



A Nonrelativistic Quantum Field Theory with Point Interactions in Three Dimensions

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Abstract. We construct a Hamiltonian for a quantum-mechanical model of nonrelativistic particles in three dimensions interacting via the creation and annihilation of a second type of nonrelativistic particles, which are bosons. The interaction between the two types of particles is a point interaction concentrated on the points in configuration space where the positions of two different particles coincide. We define the operator, and its domain of self-adjointness, in terms of co-dimension-three boundary conditions on the set of collision configurations relating sectors with different numbers of particles.

1. Introduction

In this article, we introduce a mathematical model for an interacting system composed of two kinds of nonrelativistic quantum particles. The number of the first kind of particles, which we call x -particles, is conserved. These particles interact by creating and annihilating bosons of a second kind, called y -particles. The interaction is supported by the set of collision configurations between at least one x -particle and one y -particle. Formally, it is given by the linear coupling $a(\delta_x) + a^*(\delta_x)$, where δ_x denotes the delta distribution at the position x of an x -particle and a^* , a are the creation and annihilation operators of the y -particles.

A very similar model is the Bogoliubov–Fröhlich polaron, used in physics to describe the interaction of impurities, the x -particles, with a dilute Bose–Einstein condensate [6, 7, 26]. In this context, it is natural to assume point interactions due to the dilute nature of the system. Furthermore, in the Bogoliubov approximation, the quasi-particles describing excitations of the condensate have a dispersion relation $\omega(k) = \sqrt{ak^2 + bk^4}$ that grows quadratically for large momenta, so they behave similarly to nonrelativistic particles.

Our model is also closely related to the local, Galilean invariant, Lee model [9, 20]. In that model, there are three types of nonrelativistic particles,

usually called V , Θ and N . The V -particles can create Θ -particles, whereupon they are transformed into N -particles. Conversely, a Θ and an N -particle can combine to form a V -particle. These processes are such that the total mass is conserved, $m_V = m_\Theta + m_N$. Since the N and Θ -particles cannot create any further ones, this model is composed of invariant sectors with a bounded total number of particles. This and the constraint on the masses are the essential differences to the model we will consider.

Because of the singular interaction, the formal Hamiltonian of such a model is ultraviolet divergent and ill defined. We will define a self-adjoint and bounded-from-below Hamiltonian for our model and describe its domain using (generalised) boundary conditions that relate the (singular) behaviour of the wavefunction near the collision configurations and the wavefunction with one fewer y -particle. Such boundary conditions have appeared independently in the literature as models for nuclear reactions and simplified models of quantum field theory (see Moshinsky [15–17], Moshinsky–Laurrabaquio [18], Thomas [25], and Yafaev [28]). More recently, they were proposed as a general approach to the ultraviolet problem by Teufel and Tumulka [23, 24], who called them interior-boundary conditions. The result most closely related to our question is the one of [25], which treats a model with one or two x -particles and at most one y -particle, and resembles one of the sectors of the Galilean Lee model. A variant of our model, where the x -particles are not dynamical but fixed at certain locations, was treated in [28], with at most one y -particle, and by Schmidt, Teufel, Tumulka and the author [12] on the whole Fock space. Schmidt and the author [11] showed that one can define the Hamiltonian for less singular models with dynamical x -particles using this type of boundary conditions. Examples include the Nelson model and the two-dimensional variant of our problem. These results were generalised by Schmidt to general dispersion relations for the x -particles [21] and massless models [22]. We expect that our analysis can be similarly generalised.

For many models in nonrelativistic quantum field theory, the Hamiltonian may be defined by a renormalisation procedure. To our knowledge, no rigorous method was known to work for our problem, so far. It was observed numerically in [7] (for a model with the same ultraviolet behaviour as ours) that, after subtracting the expected linearly divergent quantity, there is still a logarithmic divergence in the ultraviolet cutoff. Concerning rigorous results, the less singular cases treated in [11] can be renormalised using a technique due to Nelson [8, 19]—but this does not seem to work for the three-dimensional model (the conditions of e.g. [8, Thm 3.3] are not satisfied, and the predicted divergence would be in conflict with the numerical results of [7]). Schrader [20] used a reordered resolvent expansion to renormalise the Hamiltonian for the Galilean Lee model. However, later generalisations of this method [5, 27] do not cover our model either. This seems to be related to the fact that the constraint on the masses in the Galilean Lee model alters the structure of the singularities in a specific way (see Remark 11).

In this article, we will adapt the techniques of [11] to define the Hamiltonian for our model. We explain how this result can be understood in the

language of renormalisation in Remark 2. We also give an explicit characterisation of the domain, which is generally not easy to obtain by renormalisation.

2. Overview and Results

In this section, we present our main results and the reasoning behind them, while leaving the technical details for the later sections. Let us first introduce the necessary notation. We consider a fixed number M of x -particles, so the Hilbert space of our problem is

$$\mathcal{H} := L^2(\mathbb{R}^{3M}) \otimes \Gamma(L^2(\mathbb{R}^3)),$$

where $\Gamma(L^2(\mathbb{R}^3))$ is the bosonic Fock space over $L^2(\mathbb{R}^3)$. We will denote the sector of \mathcal{H} with n y -particles by $\mathcal{H}^{(n)}$, by $\psi^{(n)}$ the component of $\psi \in \mathcal{H}$ in this sector and by $N = d\Gamma(1)$ the number operator. The Hamiltonian for the non-interacting model is

$$L = -\frac{1}{2m} \sum_{\mu=1}^M \Delta_{x_\mu} + d\Gamma(-\Delta_y + 1), \tag{1}$$

where m is the mass of the x -particles, and we have set the mass of the y -particles to $\frac{1}{2}$ and their rest energy to one. Its domain $D(L)$ is given by

$$D(L) = \left\{ \psi \in D(N) \mid \forall n \in \mathbb{N} : \psi^{(n)} \in H^2(\mathbb{R}^{3(M+n)}) \right\}.$$

The interaction operator is formally given by

$$\sum_{\mu=1}^M (a^*(\delta_{x_\mu}) + a(\delta_{x_\mu})).$$

The obvious problem with this operator is that the creation operator is not a densely defined operator on \mathcal{H} , as

$$a^*(\delta_{x_\mu})\psi^{(n)} = \frac{1}{\sqrt{n+1}} \sum_{i=1}^n \delta_{x_\mu}(y_i)\psi^{(n)}(X, \hat{Y}_i)$$

(where $X = (x_1, \dots, x_M)$, $\hat{Y}_i = (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_{n+1}) \in \mathbb{R}^{3n}$) is not an element of $\mathcal{H}^{(n+1)}$ for any nonzero $\psi^{(n)} \in \mathcal{H}^{(n)}$. The annihilation operator is less problematic, since

$$a(\delta_{x_\mu})\psi^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \psi^{(n)}(X, Y)|_{y_i=x_\mu} = \sqrt{n}\psi^{(n)}(X, Y)|_{y_n=x_\mu}, \tag{2}$$

is a well-defined element of \mathcal{H} for $\psi \in D(L)$. Note, however, that the creation operator is a well-defined operator from \mathcal{H} to a space of distributions. In this sense, it is the adjoint of the annihilation operator, $(a(\delta_x))^* : \mathcal{H}^{(n)} \rightarrow D(L)'$.

Our approach is not to give a meaning to the interaction operator directly, but rather to implement an interaction using boundary conditions on the sets

$$\mathcal{E}^n = \left\{ (X, Y) \in \mathbb{R}^{3M} \times \mathbb{R}^{3n} \mid \prod_{\mu=1}^M \prod_{i=1}^n |x_\mu - y_i| = 0 \right\} \tag{3}$$

of collision configurations between the x -particles and the y -particles. We will then see that the resulting operator includes an interaction of the desired form when interpreted in the sense of distributions. This is similar to the representation of point interactions, with fixed particle number, by boundary conditions (see e.g. [2–4, 13, 14]).

In order to impose boundary conditions for L , we first restrict L to the domain

$$D(L_0) = \left\{ \psi \in D(N) \mid \forall n \in \mathbb{N} : \psi^{(n)} \in H_0^2(\mathbb{R}^{3(M+n)} \setminus \mathcal{E}^n) \right\} \tag{4}$$

of functions that vanish on \mathcal{E}^n . The adjoint L_0^* of $L_0 = L|_{D(L_0)}$ is then an extension of L . Its domain contains functions of the form

$$G\varphi^{(n)} := - \sum_{\mu=1}^M L^{-1} a^*(\delta_{x_\mu}) \varphi^{(n)} \in \mathcal{H}^{(n+1)}, \tag{5}$$

with $\varphi \in \mathcal{H}$. These functions diverge like $|x_\mu - y_i|^{-1}$ near the set where $|x_\mu - y_i| = 0$ (see Eq. (14)). We interpret

$$(B_\mu \psi^{(n+1)})(X, Y) = \beta \lim_{y_{n+1} \rightarrow x_\mu} |x_\mu - y_{n+1}| \psi^{(n+1)}(X, Y, y_{n+1}),$$

with some constant β (depending on m and n), as a singular boundary value of such functions (the limit exists almost everywhere and defines an element of $\mathcal{H}^{(n)}$; see Lemma 7). Note that, by symmetry, the limit is the same for $y_j \rightarrow x$, for any $j = 1, \dots, n + 1$. We set $B = \sum_{\mu=1}^M B_\mu$ and choose β so that $BG\varphi = M\varphi$. The free operator L then corresponds to the operator with the Dirichlet-type boundary condition $B\psi = 0$.

There is a second relevant boundary value operator, given by the finite part of $\psi^{(n+1)}$ at $y_{n+1} = x_\mu$. More precisely

$$\begin{aligned} & (A_\mu \psi^{(n+1)})(X, Y) \\ &= \lim_{r \rightarrow 0} \frac{\sqrt{n+1}}{4\pi} \int_{S^2} \left(\psi^{(n+1)}(X, Y, x_\mu + r\omega) - \frac{\alpha}{r} (B\psi^{(n+1)})(X, Y) \right) d\omega, \end{aligned}$$

where $\alpha = (M\beta)^{-1}$ and the limit is taken in the distributional sense (see Lemma 9). Note that this is a local boundary value operator extending the evaluation at $y_{n+1} = x_\mu$. We set $A = \sum_{\mu=1}^M A_\mu$, which then extends the annihilation operator.

It has been shown for variants of our model (with fixed x -particles in [12], and in two dimensions in [11]) that the operator $L_0^* + A$ is self-adjoint on a domain characterised by the boundary conditions $B\psi^{(n+1)} = M\psi^{(n)}$, for all $n \in \mathbb{N}$. These conditions should be viewed as relations on the space $D(L) \oplus G\mathcal{H} \subset D(L_0^*)$. Since $G\mathcal{H} \cap D(L) = \{0\}$, any vector in that space can be written as $\psi = \varphi + G\xi$ with unique $\varphi \in D(L)$, $\xi \in \mathcal{H}$. Because $B\varphi = 0$ and $BG\xi = M\xi$, the condition $B\psi = M\psi$ is then equivalent to $\psi = \xi$, or $\varphi = \psi - G\psi \in D(L)$. To see how this boundary condition is related to the creation and annihilation operators, first observe that $G\psi$ is in the kernel of

L_0^* , for every $\psi \in \mathcal{H}$. That is, for any $\varphi \in D(L_0)$, which vanishes on the set \mathcal{C}^n of collision configurations by definition, we have

$$\begin{aligned} \langle L_0^* G\psi, \varphi \rangle_{\mathcal{H}} &= \langle \psi, G^* L_0 \varphi \rangle_{\mathcal{H}} \\ &= - \sum_{n=0}^{\infty} \sum_{\mu=1}^M \langle \psi^{(n)}, a(\delta_{x_\mu}) \varphi^{(n+1)} \rangle_{\mathcal{H}^{(n)}} = 0. \end{aligned} \tag{6}$$

Then, taking any $\psi \in \mathcal{H}$ such that $\psi - G\psi \in D(L)$ gives

$$L_0^* \psi + A\psi = L(\psi - G\psi) + A\psi = L\psi + \sum_{\mu=1}^M a^*(\delta_{x_\mu})\psi + A\psi, \tag{7}$$

in the sense of distributions. Thus, $L_0^* + A$ represents an operator with creation and annihilation operators up to a choice of domain, including the boundary condition, and the choice of the extension A . The latter choice is made in a natural way such that the operator is local.

In the proof of self-adjointness for the two-dimensional model [11], a key step is then to use again that $G^*L = -\sum_{\mu=1}^M a(\delta_{x_\mu})$ and that A extends the annihilation operator, to rewrite the Hamiltonian as

$$\begin{aligned} L_0^* \psi + A\psi &= L(1 - G)\psi + A\psi \\ &= (1 - G)^* L(1 - G)\psi + A\psi - \sum_{\mu=1}^M a(\delta_{x_\mu})(1 - G)\psi \\ &= (1 - G)^* L(1 - G)\psi + AG\psi. \end{aligned} \tag{8}$$

The operator $T = AG$ is a symmetric operator on a domain $D(T) \supset D(L)$, and the proof is then essentially reduced to proving appropriate bounds for this operator on the domain of L_0^* with the boundary condition $\psi - G\psi \in D(L)$. This operator also appears in the theory of point interactions, where it is known as the Skornyakov–Ter-Martirosyan operator.

