



Embeddings, Immersions and the Bartnik Quasi-Local Mass Conjectures

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Abstract. Given a Riemannian 3-ball (\bar{B}, g) of nonnegative scalar curvature, Bartnik conjectured that (\bar{B}, g) admits an asymptotically flat (AF) extension (without horizons) of the least possible ADM mass and that such a mass minimizer is an AF solution to the static vacuum Einstein equations, uniquely determined by natural geometric conditions on the boundary data of (\bar{B}, g) . We prove the validity of the second statement, i.e., such mass minimizers, if they exist, are indeed AF solutions of the static vacuum equations. On the other hand, we prove that the first statement is not true in general; there is a rather large class of bodies (\bar{B}, g) for which a minimal mass extension does not exist.

1. Introduction

A fundamental problem in general relativity is the formulation of a “suitable” definition of quasi-local mass (cf. [49, Problem 1]). To motivate this concept, consider for instance a time-symmetric, asymptotically flat (AF) initial data set (M, g) without boundary for the Einstein equations, i.e., a Riemannian 3-manifold viewed as a totally geodesic space-like hypersurface in a Lorentzian $(3+1)$ -dimensional space-time. Assuming the space-time obeys the dominant energy condition, the submanifold (M, g) has nonnegative scalar curvature. The quasi-local mass of a compact region $\Omega \subset (M, g)$ should be a real number that represents the mass contained within Ω .

Many definitions of quasi-local mass have been put forth in the last several decades, though we make no attempt here to give a comprehensive history; see [55] for an excellent review. Some of the “classical” examples include the Hawking mass [26], the Brown–York mass [16] and the Bartnik mass [7]. More recently, Wang–Yau proposed a very interesting definition that generalizes the approach of Brown–York [57].

In this paper, we are interested in the Bartnik mass, whose setup we now recall. Let Ω be a smooth 3-manifold, with boundary, diffeomorphic to

the closed 3-ball \bar{B} in \mathbb{R}^3 , and let g_Ω be a Riemannian metric on Ω with nonnegative scalar curvature. The Bartnik mass was originally defined as

$$m_B(\Omega, g_\Omega) = \inf_g \{m_{\text{ADM}}(g)\}, \tag{1.1}$$

where the infimum is taken over the set of smooth AF metrics g on \mathbb{R}^3 such that (Ω, g_Ω) embeds isometrically into (\mathbb{R}^3, g) , and (\mathbb{R}^3, g) has nonnegative scalar curvature and contains no horizons [7]. (Note that a smooth AF 3-manifold of nonnegative scalar curvature with no horizons is diffeomorphic to \mathbb{R}^3 [43].) Bartnik defined a horizon to be a stable minimal 2-sphere, but a number of variants have since been considered in the literature. Among these, we will take a horizon to be an immersed compact minimal surface that surrounds Ω ; this choice is discussed further in Sect. 2.

The Bartnik mass satisfies many of the generally desired properties of a quasi-local mass (cf. [7]). For instance, $m_B(\Omega, g_\Omega)$ is nonnegative, by the positive mass theorem [52, 58]. Furthermore, if (Ω, g_Ω) is isometric to a smooth region in Euclidean space $(\mathbb{R}^3, g_{\text{Eucl}})$, then $m_B(\Omega, g_\Omega)$ vanishes. Bartnik conjectured that the converse holds (“strict positivity of m_B ”), i.e., if $m_B(\Omega, g_\Omega) = 0$, then (Ω, g_Ω) is a Euclidean region. A key result of Huisken and Ilmanen [28] shows that if $m_B(\Omega, g_\Omega) = 0$, then (Ω, g_Ω) is *locally flat*, i.e., locally isometric to Euclidean space (although this result applies to a slightly different definition of Bartnik mass). The Bartnik mass also enjoys monotonicity (i.e., a region contained in (Ω, g_Ω) cannot have a greater value of m_B ; this follows from the definition), and the Bartnik mass limits to the ADM mass for an exhausting sequence of large balls in an AF manifold of nonnegative scalar curvature [28]. The most fundamental open questions regarding the Bartnik mass are to determine under which general conditions the infimum in (1.1) is achieved, to understand the structure of the space of such minimizers and to describe the behavior of the corresponding mass functional on the space of minimizers. Before proceeding further, we recast the Bartnik mass in a slightly different manner, by focusing on the role played by the boundary geometry on the two sides of $\partial\Omega$.

For a pair (Ω, g_Ω) as above, let $\gamma = g_\Omega|_{T(\partial\Omega)}$ be the induced metric on $\partial\Omega \cong S^2$, and let H be the mean curvature of $\partial\Omega$, (with respect to the unit outward normal, i.e., positive for round spheres in \mathbb{R}^3). The pair (γ, H) will be called the (geometric) *Bartnik boundary data* of (Ω, g_Ω) . More generally, any pair (γ, H) , where γ is a smooth Riemannian metric on S^2 and H is a smooth function on S^2 , will be called Bartnik boundary data.

