^{\text{even}}(\mathbf{0})$  is

$$Z_N^{\text{even}}(\mathbf{0}) = \sqrt{2^N \pi^{N^2}}. \tag{4.10}$$

The infinite sum in (4.8) can be given a rigorous formal meaning in the algebra  $\mathbb{C}[N, \alpha][[s]]$ ; introducing the grading  $\text{deg } s_k := k$ , the latter algebra is obtained taking the inductive limit  $K \rightarrow \infty$  from the algebras of polynomials in  $\mathbf{s}$  of degree  $< K$ , with coefficients in  $\mathbb{C}[N, \alpha]$ . Equivalently, this grading can be encoded, up to an inessential shift, by a (small) variable  $\epsilon$  via the transformation  $s_k \mapsto \epsilon^{k-1} s_k$ , which is the same as considering the matrix model  $\int_{\mathbb{H}_N} \exp \left[ -\frac{1}{\epsilon} \left( \frac{X^2}{2} - \sum_{k \geq 1} s_k X^{2k} \right) \right] dX$ . For simplicity, we have preferred to avoid the explicit  $\epsilon$  dependence, even though we shall restore it for the statement of the Hodge-GUE/LUE correspondence (Theorem 4.4, Corollary 4.5).

It must be stressed that (4.8) makes sense for any complex  $N$ , and not just for positive integers as it would be required by the genuine matrix integral interpretation; indeed, the correlators are polynomials in  $N$ .

For the purposes of this section, it is convenient to apply the same arguments to the Laguerre partition function (with  $\mathbf{t}_- = \mathbf{0}$ ) and similarly identify the latter with the formal series

$$\log Z_N(\alpha; \mathbf{t}_+) = \log Z_N(\alpha; \mathbf{0}) + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{t_{k_1} \cdots t_{k_r}}{r!} \langle \text{tr } X^{k_1} \cdots \text{tr } X^{k_r} \rangle_c \tag{4.11}$$

where  $Z_N(\alpha; \mathbf{0})$  is given in (1.3) and the correlators are as in (1.27); using the last expression provided in (1.3) and the fact that the correlators are polynomials in  $N, \alpha$ , expression (4.11) makes sense also for  $N$  complex. This remark is crucial for a correct understanding of formulae (4.14) and (4.15).

Let us finally recall from the introduction and [28] that the mGUE partition function is introduced by (1.29), the left side of which being interpreted formally as in (4.8). Of course in the identification of Theorem 1.5, the right side must be interpreted formally as in (4.11).

### 4.3. Proof of Theorem 1.5

The proof of Theorem 1.5 relies on two main ingredients: on the one side the factorization property (4.7) and a symmetry property of the formal positive LUE partition function (4.11), which we now describe.

**Lemma 4.2.** *The LUE connected correlator  $\langle \text{tr } X^{k_1} \cdots \text{tr } X^{k_r} \rangle_c$  with  $k_1, \dots, k_r > 0$  is a polynomial in  $N, \alpha$ , and it is invariant under the involution  $(N, \alpha) \mapsto (N + \alpha, -\alpha)$ .*

*Proof.* It follows directly from Theorem 1.1, as the coefficients of  $R_+$ , defined in (1.10), are polynomials in  $N, \alpha$  which are manifestly symmetric under the aforementioned transformation. Indeed, from (1.12) and (1.13) we see that all the coefficients  $A_\ell(N, M), B_\ell(N, M)$  are symmetric in  $N, M$  but  $A_0(N, M) = N$ ; however,  $R_+$  only contains the combination  $\ell A_\ell(N, M)$ , which is always symmetric in  $N, M$ . □

*Remark 4.3.* As already pointed out in “Introduction,” the symmetry of the above lemma is equivalent, by (1.20), to the symmetry property (1.39) of Hurwitz numbers. An alternative, purely combinatorial derivation of the symmetry in the latter formulation can be given as follows. In the group algebra of the group of permutations of  $\{1, \dots, d\}$  (consisting of formal  $\mathbb{C}$ -linear combinations of permutations of  $\{1, \dots, d\}$ ), we consider the distinguished elements

- $\mathcal{C}_\lambda$  for any partition  $\lambda$  of  $d$ , which is the sum of all permutations of cycle type  $\lambda$ , and
- $\mathcal{J}_m := (1, m) + \cdots + (m-1, m)$  (*Jucys–Murphy elements*) for  $m = 2, \dots, d$ .

Such elements commute among themselves and are related by

$$(1 + \xi \mathcal{J}_2) \cdots (1 + \xi \mathcal{J}_d) = \sum_{|\nu|=d} \xi^{d-\ell(\nu)} \mathcal{C}_\nu. \tag{4.12}$$

From this relation, we deduce that for every partition  $\mu$  of  $d$  of length  $\ell$  we have

$$C_\mu \sum_{|\nu|=d} y^{d-\ell(\nu)} C_\nu (1+z\mathcal{J}_2) \cdots (1+z\mathcal{J}_d) = C_\mu \sum_{|\lambda|=d} z^{d-\ell(\lambda)} (1+y\mathcal{J}_2) \cdots (1+y\mathcal{J}_d) C_\lambda. \tag{4.13}$$

From the definition of Hurwitz numbers, recalled in Sect. 1.2, the coefficient in front of  $C_{[1^d]} y^{d-s} z^{2g-2+\ell+s}$  on the left side of (4.13) is  $H_g^>(\mu; s)$  (up to the normalization factor  $\frac{z\mu}{d!}$ ); the coefficient of the same element  $C_{[1^d]} y^{d-s} z^{2g-2+\ell+s}$  on the right side of (4.13) is  $H_g^>(\mu; 2-2g+d-\ell-s)$  (up to the same normalization factor  $\frac{z\mu}{d!}$ ), yielding (1.39).

Let us restate Lemma 4.2, in view of the formal expansion (4.11), as the following identity

$$\frac{Z_N(\alpha; \mathbf{t}_+)}{Z_N(\alpha; \mathbf{0})} = \frac{Z_{N+\alpha}(-\alpha; \mathbf{t}_+)}{Z_{N+\alpha}(-\alpha; \mathbf{0})}. \tag{4.14}$$

The following special case ( $\alpha = \frac{1}{2}$ ) of (4.14)

$$Z_{N+\frac{1}{2}}\left(-\frac{1}{2}; \mathbf{t}_+\right) = \frac{\pi^{\frac{3}{8}+\frac{N}{2}} G(N+1)}{G\left(N+\frac{3}{2}\right)} Z_N\left(\frac{1}{2}; \mathbf{t}_+\right) \tag{4.15}$$

plays a key role in the proof of Theorem 1.5, which we are now ready to give.

*Proof of Theorem 1.5.* We use the uniqueness of the decomposition (1.29) which defines the mGUE partition function; rewriting it under the substitution  $N \mapsto 2N$  we have

$$\frac{Z_{2N}^{\text{even}}(\mathbf{s})}{(2\pi)^{2N} \text{Vol}(2N)} = \tilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) \tilde{Z}_{2N+\frac{1}{2}}(\mathbf{s}). \tag{4.16}$$

On the other hand, from (4.7) we have

$$Z_{2N}^{\text{even}}(\mathbf{s}) = D_N Z_N\left(-\frac{1}{2}; \mathbf{t}_+\right) Z_N\left(\frac{1}{2}; \mathbf{t}_+\right), \tag{4.17}$$

where here and below we are identifying  $t_k = 2^k s_k$ . The proportionality constant  $D_N$  is explicitly evaluated from (1.3) and (4.10) as

$$D_N = \frac{Z_{2N}^{\text{even}}(\mathbf{0})}{Z_N\left(-\frac{1}{2}; \mathbf{0}\right) Z_N\left(\frac{1}{2}; \mathbf{0}\right)} = \frac{2^N \pi^{N^2+N+\frac{1}{2}} G\left(\frac{1}{2}\right)^2}{G\left(N+\frac{1}{2}\right) G\left(N+\frac{3}{2}\right)}. \tag{4.18}$$