In the three-dimensional case, the range of G is not contained in the domain $D(T) = D(L^{1/2})$ of T . Since a vector ψ satisfying $\psi - G\psi \in D(L)$ is in $D(L^{1/2})$ if and only if $G\psi$ is, T and thus also A do not map all vectors satisfying $(1 - G)\psi \in D(L)$ to \mathcal{H} (see also Lemma 8 and Remark 10). Consequently, this is not a good domain for $L_0^* + A$. This forces us to modify our approach, by treating T as a perturbation to the free part L . We let

$$K = L + T \tag{9}$$

with $D(K) = D(L)$. This operator is self-adjoint and bounded from below (see Sect. 3.3). We may restrict it to the kernel of $a(\delta_x)$, as above, and define

$$K_0 = K|_{D(L_0)}.$$

The domain of the adjoint K_0^* contains functions of the form

$$G_T \varphi^{(n)} := - \sum_{\mu=1}^M (K + c_0)^{-1} a^*(\delta_{x_\mu}) \varphi^{(n)} \in \mathcal{H}^{(n+1)},$$

where $c_0 > 0$ is chosen appropriately. These functions are in the kernel of $K_0^* + c_0$, by the reasoning of Eq. (6). Their main divergence is still proportional to $|x_\mu - y_i|^{-1}$, but their asymptotic expansion also contains a logarithmically divergent term, as we will show. Thus, the boundary operator B will still be defined on such functions and we can extend $a(\delta_{x_\mu})$, by taking the finite part in this expansion, which yields an operator A_T .

$$(A_T\psi^{(n+1)})(X, Y) = \lim_{r \rightarrow 0} \sum_{\mu=1}^M \frac{1}{4\pi} \int_{S^2} (\sqrt{n+1}\psi^{(n+1)}(X, Y, x_\mu + r\omega) + f(r)(B\psi^{(n+1)})(X, Y)) d\omega,$$

where $f(r) = \alpha r^{-1} + \gamma \log r$ is a fixed function, with explicit constants α , as for A , and γ (depending on m , see Eq. (32)). This local boundary value operator then extends the sum of the $a(\delta_{x_\mu})$ to elements of the range of G_T (see Proposition 13). Its action there is given by

$$A_T G_T = T + S,$$

with an operator S that is symmetric on $D(S) = D(L^\varepsilon)$ for any $\varepsilon > 0$.

For any $\psi \in D(K_0^*)$ such that $\psi - G_T\psi \in D(L)$, we then have, as in (7),

$$\begin{aligned} L\psi + \sum_{\mu=1}^M a^*(\delta_{x_\mu}) + A_T\psi &= (K + c_0)(1 - G_T)\psi - c_0\psi - T\psi + A_T\psi \\ &= K_0^*\psi + (A_T - T)\psi. \end{aligned} \tag{10}$$

The left-hand side is similar to (7), but we have chosen a different domain $D(K_0^*)$ and thus also a different extension A_T . This is the expression that will replace $L_0^* + A$ in our case of the three-dimensional model. Note that $A_T - T = A_T(1 - G_T) + S$, so this operator is better behaved than A_T alone and it will map vectors satisfying $(1 - G_T)\psi \in D(L) = D(K)$ to \mathcal{H} .

Using $G_T^*(K + c_0) = -\sum_{\mu=1}^M a(\delta_{x_\mu})$, we can further rewrite the Hamiltonian, as in (8), and obtain

$$\begin{aligned} K_0^* + (A_T - T) &= (K + c_0)(1 - G_T) + \sum_{\mu=1}^M a(\delta_{x_\mu})(1 - G_T) + S - c_0 \\ &= (1 - G_T)^*(K + c_0)(1 - G_T) + S - c_0. \end{aligned} \tag{11}$$

Our main result is:

Theorem 1. *Let A and A_T , be the extensions of $\sum_{\mu=1}^M a(\delta_{x_\mu})$ defined above, $AG = T$ and $A_T G_T - T = S$. Let*

$$D(H) = \left\{ \psi \in D(N) \mid \forall n \in \mathbb{N} : \psi^{(n+1)} - G_T\psi^{(n)} \in H^2 \left(\mathbb{R}^{3(M+n+1)} \right) \right\},$$

then the operator

$$\begin{aligned} H &= K_0^* + (A_T - T) \\ &= (1 - G_T)^*(K + c_0)(1 - G_T) + S - c_0 \end{aligned}$$

is self-adjoint on $D(H)$ and bounded from below. Furthermore, for any $n \in \mathbb{N}$ and $\psi \in D(H)$, we have (setting $\psi^{(-1)} = 0$)

$$H\psi^{(n)} = L\psi^{(n)} + \sum_{\mu=1}^M a^*(\delta_{x_\mu})\psi^{(n-1)} + A_T\psi^{(n+1)}$$

as elements of $H^{-2}(\mathbb{R}^{3(M+n)})$.

The condition $\psi^{(n+1)} - G_T\psi^{(n)} \in H^2(\mathbb{R}^{3(M+n+1)})$ means that functions in $D(H)$ satisfy the boundary condition $B\psi = M\psi$, among all functions of the form $\varphi + G_T\xi$ with $\varphi \in D(L)$ and $\xi \in \mathcal{H}$.

Remark 2. The transformation $(1 - G_T)$ achieves something similar to the Gross transformation in the Nelson model, in that it provides a transformation relating the interacting operator and a perturbation of the free operator. Applying the inverse of $(1 - G_T)$ and its adjoint (which exist by Proposition 12) to expression (11), we have explicitly

$$(1 - G_T^*)^{-1}H(1 - G_T)^{-1} = L + T + c_0 - c_0(1 - G_T^*)^{-1}(1 - G_T)^{-1} + (1 - G_T^*)^{-1}S(1 - G_T)^{-1}.$$

Our proof of Theorem 1 implies that this is indeed a self-adjoint operator on $D(L)$. The key difference is that the Gross transformation is a Weyl operator, and thus constructed starting from a one-particle function, while G_T contains in T an n -body interaction that cannot be expressed in such a way. Note also that $(1 - G_T)$ is not unitary.

Applying the corresponding transformation to the model with an ultra-violet cutoff $\Lambda > 0$ allows for a reformulation of our result in the language of renormalisation, as in [11]. More precisely, one introduces the cutoff interaction v_Λ and the associated objects (here for $M = 1$)

$$\begin{aligned} G_\Lambda &= -L^{-1}a^*(v_\Lambda), \\ T_\Lambda &= -a(v_\Lambda)L^{-1}a^*(v_\Lambda) + \frac{m\Lambda}{\pi^2(2m + 1)}, \\ K_\Lambda &= L + T_\Lambda, \\ G_{T_\Lambda} &= -(K_\Lambda + c_0)^{-1}a^*(v_\Lambda). \end{aligned}$$

Then, by the same algebra that leads to Eqs. (10), (11),

$$L + a^*(v_\Lambda) + a(v_\Lambda) = (1 - G_{T_\Lambda})^*K_\Lambda(1 - G_{T_\Lambda}) + S_\Lambda - c_0 - \frac{m\Lambda}{\pi^2(2m + 1)},$$

with

$$S_\Lambda = -a(v_\Lambda)(K_\Lambda^{-1} - L^{-1})a^*(v_\Lambda) = a(v_\Lambda)K_\Lambda^{-1}T_\Lambda L^{-1}a^*(v_\Lambda).$$

Along the lines of [11, Sect. 3.4], one can then obtain a renormalisation procedure as follows. From our estimates on T and S (given in Lemma 8 and Lemma 19), one deduces that, as $\Lambda \rightarrow \infty$, T_Λ converges to T , and $S_\Lambda - c\gamma_m \log \Lambda$ converges to S (for some $c \in \mathbb{R}$, γ_m as in (30)), strongly as operators from

$D(T)$, respectively $D(S)$, to \mathcal{H} . Together with similar convergence results for G_Λ, G_{T_Λ} , one then obtains strong resolvent convergence of

$$L + a^*(v_\Lambda) + a(v_\Lambda) + \frac{m\Lambda}{\pi^2(2m + 1)} - c\gamma_m \log \Lambda$$

to the operator H . This shows exactly the divergence in Λ , with a linear and a logarithmic term, observed numerically in [7] (confirming this observation in view of earlier work [26], where convergence without the logarithmic term was claimed). Additionally, our analysis for $M > 1$ shows that the logarithmic term is proportional to M , even though one might naively expect it to be of order M^2 , given the form of S_Λ . This point of view is developed in more detail for the Bogoliubov–Fröhlich Hamiltonian in [10].

The rest of the article is devoted to the proof of Theorem 1. We start by discussing in detail the properties of the maps G and G_T as well as the domain of H . We then define the operator A_T as a (distribution-valued) extension of the annihilation operator to $D(L) \oplus G_T\mathcal{H}$ and give some estimates on its regularity and growth in the particle number in Sect. 4. These results are put together in the proof of Theorem 1 in Sect. 5.

3. The Domain

Before discussing the domain of H , let us recall some properties of extensions of the Laplacian L_0 , the restriction of L to vectors vanishing on the collision configurations \mathcal{C}^n (cf. Eq. (4)). Spelling out the definition of the operator G , Eq. (5), we have

$$G\psi^{(n)} = - \left(-\frac{1}{2m}\Delta_X - \Delta_Y + n + 1 \right)^{-1} \frac{1}{\sqrt{n+1}} \sum_{\mu=1}^M \sum_{i=1}^{n+1} \delta_{x_\mu}(y_i)\psi^{(n)}(X, \hat{Y}_i),$$

where we recall that $X = (x_1, \dots, x_M) \in \mathbb{R}^{3M}$ stands for the positions of the x -particles, $Y = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{3(n+1)}$ for those of the y -particles, and $\hat{Y}_i \in \mathbb{R}^{3n}$ is Y without the i th entry. We view G both as an operator on \mathcal{H} and an operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(n+1)}$, without distinguishing these by the notation.

The functions in the range of G have a specific singular behaviour on the set \mathcal{C}^{n+1} (defined in Eq. (3)) of collision configurations. Since

$$(-\Delta_x + \lambda^2)^{-1}\delta = \frac{e^{-\lambda|x|}}{4\pi|x|},$$

the function $G\psi^{(n)}$ will diverge like $|x_\mu - y_i|^{-1}$ on the plane $\{x_\mu = y_i\}$. The singular set \mathcal{C}^{n+1} is just the union of these planes, and due to symmetry, the divergence on each of these is exactly the same. To be more explicit, first observe that the Fourier transform of $\delta_{x_\mu}(y_i)\psi^{(n)}(X, \hat{Y}_i)$ equals

$$\frac{1}{(2\pi)^{3/2}} \hat{\psi}^{(n)}(P + e_\mu k_i, \hat{K}_i), \tag{12}$$

where $e_\mu k_i$ is the vector in $\mathbb{R}^{3M} = \bigoplus_{\mu=1}^M \mathbb{R}^3$ with the μ th component equal to k_i and all other components equal to zero, and $p_\mu, x_\mu; k_i, y_i$ are conjugate Fourier variables. Now consider, for simplicity, the case $M = 1$. Choosing the centre of mass coordinate $s = \frac{2m}{2m+1}x + \frac{1}{2m+1}y_i$ and the (weighted) relative coordinate $r = \sqrt{\frac{2m}{2m+1}}(y_i - x)$ for the pair x, y_i gives

$$\begin{aligned} & \left(-\frac{1}{2m}\Delta_x - \Delta_Y + n + 1\right)^{-1} \delta_x(y_i)\psi^{(n)}(x, \hat{Y}_i) \\ &= \frac{1}{(2\pi)^{3(n+1)/2+3}} \int_{\mathbb{R}^{3(n+2)}} \frac{e^{ipx+iKY} \widehat{\psi}^{(n)}(p + k_i, \hat{K}_i)}{\frac{1}{2m}p^2 + K^2 + n + 1} dp dK \tag{13} \\ &= \frac{1}{(2\pi)^{3(n+1)/2}} \left(\frac{2m}{2m+1}\right)^{3/2} \frac{1}{4\pi|r|} \int_{\mathbb{R}^{3(n+1)}} e^{-|r|\sqrt{n+1+\frac{1}{2m+1}\xi^2+\hat{K}_i^2}} \\ &\quad \times e^{i\hat{K}_i\hat{Y}_i+i\xi s} \widehat{\psi}^{(n)}(\xi, \hat{K}_i) d\xi d\hat{K}_i. \tag{14} \end{aligned}$$

This function diverges like $\frac{m}{2\pi(2m+1)|x-y_i|} \psi^{(n)}\left(\frac{2m}{2m+1}x + \frac{1}{2m+1}y_i, \hat{Y}_i\right)$ as $r \rightarrow 0$, i.e. $|x - y_i| \rightarrow 0$ (note that r is the weighted relative coordinate). Observe also that (14) is a smooth function of s and \hat{Y}_i for $|r| > 0$ because of the exponential decay of its Fourier transform.

3.1. Properties of G

We will now discuss the mapping properties of the operator G . As already noted in Eq. (6), G maps elements of \mathcal{H} to the kernel of L_0^* . Furthermore, G is bounded from \mathcal{H} to $D(L^s)$, $s < \frac{1}{4}$:

Lemma 3. *Let $0 \leq s < \frac{1}{4}$. There exists a constant C such that for all $\psi \in \mathcal{H}$*

$$\|L^s G\psi\|_{\mathcal{H}} \leq C \|N^{s-\frac{1}{4}}\psi\|_{\mathcal{H}}.$$

Proof. Since we will not aim for optimal estimates w.r.t. the number M of x -particles, we can just treat the case $M = 1$ and then bound the sum over the x -particles by the Cauchy–Schwarz inequality.