Bartnik pointed out that a minimizer g of (1.1) would only be expected to be Lipschitz along the “seam” $\partial\Omega$, obeying the boundary conditions [9, 10]

$$\gamma_{\partial\Omega} = \gamma_{\partial M}, \quad H_{\partial\Omega} = H_{\partial M}, \tag{1.2}$$

where M is the closure of the complement of the embedded image of Ω in \mathbb{R}^3 and $H_{\partial M}$ is the mean curvature of ∂M with respect to the unit normal pointing into M . The significance of matching the mean curvatures on both sides is that it assures the scalar curvature is distributionally nonnegative across the seam. The scalar curvature is also well known to be distributionally nonnegative if

$$\gamma_{\partial\Omega} = \gamma_{\partial M}, \quad H_{\partial\Omega} \geq H_{\partial M}, \quad (1.3)$$

are satisfied; we discuss this point further in Remark 2.9. The boundary condition (1.3) was also considered by Miao [44] and Shi and Tam [53]. Thus, we consider the following reformulation of the Bartnik mass. Fix M as a smooth manifold with boundary diffeomorphic to the closure of $\mathbb{R}^3 \setminus \bar{B}$, and consider the space $\mathcal{P}(M)$ of smooth, AF Riemannian metrics g on M (see (2.1)) with nonnegative scalar curvature, with $\mathring{\mathcal{P}}(M)$ being the subset such that (M, g) contains no horizons (defined as above: immersed compact minimal surfaces that surround Ω). We call $g \in \mathcal{P}(M)$ an *admissible extension* of a region (Ω, g_Ω) as above if (1.3) holds. We take as our definition of the Bartnik mass:

$$m_B(\Omega, g_\Omega) = \inf\{m_{\text{ADM}}(g) : g \in \mathring{\mathcal{P}}(M) \text{ is an admissible extension of } (\Omega, g_\Omega)\}. \quad (1.4)$$

One might also consider the mass defined by the equality condition (1.2). Both of these versions have previously appeared in the literature. For further discussion on the numerous variations in the definition of Bartnik mass, and some progress on reconciling them, see [32, 42].

These three definitions, based on (1.1)–(1.3), all require a precise choice among the various possible definitions of horizon. A major reason a horizon is defined here to be a surrounding minimal surface (as opposed to an arbitrary minimal surface in M) is that $\mathring{\mathcal{P}}(M)$ is then open in $\mathcal{P}(M)$, cf. Lemma 2.1, so that this condition is stable. (This is unknown for other definitions of the horizon condition.)

Regarding then the boundary conditions (1.1)–(1.3) themselves, we prove in Theorem 2.10 that if a minimizer subject to (1.3) exists in $\mathring{\mathcal{P}}(M)$, it necessarily satisfies (1.2) (cf. also prior work of Miao on this issue [45]). This result strongly suggests the two definitions of Bartnik mass based on (1.2) and (1.3) are equivalent and also very likely equivalent to (1.1), cf. Remark 2.9 and [32, 42]. Henceforth, we adopt (1.4) as the definition of the Bartnik mass.

The following three conjectures are due to Bartnik; they are discussed in [7, 10] and in most detail in [9].

Conjecture I. *Any region (Ω, g_Ω) with $\Omega \cong \bar{B}$, $H_{\partial\Omega} > 0$ and g_Ω of nonnegative scalar curvature admits an admissible extension in $\mathring{\mathcal{P}}(M)$.*

Thus, conjecturally, any metric of nonnegative scalar curvature on a ball, with positive boundary mean curvature, can be extended to an AF manifold with nonnegative scalar curvature, where the extension has no horizons and (1.3) is satisfied. (The hypothesis of positive boundary mean curvature is imposed because if, for instance, $H_{\partial\Omega}$ were negative everywhere, then any AF extension would contain a horizon.) This general extension conjecture essentially appears in [9, Problem 1]. It implies that any region (Ω, g_Ω) as above has a well-defined Bartnik mass (1.4).

Conjecture I is known as the Bartnik extension conjecture and remains open in general (even allowing extensions in $\mathcal{P}(M)$). Further discussion of the conjecture and some partial results are given in Sect. 3.

Conjecture II. *For any region (Ω, g_Ω) with $\Omega \cong \bar{B}$, $H_{\partial\Omega} > 0$ and g_Ω of nonnegative scalar curvature, there exists an admissible extension $g \in \mathring{\mathcal{P}}(M)$ realizing the Bartnik mass (1.4). Moreover, g satisfies the boundary conditions (1.2).*

Conjecture II is known as the Bartnik mass-minimization conjecture. Bartnik [9, 11] developed a heuristic program suggesting that a metric g realizing the Bartnik mass (1.4) on M yields an asymptotically flat (AF) solution of the static vacuum Einstein equations, i.e., there is a potential function $u : M \rightarrow \mathbb{R}$, with $u \rightarrow 1$ at infinity, such that

$$u \operatorname{Ric}_g = D^2u, \quad \Delta u = 0. \tag{1.5}$$

(Moreover, the AF potential function u is expected to be positive). This has been partially verified, using quite different methods, by Corvino [21, 22], cf. Remark 2.11 for further discussion. We give a full proof of this proposal, thus completely implementing Bartnik’s program:

Theorem 1.1. *For a region (Ω, g_Ω) as above with $H_{\partial\Omega} > 0$, a metric g minimizing the Bartnik mass (1.4) admits an AF potential function $u > 0$ such that (g, u) is an AF solution of the static vacuum Einstein Eqs. (1.5). Moreover, such a minimizer g satisfies (1.2).*

We refer to Theorem 2.8 and Theorem 2.10 for further details.

Conjecture III. *For any geometric Bartnik boundary data (γ, H) on S^2 , with $H > 0$, there exist a unique extension $g \in \mathring{\mathcal{P}}(M)$ of (γ, H) and a function $u > 0$ with $u \rightarrow 1$ at infinity, such that the pair (g, u) is an AF solution of the static vacuum Einstein Eqs. (1.5).*

Conjecture III is known as the Bartnik static metric extension conjecture.

In addition to the horizon issue, the assumption $H > 0$ in Conjectures II and III is made due to the black hole uniqueness theorem, cf. [17, 29], and also [46]. Namely, the data $(\gamma, 0)$ are boundary data of a static vacuum solution only for γ a round, constant curvature metric on S^2 , realized by the family of Schwarzschild metrics. Thus, Conjectures II and III are well known to fail for $H = 0$ boundary data.

