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Classification of String Solutions for the Self-Dual Einstein–Maxwell–Higgs Model

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Abstract. In this paper, we are concerned with an elliptic system arising from the Einstein–Maxwell–Higgs model which describes electromagnetic dynamics coupled with gravitational fields in spacetime. Reducing this system to a single equation and setting up the radial ansatz, we classify solutions into three cases: topological solutions, nontopological solutions of type I, and nontopological solutions of type II. There are two important constants: $a > 0$ representing the gravitational constant and $N \geq 0$ representing the total string number. When $0 \leq aN < 2$, we give a complete classification of all possible solutions and prove the uniqueness of solutions for a given decay rate. In particular, we obtain a new class of topological solitons, with nonstandard asymptotic value $\sigma < 0$ at infinity, when the total string number is sufficiently large such that $1 < aN < 2$. We also prove the multiple existence of solutions for a given decay rate in the case $aN \geq 2$. Our classification improves previous results which are known only for the case $0 < aN < 1$.

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1. Introduction

The classical Ginzburg–Landau model was proposed in 1950s to provide phenomenological descriptions on superconductivity at low temperature. In the framework of special relativity, this model is called the Maxwell–Higgs model in the sense that it illustrates a magnetically charged scalar field which interacts with a $U(1)$ gauge field and the gauge fields obey the Maxwell dynamics. The complex-valued order parameter ϕ and the gauge fields A_μ are coupled on the Minkowski space with the metric $\text{diag}(-1, 1, 1, 1)$. The metric is used to raise or lower indices. Then, the Lagrangian is given by

$$\mathcal{L}_0 = \frac{\varepsilon^2}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)(D^\mu \phi)^* + \frac{\lambda}{8} (|\phi|^2 - \tau^2)^2. \tag{1.1}$$

Here, $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the electromagnetic field with $\mu, \nu = 0, 1, 2, 3$. We denote the complex conjugate of ϕ by ϕ^*. The constant $\varepsilon > 0$ represents the strength of the electromagnetic interaction. The dimensionless constant $\lambda > 0$ is the Higgs coupling constant which characterizes the type of superconductivity. If $\lambda < \varepsilon^{-2}(\lambda > \varepsilon^{-2}$, resp.), we are led to type I (type II, resp.) superconductivity such that the vortices attract (repel, resp.). The constant $\tau > 0$ is called the symmetry-breaking parameter in the sense that it breaks the symmetry of vacuum states. Indeed, the Lagrangian $\mathcal{L}_0$ is invariant under the local gauge transformation

$$\phi \mapsto \phi e^{i\chi}, \quad A_\mu \mapsto \partial_\mu \chi + A_\mu \tag{1.2}$$

for any smooth real-valued function χ, that is, $\mathcal{L}_0$ is invariant under the local $U(1)$ gauge transformation. The minimum states of the potential is given by the circle $\phi = \tau e^{i\theta}$ for $\theta \in \mathbb{R}$ and each point on this circle is not invariant under the $U(1)$ gauge transformation although the set of total vacuum states is invariant.

Among the solutions of the static Euler–Lagrange equations, special interest has been taken on the vortex solutions. Vortex solutions are realized under the assumption that ϕ and A_μ are independent on x^3 and $A_0, A_3 \equiv 0$. Such a configuration is useful when we consider vortex lines in type II superconductors. In particular, if $\lambda = \varepsilon^{-2}$, then it allows static multi-vortex solutions with the vortex location at arbitrary points of the plane. Moreover, the vortices are in the equilibrium state without interacting each other. These solutions give the minimum of the static energy in the Bogomol'nyi limit and can be obtained by solving the self-dual equations [1]:

$$(D_1 \pm iD_2)\phi = 0, \tag{1.3}$$

$$\varepsilon F_{12} \pm \frac{1}{2\varepsilon}(|\phi|^2 - \tau^2) = 0. \tag{1.4}$$

The solution structures of (1.3) and (1.4) are well established in [26,27]. One may refer to [17,22] for the above contents.

If we consider the effect of gravity in the electromagnetic dynamics, we need to treat the model in the general relativity frame. This means that the Maxwell–Higgs model is considered on a $(3+1)$-dimensional Lorentzian manifold $\mathcal{M}$ and the metric of $\mathcal{M}$ satisfies the Einstein equations. We call this model the Einstein–Maxwell–Higgs (EMH) model. As in the above Maxwell–Higgs model, if we assume that $\mathcal{M} \cong \mathbb{R}^{1,1} \times \mathcal{S}$, namely, $\mathcal{M}$ is uniform in the time direction and one space direction, then the EMH model allows vortex-like solutions [13,21] which are called the cosmic string solutions. Cosmic strings are understood as one-dimensional topological defects formed in symmetry-breaking phase transitions and believed to be relevant in the theory of galaxy formation in the early universe [14]. Cosmic strings play similar roles as vortex lines in type II superconductors.

In this paper, we study the self-dual equations for the EMH model which come from the Bogomol'nyi limit, i.e., the energy minimizing configuration of the static energy functional of the EMH model. Solutions of the self-dual equations are characterized by the zeros of the scalar field ϕ which are called strings. By assuming that the reduced manifold $\mathcal{S}$ is conformal to $\mathbb{R}^2$, we will find string solutions with the string locations at one point. Utilizing the Jaffe–Taubes argument [17], we will reduce the self-dual equations to an equivalent elliptic equation. Then, finding string solutions with the string locations at one point, say at the origin, leads to finding radially symmetric solutions on $\mathbb{R}^2$ which have singularities at the origin. We will classify all the possible radially symmetric solutions of the reduced elliptic equation and give the physical interpretation of them by computing some physical quantities.

In the following, we take the approach of [13,21,25] for the derivation of the self-dual equations of the EMH model. Let $\mathcal{M}$ be a four-dimensional Lorentzian manifold with a metric $(g_{\mu\nu})$ for $\mu,\nu = 0,1,2,3$. The metric signature is given by $(-,+,+,+)$ and the inverse matrix of $(g_{\mu\nu})$ is denoted by $(g^{\mu\nu})$. Let $\mathcal{P}$ be a $U(1)$-line bundle over $\mathcal{M}$ of positive degree N which is determined by the first Chern class of $\mathcal{P}$. The Lagrangian density of the EMH model on $\mathcal{M}$ is given by

$$\mathcal{L} = \frac{\varepsilon^2}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} + \frac{1}{2} g^{\mu\nu} (D_\mu \phi)(D_\nu \phi)^* + \frac{\lambda}{8} (|\phi|^2 - \tau^2)^2. \tag{1.5}$$

In the situation on the curved spacetime, we have the following interpretation of each quantity. The Higgs field ϕ is a smooth section of $\mathcal{P}$, $A = A_\mu \mathrm{d}x^\mu$ is a unitary connection on $\mathcal{P}$, $F = \mathrm{d}A = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curvature 2-form of A, and $D = d - iA$ is the covariant derivative. The Einstein–Hilbert action is given by

$$S = \int_{\mathcal{M}} \Big(\frac{R}{16\pi G} + \mathcal{L} \Big) \sqrt{-g} \mathrm{d}x,$$

where R is the scalar curvature of $\mathcal{M}$, $g = \det(g_{\mu\nu})$ and G is the gravitational constant. The Euler–Lagrange equations of the action are

$$\frac{1}{\sqrt{|g|}} D_\mu (g^{\mu\nu} \sqrt{|g|} D_\nu \phi) = \frac{\lambda}{2} (|\phi|^2 - \tau^2)\phi \tag{1.6}$$

$$\frac{\varepsilon^2}{\sqrt{|g|}} \partial_\alpha (g^{\mu\nu} g^{\alpha\beta} \sqrt{|g|} F_{\nu\beta}) = \frac{i}{2} g^{\mu\nu} [\phi (D_\nu \phi)^* - \phi^* (D_\nu \phi)], \tag{1.7}$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{1.8}$$

where $R_{\mu\nu}$ is the Ricci tensor of $\mathcal{M}$ and $T_{\mu\nu}$ is the energy–momentum tensor given by

$$T_{\mu\nu} = \varepsilon^2 g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} + \frac{1}{2} [(D_\mu \phi)(D_\nu \phi)^* + (D_\nu \phi)(D_\mu \phi)^*] - g_{\mu\nu} \mathcal{L}.$$

In this paper, we are interested in the static solutions of (1.6), (1.7) and (1.8). Since the full problem is quite difficult, we make some assumptions on the metric and the gauge fields in order to find physically meaningful solutions. In particular, under the assumption that the spacetime metric takes a special form, we obtain the self-dual equations as follows. We assume that the metric is uniform in the direction of time and the third spacial component x^3. Then, $\mathcal{M} = \mathbb{R}^2 \times \mathcal{S}$ and gravity occurs from the curved structure of a two-dimensional manifold $\mathcal{S}$. That is, the line element takes the form

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^3)^2 + g_{jk}\mathrm{d}x^j\mathrm{d}x^k.$$

Here, (g_{jk}) with $j,k = 1,2$ is the metric tensor of an unknown two-dimensional Riemannian manifold $\mathcal{S}$. Furthermore, we also assume that ϕ and A_μ are fields on $\mathcal{S}$, that is, ϕ and A_μ depend only on x_1 and x_2 and $A_0 = A_3 = 0$. This is a reasonable assumption since the curvedness structure of $\mathcal{M}$ comes from $\mathcal{S}$. Moreover, such a configuration produces cosmic string solutions and suggests an explanation of galaxy formation in the early universe [14].

In the following, we denote $A = (A_1, A_2)$. Now the Euler–Lagrange equations (1.6) and (1.7) are reduced to

$$\frac{1}{\sqrt{|g|}}D_j(g^{jk}\sqrt{|g|}D_k\phi) = \frac{\lambda}{2}(|\phi|^2 - \tau^2)\phi \tag{1.9}$$

$$\frac{\varepsilon^2}{\sqrt{|g|}}\partial_l(g^{jk}g^{lm}\sqrt{|g|}F_{km}) = \frac{i}{2}g^{jk}[\phi(D_k\phi)^* - \phi^*(D_k\phi)], \tag{1.10}$$

where $j,k,l,m = 1,2$. Moreover, the Einstein equation (1.8) can be written as

$$K_g = 8\pi G\mathcal{E}, \tag{1.11}$$

where K_g is the Gaussian curvature of the manifold $(\mathcal{S}, g_{jk})$ and $\mathcal{E}$ is the energy density given by

$$\mathcal{E} = \frac{\varepsilon^2}{4}g^{jl}g^{km}F_{jk}F_{lm} + \frac{1}{2}g^{jk}(D_j\phi)(D_k\phi)^* + \frac{\lambda}{8}(|\phi|^2 - \tau^2)^2. \tag{1.12}$$

If $\lambda = \varepsilon^{-2}$, then we obtain

$$\begin{aligned}\mathcal{E} = {} & \frac{1}{4}g^{jl}g^{km}\left[\varepsilon F_{jk} \pm \frac{1}{2\varepsilon}\gamma_{jk}(|\phi|^2 - \tau^2)\right]\left[\varepsilon F_{lm} \pm \frac{1}{2\varepsilon}\gamma_{lm}(|\phi|^2 - \tau^2)\right] \\ & + \frac{1}{4}g^{jk}(D_j\phi \pm i\gamma_j^l D_l\phi)(D_k\phi \pm i\gamma_k^m D_m\phi)^* \\ & \pm \frac{1}{4}\tau^2\gamma^{jk}F_{jk} \pm \nabla_j(\gamma^{jk}J_k),\end{aligned} \tag{1.13}$$

where γ_{jk} is a skew-symmetric tensor with $\gamma_{12} = \sqrt{|g|}$, ∇_j denotes the covariant derivative with respect to the metric (g_{jk}), and J_k is the current vector given by

$$J_k = -\frac{i}{4}\Big[\phi^*(D_k\phi) - \phi(D_k\phi)^*\Big]. \tag{1.14}$$

Thus, if $\lambda = \varepsilon^{-2}$, then the minimum energy is saturated by the following self-dual equations

$$(D_j \pm iD_k)\phi = 0, \tag{1.15}$$

$$\varepsilon F_{jk} \pm \frac{1}{2\varepsilon}\gamma_{jk}(|\phi|^2 - \tau^2) = 0. \tag{1.16}$$

In this paper, we will concentrate on the self-dual equations (1.15) and (1.16) coupled to the Einstein equation (1.11). It is obvious that if (ϕ, A, g_{jk}) is a solution of (1.11)–(1.16), then it is also a solution of (1.6)–(1.8). Under some assumptions on the manifold $\mathcal{S}$, the converse turns out to be true, that is, (1.11)–(1.16) and (1.6)–(1.8) are equivalent [29]. Hereafter, we will take the upper signs in both (1.15) and (1.16). The analysis for the lower sign case is parallel to that for the upper sign case [17].

By following the Jaffe–Taubes argument [17] which was originated from Witten [28], we have the representation formula of ϕ from (1.15) as

$$\phi = \exp\left(\frac{\tilde{u}}{2} + i\Theta\right), \quad \Theta = \sum_{j=1}^{d} n_j \mathrm{Arg}(x - p_j). \tag{1.17}$$

Here, $\{p_j\}_{j=1}^d \subset \mathbb{R}^2$ denotes the set of distinct zeros of ϕ with their multiplicities $\{n_j\}_{j=1}^d$. We define the total string number N by

$$N = n_1 + \cdots + n_d.$$

It is standard to represent A in terms of $\tilde{u}$ by use of (1.15):

$$A_1 = \frac{1}{2}\partial_2 \tilde{u} - \frac{x_2}{|x|^2}, \quad A_2 = -\frac{1}{2}\partial_1 \tilde{u} + \frac{x_1}{|x|^2}. \tag{1.18}$$

By plugging (1.17) and (1.18) in (1.11) and (1.16), we derive that

$$K_g = -2\pi G\left[\frac{\tau^2}{\varepsilon^2}(e^{\tilde{u}} - \tau^2) - \Delta_g e^{\tilde{u}}\right], \tag{1.19}$$

$$\Delta_g \tilde{u} = \frac{1}{\varepsilon^2}(e^{\tilde{u}} - \tau^2) + 4\pi \sum_{j=1}^{d} n_j \delta_{p_j}. \tag{1.20}$$

Now, we further assume that the manifold $\mathcal{S}$ is conformal to the Euclidean metric on $\mathbb{R}^2$. In other words, we assume that

$$g_{ij} = e^{\tilde{\eta}}\delta_{ij} \tag{1.21}$$

for some smooth function $\tilde{\eta}$. Let $a = 4\pi G\tau^2$. Then, (1.19) and (1.20) are finally transformed into the following elliptic system in $\mathbb{R}^2$:

$$\Delta \tilde{u} = \frac{1}{\varepsilon^2} e^{\tilde{\eta}}(e^{\tilde{u}} - \tau^2) + 4\pi \sum_{j=1}^{d} n_j \delta(z - p_j) \tag{1.22}$$

$$\Delta\left(\tilde{\eta} + \frac{a}{\tau^2}e^{\tilde{u}}\right) = \frac{a}{\varepsilon^2} e^{\tilde{\eta}}(e^{\tilde{u}} - \tau^2). \tag{1.23}$$

By further reduction, we can draw a single elliptic equation as follows. Let

$$u(x) = \tilde{u}(x/\tau) - 2\ln\tau, \quad \eta(x) = \tilde{\eta}(x/\tau). \tag{1.24}$$

Then, by subtracting (1.23) from (1.22), we see that

$$\Delta\left(u - \frac{1}{a}\eta - e^u\right) = 4\pi \sum_{j=1}^{d} n_j \delta_{p_j}.$$

Thus,

$$H(x) = u(x) - \frac{1}{a}\eta(x) - e^{u(x)} - \sum_{j=1}^{d} n_j \ln|x - p_j|^2 \tag{1.25}$$

is a harmonic function. We set $H(x) = 0$. Then, (1.22) and (1.23) are equivalent to the following single equation:

$$\Delta u = \frac{1}{\varepsilon^2}\left(\prod_{j=1}^{d}|x - p_j|^{2n_j}\right)^{-a} e^{a(u-e^u)}(e^u - 1) + 4\pi\sum_{j=1}^{d} n_j\delta_{p_j}. \tag{1.26}$$

This is the main equation to be dealt with in this paper. We are looking for solutions of (1.26) for which $K(x)f(u(x), a, \varepsilon) \in L^1(\mathbb{R}^2)$, where

$$K(x) = \left(\prod_{j=1}^{d}|x - p_j|^{2n_j}\right)^{-a}, \quad f(u, a, \varepsilon) = \frac{1}{\varepsilon^2}e^{a(u-e^u)}(1 - e^u).$$

Such solutions are important in the physical literature in that they produce finite values of several important physical quantities. We shall discuss this issue in Sect. 5. Then, the integrability condition gives the following three kinds of boundary conditions:

Topological condition: $u(x) \to \sigma$ as $|x| \to \infty$,

Nontopological condition of type I: $u(x) \to -\infty$ as $|x| \to \infty$,

Nontopological condition of type II: $u(x) \to \infty$ as $|x| \to \infty$.

Solutions for each boundary condition are called topological solutions and nontopological solutions of type I and II, respectively. In this paper, by assuming that $p_1 = \cdots = p_d = 0$, we study the existence and properties of topological solutions and nontopological solutions of type I and II which are radially symmetric about the origin.

Once we find a solution u of (1.29), we can recover the solution $(\tilde{u}, \tilde{\eta})$ of (1.22) and (1.23) by (1.24) and (1.25). Then, we get a solution pair (ϕ, A, g_{ij}) of (1.11), (1.15) and (1.16) by the formula (1.17), (1.18) and (1.21). We say that (ϕ, A, g_{ij}) is a topological solution, a nontopological solution of type I and a nontopological solution of type II if u is a topological solution, a nontopological solution of type I and a nontopological solution of type II, respectively. In the last section, we will show that the static energy and the magnetic flux are quantized for topological solutions but not quantized for nontopological solutions.

Before proceeding further, we make a remark on the value σ for the topological boundary condition. It turns out that $\sigma \leq 0$ as we shall see. At the first glance, since the nonlinear term of (1.26) contains $(e^u - 1)$, one may expect that $\sigma = 0$ for topological solutions. This is true for the case $0 < aN < 1$. However, if $1 < aN < 2$, then $\sigma < 0$ as we shall see. This fact is due to the function $K(x)$ which behaves like a potential term coupled with the nonlinear term $f(u, a, \varepsilon)$. In fact, in the case $aN > 1$, the integrability of $K(x)f(u, a, \varepsilon)$ in (1.26) may follow from the decay rate of $K(x)$ although the nonlinear term

$f(u, a, \varepsilon)$ does not vanish at infinity. The function $K(x)$ appears from the gravitational effect in the EMH model. Thus, the possibility of $\sigma \neq 0$ is a consequence of gravity in the physical model and does not appear in self-dual gauge field models without gravity. For instance, as typical examples of self-dual equations in gauge field theories without gravity, we may consider the following two equations:

$$\Delta u = \frac{1}{\varepsilon^2} e^u (e^u - 1) + 4\pi \sum_{j=1}^{d} n_j \delta_{p_j} \tag{1.27}$$

and

$$\Delta u = \frac{1}{\varepsilon^2} \frac{e^u(e^u - 1)}{(\tau + e^u)^3} + 4\pi \sum_{j=1}^{d_1} n_{j,1} \delta_{p_{j,1}} - 4\pi \sum_{j=1}^{d_2} n_{j,2} \delta_{p_{j,2}}, \tag{1.28}$$

where τ is a positive constant. Equation (1.27) comes from the Abelian Chern–Simons model [15,16] and allows the topological boundary condition and nontopological boundary condition of type I, while (1.28) arises from the Chern–Simons gauged $O(3)$ sigma model [18] and has the topological boundary condition and the nontopological boundary condition of type I and II. In both (1.27) and (1.28), if there exists a topological solution, then $\sigma = 0$. One can find the construction of such solutions in [10,24]. For the existence of nontopological solutions of (1.27) and (1.28), one may refer to [5,6,9,11,19,20]. One of the main contributions of this paper is to show that $\sigma < 0$ for topological solutions of (1.26) when $1 < aN < 2$. This is a new class of topological solitons, with nonstandard asymptotic value $\sigma < 0$ at infinity, when the total string number is sufficiently large such that $1 < aN < 2$. This also manifests the effect of the gravity term $K(x)$ in (1.26).

Now, we take into account the simplest case of (1.26), that is, we suppose $p_1 = \cdots = p_d = 0$. We consider the radially symmetric solution $u(x) = u(|x|)$. Then, we can rewrite (1.26) as

$$\begin{cases} u'' + \frac{1}{r} u' = -\, r^{-2aN} f(u, a, \varepsilon), & r > 0, \\ u(r) = 2N \ln r + s + o(1) \quad \text{near} & r = 0. \end{cases} \tag{1.29}$$

Here, $r = |x|$ and $'$ denotes the derivative with respect to r. From now on, we will write $f(u, a)$ instead of $f(u, a, \varepsilon)$ for simplicity unless we need to emphasize the dependence of a quantity on ε. We note that the initial condition in (1.29) means that if $\phi(x, s)$ corresponds to the solution $u(r, s)$ by the relation (1.17), $\phi(x, s)$ enjoys the following behavior near the unique zero at the origin:

$$|\phi(x, s)|^2 = e^s |x|^{2N} + o(1) \tag{1.30}$$

as $|x| \to 0$. Since the function $r^{-2aN} f(u, a)$ is uniformly bounded in both $r > 0$ and $u \in \mathbb{R}$, it is standard to show the global existence and the uniqueness of solutions to (1.29). We denote by $u(r, s)$ the unique global solution of (1.29). We will show later that for any $s \in \mathbb{R}$,

$$\beta(s) = \beta(s; a, N) = \int_0^\infty r^{1-2aN} f\big(u(r, s), a\big) \mathrm{d}r$$

exists and

$$u(r,s) = \big[2N - \beta(s)\big] \ln r + O(1) \quad \text{as} \quad r \to \infty. \tag{1.31}$$

The value $\beta(s)$ plays an important role in classifying the radially symmetric nontopological solutions of types I and II.

The first result for (1.29) was given by Spruck and Yang in [25] where they showed the existence of nontopological solutions of type I for the case $0 < aN < 1$ and $s < -\ln(1 + a^{-1})$. This result was improved in [7] such that if $0 < aN < 1$, then there exists a unique number $s_* \in \mathbb{R}$ such that we have a unique nontopological solution of type I for each $s < s_*$ and topological solution with $\sigma = 0$ for $s = s_*$. For the multi-vortex case (1.26), Yang proved in [30] that if $0 < aN < 1$, then there exists a topological solution with $\sigma = 0$ for all small $\varepsilon > 0$. Moreover, when $0 < aN < 1$, Chae proved in [3] that there exists $\delta_0 > 0$ such that for any $0 < \delta < \delta_0$ (1.26) possesses nontopological solutions u_δ satisfying that

$$u_\delta(x) = \left[2N - \left(\frac{4}{a} + \delta\right)\right] \ln|x| + o(\ln|x|) \quad \text{as} \quad |x| \to \infty.$$

Up to now, the existence and properties of solutions to (1.26) or (1.29) have been focused on the case $0 < aN < 1$ with topological and type I nontopological boundary conditions. However, if $aN \geq 1$, then the arguments previously used in the case $0 < aN < 1$ face severe obstacles and are no longer valid. Indeed, the sign of $1 - aN$ is very important in classifying solutions and verifying the properties of solutions. Therefore, the case $aN \geq 1$ causes mathematical challenges. The purpose of this paper is to classify all radial solutions of (1.29) *for any $a > 0$ and $N \geq 0$ having topological and nontopological boundary conditions of type I and II.* We provide the complete description of all possible types of solutions with the explicit decay rates of them at infinity. Such results may shed a light to the study of full multi-string Eq. (1.26) such as the existence of bubbling solutions made by gluing radial solutions. We state the main result as follows.