It is then enough to show that two factorizations (4.16) and (4.17) are consistent once we identify  $\tilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) = C_N Z_N\left(-\frac{1}{2}; \mathbf{t}_+\right)$  with  $C_N$  a constant depending on  $N$  only. Such consistency follows from the chain of equalities

$$\begin{aligned} \frac{Z_{2N}^{\text{even}}(\mathbf{s})}{(2\pi)^{2N} \text{Vol}(2N)} &= \tilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) \tilde{Z}_{2N+\frac{1}{2}}(\mathbf{s}) \\ &= \tilde{Z}_{2N-\frac{1}{2}}(\mathbf{s}) \tilde{Z}_{(N+\frac{1}{2})-\frac{1}{2}}(\mathbf{s}) \end{aligned}$$

$$\begin{aligned}
 &= C_N Z_N \left( -\frac{1}{2}; \mathbf{t}_+ \right) C_{N+\frac{1}{2}} Z_{N+\frac{1}{2}} \left( -\frac{1}{2}; \mathbf{t}_+ \right) \\
 &= C_N C_{N+\frac{1}{2}} \frac{\pi^{\frac{3}{8}+\frac{N}{2}} \mathbf{G}(N+1)}{\mathbf{G}(N+\frac{3}{2})} Z_N \left( -\frac{1}{2}; \mathbf{t}_+ \right) Z_N \left( \frac{1}{2}; \mathbf{t}_+ \right),
 \end{aligned} \tag{4.19}$$

where we have used symmetry property (4.15). This shows that two factorizations (4.16) and (4.17) are consistent, provided we also identify the proportionality constants (4.18) and (4.19)

$$C_N C_{N+\frac{1}{2}} \frac{\pi^{\frac{3}{8}+\frac{N}{2}} \mathbf{G}(N+1)}{\mathbf{G}(N+\frac{3}{2})} = D_N = \frac{2^N \pi^{N^2+N+\frac{1}{2}} \mathbf{G}(\frac{1}{2})^2}{\mathbf{G}(N+\frac{1}{2}) \mathbf{G}(N+\frac{3}{2})}, \tag{4.20}$$

and this uniquely fixes

$$C_N = \frac{2^{\frac{N}{2}-\frac{1}{8}} \pi^{\frac{N^2}{2}} \mathbf{G}(\frac{1}{2})}{\mathbf{G}(N+\frac{1}{2})}, \tag{4.21}$$

as stated in (1.31). □

We conclude this section with a couple of remarks.

First, the identification of the mGUE and LUE partition functions is manifest also from the *Virasoro constraints* of the two models. Indeed, Virasoro constraints for the modified GUE partition function have been derived in [28], directly from those of the GUE partition function, and they assume the form

$$\tilde{\mathcal{L}}_n \tilde{Z}_N(\mathbf{s}) = 0, \tag{4.22}$$

for  $n \geq 0$ , where  $\tilde{\mathcal{L}}_n = \tilde{\mathcal{L}}_n(N, \mathbf{s})$  and

$$\tilde{\mathcal{L}}_n(N, \mathbf{s}) := \begin{cases} \sum_{k \geq 1} k (s_k - \frac{1}{2} \delta_{k,1}) \frac{\partial}{\partial s_k} + \frac{N^2}{4} - \frac{1}{16}, & n = 0, \\ \sum_{k=1}^{n-1} \frac{\partial^2}{\partial s_k \partial s_{n-k}} + \sum_{k \geq 1} k (s_k - \frac{1}{2} \delta_{k,1}) \frac{\partial}{\partial s_{k+n}} + N \frac{\partial}{\partial s_n}, & n \geq 1. \end{cases} \tag{4.23}$$

On the other hand, it is well known [1, 43] that the LUE partition function with only positive couplings  $\mathbf{t}_+$  satisfies the Virasoro constraints

$$\mathcal{L}_n^{(\alpha)} Z_N(\alpha; \mathbf{t}_+) = 0, \tag{4.24}$$

for  $n \geq 0$ , where  $\mathcal{L}_n^{(\alpha)} = \mathcal{L}_n^{(\alpha)}(N, \mathbf{t}_+)$  and

$$\mathcal{L}_n^{(\alpha)}(N, \mathbf{t}_+) := \begin{cases} \sum_{k \geq 1} k (t_k - \delta_{k,1}) \frac{\partial}{\partial t_k} + N(N + \alpha), & n = 0, \\ \sum_{k=1}^{n-1} \frac{\partial^2}{\partial t_k \partial t_{n-k}} + \sum_{k \geq 1} k (t_k - \delta_{k,1}) \frac{\partial}{\partial t_{k+n}} + (2N + \alpha) \frac{\partial}{\partial t_n}, & n \geq 1. \end{cases} \tag{4.25}$$

The Virasoro constraints  $\tilde{\mathcal{L}}_n(N, \mathbf{s})$  in (4.23) and  $\mathcal{L}_n^{(\alpha)}(N, \mathbf{t}_+)$  in (4.25) satisfy

$$2^n \tilde{\mathcal{L}}_n \left( 2N - \frac{1}{2}, \mathbf{s} \right) = \mathcal{L}_n^{(-\frac{1}{2})}(N, \mathbf{t}_+) \tag{4.26}$$

under the identification  $t_k = 2^k s_k$ , in agreement with Theorem 1.5.

Second, in [29] formulae of similar nature as those of Theorem 1.1 are derived for the modified GUE partition function. It can be checked that such formulae match with those of Theorem 1.1 restricted to  $\alpha = -\frac{1}{2}$  under the identifications of times made explicit in the statement of Theorem 1.5.

**4.4. Proof of Corollary 1.6**

From Theorem 1.5 and the Hodge-GUE correspondence of [28], which we now recall, we are able to deduce a *Hodge-LUE correspondence*; to state this result (Corollary 4.5), let us introduce the generating function

$$\mathcal{H}(\mathbf{p}; \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} \sum_{n \geq 0} \sum_{k_1, \dots, k_n \geq 0} \frac{p_{k_1} \cdots p_{k_n}}{n!} \int_{\mathcal{M}_{g,n}} \Lambda^2(-1)\Lambda \left(\frac{1}{2}\right) \prod_{i=1}^n \psi_i^{k_i}, \tag{4.27}$$

for *special cubic Hodge integrals* (with the standard notations recalled before the statement of Corollary 1.6); here,  $\mathbf{p} = (p_0, p_1, \dots)$ .

**Theorem 4.4.** (Hodge-GUE correspondence [28]) *Introduce the formal series*

$$A(\lambda, \mathbf{s}) := \frac{1}{4} \sum_{j_1, j_2 \geq 1} \frac{j_1 j_2}{j_1 + j_2} \binom{2j_1}{j_1} \binom{2j_2}{j_2} s_{j_1} s_{j_2} + \frac{1}{2} \sum_{j \geq 1} \left(\lambda - \frac{j}{j+1}\right) \binom{2j}{j} s_j, \tag{4.28}$$

and a transformation of an infinite vector of times  $\mathbf{s} = (s_1, s_2, \dots) \mapsto \mathbf{p} = (p_0, p_1, \dots)$  depending on a parameter  $\lambda$  as

$$p_k(\lambda, \mathbf{s}) := \sum_{j \geq 1} j^{k+1} \binom{2j}{j} s_j + \delta_{k,1} + \lambda \delta_{k,0} - 1, \quad k \geq 0. \tag{4.29}$$

Then, we have

$$\mathcal{H}(\mathbf{p}(\lambda, \mathbf{s}); \sqrt{2}\epsilon) + \epsilon^{-2} A(\lambda, \mathbf{s}) = \log \tilde{Z}_{\frac{\lambda}{\epsilon}}((s_1, \epsilon s_2, \epsilon^2 s_3, \dots)) + B(\lambda, \epsilon), \tag{4.30}$$

where  $B(\lambda, \epsilon)$  is a constant depending on  $\lambda$  and  $\epsilon$  only and  $\tilde{Z}_{\frac{\lambda}{\epsilon}}$  is the mGUE partition function in (1.29).

**Corollary 4.5.** (Hodge-LUE correspondence) *Let  $\mathcal{H}(\mathbf{p}(\lambda, \mathbf{s}); \sqrt{2}\epsilon)$  as above in (4.30) and  $Z_N(-\frac{1}{2}; \mathbf{t}_+)$  the Laguerre partition function (1.26) with parameter  $\alpha = -\frac{1}{2}$  and times  $\mathbf{t}_+$  and  $\mathbf{t}_- = 0$ . We have*

$$\mathcal{H}(\mathbf{p}(\lambda, \mathbf{s}); \sqrt{2}\epsilon) + \epsilon^{-2} A(\lambda, \mathbf{s}) = \log Z_N\left(-\frac{1}{2}; \mathbf{t}_+\right) + C(N, \epsilon), \tag{4.31}$$

where we identify

$$\lambda = \epsilon \left(2N - \frac{1}{2}\right), \quad t_k = 2^k \epsilon^{k-1} s_k, \tag{4.32}$$

and  $A(\lambda, \mathbf{s})$  is defined in (4.28),  $\mathbf{p}(\lambda, \mathbf{s})$  is defined in (4.29), and  $C(N, \epsilon)$  is a constant depending on  $N$  and  $\epsilon$  only.