It is clear that Conjectures II and III each imply Conjecture I. Using Theorem 1.1, Conjecture II implies the existence part of Conjecture III for the special case of boundary data (γ, H) obtained from a region Ω with nonnegative scalar curvature and $H > 0$. On the other hand, even for this special class of boundary data, Conjecture III does not imply Conjecture II, since all mass-minimizing sequences for a given body Ω may fail to converge to a limit. As discussed in Proposition 2.7, the static vacuum solutions given in Conjecture III are precisely the critical points of the ADM mass m_{ADM} (with fixed boundary conditions), but it is not clear that these are minimizers. If Conjecture II holds, so minimizers exist, then the uniqueness of Conjecture III would imply that all critical points are minimizers.

Given this background, the main purpose of this work is to prove that Conjecture II is not true in the generality stated, so that further hypotheses

are required to maintain its validity (see the discussion at the end of Sect. 5). As discussed below, similar remarks apply to Conjecture III. This failure is related to the degeneration of the exterior manifold-with-boundary structure on M , given control on the boundary data in (1.2) or (1.3). This is most simply described in the passage from embedded spheres to immersed spheres in \mathbb{R}^3 , which we now discuss.

Let $\text{Imm}(\bar{B}, \mathbb{R}^3)$ be the space of smooth immersions

$$F: \bar{B} \rightarrow \mathbb{R}^3,$$

of the closed 3-ball $\bar{B} \subset \mathbb{R}^3$; thus, F extends to an immersion of an open neighborhood of \bar{B} . Let $\mathcal{F} \subset \text{Imm}(\bar{B}, \mathbb{R}^3)$ denote the subspace of immersions that restrict to embeddings on the interior of \bar{B} and on which the self-intersection set Z of $F|_{\partial\bar{B}} = F|_{S^2}$ consists of a finite, nonzero number of double points. Thus, there is a finite set $Z \subset S^2$ given as a disjoint union $(\cup z_i) \cup (\cup z'_i)$ such that F is injective on $\bar{B} \setminus Z$, $F(z_i) = F(z'_i)$ for each i , and $F(z_i) \neq F(z_j)$ for $i \neq j$. For $F \in \mathcal{F}$, the set $F(\bar{B})$ is not a smooth region in \mathbb{R}^3 . However, the pull-back $(\bar{B}, F^*(g_{\text{Eucl}}))$ is obviously a smooth, locally flat Riemannian manifold with boundary. It is easy to see that \mathcal{F} provides a large, infinite-dimensional space of such locally flat domains. We also remark that there is a large class of immersions $F \in \mathcal{F}$ such that $(\bar{B}, F^*(g_{\text{Eucl}}))$ has positive boundary mean curvature (consider, for example, a torus in \mathbb{R}^3 with positive mean curvature, appropriately cut with spherical caps glued back in, tangent at their respective poles — see [4, Figure 1]).

Theorem 1.2. *Conjecture II is false for any region $(\bar{B}, F^*(g_{\text{Eucl}}))$ for $F \in \mathcal{F}$ as above. In particular, there is no admissible extension of $(\bar{B}, F^*(g_{\text{Eucl}}))$ whose ADM mass attains the Bartnik mass (which equals zero).*

In particular, this also shows that strict positivity of the Bartnik mass fails, i.e., the result of Huisken and Ilmanen [28] that $m_B(\Omega, g_\Omega) = 0$ implies local flatness is optimal. This is because the proof of Theorem 1.2 will show that $(\bar{B}, F^*(g_{\text{Eucl}}))$ has zero Bartnik mass and does not embed isometrically in Euclidean 3-space, cf. also Remark 4.4.

Note that for $F \in \mathcal{F}$, there is a sequence of embeddings F_i of the closed 3-ball into \mathbb{R}^3 with $F_i \rightarrow F$ smoothly, with the corresponding embedded spheres $F_i(S^2)$ converging smoothly to an immersed sphere. In particular, the class \mathcal{F} of immersions is at the boundary of the space of embeddings. Of course, Conjecture II holds for regions Ω isometrically embedded in \mathbb{R}^3 .

As noted above, the pulled-back Euclidean metrics $F_i^*(g_{\text{Eucl}})$ converge smoothly to a limiting smooth flat metric on the abstract 3-ball \bar{B} with limit boundary data (γ, H) . However, the flat metrics on the complementary manifolds $M_i = \mathbb{R}^3 \setminus F_i(\bar{B})$ degenerate in the limit. It is a priori possible that there is a distinct sequence of (nonflat) admissible extensions g_i of the boundary data (γ, H) with $m_{\text{ADM}}(g_i)$ converging to the infimum of the mass of such extensions, which do not degenerate and so give a limit realizing the Bartnik mass. The main content of Theorem 1.2 is to prove that in fact this does not occur.

We conjecture that this phenomenon is quite general, i.e., that Conjecture II is false for any domain $(\tilde{B}, F^*(g_{\text{Eucl}}))$ obtained from an immersion $F : \tilde{B} \rightarrow \mathbb{R}^3$ that is not an embedding (even if F is not at the boundary of the space of embeddings), cf. Conjecture 4.11.

A version of the discussion above also holds with respect to Conjecture III. Namely, let $\mathcal{E}^{m,\alpha}$ be the moduli space of AF static vacuum solutions (g, u) , $u > 0$, on $M = \mathbb{R}^3 \setminus B$. The moduli space $\mathcal{E}^{m,\alpha}$ is the space of all static vacuum metrics (g, u) which are $C^{m,\alpha}$ smooth up to ∂M , modulo the action of the $C^{m+1,\alpha}$ diffeomorphisms $\text{Diff}_1^{m+1,\alpha}(M)$ of M equal to the identity on ∂M (and asymptotic to the identity at infinity). It is proved in [2] (cf. also [4]) that $\mathcal{E}^{m,\alpha}$ is a smooth Banach manifold, and moreover, the map to Bartnik boundary data

$$\begin{aligned} \Pi_B : \mathcal{E}^{m,\alpha} &\rightarrow \text{Met}^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2), \\ \Pi_B(g, u) &= (\gamma, H), \end{aligned} \tag{1.6}$$

is a smooth Fredholm map, of Fredholm index 0. Here, $\text{Met}^{m,\alpha}(S^2)$ is the space of $C^{m,\alpha}$ Riemannian metrics on S^2 with the $C^{m,\alpha}$ topology.