Theorem 1.1. *Let a be a positive real number and N be a nonnegative integer. Regarding the existence of solutions of (1.29) and its properties, we have the following.*

- (i) *Suppose that $0 \leq aN < 2$ and $aN \neq 1$.*
  - (i-a) *There exists a unique $s_* = s_*(a,N)$ and $\sigma_{a,N} \in (-\infty, 0]$ such that $u(r,s_*) \to \sigma_{a,N}$ as $r \to \infty$. In other words, $u(r,s_*)$ is a topological solution with $\beta(s_*) = 2N$. If $0 < aN < 1$, then $\sigma_{a,N} = 0$ and $u(r,s_*) = O(r^{-\alpha})$ for any $\alpha > 0$ as $r \to \infty$. If $1 < aN < 2$, then $\sigma_{a,N} < 0$ and $u(r,s_*) = \sigma_{a,N} + O(r^{2-2aN})$ as $r \to \infty$. If $N = 0$, then $s_* = 0$ and $u(r,s_*) \equiv 0$.*
  - (i-b) *If $s < s_*$, $u(r,s)$ is a nontopological solution of type I. For $0 \leq aN < 1$, $\beta : (-\infty, s_*) \to (4/a, \infty)$ is continuous, onto, and strictly increasing. For $1 < aN < 2$, $\beta : (-\infty, s_*) \to (2N, 4/a)$ is continuous, onto, and strictly decreasing. Furthermore, $u(r,s) =$*

$(2N-\beta)\ln r + I_{a,N}(s) + O(r^{2-a\beta})$ for some constant $I_{a,N}(s)$ as $r \to \infty$.

(i-c) *If $s > s_*$, $u(r,s)$ is a nontopological solution of type II. For $0 \le aN < 1$, $\beta : (s_*, \infty) \to (-\infty, 0)$ is continuous, onto, and strictly increasing. For $1 < aN < 2$, $\beta : (s_*, \infty) \to (0, 2N)$ is continuous, onto, and strictly decreasing. Furthermore, $u(r,s) = (2N-\beta)\ln r + I_{a,N}(s) + O(r^{2-2aN-m})$ for some constant $I_{a,N}(s)$ and any $m > 2-2aN$ as $r \to \infty$.*

(ii) *If $aN = 1$ or $aN \ge 2$, (1.29) allows only nontopological solutions of type II.*

(ii-a) *If $aN = 1$, $\beta(s) = 0$ for all $s \in \mathbb{R}$.*

(ii-b) *If $aN \ge 2$, $\beta : (-\infty, \infty) \to (0, 4/a)$ is continuous such that*

$$\lim_{s\to\infty} \beta(s) = \lim_{s\to-\infty} \beta(s) = 0. \tag{1.32}$$

In particular, if $\beta_*(a,N) = \sup_{s\in\mathbb{R}} \beta(s;a,N) < 4/a$, then for each $\beta_0 \in (0, \beta_*)$, there exist at least two solutions $u(r,s)$ for which $\beta(s) = \beta_0$.

In both cases, $u(r,s) = (2N-\beta)\ln r + I_{a,N}(s) + O(r^{2-2aN-m})$ for some constant $I_{a,N}(s)$ and any $m > 2-2aN$ as $r \to \infty$.

Theorem 1.1 follows from Propositions 2.1, 3.1, 3.2, 3.3 and 3.4. Once we find solutions u of (1.26), we can recover solutions (ϕ, A, η) of (1.11), (1.15) and (1.16) by the formula (1.17), (1.18) and (1.25). In the last section, by using Theorem 1.1, we will compute some physical quantities for the recovered solutions (ϕ, A, η) related to the static energy functional (1.12).

In the following, we briefly explain basic ideas of the proof and the role of the value aN. For the case $0 \le aN < 1$, we employ an argument of shooting method. We classify all radial solutions of (1.29) according to the shooting parameter s and find the range of the decreasing/increasing rate $\beta(s)$ in (1.31) by using the Pohozaev identity. Then, using the method of [6], we obtain the uniqueness of solutions for given β in the range. If $aN > 1$, the argument for the case $0 \le aN < 1$ is no longer valid. Indeed, we get the factor $(2-2aN)$ in front of several integral formulas for the proof and the sign of this value is very important in the argument. To overcome this difficulty, by introducing the Kelvin transform $\hat{u}(r) = u(r^{-1})$, we see that if $1 < aN < 2$, then $\hat{u}$ satisfies

$$\begin{cases} \hat{u}'' + \frac{1}{r}\hat{u}' + r^{-2\hat{a}\hat{N}} f(\hat{u}, a) = 0, & r > 0, \\ \hat{u}(r,\hat{s}) = 2\hat{N}\ln r + \hat{s} + o(1) \quad \text{as} & r \to 0, \end{cases} \tag{1.33}$$

where $\hat{a}$ and $\hat{N}$ are positive real numbers so that $0 < \hat{a}\hat{N} < 1$. Since we know the existence and properties of solutions to (1.33), we can construct the solutions for the case $1 < aN < 2$ via the reverse transformation. In particular, by solving an auxiliary problem, we prove the uniqueness of the topological solution. We study this auxiliary problem in the appendix. For the case $aN \ge 2$, we show the limit (1.32) by utilizing the Pohozaev identity and obtain the multiple existence of solutions to given increasing rate β.

Theorem 1.1 tells us that the value aN plays an important role in the proof of existence of solutions and determines the solution structures. As the following two examples show, this phenomenon seems to be a general principle in the self-dual gauge theories coupled with the Einstein equations in account of gravity. The first example is

$$\begin{cases} -u'' - \dfrac{1}{r}u' = \lambda e^{au} + r^{2N}e^{u}, \\ u(0) = s \in \mathbb{R}, \quad u'(0) = 0. \end{cases} \tag{1.34}$$

This equation appears in a massive W-boson model which is in consideration under gravitational effect [4,23,32]. As before, the constant $a > 0$ represents the scaled gravitational constant and $\lambda > 0$. Poliakovsky and Tarantello [23] classified the solutions of (1.34) according to the value $a(N+1)$. They showed that for each $\beta \in I$, (1.34) allows a solution u which behaves like $u(r) = -\beta \ln r + O(1)$ as $r \to \infty$, where

$$\begin{cases} a(N+1) < 1 \quad \Rightarrow \quad I = \left(\max\left\{ 4(N+1), \dfrac{4}{a} - 4(N+1) \right\}, \dfrac{4}{a} \right), \\ a(N+1) > 1 \quad \Rightarrow \quad I = \left(\dfrac{4}{a}, \max\left\{ \dfrac{4}{a}, 4(N+1) - \dfrac{4}{a} \right\} \right). \end{cases} \tag{1.35}$$

Here, one can see that the value $a(N+1)$ gives a criterion on the asymptotic behavior of solutions.

The second example is

$$\begin{cases} u'' + \dfrac{1}{r}u' = \dfrac{1}{\varepsilon^2} r^{-2aN} \left[\dfrac{e^u}{(e^u+1)^2} \right]^a \left(\dfrac{e^u - 1}{e^u + 1} \right), \\ u(r) = 2N \ln r + s + o(1) \quad \text{as} \quad r \searrow 0. \end{cases} \tag{1.36}$$

This equation is the reduction of self-dual equations for a Maxwell gauged $O(3)$ sigma model coupled with the Einstein equations [31,33]. Recently, Chern and Yang showed in [8] that if $0 < aN < 1$, then (1.36) allows a unique topological solution and one parameter family of nontopological solutions of type I and II. Indeed, they obtained a similar result to Theorem 1.1 (i) of this paper. They also proved that if $aN \geq 2$, then (1.36) possesses only nontopological solutions of type II. However, the case $1 < aN < 2$ has remained unsolved yet as far as we know. It turns out that the method employed in the proof of Theorem 1.1 is still valid for (1.36) in the case $1 < aN < 2$. We will report this aspect briefly in Sect. 6.

In conclusion, previous two examples tell us the importance of the number aN, the product of the scaled gravitational constant and the total string number N. When there are no topological solutions in a self-dual system coupled with gravity, (1.34) can be a model problem and the work of [23] provides the complete structure of solutions according to the value $a(N+1)$. On the other hand, (1.29) may serve as a model problem for self-dual equations with gravity which allow topological solutions and nontopological solutions of type I and II. In this case, Theorem 1.1 gives us the complete understanding of all possible solutions according to the value aN, and this is the main contribution of the present paper.

Here is an outline of the rest of this paper. In Sect. 2, when $0 \leq aN < 1$, we establish the existence of topological solutions and nontopological solutions of type I and II. The main tool is the shooting method with Pohozaev-type identities and Sturm–Liouville-type comparison argument. In Sect. 3, we classify all solutions of (1.29) in the case $aN \geq 1$. The argument of shooting in Sect. 2 may not work in the same way because of the condition $aN \geq 1$. To avoid this obstruction, we consider the Kelvin transform (1.33) for the case $1 < aN < 2$, which reduces the problem to the known result of Sect. 1. Regarding the case $aN \geq 2$, we prove the multiple existence of solutions enjoying given decay rates. In Sect. 4, we study an auxiliary problem arising from the case $1 < aN < 2$ in Sect. 3. In Sect. 5, returning to the original physical model, we compute several important quantities such as energy, magnetic flux, and the total Gaussian curvature by utilizing Theorem 1.1. In Sect. 6, we study equation (1.36) for the case $aN \geq 1$. By using the argument in previous sections, we obtain a classification of solutions for the case $aN \geq 1$ and improve known results.

We close this section with a remark on the solvability of the full Cauchy problems (1.6), (1.7) and (1.8). It was reported in [2] that there exists a global unique classical solution under the assumption of spherically symmetric spacetime. In the light of [12], such assumption implies that $\mathcal{M}$ is diffeomorphic to $\mathbb{R}^4$, the group $SO(3)$ acts on $\mathcal{M}$ as an isometry, and the group orbits are the metric spacelike 2-spheres. See [2,12] for more details.

2. Radial Solutions for $0 \leq aN < 1$

In this section, we prove Theorem 1.1 for the case

$$a > 0, \quad N \geq 0, \quad 0 \leq aN < 1.$$

Given parameters of $a > 0$ and $N \geq 0$, we need to verify each statement in Theorem 1.1 for topological solutions, nontopological solutions of type I, and nontopological solutions of type II. It is useful to consider a slightly generalized version of (1.29):

$$\begin{cases} u'' + \frac{1}{r}u' = \frac{1}{\varepsilon^2} r^{-2aN} e^{b(u-e^u)}(e^u - 1), & r > 0, \\ u(r) = 2N \ln r + s + o(1) \quad \text{near} & r = 0. \end{cases} \tag{2.1}$$

Here, a, b are positive real numbers and N is a nonnegative real number such that $0 \leq aN < 1$. We will use the result for this generalized equation in the construction of solutions of (1.29) for the case $1 < aN < 2$. By letting $v(r) = u(r) - 2N \ln r$ and integrating (2.1) twice, we obtain

$$v(r) = s - \int_0^r \frac{1}{t} \int_0^t \tau^{1+2bN-2aN} h(\tau, v, b) \mathrm{d}\tau \mathrm{d}t,$$

where $h(r, v, b) = e^{b(v - r^{2N} e^v)}(1 - r^{2N} e^v)/\varepsilon^2$. Since $1 + 2bN - 2aN > -1$, this integral is well defined. Since $h(r, v, b)$ is uniformly bounded in both $r > 0$

and $v \in \mathbb{R}$, it is standard to show the global existence and the uniqueness of solutions to (2.1). We denote by $u(r, s, a, N, b)$ the unique global solution of (2.1). If there is no risk of confusion, we write $u(r, s)$ or $u(r)$ for simplicity.

The main result of this section is to provide the complete classification of solutions for given real numbers $a > 0, N \geq 0, b > 0$ with $0 \leq aN < 1$. We want to do that according to the value

$$\beta_{a,N,b}(s) = \int_0^\infty r^{1-2aN} f\big(u(r,s), b\big) \mathrm{d}r.$$

Here, we recall

$$f(u,b) = \frac{1}{\varepsilon^2} e^{b(u-e^u)}(1-e^u).$$

By Lemma 2.3 below, it turns out that $\beta_{a,N,b}(s)$ has finite values for all $s \in \mathbb{R}$. Moreover, β is continuous except at one point s_* where it has infinite one sided limits. Let us write simply $\beta(s) = \beta_{a,N,b}(s)$ if there is no confusion. By (2.5) below, we obtain

$$\lim_{r\to\infty} ru'(r,s) = 2N - \beta(s). \tag{2.2}$$

Now, we state the main result of this section as follows.

Proposition 2.1. *Let a, N, b be real numbers such that*

$$a > 0, \quad b > 0, \quad N \geq 0, \quad 0 \leq aN < 1.$$

Then, the following statements hold true.

(i) *There exists a unique $s_* = s_*(a, N)$ such that $\beta(s_*) = 2N$, i.e., (2.1) has a unique topological solution. If $N > 0$, then $u(r, s_*) = O(r^{-\alpha})$ for any $\alpha > 0$ as $r \to \infty$. If $N = 0$, then $s_* = 0$ and $u(r, s_*) \equiv 0$.*

(ii) *If $s < s_*$, $u(r, s)$ is a nontopological solution of type I. The function $\beta : (-\infty, s_*) \to (\bar{\beta}, \infty)$ is continuous, onto, and strictly increasing, where*

$$\bar{\beta} = \bar{\beta}_{a,N,b} = \frac{4 + 4(b-a)N}{b} > 0.$$

We have

$$\lim_{s \nearrow s_*} \beta(s) = \infty, \qquad \lim_{s\to-\infty} \beta(s) = \bar{\beta}_{a,N,b}. \tag{2.3}$$

In addition, $u(r, s) = (2N - \beta) \ln r + I + O(r^{2+2(b-a)N-b\beta})$ for some constant $I = I(s)$ as $r \to \infty$.

(iii) *If $s > s_*$, $u(r, s)$ is a nontopological solution of type II. The function $\beta : (s_*, \infty) \to (-\infty, 0)$ is continuous, onto, and strictly increasing. We have*

$$\lim_{s \searrow s_*} \beta(s) = -\infty, \qquad \lim_{s\to\infty} \beta(s) = 0. \tag{2.4}$$

Furthermore, $u(r, s) = (2N-\beta) \ln r+J+O(r^{2-2aN-m})$ for some constant $J = J(s)$ as $r \to \infty$, where $m > 2 - 2aN$ is any number.

We remark that if $a = b$, then Proposition 2.1 establishes the statement (i) of Theorem 1.1 for the case $0 \leq aN < 1$. Parts of Proposition 2.1 are established in [7], where the existence of *topological solutions and nontopological solutions of type I* and the limit (2.3) were proved. So, the main points of the proof of Proposition 2.1 are the verification of existence of nontopological solutions of type II, properties for nontopological solutions of type II including the limit (2.4), and the monotonicity of β for nontopological solutions of type I and type II. The proof of the monotonicity of β is nontrivial and becomes one of the main contributions of this paper. Here, we provide the full proof of Proposition 2.1 containing the case of topological solutions and nontopological solutions of type I for two reasons: (i) for the sake of completeness and (ii) the proof for $0 \leq aN < 1$ is used for the case $aN \geq 1$. The proof of Proposition 2.1 follows from a series of lemmas and is given at the end of this section.

Integrating (2.1), we get useful formula

$$ru'(r,s) = 2N - \int_0^r \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau, \tag{2.5}$$

$$u(r,s) = 2N\ln r + s - \int_0^r \frac{1}{t}\int_0^t \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau\mathrm{d}t. \tag{2.6}$$

To classify radial solutions, we define

$$\begin{aligned}
S^{+}_{a,N,b} &= \{s \in \mathbb{R} : u(r_0,s) > 0 \text{ for some } r_0 > 0\},\\
S^{0}_{a,N,b} &= \{s \in \mathbb{R} : u(r,s) \leq 0 \text{ and } u'(r,s) \geq 0 \text{ for all } r > 0\},\\
S^{-}_{a,N,b} &= \{s \in \mathbb{R} : u(r,s) \leq 0 \text{ for all } r > 0 \text{ and } u'(r_0,s) < 0 \text{ for some } r_0 > 0\}.
\end{aligned}$$

In what follows we shall often drop the indices a, N, b in $S^{\pm,0}_{a,N,b}$ for simplicity. By the continuous dependence of solutions on s, it is obvious that S^+ and S^- are open. Hence, $S^0 = \mathbb{R}\backslash(S^+ \cup S^-)$ is closed. In the following, we assume that $N > 0$. The case $N = 0$ will be treated at the end of this section.

Lemma 2.2. *Let $0 < aN < 1$. Then, we have the following.*

(i) *If $s \in S^{+}_{a,N,b}$, then $u'(r,s) > 0$ for $u < 0$. Moreover, there is a unique point $y(s) = y(s,a,N,b)$ such that $u(y(s),s) = 0$.*
(ii) *If $s \in S^{0}_{a,N,b}$, then $\lim_{r\to\infty} u(r,s) = 0$.*
(iii) *If $s \in S^{-}_{a,N,b}$, then $(ru')' < 0$, $u(r,s) < 0$ and $\lim_{r\to\infty} u(r,s) = -\infty$. Moreover, u has a unique maximum point $z(s) = z(s,a,N,b)$ such that $u'(r,s) > 0$ for $0 < r < z$ and $u'(r,s) < 0$ for $z < r < \infty$.*

Proof. (i) Let $s \in S^+$ and $y(s) = \inf\{r : u(r,s) \geq 0\}$ so that $u'(y,s) > 0$. Since $(ru')' < 0$ for $u < 0$, $ru'(r)$ is decreasing for $u < 0$. Hence, $ru'(r) > yu'(y) > 0$ for $r < y$. Since $(ru')' > 0$ for $r > y$, $u(r) > 0$ for all $r > y$.
(ii) Let $s \in S^0$. Since $u(r) \leq 0$ and $u'(r) \geq 0$, there exists $\lim_{r\to\infty} u(r) = c \in (-\infty, 0]$. If $c \neq 0$ and $u(r_0) = c - 1$, then (2.5) implies that

$$\begin{aligned}\lim_{r\to\infty} ru'(r) &= 2N - \int_0^\infty r^{1-2aN} f(u,b)\mathrm{d}r \\ &\le 2N - M\int_{r_0}^\infty r^{1-2aN}\mathrm{d}r = -\infty,\end{aligned}$$

which is a contradiction. Here, $M = \inf\{f(u,b) : c-1 \le u \le c\} > 0$.

(iii) For $s \in S^-$, $u(r) < 0$ for all $r > 0$ by the strong maximum principle. Then $(ru')' < 0$ for all $r > 0$, which implies that u has a unique maximum point $z(s)$. Since $u'(r) < 0$ for $r > z$, there exists $\lim_{r\to\infty} u(r) = c \in [-\infty, 0)$. If $c \ne -\infty$, then there exists $R > z$ such that

$$(ru')' < -c_0 r^{1-2aN} \tag{2.7}$$

for all $r > R$ and $u'(R) < 0$. Here, $c_0 = \inf\{f(u,b) : c \le u \le u(z(s),s)\} > 0$. Integrating (2.7) twice, we obtain that

$$u(r) < u(R) + \Big(Ru'(R) + \frac{c_0R^{2-2aN}}{2-2aN}\Big)\ln\frac{r}{R} - \frac{c_0(r^{2-2aN}-R^{2-2aN})}{(2-2aN)^2},$$

which implies that $u(r) \to -\infty$ as $r \to \infty$, a contradiction. So $c = -\infty$. It is obvious that $u'(r) > 0$ for $0 < r < z$ and $u'(r) < 0$ for $z < r < \infty$.

□

Lemma 2.3. *Let $0 < aN < 1$. The function $\beta(s)$ is finite for $s \in \mathbb{R}$ and continuous on $S^-_{a,N,b} \cup S^+_{a,N,b}$. Moreover, $\beta(s) = 2N$ for $s \in S^0_{a,N,b}$, $\beta(s) > \big[2 + 2(b-a)N\big]/b > 0$ for $s \in S^-_{a,N,b}$, and $\beta(s) < 2N$ for $s \in S^+_{a,N,b}$.*

Proof. By Lemma 2.2, the limit $c_s := \lim_{r\to\infty} ru'(r,s)$ exists for all $s \in \mathbb{R}$. It is easy to see that if $c_s = \pm\infty$, then $f\big(u(r,s)\big) \in L^1(\mathbb{R})$ so that $\beta(s)$ is finite, which in turn implies that c_s is finite, a contradiction. Hence, $\beta(s) = 2N - \lim_{r\to\infty} ru'(r,s)$ is finite. Since $\beta(s)$ is finite, from the integrability condition $f\big(u(r,s)\big) \in L^1(\mathbb{R})$, it is easy to check that $\beta(s) > \big[2+2(b-a)N\big]/b$ for $s \in S^-$ and $\beta(s) < 2N$ for $s \in S^+$. Since $c_s = 0$ for $s \in S^0$, we have $\beta(s) = 2N$ for $s \in S^0$.

For continuity on $S^- \cup S^+$, let $s_0 \in S^-$ and choose a number λ such that $\beta(s_0) > \lambda > [2 + 2(b-a)N]/b$. Then, there exists R such that $ru'(r,s_0) < -\lambda + 2N$ for $r \ge R$. By the continuous dependence of solutions on s, there exists $\delta > 0$ so that $Ru'(R,s) < -\lambda + 2N$ for all $|s-s_0| \le \delta$. Since $(ru')' < 0$, it follows that $ru'(r,s) < -\lambda + 2N$ for all $r \ge R$ and $|s-s_0| \le \delta$. Then,

$$r^{1-2aN} f\big(u(r,s),b\big) \le Cr^{1+2(b-a)N-b\lambda}, \quad \forall r \ge R,\ |s-s_0| \le \delta.$$

Now, the continuity of $\beta(s)$ at s_0 comes from the Lebesgue convergence theorem. The proof for the case $s_0 \in S^+$ follows similarly. □

Lemma 2.4. *Let $0 < aN < 1$. If $s \in S^-_{a,N,b} \cup S^+_{a,N,b}$, then*

$$\beta(\beta - 4N) = \frac{4-4aN}{b\varepsilon^2}\int_0^\infty r^{1-2aN} e^{b(u-e^u)}\mathrm{d}r, \tag{2.8}$$

$$\beta(\beta - \bar\beta) = \frac{4-4aN}{b\varepsilon^2}\int_0^\infty r^{1-2aN} e^{b(u-e^u)} e^u\mathrm{d}r, \tag{2.9}$$

where $\bar{\beta} = \bar{\beta}_{a,N,b} = \big[4+4(b-a)N\big]/b > 0$. *Furthermore,* $\beta(s) > \bar{\beta}$ *for* $s \in S^-_{a,N,b}$ *and* $\beta(s) < 0$ *for* $s \in S^+_{a,N,b}$.