*Proof.* It follows from (4.30) upon the substitution  $\lambda \mapsto \epsilon(2N - \frac{1}{2})$  and applying Theorem 1.5 for the set of times  $\epsilon^{k-1} s_k, k \geq 1$ .  $\square$

It would be interesting to construct the Double Ramification hierarchy (see [15, 16]) for cubic Hodge integrals and then check in this case the conjecture formulated in [17] by which the logarithm of the corresponding tau function should coincide with the LUE partition function, after the change of variables described in [18].

Finally, Corollary 1.6 is obtained matching the coefficients in (4.30) using (1.20).

*Proof of Corollary 1.6.* We apply  $\frac{\partial^\ell}{\partial s_{\mu_1} \cdots \partial s_{\mu_\ell}} \Big|_{\mathbf{s}=\mathbf{0}}$ , for  $\ell > 0$ , on both sides of (4.30). On the right side, we get, in view of Theorem 1.5

$$\begin{aligned} & \frac{\partial^\ell}{\partial s_{\mu_1} \cdots \partial s_{\mu_\ell}} \Big|_{\mathbf{s}=\mathbf{0}} \log \tilde{Z}_{\frac{\lambda}{\epsilon}} \left( (s_1, \epsilon s_2, \epsilon^2 s_3, \dots) \right) \\ &= \epsilon^{|\mu|-\ell} 2^{|\mu|} \langle \text{tr } X^{\mu_1} \cdots \text{tr } X^{\mu_\ell} \rangle_c \Big|_{N=\frac{\lambda}{2\epsilon} + \frac{1}{4}, \alpha=-\frac{1}{2}} \\ &= \epsilon^{|\mu|-\ell} 2^{|\mu|} \sum_{\gamma \geq 0} \sum_{s=1}^{1-2\gamma+|\mu|-\ell} \left( \frac{\lambda + \frac{\epsilon}{2}}{2\epsilon} \right)^{2-2\gamma+|\mu|-\ell} \left( \frac{\lambda - \frac{\epsilon}{2}}{\lambda + \frac{\epsilon}{2}} \right)^s H_\gamma^>(\mu; s), \end{aligned} \tag{4.33}$$

where in the last step we have used (1.20); we also note that the substitutions  $2N - \frac{1}{2} = \frac{\lambda}{\epsilon}$ ,  $\alpha = -\frac{1}{2}$ , from Theorem 1.5, yield  $N = \frac{\lambda + \frac{\epsilon}{2}}{2\epsilon}$ ,  $c = \frac{\lambda - \frac{\epsilon}{2}}{\lambda + \frac{\epsilon}{2}}$ . On the other side, we get

$$\frac{\partial^\ell}{\partial s_{\mu_1} \cdots \partial s_{\mu_\ell}} \Big|_{\mathbf{s}=\mathbf{0}} \mathcal{H}(\mathbf{p}(\lambda, \mathbf{s}); \sqrt{2\epsilon}) + \epsilon^{-2} \frac{\partial^\ell}{\partial s_{\mu_1} \cdots \partial s_{\mu_\ell}} \Big|_{\mathbf{s}=\mathbf{0}} A(\lambda, \mathbf{s}). \tag{4.34}$$

The contributions from the last term are directly evaluated from (4.28) and give the second line of (1.36). For the first term, we recall the affine change of variable (4.29) and compute

$$\begin{aligned} & \frac{\partial^\ell}{\partial s_{\mu_1} \cdots \partial s_{\mu_\ell}} \mathcal{H}(\mathbf{p}(\lambda, \mathbf{s}); \sqrt{2\epsilon}) \\ &= \sum_{i_1, \dots, i_\ell \geq 0} \prod_{b=1}^\ell \mu_b^{i_b+1} \binom{2\mu_b}{\mu_b} \frac{\partial^\ell}{\partial p_{i_1} \cdots \partial p_{i_\ell}} \mathcal{H}(\mathbf{p}(\lambda, \mathbf{s}); \sqrt{2\epsilon}) \\ &= \sum_{g, n \geq 0} \frac{(\sqrt{2\epsilon})^{2g-2}}{n!} \sum_{\substack{k_1, \dots, k_n \geq 0 \\ i_1, \dots, i_\ell \geq 0}} \int_{\mathcal{M}_{g, n+\ell}} \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \prod_{a=1}^n p_{k_a}(\lambda, \mathbf{s}) \psi_a^{k_a} \\ &\times \prod_{b=1}^\ell \mu_b^{i_b+1} \binom{2\mu_b}{\mu_b} \psi_{n+b}^{i_b}. \end{aligned} \tag{4.35}$$

Evaluation at  $\mathbf{s} = \mathbf{0}$  corresponds to  $p_k = \delta_{k,1} + \lambda \delta_{k,0} - 1$ ; thus, in the previous expression, we set  $n = m + r$ , where  $m$  is the number of  $k_a$ 's equal to zero, and the remaining  $k_1, \dots, k_r$ 's are all  $\geq 2$  (we are evaluating at  $p_1 = 0$ ), and so the evaluation of the (4.35) at  $p_k = \delta_{k,1} + \lambda \delta_{k,0} - 1$  reads

$$\begin{aligned} & \sum_{g, m, r \geq 0} (\sqrt{2\epsilon})^{2g-2} \sum_{\substack{k_1, \dots, k_r \geq 2 \\ i_1, \dots, i_\ell \geq 0}} \frac{(\lambda - 1)^m (-1)^r}{m! r!} \\ & \times \int_{\mathcal{M}_{g, \ell+m+r}} \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \prod_{a=1}^r \psi_a^{k_a} \prod_{b=1}^\ell \mu_b^{i_b+1} \binom{2\mu_b}{\mu_b} \psi_{m+r+b}^{i_b} \end{aligned}$$



$$\begin{aligned}
 &= \sum_{g,m,r \geq 0} \left(\sqrt{2}\epsilon\right)^{2g-2} \sum_{d_1, \dots, d_r \geq 1} \frac{(\lambda - 1)^m (-1)^r}{m! r!} \\
 &\quad \times \int_{\overline{\mathcal{M}}_{g, \ell+m+r}} \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \prod_{a=1}^r \psi_a^{d_a+1} \prod_{b=1}^{\ell} \frac{\mu_b \binom{2\mu_b}{\mu_b}}{1 - \mu_b \psi_{m+r+b}}, \tag{4.36}
 \end{aligned}$$

where in the last step we rename  $k_a = d_a + 1$ ,  $d_a \geq 1$ .

We can trade the  $\psi_1, \dots, \psi_r$  classes in (4.36) for a suitable combination of Mumford  $\kappa$  classes, following ideas from [13]. Let  $\pi : \overline{\mathcal{M}}_{g, \ell+m+r} \rightarrow \overline{\mathcal{M}}_{g, \ell+m}$  be the map forgetting the first  $r$  marked points (and contracting the resulting unstable components); then, we have the following iterated version of the *dilation equation*

$$\pi_* \left( (\pi^* \mathcal{X}) \prod_{a=1}^r \psi_a^{d_a+1} \right) = \mathcal{X} \sum_{\sigma \in \mathfrak{S}_r} \prod_{\gamma \in \text{Cycles}(\sigma)} \kappa_{\sum_{a \in \gamma} d_a}, \quad d_a, \dots, d_r \geq 1, \tag{4.37}$$

for any  $\mathcal{X} \in H^\bullet(\overline{\mathcal{M}}_{g, \ell+m}, \mathbb{Q})$ . Here and below,  $\mathfrak{S}_r$  is the group of permutations of  $\{1, \dots, r\}$  and  $\text{Cycles}(\sigma)$  is the set of disjoint cycles in the permutation  $\sigma$ ,  $\sigma = \prod_{\gamma \in \text{Cycles}(\sigma)} \gamma$ . In our case, it is convenient to set

$$\mathcal{X} = \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \prod_{b=1}^{\ell} \frac{\mu_b \binom{2\mu_b}{\mu_b}}{1 - \mu_b \psi_{m+b}}, \quad \pi^* \mathcal{X} = \Lambda^2(-1) \Lambda\left(\frac{1}{2}\right) \prod_{b=1}^{\ell} \frac{\mu_b \binom{2\mu_b}{\mu_b}}{1 - \mu_b \psi_{m+r+b}}, \tag{4.38}$$

so that the sum over  $r \geq 0$  and  $d_1, \dots, d_r \geq 1$  in (4.36) can be expressed as

$$\begin{aligned}
 &\sum_{r \geq 0} \frac{(-1)^r}{r!} \sum_{d_1, \dots, d_r \geq 1} \int_{\overline{\mathcal{M}}_{g, \ell+m+r}} (\pi^* \mathcal{X}) \prod_{a=1}^r \psi_a^{d_a+1} \\
 &= \sum_{r \geq 0} \frac{(-1)^r}{r!} \sum_{d_1, \dots, d_r \geq 1} \int_{\overline{\mathcal{M}}_{g, \ell+m}} \mathcal{X} \sum_{\sigma \in \mathfrak{S}_r} \prod_{\gamma \in \text{Cycles}(\sigma)} \kappa_{\sum_{a \in \gamma} d_a}. \tag{4.39}
 \end{aligned}$$