Now consider the map Π_B^+ , the restriction of Π_B to the open subspace $\mathcal{E}_+^{m,\alpha}$ of static vacuum metrics with $H > 0$ at ∂M :

$$\Pi_B^+ : \mathcal{E}_+^{m,\alpha} \rightarrow \text{Met}^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(M). \tag{1.7}$$

Conjecture III is equivalent to the statement that Π_B^+ in (1.7) is a bijection. However, it is proved in [4] that Π_B^+ is not a homeomorphism; in fact, the inverse map to Π_B^+ , if it exists, is not continuous in general. The failure of the homeomorphism property is closely related to the behavior of Π_B^+ at the boundary of the space of (flat) embeddings within the larger space of immersions, discussed above in connection with Conjecture II.

Theorem 1.2 shows that a major obstacle in establishing the validity of Conjecture II is controlling the behavior of mass-minimizing sequences arbitrarily close to the boundary $\partial M = \partial\Omega$, given control on the Bartnik boundary data (γ, H) , so that the manifold-with-boundary structure of M does not degenerate. A similar difficulty arises in proving Conjecture III; for example, it is much simpler to control the behavior of sequences of static vacuum solutions in the interior of M (away from ∂M) compared with controlling the behavior near the boundary; see, for example, the analysis in [1]. We expect a similar phenomenon for more general mass-minimizing sequences.

In contrast to the negative results above on Conjectures II and III, we present positive evidence for the validity of Conjecture I in Sect. 3. We prove in Proposition 3.2 that if the boundary data (γ, H) admit an extension to an AF metric of nonnegative scalar curvature, then so do (γ, \tilde{H}) , for any $\tilde{H} \geq H$. Combining this with previous results in [3,38] leads to the verification of Conjecture I for a wide variety of boundary data (γ, H) , although without addressing the issue of horizons.

The contents of the paper are briefly as follows. In Sect. 2, we discuss the various possible definitions of horizon as well as the boundary conditions (1.2)–(1.3), and the relations of minimizers of the Bartnik mass with the static vacuum Einstein equations. The main results are Theorems 2.8 and 2.10, which imply Theorem 1.1. In Sect. 3, we discuss Conjecture I and present new evidence for its validity in general. Section 4 is devoted to the proof of Theorem 1.2, while Conjecture III is discussed further in Sect. 5. We note that although the topics of these sections are of course inherently related, the sections themselves are essentially independent of each other.

2. Mass Minimizers and the Static Vacuum Einstein Equations

In this section, we discuss relations between the various notions of Bartnik mass from Introduction and their relations with the static vacuum Einstein equations.

Starting with an idea suggested by Brill et al. [15], Bartnik [9, 11] presented a heuristic argument that critical points of the mass on the space of solutions of the (time-symmetric) four-dimensional vacuum Einstein constraint equations, with fixed boundary data (γ, H) , should be given by solutions of the static vacuum Einstein equations. This strongly suggested that minimizers of the Bartnik mass (with respect to a suitable horizon condition) should then also be static vacuum Einstein solutions. Some recent work along these lines has also been carried out by McCormick [40, 41].

The main results of this section are a full proof of Bartnik's proposal for Bartnik mass minimizers, cf. Theorems 2.8 and 2.10. In addition, Theorem 2.10 shows that a Bartnik mass minimizer defined according to (1.3) actually satisfies (1.2), leading to a corresponding strong monotonicity result in Corollary 2.12. We refer the reader to Remark 2.11 for relevant prior results.

Throughout, M will be a smooth 3-manifold with boundary, diffeomorphic to $\mathbb{R}^3 \setminus B$, where B is an open ball. A $C^{m,\alpha}$ Riemannian metric g on M (i.e., $C^{m,\alpha}$ up to and on ∂M) will be called asymptotically flat (AF) if

$$g_{ij} - \delta_{ij} \in C_\delta^{m,\alpha}(M), \quad (2.1)$$

where $C_\delta^{m,\alpha}(M)$ is the weighted Hölder space of functions on M that decay to 0 at a rate $r^{-\delta}$ with k th derivatives decaying at the rate $r^{-\delta-k}$, $k \leq m$, and with appropriately weighted Hölder α -seminorms of the m th derivatives bounded, cf. [19] for example. The rate δ is fixed throughout and assumed to satisfy

$$\frac{1}{2} < \delta < 1.$$

We also fix any $m \geq 3$ and $\alpha \in (0, 1)$.

Given a fixed δ as above, let $\mathcal{P}^{m,\alpha}(M) = \mathcal{P}_\delta^{m,\alpha}(M)$ be the space of AF metrics g on M with nonnegative scalar curvature $s = s_g$. Recall that the ADM mass m_{ADM} of $g \in \mathcal{P}^{m,\alpha}(M)$ is only defined [6] for metrics with

$$s \in L^1(M). \tag{2.2}$$

The Bartnik mass (1.4) is then obtained by minimizing the ADM mass on $\mathcal{P}^{m,\alpha}(M)$ subject to the boundary conditions (1.3) on ∂M and subject to the no-horizon condition. Alternatively, one might consider minimizing the mass subject to the stronger condition (1.2).

As mentioned in Introduction, there are several notions of horizon appearing in the literature without a current general consensus. The most strict condition is that (M, g) has no immersed compact minimal surfaces; let m_B^s denote the corresponding Bartnik mass. Variations of this condition such as no stable compact minimal surfaces or no embedded compact minimal surfaces have also been considered. In some cases, the minimal surfaces are required to be topological spheres.