Proof. Multiplying (2.1) by ru', we obtain the following Pohozaev-type identity:

$$\begin{aligned} E(r,s,a,N,b) &:= \frac{1}{2}|ru'(r,s)|^2 - 2N^2 + r^{2-2aN}F\big(u(r,s),b\big) \\ &= (2-2aN)\int_0^r \tau^{1-2aN}F\big(u(\tau,s),b\big)\mathrm{d}\tau. \end{aligned} \tag{2.10}$$

Here, $F(u,b) = e^{b(u-e^u)}/(\varepsilon^2 b)$. Letting $r \to \infty$ in (2.10), we obtain (2.8). Moreover, (2.8) can be rewritten as

$$\beta(\beta - 4N) = \frac{\beta(4-4aN)}{b} + \frac{4-4aN}{b\varepsilon^2}\int_0^\infty r^{1-2aN}e^{b(u-e^u)}e^u\mathrm{d}r,$$

which leads us to (2.9). We deduce from (2.8) and (2.9) that $\beta(s) > \max\{\bar{\beta}, 4N\} = \bar{\beta}$ for $s \in S^-$ and $\beta(s) < 0$ for $s \in S^+$. □

Set $\varphi(r,s) = \frac{\partial}{\partial s}u(r,s)$. Then φ satisfies the linearized equation

$$\begin{cases} \varphi'' + \dfrac{1}{r}\varphi' = -r^{-2aN}f'(u,b)\varphi, \\ \varphi(0,s) = 1, \quad \varphi'(0,s) = 0. \end{cases}$$

Here, $f'(u,b) := \partial f/\partial u = e^{b(u-e^u)}\{b(1-e^u)^2 - e^u\}/\varepsilon^2$. For the Sturm–Liouville-type comparison argument, we set $w_c(r,s) = ru'(r,s) + c$. Then w_c satisfies that

$$\begin{cases} w_c'' + \dfrac{1}{r}w_c' = -r^{-2aN}f'(u,b)w_c - (2-2aN)r^{-2aN}f(u,b) \\ \qquad\qquad\qquad + cr^{-2aN}f'(u,b), \\ w_c(0,s) = 2N + c, \quad w_c'(0,s) = 0. \end{cases} \tag{2.11}$$

Here, $w_c' = \partial w_c/\partial r$. For simplicity, let

$$\varepsilon^2\Phi_c(r,s) = (2-2aN)(1-e^u) - c[b(1-e^u)^2 - e^u] \tag{2.12}$$

so that

$$(rw_c')' = -r^{1-2aN}f'(u,b)w_c - r^{1-2aN}e^{b(u-e^u)}\Phi_c(r,s).$$

In what follows we shall often write $\varphi(r)$ instead of $\varphi(r,s)$, and so on. The following lemma deals with the monotonicity of the solution with respect to the parameter s.

Lemma 2.5. *Let* $0 < aN < 1$. *If* $u(r,s) < 0$ *and* $u'(r,s) > 0$ *on* $(0,R)$ *for some* $R > 0$, *then* $\varphi(r,s) > 0$ *for all* $r \in (0,R)$. *Moreover, if* $s \in S^0_{a,N,b}$, *then* $\varphi(r,s) \to \infty$ *as* $r \to \infty$.

Proof. Let $s \in \mathbb{R}$ be fixed. Suppose that $\varphi(r,s)$ has the first zero at $r_0 < R$. Then, $r_0 < y(s)$ for $s \in S^+$ and $r_0 < z(s)$ for $s \in S^-$. Hence, $f\big(u(r,s)\big) > 0$ on $(0,r_0)$. Comparing φ and w_0, we obtain

$$0 > r_0\varphi'(r_0,s)w_0(r_0,s) = (2-2aN)\int_0^{r_0} r^{1-2aN}f(u,b)\varphi(r)\mathrm{d}r > 0,$$

a contradiction. Thus, $\varphi(r,s) > 0$ for all $r \in (0,R)$ if $s \in \mathbb{R}$.

Now, let $s \in S^0$. Then $\varphi(r,s) > 0$ for all $r \in (0,\infty)$. Since

$$r\varphi'(r,s)w_0(r,s) - rw_0'(r,s)\varphi(r,s) = (2-2aN)\int_0^{r} t^{1-2aN}f(u,b)\varphi(t)\mathrm{d}t > 0,$$

$\varphi'(r,s) > 0$ for all large $r > 0$. Moreover, since $u(r,s) \to 0$ as $r \to \infty$, there exists R_0 such that $(r\varphi')' \geq 0$ for all $r \geq R_0$. Integrating this inequality twice, we deduce that for all $r \geq R_0$

$$\varphi(r,s) \geq \varphi(R_0,s) + R_0\varphi'(R_0,s)\ln\frac{r}{R_0}.$$

Therefore, $\lim_{r\to\infty}\varphi(r,s) = \infty$. □

Lemma 2.6. *Let* $0 < aN < 1$, $s \in S^-_{a,N,b}$ *and*

$$m(s) = m(s,a,N,b) = u\big(z(s),s,a,N,b\big) = \sup\{u(r,s,a,N,b)|r > 0\}.$$

Then $m(s)$ *is increasing and* $\lim_{s\to-\infty} m(s) = -\infty$.

Proof. For $s \in S^-$, $u'(z(s),s) = 0$ and $(ru')'(z(s),s) < 0$. Then the implicit function theorem tells us that $z(s)$ is continuously differentiable with respect to s. We note from Lemma 2.5 that

$$\frac{\mathrm{d}}{\mathrm{d}s}m(s) = u'(z(s),s)z'(s) + \frac{\mathrm{d}u}{\mathrm{d}s}(z(s),s) = \varphi(z(s),s) \geq 0. \tag{2.13}$$

Hence, $m(s)$ is increasing in s.

Let s_n be any decreasing sequence such that $s_n \in S^-$ and $s_n \to -\infty$. Assume that $\xi = \inf_n m(s_n) > -\infty$. For simplicity, let $m_n = m(s_n)$ and $z_n = z(s_n)$. It comes from (2.6) that $\xi \leq m_n \leq 2N\ln z_n + s_n$, which implies $z_n \to \infty$. Let $r_n < z_n$ be the unique number such that $u(r_n,s_n) = m_n - 1$. Since $ru'(r,s_n) \leq 2N$ for all $r > 0$, it follows that $1 \leq 2N\ln(z_n/r_n)$. We infer from (2.13) that $\xi - 1 \leq u(r,s_n) \leq m_1$ for $r_n \leq r \leq z_n$. Then, we are led to a contradiction:

$$\begin{aligned} 2N &= \int_0^{z_n} r^{1-2aN}f\big(u(r,s_n),b\big)\mathrm{d}r \geq c_0\int_{r_n}^{z_n} r^{1-2aN}\mathrm{d}r \\ &\geq \frac{c_0\cdot z_n^{2-2aN}}{2-2aN}\Big(1-e^{-\frac{2-2aN}{2N}}\Big) \quad\to\quad \infty, \end{aligned}$$

where $c_0 = \inf\{f(u,b) \mid \xi - 1 \leq u \leq m_1\}$. □

Now, we are in a position to classify the solutions of (2.1) according to the shooting parameter s.

Lemma 2.7. *If* $0 < aN < 1$, *then there exists* $s_* = s_*(a,N,b) \in \mathbb{R}$ *such that* $S^+_{a,N,b} = (s_*,\infty)$, $S^-_{a,N,b} = (-\infty,s_*)$, *and* $S^0_{a,N,b} = \{s_*\}$.

Proof. First, we claim that

$$s \in S^- \text{ for all } s \ll -1 \text{ and } s \in S^+ \text{ for all } s \gg 1. \tag{2.14}$$

If $s \in S^+ \cup S^0$, then we can take $r_1 < r_2$ such that $u(r_1, s) = -2$ and $u(r_2, s) = -1$. Since $ru' \leq 2N$ for $u < 0$, we have $1 \leq 2N \ln(r_2/r_1)$. By (2.6), it holds that $-1 < 2N \ln r_2 + s$. Thus, $r_2 \to \infty$ as $s \to -\infty$. Then, as $s \to -\infty$,

$$\begin{aligned} 0 < r_2 u'(r_2) &= r_1 u'(r_1) - \int_{r_1}^{r_2} r^{1-2aN} f\big(u(r,s), b\big)\mathrm{d}r \\ &\leq 2N - \frac{C_0 r_2^{2-2aN}}{2-2aN}\big(1 - e^{-\frac{2-2aN}{2N}}\big) \ \to \ -\infty, \end{aligned}$$

which yields a contradiction and the first part of (2.14) follows. Here, $C_0 = \inf\{f(u,b) : -2 \leq u \leq -1\}$.

For the second part of (2.14), let $v(r,s) = u(r,s) - 2N \ln r$. Then for all large s, it holds that

$$\begin{aligned} v(1,s) &= s - \int_0^1 \frac{1}{t} \int_0^t \tau^{1+2bN-2aN} h\big(v(\tau,s), \tau, b\big)\mathrm{d}\tau \mathrm{d}t \\ &\geq s - \frac{\|h\|_\infty}{4(1+bN-aN)^2} > 0, \end{aligned}$$

where $h(v,r,b) = e^{b(v - r^{2N}e^v)}(1 - r^{2N}e^v)/\varepsilon^2$ such that $\|h(\cdot,\cdot,b)\|_{L^\infty(\mathbb{R}^2)} < \infty$. Hence, $s \in S^+$ for all $s \gg 1$.

Next, we also claim that

$$\begin{cases} \text{if } (s_1, s_2) \subset S^-, & \text{then } s_1 \in S^-; \\ \text{if } (s_1, s_2) \subset S^+, & \text{then } s_2 \in S^+. \end{cases} \tag{2.15}$$

We recall from (2.13) that $m(s)$ is increasing for $s \in S^-$. Given $(s_1, s_2) \subset S^-_{a,N,b}$, fix any $s_0 \in (s_1, s_2)$. Then $m(s) \leq m(s_0) < 0$ for all $s \in (s_1, s_0)$. Now, the continuous dependence of solutions on s implies that $\sup_{r>0} u(r, s_1) \leq m(s_1) < 0$, and hence $s_1 \in S^-$.

On the other hand, if $s \in (s_1, s_2) \subset S^+$, then $u'(r,s) > 0$ and $\varphi(r,s) > 0$ on $(0, y(s)]$. The implicit function theorem implies that $y(s)$ is continuously differentiable with respect to s. Since

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} u(y(s), s) = u'(y(s), s) y'(s) + \frac{\mathrm{d}u}{\mathrm{d}s}(y(s), s),$$

it holds that $y'(s) = -\varphi(y(s), s)/u'(y(s), s) < 0$, which implies that there exists $\lim_{s \nearrow s_2} y(s) = y_0$. Since $0 = 2N \ln y(s) + s + O(1)$ for $s \in (s_1, s_2)$ by (2.6), we have $y_0 > 0$. Then, from continuous dependence of solutions on s we deduce that $u(y_0, s_2) = 0$. Thus $s_2 \in S^+$ and the claim (2.15) is proved.

Now, by (2.14) and (2.15), there exist two numbers $s_*(a,N) \leq s^*(a,N)$ such that $S^+ = (s^*, \infty)$, $S^- = (-\infty, s_*)$, and $S^0 = [s_*, s^*]$. It remains to show that $s_* = s^*$. If $s_* < s^*$, then we infer from Lemma 2.5 that

$$
\begin{aligned}
0 = \lim_{r\to\infty}(u(r,s^*) - u(r,s_*)) &= \lim_{r\to\infty}\int_{s_*}^{s^*}\frac{\partial}{\partial s}u(r,s)\mathrm{d}s \\
&= \lim_{r\to\infty}\int_{s_*}^{s^*}\varphi(r,s)\mathrm{d}s \ \geq \ \int_{s_*}^{s^*}\liminf_{r\to\infty}\varphi(r,s)\mathrm{d}s = \infty,
\end{aligned}
$$

which is a contradiction. □

Remark 2.8. If $a = b$, then for any $a > 0$ and $N > 0$ it follows that $s \in S^+_{a,N,a}$ for all $s \gg 1$. One can easily check this fact from the proof of Lemma 2.7.

Lemma 2.9. *Let* $0 < aN < 1$.

(i) $\beta_{a,N,b} : (-\infty, s_*) \to (\bar{\beta}_{a,N,b}, \infty)$ *is onto such that* $\beta_{a,N,b}(s) \to \infty$ *as* $s \nearrow s_*$, *and* $\beta_{a,N,b}(s) \to \bar{\beta}_{a,N,b}$ *as* $s \to -\infty$.

(ii) $\beta_{a,N,b} : (s_*, \infty) \to (-\infty, 0)$ *is onto such that* $\beta_{a,N,b}(s) \to -\infty$ *as* $s \searrow s_*$, *and* $\beta_{a,N,b}(s) \to 0$ *as* $s \to \infty$.

Proof. By Lemma 2.4, we know that $\beta(s) > \bar{\beta}$ for $s < s_*$ and $\beta(s) < 0$ for $s > s_*$. Using (2.9), we observe by Fatou's lemma that

$$
\begin{aligned}
\liminf_{s\to s_*}\beta(\beta-\bar{\beta}) &= \liminf_{s\to s_*}\frac{4-4aN}{b\varepsilon^2}\int_0^\infty r^{1-2aN}e^{b\left(u(r,s)-e^{u(r,s)}\right)}e^{u(r,s)}\mathrm{d}r \\
&\geq \frac{4-4aN}{b\varepsilon^2}\int_0^\infty r^{1-2aN}e^{b\left(u(r,s_*)-e^{u(r,s_*)}\right)}e^{u(r,s_*)}\mathrm{d}r = \infty.
\end{aligned}
$$

Hence, $\beta(s) \to \infty$ as $s \nearrow s_*$ and $\beta(s) \to -\infty$ as $s \searrow s_*$. We also derive from (2.9) that for $s \in S^-$

$$
\begin{aligned}
\beta(\beta-\bar{\beta}) &\leq \frac{e^{m(s)}(4-4aN)}{b\varepsilon^2}\int_0^\infty r^{1-2aN}e^{b(u-e^u)}\mathrm{d}r \\
&\leq e^{m(s)}\cdot\beta(\beta-4N).
\end{aligned}
$$

Since $m(s) \to -\infty$ as $s \to -\infty$, it holds that $\beta(s) \to \bar{\beta}$ as $s \to -\infty$. Since β is a continuous function of s, $\beta : (-\infty, s_*) \to (\bar{\beta}, \infty)$ is onto.

On the other hand, if $s \in S^+$, then $u(1,s) = s + O(1) \to \infty$ as $s \to \infty$. We define the number $t_s < 1$ such that $u(t_s, s) = \frac{1}{2}u(1,s)$. Since $2N\ln t_s = -\frac{s}{2} + O(1)$ by (2.6), $t_s \to 0$ as $s \to \infty$. Then, as $s \to \infty$,

$$
\begin{aligned}
\beta(\beta-4N) &= \frac{4-4aN}{b\varepsilon^2}\int_0^\infty r^{1-2aN}e^{b(u-e^u)}\mathrm{d}r \\
&\leq o(1) + \frac{ce^{-s/2}(4-4aN)}{b\varepsilon^2}\int_{t_s}^\infty r^{1-2aN}e^{b(u-e^u)}e^u\mathrm{d}r \\
&\leq o(1) + o(1)\cdot\beta\Big(\beta-\bar{\beta}\Big).
\end{aligned}
$$

Since $\beta(s) < 0$ for $s > s_*$, this implies that $\beta(s) \to 0$ as $s \to \infty$. □

In the remaining part of this section, we pay attention to the proof of monotonicity of β following the idea of [6,10]. Indeed, we will show that $\beta'(s) \neq 0$ for $s \in S^{\pm}$. It is easy to see by the Lebesgue convergence theorem that

$$\beta'(s) = \int_0^\infty r^{1-2aN} f'\big(u(r,s)\big)\varphi(r,s)\mathrm{d}r = -\lim_{r\to\infty} r\varphi'(r,s). \tag{2.16}$$

Thus, the monotonicity of $\beta(s)$ is closely related to the behavior of $\varphi(r,s)$ at infinity. For comparison arguments, we recall $w_c(r)$ defined by (2.11).

Lemma 2.10. *If* $0 < aN < 1$ *and* $s \in S^+_{a,N,b}$, *then* $\varphi(r,s)$ *has exactly one zero and* $\lim_{r\to\infty}\varphi(r,s) = -\infty$. *Moreover,* $\beta'_{a,N,b}(s) > 0$ *such that* $\beta_{a,N,b}(s)$ *is strictly increasing.*

Proof. For simplicity, we write $u(r) = u(r,s)$, $\varphi(r) = \varphi(r,s)$, $f(u) = f(u,b)$ and so on. We recall from Lemma 2.2 that $y = y(s)$ is the unique point such that $u\big(y(s),s\big) = 0$. Suppose $\varphi(r) > 0$ for all $r > 0$. Then, $(r\varphi')' < 0$ for all large r so that $\varphi'(r) > 0$ as $r \to \infty$. Comparing w_0 with φ, we obtain that

$$(2-2aN)\int_0^\infty r^{1-2aN} f(u)\varphi(r)\mathrm{d}r = \lim_{r\to\infty} r\varphi'(r)w_0(r) \geq 0. \tag{2.17}$$

Now, we note that

$$\left(\frac{\varphi}{w_0}\right)'(r) = \frac{(2-2aN)}{rw_0^2}\int_0^r t^{1-2aN} f(u)\varphi(t)\mathrm{d}t. \tag{2.18}$$

Since the map

$$r \mapsto (2-2aN)\int_0^r t^{1-2aN} f(u)\varphi(t)\mathrm{d}t \tag{2.19}$$

is increasing on $(0,y)$ and decreasing on (y,∞), we deduce from (2.17) and (2.18) that $(\varphi/w_0)'(r) > 0$ for all $r > 0$. Set $c_0 = (\varphi/w_0)(y) > 0$. Then $(\varphi - c_0w_0)(r) < 0$ on $(0,y)$ and $(\varphi - c_0w_0)(r) > 0$ on (y,∞) such that $f(u)(\varphi - c_0w_0)(r) < 0$ for all $r \neq y$. Hence, we obtain that

$$\begin{aligned} 0 &\leq (2-2aN)\int_0^\infty r^{1-2aN} f(u)\varphi(r)\mathrm{d}r \\ &< c_0(2-2aN)\int_0^\infty r^{1-2aN} f(u)w_0(r)\mathrm{d}r \\ &= -\frac{c_0(2-2aN)^2}{b\varepsilon^2}\int_0^\infty r^{1-2aN} e^{b(u-e^u)}\mathrm{d}r \ < \ 0, \end{aligned}$$

which is a contradiction. As a consequence, $\varphi(r)$ must have the first zero at $r_1 = r_1(s) > 0$ such that $y < r_1$ by Lemma 2.5.

If $\varphi(r)$ has the second zero at r_2, then $\varphi(r) < 0$ on (r_1,r_2). Since $(\varphi/w_0)'(r) > 0$ for $r \leq y$ by (2.18) and $(\varphi/w_0)(r_1) = 0$, there exists $\eta_1 \in (y,r_1)$ such that

$$0 = \left(\frac{\varphi}{w_0}\right)'(\eta_1) = \frac{r\varphi' w_0 - rw_0'\varphi}{rw_0^2}(\eta_1). \tag{2.20}$$

In particular, $\varphi'(\eta_1) > 0$ since $u(\eta_1) > 0$ such that $w_0'(\eta_1) > 0$. Let

$$\inf\{(\varphi/w_0)(r) | r \in (r_1,r_2)\} = (\varphi/w_0)(\eta_2).$$

Then, $(\varphi/w_0)'(\eta_2) = 0$ and hence $\varphi'(\eta_2) < 0$ by a similar computation as (2.20). As a consequence,

$$\int_{\eta_1}^{\eta_2} r^{1-2aN} f'(u)\varphi(r)\mathrm{d}r = \eta_1\varphi'(\eta_1) - \eta_2\varphi'(\eta_2) > 0.$$

Moreover, it follows from (2.18) that

$$\int_{\eta_1}^{\eta_2} r^{1-2aN} f(u)\varphi(r)\mathrm{d}r = 0.$$

A simple calculation gives

$$\left(\frac{f'(u)}{f(u)}\right)' = \frac{-\,e^u\left\{b(1-e^u)^2+1\right\}}{(1-e^u)^2} < 0. \tag{2.21}$$

Then, one can check that $\left[f'\big(u(r)\big) - c_1 f\big(u(r)\big)\right]\varphi(r) < 0$ for $r \in (\eta_1, \eta_2)$, where $c_1 = f'\big(u(r_1)\big)/f\big(u(r_1)\big)$. This leads us to a contradiction:

$$0 < \int_{\eta_1}^{\eta_2} r^{1-2aN} f'(u)\varphi(r)\mathrm{d}r = \int_{\eta_1}^{\eta_2} r^{1-2aN}[f'(u) - c_1 f(u)]\varphi(r)\mathrm{d}r < 0.$$

Therefore, $\varphi(r)$ has exactly one zero.

Now, r_1 is the unique zero of $\varphi(r)$ with $y < r_1$. Since

$$0 < (2-2aN)\int_0^y r^{1-2aN} f(u)\varphi(r)\mathrm{d}r = y\varphi'(y)w_0(y),$$

it holds that $\varphi'(y) > 0$. Since $\varphi'(r_1) < 0$ and $\varphi'(r) < 0$ near $r = 0$, we deduce that $\varphi'(r)$ has at least two zeros in $(0, r_1)$. In particular, $(r\varphi')'$ has at least two zeros in $(0, r_1)$. Meanwhile, $f'\big(u(r)\big)$ has exactly two zeros $t_1 < t_2$ in $(0,\infty)$ such that $f'\big(u(r)\big) < 0$ on (t_1, t_2) and $f'\big(u(r)\big) > 0$ on $(0, t_1) \cup (t_2, \infty)$. Thus, $(r\varphi')'$ has exactly three zeros $t_1 < t_2 < r_1$ in $(0,\infty)$, and $\varphi'(r)$ has two zeros $R_1 < R_2$ such that

$$R_1 < y < R_2 < r_1 \quad \text{and} \quad t_1 < R_1 < t_2 < R_2 < r_1.$$

Since $\varphi(r) < 0$ on (r_1, ∞) and $(r\varphi')'(r) > 0$ for $r > r_1$, $r\varphi'(r)$ is increasing on (r_1, ∞) and there exists $\lim_{r\to\infty} r\varphi'(r, s) = \delta_s \le 0$.