Let us now recall that for any set of variables  $F_1, F_2, \dots$ , we have the identity of symmetric functions

$$\exp \left( \sum_{r \geq 1} \frac{\xi^r}{r} F_r \right) = \sum_{\nu} \frac{\xi^{|\nu|}}{z_{\nu}} F_{\nu_1} \cdots F_{\nu_{\ell(\nu)}}, \tag{4.40}$$

where the sum on the right side extends over the set of all partitions  $\nu = (\nu_1, \dots, \nu_{\ell(\nu)})$ ,  $|\nu| = \nu_1 + \dots + \nu_{\ell(\nu)}$ , and  $z_{\nu} := \prod_{i \geq 1} (i^{m_i}) m_i!$ ,  $m_i$  being the multiplicity of  $i$  in the partition  $\nu$ , as above. Applying this relation to

$$F_r = \sum_{d_1, \dots, d_r \geq 1} \kappa_{\sum_{a=1}^r d_a} = \sum_{d \geq r} \binom{d-1}{r-1} \kappa_d, \quad \xi = -1, \tag{4.41}$$

since for any partition  $\nu$  of  $r$  the quantity  $r!/z_{\nu}$  is the cardinality of the conjugacy class labeled by  $\nu$  in  $\mathfrak{S}_r$ , we deduce that

$$\begin{aligned} \sum_{r \geq 0} \frac{(-1)^r}{r!} \sum_{d_1, \dots, d_r \geq 1} \int_{\mathcal{M}_{g, \ell+m}} \mathcal{X} \sum_{\sigma \in \mathfrak{S}_r} \prod_{\gamma \in \text{Cycles}(\sigma)} \kappa_{\sum_{a \in \gamma} d_a} \\ = \int_{\mathcal{M}_{g, \ell+m}} \mathcal{X} \exp\left(-\sum_{d \geq 1} \frac{\kappa_d}{d}\right), \end{aligned} \tag{4.42}$$

where we also use the identity  $\sum_{r \geq 1} \frac{(-1)^r}{r} \binom{d-1}{r-1} = -\frac{1}{d}$ . The proof is complete.  $\square$

*Example 4.6.* Comparing the coefficients of  $\epsilon^{-2}$  on both sides of (1.35), we obtain the following relation in genus zero

$$\mathcal{H}_{0, \mu} = 2^{\ell-2} \lambda^{|\mu|+2-\ell} \sum_{s=1}^{|\mu|+1-\ell} H_0^>(\mu; s) \tag{4.43}$$

valid for any partition  $\mu$  of length  $\ell$ . One can check that (4.43) is consistent with the computations of Hurwitz numbers in genus zero performed in Example 3.3.

For example, for  $\ell = 1$ , namely  $\mu = (\mu_1)$  we compute the first terms in the  $(\lambda - 1)$ -expansion of the left side of (4.43), directly from (1.35),

$$\mathcal{H}_{0, (\mu_1)} = \frac{1}{2} \frac{1}{\mu_1 + 1} \binom{2\mu_1}{\mu_1} + \frac{(\lambda - 1)}{2} \binom{2\mu_1}{\mu_1} + \frac{(\lambda - 1)^2}{4} \mu_1 \binom{2\mu_1}{\mu_1} + \mathcal{O}((\lambda - 1)^3). \tag{4.44}$$

On the other hand, the right side of (4.43) is computed as

$$\begin{aligned} \frac{1}{2} \lambda^{\mu_1+1} \sum_{s=1}^{\mu_1} H_0^>((\mu_1); s) &= \frac{1}{2\mu_1} \lambda^{\mu_1+1} \sum_{s=1}^{\mu_1} \binom{\mu_1}{s} \binom{\mu_1}{s-1} \\ &= \frac{1}{2\mu_1} \lambda^{\mu_1+1} \binom{2\mu_1}{\mu_1 - 1} \\ &= \frac{1}{2(\mu_1 + 1)} \binom{2\mu_1}{\mu_1} \sum_{b=0}^{\mu_1+1} \binom{\mu_1 + 1}{b} (\lambda - 1)^b, \end{aligned} \tag{4.45}$$

where we use (3.33) and the identity

$$\sum_{s=1}^{\mu_1} \binom{\mu_1}{s-1} \binom{\mu_1}{s} = \sum_{s=0}^{\mu_1-1} \binom{\mu_1}{s} \binom{\mu_1}{\mu_1 - 1 - s} = \binom{2\mu_1}{\mu_1 - 1}, \tag{4.46}$$

which follows from the *Chu-Vandermonde identity*  $\sum_{s=0}^{k-1} \binom{a}{s} \binom{b}{k-1-s} = \binom{a+b}{k-1}$  for  $a = b = k = \mu_1$ . Expressions (4.44) and (4.45) match.

### Acknowledgements

We are grateful to Marco Bertola, John Harnad and Di Yang for very useful discussions. This project has received funding from the European Union’s H2020 Research and Innovation Programme under the Marie Skłodowska-Curie Grant No. 778010 *IPaDEGAN*. G.R. wishes to thank the School of Mathematical Sciences at the University of Science and Technology of China

in Hefei for hospitality during which part of this work was completed; the research of G.R. is supported by the Fonds de la Recherche Scientifique—FNRS, under EOS Project O013018F. The authors wish to thank the anonymous referee for careful reading the manuscript and suggesting important improvements to the presentation.

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## Appendix A: Numerical Tables

### A.1. Tables of Some Weighted Strictly Monotone Double Hurwitz Numbers

We recall that  $H_g^>(\mu; s) = \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^>(\mu; \nu)$ , where  $h_g^>(\mu; \nu)$  is the strictly monotone double Hurwitz number with partitions  $\mu$  and  $\nu$ , see (1.22).

$\mu = (3, 1)$	$g = 0$	$g = 1$
$s = 1$	3	3
$s = 2$	9	0
$s = 3$	3	0

$\mu = (3, 2)$	$g = 0$	$g = 1$
$s = 1$	6	18
$s = 2$	30	18
$s = 3$	30	0
$s = 4$	6	0

$\mu = (3, 3)$	$g = 0$	$g = 1$	$g = 2$
$s = 1$	9	75	36
$s = 2$	72	198	0
$s = 3$	138	75	0
$s = 4$	72	0	0
$s = 5$	9	0	0

$\mu = (4, 4)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$s = 1$	16	616	3304	1104
$s = 2$	264	4636	8132	0
$s = 3$	1200	8496	3304	0
$s = 4$	1940	4636	0	0
$s = 5$	1200	616	0	0
$s = 6$	264	0	0	0
$s = 7$	16	0	0	0

$\mu = (6, 3)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$s = 1$	18	1428	16002	22872
$s = 2$	414	15120	70938	22872
$s = 3$	2598	43680	70938	0
$s = 4$	6210	43680	16002	0
$s = 5$	6210	15120	0	0
$s = 6$	2598	1428	0	0
$s = 7$	414	0	0	0
$s = 8$	18	0	0	0

$\mu = (2, 1, 1)$	$g = 0$	$\mu = (2, 2, 1)$	$g = 0$	$g = 1$	$\mu = (2, 2, 2)$	$g = 0$	$g = 1$
$s = 1$	6	$s = 1$	16	8	$s = 1$	40	80
$s = 2$	6	$s = 2$	40	0	$s = 2$	176	80
		$s = 3$	16	0	$s = 3$	176	0
					$s = 4$	40	0

$\mu = (4, 4, 4)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$
$s = 1$	704	89760	2631552	18161440	19033344
$s = 2$	21312	1568640	24587904	75241920	19033344
$s = 3$	204480	8507520	66562944	75241920	0
$s = 4$	843648	18934080	66562944	18161440	0
$s = 5$	1673856	18934080	24587904	0	0
$s = 6$	1673856	8507520	2631552	0	0
$s = 7$	843648	1568640	0	0	0
$s = 8$	204480	89760	0	0	0
$s = 9$	21312	0	0	0	0
$s = 10$	704	0	0	0	0