For $M = 1$ and arbitrary $n \in \mathbb{N}$, we start from expression (13) and use the simple inequality

$$\begin{aligned} & \operatorname{Re} \left(\overline{\widehat{\psi}^{(n)}(p + k_i, \hat{K}_i)} \widehat{\psi}^{(n)}(p + k_j, \hat{K}_j) \right) \\ & \leq \frac{1}{2} \left(\frac{|\widehat{\psi}^{(n)}(p + k_i, \hat{K}_i)|^2 k_j^2}{k_i^2} + \frac{|\widehat{\psi}^{(n)}(p + k_j, \hat{K}_j)|^2 k_i^2}{k_j^2} \right). \end{aligned}$$

This implies (writing $L(p, K)$ for the Fourier representation of L)

$$\begin{aligned} \left\| L^s G \psi^{(n)} \right\|_{\mathcal{H}^{(n+1)}}^2 &\leq \frac{1}{(2\pi)^3(n+1)} \sum_{i,j=1}^{n+1} \int \frac{|\hat{\psi}^{(n)}(p+k_i, \hat{K}_i)|^2 k_j^2}{L(p, K)^{2-2s} k_i^2} dp dK \\ &\leq \frac{\left\| \psi^{(n)} \right\|_{\mathcal{H}^{(n)}}^2}{(2\pi)^3} \sup_{(p, \hat{K}_{n+1}) \in \mathbb{R}^{3(n+1)}} \sum_{j=1}^{n+1} \int \frac{k_j^2 dk_{n+1}}{\left(\frac{1}{2m}(p-k_{n+1})^2 + K^2 + n+1 \right)^{2-2s} k_{n+1}^2}. \end{aligned}$$

By splitting the summand with $j = n + 1$ from the rest and changing variables $k_{n+1} \mapsto k_{n+1}/\sqrt{n+1}$, respectively $k_{n+1} \mapsto k_{n+1}/\sqrt{n+1 + \hat{K}_{n+1}^2}$, we can further bound this by

$$\begin{aligned} \left\| L^s G \psi^{(n)} \right\|_{\mathcal{H}^{(n+1)}}^2 &\leq \frac{\left\| \psi^{(n)} \right\|_{\mathcal{H}^{(n)}}^2}{(2\pi)^3} \left(\int \frac{dk_{n+1}}{(k_{n+1}^2 + n+1)^{2-2s}} + \sup_{\hat{K}_{n+1}} \hat{K}_{n+1}^2 \int \frac{dk_{n+1}}{(K^2 + n+1)^{2-2s} k_{n+1}^2} \right) \\ &\leq C^2(n+1)^{2s-\frac{1}{2}} \left\| \psi^{(n)} \right\|_{\mathcal{H}^{(n)}}^2. \end{aligned} \tag{15}$$

This proves the claim. □

As an immediate Corollary, we have that $\sum_{\mu=1}^M a(\delta_{x_\mu})$ maps $D(L)$ to \mathcal{H} , continuously.

Corollary 4. *The operator $-G^* = \sum_{\mu=1}^M a(\delta_{x_\mu})L^{-1}$ is bounded on \mathcal{H} .*

Remark 5 (The domain of L_0^*). The operator G can be used to parameterise the domain of L_0^* . From the proof of Lemma 3, one easily infers that $G : \mathcal{H} \rightarrow \ker L_0^*$ can be extended to $D(L^{-1/4}) := D(L^{1/4})'$. We then have a characterisation of $D(L_0^*)$ by

$$D(L_0^*) = D(L) \oplus GD(L^{-1/4}).$$

Since we only work on the subspace of $D(L_0^*)$ where G acts on \mathcal{H} , we do not need this parameterisation and we will not give a detailed proof. The central argument to obtain this parameterisation is to show that the norm $\|G\psi\|_{\mathcal{H}^{(n+1)}}$ is equivalent to the norm on $H^{-1/2}(\mathbb{R}^{3(M+n)})$ by a generalisation of [3, Lem B.2] to an arbitrary number of particles. This implies the parameterisation above by [1, Prop. 2.9].

On this domain, one could consider models with more general boundary conditions of the form

$$aA\psi + bB\psi = \psi.$$

These correspond to additional direct point interactions between the x - and y -particles (see [12, Sect. 4]) for the case with static x -particles. We will not pursue this here.

As an operator on \mathcal{H} , G has another important property:

Proposition 6. *The operator $1 - G$ is invertible on \mathcal{H} , and both $1 - G$ and $(1 - G)^{-1}$ map the domain $D(N)$ of the number operator to itself.*

Proof. We claim that

$$(1 - G)^{-1} = \sum_{j=0}^{\infty} G^j$$

is a bounded operator on \mathcal{H} . To prove this, we first note that, since G maps $\mathcal{H}^{(j)}$ to $\mathcal{H}^{(j+1)}$, the sum on $\mathcal{H}^{(n)}$ is actually finite,

$$((1 - G)^{-1}\psi)^{(n)} = \sum_{j=0}^n G^j \psi^{(n-j)}.$$

The operator is thus well defined and one easily checks that it is an inverse to $1 - G$. To show boundedness, we use Lemma 3 to obtain

$$\left\| G^j \psi^{(n-j)} \right\|_{\mathcal{H}^{(n)}} \leq C^j \left(\prod_{i=0}^{j-1} (n - i) \right)^{-\frac{1}{4}} \left\| \psi^{(n-j)} \right\| \leq \frac{C^j}{(j!)^{\frac{1}{4}}} \left\| \psi^{(n-j)} \right\|_{\mathcal{H}^{(n-j)}}. \tag{16}$$

This gives a bound on $(1 - G)^{-1}$ by

$$\left\| (1 - G)^{-1} \psi \right\|_{\mathcal{H}} \leq \sum_{j=0}^{\infty} \left\| G^j \psi \right\| \leq \left\| \psi \right\|_{\mathcal{H}} \left(\sum_{j=0}^{\infty} \frac{C^j}{(j!)^{\frac{1}{4}}} \right).$$

The operator $1 - G$ maps $D(N)$ to itself if G does. From the estimate of Eq. (15), we see that

$$\left\| n(G\psi)^{(n)} \right\| \leq C \left\| n^{\frac{3}{4}} \psi^{(n-1)} \right\| \leq 2C \left\| (n - 1) \psi^{(n-1)} \right\|,$$

which proves that G leaves $D(N)$ invariant.

To prove the same for $(1 - G)^{-1}$, observe that G^j maps $D(N)$ to itself for any finite j , because G does. It is thus sufficient to prove the claim for $\sum_{j=j_0}^{\infty} G^j$, for some $j_0 \in \mathbb{N}$. From Lemma 3, we obtain for $j \geq 4$, as in Eq. (16),

$$\begin{aligned} \left\| nG^j \psi^{(n-j)} \right\|_{\mathcal{H}^{(n)}} &\leq \frac{n}{((n - 3)(n - 2)(n - 1)n)^{\frac{1}{4}}} \frac{C^j}{((j - 4)!)^{\frac{1}{4}}} \left\| \psi^{(n-j)} \right\|_{\mathcal{H}^{(n-j)}} \\ &\leq \tilde{C} \frac{C^j}{((j - 4)!)^{\frac{1}{4}}} \left\| \psi^{(n-j)} \right\|_{\mathcal{H}^{(n-j)}}. \end{aligned}$$

This shows that $\sum_{j=4}^{\infty} G^j$ maps \mathcal{H} to $D(N)$ and completes the proof. \square

3.2. Singular Boundary Values

The asymptotic behaviour on \mathcal{C}^{n+1} can be used to define local boundary operators on $D(L_0^*)$. We will define these only on the range of G (acting on \mathcal{H}). These boundary values will be defined by certain limits as $x_\mu - y_{n+1} \rightarrow 0$. These limits exist only almost everywhere or in the sense of distributions. In the following, an expression such as $\lim_{|x_\mu - y_{n+1}| \rightarrow 0} \Phi(X, Y, y_{n+1})$ should thus be read as the limit for $\varepsilon \rightarrow 0$ of the functions $\Phi_\varepsilon(X, Y)$ given by restricting $\Phi(X, Y, y_{n+1})$ to the plane where $\varepsilon = y_{n+1} - x_\mu \in \mathbb{R}^3$ is fixed. The boundary values we define can be extended to $D(L_0^*)$ (cf. Remark 5) in the sense of

distributions on the boundary without the singular set \mathcal{C}^n (see [12, Lem. 6] for the case of a fixed source), but we will not need this here. Set, for suitable $\varphi \in \mathcal{H}^{(n+1)}$,

$$(B\varphi)(X, Y) := -\frac{2\pi(2m+1)}{m} \sqrt{n+1} \sum_{\mu=1}^M \lim_{|x_\mu - y_{n+1}| \rightarrow 0} |x_\mu - y_{n+1}| \varphi(X, Y, y_{n+1}). \tag{17}$$

We then have:

Lemma 7. *Let $n \in \mathbb{N}$ and $\psi \in \mathcal{H}^{(n)}$. The element $G\psi \in \mathcal{H}^{(n+1)}$ has a representative such that the limit in Eq. (17) exists for almost every (X, Y) and the equality*

$$(BG\psi)(X, Y) = M\psi(X, Y)$$

holds.

Proof. The fact that $G\psi \in \mathcal{H}^{(n+1)}$ was proved in Lemma 3. It follows from the analogue of Eq. (14) for arbitrary M that $L^{-1}\delta_{x_\nu}(y_i)\psi(X, \hat{Y}_i)$ has a smooth representative outside of the plane $\{x_\nu = y_i\}$. After multiplying by $|x_\mu - y_{n+1}|$, the limit as $|x_\mu - y_{n+1}| \rightarrow 0$ is thus zero a.e., except if $\mu = \nu$ and $i = n + 1$. In the latter case, the limit equals $\psi(X, Y)$, in L^2 and thus almost everywhere, by strong continuity of the semigroup $e^{-t\sqrt{-\Delta+\lambda^2}}$. There is one such contribution for every μ , giving the prefactor M in the statement. \square

The limit in (17) defining B vanishes on $D(L)$, because $\psi^{(n)} \in H^2(\mathbb{R}^{3(M+n)})$ has a well-defined evaluation in the L^2 -sense, the Sobolev trace, on the set $\{x_\mu = y_{n+1}\}$. We can thus define an operator

$$B : D(L) \oplus G\mathcal{H} \rightarrow \mathcal{H}, \quad \psi + G\varphi \mapsto M\varphi,$$

which can be calculated using the local expression (17) outside of the singular sets.

The annihilation operator

$$a(\delta_{x_\mu})\psi^{(n+1)} = \sqrt{n+1}\psi^{(n)}(X, Y)|_{y_{n+1}=x_\mu},$$

is not defined on the singular functions in the range of G . It can, however, be extended by considering only the finite part at $y_{n+1} = x_\mu$, i.e. subtracting the explicit divergence before evaluation:

$$(A\varphi)(X, Y) := \lim_{r \rightarrow 0} \sum_{\mu=1}^M \frac{1}{4\pi} \int_{S^2} \left(\sqrt{n+1}\varphi(X, Y, x_\mu + r\omega) + \frac{m}{2\pi(2m+1)} \frac{1}{r} \frac{(B\varphi)(X, Y)}{M} \right) d\omega. \tag{18}$$

At least formally, $T\psi = AG\psi$ then defines an operator which preserves the number of particles (we will define an operator T below, and then prove equality in Lemma 9). This operator also appears in the theory of point interactions, where it is known as the Skornyakov–Ter-Martyrosyan operator. For fixed $n \in \mathbb{N}$, T is composed of two contributions. The first is the finite part of

the term $L^{-1}\delta_{x_\mu}(y_{n+1})\psi(X, Y)$, which we call the diagonal part T_d . The second contribution is just the evaluation of terms of the form $L^{-1}\delta_{x_\nu}(y_i)\psi(X, \hat{Y}_i)$ with either $\nu \neq \mu$ or $i \neq n+1$. These diverge on different planes and are smooth on $\{x_\mu = y_{n+1}\}$ outside of the lower-dimensional set \mathcal{C}^n . We call this the off-diagonal part T_{od} . The action of T can be read off from Eqs. (13), (14) and is conveniently expressed in Fourier variables. For an n -particle wavefunction $\psi^{(n)}$, we have

$$\widehat{T_d\psi^{(n)}}(P, K) = \frac{1}{4\pi} \left(\frac{2m}{2m+1} \right)^{\frac{3}{2}} \sum_{\mu=1}^M \sqrt{n+1 + \frac{1}{2m+1}p_\mu^2 + \frac{1}{2m}\hat{P}_\mu^2 + K^2} \hat{\psi}^{(n)}(P, K), \tag{19}$$

which is obtained from (14) by developing the exponential, and

$$\begin{aligned} \widehat{T_{od}\psi^{(n)}}(P, K) = & -\frac{1}{(2\pi)^3} \left(\sum_{\mu,\nu=1}^M \sum_{i=1}^n \int \frac{\hat{\psi}^{(n)}(P - e_\mu\xi + e_\nu k_i, \hat{K}_i, \xi)}{n+1 + \frac{1}{2m}(P - e_\mu\xi)^2 + K^2 + \xi^2} d\xi \right. \\ & \left. + \sum_{\mu \neq \nu=1}^M \int \frac{\hat{\psi}^{(n)}(P - e_\mu\xi + e_\nu\xi, K)}{n+1 + \frac{1}{2m}(P - e_\mu\xi)^2 + K^2 + \xi^2} d\xi \right), \end{aligned} \tag{20}$$

which corresponds to the evaluation of (13) at $y_{n+1} = x$.