A somewhat weaker condition (and the one that we adopt) is that there are no immersed compact minimal surfaces surrounding ∂M in M , i.e., any path from ∂M to infinity must pass through the surface. (Again one might consider variations such as no stable or no embedded surrounding compact minimal surfaces). Let m_B^w denote the corresponding Bartnik mass; then, one clearly has

$$m_B^w \leq m_B^s. \tag{2.3}$$

The same relation holds with respect to the weaker and stronger boundary conditions (1.2) and (1.3), respectively.

Moreover, a third definition was suggested by Bray [14], requiring that ∂M be outer-minimizing in (M, g) . This version of the mass will be discussed briefly in Sect. 5, but not used before then.

One of the main reasons for preferring the weaker condition is the following stability result. Let

$$\mathring{\mathcal{P}}^{m,\alpha}(M) \subset \mathcal{P}^{m,\alpha}(M) \tag{2.4}$$

be the subset of metrics that have no immersed minimal surface surrounding ∂M .

Lemma 2.1. $\mathring{\mathcal{P}}^{m,\alpha}(M)$ is an open subset of $\mathcal{P}^{m,\alpha}(M)$.

Proof. We show that the complement is closed. Let $\{g_i\}$ be a sequence in $\mathcal{P}^{m,\alpha}(M) \setminus \mathring{\mathcal{P}}^{m,\alpha}(M)$ converging to some $g \in \mathcal{P}^{m,\alpha}(M)$. Each g_i is an AF metric on M such that (M, g_i) contains an immersed minimal surface Σ_i surrounding ∂M . The unbounded component \widehat{M}_i of $M \setminus \Sigma_i$ is then AF; one may then minimize the area functional for surfaces in \widehat{M}_i homologous to infinity. Since $\partial \widehat{M}_i$ together with a large sphere S near infinity (independent of i) serve as well-defined barriers, it follows from well-known results of Meeks et al. [43] that \widehat{M}_i contains a minimal surface $\widehat{\Sigma}_i$ that has the least area among surfaces surrounding $\partial \widehat{M}_i$.

In particular, $\widehat{\Sigma}_i$ is stable. Further, the area of $\widehat{\Sigma}_i$ with respect to g_i is uniformly bounded, since $\widehat{\Sigma}_i$ has less g_i -area than S , and $\text{area}_{g_i}(S) \rightarrow \text{area}_g(S)$. Using the well-known curvature estimates of Schoen, it is then standard, (cf. [20] for example) that a subsequence of $\widehat{\Sigma}_i$ converges to a

stable minimal surface Σ in (M, g) . Clearly, Σ surrounds ∂M , so that $g \in \mathcal{P}^{m, \alpha}(M) \setminus \mathring{\mathcal{P}}^{m, \alpha}(M)$. □

It is mainly for this stability behavior that we choose a horizon to be a surrounding minimal surface. (Such stability is unknown for other definitions of horizon in an AF manifold with boundary.) A further reason is that static vacuum Einstein metrics have no horizons in this sense (except for Schwarzschild metrics), by a result of Miao [46]. (This is again unknown for general minimal surfaces). We prove Theorem 1.2 for the version of the Bartnik mass we adopt (i.e., for horizons as surrounding minimal surfaces and for the weaker boundary condition (1.3)). In Remark 4.10, we describe a possible modification to the proof that may allow for the stronger boundary condition (1.2). Theorem 2.10 strongly suggests that with respect to $\mathring{\mathcal{P}}^{m, \alpha}(M)$ as in (2.4), the boundary conditions (1.2) and (1.3) give equal Bartnik masses; this is less clear for the stronger definition m_B^s .

To summarize using the current notation, as in (1.4), we set

$$m_B(\Omega, g_\Omega) = m_B(\gamma, H) = \inf \{ m_{\text{ADM}}(g) : g \in \mathring{\mathcal{P}}^{m, \alpha}(M), g|_{\partial M} = \gamma, H_{\partial M} \leq H \}. \tag{2.5}$$

Note that an immediate consequence of the definition is the following (weak) inverse monotonicity property: If $H' \leq H$, then

$$m_B(\gamma, H) \leq m_B(\gamma, H'). \tag{2.6}$$

A strong monotonicity will be proved in Corollary 2.12 below.

Returning to the discussion prior to (2.2), let $\mathcal{S}^{m, \alpha}(M) = \mathcal{S}_\delta^{m, \alpha}(M)$ be the space of pairs (g, u) , with g a $C_\delta^{m, \alpha}$ AF metric on M and u an AF function, i.e., $u - 1 \in C_\delta^{m, \alpha}(M)$, so that $u \rightarrow 1$ at infinity. We write $\mathcal{S}^{m, \alpha}(M) = \text{Met}_{AF}^{m, \alpha}(M) \times C_{AF}^{m, \alpha}(M)$. Data (g, u) for which $u > 0$ correspond to AF Lorentzian metrics on $\mathcal{M} = \mathbb{R} \times M$ of the form

$$g_{\mathcal{M}} = -u^2 dt^2 + g. \tag{2.7}$$

Metrics of the form (2.7) (or such pairs (g, u)) will be referred to as *static*; this is not to be confused with other notions of static (e.g., static vacuum or Corvino’s definition of static in [21]).

Clearly, $\mathcal{S}^{m, \alpha}(M) = \text{Met}_{AF}^{m, \alpha}(M) \times C_{AF}^{m, \alpha}(M)$ is a smooth Banach manifold. Let $\mathcal{S}_+^{m, \alpha}(M) \subset \mathcal{S}^{m, \alpha}(M)$ be the subset such that

$$s_g \geq 0.$$

Thus, $\mathcal{S}_+^{m, \alpha}(M) = \mathcal{P}^{m, \alpha}(M) \times C_{AF}^{m, \alpha}(M)$. Note that if $(g, u) \in \partial(\mathcal{S}_+^{m, \alpha}(M))$, then the scalar curvature of g is nonnegative and vanishes at some point in M . We point out that the condition (2.2) is not assumed a priori on $\mathcal{S}_+^{m, \alpha}$.