$\mu = (4, 3, 2, 1)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$\mu = (2, 2, 2, 2)$	$g = 0$	$g = 1$	$g = 2$
$s = 1$	1728	54432	235872	70848	$s = 1$	672	3360	1008
$s = 2$	26136	379512	570672	0	$s = 2$	4464	8016	0
$s = 3$	111024	680832	235872	0	$s = 3$	7872	3360	0
$s = 4$	175824	379512	0	0	$s = 4$	4464	0	0
$s = 5$	111024	54432	0	0	$s = 5$	672	0	0
$s = 6$	26136	0	0	0				
$s = 7$	1728	0	0	0				

$\mu = (5, 4, 4, 2)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
$s = 1$	29120	7047040	444924480	8434666240	42317475200	35974149120
$s = 2$	1212800	180513600	6829912320	71893480000	168041817600	35974149120
$s = 3$	16616960	1529449920	33913376640	186374568640	168041817600	0
$s = 4$	103248000	5796138240	72317482560	186374568640	42317475200	0
$s = 5$	331189440	11030467200	72317482560	71893480000	0	0
$s = 6$	584935680	11030467200	33913376640	8434666240	0	0
$s = 7$	584935680	5796138240	6829912320	0	0	0
$s = 8$	331189440	1529449920	444924480	0	0	0
$s = 9$	103248000	180513600	0	0	0	0
$s = 10$	16616960	7047040	0	0	0	0
$s = 11$	1212800	0	0	0	0	0
$s = 12$	29120	0	0	0	0	0

$\mu = (2, 2, 2, 1, 1)$	$g = 0$	$g = 1$	$\mu = (3, 3, 2, 2, 2)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$s = 1$	1680	3360	$s = 1$	71280	2661120	18461520	18722880
$s = 2$	7392	3360	$s = 2$	1206144	23973840	75182256	18722880
$s = 3$	7392	0	$s = 3$	6314976	63697968	75182256	0
$s = 4$	1680	0	$s = 4$	13791600	63697968	18461520	0
			$s = 5$	13791600	23973840	0	0
			$s = 6$	6314976	2661120	0	0
			$s = 7$	1206144	0	0	0
			$s = 8$	71280	0	0	0

**A.2. Tables of Some Weighted Weakly Monotone Double Hurwitz Numbers**

We recall that  $H_g^{\geq}(\mu; s) = \frac{z_{\mu}^{\mu}}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{\geq}(\mu; \nu)$ , where  $h_g^{\geq}(\mu; \nu)$  is the weakly monotone double Hurwitz number with partitions  $\mu$  and  $\nu$ , see (1.22).

In general,  $H_g^{\geq}(\mu; s) \neq 0$  for every  $s \leq |\mu|$  and  $g \geq 0$ ; therefore, we calculate  $H_g^{\geq}(\mu; s)$  for the first few values of  $g$ .

$\mu = (3, 1)$	$g = 0$	$g = 1$	$g = 2$
$s = 1$	3	45	483
$s = 2$	18	255	2688
$s = 3$	30	420	4410
$s = 4$	15	210	2205

$\mu = (3, 2)$	$g = 0$	$g = 1$	$g = 2$
$s = 1$	6	168	3402
$s = 2$	54	1464	29058
$s = 3$	156	4176	82212
$s = 4$	180	4800	94260
$s = 5$	72	1920	37704

$\mu = (3, 3)$	$g = 0$	$g = 1$	$g = 2$
$s = 1$	9	462	16443
$s = 2$	117	5742	197559
$s = 3$	516	24660	833472
$s = 4$	1008	47580	1594836
$s = 5$	900	42300	1413720
$s = 6$	300	14100	471240

$\mu = (1, 1, 1)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$
$s = 1$	4	20	84	340	1364
$s = 2$	12	60	252	1020	4092
$s = 3$	8	40	168	680	2728

$\mu = (3, 2, 1)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$
$s = 1$	42	2268	81774	2498496
$s = 2$	558	28248	982326	29405736
$s = 3$	2472	121320	4143024	122714160
$s = 4$	4836	234060	7926312	233606280
$s = 5$	4320	208080	7025760	206699040
$s = 6$	1440	69360	2341920	68899680

$\mu = (5, 3, 2)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
$s = 1$	330	98670	17117100	2288397540	262779844470	27370788935490
$s = 2$	11790	3139530	508126980	64989626220	7244914364850	739259601861510
$s = 3$	151140	37555800	5814501240	722008428240	78865374260700	793209599173640
$s = 4$	973200	231506100	34809669720	4236585517200	456285210221400	45429895491347220
$s = 5$	3600180	832748640	122812524600	14745786668160	1572851081541420	155505293985110400
$s = 6$	8126700	1346504080	268910866680	31999520486160	3391243294051140	333707416656660000
$s = 7$	11380320	2557716000	369587047200	43733298023520	4615886297332800	452853891923025600
$s = 8$	9649080	2155587000	310123401000	36581098895880	3852087017209200	377274782175656400
$s = 9$	4536000	1010772000	145151092800	17098516260000	1798743628584000	176040872796600000
$s = 10$	907200	202154400	29030218560	3419703252000	359748725716800	35208174559320000

$\mu = (1, 1, 1, 1)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
$s = 1$	30	420	4410	42240	390390	3554460
$s = 2$	174	2364	24498	233328	2151222	19565892
$s = 3$	288	3888	40176	382176	3521664	32022864
$s = 4$	144	1944	20088	191088	1760832	16011432

$\mu = (2, 2, 1, 1)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
$s = 1$	224	11760	417648	12652640	353825472	9465041040
$s = 2$	2936	145560	5001792	148676240	4111488168	109250057640
$s = 3$	12912	623088	21061152	619916064	17042443920	451231651728
$s = 4$	25176	1200264	40262736	1179630192	32339018280	854769872184
$s = 5$	22464	1066464	35678592	1043606592	28581355584	754984855584
$s = 6$	7488	355488	11892864	347868864	9527118528	2516661618528

$\mu = (3, 2, 2, 1)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$	$g = 5$
$s = 1$	1080	142560	11891880	808030080	49030839000	2777130588960
$s = 2$	24408	2975688	236613384	15604156944	928759785048	51934912866648
$s = 3$	195696	22833936	1764985248	114273524448	6718979907216	372620872120176
$s = 4$	764208	86946408	6607836864	423012867984	24682857466608	1361716707058488
$s = 5$	1622160	181944000	13692581280	870735528000	50576815946160	2781487931040000
$s = 6$	1911600	212829120	15934474080	1009718844480	58506896866320	3212163320083200
$s = 7$	1175040	130440960	9745954560	616691715840	35698249900800	1958572008345600
$s = 8$	293760	32610240	2436488640	154172928960	8924562475200	489643002086000

$\mu = (3, 3, 3, 3)$	$g = 0$	$g = 1$	$g = 2$	$g = 3$	$g = 4$
$s = 1$	14742	6781320	1863064476	397980044280	73027276324002
$s = 2$	684774	286543656	73938326364	15124478632344	2690423275640562
$s = 3$	11927088	4700315952	1162209509712	230530176869328	40089332784598560
$s = 4$	108506304	41049414576	9847619855856	1910059732782864	326635075616752080
$s = 5$	591049872	217264375440	50997568912848	9730568084094800	1643434518194147520
$s = 6$	2065978224	744104821680	171941934622896	32417467690208400	5425295582074933440
$s = 7$	4798180800	1703613513600	389301061256640	72772493528332800	12099023079466665600
$s = 8$	7485955200	2632114985400	596891523260160	110918372096491200	18356651181359395200
$s = 9$	7754940000	2709582840000	611410862412000	11317727916388000	18674140608815688000
$s = 10$	5114988000	1780691688000	400648862930400	73995902520393600	12187705122917006400
$s = 11$	1944000000	675695520000	151836376608000	28014789102336000	4610660182447564800
$s = 12$	324000000	112615920000	25306062768000	4669131517056000	768443363741260800

### A.3. Tables of Some Positive LUE Correlators

We write the correlators in terms of  $N$  and the parameter  $\alpha = N(c - 1)$ .