It is a well-known result in the theory of point interactions (see e.g. [3, 13]) that, for a fixed number n of particles, $T = T_d + T_{od}$ is bounded from $H^1(\mathbb{R}^{3(M+n)})$ to $\mathcal{H}^{(n)}$, and symmetric. Since, in our problem, n is not fixed, the dependence of the bound on n is important. A bound on T_{od} which is independent of n , even though the number of terms grows linearly in n , was proved for the norm of T_{od} as an operator from $H^{1/2}$ to $H^{-1/2}$ by Moser and Seiringer [13] (with $M = 1$). We adapt their method to prove the same for the operator from H^1 to $\mathcal{H}^{(n)}$, and arbitrary M , in Lemma 17. This gives:

Lemma 8. *For all $n \in \mathbb{N}$, define the operator $T = T_d + T_{od}$ by (19), (20) on the domain $D(T)^{(n)} = \mathcal{H}^{(n)} \cap H^1(\mathbb{R}^{3(M+n)})$ and denote the induced operator on \mathcal{H} by the same symbol. The operator T is symmetric on $D(T) = D(L^{1/2})$, and there exists a constant C such that for all $\psi \in D(L^{1/2})$*

$$\|T\psi\|_{\mathcal{H}} \leq C\|(1 + L^{1/2})\psi\|_{\mathcal{H}}.$$

Proof. The statement is trivial for T_d . The bound on T_{od} is proved in Lemma 17. To show symmetry of T_{od} on its domain, one may use the representation

$$T_{od}\psi^{(n)}(X, \hat{Y}_{n+1}) = - \sum_{(\mu, n+1) \neq (\nu, i)} \tau_{x_\mu}(y_{n+1}) L^{-1}\delta_{x_\nu}(y_i)\psi^{(n)}(X, \hat{Y}_i), \tag{21}$$

where $\tau_{x_\mu}(y_{n+1})$ denotes evaluation at $y_{n+1} = x_\mu$ (outside of \mathcal{C}^n). This proves the claim, because of the bounds on T_{od} obtained before and because $\tau_x^* = \delta_x$, as a map from smooth functions to distributions. \square

For the boundary operator A , we can now prove:

Lemma 9. *Let $\psi \in \mathcal{H}^{(n)}$. The limit in Eq. (18) with $\varphi = G\psi$ exists in $H^{-1}(\mathbb{R}^{3(M+n)})$ and $AG\psi = T\psi$.*

Proof. It is sufficient to prove the claim for a fixed $\mu \in \{1, \dots, M\}$. We start with the diagonal part T_d , which is clearly an operator from $\mathcal{H}^{(n)}$ to $H^{-1}(\mathbb{R}^{3(M+n)})$. Consider the representation of G in Eq. (14). This shows, for $M = 1$, that the limit for $|x - y_{n+1}| \rightarrow 0$ of

$$(L^{-1}\delta_x(y_{n+1})\psi)(x, Y) - \frac{m}{2\pi(m+1)} \frac{1}{|x - y_{n+1}|} \psi\left(\frac{2m}{2m+1}x + \frac{1}{2m+1}y_{n+1}, Y\right)$$

exists in H^{-1} for $\psi \in L^2$ and equals $-(2\pi)^3 T_d \psi$ because this limit is (up to a prefactor) just the derivative at $r = 0$ of the semigroup e^{-rT_d} . To complete the proof for $M = 1$, we need to show that the error made by replacing $\psi(x, Y)$ with $\psi(\frac{2m}{2m+1}x + \frac{1}{2m+1}y_{n+1}, Y)$ converges to zero as $r \rightarrow 0$. This follows, by duality, from the fact that for $f \in H^1(\mathbb{R}^{3(1+n)})$

$$\lim_{r \rightarrow 0} \frac{1}{r} \int_{S^2} \left(f\left(x - \frac{1}{2m+1}r\omega, Y\right) - f(x, Y) \right) d\omega = -\frac{1}{2m+1} \int_{S^2} \omega \cdot \nabla f(x, Y) d\omega = 0 \tag{22}$$

in $\mathcal{H}^{(n)}$. The generalisation to arbitrary M is straightforward, completing the argument for T_d .

The off-diagonal part T_{od} is a bounded operator from $\mathcal{H}^{(n)}$ to $H^{-1}(\mathbb{R}^{3(M+n)})$ by Lemma 8 and duality. The estimates of Lemma 17 also yield a uniform bound for the operator obtained by evaluation at $y_{n+1} - x_\mu = \varepsilon$, whose integral kernel differs from that of T_{od} by a factor $e^{i\varepsilon\xi}$. This implies dominated convergence for $\varepsilon \rightarrow 0$. □

We have thus defined a second boundary value operator

$$A : D(L) \oplus G\mathcal{H} \rightarrow D(L^{-1/2}), \quad \psi + G\varphi \mapsto \sum_{\mu=1}^M a(\delta_{x_\mu})\psi + T\varphi,$$

where T is extended to $T : \mathcal{H}^{(n)} \rightarrow H^{-1}(\mathbb{R}^{3(M+n)})$ by duality. This is an extension of the annihilation operator, originally defined on $D(L)$. There are of course many such extensions, e.g. the map $G\varphi \mapsto 0$ provides an example. The extension A is special (though still not unique) in that it is also a sum of local operators, in the sense that $(A\psi)^{(n)}(X, Y)$ is a sum over μ in which each term is completely determined by ψ restricted to any neighbourhood of the point $(X, Y, x_\mu) \in \mathcal{C}^{n+1}$.

Remark 10 (The model with one y -particle). The results we have obtained so far are sufficient to discuss the model with at most one y -particle. For the cases $M = 1, 2$ this is essentially the model introduced in [25]. Certain sectors of the Galilean Lee model [9, 20] can also be described in a very similar way.

Consider the subspace D of $\mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)}$ formed by elements $\psi = (\psi^{(0)}, \psi^{(1)})$ with $\psi^{(0)} \in D(L) = H^2(\mathbb{R}^{3M})$, $\psi^{(1)} \in D(L) \oplus G\mathcal{H}^{(0)}$. On this

space, both $B\psi^{(1)}$ and $A\psi^{(1)}$ are well defined. Since $BG\varphi = M\varphi$, we also have

$$\psi^{(1)} - GB\psi^{(1)}/M \in D(L) = H^2(\mathbb{R}^{3(M+1)}).$$

Using first that $L_0^*G = 0$ and then the symmetry of $(L, D(L))$, we find the identity for $\psi, \varphi \in D$

$$\begin{aligned} \langle L_0^*\psi^{(1)}, \varphi^{(1)} \rangle - \langle \psi^{(1)}, L_0^*\varphi^{(1)} \rangle &= \langle L_0^*(\psi^{(1)} - GB\psi^{(1)}/M), \varphi^{(1)} \rangle - \langle \psi^{(1)}, L_0^*(\varphi^{(1)} - GB\varphi^{(1)}/M) \rangle \\ &= \langle L(\psi^{(1)} - GB\psi^{(1)}/M), GB\varphi^{(1)}/M \rangle - \langle GB\psi^{(1)}/M, L(\varphi^{(1)} - GB\varphi^{(1)}/M) \rangle. \end{aligned}$$

Since $LG = -\sum_{\mu=1}^M a^*(\delta_{x_\mu})$, this equals

$$\begin{aligned} \frac{1}{M} \sum_{\mu=1}^M \left(-\langle a(\delta_{x_\mu})(\psi^{(1)} - GB\psi^{(1)}/M), B\varphi^{(1)} \rangle \right. \\ \left. + \langle B\psi^{(1)}, a(\delta_{x_\mu})(\varphi^{(1)} - GB\varphi^{(1)}/M) \rangle \right). \end{aligned}$$

If $B\psi^{(1)}$ and $B\varphi^{(1)}$ are elements of the domain $D(T)^{(0)} = H^1(\mathbb{R}^{3M})$ of the symmetric operator T , we can add the term $(\langle TB\psi^{(1)}, B\varphi^{(1)} \rangle - \langle B\psi^{(1)}, TB\varphi^{(1)} \rangle)/M^2 = 0$ to this equation. Since $AG = T$, it then becomes

$$\langle L_0^*\psi^{(1)}, \varphi^{(1)} \rangle - \langle \psi^{(1)}, L_0^*\varphi^{(1)} \rangle = \langle B\psi^{(1)}/M, A\varphi^{(1)} \rangle - \langle A\psi^{(1)}, B\varphi^{(1)}/M \rangle \tag{23}$$

This implies that the operator

$$H^{(1)}(\psi^{(0)}, \psi^{(1)}) = (L\psi^{(0)} + A\psi^{(1)}, L_0^*\psi^{(1)})$$

is symmetric if we impose the boundary condition $B\psi^{(1)} = M\psi^{(0)}$, i.e. on the domain

$$D(H^{(1)}) = \left\{ \psi \in \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \mid \psi^{(0)} \in D(L), \psi^{(1)} \in D(L) \oplus G\mathcal{H}^{(0)}, B\psi^{(1)} = M\psi^{(0)} \right\}.$$

One can prove that $H^{(1)}$ is self-adjoint, for example by constructing its resolvent along the lines of [9, 25] or by adapting our proof in Sect. 5. For $\psi \in D(H^{(1)})$, we have

$$L_0^*\psi^{(1)} = L(\psi^{(1)} - G\psi^{(0)}) = L\psi^{(1)} + \sum_{\mu=1}^M a^*(\delta_{x_\mu})\psi^{(0)}, \tag{24}$$

where the right-hand side is a sum in $H^{-2}(\mathbb{R}^{3(M+1)})$. We can thus also write

$$H^{(1)}\psi = L\psi + A\psi^{(1)} + \sum_{\mu=1}^M a^*(\delta_{x_\mu})\psi^{(0)}.$$

It is important to note that we have used the fact that $\psi^{(0)} \in H^2(\mathbb{R}^{3M}) \subset D(T)^{(0)}$. This does not carry over to cases with more y -particles, since $B\psi^{(1)} = \psi^{(0)}$ implies that $\psi^{(1)}$ is *not* in $D(T)^{(1)} = H^1(\mathbb{R}^{3(M+1)})$ (or even $H^{1/2}(\mathbb{R}^{3(M+1)})$), if $\psi^{(0)} \neq 0$. Consequently, $A\psi^{(2)} = a(\delta_x)(\psi^{(2)} - G\psi^{(1)}) +$

$T\psi^{(1)}$ does not define an element of $\mathcal{H}^{(1)}$, and the definition $H^{(2)}$ by analogy with $H^{(1)}$ does not yield a well-defined operator from $D(H^{(2)})$ to \mathcal{H} .

3.3. The Operator K and Its Extension

Up to now, we have developed the theory very much in parallel to [11], where we treated in particular the two-dimensional variant of our model. There, we proved that $L_0^* + A$ is self-adjoint of the domain $D(L) \oplus G\mathcal{H}$ with the boundary condition $B\psi = M\psi$. For the three-dimensional model, this cannot be true as such. In this case, $AG = T$ is defined on $D(L^{1/2})$ but if $B\psi^{(n)} = M\psi^{(n-1)} \neq 0$, then $\psi^{(n)} \notin H^1(\mathbb{R}^{3(M+n)}) \cap \mathcal{H}^{(n)} = D(T)^{(n)}$, since such a function must diverge like $\psi^{(n-1)}(X, \hat{Y}_n)|x_\mu - y_n|^{-1}$ as $y_n \rightarrow x_\mu$. Consequently, we should not expect $L_0^* + A$ to map this domain to \mathcal{H} . This problem cannot be remedied by simply interpreting the operators as quadratic forms, since one can also show that $G\mathcal{H}^{(n-1)} \cap H^{1/2}(\mathbb{R}^{3(M+n)}) = \{0\}$ (see [11, Prop. 4.2]).

Remark 11. Of course, the fact that $G\psi$ is not in the domain $D(T) = D(L^{1/2})$ of T does not immediately imply that $TG\psi \notin \mathcal{H}$, but only that $T_dG\psi \notin \mathcal{H}$ and $T_{\text{od}}G\psi \notin \mathcal{H}$, separately. Cancellations between these two terms can make TG well defined on $D(L) \oplus G\mathcal{H}$. In this case, one can proceed with the operator $L_0^* + A$. This happens in the variant of our model with fixed x -particles [12], which formally corresponds to taking $m = \infty$ (see Eq. (32)). Similar cancellations occur in the Galilean Lee model [9, 20], when formulated in our language, due to the constraint on the masses of the different particles. Proposition 13 shows that, in our model, there are no such cancellations.

Our solution to the problem of defining AG is to change the regularity of elements of the domain in such a way that the singularities of A and L_0^* cancel, similar to the cancellation in $L + a^*$ on $D(H^{(1)}) \subset D(L_0^*)$ in the case of one y -particle (see Eq. (24)). Let K be the operator

$$K = L + T$$

with $D(K) = D(L)$ and T given by (19), (20). By Lemma 8 and the Kato–Rellich theorem, K is self-adjoint and bounded from below. Let K_0 be the restriction of K to $D(L_0)$, the vectors $\psi \in D(L)$ with $\psi^{(n)} \in H_0^2(\mathbb{R}^{3(M+n)} \setminus \mathcal{C}^n)$. In analogy with G , we define a map

$$G_T : \mathcal{H} \rightarrow \ker(K_0^* + c_0), \quad G_T\psi := - \sum_{\mu=1}^M (K + c_0)^{-1} a^*(\delta_{x_\mu})\psi, \quad (25)$$

where $c_0 > -\min \sigma(K)$ is a fixed constant. The important properties of G carry over to G_T by perturbation theory.

Proposition 12. *Let $c_0 > -\min \sigma(K)$ and G_T be defined by (25). For $0 \leq s < \frac{1}{4}$, there exists a constant C such that*

$$\|L^s G_T\psi\|_{\mathcal{H}} \leq C \|N^{s-\frac{1}{4}}\psi\|_{\mathcal{H}}.$$

Moreover, $1 - G_T$ is invertible on \mathcal{H} and $1 - G_T$ as well as $(1 - G_T)^{-1}$ map the domain of the number operator $D(N)$ to itself.

Proof. By the resolvent formula, we have

$$G_T\psi = G\psi - L^{-1}(T + c_0)G_T\psi. \tag{26}$$

The bound on $L^s G_T$ then follows from the bound on $L^s G$, Lemma 3, and the fact that $L^{-1/2}T$ is bounded on \mathcal{H} , by Lemma 8. The continuity and the mapping properties of

$$(1 - G_T)^{-1} = \sum_{j=0}^{\infty} G_T^j$$

follow from this by exactly the same proof as in Proposition 6. □

This proposition implies that $(1 - G_T)\psi \in D(N)$ if and only if $\psi \in D(N)$, so we have

$$\begin{aligned} D(H) &= \{\psi \in D(N) : (1 - G_T)\psi \in D(L)\} \\ &= \{\psi \in \mathcal{H} : (1 - G_T)\psi \in D(L)\} \\ &= (1 - G_T)^{-1}D(L). \end{aligned} \tag{27}$$

This also shows that $D(H)$ is dense in \mathcal{H} , by continuity and surjectivity of $(1 - G_T)^{-1}$. Concerning the regularity of vectors in the range of G_T , applying identity (26) twice, we find for any $n \geq 1$

$$G_T\psi^{(n)} = G\psi^{(n)} - L^{-1}(T + c_0)G\psi^{(n)} + (L^{-1}(T + c_0))^2G_T\psi^{(n)}. \tag{28}$$

The first term is an element of $H^s(\mathbb{R}^{3(M+n+1)})$, $s < \frac{1}{2}$, by Lemma 3 and diverges like $|x_\mu - y_i|^{-1}$ near \mathcal{E}^{n+1} by Lemma 7. The operator $TL^{-1}T$ is bounded on $\mathcal{H}^{(n)}$ by Lemma 8, so the last term above is an element of $H^2(\mathbb{R}^{3(M+n+1)})$. It can thus be evaluated on the co-dimension-three set \mathcal{E}^{n+1} of collision configurations. The term $L^{-1}(T + c_0)G\psi^{(n)}$ is in $H^1(\mathbb{R}^{3(M+n+1)})$ (even in H^s , $s < 3/2$), so it should have a less pronounced divergence on \mathcal{E}^{n+1} than the $|x_\mu - y_i|^{-1}$ -divergence of $G\psi^{(n)}$. In fact, we will show in Proposition 13 that this term diverges logarithmically. We can thus define $BG_T\psi = M\psi$ by the same limit (17) as for G . We will define a modification A_T of the operator A on the range of G_T in the next section (see Proposition 13).