We claim that $\delta_s \ne 0$. To see this, let $c_1 = f'(u(r_1))/f(u(r_1)) < 0$ as before. Then $f'(u) - c_1 f(u)$ is negative on (R_2, r_1) and positive on (r_1, ∞). Hence $[f'(u) - c_1 f(u)]\varphi(r) < 0$ on $(R_2, r_1) \cup (r_1, \infty)$. Since $\varphi(R_2) > 0$, if $\delta_s = 0$, then

$$\begin{aligned}
0 = -\lim_{r\to\infty} r\varphi'(r) &= \int_2^{\infty} r^{1-2aN} f'(u)\varphi(r)\mathrm{d}r \\
&< c_1 \int_{R_2}^{\infty} r^{1-2aN} f(u)\varphi(r)\mathrm{d}r \\
&= \frac{c_1}{2-2aN}\lim_{r\to\infty}\left[r\varphi'(r)w_0(r) - rw_0'(r)\varphi(r)\right] \\
&\quad - \frac{c_1}{2-2aN}\left[R_2\varphi'(R_2)w_0(R_2) - R_2 w_0'(R_2)\varphi(R_2)\right] \\
&= \frac{c_1}{2-2aN} R_2 w_0'(R_2)\varphi(R_2) \; < \; 0,
\end{aligned}$$

which is a contradiction. Consequently, $r\varphi'(r) \le \delta_s < 0$ for all large r and thus $\lim_{r\to\infty}\varphi(r) = -\infty$. Hence, $\beta(s)$ is strictly increasing by (2.16). $\square$

Lemma 2.11. *If* $0 < aN < 1$ *and* $s \in S^-_{a,N,b}$*, then* $\varphi(r,s)$ *has exactly one zero and* $\lim_{r\to\infty}\varphi(r,s) = -\infty$*. Moreover,* $\beta'_{a,N,b}(s) > 0$ *such that* $\beta_{a,N,b}(s)$ *is strictly increasing.*

Proof. For simplicity, we write $u(r) = u(r,s)$, $\varphi(r) = \varphi(r,s)$, $f(u) = f(u,b)$ and so on. We write $\Phi_c(r) = G_c\big(e^{u(r)}\big)$, where

$$\varepsilon^2 G_c(t) = g(t) - ch(t), \quad g(t) = (2-2aN)(1-t), \quad h(t) = b(1-t)^2 - t. \tag{2.22}$$

We note that $h(t)$ has two zeros T_1, T_2 with $T_1 < 1 < T_2$. Moreover, $G_c(t) \geq 0$ for all $t \in (0,1)$ if and only if $c \leq c_1 := (2-2aN)/b$.

We recall that $z(s)$ is the unique point such that $u'(z(s)) = 0$. Suppose $\varphi(r) \geq 0$ for all $r \geq 0$ so that $\varphi'(r) > 0$ as $r \to \infty$. Then

$$0 < (2-2aN)\int_0^\infty r^{1-2aN} f(u)\varphi(r)\mathrm{d}r = \lim_{r\to\infty} r\varphi'(r)w_0(r) < 0,$$

which yields a contradiction. As a consequence, $\varphi(r)$ must have the first zero at $r_1 > 0$ such that $z(s) < r_1$ by Lemma 2.5.

Assume that $\varphi(r)$ changes its signs more than twice. Let r_2 be the second zero of $\varphi(r)$. If r_3 is the third zero, then there exist R_1 and R_2 with $R_1 < r_2 < R_2 < r_3$ such that $\varphi(r)$ has a local minimum and a local maximum at $r = R_1, R_2$ respectively. Set $\sigma_1 = e^{u(R_1)}$, $\xi = e^{u(r_2)}$ and $\sigma_2 = e^{u(R_2)}$. Then, $\sigma_2 < \xi < \sigma_1 < 1$. Since

$$0 \leq (r\varphi')'(R_1) = \Big[-\frac{1}{\varepsilon^2} r^{1-2aN} e^{b(u-e^u)} h(e^u)\varphi\Big](R_1),$$

it follows that $h(\sigma_1) \geq 0$ which says that $\sigma_1 \leq T_1$. Hence, we can choose $c_0 > c_1$ such that $g(\xi) = c_0 h(\xi)$, namely, $\Phi_{c_0}(r_2) = 0$. See Fig. 1. In particular, $\Phi_{c_0}(r) > 0$ for $R_1 < r < r_2$ and $\Phi_{c_0}(r) < 0$ for $r_2 < r < R_2$. As a consequence, $\Phi_{c_0}(r)\varphi(r) < 0$ on $(R_1, r_2) \cup (r_2, R_2)$. Therefore, we obtain that

$$\begin{aligned} 0 &> \int_{R_1}^{R_2} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_0}(r)\varphi(r)\mathrm{d}r \\ &= -R_2 w'_{c_0}(R_2)\varphi(R_2) + R_1 w'_{c_0}(R_1)\varphi(R_1) > 0, \end{aligned}$$

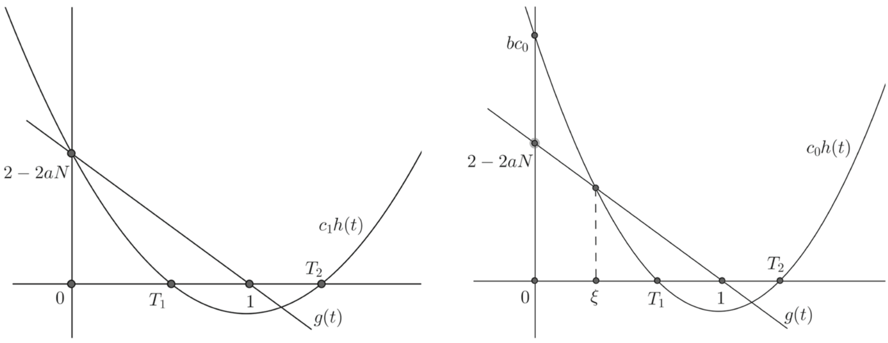
which is a contradiction. In the sequel, $\varphi(r)$ has at most two zeros. Therefore, either $\varphi(r)$ has only one zero or $\varphi(r)$ has exactly two zeros.

If $\varphi(r)$ has two zeros at $r_1 < r_2$, then $\varphi(r) > 0$ on (r_2, ∞). Since $u(r)$ is decreasing for $r > r_2$ and $\Phi_{c_0}(r_2) = 0$, we see that $\Phi_{c_0}(r) < 0$ for $r > r_2$. Since $(r\varphi')' < 0$ for all large r, there exists $\mu_s = \lim_{r\to\infty} r\varphi'(r,s) \geq 0$. If $\mu_s = 0$, then

$$0 > \int_{r_2}^\infty r^{1-2aN} e^{b(u-e^u)} \Phi_{c_0}(r)\varphi(r)\mathrm{d}r = -r_2\varphi'(r_2)w_{c_0}(r_2).$$

Hence, $w_{c_0}(r_2) > 0$ which also implies that $w_{c_0}(r_1) > 0$. Therefore,

$$\begin{aligned} 0 &> \int_{r_1}^{r_2} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_0}(r)\varphi(r)\mathrm{d}r \\ &= r_2\varphi'(r_2)w_{c_0}(r_2) - r_1\varphi'(r_1)w_{c_0}(r_1) > 0, \end{aligned}$$



(a) $c = c_1 = (2 - 2aN)/b$. **(b)** $c = c_0 > c_1$.

FIGURE 1. Graph of $g(t)$ and $ch(t)$

a contradiction. Thus, $\mu_s > 0$. In particular, $\beta(s)$ is decreasing by (2.16). However, this is impossible by Lemma 2.9. So, we conclude that $\varphi(r)$ has only one zero at r_1.

We note that $\varphi(r) < 0$ on (r_1, ∞) and thus $(r\varphi')' > 0$ for all large r. Hence, there exists $\lim_{r\to\infty} r\varphi'(r, s) = \delta_s \le 0$. If $\delta_s < 0$, then $\lim_{r\to\infty} \varphi(r, s) = -\infty$ and the proof is complete by (2.16). On the contrary, let us suppose that $\delta_s = 0$. Note that if $c_1 = (2 - 2aN)/b$ as before, then $\Phi_{c_1}(r) > 0$ for all $r > 0$. If $w_{c_1}(r) > 0$ on $(0, r_1)$, then

$$0 < \int_0^{r_1} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_1}(r)\varphi(r)\mathrm{d}r = r_1\varphi'(r_1)w_{c_1}(r_1) \le 0,$$

a contradiction. Hence, $w_{c_1}(r)$ has a zero before r_1. In particular, $0 > w_{c_1}(r_1) = r_1u'(r_1) + c_1$. So, if we set $c_2 = -r_1u'(r_1) > c_1$, then $w_{c_2}(r_1) = 0$ and $\{r > 0 : \Phi_{c_2}(r) < 0\} \neq \emptyset$. We note that there exists $T_0 \in (0, 1)$ such that $G_{c_2}(t) < 0$ for $0 < t < T_0$ and $G_{c_2}(t) > 0$ for $T_0 < t < 1$. Refer to Fig. 1 by replacing c_0 and ξ by c_2 and T_0, respectively. There are two possibilities. First, if $\Phi_{c_2}(r_1) \le 0$, then $e^{u(r_1)} \le T_0$. Since $u(r)$ is decreasing for $r > r_1$, it holds that $\Phi_{c_2}(r) < 0$ on (r_1, ∞). Hence,

$$0 < \int_{r_1}^{\infty} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = \lim_{r\to\infty} (r\varphi' w_{c_2} - r w'_{c_2}\varphi) = 0,$$

a contradiction. Here, we used the fact that $\delta_s = 0$.

Next, we assume that $\Phi_{c_2}(r_1) > 0$. Then, there exist two numbers η_1, η_2 such that

$$\begin{cases} \eta_1 < z < r_1 < \eta_2, \\ \Phi_{c_2}(\eta_1) = \Phi_{c_2}(\eta_2) = 0, \quad \text{i.e.,} \quad e^{u(\eta_1)} = e^{u(\eta_2)} = T_0, \\ \Phi_{c_2}(r) < 0 \quad \text{on} \quad (0, \eta_1) \cup (\eta_2, \infty), \\ \Phi_{c_2}(r) > 0 \quad \text{on} \quad (\eta_1, \eta_2). \end{cases}$$

We note that

$$\left[r(w_0'\varphi-\varphi' w_0)\right]'=-(2-2aN)r^{1-2aN}f(u)\varphi(r)\begin{cases}<0 & \text{for} \quad 0<r<r_1,\\ >0 & \text{for} \quad r>r_1,\end{cases}$$

and by $\delta_s=0$

$$\lim_{r\to 0} r(w_0'\varphi-\varphi' w_0)=\lim_{r\to\infty} r(w_0'\varphi-\varphi' w_0)=0.$$

So, it follows that for all $r\in(0,r_1)\cup(r_1,\infty)$

$$\left(\frac{w_0}{\varphi}\right)'(r)=\frac{r(w_0'\varphi-\varphi' w_0)}{r\varphi^2}<0.$$

Set

$$\lambda_1:=\frac{w_0}{\varphi}(\eta_1)>0,\quad \lambda_2:=\frac{w_0}{\varphi}(\eta_2)>0.$$

Then, we have

$$(\lambda_1\varphi-w_0)(r)\begin{cases}<0 & \text{for} \quad 0<r<\eta_1,\\ >0 & \text{for} \quad \eta_1<r<z,\end{cases} \tag{2.23}$$

and

$$(\lambda_2\varphi-w_0)(r)\begin{cases}>0 & \text{for} \quad r_1<r<\eta_2,\\ <0 & \text{for} \quad r>\eta_2.\end{cases} \tag{2.24}$$

We note that for $z<r<r_1$,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r\varphi' w_{c_2}-rw_{c_2}'\varphi\right]=r^{1-2aN}e^{b(u-e^u)}\Phi_{c_2}(r)\varphi(r)>0.$$

Hence,

$$\begin{aligned}\int_0^z r^{1-2aN}e^{b(u-e^u)}\Phi_{c_2}(r)\varphi(r)\mathrm{d}r&=\left[r\varphi' w_{c_2}-rw_{c_2}'\varphi\right](z)\\ &<\left[r\varphi' w_{c_2}-rw_{c_2}'\varphi\right](r_1)\;=\;0.\end{aligned} \tag{2.25}$$

Since $\Phi_{c_2}(r)(r^{2-2aN}-\eta_1^{2-2aN})>0$ on $(0,z)$, (2.23) and (2.25) imply that

$$\begin{aligned}0&>\lambda_1\int_0^z e^{b(u-e^u)}\Phi_{c_2}(r)\varphi(r)r^{1-2aN}\mathrm{d}r\\ &>\int_0^z e^{b(u-e^u)}\Phi_{c_2}(r)u'(r)r^{2-2aN}\mathrm{d}r\\ &>\eta_1^{2-2aN}\int_0^z e^{b(u-e^u)}\Phi_{c_2}(r)u'(r)\mathrm{d}r\\ &=\eta_1^{2-2aN}\int_{-\infty}^{u(z)} e^{b(t-e^t)}G_{c_2}(e^t)\mathrm{d}t=\eta_1^{2-2aN}H_{c_2}\big(u(z)\big).\end{aligned}$$

Here,

$$H_c(u)=\int_{-\infty}^{u} e^{b(t-e^t)}G_c(e^t)\mathrm{d}t.$$

Similarly, since $\Phi_{c_2}(r)(r^{2-2aN} - \eta_2^{2-2aN}) < 0$ on (r_1, ∞) and $\delta_s = 0$, (2.24) yields that

$$\begin{aligned} 0 &= \lambda_2 \int_{r_1}^{\infty} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r) r^{1-2aN} \mathrm{d}r \\ &> \int_{r_1}^{\infty} e^{b(u-e^u)} \Phi_{c_2}(r) u'(r) r^{2-2aN} \mathrm{d}r \\ &> \eta_2^{2-2aN} \int_{r_1}^{\infty} e^{b(u-e^u)} \Phi_{c_2}(r) u'(r) \mathrm{d}r \\ &= \eta_2^{2-2aN} \int_{u(r_1)}^{-\infty} e^{b(t-e^t)} G_{c_2}(e^t) \mathrm{d}t = -\eta_2^{2-2aN} H_{c_2}\big(u(r_1)\big). \end{aligned}$$

In the sequel, since

$$\frac{\mathrm{d}}{\mathrm{d}r} H_{c_2}\big(u(r)\big) = e^{b(u-e^u)} \Phi_{c_2}(r) u'(r) < 0 \quad \text{for} \quad r \in (z, r_1),$$

we arrive at a contradiction:

$$0 > H_{c_2}\big(u(z)\big) > H_{c_2}\big(u(r_1)\big) > 0.$$

This finishes the proof. □

Lemma 2.12. *Let* $0 < aN < 1$.

(i) *If* $s \in S^-_{a,N,b}$, *then* $u(r,s) = (2N - \beta)\ln r + I + O(r^{2+2(b-a)N-b\beta})$ *for some constant* $I = I(s)$ *as* $r \to \infty$.

(ii) *If* $s \in S^+_{a,N,b}$, *then* $u(r,s) = (2N - \beta)\ln r + J + O(r^{2-2aN-m})$ *for some constant* $J = J(s)$ *as* $r \to \infty$, *where* $m > 2 - 2aN$ *is any number.*

(iii) *If* $s \in S^0_{a,N,b}$, *then* $u(r,s) = O(r^{-\alpha})$ *for any* $\alpha > 0$.

Proof. (i) Let $s \in S^-$ and set $u(r,s) = (2N - \beta)\ln r + v(r,s)$. Then, v satisfies

$$(rv')' = \frac{1}{\varepsilon^2} r^{1-2aN+2bN-b\beta} e^{b(v - r^{2N-\beta} e^v)} (r^{2N-\beta} e^v - 1). \tag{2.26}$$

By (2.2), $v(r,s) = o(\ln r)$ as $r \to \infty$. Given any small $0 < \delta < \beta - \bar{\beta}/2$, let us fix a large number $R_0 > 0$ such that for all $r \geq R_0$,

$$|v(r,s)| \leq \delta \ln r. \tag{2.27}$$

Hence, $f\big(u(r,s), b\big) \leq c_0 r^{2bN-b\beta+b\delta}$ for all $r > R_0$. Integrating (2.26) twice, we obtain that for $r \geq R_0$,

$$v(r,s) - v(R_0,s) = \int_{R_0}^{r} \frac{1}{t} \int_t^{\infty} \tau^{1-2aN} f\big(u(\tau,s),b\big) \mathrm{d}\tau \mathrm{d}t.$$

Since $2 - 2aN + 2bN - b\beta + b\delta = b(\bar{\beta}/2 - \beta + \delta) < 0$,

$$\int_{R_0}^{\infty} \frac{1}{t} \int_t^{\infty} \tau^{1-2aN} f\big(u(\tau,s),b\big) \mathrm{d}\tau \mathrm{d}t \leq \frac{c_0 R_0^{2-2aN+2bN-b\beta+b\delta}}{(2-2aN+2bN-b\beta+b\delta)^2} < \infty.$$

Consequently, $v(r,s) = O(1)$ as $r \to \infty$.
Now, replacing (2.27) by $v(r,s) = O(1)$ and employing the same argument again, we conclude that as $r \to \infty$,

$$u(r,s) = (2N - \beta)\ln r + I(s) + O(r^{2-2aN+2bN-b\beta}).$$

Indeed, since $v(r,s) = O(1)$ as $r \to \infty$, there exist $R_1 > 0$ and $c_1 > 0$ such that $f\big(u(r,s),b\big) \le c_1 r^{b(2N-\beta)}$ for all $r > R_1$. Integrating (2.1) twice, we obtain that for $r > R_1$

$$\begin{aligned} u(r,s) &= (2N-\beta)\ln r + I(s) - \int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t \\ &= (2N-\beta)\ln r + I(s) + O(r^{2+2(b-a)N-b\beta}), \end{aligned}$$

where

$$I(s) = u(R_1,s) - (2N-\beta)\ln R_1 + \int_{R_1}^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t$$

and

$$\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t \le \frac{c_1 r^{2+2(b-a)N-b\beta}}{(2+2(b-a)N-b\beta)^2}.$$

Since $2 + 2(b-a)N - b\beta < -b\bar{\beta}/2 < 0$, the last integral is well defined. This establishes the assertion (i).

(ii) Let $s \in S^+$ and $m \in (2 - 2aN, \infty)$ be given. Since $2N - \beta > 0$, by (2.2) there exists a positive number b_0 such that $f\big(u(r,s),b\big) = O(e^{-b_0 r^{2N-\beta}})$ for all large r. Hence, there exist constants $c_2 = c_2(s) > 0$ and $R_2 = R_2(s) > 0$ such that $f\big(u(r,s),b\big) \le c_2 r^{-m}$ for all $r > R_2$. We note that c_2 and R_2 are independent of m. Integrating (2.1) twice, we obtain that for $r > R_2$

$$u(r,s) - u(R_2,s) = (2N-\beta)\ln\frac{r}{R_2} + \int_{R_2}^r \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t.$$

Since $m > 2 - 2aN$, we see that for any $r \ge R_2$

$$\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t \le \frac{c_2 r^{2-2aN-m}}{(2-2aN-m)^2} < \infty.$$

Hence, for all $r \ge R_2$

$$\begin{aligned} u(r,s) &= (2N-\beta)\ln r + J(s) - \int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t \\ &= (2N-\beta)\ln r + J(s) + O(r^{2-2aN-m}), \end{aligned}$$

where

$$J(s) = u(R_2,s) - (2N-\beta)\ln R_2 + \int_{R_2}^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),b\big)\mathrm{d}\tau \mathrm{d}t.$$

(iii) Suppose that $s \in S^0$. Choose $R_1 > 1$ such that $f\big(u(r,s),b\big) \ge -c_0 u$ for some $c_0 > 0$ and for all $r \ge R_1$. Thus, $\Delta u \le c_0 r^{-2aN} u$ for all $r \ge R_1$. Given $\alpha > 0$, choose R_2 such that $\alpha^2 r^{-2+2aN} \le c_0$ for all $r \ge R_2$. In particular, $\alpha^2 R_2^{-2+2aN} \le c_0$. Set $R = \max\{R_1, R_2\}$ and define $h(r) =$

$c_1 r^{-\alpha}$ where c_1 is a constant chosen such that $c_1 \geq -R^\alpha u(R,s) > 0$. Then, for any $r \geq R$

$$\Delta h = \alpha^2 r^{-2aN} r^{-2+2aN} h(r) \leq c_0 r^{-2aN} h(r)$$

such that

$$\Delta(u+h) \leq c_0 r^{-2aN}(u+h).$$

Then, it comes from the maximum principle that $u + h \geq 0$ and the statement (iii) follows as desired.

□

Proof of Proposition 2.1. First, let $N > 0$. By Lemma 2.7, there exists s_* such that $u(r, s_*)$ is a unique topological solution, $u(r,s)$ is a nontopological solution of type I for $s < s_*$, and $u(r,s)$ is a nontopological solution of type II for $s > s_*$. Since $\lim_{r\to\infty} ru'(r,s_*) = 0$, $\beta(s_*) = 2N$. By Lemmas 2.3 and 2.9, the function $\beta(s)$ is continuous and onto. By Lemmas 2.10 and 2.11, $\beta'(s) > 0$ and so $\beta(s)$ is strictly increasing. The asymptotic behaviors are consequences of Lemma 2.12.

Next, let us suppose that $N = 0$. Since $u(r) = s + o(1)$ near $r = 0$, we easily obtain that $S^+ = (0,\infty)$, $S^0 = \{0\}$, and $S^- = (-\infty, 0)$. It is obvious that if $s \in S^+$, then $u'(r,s) > 0$ and $\lim_{r\to\infty} u(r,s) = \infty$. If $s \in S^-$, then $u'(r,s) < 0$ and $\lim_{r\to\infty} u(r,s) = -\infty$. It follows from the Pohozaev-type identities

$$\beta^2 = \frac{4}{b\varepsilon^2}\int_0^\infty re^{b(u-e^u)}\mathrm{d}r$$

$$\beta\Big(\beta - \frac{4}{b}\Big) = \frac{4}{b\varepsilon^2}\int_0^\infty re^{b(u-e^u)}e^u\mathrm{d}r$$

that $\beta(s) > 4/b$ for $s \in S^-$ and $\beta(s) < 0$ for $s \in S^+$. Moreover, proceeding as in the proof of Lemma 2.9, one can show that $\beta : (-\infty, 0) \to (4/b, \infty)$ is onto such that $\lim_{s\to-\infty} \beta(s) = 4/b$ and $\lim_{s\nearrow 0} \beta(s) = \infty$. Also, $\beta : (0,\infty) \to (-\infty, 0)$ is onto such that $\lim_{s\to\infty} \beta(s) = 0$ and $\lim_{s\searrow 0} \beta(s) = -\infty$.