$$\begin{aligned}
\langle \text{tr } X \text{ tr } X \rangle_c &= N(\alpha + N), \\
\langle \text{tr } X^2 \text{ tr } X \rangle_c &= 2N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^2 \text{ tr } X^2 \rangle_c &= 2\alpha(1 + 2\alpha^2)N + 2(1 + 11\alpha^2)N^2 + 36\alpha N^3 + 18N^4, \\
\langle \text{tr } X^3 \text{ tr } X^1 \rangle_c &= 3N(\alpha + N)(1 + \alpha^2 + 5\alpha N + 5N^2) \\
\langle \text{tr } X^3 \text{ tr } X^2 \rangle_c &= 6N(\alpha + N)(\alpha + 2N)(3 + \alpha^2 + 6\alpha N + 6N^2) \\
\langle \text{tr } X^3 \text{ tr } X^3 \rangle_c &= 3N(\alpha + N)(12 + 25\alpha^2 + 3\alpha^4 + 4\alpha(29 + 9\alpha^2)N \\
&\quad + 4(29 + 34\alpha^2)N^2 + 200\alpha N^3 + 100N^4), \\
\langle \text{tr } X \text{ tr } X \text{ tr } X \rangle_c &= 2N(\alpha + N), \\
\langle \text{tr } X^2 \text{ tr } X \text{ tr } X \rangle_c &= 6N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^2 \text{ tr } X^2 \text{ tr } X \rangle_c &= 8\alpha(1 + 2\alpha^2)N + 8(1 + 11\alpha^2)N^2 + 144\alpha N^3 + 72N^4, \\
\langle \text{tr } X^2 \text{ tr } X^2 \text{ tr } X^2 \rangle_c &= 40\alpha^2(2 + \alpha^2)N + 48(5\alpha + 7\alpha^3)N^2 \\
&\quad + 16(10 + 59\alpha^2)N^3 + 1080\alpha N^4 + 432N^5, \\
\langle \text{tr } X^4 \text{ tr } X^3 \text{ tr } X^2 \rangle_c &= 24\alpha(328 + 1092\alpha^2 + 252\alpha^4 + 8\alpha^6)N \\
&\quad + 24(328 + 1092\alpha^2 + 252\alpha^4 + 8\alpha^6 \\
&\quad + \alpha(4826\alpha + 2765\alpha^3 + 169\alpha^5))N^2 + \\
&\quad + 24(4826\alpha + 2765\alpha^3 + 169\alpha^5 \\
&\quad + \alpha(4826 + 9935\alpha^2 + 1239\alpha^4))N^3 \\
&\quad + 24(4826 + 9935\alpha^2 + 1239\alpha^4 + \\
&\quad + \alpha(14340\alpha + 4240\alpha^3))N^4 \\
&\quad + 24(14340\alpha + 4240\alpha^3 + \alpha(7170 + 7370\alpha^2))N^5 \\
&\quad + 24(7170 + 13670\alpha^2)N^6 + 201600\alpha N^7 + 50400N^8, \\
\langle \text{tr } X \text{ tr } X \text{ tr } X \text{ tr } X \rangle_c &= 6N(\alpha + N), \\
\langle \text{tr } X^2 \text{ tr } X \text{ tr } X \text{ tr } X \rangle_c &= 24N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^2 \text{ tr } X^2 \text{ tr } X \text{ tr } X \rangle_c &= 40\alpha(1 + 2\alpha^2)N + 40(1 + 11\alpha^2)N^2 \\
&\quad + 720\alpha N^3 + 360N^4, \\
\langle \text{tr } X^2 \text{ tr } X^2 \text{ tr } X^2 \text{ tr } X \rangle_c &= 240\alpha^2(2 + \alpha^2)N + 288(5\alpha + 7\alpha^3)N^2 \\
&\quad + 96(10 + 59\alpha^2)N^3 + 6480\alpha N^4 + 2592N^5, \\
\langle \text{tr } X^2 \text{ tr } X^2 \text{ tr } X^2 \text{ tr } X^2 \rangle_c &= 48\alpha(21 + 14\alpha^2(5 + \alpha^2))N \\
&\quad + 48(21 + 377\alpha^2 + 163\alpha^4)N^2 \\
&\quad + 96(307\alpha + 338\alpha^3)N^3 + \\
&\quad + 48(307 + 1283\alpha^2)N^4 + 54432\alpha N^5 + 18144N^6, \\
\langle \text{tr } X \text{ tr } X \text{ tr } X \text{ tr } X \text{ tr } X \rangle_c &= 24N(\alpha + N), \\
\langle \text{tr } X^2 \text{ tr } X \text{ tr } X \text{ tr } X \text{ tr } X \rangle_c &= 120N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^2 \text{ tr } X^2 \text{ tr } X \text{ tr } X \text{ tr } X \rangle_c &= 240\alpha(1 + 2\alpha^2)N + 240(1 + 11\alpha^2)N^2 + 4320\alpha N^3 + 2160N^4,
\end{aligned}$$

$$\begin{aligned}
 \langle \text{tr } X^3 \text{tr } X^2 \text{tr } X^2 \text{tr } X^2 \text{tr } X \rangle_c &= 6048\alpha^2 (11 + \alpha^2) (3 + 2\alpha^2) N \\
 &+ 432 (42\alpha (11 + \alpha^2) (3 + 2\alpha^2) \\
 &+ 3\alpha^3 (611 + 121\alpha^2)) N^2 + \\
 &+ 432 (28 (11 + \alpha^2) (3 + 2\alpha^2) + 39\alpha^2 (47 + 37\alpha^2) \\
 &+ 9\alpha^2 (611 + 121\alpha^2)) N^3 + 432 (2160\alpha^3 + \\
 &+ 117\alpha (47 + 37\alpha^2) + 6\alpha (611 + 121\alpha^2)) N^4 \\
 &+ 2592 (611 + 1741\alpha^2) N^5 + 3265920\alpha N^6 \\
 &+ 933120N^7,
 \end{aligned}$$

**A.4. Tables of Some Negative LUE Correlators**

In the following formulæ we denote  $a_j := (\alpha - j)_{2j+1}$ ,  $\alpha = N(c - 1)$ .

$$\begin{aligned}
 \langle \text{tr } X^{-1} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_1 a_0} N(\alpha + N), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_2 a_0} 2N(\alpha + N)(\alpha + 2N), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \rangle_c &= \frac{1}{a_3 a_1} 2N(\alpha + N) (2\alpha^4 + 9\alpha^3 N + \alpha^2 (9N^2 - 5) \\
 &\quad - 21\alpha N - 21N^2 + 3), \\
 \langle \text{tr } X^{-3} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_3 a_0} 3N(\alpha + N) (\alpha^2 + 5\alpha N + 5N^2 + 1), \\
 \langle \text{tr } X^{-3} \text{tr } X^{-2} \rangle_c &= \frac{1}{a_4 a_1} 6N (\alpha + N) (\alpha + 2N) \\
 &\quad (2 + \alpha^4 - 26\alpha N + 6\alpha^3 N - 26N^2 \\
 &\quad + \alpha^2 (-3 + 6N^2)), \\
 \langle \text{tr } X^{-3} \text{tr } X^{-3} \rangle_c &= \frac{1}{a_5 a_2} 3N (\alpha + N) (320 - 444\alpha^2 + 147\alpha^4 \\
 &\quad - 26\alpha^6 + 3\alpha^8 + 4\alpha (-200 + 411\alpha^2 - 100\alpha^4 + 9\alpha^6) N \\
 &\quad + 4 (-200 + 1731\alpha^2 - 425\alpha^4 + 34\alpha^6) N^2 \\
 &\quad + 40\alpha (264 - 65\alpha^2 + 5\alpha^4) N^3 \\
 &\quad + 20 (264 - 65\alpha^2 + 5\alpha^4) N^4), \\
 \langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_2 a_0^2} 4N(\alpha + N)(\alpha + 2N), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_3 a_1 a_0} 2N(\alpha + N) (5\alpha^4 + 24\alpha^3 N \\
 &\quad + \alpha^2 (24N^2 - 5) - 36\alpha N - 36N^2), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_4 a_1 a_0} 24N(\alpha + N)(\alpha + 2N) (\alpha^4 + 6\alpha^3 N \\
 &\quad + \alpha^2 (6N^2 - 3) - 26\alpha N - 26N^2 + 2), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^{-2} \rangle_c &= \frac{1}{a_5 a_1^2} 8N(\alpha + N) (7\alpha^2 (\alpha^2 - 7) (\alpha^2 - 1)^2 \\
 &\quad + 36 (6\alpha^4 - 71\alpha^2 + 125) N^4
 \end{aligned}$$