4. Extension of the Annihilation Operator

In this section, we will analyse the divergence of $G_T\psi$ on the sets \mathcal{E}^n in order to define a local boundary operator A_T that extends $\sum_{\mu=1}^M a(\delta_{x_\mu})$ to $D(L) \oplus G_T\mathcal{H}$. One should think of functions in the range of G_T as having an expansion of the form

$$(G_T\psi)^{(n+1)}(X, Y) \sim \psi^{(n)}(X, \hat{Y}_{n+1}) \left(\frac{b_1}{|x_\mu - y_{n+1}|} + b_2 \log|x_\mu - y_{n+1}| \right) + F(X, Y)$$

near $x_\mu = y_{n+1}$. Here, the constant b_1 comes from the expansion of G given in Eq. (14), b_2 is determined by the second term in Eq. (28), and $F(X, Y)$ has an appropriate limit as $|x_\mu - y_{n+1}| \rightarrow 0$. As in Sect. 3.2, we would then define

boundary value operators B, A_T such that any $\varphi \in D(L) \oplus G_T \mathcal{H}$ has the expansion for $|x_\mu - y_{n+1}| \rightarrow 0$

$$\varphi^{(n+1)}(X, Y) \sim \frac{(B\varphi^{(n+1)})(X, \hat{Y}_{n+1})}{M} f(|x_\mu - y_{n+1}|) + (A_T \varphi^{(n+1)})(X, \hat{Y}_{n+1}) + o(1),$$

where $f(|x_\mu - y_{n+1}|)$ is the divergent function in the expansion above. We will justify this intuition by defining the operators B, A_T and showing that they are given by appropriate limits, in the sense of distributions.

We define the map

$$B : D(L) \oplus G_T \mathcal{H} \rightarrow \mathcal{H}, \quad \varphi + G_T \psi \mapsto M\psi.$$

As a consequence of Proposition 13, B is a sum of local boundary operators given by the same expression, Eq. (17) (where the limit is taken in H^{-1}), as the corresponding operator on $D(L) \oplus G \mathcal{H}$. To define A_T , we set for appropriate $\varphi \in \mathcal{H}^{(n+1)}$

$$\begin{aligned} & (A_T \varphi)(X, Y) \\ & := \lim_{r \rightarrow 0} \sum_{\mu=1}^M \frac{1}{4\pi} \int_{S^2} (\sqrt{n+1} \varphi(X, Y, x_\mu + r\omega) + f_m(r)(B\varphi)(X, Y)) \, d\omega, \end{aligned} \quad (29)$$

where $r > 0$,

$$f_m(r) = \frac{1}{M} \left(\frac{m}{2\pi(2m+1)} \frac{1}{r} + \gamma_m \log(r) \right),$$

and γ_m is the constant

$$\gamma_m = \frac{1}{(2\pi)^3} \left(\frac{2m}{2m+1} \right)^3 \left(\frac{2\sqrt{m(m+1)}}{2m+1} - (2m+1) \tan^{-1} \left(\frac{1}{2\sqrt{m(m+1)}} \right) \right). \quad (30)$$

Note that for $\varphi \in H^2(\mathbb{R}^{3(M+n+1)})$, $A_T \varphi$ equals the usual annihilation operator. We will show that the formula for A_T defines a map

$$A_T : D(L) \oplus G_T \mathcal{H} \rightarrow D(L^{-1/2}), \quad \varphi + G_T \psi \mapsto \sum_{\mu=1}^M a(\delta_{x_\mu}) \varphi + T\psi + S\psi, \quad (31)$$

where $T = AG$ as before and $S : D(L^\varepsilon) \rightarrow \mathcal{H}$ is symmetric, for any $\varepsilon > 0$.

Observe also that

$$\lim_{m \rightarrow \infty} \gamma_m = \frac{1}{(2\pi)^3} (1 - (\tan^{-1})'(0)) = 0. \quad (32)$$

which explains why there is no logarithmically divergent term in the case of fixed x -particles, treated in [12]. The result of this section is:

Proposition 13. *Let $\psi \in \mathcal{H}^{(n)}$. Then for $\varphi = G_T \psi$ the limit in Eq. (29) exists in $H^{-1}(\mathbb{R}^{3(M+n)})$ and $S\psi := (A_T G_T - T)\psi$ defines a symmetric operator on $D(S) = \mathcal{H}^{(n)} \cap H^\varepsilon(\mathbb{R}^{3(M+n)})$, for any $\varepsilon > 0$.*

We will give an outline of the proof here and provide some of the more technical points as separate lemmas in the appendix.

Proof. Let $R\psi := -L^{-1}TG\psi$. Then, in view of Eq. (28), we have

$$G_T\psi = G\psi + R\psi + ((L^{-1}(T + c_0))^2G_T - c_0L^{-1}G)\psi. \tag{33}$$

The sum of $G\psi$ and the $1/r$ -term in $f_m(r)$ converges to $AG\psi = T\psi \in H^{-1}$ by Lemma 9. Since the last term in Eq. (33) is an element of $H^2(\mathbb{R}^{3(M+n+1)})$, it has a Sobolev trace on $\{x_\mu = y_{n+1}\}$ and the usual annihilation operator is well defined on this term. We denote this evaluation by

$$S_{\text{reg}}\psi := \sum_{\mu=1}^M a(\delta_{x_\mu}) ((L^{-1}(T + c_0))^2G_T - c_0L^{-1}G)\psi. \tag{34}$$

It then remains to show the convergence of the sum of $R\psi$ and the logarithmic term in $f_m(r)$. It is sufficient to prove convergence in $\mathcal{H}^{(n)}$ for $\psi \in H^\varepsilon(\mathbb{R}^{3(M+n)})$, $\varepsilon > 0$, convergence in $H^{-\varepsilon}$ for $\psi \in \mathcal{H}^{(n)}$ then follows by duality.

We will focus on the calculation of the asymptotic behaviour at $r = 0$ in the case $M = 1, n = 0$; here, the full argument is provided in Lemma 18. We set $R_d = -L^{-1}T_dG, R_{od} = -L^{-1}T_{od}G$ and start by analysing $(R_d\psi)(x, y)$ at $x = y$. By Eqs. (13), (19), we have, with a change of variables $\sigma = p + k, \rho = k - \frac{1}{(2m+1)}\sigma$,

$$\begin{aligned} & (R_d\psi)(x, y) \\ &= \frac{1}{(2\pi)^{9/2}} \frac{1}{4\pi} \left(\frac{2m}{2m+1}\right)^{3/2} \int \frac{e^{ipx+iky} \sqrt{1 + \frac{1}{2m+1}p + k^2}}{L(p, k)^2} \hat{\psi}(p + k) dp dk \\ &= \frac{1}{2(2\pi)^4} \left(\frac{2m}{2m+1}\right)^{3/2} \\ & \times \frac{1}{(2\pi)^{3/2}} \int \frac{e^{i\sigma s + i\rho r} \sqrt{1 + \frac{2m+2}{2m+1}\rho^2 + b_1\sigma^2 + b_2\rho\sigma}}{\left(1 + \frac{1}{2m+1}\sigma^2 + \frac{2m+1}{2m}\rho^2\right)^2} \hat{\psi}(\sigma) d\sigma d\rho, \end{aligned} \tag{35}$$

where s, r are the centre of mass and relative coordinate and $b_1 = \frac{4m^2+2m+1}{(2m+1)^3}, b_2 = \frac{2}{(2m+1)^2}$. This acts on ψ as a Fourier multiplier with the function given by the ρ -integral. The singularity of this integral depends only on the behaviour of the integrand at infinity, so we replace the square root in the numerator by $\sqrt{(2m+2)/(2m+1)}|\rho|$. The error we make by this replacement is integrable in ρ , and the evaluation at $x = y$ gives rise to a Fourier multiplier with a function that grows no faster than $|\sigma|^\varepsilon$ (see Lemma 19). We thus have to calculate the asymptotic behaviour as $r \rightarrow 0$ of

$$\frac{1}{2(2\pi)^4} \left(\frac{2m}{2m+1}\right)^{3/2} \sqrt{\frac{2m+2}{2m+1}} \int \frac{e^{i\rho r} |\rho|}{\left(1 + \frac{1}{2m+1}\sigma^2 + \frac{2m+1}{2m}\rho^2\right)^2} d\rho. \tag{36}$$

The integral

$$\int_{\mathbb{R}^3} \frac{e^{i\lambda\xi}|\xi|}{(1+\xi^2)^2} d\xi = 4\pi \int_0^\infty \frac{\sin(t)t^2}{(\lambda^2+t^2)^2} dt = 2\pi \int_0^\infty \frac{\sin(t)+t\cos(t)}{\lambda^2+t^2} dt$$

has an expansion given by $-4\pi \log(\lambda) + \mathcal{O}(1)$ as $\lambda \rightarrow 0$. We thus find that expression (36) behaves like

$$-\frac{1}{(2\pi)^3} \left(\frac{2m}{2m+1}\right)^{7/2} \sqrt{\frac{2m+2}{2m+1}} \log\left(r\sqrt{\frac{2m}{2m+1}}\left(1+\frac{1}{2m+1}\sigma^2\right)\right),$$

up to remainders that are uniformly bounded in σ as $r \rightarrow 0$.

We now turn to $R_{\text{od}} = L^{-1}T_{\text{od}}L^{-1}\delta_x\psi$. We have (cf. (20)) and note that T_{od} here is the operator on the one-particle space)

$$\begin{aligned} & \left(T_{\text{od}}\widehat{L^{-1}\delta_x\psi}\right)(p, k) \\ &= -\frac{1}{(2\pi)^{9/2}} \int \frac{1}{2+\frac{1}{2m}(p-\xi)^2+\xi^2+k^2} \frac{\widehat{\psi}(p+k)}{1+\frac{1}{2m}(p+k-\xi)^2+\xi^2} d\xi. \end{aligned}$$

Using the same variables ρ, σ, s, r as before, this gives

$$\begin{aligned} & (R_{\text{od}}\psi)(s, r) \\ &= -\frac{1}{(2\pi)^{6+3/2}} \int \frac{e^{i\sigma s}e^{i\rho r}}{\left(1+\frac{1}{2m+1}\sigma^2+\frac{2m+1}{2m}\rho^2\right)\left(1+\frac{1}{2m}(\sigma-\xi)^2+\xi^2\right)} \\ & \quad \frac{\widehat{\psi}(\sigma)}{2+\xi^2+\frac{1}{2m+1}(\sigma-\xi)^2+\frac{2m+1}{2m}\left(\rho+\frac{1}{2m+1}\xi\right)^2} d\xi d\sigma d\rho. \end{aligned} \tag{37}$$

Similar to the case of R_{d} , this acts as a Fourier multiplier by the function given by the integral over ξ and ρ . To simplify the calculation of this integral, we replace $(\sigma-\xi)^2$ in the denominator by $\sigma^2+\xi^2$ (the error again has better decay in ξ and ρ (see Eq. (46))). For the expression resulting from the denominator of the last line, we then gather the terms

$$\xi^2 + \frac{1}{2m+1}\xi^2 + \frac{2m+1}{2m}\left(\rho + \frac{1}{2m+1}\xi\right)^2 = \frac{2m+1}{2m}\left(\xi + \frac{1}{2m+1}\rho\right)^2 + \frac{2m+2}{2m+1}\rho^2,$$

making apparent that the ξ -integral is now a convolution. This can be evaluated using the Fourier transform

$$\int_{\mathbb{R}^3} \frac{e^{-ikx}e^{-\lambda|x|}}{|x|^2} dx = 4\pi \int_0^\infty \frac{\sin(|k|t)}{|k|t} e^{-\lambda t} dt = \frac{4\pi}{|k|} \tan^{-1}\left(\frac{|k|}{\lambda}\right).$$

The result is

$$\begin{aligned} & \int \frac{1}{\left(\beta + \frac{2m+1}{2m}\xi^2\right)\left(\gamma + \frac{2m+1}{2m}\left(\xi + \frac{1}{2m+1}\rho\right)^2\right)} d\xi \\ &= 2\pi^2 \left(\frac{2m}{2m+1}\right)^2 \frac{2m+1}{|\rho|} \tan^{-1}\left(\frac{|\rho|}{\sqrt{2m(2m+1)}(\sqrt{\beta} + \sqrt{\gamma})}\right), \end{aligned} \tag{38}$$

where $\beta = 1 + \frac{1}{2m}\sigma^2$, $\gamma = 2 + \frac{1}{2m+1}\sigma^2 + \frac{2m+2}{2m+1}\rho^2$. >From this, we can see that the divergence of $R_{\text{od}}\psi$ stems from the insufficient (cubic) decay of the integrand

for large $|\rho|$, as for R_d . For the analysis of this divergence, we can thus replace the \tan^{-1} by its limit as $\rho \rightarrow \infty$, which equals (note the ρ -dependence of γ)

$$\lim_{\rho \rightarrow \infty} \tan^{-1} \left(\frac{|\rho|}{\sqrt{2m(2m+1)}(\sqrt{\beta} + \sqrt{\gamma})} \right) = \tan^{-1} \left(\frac{1}{2\sqrt{m(m+1)}} \right).$$

The asymptotics of the remaining $d\rho$ -integral can be evaluated as for R_d . One finds that $(R_{\text{od}}\psi)(s, r)$ has the asymptotic behaviour

$$\frac{1}{(2\pi)^3} \left(\frac{2m}{2m+1} \right)^2 2m \tan^{-1} \left(\frac{1}{2\sqrt{m(m+1)}} \right) \log(r)\psi(s),$$

with a convergent remainder, as $r \rightarrow 0$. Bounds on the convergent part as an operator on H^ε are provided in Lemma 19. Consequently,

$$(R\psi)(s, r) = -\gamma_m \psi(s) \log(r) + \mathcal{O}(1),$$

with a remainder that converges in $\mathcal{H}^{(n)}$ as $r \rightarrow 0$.