Now, we shall prove that $\beta'(s) \neq 0$ for $s \in S^\pm$. As before, we write $u(r) = u(r,s)$, $\varphi(r) = \varphi(r,s)$, $f(u) = f(u,b)$ and so on. First, consider the case $s < 0$. By using the same argument of the proof of Lemma 2.11, we can show that $\varphi(r)$ has exactly one zero at r_1. In addition, if $c_1 = 2/b$, then $w_{c_1}(r)$ has a zero before r_1 and $w_{c_2}(r_1) = 0$ where $c_2 := -r_1 u'(r_1) > c_1$. If $\Phi_{c_2}(r_1) > 0$, then $\Phi_{c_2}(r) > 0$ on $(0, r_1)$ such that

$$0 < \int_0^{r_1} re^{b(u-e^u)}\Phi_{c_2}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. Thus $\Phi_{c_2}(r_1) \leq 0$ such that $\Phi_{c_2}(r) < 0$ on (r_1, ∞). Since $\varphi(r) < 0$ on (r_1,∞) such that $(r\varphi')'(r) > 0$ for large r, there exists

$$\lim_{r\to\infty} r\varphi'(r,s) = \delta_s \leq 0.$$

If $\delta_s = 0$, then

$$0 < \int_{r_1}^{\infty} re^{b(u-e^u)}\Phi_{c_2}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. Therefore, $\delta_s < 0$ and thus $\beta(s)$ is strictly increasing by (2.16).

Similarly, for $s > 0$, $\varphi(r)$ must have zeros. In fact, if $\varphi(r) > 0$ for all r, then $r\varphi'(r) > 0$ for all large $r > 0$ such that

$$0 > \int_0^{\infty} 2rf(u)\varphi(r)\mathrm{d}r = \lim_{r\to\infty} r\varphi'(r)w_0(r) \geq 0.$$

Suppose that r_1 and r_2 are the first and the second zeros of $\varphi(r)$. Let $R \in (r_1, r_2)$ be the unique minimum point of $(\varphi/w_0)(r)$. From

$$0 = \left(\frac{\varphi}{w_0}\right)'(R) = \frac{(\varphi' w_0 - w_0'\varphi)(R)}{w_0^2(R)},$$

it follows that $\varphi'(R) < 0$. Moreover, by the formula

$$\left(\frac{\varphi}{w_0}\right)'(r) = \frac{1}{rw_0^2}\int_0^r 2tf(u)\varphi(t)\mathrm{d}t, \tag{2.28}$$

we obtain

$$\int_0^R 2rf(u)\varphi(r)\mathrm{d}r = 0.$$

Since $f(u) < 0$, we deduce from (2.21) that $\left[f'(u) - \mu_1 f(u)\right]\varphi(r) < 0$ for $r \in (0, R)$ where $\mu_1 = f'\big(u(r_1)\big)/f\big(u(r_1)\big)$. Gathering these information, we get a contradiction:

$$0 > \int_0^R r[f'(u) - c_1 f(u)]\varphi(r)\mathrm{d}r = \int_0^R rf'(u)\varphi(r)\mathrm{d}r = -R\varphi'(R) > 0.$$

As a consequence, $\varphi(r)$ has exactly one zero at r_1 and there exists $\lim_{r\to\infty} r\varphi'(r,s) = \delta_s \leq 0$.

We note that $w_0(r)$ is a positive increasing function. Hence, we can choose $c_1 < 0$ such that $w_{c_1}(r_1) = 0$. Since $u(r)$ is strictly increasing and $e^{u(r)} > 1$ for all $r > 0$, if $\Phi_{c_1}(r_1) < 0$, then $\Phi_{c_1}(r) < 0$ on $(0, r_1)$. This yields a contradiction:

$$0 > \int_0^{r_1} re^{b(u-e^u)}\Phi_{c_1}(r)\varphi(r)\mathrm{d}r = 0.$$

Thus, $\Phi_{c_1}(r_1) \geq 0$. Again, thanks to the monotonicity of u, it holds that $\Phi_{c_1}(r) > 0$ on (r_1, ∞). If $\delta_s = 0$, then

$$0 > \int_{r_1}^{\infty} re^{b(u-e^u)}\Phi_{c_1}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. Consequently, $\delta_s < 0$ such that $\beta(s)$ is strictly increasing by (2.16). This finishes the proof of Proposition 2.1. $\square$

3. Radial Solutions for $aN \geq 1$

This section deals with the case $aN \geq 1$. For the set of shooting parameter s, we use the same notation $S^{0,\pm}_{a,N,b}$ as in the previous section. We divide the proof into three cases: $1 < aN < 2$, $aN = 1$ and $aN \geq 2$. For the proof for the case $1 < aN < 2$, we need to study two auxiliary problems (see Step 2 and Step 4 of the proof of Proposition 3.1). In the next section, we set up these problems and establish basic results for these problems which are used in the proof of Proposition 3.1. The first main result of this section is the following:

Proposition 3.1. *Let a and N be positive real numbers such that $1 < aN < 2$.*

(i) *There exists a unique $s_* = s_*(a, N)$ such that $\beta(s_*) = 2N$, i.e., (1.29) has a unique topological solution. If $s < s_*$, $u(r, s)$ is a nontopological solution of type I. If $s > s_*$, $u(r, s)$ is a nontopological solution of type II. Furthermore,*

$$\lim_{r\to\infty} u(r, s_*) = \sigma_{a,N} < 0.$$

(ii) *The function $\beta(s)$ is continuous and onto such that $\beta : (-\infty, s_*) \to (2N, 4/a)$ and $\beta : (s_*, \infty) \to (0, 2N)$. Moreover, $\beta(s)$ is strictly decreasing on $\mathbb{R}\backslash\{s_*\}$.*

(iii) *We have*

$$\lim_{s\to s_*} \beta(s) = 2N, \quad \lim_{s\to-\infty} \beta(s) = \frac{4}{a}, \quad \lim_{s\to\infty} \beta(s) = 0. \tag{3.1}$$

Proof. Let $1 < aN < 2$ and $u(r, s)$ be a solution of (1.29). We will show that there exists $s_* = s_*(a, N) \in \mathbb{R}$ such that $S^+_{a,N,a} = (s_*, \infty)$, $S^-_{a,N,a} = (-\infty, s_*)$, and $S^0_{a,N,a} = \{s_*\}$. By Remark 2.8, $s \in S^+_{a,N,a}$ for all large s. Then, proceeding as in the proof of Lemma 2.7, we can see that $S^+_{a,N,a} = (s_*, \infty)$ for some $s_* \in [-\infty, \infty)$. We divide the proof into several steps.

Step 1 There exists a sequence $s_n \to -\infty$ such that $s_n \in S^-_{a,N,a}$ and $m(s_n) \to -\infty$ as $s_n \to -\infty$, where $m(s) = \sup\{u(r, s)|r > 0\}$.

Given $a, N > 0$ with $1 < aN < 2$, we put $N_0 = (2 - aN)/a$ such that $0 < aN_0 < 1$. Let us take any $\hat{N} > 0$ with $0 < \hat{N} < N_0$ and set $\hat{a} = (2-aN)/\hat{N}$. We note that

$$\frac{\hat{a}}{a} = \frac{2 - aN}{a\hat{N}} > \frac{2 - aN}{aN_0} = 1.$$

Consider the following initial value problem:

$$\begin{cases} \hat{u}'' + \dfrac{1}{r}\hat{u}' = -r^{-2\hat{a}\hat{N}} f(\hat{u}, a), & r > 0, \\ \hat{u}(r, \hat{s}) = 2\hat{N} \ln r + \hat{s} + o(1) & \text{as} \quad r \to 0. \end{cases} \tag{3.2}$$

Since $0 < \hat{a}\hat{N} < 1$, there exists $\hat{s}_*$ such that $S^-_{\hat{a},\hat{N},a} = (-\infty, \hat{s}_*)$ by Proposition 2.1. Moreover, it follows from Lemmas 2.4 and 2.12 that for each $\hat{s} < \hat{s}_*$,

$$\hat{u}(r, \hat{s}) = \left[2\hat{N} - \hat{\beta}(\hat{s})\right] \ln r + \hat{I} + o(1) \quad \text{as} \quad r \to \infty,$$

where

$$\hat{\beta}(\hat{s}) = \int_0^\infty r^{1-2\hat{a}\hat{N}} f\big(\hat{u}(r,\hat{s}),a\big)\mathrm{d}r$$
$$> \bar{\beta}_{\hat{a},\hat{N},a} = \frac{4}{a}(1 + a\hat{N} - \hat{a}\hat{N}) = \frac{4}{a}(a\hat{N} + aN - 1).$$

It follows from the choice of $\hat{N}$ that

$$2N + 2\hat{N} - \bar{\beta}_{\hat{a},\hat{N},a} = \frac{2}{a}(2 - a\hat{N} - aN) > \frac{2}{a}(2 - aN_0 - aN) = 0. \quad (3.3)$$

Thus, by Proposition 2.1 and Lemma 2.12, there exists $\hat{s}_0 = \hat{s}_0(\hat{N}) \in S^-_{\hat{a},\hat{N},a}$ such that $\hat{\beta}(\hat{s}_0) = 2N + 2\hat{N}$ and

$$\hat{u}(r,\hat{s}_0) = -2N\ln r + \hat{I}_0 + o(1) \quad \text{as} \quad r \to \infty,$$

for some $\hat{I}_0 = \hat{I}_0(\hat{N})$. Now, we set $u(r,\hat{I}_0) = \hat{u}(r^{-1},\hat{s}_0)$. Then, we obtain

$$\begin{cases} u'' + \frac{1}{r}u' = -\,r^{-2aN} f(u,a), & r > 0, \\ u(r,\hat{I}_0) = 2N\ln r + \hat{I}_0 + o(1) \quad \text{as} & r \to 0, \\ u(r,\hat{I}_0) = -\,2\hat{N}\ln r + \hat{s}_0 + o(1) \quad \text{as} & r \to \infty, \end{cases}$$

which implies that $\hat{I}_0 \in S^-_{a,N,a}$. Thus, $S^-_{a,N,a}$ is a nonempty open set.

We claim that $\hat{m} = \hat{m}(\hat{N}) = \max_{r>0}\hat{u}(r,\hat{s}_0) = \max_{r>0} u(r,\hat{I}_0) \to -\infty$ as $\hat{N} \nearrow N_0$. To see this, let $\hat{z} = \hat{z}(\hat{N})$ be the unique maximum point of $\hat{u}$. Suppose that $\liminf_{\hat{N}\to N_0}\hat{m} \geq -\xi$ for some $\xi > 0$. We note that $\hat{z}$ is bounded below as $\hat{N} \nearrow N_0$. Indeed, if $\hat{z} \to 0$, then we have a contradiction: as $\hat{N} \nearrow N_0$,

$$2\hat{N} = \int_0^{\hat{z}} r^{1-2\hat{a}\hat{N}} f(\hat{u},a)\mathrm{d}r \leq \frac{\|f\|_\infty \hat{z}^{2-2\hat{a}\hat{N}}}{2 - 2\hat{a}\hat{N}} \quad \to \quad 0.$$

Let r_1 be the unique point such that $r_1 < \hat{z}$ and $\hat{u}(r_1) = \hat{m} - 1$. Since $r\hat{u}' \leq 2\hat{N} < 2N_0$, $1 < 2N_0\ln(\hat{z}/r_1)$. We note from (3.3) that $0 < \hat{\beta}(\hat{s}_0) - \bar{\beta}_{\hat{a},\hat{N},a} \to 0$ as $\hat{N} \nearrow N_0$. Hence, by the Pohozaev identity (2.9), as $\hat{N} \to N_0$,

$$\begin{aligned} o(1) = \hat{\beta}(\hat{s}_0)\big(\hat{\beta}(\hat{s}_0) - \bar{\beta}_{\hat{a},\hat{N},a}\big) &= \frac{4 - 4\hat{a}\hat{N}}{a\varepsilon^2}\int_0^\infty r^{1-2\hat{a}\hat{N}} e^{a(\hat{u}-e^{\hat{u}})}e^{\hat{u}}\mathrm{d}r \\ &\geq \frac{4 - 4\hat{a}\hat{N}}{a\varepsilon^2}\int_{r_1}^{\hat{z}} r^{1-2\hat{a}\hat{N}} e^{a(\hat{u}-e^{\hat{u}})}e^{\hat{u}}\mathrm{d}r \\ &\geq \frac{2}{a\varepsilon^2}\hat{z}^{2aN-2}\big(1 - e^{(2-2aN)/2N_0}\big)\cdot\Big(\inf_{-\xi-1\leq\hat{u}\leq 0}\big[e^{a(\hat{u}-e^{\hat{u}})}e^{\hat{u}}\big]\Big) \geq C > 0, \end{aligned}$$

which yields a contradiction.

Now, choose a sequence $\hat{N}_n \nearrow N_0$ and let $s_n = \hat{I}_0(\hat{N}_n) \in S^-_{a,N,a}$ such that $m(s_n) = \sup_{r>0} u(r,s_n) \to -\infty$. If $s_n \to s' \in \mathbb{R}$, then either $s' \in S^-_{a,N,a}$ or $s' \in S^0_{a,N,a}$. In each case, $\lim_{s_n\to s'} m(s_n) = m(s') > -\infty$, a contradiction. Hence, either $s_n \to \infty$ or $s_n \to -\infty$. Since $s \in S^+_{a,N,a}$ for all large s, we conclude that $s_n \to -\infty$.

Step 2 $S^-_{a,N,a} = (-\infty, s_*)$ *and* $S^0_{a,N,a} = \{s_*\}$. *Furthermore,* $\lim_{r\to\infty} u(r, s_*) = \sigma_{a,N} < 0$.

Since $S^+_{a,N,a}$ and $S^-_{a,N,a}$ are nonempty open sets, $S^0_{a,N,a}$ is a nonempty closed set. To see that $S^0_{a,N,a}$ has only one element, suppose that $s \in S^0_{a,N,a}$ and $\lim_{r\to\infty} u(r, s) = I(s) \le 0$. If $I(s) = 0$, then we can choose $R > 0$ such that $-1 < u(r, s) < 0$ for all $r > R$ and

$$\frac{MR^{2-2aN}}{(2-2aN)^2} < 1,$$

where $M = \sup\{e^{a(u-e^u)}e^u/\varepsilon^2 : -1 < u < 0\}$. Since $u(r)$ is increasing, by the mean value theorem, we obtain that for $r > R$

$$\begin{aligned}
u(r, s) &= \frac{1}{\varepsilon^2}\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} e^{a(u(\tau,s)-e^{u(\tau,s)})}(e^{u(\tau,s)} - 1)\mathrm{d}\tau\mathrm{d}t\\
&= \frac{1}{\varepsilon^2}\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} e^{a(u(\tau,s)-e^{u(\tau,s)})}e^{\zeta(\tau,s)}u(\tau, s)\mathrm{d}\tau\mathrm{d}t\\
&\ge Mu(r, s)\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN}\mathrm{d}\tau\mathrm{d}t\\
&= \frac{Mr^{2-2aN}}{(2-2aN)^2}u(r, s) \ \ge\ \frac{MR^{2-2aN}}{(2-2aN)^2}u(r, s) \ >\ u(r, s),
\end{aligned}$$

where $u(r, s) < \zeta(r, s) < 0$. Hence, $I(s) < 0$ for $s \in S^0_{a,N,a}$.

Let us take a positive number $\hat{a} > (2aN - 2)/N$ and define a positive number $\hat{N} = (2 - aN)/\hat{a}$. Then, $0 < \hat{a}\hat{N} < 1$. If we set $\hat{u}(r, I) = u(r^{-1}, s)$ with $I = I(s)$, then $\hat{u}$ satisfies that

$$\begin{cases}
\hat{u}'' + \dfrac{1}{r}\hat{u}' = \dfrac{1}{\varepsilon^2}r^{-2\hat{a}\hat{N}}e^{a(\hat{u}-e^{\hat{u}})}(e^{\hat{u}} - 1), \quad r > 0,\\
\hat{u}(0, I) = I(s), \quad \hat{u}'(0, I) = 0,\\
\lim\limits_{r\to\infty} r\hat{u}'(r, I) = -2N.
\end{cases}$$

Thus, $\hat{u}$ is a solution of (4.1) in the next section. By the choice of $\hat{a}$ and $\hat{N}$, it follows that $2N > \bar{\gamma}_{\hat{a},\hat{N},a}$, where $\bar{\gamma}_{\hat{a},\hat{N},a}$ is defined in Proposition 4.2. Precisely,

$$\bar{\gamma}_{\hat{a},\hat{N},a} = \frac{4 - 4\hat{a}\hat{N}}{a} = \frac{4aN - 4}{a}.$$

Since the function $\gamma_{\hat{a},\hat{N},a}$ is a bijective map by Proposition 4.2, $I(s) = \gamma^{-1}_{\hat{a},\hat{N},a}(2N)$ for each $s \in S^0_{a,N,a}$. Thus, $S^0_{a,N,a}$ consists of exactly one element. Since $S^+_{a,N,a} = (s_*, \infty)$ for some $s_* \in [-\infty, \infty)$, we conclude that $S^0_{a,N,a} = \{s_*\}$ and $s_* \in (-\infty, \infty)$. In particular, $S^-_{a,N,a} = (-\infty, s_*)$.

Step 3 For $s \in S^-_{a,N,a}$, $2N < \beta(s) < 4/a$. *Moreover,* $\beta(s)$ *is continuous and one-to-one on* $S^-_{a,N,a}$.

The continuity of β follows from the Lebesgue convergence theorem. See Lemma 2.3. By the Pohozaev identities (2.8) and (2.9) with $a = b$, we see that $0 < \beta < \min\{4N, 4/a\} = 4/a$. Since u enjoys the asymptotic behavior

(1.31) and u is a nontopological solution of type I, it follows that $2N-\beta<0$. Consequently, $2N<\beta<4/a$.

Next, we show that β is one-to-one on $S^-_{a,N,a}$. Let $s_1,s_2\in S^-_{a,N,a}$ such that $\beta(s_1)=\beta(s_2)=\beta_0$. If $u_k(r)=u(r,s_k)$, then u_k satisfies

$$\begin{cases} u_k''+\frac{1}{r}u_k'=-r^{-2aN}f(u_k,a), & r>0,\\ u_k(r)=2N\ln r+s_k+o(1) \quad \text{near} & r=0,\\ u_k(r)=(2N-\beta_0)\ln r+I_k+o(1) \quad \text{near} & r\to\infty, \end{cases} \tag{3.4}$$

where the behavior at infinity comes from Proposition 3.2 below. Let $\hat{N}=(\beta_0-2N)/2$ and $\hat{a}=(2-aN)/\hat{N}$. Then, $\hat{u}_k(r)=u_k(r^{-1})$ satisfies

$$\begin{cases} \hat{u}_k''+\frac{1}{r}\hat{u}_k'=-r^{-2\hat{a}\hat{N}}f(\hat{u}_k,a), & r>0,\\ \hat{u}_k(r)=2\hat{N}\ln r+I_k+o(1) \quad \text{as} & r\to0,\\ \hat{u}_k(r)=-2N\ln r+s_k+o(1) \quad \text{as} & r\to\infty. \end{cases}$$

Moreover, if we put

$$\hat{\beta}_k=\hat{\beta}(I_k)=\int_0^\infty r^{1-2\hat{a}\hat{N}}f\big(\hat{u}_k(r),a\big)\mathrm{d}r,$$

then $\hat{\beta}_1=\hat{\beta}_2=\beta_0$. We note that $0<\hat{a}\hat{N}<1$ and $I_k\in S^-_{\hat{a},\hat{N},a}$. Since $\hat{\beta}$ is one-to-one on $S^-_{\hat{a},\hat{N},a}$ by Proposition 2.1, we conclude that $I_1=I_2$. So, $\hat{u}_1=\hat{u}_2$ and hence $s_1=s_2$.

Step 4 For $s\in S^+_{a,N,a}$, $0<\beta(s)<2N$. Furthermore, $\beta(s)$ is continuous and one-to-one on $S^+_{a,N,a}$.

The continuity and the range of β follow from a similar argument as in Step 3. Suppose that $s_1,s_2\in S^+_{a,N,a}$ and $\beta(s_1)=\beta(s_2)=\beta_0$. Then, $u_k(r)=u(r,s_k)$ satisfies (3.4) and $\hat{u}_k(r)=u_k(r^{-1})$ satisfies

$$\begin{cases} \hat{u}_k''+\frac{1}{r}\hat{u}_k'=-r^{-2\hat{a}\hat{N}}f(\hat{u}_k,a), & r>0,\\ \hat{u}_k(r)=-2\hat{N}\ln r+I_k+o(1) \quad \text{as} & r\to0,\\ \hat{u}_k(r)=-2N\ln r+s_k+o(1) \quad \text{as} & r\to\infty, \end{cases}$$

where $\hat{N}=(2N-\beta_0)/2$, $\hat{a}=(2-aN)/\hat{N}$ and $0<\hat{a}\hat{N}<1$. Thus, $\hat{u}$ is a solution of (4.7) in the next section such that $I_k\in A^-_{\hat{a},\hat{N},a}$, where the set $A^-_{\hat{a},\hat{N},a}$ is defined by (4.9). We note that $\lambda(I_1)=\lambda(I_2)=\beta_0$ where λ is defined by (4.8). Moreover, it comes from Proposition 4.3 that $\lambda:(-\infty,s_*)\to(\max\{\bar{\lambda},0\},\infty)$ is continuous, onto, and strictly increasing, where

$$\bar{\lambda}=\bar{\lambda}_{\hat{a},\hat{N},a}=\frac{4-4(\hat{a}+a)\hat{N}}{a}=2\beta_0-\frac{4}{a}.$$

If $0<\beta_0\le2/a$, then $\bar{\lambda}\le0$. If $2/a<\beta_0<2N<4/a$, then $\bar{\lambda}>0$ and

$$\beta_0-\bar{\lambda}=\frac{4}{a}-\beta_0>0.$$

Thus, in both cases, we conclude that $\beta_0 > \max\{\bar\lambda, 0\}$. Consequently, $I_1 = I_2$ by the monotonicity of λ and hence $s_1 = s_2$.

Step 5 $\lim_{s\to\infty}\beta(s) = 0$, $\lim_{s\to-\infty}\beta(s) = 4/a$, *and* $\lim_{s\to s_*}\beta(s) = 2N$.

The first limit can be proved exactly in the same way for the case $0 < aN < 1$ which is done in the proof of Lemma 2.9. By Step 1, there exists $s_n \to -\infty$ such that $m(s_n) \to -\infty$. It follows from (2.8) and (2.9) that

$$\begin{aligned}0 \ge \beta(s_n)\Big(\beta(s_n) - \frac{4}{a}\Big) &= \frac{4-4aN}{a\varepsilon^2}\int_0^\infty r^{1-2aN}e^{a(u-e^u)}e^u\mathrm{d}r\\ &\ge \frac{e^{m(s_n)}(4-4aN)}{a\varepsilon^2}\int_0^\infty r^{1-2aN}e^{a(u-e^u)}\mathrm{d}r\\ &= o(1)\cdot\beta(s_n)(\beta(s_n)-4N).\end{aligned}$$

Hence, $\beta(s_n) \to 4/a$. The monotonicity of β implies that $\beta(s) \to 4/a$ as $s \to -\infty$. Meanwhile, applying Fatou's Lemma to (2.8), we are led to

$$\begin{aligned}\limsup_{s\to s_*}\beta(s)(\beta(s)-4N) &= \limsup_{s\to s_*}\frac{4-4aN}{a\varepsilon^2}\int_0^\infty r^{1-2aN}e^{a(u-e^u)}\mathrm{d}r\\ &\le \frac{4-4aN}{a\varepsilon^2}\int_0^\infty r^{1-2aN}e^{a(u(r,s_*)-e^{u(r,s_*)})}\mathrm{d}r = -4N^2,\end{aligned}$$

where the last equality comes from (2.10). Hence, $\lim_{s\to s_*}\beta(s) = 2N$. □

Proposition 3.2. *Suppose that* $1 < aN < 2$.