Consider the Regge–Teitelboim Hamiltonian [51] in this setting:

$$\begin{aligned} \mathcal{H} : \mathcal{S}^{m, \alpha}(M) &\rightarrow \mathbb{R}, \\ \mathcal{H}(g, u) &= \int_M u s dv_g - 16\pi m_{\text{ADM}}(g), \end{aligned} \tag{2.8}$$

where $s = s_g$ is the scalar curvature of g . If $u > 0$, note that since $s_{g_{\mathcal{M}}} = s_g - 2\frac{\Delta u}{u}$ and $dv_{g_{\mathcal{M}}} = u dv_g$, the first term gives the Einstein–Hilbert action on the 4-manifold \mathcal{M} modulo a divergence term (namely $-2\Delta u$). The reason for this modification of the Einstein–Hilbert action is to obtain a well-defined variational problem for the ADM Hamiltonian; we refer to [51] for details.

If $s \notin L^1(M)$, then the individual terms in (2.8) are ill-defined although the combination is well defined. Explicitly, following [11], (2.8) may be rewritten in the form

$$\mathcal{H}(g, u) = \int_M (u - 1) s dv_g - \int_M (\mathcal{R}_0 - s) dv_g, \tag{2.9}$$

where $u - 1 \in C^{m,\alpha}_\delta(M)$ and \mathcal{R}_0 is the bulk integral for the mass, given by

$$\mathcal{R}_0 = (\delta_0 \delta_0 g - \Delta_{g_0}(\text{tr}_{g_0} g)) \frac{dv_{g_0}}{dv_g},$$

where g_0 is any background metric agreeing with g near ∂M and is Euclidean outside a compact set, and δ_0 is the corresponding divergence. By the divergence theorem, $\int_M \mathcal{R}_0 dv_g = 16\pi m_{\text{ADM}}(g)$, when the ADM mass is defined, cf. [11]. Thus, the Regge–Teitelboim Hamiltonian (2.9) is well defined and a smooth functional on the full Banach manifold $\mathcal{S}^{m,\alpha}(M)$. Of course, the definitions (2.8) and (2.9) agree when $s \in L^1(M)$.

Let

$$S^*u = D^2u - (\Delta u)g - u \text{Ric}, \tag{2.10}$$

be the formal L^2 -adjoint of the linearization $s' = Ds_g$ of the scalar curvature. The static vacuum Einstein Eqs. (1.5) are equivalent to the following system for $(g, u) \in \mathcal{S}^{m,\alpha}(M)$:

$$S^*u = 0, \quad \Delta u = 0. \tag{2.11}$$

Note that static vacuum metrics are necessarily scalar flat, $s = 0$, and are also necessarily in $\mathring{\mathcal{P}}^{m,\alpha}(M)$, i.e., as noted above, have no horizons (except for Schwarzschild metrics), cf. [46]. The relation $\Delta u = 0$ in (2.11) follows from $S^*u = 0$ by taking the trace, and using the Bianchi identity to establish constant scalar curvature; then, asymptotic flatness gives $s = 0$ and hence $\Delta u = 0$.

The following result is essentially classical and is a version of results proved in [11, 23, 51]; a simple proof in this notation is also given in [4]. Let N be the unit normal at ∂M pointing into M , and let A be the second fundamental form of ∂M in M .

Proposition 2.2. *The L^2 -gradient of \mathcal{H} on $\mathcal{S}^{m,\alpha}(M)$ is given by*

$$\nabla \mathcal{H} = (S^*u + \frac{1}{2}usg, s, uA - N(u)\gamma, 2u) \tag{2.12}$$

in the sense that, if (h, u') is any variation of (g, u) inducing the variation (h^T, H'_h) of boundary data (γ, H) , then

$$\begin{aligned} d\mathcal{H}_{(g,u)}(h, u', h^T, H'_h) &= \int_M [\langle S^*u + \frac{1}{2}usg, h \rangle + su'] \\ &\quad + \int_{\partial M} [\langle uA - N(u)\gamma, h^T \rangle + 2uH'_h]. \end{aligned} \tag{2.13}$$

(The volume forms associated with the metrics on M and ∂M are omitted, to simplify the notation).

Let $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ be the space of static metrics with fixed Bartnik boundary conditions; thus, $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ consists of pairs $(g, u) \in \mathcal{S}^{m,\alpha}(M)$ with the metric g having fixed boundary data equal to (γ, H) at ∂M . It is straightforward (using the implicit function theorem) to show that $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ is a smooth, closed Banach submanifold of $\mathcal{S}^{m,\alpha}(M)$, for all choices of $(\gamma, H) \in \text{Met}^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2)$. Tangent vectors to $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ are variations (h, u') of (g, u) such that $(h^T, H'_h) = (0, 0)$ at ∂M , where h^T is the restriction of h to $T(\partial M)$ and H'_h is the variation of the mean curvature in the direction of h .

Proposition 2.2 thus shows that critical points of the Hamiltonian \mathcal{H} on $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ are given exactly by static vacuum Einstein metrics realizing the given boundary data (γ, H) .

In contrast, we show next that there are no critical points of the mass

$$m_{\text{ADM}} : \mathcal{D} \subset \mathcal{S}_{(\gamma,H)}^{m,\alpha}(M) \rightarrow \mathbb{R},$$

where \mathcal{D} is the domain on which m_{ADM} is well defined, i.e., the subset of $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ consisting of metrics g with integrable scalar curvature. Given H , let $\mathcal{S}_{(\gamma,H \leq)}^{m,\alpha}(M)$ be the space of static metrics with boundary metric γ and mean curvature $\leq H$ at ∂M .