This remains true with $\psi(s)$ replaced by $\psi(x) = \psi(s + \frac{1}{2m+1}r)$ after averaging $\omega = (y - x)/|y - x|$, as in Eq. (22). This completes the proof for $M = 1, n = 0$.

For arbitrary M and n , the key observation is that although R is given in terms of sums corresponding to different combinations of creating and annihilating a particle on the planes $\{x_\nu = y_i\}, \nu \in \{1, \dots, M\}, i \in \{1, \dots, n\}$, only some of the contributions are actually singular. These behave in a similar way as for $M = 1, n = 0$ (see Lemma 18 for details).

We then have

$$A_T G_T \psi = T\psi + S\psi.$$

Similarly to T , the operator S is a sum of real Fourier multipliers (in this case of logarithmic growth) and integral operators. Its symmetry on $D(S)$ is shown in Lemma 20. □

5. Proof of Theorem 1

We will now prove the self-adjointness of the operator

$$H = (1 - G_T)^*(K + c_0)(1 - G_T) - c_0 + S$$

on the domain

$$D(H) = \left\{ \psi \in \mathcal{H} \mid \psi - G_T \psi \in D(L) \right\}.$$

Equality of this domain and the one given in Theorem 1 is shown in Eq. (27). The remaining statements of the theorem follow from (11) and (10) in view of the results of Sect. 3.

Lemma 14. *The operator $H_0 = (1 - G_T)^*K(1 - G_T)$ is self-adjoint on $D(H)$ and bounded from below.*

Proof. This follows directly from the invertibility of $(1 - G_T)$. □

Lemma 15. *Let $(S, D(S))$ be the symmetric operator on $\mathcal{H}^{(n)}$ defined in Proposition 13 and denote its extension to \mathcal{H} by the same symbol. Then $(S, D(S))$ is infinitesimally H_0 -bounded.*

Proof. We decompose $S\psi = SG_T\psi + S(1 - G_T)\psi$ and estimate both terms separately. For the action of S on the singular part, $G_T\psi$, we have by Lemma 19 and Proposition 12

$$\|SG_T\psi\|_{\mathcal{H}} \leq C_\varepsilon \|L^\varepsilon G_T\psi\|_{\mathcal{H}} \leq C'_\varepsilon \|\psi\|_{\mathcal{H}},$$

for any $0 < \varepsilon < \frac{1}{4}$. On the regular part, we have, again by Lemma 19,

$$\begin{aligned} \|S(1 - G_T)\psi\|_{\mathcal{H}} &\leq C_\varepsilon \|L^\varepsilon(1 - G_T)\psi\|_{\mathcal{H}} \\ &\leq \delta \|L(1 - G_T)\psi\|_{\mathcal{H}} + C_\delta \|(1 - G_T)\psi\| \\ &\leq \delta \|(1 - G_T)^{-1}\| \|H_0\psi\|_{\mathcal{H}} + C_\delta \|(1 - G_T)\| \|\psi\|_{\mathcal{H}}, \end{aligned}$$

for arbitrary $\delta > 0$. This proves the claim. □

In view of Eq. (11), this proves that H is self-adjoint on $D(H)$, by the Kato–Rellich theorem. It is also immediate that H is bounded from below.

Acknowledgements

I am grateful to Stefan Keppeler, Julian Schmidt, Stefan Teufel and Roderich Tumulka for many interesting discussions on the subject of interior-boundary conditions.

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A. Technical Lemmas

In this appendix, we spell out the details concerning the bounds on T and S . These bounds are obtained using variants of the Schur test, similar to those derived in [13], for sums of integral operators that give control on the growth in n as the number of summands increases. Applying the basic Schur test to every summand would yield a bound that grows like the number of summands. In the following lemma, we use the symmetry of the functions in $\mathcal{H}^{(n)}$ to obtain an improvement that is reflected in the order of the sum and the supremum in the constants Λ, Λ' below. In the cases relevant to us, this will lead to bounds that are independent of the number of summands. We also remark that the same lemma holds for antisymmetric wavefunctions, since only the symmetry of $|\psi|^2$ is used.

Lemma 16. *Let $\ell < n$ and d be positive integers, $M \in \mathbb{N}$, and*

$$\mathcal{J} := \left\{ J : \{1, \dots, \ell\} \rightarrow \{1, \dots, n\} \mid J \text{ one-to-one} \right\}.$$

For every $J \in \mathcal{J}$ let $\kappa_J \in L^1_{\text{loc}}(\mathbb{R}^{dM} \times \mathbb{R}^{dn} \times \mathbb{R}^{d\ell})$ be a real, nonnegative function and let $F : \mathbb{R}^{d\ell} \times \mathbb{R}^{d\ell} \rightarrow \mathbb{R}^{dM}$ be measurable.

Define the operator $I : \mathcal{D}(\mathbb{R}^{d(M+n)}) \rightarrow \mathcal{D}'(\mathbb{R}^{d(M+n)})$ by

$$(I\psi)(P, Q) = \sum_{J \in \mathcal{J}} \int_{\mathbb{R}^{d\ell}} \kappa_J(P, Q, R) \psi(P + F(Q_J, R), \hat{Q}_J, R) dR,$$

where $Q_J = (q_{J(1)}, \dots, q_{J(\ell)}) = (q_{j_1}, \dots, q_{j_\ell})$ and \hat{Q}_J denotes the vector in $\mathbb{R}^{d(n-\ell)}$ formed by $q_1, \dots, q_n \in \mathbb{R}^d$ without the entries of Q_J . Denote by $\kappa_J^t(Q, R)$ the kernel obtained from $\kappa_J(Q, R)$ by exchanging r_i with q_{j_i} for $i = 1, \dots, \ell$.

If there exists a positive function $g \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ for which the quantities

$$\Lambda := \sup_{Q \in \mathbb{R}^{dn}} \sum_{J \in \mathcal{J}} \int_{\mathbb{R}^{d\ell}} \prod_{i=1}^{\ell} \frac{g(q_{j_i})}{g(r_i)} \sup_{P \in \mathbb{R}^{dM}} \kappa_J(P, Q, R) dR$$

and

$$\Lambda' := \sup_{Q \in \mathbb{R}^{dn}} \sum_{J \in \mathcal{J}} \int_{\mathbb{R}^{d\ell}} \prod_{i=1}^{\ell} \frac{g(q_{j_i})}{g(r_i)} \sup_{P \in \mathbb{R}^{dM}} \kappa_J^t(P, Q, R) dR$$

are finite, then I extends to a bounded operator from $\mathcal{H}^{(n)} = L^2(\mathbb{R}^{dM}) \otimes L^2(\mathbb{R}^d)^{\otimes_{\text{sym}} n}$ to $L^2(\mathbb{R}^{d(M+n)})$ with norm at most $\sqrt{\Lambda\Lambda'}$.

Proof. Since κ_J is nonnegative, we have for any $\varphi, \psi \in \mathcal{D}(\mathbb{R}^{d(M+n)})$ with $\varphi, \psi \in \mathcal{H}^{(n)}$ and any $\delta > 0$

$$0 \leq \sum_{J \in \mathcal{J}} \int \left| \delta \varphi(P, Q) \prod_{i=1}^{\ell} \sqrt{\frac{g(q_{j_i})}{g(r_i)}} - \frac{1}{\delta} \psi(P + F(Q_J, R), \hat{Q}_J, R) \prod_{i=1}^{\ell} \sqrt{\frac{g(r_i)}{g(q_{j_i})}} \right|^2 \times \kappa_J(P, Q, R) dP dQ dR.$$

After expanding the square, the quadratic term in φ can be estimated by

$$\delta^2 \int \left(\sum_{J \in \mathcal{J}} \int \kappa_J(P, Q, R) \prod_{i=1}^{\ell} \frac{g(q_{j_i})}{g(r_i)} dR \right) |\varphi(P, Q)|^2 dP dQ \leq \delta^2 \Lambda \|\varphi\|_{L^2}^2.$$

For the term with $|\psi|^2$, changing variables to $\hat{Q}'_J = \hat{Q}_J$, $Q'_J = R$, $R' = Q_J$, and using the permutation symmetry of ψ , gives

$$\begin{aligned} & \sum_{J \in \mathcal{J}} \int \kappa_J(P, Q, R) |\psi(P + F(Q_J, R), \hat{Q}_J, R)|^2 \prod_{i=1}^{\ell} \frac{g(r_i)}{g(q_{j_i})} dP dQ dR \\ &= \sum_{J \in \mathcal{J}} \int \kappa_J^t(P, Q', R') |\psi(P + F(R', Q'_J), Q')|^2 \prod_{i=1}^{\ell} \frac{g(q'_{j_i})}{g(r'_i)} dP dQ' dR'. \end{aligned}$$

Using first the Hölder inequality in P and then changing variables to $P' = P + F(R', Q'_J)$, we can bound this by

$$\begin{aligned} & \sum_{J \in \mathcal{J}} \int \left(\sup_{P \in \mathbb{R}^{dM}} \kappa_J^t(P, Q', R') \right) |\psi(P', Q')|^2 \prod_{i=1}^{\ell} \frac{g(q'_{ji})}{g(r'_i)} dP' dQ' dR' \\ & \leq \Lambda' \|\psi\|_{\mathcal{H}^{(n)}}^2. \end{aligned}$$

Together, these estimates imply that

$$\begin{aligned} & 2\text{Re}(\langle \varphi, I\psi \rangle) \\ & = \sum_{J \in \mathcal{J}} \int \kappa_J(P, Q, R) 2\text{Re}(\overline{\varphi}(P, Q)\psi(P + F(Q_J, R), \hat{Q}_J, R)) dP dQ dR \\ & \leq \delta^2 \Lambda \|\varphi\|_{L^2}^2 + \frac{\Lambda'}{\delta^2} \|\psi\|_{\mathcal{H}^{(n)}}^2. \end{aligned}$$

The same holds true for the negative of the real part of $\langle \varphi, I\psi \rangle$, by replacing ψ by $-\psi$, and the imaginary part, replacing ψ by $i\psi$. This yields

$$\sup_{\substack{\|\varphi\|_{L^2}=1 \\ \|\psi\|_{\mathcal{H}^{(n)}}=1}} |\langle \varphi, I\psi \rangle| \leq \frac{1}{2} \sqrt{2(\delta^2 \Lambda^2 + \Lambda'/\delta^2)},$$

so I is bounded from $\mathcal{H}^{(n)}$ to $L^2(\mathbb{R}^{d(M+n)})$. Choosing $\delta = \sqrt{\Lambda'/\Lambda}$ gives the bound on the norm $\|I\| \leq \sqrt{\Lambda\Lambda'}$. □

Since the operator T of Lemma 8 is not bounded on $\mathcal{H}^{(n)}$, we need to slightly adapt the technique of Lemma 16 for this case.

Lemma 17. *There exists a constant C such that for all $n \in \mathbb{N}$ and $\psi \in \mathcal{H}^{(n)} \cap H^1(\mathbb{R}^{3(M+n)})$*

$$\|T_{\text{od}}\psi\|_{\mathcal{H}^{(n)}} \leq C \|\psi\|_{H^1(\mathbb{R}^{3(M+n)})}.$$

Proof. Recall the definition of T_{od} from Eq. (20)

$$\widehat{T_{\text{od}}\psi}(P, K) = - \frac{1}{(2\pi)^3} \left(\sum_{\mu, \nu=1}^M \sum_{i=1}^n \int \frac{\hat{\psi}(P - e_{\mu}\xi + e_{\nu}k_i, \hat{K}_i, \xi)}{L(P - e_{\mu}\xi, K, \xi)} d\xi \right) \tag{39}$$

$$+ \sum_{\mu \neq \nu=1}^M \int \frac{\hat{\psi}(P - e_{\mu}\xi + e_{\nu}\xi, K)}{L(P - e_{\mu}\xi, K, \xi)} d\xi. \tag{40}$$

Since we are not interested in the exact dependence of the norm of T_{od} on M and m , we will just estimate the operator for fixed indices μ, ν and $m = \frac{1}{2}$. Set $\kappa(K, \xi) = \frac{1}{n+1+K^2+\xi^2}$. To bound the sum over i , we argue as in Lemma 16 and obtain for $\psi \in \mathcal{H}^{(n)} \cap H^1(\mathbb{R}^{3(M+n)})$, $\varphi \in \mathcal{H}^{(n)}$ and $0 < \varepsilon < \frac{1}{2}$ the inequality

$$\begin{aligned} & \sum_{i=1}^n \int \frac{2\text{Re} \left(\overline{\varphi}(P, K) \hat{\psi}(P - e_{\mu}\xi + e_{\nu}k_i, \hat{K}_i, \xi) \right)}{n + 1 + (P - e_{\mu}\xi)^2 + K^2 + \xi^2} dP dK d\xi \\ & \leq \sum_{i=1}^n \int \kappa(K, \xi) \left(\kappa(K, \xi)^{-1/2+\varepsilon} |\hat{\psi}|^2(P - e_{\mu}\xi + e_{\nu}k_i, \hat{K}_i, \xi) \frac{(\frac{1}{n} + \xi^2)^{1+\varepsilon}}{(1 + k_i^2)} \right) \end{aligned}$$

$$+ \kappa(K, \xi)^{1/2-\varepsilon} |\varphi|^2(P, K) \frac{(1 + k_i^2)}{\left(\frac{1}{n} + \xi^2\right)^{1+\varepsilon}} \Big) dPdKd\xi.$$