(i) *If* $s \in S^-_{a,N,a}$, *then* $u(r,s) = (2N-\beta)\ln r + I + O(r^{2-a\beta})$ *for some constant* $I = I(s)$.

(ii) *If* $s \in S^+_{a,N,a}$, *then* $u(r,s) = (2N-\beta)\ln r + J + O(r^{2-2aN-m})$ *for some constant* $J = J(s)$, *where* $m > 2-2aN$ *is any large number.*

(iii) *If* $s \in S^0_{a,N,a}$, *then* $u(r,s) = \sigma_{a,N} + O(r^{2-2aN})$ *where* $\sigma_{a,N}$ *is given by Proposition* 3.1.

Proof. (i) Let $s \in S^-$. By (2.2), letting $u(r,s) = (2N-\beta)\ln r + v(r,s)$ yields that $v(r,s) = o(\ln r)$ as $r \to \infty$. Given any small $0 < \delta < \beta - 2N$, let us fix a large number $R > 0$ such that for all $r \ge R$,

$$|v(r,s)| \le \delta\ln r. \tag{3.5}$$

Then

$$(rv')' = \frac{1}{\varepsilon^2}r^{1-a\beta}e^{a(v-r^{2N-\beta}e^v)}(r^{2N-\beta}e^v - 1)$$

and for all $r \ge R$, $f\big(u(r,s),a\big) \le c_0r^{2aN-a\beta+a\delta}$. Integrating above equation for v twice, we obtain that for $r \ge R$,

$$v(r,s) - v(R,s) = \int_R^r \frac{1}{t}\int_t^\infty \tau^{1-2aN}f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t.$$

Since $2 - a\beta + a\delta < 2aN - a\beta + a\delta < 0$, for any $r \ge R$

$$\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN}f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t \le c_0\frac{r^{2-a\beta+a\delta}}{(2-a\beta+a\delta)^2} < \infty.$$

Consequently, we obtain that for $r > R$,

$$v(r,s) = I(s) + O(r^{2-a\beta+a\delta}),$$

where

$$I(s) = v(R,s) + \int_R^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t.$$

Therefore, $v(r,s) = O(1)$ as $r \to \infty$.

Now, replacing (3.5) by $v(r) = O(1)$ and following the above argument again, we conclude that as $r \to \infty$,

$$u(r,s) = (2N-\beta)\ln r + I(s) + O(r^{2-a\beta}).$$

Indeed, for a fixed a large number $R > 0$ such that $f\big(u(r,s),a\big) = O(e^{au(r,s)})$ for all $r > R$, there exists a constant $c_0 > 0$ such that $f\big(u(r,s),a\big) \leq c_0 r^{a(2N-\beta)}$ for all $r > R$. Integrating (1.29) twice, we obtain that for $r > R$

$$u(r,s) - u(R,s) = (2N-\beta)\ln\frac{r}{R} + \int_R^r \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t.$$

Since $a\beta > 2aN > 2$, we see that for any $r \geq R$

$$\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t \leq \frac{c_0 r^{2-a\beta}}{(a\beta-2)^2} < \infty.$$

Hence, for all $r \geq R$

$$\begin{aligned} u(r,s) &= (2N-\beta)\ln r + I(s) - \int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t \\ &= (2N-\beta)\ln r + I(s) + O(r^{2-a\beta}), \end{aligned}$$

where

$$I(s) = u(R,s) - (2N-\beta)\ln R + \int_R^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t.$$

This establishes the assertion (i).

(ii) If $s \in S^+$, following the same way as in the proof of Lemma 2.12 , we obtain the desired decay rate for any $m > 2 - 2aN$.

(iii) If $s \in S^0$, then there exists a constant $c_1 > 0$ such that $f\big(u(r,s),a\big) \leq c_1$ for all $r > 0$. Choosing any $R > 0$ and integrating (1.29) twice, we obtain that for $r > R$

$$u(r,s) = u(R,s) + \int_R^r \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t.$$

We note that for any $r \geq R$

$$\int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t \leq \frac{c_1 r^{2-2aN}}{(2-2aN)^2} < \infty.$$

Thus, for any $r \geq R$

$$u(r,s) = \sigma_{a,N} - \int_r^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t = \sigma_{a,N} + O(r^{2-2aN}).$$

Here, by Proposition 3.1

$$\sigma_{a,N} = u(R,s) + \int_R^\infty \frac{1}{t}\int_t^\infty \tau^{1-2aN} f\big(u(\tau,s),a\big)\mathrm{d}\tau\mathrm{d}t.$$

This completes the proof.

□

Proposition 3.3. *If $aN = 1$, (1.29) allows only nontopological solutions of type II such that $\beta(s) = 0$ for all $s \in \mathbb{R}$. Moreover, $u(r,s) = (2N-\beta)\ln r + J + O(r^{2-2aN-m})$ for some constant $J = J(s)$, where $m > 2-2aN$ is any number.*

Proof. By multiplying (1.29) by ru' and integrating it, we have

$$(ru')^2 = 4N^2 - \frac{2}{a}F(u) = 4N^2 - \frac{2N}{\varepsilon^2}e^{(u-e^u)/N}.$$

If there exists $s \in S^0$, then as $r \to \infty$,

$$0 = 2N\Big(2N - \frac{1}{\varepsilon^2}e^{-1/N}\Big),$$

which implies that ε should satisfy $\varepsilon^2 = \varepsilon_0^2 := (2Ne^{1/N})^{-1}$. In the sequel, for $\varepsilon \neq \varepsilon_0$, S^0 is an empty set which in turn implies $S^- = \emptyset$.

Now, we consider the case $\varepsilon = \varepsilon_0$ and suppose that (1.29) possesses a topological solutionu_0 for $\varepsilon = \varepsilon_0$. From the finiteness condition of β, it follows that $\sigma = 0$, namely, $u_0(x) \to 0$ as $|x| \to \infty$. If we choose any $\varepsilon_1 \in (0, \varepsilon_0)$, then

$$-\Delta u_0 = \frac{1}{\varepsilon_0^2}r^{-2}e^{a(u_0-e^{u_0})}(1-e^{u_0}) < \frac{1}{\varepsilon_1^2}r^{-2}e^{a(u_0-e^{u_0})}(1-e^{u_0})$$

in the sense of distribution. This implies that u_0 is a subsolution of (1.29) for $\varepsilon = \varepsilon_1$. Meanwhile, it is easy to check that for any $\varepsilon > 0$, $u \equiv 0$ is a supersolution of (1.29). Then the super-/subsolution method leads us to obtain a topological solution of (1.29) with $aN = 1$ and $\varepsilon = \varepsilon_1$, which is a contradiction. In the sequel, we conclude that $S^0 = S^- = \emptyset$ for $\varepsilon = \varepsilon_0$.

We have seen that $S^+ = \mathbb{R}$ if $aN = 1$. The Pohozaev identity (2.8) yields that $\beta(\beta - 4N) = 0$ and so either $\beta = 0$ or $\beta = 4N$. Since $2N - \beta > 0$ by the integrability condition for $f(u,a)$, it holds that $\beta(s) = 0$ for all $s \in \mathbb{R}$. The proof of the asymptotic behavior of solutions is exactly the same as in the proof of Lemma 2.12. □

Proposition 3.4. *Let $aN \geq 2$. Then, (1.29) allows only nontopological solutions of type II. The function $\beta : (-\infty,\infty) \to (0, 4/a)$ is continuous. Moreover,*

$$\lim_{s\to\infty}\beta(s) = \lim_{s\to-\infty}\beta(s) = 0. \tag{3.6}$$

In particular, if $\beta_* = \sup_{s\in\mathbb{R}}\beta(s)$, then for each $\beta_0 \in (0,\beta_*)$ there exist at least two solutions $u(r,s)$ for which $\beta(s) = \beta_0$. Moreover, $u(r,s) = (2N-\beta)\ln r + J + O(r^{2-2aN-m})$ for some constant $J = J(s)$, where $m > 2-2aN$ is any number.

Proof. By Remark 2.8, $S^+ \neq \emptyset$. Proceeding as in the proof of Lemma 2.3, one can show that $\beta(s)$ is finite for all $s \in S^\pm$. For $s \in S^\pm$, we also obtain the Pohozaev identities as in Lemma 2.4:

$$\beta(\beta - 4N) = \frac{4 - 4aN}{a\varepsilon^2} \int_0^\infty r^{1-2aN} e^{a(u-e^u)} \mathrm{d}r, \tag{3.7}$$

$$\beta\Big(\beta - \frac{4}{a}\Big) = \frac{4 - 4aN}{a\varepsilon^2} \int_0^\infty r^{1-2aN} e^{a(u-e^u)} e^u \mathrm{d}r. \tag{3.8}$$

These identities imply that

$$0 < \beta(s) < \frac{4}{a} \leq 2N \quad \text{for} \quad s \in S^\pm. \tag{3.9}$$

However, if $s \in S^-$, it is necessary that $\beta(s) > 2N$. So, $S^- = \emptyset$. Meanwhile, if $s_* \in S^0$, then we can choose a sequence $s_n \in S^+$ such that $s_n \to s_*$. Then it comes from (3.7) and Fatou's Lemma that

$$\begin{aligned} \limsup_{s_n \to s_*} \Big[\beta(s_n)\big(\beta(s_n) - 4N\big)\Big] &\leq \frac{4 - 4aN}{a\varepsilon^2} \int_0^\infty r^{1-2aN} e^{a(u(r,s_*) - e^{u(r,s_*)})} \mathrm{d}r \\ &= -4N^2, \end{aligned}$$

where the last equality comes from (2.10) by letting $r \to \infty$. Thus, $\beta(s_n) \to 2N$ which is impossible for $aN > 2$ by (3.9). As a consequence, if $aN > 2$, then $S^0 = \emptyset$ and $S^+ = \mathbb{R}$.

Next, let $aN = 2$. If $s \in S^0$, then $u(r, s) \to I_s \leq 0$ as $r \to \infty$ and $\lim_{r\to\infty} u'(r, s) = 0$. If we set $\hat{u}(r, I_s) = u(r^{-1}, s)$, then $\hat{u}$ satisfies

$$\begin{cases} \hat{u}'' + \frac{1}{r}\hat{u}' = \frac{1}{\varepsilon^2} e^{a(\hat{u} - e^{\hat{u}})}(e^{\hat{u}} - 1), & r > 0, \\ \hat{u}(r, I) = I_s + o(1) \quad \text{as} & r \to 0, \\ \hat{u}(r, I) = -2N \ln r + O(1) \quad \text{as} & r \to \infty. \end{cases}$$

This equation is the same as (1.29) with $N = 0$. Thus, Proposition 2.1 implies that $2N > 4/a$, which is a contradiction. Therefore, $S^+ = \mathbb{R}$.

Now, we show (3.6). Let $t_s < 1$ be such that $u(t_s, s) = \frac{1}{2}u(1, s) = \frac{s}{2} + O(1)$ as $s \to \infty$. Then, $2N \ln t_s = -\frac{s}{2} + O(1)$ so that $t_s \to 0$ as $s \to \infty$. We obtain that

$$\begin{aligned} \int_0^\infty r^{1-2aN} e^{a(u-e^u)} \mathrm{d}r &= \left(\int_0^{t_s} + \int_{t_s}^\infty\right) r^{1-2aN} e^{a(u-e^u)} \mathrm{d}r \\ &\leq o(1) + Ce^{-s/2} \int_{t_s}^\infty r^{1-2aN} e^{a(u-e^u)} e^u \mathrm{d}r \\ &= o(1) + o(1) \cdot \beta\Big(\beta - \frac{4}{a}\Big) = o(1), \end{aligned}$$

as $s \to \infty$. Hence, by (3.7) $0 \geq \beta(\beta - 4N) = o(1)$ as $s \to \infty$. Consequently, we deduce from (3.9) that $\beta(s) \to 0$ as $s \to \infty$.

Let $s \to -\infty$ be a sequence and let r_s be such that $u(r_s, s) = s/2$. By (2.6), it is easy to see that $r_s \to \infty$. Then, by (3.7)

$$\beta\Big(\beta - 4N\Big) = \frac{4-4aN}{a\varepsilon^2}\left(\int_0^1 + \int_1^{r_s} + \int_{r_s}^{\infty}\right) r^{1-2aN} e^{a(u-e^u)}\mathrm{d}r$$
$$= \frac{4-4aN}{a\varepsilon^2}\cdot(\mathrm{I}+\mathrm{II}+\mathrm{III}).$$

We note that

$$\mathrm{I} \le C\int_0^1 r^{1-2aN} e^{2aN\ln r + as}\mathrm{d}r = \frac{Ce^{as}}{2} = o(1),$$
$$\mathrm{II} \le e^{\frac{as}{2}}\int_1^{r_s} r^{1-2aN}\mathrm{d}r \le \frac{e^{\frac{as}{2}}\left(1 - r_s^{2-2aN}\right)}{2aN-2} = o(1),$$
$$\mathrm{III} \le \left(\sup_{u\in\mathbb{R}} e^{a(u-e^u)}\right)\cdot\int_{r_s}^{\infty} r^{1-2aN}\mathrm{d}r = \left(\sup_{u\in\mathbb{R}} e^{a(u-e^u)}\right)\cdot\frac{r_s^{2-2aN}}{2aN-2} = o(1),$$

as $s \to -\infty$. So, $0 \ge \beta(s)\big(\beta(s) - 4N\big) = o(1)$ such that $\beta(s) \to 0$ as $s \to -\infty$.

The asymptotic behavior of $u(r, s)$ as $r \to \infty$ follows from the same argument of the proof of Lemma 2.12. □

Remark 3.5. Suppose $1 < aN < 2$ and let $m(s) = \sup_{r>0} u(r, s)$. Then, $m(s) < (1-\delta)s$ for any small $0 < \delta < 1$ as $s \to -\infty$. Otherwise, there exist $s_n \to -\infty$ and $r_n \to \infty$ such that $u(r_n, s_n) = (1-2\delta)s_n$ and $s_n < z(s_n)$. Then, as in the proof of Proposition 3.4, we can show that $\beta(s_n)\big(\beta(s_n) - 4N\big) \to 0$, namely, either $\beta(s_n) \to 0$ or $\beta(s_n) \to 4N$. This contradicts to Proposition 3.1 since $0 < 2N < \beta(s_n) < 4/a < 4N$.

4. Auxiliary Problems

In this section, we consider two auxiliary problems for Eq. (2.1) which are used in Proposition 3.1. In fact, Propositions 4.1 and 4.3 are used in Step 2 and Step 3 of Proposition 3.1, respectively. Throughout this section, let $a, N, b > 0$ such that $0 < aN < 1$. First, we consider the following equation: for $s < 0$,

$$\begin{cases} u'' + \frac{1}{r}u' = \frac{1}{\varepsilon^2}r^{-2aN}e^{b(u-e^u)}(e^u - 1) = -r^{-2aN}f(u, b), & r > 0, \\ u(0) = s, & u'(0) = 0. \end{cases} \tag{4.1}$$

We note that (4.1) is different from (2.1) with $N = 0$. Let us denote the solution of (4.1) by $u(r, s, a, N, b)$, or simply $u(r, s)$. We define

$$\gamma(s) = \gamma_{a,N,b}(s) = \int_0^{\infty} r^{1-2aN} f\big(u(r, s), b\big)\mathrm{d}r > 0.$$

Then, $\gamma(s) = -\lim_{r\to\infty} ru'(r, s)$. It is not difficult to see that $u(r)$ is strictly decreasing in $r > 0$, $\lim_{r\to\infty} u(r, s) = -\infty$, and $\gamma(s) \in (0, \infty)$. Then, the integrability condition for $f(u, b)$ implies that

$$\gamma(s) > \frac{2-2aN}{b}.$$

Regarding the properties of solutions of (4.1), we establish Proposition 4.2 below. We start with the following lemma.

Lemma 4.1. *Let a, N, b be positive real numbers such that $0 < aN < 1$. Then, for any $R > 0$, $u(r,s) \to 0$ uniformly on $[0,R]$ as $s \nearrow 0$.*

Proof. Set $c_0 = \sup\{e^{b(u-e^u)}/\varepsilon^2 : -1 \le u \le 0\}$ and let $R > 0$ be given. Given a small $\eta \in (0,1)$, let us choose $\delta \in (0, \eta/2)$ such that

$$\frac{c_0 R^{2-2aN}}{(2-2aN)^2}(e^{-\delta}-1) > -\frac{\eta}{2}.$$

Let $r_s = r_s(\delta)$ be the number such that $u(r_s,s) = -\delta$. We claim that $r_s \to \infty$ as $s \nearrow 0$. To see this, for $|s| \ll \delta$, let $r_k = r_k(s,\delta)$ be the number such that $u(r_k,s) = ks$ for $k = 1,2,\ldots,n_s$ where $n_s = n_s(\delta)$ is the greatest integer less than $-\delta/s$. It is obvious that $r_1 = 0$, $r_{n_s} \le r_s$ and $n_s \to \infty$ as $s \nearrow 0$. Since u is strictly decreasing, we see that for $0 < \tau < r < r_{n_s}$

$$\begin{aligned}\big(\tau u'(\tau)\big)' &= \frac{1}{\varepsilon^2}\tau^{1-2aN} e^{b(u(\tau)-e^{u(\tau)})}(e^{u(\tau)}-1)\\ &= \frac{1}{\varepsilon^2}\tau^{1-2aN} e^{b(u(\tau)-e^{u(\tau)})} e^{\zeta(\tau)} u(\tau) \ge c_0 \tau^{1-2aN} u(r),\end{aligned}$$

where $u(\tau) \le \zeta(\tau) \le s$. Integrating this inequality on $(0,r)$, we obtain that for $0 < r < r_{n_s}$

$$u'(r) \ge \frac{c_0 r^{1-2aN}}{2-2aN}\, u(r).$$

Integrating the latter inequality on (r_k, r_{k+1}), we get

$$r_{k+1}^{2-2aN} - r_k^{2-2aN} \ge \frac{1}{k+1}\cdot\frac{(2-2aN)^2}{c_0},$$

which implies that as $s \nearrow 0$,

$$r_{n_s}^{2-2aN} = \sum_{k=1}^{n_s-1}\left(r_{k+1}^{2-2aN} - r_k^{2-2aN}\right) \ge \frac{(2-2aN)^2}{c_0}\sum_{k=1}^{n_s-1}\frac{1}{k+1} \;\to\; \infty.$$

Thus, the claim follows.

Now, for all $|s| \ll \delta$, we have $R < r_s$ such that $u(R,s) > -\delta$. Then, for each $r \in [0,R]$,

$$\begin{aligned}0 > u(r,s) &= s + \frac{1}{\varepsilon^2}\int_0^r \frac{1}{t}\int_0^t \tau^{1-2aN} e^{b(u-e^u)}(e^u-1)\mathrm{d}\tau\mathrm{d}t\\ &\ge -\delta + c_0(e^{-\delta}-1)\int_0^R \frac{1}{t}\int_0^t \tau^{1-2aN}\mathrm{d}\tau\mathrm{d}t\\ &= -\delta + \frac{c_0 R^{2-2aN}}{(2-2aN)^2}(e^{-\delta}-1) \;>\; -\eta.\end{aligned}$$

Therefore, $u(r,s) \to 0$ uniformly on $[0,R]$ as $s \nearrow 0$. $\square$

Proposition 4.2. *Let a, N, b be positive real numbers such that $0 < aN < 1$. The function $\gamma : (-\infty, 0) \to (\bar{\gamma}, \infty)$ is continuous, onto, and strictly increasing, where $\bar{\gamma} = \bar{\gamma}_{a,N,b} = (4 - 4aN)/b$.*

Proof. First, analogous to Lemma 2.4, we obtain the Pohozaev type identities:

$$\gamma^2 = \frac{4-4aN}{b\varepsilon^2} \int_0^\infty r^{1-2aN} e^{b(u-e^u)} \mathrm{d}r, \tag{4.2}$$

$$\gamma(\gamma - \bar{\gamma}) = \frac{4-4aN}{b\varepsilon^2} \int_0^\infty r^{1-2aN} e^{b(u-e^u)} e^u \mathrm{d}r. \tag{4.3}$$

Thus, $\gamma(s) > \bar{\gamma}$. Moreover, we obtain

$$\lim_{s \to -\infty} \gamma(s) = \bar{\gamma} \qquad \text{and} \qquad \lim_{s \nearrow 0} \gamma(s) = \infty. \tag{4.4}$$

Indeed, it follows from (4.3) that

$$\gamma(\gamma - \bar{\gamma}) \le e^s \cdot \frac{4-4aN}{b\varepsilon^2} \int_0^\infty r^{1-2aN} e^{b(u-e^u)} \mathrm{d}r = e^s \gamma^2,$$

which implies the first part of (4.4). Meanwhile, by Lemma 4.1, $u(r, s) \to 0$ locally uniformly as $s \nearrow 0$. Thus, (4.2) yields the second part of (4.4). As a consequence, since $\gamma(s)$ is continuous by the same argument as in Lemma 2.3, $\gamma : (-\infty, 0) \to (\bar{\gamma}, \infty)$ is a surjective continuous function.

It remains to show that $\gamma(s)$ is increasing. To this end, let $\varphi(r, s) = \frac{\partial}{\partial s} u(r, s)$ which satisfies

$$\begin{cases} \varphi'' + \frac{1}{r}\varphi' = -\, r^{-2aN} f'(u, b)\varphi, & r > 0, \\ \varphi(0, s) = 1, & \varphi'(0, s) = 0. \end{cases} \tag{4.5}$$

If $\varphi(r) > 0$ for all $r > 0$, then $(r\varphi')'(r) < 0$ for all large r, which implies that $\varphi'(r) > 0$ for all large r. Thus, if we set $w(r) = ru'(r)$, then

$$0 < \int_0^\infty (2 - 2aN) r^{1-2aN} f(u, b)\varphi(r) \mathrm{d}r = \lim_{r \to \infty} r\varphi'(r) w(r) \le 0,$$

a contradiction. Hence, $\varphi(r)$ has a zero and we denote by r_1 the first zero of $\varphi(r)$.

Let $w_c(r, s) = ru'(r, s) + c$. Then w_c satisfies that

$$\begin{cases} w_c'' + \dfrac{1}{r} w_c' = -r^{-2aN} f'(u, b) w_c - r^{-2aN} e^{b(u-e^u)} \Phi_c(r, s), \\ w_c(0, s) = c, \quad w_c'(0, s) = 0, \end{cases} \tag{4.6}$$

where $\Phi_c(r, s)$ is defined by (2.12). Let $c_0 = (2 - 2aN)/b > 0$. Since $\Phi_{c_0}(r) > 0$ for all $r > 0$, $w_{c_0}(r)$ has a zero before r_1. Indeed, otherwise, $w_{c_0}(r) > 0$ on $(0, r_1)$ such that

$$0 < \int_0^{r_1} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_0}(r)\varphi(r) \mathrm{d}r = r_1 \varphi'(r_1) w_{c_0}(r_1) \le 0,$$

a contradiction. In particular, $r_1 u'(r_1) + c_0 = w_{c_0}(r_1) < 0$.