Lemma 2.3. *For any $(g, u) \in \mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ for which m_{ADM} is defined, one has*

$$(Dm_{\text{ADM}})_g \neq 0. \tag{2.14}$$

(i.e., there exists an appropriate variation h of g with $(Dm_{\text{ADM}})_g(h) \neq 0$). If, in addition, $(g, u) \in \mathcal{S}_+^{m,\alpha}(M)$, then $(Dm_{\text{ADM}})_g$ is nonvanishing in the directions of $\mathcal{S}_+^{m,\alpha}(M) \cap \mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ (again in the sense that there exists an appropriate variation). Furthermore, if $(g, u) \in \mathcal{S}_+^{m,\alpha}(M)$ and

$$s_g \neq 0,$$

then there is an infinitesimal deformation $(h, 0)$ of (g, u) in the direction of $\mathcal{S}_+^{m,\alpha}(M) \cap \mathcal{S}_{(\gamma,H \leq)}^{m,\alpha}(M)$ such that

$$(Dm_{\text{ADM}})_g(h) < 0, \tag{2.15}$$

so that there are metrics $g' \in \mathcal{S}_+^{m,\alpha}(M) \cap \mathcal{S}_{(\gamma,H \leq)}^{m,\alpha}(M)$ with $m_{\text{ADM}}(g') < m_{\text{ADM}}(g)$.

Proof. We use a well-known conformal argument, cf. [21] for example. Suppose g is an AF metric and $\tilde{g} = v^4 g$ is a conformal deformation of g , with $v > 0$ in $C_{AF}^{m,\alpha}$, so that \tilde{g} is AF. The scalar curvatures of \tilde{g} and g are related by

$$v^5 \tilde{s} = -8\Delta v + sv.$$

Suppose the ADM mass m of g is defined and that $\Delta v \in L^1(M)$. Then, the ADM mass \tilde{m} of \tilde{g} is also defined, and a well-known formula (cf. [44], eqn. (46)) relating m and \tilde{m} reads

$$\tilde{m} = m - \frac{1}{2\pi} \lim_{r \rightarrow \infty} \int_{S(r)} N(v) dA, \tag{2.16}$$

where N is the outward unit normal at the coordinate sphere $S(r)$, and dA is the induced area form on $S(r)$ with respect to g .

Let φ be a superharmonic function on (M, g) , $\Delta\varphi \leq 0$, with $\varphi = 0$ in a neighborhood of ∂M and with φ harmonic outside a large compact set, tending to a constant $-c < 0$ at infinity. It is easy to see such functions exist. Let $F_t : M \rightarrow M$ be a smooth family of diffeomorphisms equal to the identity near ∂M and equal to the map $x \rightarrow (1 - tc)^{-2}x$ near infinity with $F_0 = Id$. We apply (2.16) to the curve of metrics $g_t = F_t^*((1 + t\varphi)^4 g)$. (The diffeomorphisms are needed to put the curve g_t in the space $Met_{AF}^{m,\alpha}(M)$). Note that $s_g \geq 0$ implies $s_{g_t} \geq 0$.

Taking the derivative of (2.16) and using the divergence theorem gives, for r sufficiently large,

$$m'_h = (Dm_{\text{ADM}})_g(h) = -\frac{1}{2\pi} \int_{S(r)} N(\varphi) dA > 0,$$

for the variation $h = \partial_t g_t|_{t=0}$. (Note the diffeomorphisms F_t may be neglected in this calculation, since the ADM mass is diffeomorphism invariant). Since h preserves the boundary conditions, this proves the first two statements.

To prove the last statement, let $v \in C_{AF}^{m,\alpha}$ be the unique solution to the equation $-8\Delta v + s\rho v = 0$ with $v = 1$ on ∂M and $v \rightarrow 1$ at infinity, where $s \geq 0$, $s \not\equiv 0$ and $0 \leq \rho \leq 1$ is smooth and compactly supported, such that $\rho s \not\equiv 0$. In particular, $v \in C_{AF}^{m,\alpha}$. By the minimum principle, $v > 0$ on M , so that $\tilde{g} = v^4 g$ is well defined asymptotically flat, and $s_{\tilde{g}} \geq 0$. By the maximum principle, $v < 1$ in the interior of M (since ρs is not identically zero) and $N(v) < 0$ at ∂M . There is an $\varepsilon > 0$ such that the level set $v^{-1}(1 - \varepsilon)$ has a compact, regular component L that is a closed surface in the AF end of M , enclosing the support of ρ . Thus, v is g -harmonic outside of L , equaling $1 - \varepsilon$ on L and approaching 1 at infinity. By the Hopf maximum principle, $N(v) > 0$ along L . Applying the divergence theorem in the region outside of L , it follows that

$$\tilde{m} = m - \frac{1}{2\pi} \lim_{r \rightarrow \infty} \int_{S(r)} N(v) dA < m.$$

Moreover, one has $\tilde{H} = H + 4N(v) < H$ on ∂M , by the Hopf maximum principle.