The term with $|\varphi|^2$ is bounded by

$$\begin{aligned} & \|\varphi\|_{\mathcal{H}^{(n)}}^2 \sup_{K \in \mathbb{R}^{3n}} \sum_{i=1}^n (1 + k_i^2) \int \kappa(K, \xi)^{3/2-\varepsilon} (1 + \xi^2)^{-1-\varepsilon} d\xi \\ & \leq \|\varphi\|_{\mathcal{H}^{(n)}}^2 \sup_{K \in \mathbb{R}^{3n}} \frac{n + K^2}{(n + 1 + K^2)} \int \frac{1}{(1 + \eta^2)^{3/2-\varepsilon} \eta^{2(1+\varepsilon)}} d\eta \\ & \leq \Lambda \|\varphi\|_{\mathcal{H}^{(n)}}^2, \end{aligned}$$

where Λ is clearly independent of n . In the i th term with $|\hat{\psi}|^2$, we perform the change of variables $(k_i, \xi) \mapsto (\eta, k_i)$ which gives a bound by

$$\begin{aligned} & \sum_{i=1}^n \int |\hat{\psi}|^2(P, K) \kappa(K, \eta)^{1/2+\varepsilon} \frac{\left(\frac{1}{n} + k_i^2\right)^{1+\varepsilon}}{1 + \eta^2} dPdKd\eta \\ & \leq \sum_{i=1}^n \left\| \sqrt{1 + K^2} \psi \right\|_{\mathcal{H}^{(n)}}^2 \sup_{K \in \mathbb{R}^{3n}} \frac{\sum_{i=1}^n \left(\frac{1}{n} + k_i^2\right)^{1+\varepsilon}}{(1 + K^2)^{1+\varepsilon}} \int \frac{1}{(1 + \eta^2)^{1/2+\varepsilon} \eta^2} d\eta \\ & \leq \Lambda' \|\psi\|_{H^1}^2. \end{aligned}$$

Here Λ' is independent of n because the $(1 + \varepsilon)$ -norm of the vector $(1/n + q_1^2, \dots, 1/n + q_n^2) \in \mathbb{R}^n$ is bounded by its 1-norm. By the argument of Lemma 16, this proves that (39) defines an operator that is bounded from H^1 to $\mathcal{H}^{(n)}$ by $\sqrt{\Lambda\Lambda'}$, which is independent of n .

The remaining operator (40) we have to estimate is

$$\psi \mapsto \int \frac{\hat{\psi}(P - e_\mu \xi + e_\nu \xi, K)}{n + 1 + (P - e_\mu \xi)^2 + K^2 + \xi^2} d\xi$$

with $\mu \neq \nu$. Since the number of these terms is independent of n , the necessary bound can be obtained by the standard Schur test. Explicitly, we have, as above,

$$\begin{aligned} & \int \frac{2\text{Re} \left(\overline{\varphi}(P, K) \hat{\psi}(P - e_\mu \xi + e_\nu \xi, K) \right)}{n + 1 + (P - e_\mu \xi)^2 + K^2 + \xi^2} d\xi dPdK \\ & \leq \int \frac{1}{1 + (P - e_\mu \xi)^2 + \xi^2} \left(|\hat{\psi}|^2(P - e_\mu \xi + e_\nu \xi, K) \frac{(1 + (p_\nu + \xi)^2)^{1+\varepsilon}}{(1 + p_\mu^2)^{1/2+\varepsilon}} \right. \\ & \quad \left. + |\varphi|^2(P, K) \frac{(1 + p_\mu^2)^{1/2+\varepsilon}}{(1 + (p_\nu + \xi)^2)^{1+\varepsilon}} \right) d\xi dK dP. \end{aligned}$$

The term with φ^2 is bounded using

$$\begin{aligned} & \int \frac{|\varphi|^2(P, K)}{1 + (P - e_\mu \xi)^2 + \xi^2} \frac{(1 + p_\mu^2)^{1/2+\varepsilon}}{(1 + (p_\nu + \xi)^2)^{1+\varepsilon}} d\xi dPdK \\ & \leq \|\varphi\|_{\mathcal{H}^{(n)}}^2 \sup_{P \in \mathbb{R}^{3M}} (1 + p_\mu^2)^{1/2+\varepsilon} \int \frac{d\xi}{\left(1 + \frac{1}{2}p_\mu^2 + 2(\xi - \frac{1}{2}p_\mu)^2\right) (1 + (p_\nu + \xi)^2)^{1+\varepsilon}} \end{aligned}$$

$$\begin{aligned} &\leq \|\varphi\|_{\mathcal{H}^{(n)}}^2 \sup_{P \in \mathbb{R}^{3M}} (1 + p_\mu^2)^{1/2+\varepsilon} \int \frac{d\xi}{(1 + \frac{1}{2}p_\mu^2 + 2\xi^2)(1 + \xi^2)^{1+\varepsilon}} \\ &\leq \|\varphi\|_{\mathcal{H}^{(n)}}^2 2 \int \frac{d\eta}{(1 + 4\eta^2)\eta^{2+2\varepsilon}}, \end{aligned}$$

where we have used the Hardy–Littlewood inequality. After changing variables to $Q = P - e_\mu \xi + e_\nu \xi$, the argument for the term with $|\psi|^2$ is essentially the same, and we conclude as before. \square

Lemma 18. *Let $\varepsilon > 0$, $\psi \in \mathcal{H}^{(n)} \cap H^\varepsilon(\mathbb{R}^{3(M+n)})$ and $R\psi = -L^{-1}TG\psi$. Then*

$$S_{\text{sing}}\psi := \lim_{r \rightarrow 0} \sum_{\mu=1}^M \frac{1}{4\pi} \int_{S^2} (\sqrt{n+1}(R\psi)(X, Y, x_\mu + r\omega) + \gamma_m \log(r)\psi(X, Y)) d\omega$$

exists in $\mathcal{H}^{(n)}$.

Proof. Let $R_d = -L^{-1}T_dG$, $R_{od} = -L^{-1}T_{od}G$.

In this proof, we will focus on the exact form of the logarithmic divergence and discard some regular terms. Precise estimates of these will be given in the proof of Lemma 19.

When expanded, the expression for $\sqrt{n+1}R_d = -\sqrt{n+1}L^{-1}T_dG$ will contain a sum over $\nu \in \{1, \dots, M\}$ coming from T_d (see Eq. (19)), and a second sum over pairs $(\lambda, i) \in \{1, \dots, M\} \times \{1, \dots, n+1\}$ coming from the creation operator in G . We then want to evaluate this expression on the plane $\{x_\mu = y_{n+1}\}$. For a fixed set of indices, we can write the corresponding term using the Fourier transform, obtaining an expression similar to (35). This is, up to a prefactor,

$$\int \frac{\sqrt{n+2 + \frac{1}{2m+1}p_\nu^2 + \frac{1}{2m}\hat{P}_\nu^2 + K^2}}{L(P, K)^2} e^{iXP+iYK} \hat{\psi}(P + e_\lambda k_i, \hat{K}_i) dPdK. \tag{41}$$

To analyse the behaviour as $x_\mu - y_{n+1} \rightarrow 0$, it is instructive to set $Q = P + e_\mu k_{n+1}$. The function ψ then appears as

$$\hat{\psi}(Q - e_\mu k_{n+1} + e_\lambda k_i, \hat{K}_i).$$

The operator defined by (41) is of a very different nature for $(\mu, n+1) = (\lambda, i)$ and all other cases. In the case of equality, it is essentially a Fourier multiplier by the value of the k_{n+1} -integral, since then $\hat{\psi}$ no longer depends on this variable. This is singular as $x_\mu - y_{n+1} \rightarrow 0$, because the integral is not absolutely convergent. On the other hand, if $(\mu, n+1) \neq (\lambda, i)$, then this is an integral operator that can be bounded on L^2 by the Schur test and depends continuously on $x_\mu - y_{n+1}$. We prove uniform bounds in the particle number on this operator in Lemma 19. Consider now the singular terms, with $(\mu, n+1) = (\lambda, i)$. For given μ , there are still M of these, indexed by ν . For $\nu \neq \mu$, the integral over k_{n+1} can be rewritten as

$$\int \frac{e^{i\rho(y_{n+1}-x_\mu)} \sqrt{n+2 + \frac{1}{2m+1}\sigma^2 + \frac{2m+1}{2m}\rho^2 + \frac{1}{2m+1}p_\nu^2 + \frac{1}{2m}\hat{P}_{\mu,\nu}^2 + \hat{K}_{n+1}^2}}{\left(n+1 + \frac{1}{2m+1}\sigma^2 + \frac{2m+1}{2m}\rho^2 + \frac{1}{2m}\hat{P}_\mu^2 + \hat{K}_{n+1}^2\right)^2} d\rho.$$

This can be analysed starting from Eq. (36), by replacing σ^2 with the appropriate expression and the prefactor $\sqrt{(2m+2)/(2m+1)}$ by $\sqrt{(2m+1)/(2m)}$. The term for $\mu = \nu$ has the same prefactor as (36), so the total prefactor of the divergent term $\log|x_\mu - y_{n+1}|$ is

$$-\frac{1}{(2\pi)^3} \left(\frac{2m}{2m+1}\right)^3 \left(\frac{2\sqrt{m(m+1)}}{2m+1} + (M-1)\right). \tag{42}$$

For R_{od} , we have a sum over $(\nu, \lambda, i) \in \{1, \dots, M\}^2 \times \{1, \dots, n+2\}$ with $(\nu, n+2) \neq (\lambda, i)$ coming from T_{od} , Eq. (20), and a sum over (ω, j) , with $\omega \in \{1, \dots, M\}$ and $i \neq j \in \{1, \dots, n+2\}$, coming from the creation operator. All the summands can be written in a form similar to (37), proportional to

$$\int \frac{e^{iPX+i\hat{K}_{n+2}\hat{Y}_{n+2}}\hat{\psi}(P - e_\nu k_{n+2} + e_\lambda k_i + e_\omega k_j, \hat{K}_{i,j})}{L(P, \hat{K}_{n+2})L(P - e_\nu k_{n+2}, K)L(P - e_\nu k_{n+2} + e_\lambda k_i, \hat{K}_i)} dP dK, \tag{43}$$

where $K = (k_1, \dots, k_{n+2})$ and L denotes the operator on the space with the number of particles corresponding to the dimension of its argument (which is either $n+1$ or $n+2$). Here, $\hat{\psi}$ occurs in the form (with $Q = P + e_\mu k_{n+1}$)

$$\psi(Q - e_\mu k_{n+1} - e_\nu k_{n+2} + e_\lambda k_i + e_\omega k_j, \hat{K}_{i,j}).$$

As in the case $n = 0$, the corresponding operator is singular only when the first argument of ψ equals Q . Since $(\nu, n+2) \neq (\lambda, i)$, this only happens for $(\nu, n+2) = (\omega, j)$ and $(\mu, n+1) = (\lambda, i)$. In the singular case, the operator is a Fourier multiplier by a function of $(Q, \hat{K}_{n+1, n+2})$ proportional to

$$\int \frac{1}{n+1 + \frac{1}{2m}(Q - e_\mu k_{n+1})^2 + \hat{K}_{n+2}^2} \frac{1}{n+1 + \frac{1}{2m}(Q - e_\nu k_{n+2})^2 + \hat{K}_{n+1}^2} \times \frac{1}{n+2 + \frac{1}{2m}(Q - e_\mu k_{n+1} - e_\nu k_{n+2})^2 + K^2} dk_{n+1} dk_{n+2}.$$

For fixed μ , this gives one term with $\mu = \nu$ that behaves exactly like expression (37) for $M = 1, n = 0$. The $M - 1$ terms with $\mu \neq \nu$ give a prefactor of $\log(r)$ that exactly cancels the $(M - 1)$ -term in Eq. (42), so $\sqrt{n+1}R\psi$ has the same singularity for $x_\mu - y_{n+1} \rightarrow 0$ as in the case $M = 1, n = 0$ that was treated in the proof of Proposition 13.

This proves that the sum of the singular terms and $\gamma_m \log(r)\psi$ converges as $r \rightarrow 0$. Convergence of the remaining terms is implied by the bounds of Lemma 19, as argued in Lemma 9. \square

Lemma 19. *Let S_{reg} be given by (34) and $S_{\text{sing}} = S - S_{\text{reg}}$ be the operator from Lemma 18. There exists a constant C such that for all $\varepsilon > 0, n \in \mathbb{N}$ and $\psi \in \mathcal{H}^{(n)} \cap H^\varepsilon(\mathbb{R}^{3(M+n)})$*

$$\|S\psi\|_{\mathcal{H}^{(n)}} \leq C \left(\frac{1}{\varepsilon} \|\psi\|_{H^\varepsilon} + (1 + \log(n+1)) \|\psi\|_{\mathcal{H}^{(n)}}\right). \tag{44}$$

Proof. For the regular part S_{reg} , we have

$$\|S_{\text{reg}}\psi\|_{\mathcal{H}^{(n)}} \leq M \left\| a(\delta_{x_1})L^{-1} \right\|_{\mathcal{H}^{(n+1)} \rightarrow \mathcal{H}^{(n)}} \times \left\| ((T + c_0)L^{-1}(T + c_0)G_T - c_0G) \psi \right\|_{\mathcal{H}^{(n+1)}}.$$

By Lemma 8, and Proposition 12, Lemma 3, we have

$$\left\| ((T + c_0)L^{-1}(T + c_0)G_T - c_0G) \psi \right\|_{\mathcal{H}^{(n+1)}} \leq C(n + 1)^{-1/4} \|\psi\|_{\mathcal{H}^{(n)}}.$$

With Corollary 4, this gives the bound

$$\|S_{\text{reg}}\psi\|_{\mathcal{H}^{(n)}} \leq C(n + 1)^{-1/4} \|\psi\|_{\mathcal{H}^{(n)}}.$$

For the singular part S_{sing} , we give quantitative improvements on the proofs of Proposition 13 and Lemma 18. Since we are not interested in the exact dependence on m here, we set the mass of the x -particles to $m = \frac{1}{2}$ during this proof.