Suppose that $\varphi(r)$ has the second zero at r_2. Let $c_i = -r_i u'(r_i) > c_0$ for $i = 1, 2$ such that $w_{c_i}(r_i) = 0$. Since $(ru')'(r) < 0$ for all $r > 0$, $c_1 < c_2$. Let $\xi_i \in (0, 1)$ be the unique point such that $G_{c_i}(\xi_i) = 0$, where G_c is defined by

(2.22). We also set t_i to be the unique point such that $e^{u(t_i)} = \xi_i$. It is obvious that $\xi_1 < \xi_2$ and $t_2 < t_1$. If $t_2 \geq r_1$, then $\Phi_{c_2}(r) \geq 0$ for $r \in (0, r_1)$ such that

$$0 < \int_0^{r_1} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = r_1\varphi'(r_1)w_{c_2}(r_1) < 0,$$

a contradiction. Hence, $t_2 < r_1$ such that $\Phi_{c_2}(r) < 0$ for all $r > r_1$. If $\varphi(r)$ has the third zero at r_3, then we obtain a contradiction:

$$0 > \int_{r_2}^{r_3} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = r_3\varphi'(r_3)w_{c_2}(r_3) > 0.$$

Since $\varphi(r) > 0$ for all $r > r_2$, $(r\varphi')'(r) < 0$ for all large r. Hence, there exists $\lim_{r\to\infty} r\varphi'(r,s) = \delta_s \geq 0$ such that $\gamma'(s) = -\delta_s$ by the Lebesgue convergence theorem. If $\delta_s = 0$, then

$$0 > \int_{r_2}^{\infty} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. As a consequence, $\gamma'(s) < 0$ which violates (4.4). So, we have proved that $\varphi(r)$ has exactly one zero at r_1.

Now, we claim that $t_1 < r_1$. Otherwise, $\Phi_{c_1}(r) > 0$ for $r \in (0, r_1)$ and hence

$$0 < \int_0^{r_1} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_1}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. Due to the fact that $t_1 < r_1$, it holds that $\Phi_{c_1}(r) < 0$ for all $r > r_1$. Since $\varphi(r) < 0$ on (r_1, ∞) and $(r\varphi')'(r) > 0$ for all large r, there exists $\lim_{r\to\infty} r\varphi'(r,s) = \rho_s \leq 0$ and $\gamma'(s) = -\rho_s$. If $\rho_s = 0$, then

$$0 < \int_{r_1}^{\infty} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_1}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. Hence, $\gamma'(s) > 0$. This completes the proof. $\square$

The second equation to be considered is

$$\begin{cases} u'' + \dfrac{1}{r}u' = \dfrac{1}{\varepsilon^2} r^{-2aN} e^{b(u-e^u)}(e^u - 1) = -\, r^{-2aN} f(u,b), \quad r > 0, \\ u(0) = -\,2N \ln r + s + o(1). \end{cases} \tag{4.7}$$

The difference between (2.1) and (4.7) is the behavior of $u(r,s)$ near $r = 0$. Let us denote the solution of (4.7) by $u(r,s,a,N,b)$, or simply $u(r,s)$. We define

$$\lambda(s) = \lambda_{a,N,b}(s) = \int_0^{\infty} r^{1-2aN} f\big(u(r,s),b\big)\mathrm{d}r. \tag{4.8}$$

Then, $2N + \lambda(s) = -\lim_{r\to\infty} ru'(r,s)$. The shooting method shows that (4.7) has three kinds of solutions which are classified into the following sets:

$$\begin{cases} A^-_{a,N,b} = \{s \in \mathbb{R} : u(r_0,s) < 0 \text{ for some } r_0 > 0\}, \\ A^0_{a,N,b} = \{s \in \mathbb{R} : u(r,s) \geq 0 \text{ and } u'(r,s) \leq 0 \text{ for all } r > 0\}, \\ A^+_{a,N,b} = \{s \in \mathbb{R} : u(r,s) \geq 0 \text{ for all } r > 0 \text{ and } u'(r_0,s) > 0 \\ \qquad\qquad \text{for some } r_0 > 0\}. \end{cases} \tag{4.9}$$

By proceeding as in Lemmas 2.2–2.7, one can see that there exists s_* such that $A^+_{a,N,b} = (s_*, \infty)$, $A^0_{a,N,b} = \{s_*\}$, and $A^-_{a,N,b} = (-\infty, s_*)$. Regarding (4.7), we are only interested in the solutions whose shooting parameters are in $A^-_{a,N,b}$.

Proposition 4.3. *Let a, N, b be positive real numbers such that $0 < aN < 1$. The function $\lambda : (-\infty, s_*) \to (\max\{\bar\lambda, 0\}, \infty)$ is continuous, onto, and strictly increasing, where*

$$\bar\lambda = \bar\lambda_{a,N,b} = \frac{4 - 4(a+b)N}{b}.$$

Proof. For simplicity, we omit the subscript by writing $A^- = A^-_{a,N,b}$, $\lambda = \lambda_{a,N,b}$ and so on. Let $s \in A^-$. It is not difficult to see that $u'(r) < 0$ for all $r > 0$. Thus, there exists a unique zero $y = y(s)$ of $u(r,s)$. It is obvious that $\lambda(s)$ is finite and the integrability condition yields that

$$\lambda(s) > \frac{2 - 2(a+b)N}{b}.$$

The Pohozaev-type identities are as follows:

$$\lambda(\lambda + 4N) = \frac{4 - 4aN}{b\varepsilon^2} \int_0^\infty r^{1-2aN} e^{b(u-e^u)} \mathrm{d}r,$$
$$\lambda(\lambda - \bar\lambda) = \frac{4 - 4aN}{b\varepsilon^2} \int_0^\infty r^{1-2aN} e^{b(u-e^u)} e^u \mathrm{d}r.$$

Then we obtain that $\lambda(s) > \max\{\bar\lambda, 0\}$. Moreover, by a similar argument as in Lemma 2.9, we obtain

$$\lim_{s\to-\infty} \lambda(s) = \max\{\bar\lambda, 0\} \quad \text{and} \quad \lim_{s\nearrow s_*} \lambda(s) = \infty. \tag{4.10}$$

It remains to show that $\lambda(s)$ is strictly increasing. To this end, we recall that $\varphi(r) = \frac{\partial}{\partial s}u(r,s)$ and $w_c(r) = ru'(r) + c$ satisfy

$$\begin{cases} \varphi'' + \frac{1}{r}\varphi' = -r^{-2aN} f'(u,b)\varphi, & r > 0, \\ \varphi(0,s) = 1, & \varphi'(0,s) = 0, \end{cases} \tag{4.11}$$

and

$$\begin{cases} w_c'' + \dfrac{1}{r} w_c' = -r^{-2aN} f'(u,b) w_c - r^{-2aN} e^{b(u-e^u)} \Phi_c(r,s), \\ w_c(0,s) = -2N + c, \quad w_c'(0,s) = 0, \end{cases} \tag{4.12}$$

where $\Phi_c(r,s)$ is defined by (2.12). Then, by proceeding as in Lemma 2.10, we can show that $\varphi(r)$ has the first zero at $r_1 = r_1(s) > 0$ such that $y < r_1$. Moreover, by the same argument as in Lemma 2.11, one can see that r_1 is the unique zero of $\varphi(r)$.

We note that $\varphi(r) < 0$ on (r_1, ∞) and thus $(r\varphi')' > 0$ for all large r. Hence, there exists $\lim_{r\to\infty} r\varphi'(r,s) = \delta_s \le 0$. If $\delta_s < 0$, the proof is complete. On the contrary, let us assume that $\delta_s = 0$. Since $w_0(y) < 0$ and

$$0 > (2 - 2aN) \int_0^y r^{1-2aN} f(u)\varphi(r) \mathrm{d}r = y\varphi'(y) w_0(y),$$

we get $\varphi'(y) > 0$. For $c_0 = (2-2aN)/b > 0$, $\Phi_{c_0}(r) > 0$ on (y, ∞). Since $u'(r) < 0$ for all $r > 0$, we can choose $c_1 > c_0$ such that $\Phi_{c_1}(r_1) = 0$. Then $\Phi_{c_1}(r) > 0$ on (y, r_1) and $\Phi_{c_1}(r) < 0$ on (r_1, ∞) which implies that $\Phi_{c_1}(r)\varphi(r) > 0$ on (y, ∞). Hence,

$$0 < \int_y^\infty r^{1-2aN} e^{b(u-e^u)} \Phi_{c_1}(r)\varphi(r)\mathrm{d}r = -y\varphi'(y)w_{c_1}(y) < 0,$$

a contradiction. Here, the last inequality holds only when $\bar{\lambda} > 0$. In fact, if $\bar{\lambda} > 0$, then $0 < \bar{\lambda}/2 = -2N + (2-2aN)/b < -2N + c_1 = w_{c_1}(0) < w_{c_1}(y)$.

Suppose that $\bar{\lambda} < 0$ such that $2N > (2-2aN)/b$. Let $c_1 = 2N$. Then $w_{c_1}(0) = 0$, $\lim_{r\to\infty} w'_{c_1}(r) = 0$, and $\lim_{r\to\infty} w_{c_1}(r) = -\lambda < 0$. We note that $w'_c(r) > 0$ on $(0, y)$ and $w'_c(r) < 0$ on (y, ∞) for any c. So, $w_{c_1}(r)$ has a zero R after y, i.e., $w_{c_1}(R) = 0$ and $R > y$. We have two choices: either $r_1 \le R$ or $r_1 > R$. First, we suppose that $r_1 \le R$. Then we can choose $c_2 \in (0, c_1)$ such that $w_{c_2}(r_1) = 0$. If $c_2 < (2-2aN)/b$, then $\Phi_{c_2}(r) > 0$ on (y, ∞), which implies that

$$0 > \int_{r_1}^\infty r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = 0,$$

a contradiction. Hence, $c_1 > c_2 > (2-2aN)/b$ such that $\Phi_{c_2}(r)$ has a unique zero on (y, ∞) at t. Let $T_0 = e^{u(t)}$. If $e^{u(r_1)} \ge T_0$, then $\Phi_{c_2}(r) > 0$ on (y, r_1) and so

$$0 < \int_y^{r_1} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = -\,y\varphi'(y)w_{c_2}(y) < 0.$$

Hence, $e^{u(r_1)} < T_0$ such that $\Phi_{c_2}(r) < 0$ on (r_1, ∞). Thus,

$$0 < \int_{r_1}^\infty r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = 0,$$

which is a contradiction. Therefore, $\delta_s \ne 0$ if $r_1 \le R$.

Next, we assume that $r_1 > R$. Then we can choose $c_2 > c_1$ such that $w_{c_2}(r_1) = 0$. As above, we denote by t the unique zero of $\Phi_{c_2}(r)$ on (y, ∞) such that $e^{u(t)} = T_0$. If $e^{u(r_1)} \ge T_0$, then $\Phi_{c_2}(r) > 0$ on (y, r_1) and so

$$0 < \int_y^{r_1} r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = -\,y\varphi'(y)w_{c_2}(y) < 0,$$

which yields that $e^{u(r_1)} < T_0$. In particular, $\Phi_{c_2}(r) < 0$ on (r_1, ∞) and we have a contradiction:

$$0 < \int_{r_1}^\infty r^{1-2aN} e^{b(u-e^u)} \Phi_{c_2}(r)\varphi(r)\mathrm{d}r = 0.$$

Thus, $\delta_s \ne 0$ and the proof is complete. $\square$

5. Calculation of Physical Quantities

In this section, we return to the original self-dual equations (1.15) and (1.16) coupled to the Einstein equation (1.11). If u is the solution of (1.29) obtained by Theorem 1.1, then we can recover the solution $(\tilde{u}, \tilde{\eta})$ of (1.22) and (1.23) by (1.24) and (1.25). Then, we get a solution pair (ϕ, A, g_{ij}) of (1.11), (1.15) and (1.16) by the formulae (1.17), (1.18) and (1.21). Regarding this solution (ϕ, A, g_{ij}), we will compute several important quantities such as energy, magnetic flux and the total Gaussian curvature. We also compute the decay/blowup rate of the energy density. The main result of this section is summarized in Theorem 5.1 at the end of this section.

Let $u(r)$ be the solution of (1.29) obtained by Theorem 1.1. We recall that $u(x)$ is a radially symmetric solution of (1.26) under the condition

$$p_1 = \cdots = p_d = 0,$$

which will be always assumed throughout this section. We define

$$\eta(x) = a\big[u(x) - e^{u(x)} - N \ln |x|^2\big],$$

and set

$$\tilde{u}(x) = u(\tau x) + \ln \tau^2, \quad \tilde{\eta}(x) = \eta(\tau x).$$

Then, $(\tilde{u}, \tilde{\eta})$ becomes a solution of (1.19) and (1.20) with $p_1 = \cdots = p_d = 0$. Since u and η are radially symmetric functions, as long as there is no confusion, we will use both expressions $u(x)$ and $u(r)$ in this section where $r = |x|$ for $x \in \mathbb{R}^2$.

In the following, using the properties proved in Theorem 1.1, we will compute the following physical quantities for the solution pair $(\phi, A, \tilde{\eta})$:

$$\begin{aligned} &\text{The static energy } E = \int_{\mathbb{R}^2} \mathcal{E} e^{\tilde{\eta}} \mathrm{d}x, \\ &\text{The magnetic flux } \Phi = \int_{\mathbb{R}^2} F_{12} \mathrm{d}x, \\ &\text{The total Gaussian curvature } \kappa = \int_{\mathbb{R}^2} K_g e^{\tilde{\eta}} \mathrm{d}x. \end{aligned} \tag{5.1}$$

To this end, we represent $\phi, e^{\tilde{\eta}}, F_{12}, |D_j\phi|^2, \mathcal{E}, K_g$ in terms of u. We note that

$$\begin{cases} |\phi(x)|^2 = e^{\tilde{u}(x)} = \tau^2 e^{u(\tau x)}, \\ e^{\tilde{\eta}(x)} = |\tau x|^{-2aN} \cdot \exp\Big(a\big[u(\tau x) - e^{u(\tau x)}\big]\Big), \\ F_{12} = \dfrac{1}{2\varepsilon^2} e^{\tilde{\eta}(x)} (\tau^2 - |\phi(x)|^2) = \dfrac{\tau^2}{2\varepsilon^2} e^{\tilde{\eta}(x)} (1 - e^{u(\tau x)}), \\ |D_1\phi|^2 + |D_2\phi|^2 = \dfrac{1}{2} e^{\tilde{u}} |\nabla \tilde{u}|^2 = \dfrac{1}{2} \tau^4 e^{u(\tau x)} |\nabla u(\tau x)|^2, \\ K_g = -\dfrac{1}{2} e^{-\tilde{\eta}} \Delta \tilde{\eta}. \end{cases} \tag{5.2}$$

We divide the calculation into two parts: for topological solutions and non-topological solutions of type I in Sect. 5.1, and for nontopological solutions of type II in Sect. 5.2. We recall that u enjoys the following asymptotic behavior

$$u(r) = \big[2N - \beta(s)\big] \ln r + O(1) \quad \text{as} \quad r \to \infty,$$

where

$$\beta = \int_0^\infty r^{1-2aN} f\big(u(r), a\big) \mathrm{d}r.$$

By Theorem 1.1, β belongs to certain domains in $\mathbb{R}$ according to the values aN. We also recall that $a = 4\pi G\tau^2$, where G is the gravitational constant.

5.1. Topological Solutions and Nontopological Solutions of Type I

We assume that $0 \leq aN < 2$ and $aN \neq 1$ and let u be a topological solution or a nontopological type I solution of (1.29). The proof is split into two parts: the decay estimates and the calculation of (5.1).

(i) Decay Estimates

We compute the decay estimates of each term in the energy density (1.12), the conformal factor $e^{\tilde{\eta}}$, and the Gaussian curvature K_g. From the decay rates of u given by Theorem 1.1, we deduce that for topological solutions, as $r = |x| \to \infty$,

$$e^{\tilde{u}} = \begin{cases} \tau^2\big(1 + O(r^{-\alpha})\big) \text{ for any } \alpha \text{ if } 0 < aN < 1, \\ \tau^2 e^{\sigma}\big(1 + O(r^{2-2aN})\big) \text{ if } 1 < aN < 2, \end{cases}$$

and hence

$$|\phi|^2 - \tau^2 = \begin{cases} O(r^{-\alpha}) \text{ for any } \alpha \text{ if } 0 < aN < 1, \\ \tau^2(e^{\sigma} - 1) + O(r^{2-2aN}) \text{ if } 1 < aN < 2. \end{cases}$$

If $N = 0$, then the topological solution is a constant solution $(\tilde{u}, A, \tilde{\eta}) = (\ln \tau^2, 0, -a)$ and hence g_{ij} is just a dilation of the standard metric on $\mathbb{R}^2$. In the following, we will not consider the trivial topological solution for the case $aN = 0$. However, the nontopological solution is still of interest for the case $aN = 0$. For a nontopological solution of type I, we obtain

$$|\phi|^2 = e^{\tilde{u}} = O(r^{2N-\beta}) \quad \text{as} \quad r \to \infty.$$

On the other hand, we note from (5.2) that as $r \to \infty$,

$$F_{12} = O(e^{\tilde{\eta}}) = \begin{cases} O\big(r^{-2aN}\big) & \text{for a topological solution,} \\ O(r^{-a\beta}) & \text{for a nontopological solution of type I.} \end{cases}$$

By the standard potential theory, we derive from (1.22) that

$$\tilde{u}(x) = \int_{\mathbb{R}^2} \frac{1}{\varepsilon^2} e^{\tilde{\eta}(y)}(e^{\tilde{u}(y)} - \tau^2)(\ln|x-y|)\,\mathrm{d}y + 2N \ln|x| + c_0$$

for some constant c_0. Thus,

$$\nabla\tilde{u}(x) = \int_{\mathbb{R}^2} \frac{1}{\varepsilon^2} e^{\tilde{\eta}(y)}(e^{\tilde{u}(y)} - \tau^2) \cdot \frac{x-y}{|x-y|^2}\,\mathrm{d}y + 2N\frac{x}{|x|^2},$$

or equivalently,

$$\nabla u(\tau x) = \frac{\tau}{\varepsilon^2}\int_{\mathbb{R}^2} e^{\tilde{\eta}(y)}(e^{u(\tau y)}-1)\cdot\frac{x-y}{|x-y|^2}\,\mathrm{d}y + \frac{2Nx}{\tau|x|^2}. \tag{5.3}$$

We note that if u is a topological solution, then

$$e^{\tilde{\eta}}(e^{\tilde{u}}-\tau^2) = \begin{cases} 0, & \text{if } aN = 0,\\ O\big(r^{-2aN-\alpha}\big) & \text{for any } \alpha>0, \text{ if } 0<aN<1,\\ O\big(r^{-2aN}\big), & \text{if } 1<aN<2. \end{cases} \tag{5.4}$$

On the other hand, if u is a nontopological solution of type I, then

$$e^{\tilde{\eta}}(e^{\tilde{u}}-\tau^2) = O\big(r^{-a\beta}\big). \tag{5.5}$$

Thus, the integral in (5.3) is finite for $0<aN<2$ with $aN\neq 1$ which implies that

$$|\nabla u(\tau r)| = O(r^{-1}) \quad \text{as} \quad r\to\infty.$$

As a consequence, we see that

$$|D_1\phi|^2+|D_2\phi|^2 = \begin{cases} O\big(r^{-2}\big) & \text{if } u \text{ is a topological solution},\\ O\big(r^{2N-\beta-2}\big) & \text{if } u \text{ is a nontopological solution of type I}. \end{cases}$$

Meanwhile, it follows from (1.22) and (1.23) that

$$\begin{aligned}\Delta\tilde{\eta} &= -\frac{a}{\tau^2}e^{\tilde{u}}|\nabla\tilde{u}|^2 - \frac{a}{\tau^2}e^{\tilde{u}}\Delta\tilde{u} + \frac{a}{\varepsilon^2}e^{\tilde{\eta}}(e^{\tilde{u}}-\tau^2)\\ &= -\frac{a}{\tau^2}e^{\tilde{u}}|\nabla\tilde{u}|^2 - \frac{a}{\tau^2\varepsilon^2}e^{\tilde{u}+\tilde{\eta}}(e^{\tilde{u}}-\tau^2) + \frac{a}{\varepsilon^2}e^{\tilde{\eta}}(e^{\tilde{u}}-\tau^2).\end{aligned}$$

Hence,

$$K_g = -\frac{1}{2}e^{-\tilde{\eta}}\Delta\tilde{\eta} = \frac{a}{2\tau^2}e^{\tilde{u}-\tilde{\eta}}|\nabla\tilde{u}|^2 + \frac{a}{2\tau^2\varepsilon^2}e^{\tilde{u}}(e^{\tilde{u}}-\tau^2) - \frac{a}{2\varepsilon^2}(e^{\tilde{u}}-\tau^2). \tag{5.6}$$

If u is a topological solution, then as $r\to\infty$,

$$K_g = \begin{cases} O(r^{2aN-2}) & \text{for } 0<aN<1,\\ O(r^{2aN-2}) + \frac{a\tau^2}{2\varepsilon^2}(e^{\sigma}-1)^2 & \text{for } 1<aN<2. \end{cases}$$

If u is a nontopological solution of type I, then

$$K_g = O(r^{2N-\beta+a\beta-2}) + O(r^{2N-\beta}) + \frac{a\tau^2}{2\varepsilon^2} = O(r^{2N-\beta+a\beta-2}) + \frac{a\tau^2}{2\varepsilon^2}$$

as $r\to\infty$.