$$\begin{aligned}
 &+ 72\alpha (6\alpha^4 - 71\alpha^2 + 125) N^3 \\
 &+ 9(\alpha - 1)\alpha(\alpha + 1) (9\alpha^4 - 89\alpha^2 + 100) N \\
 &+ 9 (33\alpha^6 - 382\alpha^4 + 689\alpha^2 - 100) N^2), \\
 \langle \text{tr } X^{-3} \text{tr } X^{-2} \text{tr } X^{-2} \rangle_c &= \frac{1}{a_6 a_2 a_1} 24N(\alpha + N)(\alpha + 2N) \left( 2\alpha^2 (\alpha^2 - 1)^2 \right. \\
 &\quad \left. (2\alpha^4 - 25\alpha^2 + 68) \right. \\
 &\quad + 60 (3\alpha^6 - 71\alpha^4 + 488\alpha^2 - 840) N^4 + \\
 &\quad + 120\alpha (3\alpha^6 - 71\alpha^4 + 488\alpha^2 - 840) N^3 \\
 &\quad + (\alpha - 1)\alpha(\alpha + 1) (57\alpha^6 - 1105\alpha^4 + 6148\alpha^2 - 7200) N + \\
 &\quad \left. + (237\alpha^8 - 5422\alpha^6 + 36533\alpha^4 - 63748\alpha^2 + 7200) N^2 \right), \\
 \langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_3 a_1 a_0^2} 6N(\alpha + N) (5\alpha^4 + 24\alpha^3 N + \alpha^2 (24N^2 - 5) \\
 &\quad - 36\alpha N - 36N^2), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_4 a_1 a_0^2} 12N(\alpha + N)(\alpha + 2N) \\
 &\quad (7\alpha^4 + 44\alpha^3 N + \alpha^2 (44N^2 - 7) \\
 &\quad - 144\alpha N - 144N^2), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^{-1} \rangle_c &= \frac{1}{a_5 a_2 a_1 a_0} 8N(\alpha + N) \\
 &\quad \left( 14\alpha^2 (\alpha^2 - 1)^2 (2\alpha^4 - 13\alpha^2 + 20) \right. \\
 &\quad + 12 (78\alpha^6 - 1055\alpha^4 + 4237\alpha^2 - 3800) N^4 + \\
 &\quad + 24\alpha (78\alpha^6 - 1055\alpha^4 + 4237\alpha^2 - 3800) N^3 \\
 &\quad + 3(\alpha - 1)\alpha(\alpha + 1) \\
 &\quad \left. (113\alpha^6 - 1205\alpha^4 + 3632\alpha^2 - 800) N + \right. \\
 &\quad \left. + 3 (425\alpha^8 - 5538\alpha^6 + 21785\alpha^4 - 19632\alpha^2 + 800) N^2 \right),
 \end{aligned}$$

**A.5. Tables of Some Mixed LUE Correlators**

In the following formulæ we denote  $a_j := (\alpha - j)_{2j+1}$ ,  $\alpha = N(c - 1)$ .

$$\begin{aligned}
 \langle \text{tr } X^{-1} \text{tr } X^1 \rangle_c &= -\frac{1}{a_0} N, \\
 \langle \text{tr } X^{-2} \text{tr } X^1 \rangle_c &= -\frac{1}{a_1} 2N(\alpha + N), \\
 \langle \text{tr } X^{-1} \text{tr } X^2 \rangle_c &= -\frac{1}{a_0} 2N(\alpha + N), \\
 \langle \text{tr } X^{-2} \text{tr } X^2 \rangle_c &= -\frac{1}{a_1} (\alpha + 2N) (\alpha^2 + 2\alpha N + 2N^2 - 1), \\
 \langle \text{tr } X^{-3} \text{tr } X^1 \rangle_c &= -\frac{1}{a_2} 3N (\alpha + N) (\alpha + 2N), \\
 \langle \text{tr } X^{-3} \text{tr } X^2 \rangle_c &= -\frac{1}{a_2} 6N (\alpha + N) (-2 + \alpha^2 + 2\alpha N + 2N^2),
 \end{aligned}$$



$$\begin{aligned}
\langle \text{tr } X^{-1} \text{tr } X^3 \rangle_c &= -\frac{1}{a_0} 3N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^{-2} \text{tr } X^3 \rangle_c &= -\frac{1}{a_1} 6N(\alpha + N)(-1 + \alpha^2 + 2\alpha N + 2N^2), \\
\langle \text{tr } X^{-3} \text{tr } X^3 \rangle_c &= -\frac{1}{a_2} 3(2\alpha(4 - 5\alpha^2 + \alpha^4) + (4 - 11\alpha^2 + 3\alpha^4)N \\
&\quad + 6\alpha(-3 + 2\alpha^2)N^2 + 4(-3 + 7\alpha^2)N^3 + 30\alpha N^4 + 12N^5), \\
\langle \text{tr } X^{-1} \text{tr } X^1 \text{tr } X^1 \rangle_c &= 0, \\
\langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^1 \rangle_c &= -\frac{1}{a_0 a_1} 2N(\alpha + N), \\
\langle \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^1 \rangle_c &= -\frac{1}{a_2 a_0} 6N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^{-1} \text{tr } X^2 \text{tr } X^1 \rangle_c &= -\frac{1}{a_0} 2N(\alpha + N), \\
\langle \text{tr } X^{-2} \text{tr } X^1 \text{tr } X^1 \rangle_c &= \frac{1}{a_1} 2N(\alpha + N), \\
\langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^2 \rangle_c &= -\frac{1}{a_0 a_1} 2N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^{-1} \text{tr } X^2 \text{tr } X^2 \rangle_c &= -\frac{1}{a_0} 8N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^{-2} \text{tr } X^2 \text{tr } X^1 \rangle_c &= 0, \\
\langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^1 \rangle_c &= -\frac{1}{a_3 a_1} 8N(\alpha + N)(2\alpha^4 + 9\alpha^3 N + \alpha^2(9N^2 - 5) \\
&\quad - 21\alpha N - 21N^2 + 3), \\
\langle \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^2 \rangle_c &= -\frac{1}{a_2 a_0} 8N(\alpha + N)(-1 + \alpha^2 + 3\alpha N + 3N^2), \\
\langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^3 \rangle_c &= -\frac{1}{a_1 a_0} 12N^2(\alpha + N)^2, \\
\langle \text{tr } X^{-1} \text{tr } X^1 \text{tr } X^3 \rangle_c &= -\frac{1}{a_0} 6N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^{-3} \text{tr } X^{-1} \text{tr } X^1 \rangle_c &= -\frac{1}{a_3 a_0} 12N(\alpha + N)(\alpha^2 + 5\alpha N + 5N^2 + 1), \\
\langle \text{tr } X^{-3} \text{tr } X^1 \text{tr } X^1 \rangle_c &= \frac{1}{a_2} 6N(\alpha + N)(\alpha + 2N), \\
\langle \text{tr } X^{-3} \text{tr } X^{-2} \text{tr } X^1 \rangle_c &= -\frac{1}{a_4 a_1} 30N(\alpha + N)(\alpha + 2N)(\alpha^4 + 6\alpha^3 N + \alpha^2(6N^2 - 3) \\
&\quad - 26\alpha N - 26N^2 + 2), \\
\langle \text{tr } X^{-3} \text{tr } X^{-2} \text{tr } X^2 \rangle_c &= -\frac{1}{a_4 a_1} 24N(\alpha + N)(2\alpha^6 - 17\alpha^4 + 10(3\alpha^2 - 13)N^4 \\
&\quad + 20(3\alpha^2 - 13)\alpha N^3 + 23\alpha^2 + \\
&\quad + 2(23\alpha^4 - 120\alpha^2 + 57)N^2 + 2(8\alpha^4 - 55\alpha^2 + 57)\alpha N - 8), \\
\langle \text{tr } X^{-1} \text{tr } X^1 \text{tr } X^1 \text{tr } X^1 \rangle_c &= 0, \\
\langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^1 \text{tr } X^1 \rangle_c &= \frac{1}{a_1 a_0} 2N(\alpha + N),
\end{aligned}$$