We first estimate the errors made by the simplifications in the calculation of the singularity for $M = 1, n = 0$ in Proposition 13. These generalise to the corresponding calculations for arbitrary M, n in Lemma 18 in a straightforward way. The replacement made from (35) to (36) produces an error given by

$$c \int \frac{e^{i\sigma s + i\rho r} \left(\sqrt{1 + \frac{3}{2}\rho^2 + \frac{3}{8}\sigma^2 - \frac{1}{2}\rho\sigma} - \sqrt{\frac{3}{2}|\rho|} \right)}{\left(1 + \frac{1}{2}\sigma^2 + 2\rho^2\right)^2} \hat{\psi}(\sigma) d\sigma d\rho. \tag{45}$$

The integrand is bounded by

$$\frac{1 + \frac{3}{8}\sigma^2 + \frac{1}{2}\sigma\rho}{\left(1 + \frac{3}{8}\sigma^2 + \frac{1}{2}\sigma\rho + \frac{3}{2}\rho^2\right)^{1/2} \left(1 + \frac{1}{2}\sigma^2 + 2\rho^2\right)^2} \leq C \frac{1 + |\sigma|^\varepsilon}{\left(1 + \frac{1}{2}\sigma^2 + 2\rho^2\right)^{3/2 + \varepsilon/2}},$$

for $0 < \varepsilon < 1/2$. This is integrable in ρ , so (45) can be evaluated at $r = 0$, leading to an estimate by

$$\|(45)|_{r=0}\|_{\mathcal{H}^{(0)}} \leq \frac{C}{\varepsilon} \|\psi\|_{H^\varepsilon}.$$

The simplified singular part of R_d , (36) contributes a Fourier multiplier by $\log(1 + \frac{1}{\sqrt{2}}|\sigma|)$ which can be estimated on H^ε in the same way. This completes the case of R_d for $M = 1, n = 0$. The reasoning for the divergent part and arbitrary n is essentially the same, except that there is a term growing like $\log(n + 1)$ due to the n -dependence of L .

In the calculation for R_{od} with $M = 1, n = 0$, we made a simplification in replacing $(\sigma - \xi)^2$ by $\sigma^2 + \xi^2$ in the denominator of (37), i.e. replacing

$$\tau(\sigma, \rho, \xi) = \frac{1}{\left(1 + \frac{1}{2}\sigma^2 + 2\rho^2\right) \left(1 + (\sigma - \xi)^2 + \xi^2\right) \left(2 + \xi^2 + \frac{1}{2}(\sigma - \xi)^2 + 2(\rho + \frac{1}{2}\xi)^2\right)}$$

by

$$\tau_0(\sigma, \rho, \xi) = \frac{1}{\left(1 + \frac{1}{2}\sigma^2 + 2\rho^2\right) \left(1 + \sigma^2 + 2\xi^2\right) \left(1 + \frac{3}{2}\xi^2 + \frac{1}{2}\sigma^2 + 2(\rho + \frac{1}{2}\xi)^2\right)}.$$

We have, with $0 < \varepsilon < \frac{1}{2}$,

$$|\tau - \tau_0| \leq C \frac{|\sigma|^\varepsilon}{\left(1 + 2\rho^2\right)^{1/2 + \varepsilon/2}} \frac{1}{\left(1 + \xi^2\right)^{3/2}} \frac{1}{2 + \xi^2 + \rho^2}. \tag{46}$$

And this implies that

$$\sup_{\sigma \in \mathbb{R}^3} (1 + \sigma^2)^{-\varepsilon/2} \int |\tau - \tau_0|(\sigma, \rho, \xi) d\rho d\xi \leq \frac{C}{\varepsilon},$$

which gives the desired estimate for the error as an operator from H^ε to $\mathcal{H}^{(0)}$. The contribution of the simplified R_{od} , with τ_0 instead of τ , can be bounded by $\log(1 + |\sigma|)$, as for R_d .

For general M and n , we still need to bound the evaluations of the terms in $\sqrt{n+1}R\psi$ that are regular at $x_\mu = y_{n+1}$. We will prove that the sum of these terms gives rise to a bounded operator on $\mathcal{H}^{(n)}$, whose norm is a bounded function of n .

For $\sqrt{n+1}R_d\psi$, the regular terms are the evaluations of (41) with $(\mu, n+1) \neq (\lambda, i)$ at $x_\mu = y_{n+1}$. Denote by $\vartheta_{\mu,\nu,\lambda,i}\psi$ the Fourier transform of this function, that is,

$$\vartheta_{\mu,\nu,\lambda,i}\psi(P, \hat{K}_{n+1}) = \frac{1}{2(2\pi)^4 2^{3/2}} \int \kappa_{\mu,\nu}(P, K) \hat{\psi}(P + e_\lambda k_i - e_\mu k_{n+1}, \hat{K}_i) dk_{n+1},$$

with

$$\kappa_{\mu,\nu}(P, K) = \begin{cases} \frac{\sqrt{n+2 + \frac{1}{2m+1}p_\nu^2 + \frac{1}{2m}(\hat{P}_\nu - e_\mu k_{n+1})^2 + K^2}}{L(P - e_\mu k_{n+1}, K)^2} & \mu \neq \nu \\ \frac{\sqrt{n+2 + \frac{1}{2m+1}(p_\mu - k_{n+1})^2 + \hat{P}_\mu^2 + K^2}}{L(P - e_\mu k_{n+1}, K)^2} & \mu = \nu \end{cases}$$

$$\leq \frac{2}{L(P - e_\mu k_{n+1}, K)^{3/2}}.$$

Applying, for fixed μ, ν, λ , Lemma 16 with kernel $\kappa_{\mu,\nu}(P, \hat{K}_{n+1}, k_{n+1})$ and weight function $g(k) = 1 + k^2$, we obtain

$$\left\| \sum_{i=1}^n \vartheta_{\mu,\nu,\lambda,i}\psi \right\|_{\mathcal{H}^{(n)}} \leq \frac{\|\psi\|_{\mathcal{H}^{(n)}}}{(2\pi)^4 2^{3/2}} \sum_{i=1}^n \int \frac{(1 + k_i^2)}{(n+1 + K^2)^{3/2} |k_{n+1}|^2} dk_{n+1}$$

$$\leq C \|\psi\|_{\mathcal{H}^{(n)}}.$$

The operators $\vartheta_{\mu,\nu,\lambda,i}$ with $i = n+1$ are bounded by the standard Schur test (see also Lemma 17). This gives a bound on the evaluation of the regular terms in $\sqrt{n+1}R_d$ that is independent of n .

For $\sqrt{n+1}R_{\text{od}}\psi$, the regular terms are given by (43) with indices $(\nu, n+2) \neq (\omega, j)$ or $(\mu, n+1) \neq (\lambda, i)$ and $x_\mu = y_{n+1}$. Their Fourier transforms are

$$\Theta_{\mu,\nu,\lambda,\omega,i,j}\psi(P, \hat{K}_{n+1,n+2})$$

$$= -\frac{1}{(2\pi)^6} \int \frac{1}{L(P - e_\mu k_{n+1}, \hat{K}_{n+1})L(P - e_\nu k_{n+2} - e_\mu k_{n+1}, K)}$$

$$\frac{\hat{\psi}(P - e_\mu k_{n+1} - e_\nu k_{n+2} + e_\lambda k_i + e_\omega k_j)}{L(P - e_\nu k_{n+2} - e_\mu k_{n+1} + e_\lambda k_i, \hat{K}_i)} dk_{n+1} dk_{n+2}.$$

For fixed $\mu, \nu, \lambda, \omega$, there are $n(n+1)$ of these terms. For $j, i < n+1$, we apply Lemma 16 with $\ell = 2$. Let $\kappa_{i,j}$ be the kernel of the operator $\Theta_{\mu,\nu,\lambda,\omega,i,j}$, then

$$\begin{aligned} \kappa_{i,j}(P, \hat{K}_{n+1,n+2}, k_{n+1}, k_{n+2}) &\leq \frac{1}{(n+1 + \hat{K}_{n+2}^2)(n+2 + K^2)(n+1 + \hat{K}_i^2)} \\ &\leq \frac{1}{(n+1 + \hat{K}_{n+2}^2)^{3/2}(n+1 + \hat{K}_{i,n+1}^2)^{3/2}}. \end{aligned}$$

Choosing again the weight function $g(k) = 1 + k^2$, Lemma 16 gives

$$\left\| \sum_{i=1}^n \sum_{i \neq j=1}^n \Theta_{\mu,\nu,\lambda,\omega,i,j} \psi \right\|_{\mathcal{H}^{(n)}} \leq \frac{\|\psi\|_{\mathcal{H}^{(n)}}}{(2\pi)^6} \sqrt{\Lambda \Lambda'},$$

with

$$\begin{aligned} \Lambda &\leq \sup_Q \sum_{i=1}^n \int \frac{(1 + q_i^2)(n-1 + \hat{Q}_i^2)}{(1 + \xi^2)(n+1 + \hat{Q}^2 + \xi^2)^{3/2}(1 + \eta^2)(n+1 + \hat{Q}_i^2 + \eta^2)^{3/2}} d\eta d\xi \\ &\leq \left(\int \frac{1}{\eta^2(1 + \eta^2)^{3/2}} d\eta \right)^2, \end{aligned}$$

and the same bound for Λ' . If $j = n+2, i < n+1$, we apply Lemma 16 with $\ell = 1, \kappa_i = \kappa_{i,n+2}$ (note that κ_i and $F = e_\lambda k_i - e_\mu k_{n+1} - e_\nu k_{n+2}$ depend on the additional variable $\xi = k_{n+2}$, but this changes nothing in the proof of Lemma 16). This gives the bound

$$\begin{aligned} &\left\| \sum_{i=1}^n \Theta_{\mu,\nu,\lambda,\omega,i,n+2} \psi \right\|_{\mathcal{H}^{(n)}} \\ &\leq \frac{\|\psi\|_{\mathcal{H}^{(n)}}}{(2\pi)^6} \sup_Q \int \frac{n + Q^2}{(1 + \eta^2)(n + Q^2 + \eta^2)(n + Q^2 + \eta^2 + \xi^2)(1 + \xi^2)} d\eta d\xi \\ &\leq \frac{\|\psi\|_{\mathcal{H}^{(n)}}}{(2\pi)^6} \left(\int \frac{1}{\eta^2(1 + \eta^2)^{3/2}} d\eta \right) \left(\int \frac{1}{\xi^2(1 + \xi^2)} d\xi \right). \end{aligned}$$

The estimate for the sum with $i = n+1, j < n+1$ is the same. The remaining operators with $(i, j) = (n+1, n+2)$ (but restrictions on $\mu, \nu, \lambda, \omega$) are again bounded by the usual Schur test. This completes the proof of the lemma. \square

Lemma 20. *For any $\varepsilon > 0$ and $n \in \mathbb{N}$, the operator S is symmetric on the domain $D(S) = \mathcal{H}^{(n)} \cap H^\varepsilon(\mathbb{R}^{3(M+n)})$.*

Proof. The operator S_{reg} , defined in (34), can be written as

$$- \sum_{\mu=1}^M \sum_{\nu=1}^M a(\delta_{x_\mu}) L^{-1} ((T + c_0) L^{-1} (T + c_0) (L + T + c_0)^{-1} - c_0 L^{-1}) a^*(\delta_{x_\nu}).$$

The operator $aL^{-1} : \mathcal{H}^{(n+1)} \rightarrow \mathcal{H}^{(n)}$ is bounded and $(aL^{-1})^* = L^{-1}a^*$, so the second term above is bounded and symmetric. For the first term, observe additionally that

$$L^{-1}(T + c_0)L^{-1}(T + c_0)(L + T + c_0)^{-1} = (L + T + c_0)^{-1}(T + c_0)L^{-1}(T + c_0)L^{-1},$$

by the resolvent formula. This implies that S_{reg} is bounded and symmetric.

As shown in Lemma 18, the divergent terms in S_{sing} give rise to real Fourier multipliers. These are bounded from $H^\varepsilon(\mathbb{R}^{3(M+n)})$ to $\mathcal{H}^{(n)}$ by Lemma 19 and thus symmetric on this domain. The regular terms in S_{sing} give rise to a bounded operator by the proof of Lemma 19. Since these terms are regular, the limit in the definition of S_{sing} just gives the evaluation at $y_{n+1} = x_\mu$. Denote this evaluation map by $\tau_{x_\mu}(y_{n+1})$. Then the contribution of the regular part of R_d to S_{sing} is

$$\sum_{\mu=1}^M \sum_{(\lambda,i) \neq (\mu,n+1)} \tau_{x_\mu}(y_{n+1}) L^{-1} T_d L^{-1} \delta_{x_\lambda}(y_i) \psi(X, \hat{Y}_i).$$

This defines a symmetric operator because $(\tau_x L^{-1})^* = L^{-1} \delta_x$, T_d is symmetric and both T_d and L commute with permutations of the y_i . Similarly, the regular terms in R_{od} are the sum of

$$\tau_{x_\mu}(y_{n+1}) L^{-1} \tau_{x_\nu}(y_{n+2}) L^{-1} \delta_{x_\lambda}(y_i) L^{-1} \delta_{x_\omega}(y_j) \psi^{(n)}(X, \hat{Y}_{i,j})$$

over all indices with $i \neq j$ and $(\mu, n+1) \neq (\lambda, i)$ or $(\nu, n+2) \neq (\omega, j)$ (see also Eq. (21)). This operator is symmetric for the same reason as in the case of R_d . \square

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Communicated by Jan Dereziński.

Received: October 15, 2018.

Accepted: July 26, 2019.