(ii) Calculation of (5.1)

From (1.13), (1.14) and (1.22), the energy density is written as

$$\begin{aligned}\mathcal{E}e^{\tilde{\eta}} &= e^{\tilde{\eta}}\Big[\frac{1}{4}\tau^2\gamma^{jk}F_{jk} + \nabla_j(\gamma^{jk}J_k)\Big] = \frac{e^{\tilde{\eta}}}{4}\Big[\tau^2(\tau^2-|\phi|^2) + e^{-\tilde{\eta}}\Delta|\phi|^2\Big]\\ &= \frac{1}{4}\Big[\tau^2 e^{\tilde{\eta}}(\tau^2-e^{\tilde{u}}) + \Delta e^{\tilde{u}}\Big] = \frac{1}{4}\Big[-\frac{\tau^2}{r}(r\tilde{u}')' + \frac{1}{r}\big(r(e^{\tilde{u}})'\big)'\Big],\end{aligned}$$

which leads us to

$$E = \Big[-\tau^2 r\tilde{u}' \Big]_{r=0}^{\infty} + \frac{\pi}{2}\Big[r\tilde{u}' e^{\tilde{u}} \Big]_{r=0}^{\infty} = \frac{\pi\tau^2}{2}\Big[\beta + \lim_{r\to\infty}(ru'e^{u}) \Big]. \tag{5.7}$$

Thus, we conclude that for $0 < aN < 2$ with $aN \neq 1$,

$$E = \begin{cases} N\pi\tau^2, & \text{if } u \text{ is a topological solution,} \\ \frac{\pi\tau^2}{2}\beta, & \text{if } u \text{ is a nontopological solution of type I.} \end{cases} \tag{5.8}$$

Meanwhile, it follows from (1.22) and (1.23) that

$$\begin{aligned} K_g &= -\frac{1}{2}e^{-\tilde{\eta}}\Delta\tilde{\eta} = -\frac{1}{2}e^{-\tilde{\eta}}\Big[\frac{a}{\varepsilon^2}e^{\tilde{\eta}}(e^{\tilde{u}} - \tau^2) - \frac{a}{\tau^2}\Delta e^{\tilde{u}} \Big] \\ &= \frac{a}{2}e^{-\tilde{\eta}}\Big[-\Delta\tilde{u} + \frac{1}{\tau^2}\Delta e^{\tilde{u}} \Big]. \end{aligned} \tag{5.9}$$

Thus, the total Gaussian curvature κ is reduced to

$$\kappa = \pi a \int_0^{\infty} \Big[-(r\tilde{u}')' + \frac{1}{\tau^2}\big(r(e^{\tilde{u}})'\big)' \Big] \mathrm{d}r = \pi a\beta + \pi a \lim_{r\to\infty}(ru')e^{u}, \tag{5.10}$$

and we have

$$\kappa = \begin{cases} 2\pi N a & \text{for a topological solution,} \\ \pi a\beta & \text{for a nontopological solution of type I.} \end{cases}$$

Finally, the total magnetic flux is given by

$$\Phi = \int_{\mathbb{R}^2} \frac{1}{2\varepsilon^2}e^{\tilde{\eta}}(\tau^2 - e^{\tilde{u}})\mathrm{d}x = -\pi\int_0^{\infty}(r\tilde{u}')'\mathrm{d}r = \pi\beta. \tag{5.11}$$

So,

$$\Phi = \begin{cases} 2\pi N, & \text{if } u \text{ is a topological solution,} \\ \pi\beta, & \text{if } u \text{ is a nontopological solution of type I,} \end{cases}$$

where $\beta > 4/a$ for $0 \leq aN < 1$ and $2N < \beta < 4/a$ for $1 < aN < 2$.

5.2. Nontopological Solutions of Type II

Let u be a nontopological solution of type II of (1.29). By Theorem 1.1,

$$|\phi|^2 = e^{\tilde{u}} = O(r^{2N-\beta}) \quad \text{as} \quad r \to \infty.$$

Hence, $e^{\tilde{u}} \to \infty$ as $r \to \infty$. We also see that as $r \to \infty$,

$$e^{\tilde{\eta}} = \begin{cases} O\big(r^{-a\beta}e^{-a(\tau r)^{2N-\beta}}\big) & \text{for } aN \neq 1, \\ O\big(e^{-a(\tau r)^{2N}}\big) & \text{for } aN = 1. \end{cases}$$

Hence, it comes from (5.2) that

$$F_{12} = \begin{cases} O\big(r^{2N-(a+1)\beta}e^{-a(\tau r)^{2N-\beta}}\big) & \text{for } aN \neq 1, \\ O\big(r^{2N}e^{-a(\tau r)^{2N}}\big) & \text{for } aN = 1. \end{cases}$$

By the blowup rate of $e^{\tilde{u}}$ and the decay rate of $e^{\tilde{\eta}}$, it is easy to see from (5.3) that

$$|\nabla u(\tau r)| = O(r^{-1}) \quad \text{as} \quad r \to \infty.$$

So, by (5.3)

$$|D_1\phi|^2 + |D_2\phi|^2 = \begin{cases} O(r^{2N-\beta-2}) & \text{for } aN \neq 1, \\ O(r^{2N-2}) & \text{for } aN = 1. \end{cases}$$

On the other hand, by (5.6)

$$K_g = O(r^{2N-\beta+a\beta-2}e^{a(\tau r)^{2N-\beta}}) + O(r^{4N-2\beta}) + \frac{a\tau^2}{2\varepsilon^2}.$$

Hence,

$$K_g = \begin{cases} O(r^{2N+(a-1)\beta-2}e^{a(\tau r)^{2N-\beta}}) & \text{for } aN \neq 1, \\ O(r^{2N-2}e^{a(\tau r)^{2N}}) & \text{for } aN = 1, \end{cases}$$

which implies that $K_g \to \infty$ as $r \to \infty$.

Finally, we compute E, Φ, and κ. We deduce from (5.7), (5.10) and (5.11) that $E = \infty$, $\Phi = \pi\beta$ and $\kappa = \infty$.

Now, gathering all the above results, we have the following theorem. It deserves attention that the uniqueness in the theorem means the uniqueness up to the gauge transformation (1.2) since Eqs. (1.11), (1.15) and (1.16) are gauge invariant.

Theorem 5.1. *Consider the self-dual equations* (1.11), (1.15) *and* (1.16) *of the EMH model coupled to the Einstein equations on the Lorentz space* $\mathcal{M} \cong \mathbb{R}^{1,1} \times \mathcal{S}$. *The metric* g_{ij} *on the two-dimensional surface* $\mathcal{S}$ *is assumed to be conformal to the standard metric on* $\mathbb{R}^2$ *with the conformal factor* $e^{\tilde{\eta}}$, *that is,* $g_{ij} = e^{\tilde{\eta}}\delta_{ij}$. *Then, given a nonnegative integer* N, *there exist disjoint sets* $\Gamma_N^-, \Gamma_N^0, \Gamma_N^+ \subset \mathbb{R}$ *such that for each* $\beta \in \Gamma_N^- \cup \Gamma_N^0 \cup \Gamma_N^+$ *we have a unique radially symmetric solution* $(\phi^\beta, A^\beta, g_{ij}^\beta)$ *and* ϕ^β *has a unique zero of order* N *at the origin. Moreover,* $(\phi^\beta, A^\beta, g_{ij}^\beta)$ *is a topological(nontopological of type I, nontopological of type II, resp.) solution if and only if* $\beta \in \Gamma_N^0$ $(\beta \in \Gamma_N^-$, $\beta \in \Gamma_N^+$, *resp.). Conversely, any radially symmetric solution coincides with one of* $(\phi^\beta, A^\beta, g_{ij}^\beta)$ *for some* β. *If we set* $a = 4\pi\tau^2 G$, *then we have the following properties according to the value* N.

(i) *Suppose that* $0 \leq N < 2a^{-1}$ *with* $N \neq a^{-1}$.

(i-a) *We have a unique topological solution such that* $\Gamma_N^0 = \{2N\}$. *If* $N = 0$, *then* $(\phi^\beta, A^\beta, e^{\tilde{\eta}}) = (\tau, 0, e^{-a})$ *and thus* g_{ij}^β *is just a dilation of the standard metric of* $\mathbb{R}^2$. *If* $N > 0$, *then as* $r = |x| \to \infty$, *we have*

$$|\phi|^2 = \tau^2 e^\sigma + o(1), \quad |g| = e^{\tilde{\eta}} = O(r^{-2aN}), \quad F_{12} = O(r^{-2aN}),$$
$$|D_j\phi|^2 = O(r^{-2}), \quad K_g = a\tau^2/(2\varepsilon^2) + O(r^{2aN-2}).$$

So, the energy density $\mathcal{E}$ *decays to 0 at infinity and the Gaussian curvature is asymptotically constant at infinity. If* $0 < N < a^{-1}$, *then* $\sigma = 0$. *If* $a^{-1} < N < 2a^{-1}$, *then* $\sigma < 0$. *The static energy* E, *magnetic flux* Φ, *and total Gaussian curvature* κ *are given by*

$$E = \pi\tau^2 N, \qquad \Phi = 2\pi N, \qquad \kappa = 2\pi a N.$$

In particular, a topological solution has a quantized static energy and a magnetic flux.

(i-b) *If* $0 \le N < a^{-1}$, *then* $\Gamma_N^- = (4a^{-1}, \infty)$. *If* $a^{-1} < N < 2a^{-1}$, *then* $\Gamma_N^- = (2N, 4a^{-1})$. *As* $r = |x| \to \infty$, *we have*

$$|\phi|^2 = O(r^{2N-\beta}), \quad |g| = e^{\tilde{\eta}} = O(r^{-a\beta}), \quad F_{12} = O(r^{-a\beta}),$$
$$|D_j\phi|^2 = O(r^{2N-\beta-2}), \quad K_g = a\tau^2/(2\varepsilon^2) + O(r^{2N-\beta+a\beta-2}).$$

So, the energy density decays to 0 at infinity and the Gaussian curvature is asymptotically constant at infinity. We also have

$$E = \frac{\pi\tau^2}{2}\beta, \qquad \Phi = \pi\beta, \qquad \kappa = \pi a\beta.$$

(i-c) *If* $0 \le N < a^{-1}$, *then* $\Gamma_N^+ = (-\infty, 0)$. *If* $a^{-1} < N < 2a^{-1}$, *then* $\Gamma_N^+ = (0, 2N)$. *As* $r = |x| \to \infty$, *we have*

$$\begin{aligned} &|\phi|^2 = O(r^{2N-\beta}), \quad |g| = e^{\tilde{\eta}} = O(r^{-a\beta} e^{-a(\tau r)^{2N-\beta}}), \\ &F_{12} = O(r^{2N-(a+1)\beta} e^{-a(\tau r)^{2N-\beta}}), \\ &|D_j\phi|^2 = O(r^{2N-\beta-2}), \quad K_g = O(r^{2N+(a-1)\beta-2} e^{a(\tau r)^{2N-\beta}}). \end{aligned} \tag{5.12}$$

So, the energy density and the Gaussian curvature blow up at infinity. Moreover, $E = \kappa = \infty$ *and* $\Phi = \pi\beta$.

(ii) *Suppose that* $N = a^{-1}$. *Then,* $\Gamma_N^- = \Gamma_N^0 = \emptyset$ *and* $\Gamma_N^+ = \{0\}$. *By Theorem 1.1, we have one parameter family of nontopological solutions of type II characterized by the behavior of* ϕ *in* (1.30). *That is, given any* $s \in \mathbb{R}$, *there is a solution pair* $(\phi(x,s), A(x,s), g_{ij}(x,s))$ *such that* $|\phi(x,s)|^2 = e^s|x|^{2N} + o(1)$ *as* $|x| \to 0$. *For any* $s \in \mathbb{R}$, *as* $r = |x| \to \infty$, *we have*

$$|\phi|^2 = O(r^{2N}), \quad |g| = e^{\tilde{\eta}} = O(e^{-a(\tau r)^{2N}}), \quad F_{12} = O(r^{2N} e^{-a(\tau r)^{2N}}),$$
$$|D_j\phi|^2 = O(r^{2N-2}), \quad K_g = O(r^{2N-2} e^{a(\tau r)^{2N}}).$$

So, the energy density and the Gaussian curvature blow up at infinity. Moreover, $E = \kappa = \infty$ *and* $\Phi = 0$.

(iii) *Suppose that* $N \ge 2a^{-1}$. *Then,* $\Gamma_N^- = \Gamma_N^0 = \emptyset$ *and* $\Gamma_N^+ = (0, \beta_*]$ *for some* $\beta_* < 4/a$. *For each* $\beta \in (0, \beta_*)$, *there exist at least two different nontopological solutions of type II satisfying* (5.12). *For these solutions, we also have* $E = \kappa = \infty$ *and* $\Phi = 0$.

6. Gravitational Maxwell Gauged $O(3)$ Sigma Model

In this section, we study Eq. (1.36) for the case $aN \ge 1$. As reported in Introduction, the existence and uniqueness of solutions have been unsolved yet. By applying the idea of this paper, we will provide the answers to this problem. Since the idea and proof are almost parallel, we will exhibit an outline of them. We rewrite (1.36) as

$$\begin{cases} u'' + \dfrac{1}{r}u' = -\,r^{-2aN}g(u,a), \\ u(r) = 2N\ln r + s + o(1) \quad \text{as} \quad r \searrow 0, \end{cases} \tag{6.1}$$

where

$$g(u,a) = \frac{1}{\varepsilon^2}\Big[\frac{e^u}{(1+e^u)^2}\Big]^a\Big(\frac{1-e^u}{1+e^u}\Big).$$

We denote the unique solution of (6.1) by $u(r,s)$. We also write

$$\beta(s) = \beta_{a,N}(s) = \int_0^\infty r^{1-2aN}g\big(u(r,s),a\big)\mathrm{d}r$$

such that $u(r,s)$ enjoys the asymptotic behavior

$$u(r,s) = \big[2N - \beta(s)\big]\ln r + O(1) \quad \text{as} \quad r \to \infty. \tag{6.2}$$

The known result for (6.1) is summarized as follows.

Theorem A ([8]). *Let a be a positive real number and N be a nonnegative integer.*

(i) *If $aN < 1$, the followings hold.*
  (i-a) *There exists a unique $s_* = s_*(a,N)$ such that $u(r,s_*)$ is a topological solution with $\beta(s_*) = 2N$ and $\sigma = 0$.*
  (i-b) *For $s < s_*$, $u(r,s)$ is a nontopological solution of type I. The function $\beta : (-\infty, s_*) \to (4/a, \infty)$ is continuous, onto, and strictly increasing. We have*

$$\lim_{s\nearrow s_*}\beta(s) = \infty, \qquad \lim_{s\to-\infty}\beta(s) = \frac{4}{a}. \tag{6.3}$$

  (i-c) *For $s > s_*$, $u(r,s)$ is a nontopological solution of type II. The function $\beta : (s_*, \infty) \to \big(-\infty, 8(aN-1)/a\big)$ is continuous, onto, and strictly increasing. We have*

$$\lim_{s\searrow s_*}\beta(s) = -\infty, \qquad \lim_{s\to\infty}\beta(s) = \frac{8(aN-1)}{a}. \tag{6.4}$$

(ii) *If $aN = 1$, then $\beta(s) \in \{0, 2N, 4N\}$.*
(iii) *If $aN \geq 2$, then (6.1) has only nontopological solutions of type II. Furthermore,*

$$0 < \beta(s) < \frac{4}{a}.$$

This theorem provides a complete classification of radial solutions of (6.1) when $0 < aN < 1$. However, it gives us no answer for $1 < aN < 2$ and incomplete answers for $aN = 1$ or $aN \geq 2$. To have a complete classification of radial solutions, we consider a generalized version of (6.1) : for $a, b, N > 0$ with $0 < aN < 1$,

$$\begin{cases} u'' + \dfrac{1}{r}u' = -r^{-2aN}g(u,b), \\ u(r) = 2N\ln r + s + o(1) \quad \text{as} \quad r \searrow 0. \end{cases} \tag{6.5}$$

Since $1+2bN-2aN > -1$, one can check easily the global existence of solutions of (6.1). If we also denote the unique solution of (6.5) by $u(r,s)$, we have the following result.

Theorem 6.1. *If a, N, b are positive real numbers such that $0 < aN < 1$, then there exists a unique $s_* = s_*(a,N,b)$ such that the following statements hold true.*

(i) *$u(r,s_*)$ is the unique topological solution of (6.5) such that $\beta(s_*) = 2N$ and $u(r,s_*) = O(r^{-\alpha})$ for any $\alpha > 0$ as $r \to \infty$.*

(ii) *For $s < s_*$, $u(r,s)$ is a nontopological solution of type I. The function $\beta : (-\infty, s_*) \to (\bar{\beta}, \infty)$ is continuous, onto, and strictly increasing, where*

$$\bar{\beta} = \bar{\beta}_{a,N,b} = \frac{4 + 4(b-a)N}{b} > 0.$$

In addition,

$$u(r,s) = (2N - \beta)\ln r + I + O(r^{2+2(b-a)N-b\beta}) \tag{6.6}$$

for some constant $I = I(s,a,N,b)$ as $r \to \infty$.

(iii) *For $s > s_*$, $u(r,s)$ is a nontopological solution of type II. The function*

$$\beta : (s_*, \infty) \to \Big(-\infty, \frac{8(aN-1)}{b}\Big)$$

is continuous, onto, and strictly increasing. Furthermore,

$$u(r,s) = (2N - \beta)\ln r + I + O(r^{2-2(a+b)N+b\beta}) \tag{6.7}$$

for some constant $I = I(s,a,N,b)$ as $r \to \infty$.

The proof of Theorem 6.1 is parallel to the proof of Theorem A and we omit it. One can also apply the same argument in Sect. 2. Furthermore, the asymptotic behaviors (6.6) and (6.7) can be proved as in Lemma 2.12. Now, the main result of this section is the following.

Theorem 6.2. *Let $1 < aN < 2$ and $u(r,s)$ be the unique solution of (6.1).*

(i) *There exists a unique $s_* = s_*(a,N)$ such that $u(r,s_*)$ is a topological solution with $\beta(s_*) = 2N$ and $\sigma < 0$. Moreover, $u(r,s_*) = \sigma + O(r^{2-2aN})$ as $r \to \infty$.*

(ii) *For $s < s_*$, $u(r,s)$ is a nontopological solution of type I. The function $\beta : (-\infty, s_*) \to (2N, 4/a)$ is continuous, onto, and strictly decreasing. We have*

$$\lim_{s \nearrow s_*} \beta(s) = 2N, \qquad \lim_{s \to -\infty} \beta(s) = \frac{4}{a}. \tag{6.8}$$

Moreover, $u(r,s) = (2N - \beta)\ln r + I + O(r^{2-a\beta})$ for some constant $I = I(s)$ as $r \to \infty$.

(iii) *For* $s > s_*$, $u(r,s)$ *is a nontopological solution of type II. The function* $\beta : (s_*, \infty) \to (0, 2N)$ *is continuous, onto, and strictly decreasing. We have*

$$\lim_{s \searrow s_*} \beta(s) = 2N, \qquad \lim_{s \to \infty} \beta(s) = 0. \tag{6.9}$$

Moreover, $u(r,s) = (2N - \beta)\ln r + J + O(r^{2-4aN+a\beta})$ *for some constant* $J = J(s)$ *as* $r \to \infty$.

Theorem 6.3. *If* $aN = 1$ *or* $aN \geq 2$, (6.1) *possesses only nontopological solutions of type II. Moreover, we have the following.*

(i) $\beta(s) = 0$ *for all* $s \in \mathbb{R}$ *provided* $aN = 1$.

(ii) *The function* $\beta : (-\infty, \infty) \to (0, 4/a)$ *is continuous provided* $aN \geq 2$. *In addition, we have the limit* (1.32), *which implies that there exist at least two solutions* $u(r,s)$ *such that* $\beta(s) = \beta_0 \in (0, \beta_*)$, *where* $\beta_* = \sup_{s\in\mathbb{R}} \beta(s)$.

Theorem 6.4. *Let* $N = 0$ *in* (6.1). *Then,*

$$\begin{cases} s = 0 \quad \Rightarrow \quad u(r,0) \equiv 0 \text{ is the unique topological solution,} \\ s < 0 \quad \Rightarrow \quad u(r,s) \text{ is a nontopological solution of type I,} \\ s > 0 \quad \Rightarrow \quad u(r,s) \text{ is a nontopological solution of type II.} \end{cases} \tag{6.10}$$

Moreover, the functions $\beta : (-\infty, 0) \to (4/a, \infty)$ *and* $\beta : (0, \infty) \to (-\infty, -4/a)$ *are continuous, onto, and strictly increasing.*

The proof of Theorem 6.2 is parallel to the argument of Sect. 3. One can just follow each step line by line and we skip the proof of Theorem 6.2. In the following, we give brief proofs of Theorems 6.3 and 6.4.

Proof of Theorem 6.3. The proof of part (ii) is almost the same as the proof of Propositions 3.1 and 3.2, and we omit the detail. For the proof of the part (i), we apply the same argument in the proof of Proposition 3.3 to Eq. (6.1) with $aN = 1$. It follows that

$$(ru')^2 = 2N\left(2N - \frac{1}{\varepsilon^2}\left[\frac{e^u}{(1+e^u)^2}\right]^{\frac{1}{N}}\right).$$

If $S^0 \neq \emptyset$, ε must satisfy $\varepsilon^2 = (2N4^{\frac{1}{N}})^{-1}$. In other words, if $\varepsilon^2 \neq (2N4^{\frac{1}{N}})^{-1}$, $S^0 = \emptyset$ and hence $S^- = \emptyset$. The remaining part comes from the super/subsolution argument as in Proposition 3.3. Consequently, $S^+ = \mathbb{R}$ and $\beta(s) = 0$. □

Proof of Theorem 6.4. It is easy to see (6.10). We observe that if $N = 0$, then $u(r,s)$ is a nontopological solution of type I if and only if $-u(r,-s)$ is a nontopological solution of type II. Thus, it is enough to show the properties of $\beta(s)$ for only $s < 0$. We have the following Pohozaev-type identities:

$$\begin{cases} \beta^2(s) = \dfrac{4}{\varepsilon^2 a}\displaystyle\int_0^\infty r\left[\frac{e^u}{(1+e^u)^2}\right]^a \mathrm{d}r, \\ \beta(s)\Big(\beta(s) - \dfrac{4}{a}\Big) = \dfrac{4}{\varepsilon^2 a}\displaystyle\int_0^\infty r\left[\frac{e^u}{(1+e^u)^2}\right]^a \frac{2e^u}{1+e^u}\mathrm{d}r. \end{cases}$$

Then, one can check that $\beta(s) > 4/a$ for $s < 0$. We want to show that $\beta : (-\infty, 0) \to (4/a, \infty)$ is bijective. First, we note that

$$\liminf_{s\to 0} \beta(s)\Big(\beta(s) - \frac{4}{a}\Big) = \liminf_{s\to 0} \frac{4}{\varepsilon^2 a}\int_0^\infty r\Big[\frac{e^u}{(1+e^u)^2}\Big]^a \frac{2e^u}{1+e^u}\mathrm{d}r$$
$$\geq \frac{4}{\varepsilon^2 a}\int_0^\infty r \liminf_{s\to 0}\Big(\Big[\frac{e^u}{(1+e^u)^2}\Big]^a \frac{2e^u}{1+e^u}\Big)\mathrm{d}r = \infty,$$

which implies that $\lim_{s\nearrow 0}\beta(s) = \infty$. We also derive that for $s < 0$,

$$\beta(s)\Big(\beta(s) - \frac{4}{a}\Big) = \frac{4}{\varepsilon^2 a}\int_0^\infty r\Big[\frac{e^u}{(1+e^u)^2}\Big]^a \frac{2e^u}{1+e^u}\mathrm{d}r$$
$$\leq \frac{8}{\varepsilon^2 a}\cdot e^s \int_0^\infty r\Big[\frac{e^u}{(1+e^u)^2}\Big]^a \mathrm{d}r \leq 2e^s\cdot\beta^2(s).$$

Thus, $\beta(s) \to 4/a$ as $s \to -\infty$. Therefore, β is onto. It remains to show that $\beta'(s) \neq 0$ for $s \neq 0$. The proof of this part is exactly the same as in the proof of Proposition 2.1 and we skip it. This completes the proof. □

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