$$\begin{aligned}
 \langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^1 \rangle_c &= -\frac{1}{a_2 a_0^2} 12N(\alpha + N)(\alpha + 2N), \\
 \langle \text{tr } X^{-2} \text{tr } X^1 \text{tr } X^1 \text{tr } X^1 \rangle_c &= 0, \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^1 \text{tr } X^1 \rangle_c &= \frac{1}{a_3 a_1} 24N(\alpha + N)(2\alpha^4 + 9\alpha^3 N + \alpha^2(9N^2 - 5) \\
 &\quad - 21\alpha N - 21N^2 + 3), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^1 \rangle_c &= -\frac{1}{a_4 a_1 a_0} 120N(\alpha + N)(\alpha + 2N)(\alpha^4 + 6\alpha^3 N \\
 &\quad + \alpha^2(6N^2 - 3) - 26\alpha N - 26N^2 + 2), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^2 \rangle_c &= -\frac{1}{a_3 a_1 a_0} 24N(\alpha + N)(-1 + \alpha + 2N)(\alpha + 2N) \\
 &\quad (1 + \alpha + 2N)(2\alpha^2 - 3), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-1} \text{tr } X^2 \text{tr } X^1 \rangle_c &= \frac{1}{a_2 a_0} 8N(\alpha + N)(-1 + \alpha^2 + 3\alpha N + 3N^2), \\
 \langle \text{tr } X^{-2} \text{tr } X^2 \text{tr } X^1 \text{tr } X^1 \rangle_c &= 0, \\
 \langle \text{tr } X^{-1} \text{tr } X^2 \text{tr } X^2 \text{tr } X^1 \rangle_c &= -\frac{1}{a_0} 24N(\alpha + N)(\alpha + 2N), \\
 \langle \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^2 \text{tr } X^2 \rangle_c &= -\frac{1}{a_1 a_0} 24N^2(\alpha + N)^2, \\
 \langle \text{tr } X^{-3} \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^2 \rangle_c &= -\frac{1}{a_4 a_1 a_0} 24N(\alpha + N)(100(\alpha^2 - 2)N^4 + 200\alpha(\alpha^2 - 2)N^3 + \\
 &\quad + 2(73\alpha^4 - 165\alpha^2 + 52)N^2 + 2\alpha(23\alpha^4 - 65\alpha^2 + 52)N \\
 &\quad + 5\alpha^2(\alpha^4 - 3\alpha^2 + 2)).
 \end{aligned}$$

**A.6. Topological Expansion of Some Mixed Correlators**

We compute the first terms in the large  $N$  expansion for some of the above mixed correlators, see with Proposition 1.2.

$$\begin{aligned}
 \langle \text{tr } X^{-2} \text{tr } X^3 \rangle_c &= -N \left( \frac{12}{(c-1)^3} + \frac{24}{(c-1)^2} + \frac{18}{c-1} + 6 \right) \\
 &\quad - \frac{1}{N} \left( \frac{12}{(c-1)^5} + \frac{24}{(c-1)^4} + \frac{12}{(c-1)^3} \right) + \\
 &\quad - \frac{1}{N^3} \left( \frac{12}{(c-1)^7} + \frac{24}{(c-1)^6} + \frac{12}{(c-1)^5} \right) \\
 &\quad - \frac{1}{N^5} \left( \frac{12}{(c-1)^9} + \frac{24}{(c-1)^8} + \frac{12}{(c-1)^7} \right) + O\left(\frac{1}{N^7}\right), \\
 \langle \text{tr } X^{-4} \text{tr } X^4 \rangle_c &= - \left( \frac{400}{(c-1)^7} + \frac{1400}{(c-1)^6} + \frac{1968}{(c-1)^5} + \frac{1420}{(c-1)^4} \right. \\
 &\quad \left. + \frac{560}{(c-1)^3} + \frac{120}{(c-1)^2} + \frac{16}{c-1} + 2 \right) + \\
 &\quad - \frac{1}{N^2} \left( \frac{5600}{(c-1)^9} + \frac{19600}{(c-1)^8} + \frac{26920}{(c-1)^7} + \frac{18300}{(c-1)^6} \right. \\
 &\quad \left. + \frac{6320}{(c-1)^5} + \frac{980}{(c-1)^4} + \frac{40}{(c-1)^3} \right) +
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{N^4} \left( \frac{58800}{(c-1)^{11}} + \frac{205800}{(c-1)^{10}} + \frac{280448}{(c-1)^9} + \frac{186620}{(c-1)^8} \right. \\
 & \left. + \frac{61560}{(c-1)^7} + \frac{8620}{(c-1)^6} + \frac{232}{(c-1)^5} \right) + O\left(\frac{1}{N^6}\right), \\
 \langle \text{tr } X^{-1} \text{tr } X^2 \text{tr } X^2 \rangle_c &= -N^2 \left( \frac{16}{c-1} + 24 + 8(c-1) \right), \\
 \langle \text{tr } X^{-2} \text{tr } X^{-2} \text{tr } X^1 \rangle_c &= -\frac{1}{N^4} \left( \frac{72}{(c-1)^8} + \frac{144}{(c-1)^7} + \frac{88}{(c-1)^6} + \frac{16}{(c-1)^5} \right) + \\
 & -\frac{1}{N^6} \left( \frac{912}{(c-1)^{10}} + \frac{1824}{(c-1)^9} + \frac{1112}{(c-1)^8} + \frac{200}{(c-1)^7} \right) + \\
 & -\frac{1}{N^8} \left( \frac{9144}{(c-1)^{12}} + \frac{18288}{(c-1)^{11}} + \frac{11160}{(c-1)^{10}} + \frac{2016}{(c-1)^9} \right) \\
 & + O\left(\frac{1}{N^{10}}\right), \\
 \langle \text{tr } X^{-3} \text{tr } X^{-2} \text{tr } X^2 \text{tr } X^2 \rangle_c &= \frac{1}{N^3} \left( \frac{720}{(c-1)^{10}} + \frac{2160}{(c-1)^9} + \frac{2544}{(c-1)^8} + \frac{1488}{(c-1)^7} \right. \\
 & \left. + \frac{432}{(c-1)^6} + \frac{48}{(c-1)^5} \right) + \frac{1}{N^5} \left( \frac{19200}{(c-1)^{12}} + \frac{57600}{(c-1)^{11}} \right. \\
 & \left. + \frac{66864}{(c-1)^{10}} + \frac{37728}{(c-1)^9} + \frac{10344}{(c-1)^8} + \frac{1080}{(c-1)^7} \right) + \\
 & + \frac{1}{N^7} \left( \frac{377040}{(c-1)^{14}} + \frac{1131120}{(c-1)^{13}} + \frac{1304688}{(c-1)^{12}} + \frac{724176}{(c-1)^{11}} \right. \\
 & \left. + \frac{193056}{(c-1)^{10}} + \frac{19488}{(c-1)^9} \right) + O\left(\frac{1}{N^9}\right), \\
 \langle \text{tr } X^{-3} \text{tr } X^{-1} \text{tr } X^{-1} \text{tr } X^2 \rangle_c &= -\frac{1}{N^5} \left( \frac{2400}{(c-1)^{11}} + \frac{7200}{(c-1)^{10}} + \frac{8304}{(c-1)^9} + \frac{4608}{(c-1)^8} \right. \\
 & \left. + \frac{1224}{(c-1)^7} + \frac{120}{(c-1)^6} \right) + \\
 & -\frac{1}{N^7} \left( \frac{69600}{(c-1)^{13}} + \frac{208800}{(c-1)^{12}} + \frac{239904}{(c-1)^{11}} \right. \\
 & \left. + \frac{131808}{(c-1)^{10}} + \frac{34464}{(c-1)^9} + \frac{3360}{(c-1)^8} \right) + \\
 & -\frac{1}{N^9} \left( \frac{1430400}{(c-1)^{15}} + \frac{4291200}{(c-1)^{14}} + \frac{4923408}{(c-1)^{13}} + \frac{2694816}{(c-1)^{12}} \right. \\
 & \left. + \frac{700248}{(c-1)^{11}} + \frac{68040}{(c-1)^{10}} \right) + O\left(\frac{1}{N^{11}}\right).
 \end{aligned}$$

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Massimo Gisonni, Tamara Grava and Giulio Ruzza  
SISSA  
via Bonomea 265  
34136 Trieste  
Italy  
e-mail: [massimo.gisonni@sissa.it](mailto:massimo.gisonni@sissa.it);  
[grava@sissa.it](mailto:grava@sissa.it);  
[giulio.ruzza@uclouvain.be](mailto:giulio.ruzza@uclouvain.be)

Tamara Grava  
School of Mathematics  
University of Bristol  
Bristol  
UK

Giulio Ruzza  
IRMP  
UCLouvain  
Chemin du Cyclotron 2  
1348 Ottignies-Louvain-la-Neuve  
Belgium

Communicated by Nikolai Kitanine.

Received: January 7, 2020.

Accepted: May 15, 2020.