One may also linearize this argument by choosing $v = v_t$ as above solving $-8\Delta v + ts\rho v = 0$, so that $-8\Delta v_t + sv_t = sv_t - tsv_t = sv_t(1 - t) \geq 0$, i.e., the conformally deformed metric has nonnegative scalar curvature. Taking the derivative at $t = 0$ gives the result. □

Lemma 2.3 shows the special role accorded to the scalar-flat metrics on M . In the context of the 4-metric (2.7) on \mathcal{M} , the equation

$$s = 0,$$

on (M, g) is exactly the set of vacuum Einstein constraint equations (since the second fundamental form K of M in \mathcal{M} vanishes). Let

$$\mathcal{C}^{m,\alpha} = \{(g, u) : s_g = 0\} \subset \mathcal{S}^{m,\alpha}(M).$$

be the space of solutions of the constraint equations on $\mathcal{S}^{m,\alpha}(M)$. Note that there is no condition on the lapse u (except $u - 1 \in C_\delta^{m,\alpha}$). Of course, $\mathcal{C}^{m,\alpha}$ is a (small) subset of the full boundary $\partial(\mathcal{S}_+^{m,\alpha}(M))$. Note also that since static vacuum metrics are scalar flat, any critical point (g, u) of \mathcal{H} on $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ must lie in $\mathcal{C}^{m,\alpha}$.

To proceed further, we need to examine the smoothness of the spaces $\mathcal{C}^{m,\alpha}$ and $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$, where the latter is the subset of $\mathcal{C}^{m,\alpha}$ consisting of pairs (g, u) where g induces Bartnik boundary data (γ, H) .

Proposition 2.4. *The scalar curvature map*

$$s : \mathcal{S}^{m,\alpha}(M) \rightarrow C_{\delta+2}^{m-2,\alpha}(M), \quad (g, u) \mapsto s(g)$$

is a smooth submersion at any $(g, u) \in \mathcal{S}^{m,\alpha}(M)$, i.e., the linearization Ds_g is surjective and its kernel splits. The same statement holds for the restricted map

$$s : \mathcal{S}_{(\gamma,H)}^{m,\alpha}(M) \rightarrow C_{\delta+2}^{m-2,\alpha}(M), \quad (g, u) \mapsto s(g). \tag{2.17}$$

Consequently, the spaces $\mathcal{C}^{m,\alpha}$ and $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$ (the latter if nonempty) are smooth Banach manifolds, (closed submanifolds of $\mathcal{S}^{m,\alpha}(M)$).

A similar result was proved by Bartnik [11, Theorem 3.7] for complete AF manifolds in a Hilbert space setting for the general constraint equations. The proof below is conceptually related. On the one hand, it is simpler than Bartnik’s since one only has to take account of the scalar constraint ($s = 0$); on the other hand, it is more difficult, due to the presence of boundary conditions.

Proof. We prove the second statement (i.e., for $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$), which implies the first (for $\mathcal{S}^{m,\alpha}(M)$). Note also that both statements are independent of u , so we may assume $u > 0$.

The proof proceeds (of course) by the implicit function theorem in Banach spaces. To apply this, one needs to prove that the linearization $s' = Ds_g$ in (2.17), i.e.,

$$s' : T\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M) \rightarrow TC_{\delta+2}^{m-2,\alpha}(M), \quad (h, u') \mapsto s'(h), \tag{2.18}$$

is surjective and the kernel $\text{Ker } s'$ splits as a subspace of $T\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ at (g, u) . The usual proofs of these properties for compact manifolds or complete AF manifolds, based either on conformal deformations, or on the structure of the formally elliptic operator $s' \circ (s')^*$, will not work in this setting due to the presence of the boundary conditions. We begin with the proof of surjectivity.

Consider the 4-manifold \mathcal{M} and static metrics $g_{\mathcal{M}}$ on \mathcal{M} , as in (2.7). The Ricci curvature of the metric $g_{\mathcal{M}}$, determined by (g, u) , is given by

$$\text{Ric}_{g_{\mathcal{M}}} = \text{Ric}_g - u^{-1}D^2u + u^{-1}\Delta u\omega \otimes \omega,$$

where $\omega = udt$ is a $g_{\mathcal{M}}$ -unit covector in the “vertical” direction and the first two terms are in the “horizontal” direction (tangent to M). Passing to the associated Einstein tensor $\text{Ric}_{g_{\mathcal{M}}} - \frac{sg_{\mathcal{M}}}{2}$ of $g_{\mathcal{M}}$ gives

$$E_{g_{\mathcal{M}}} = E_g - u^{-1}(D^2u - \Delta u g) + \frac{s}{2}\omega \otimes \omega, \tag{2.19}$$

where E_g is the Einstein tensor of g . Consider now the divergence-gauged Einstein operator at a background metric $\tilde{g}_{\mathcal{M}} \in \mathcal{S}^{m,\alpha}(M)$:

$$\begin{aligned} \Phi_{\tilde{g}_{\mathcal{M}}} &: \mathcal{S}_{\delta}^{m,\alpha}(M) \rightarrow \mathcal{S}_{\delta+2}^{m-2,\alpha}(M), \\ \Phi_{\tilde{g}_{\mathcal{M}}}(g_{\mathcal{M}}) &= E_{g_{\mathcal{M}}} + 2\delta_{g_{\mathcal{M}}}^* \delta_{\tilde{g}_{\mathcal{M}}}(g_{\mathcal{M}}), \end{aligned} \tag{2.20}$$

where $\delta_{\tilde{g}_{\mathcal{M}}}$ is the divergence operator with respect to $\tilde{g}_{\mathcal{M}}$ and $\delta_{g_{\mathcal{M}}}^*$ is the formal L^2 -adjoint of $\delta_{g_{\mathcal{M}}}$. (Here, we view $\mathcal{S}_{\delta+2}^{m-2,\alpha}(M)$ as the Banach space of pairs (τ, f) corresponding to $\tau + f\omega \otimes \omega$, where τ is a symmetric 2-tensor on M and f is a function on M with τ_{ij} and f in $C_{\delta+2}^{m-2,\alpha}(M)$). Let $L = D_{\tilde{g}_{\mathcal{M}}}\Phi$ be linearization of $\Phi := \Phi_{\tilde{g}_{\mathcal{M}}}$ at $g_{\mathcal{M}} = \tilde{g}_{\mathcal{M}}$, and let $\hat{h} = (h, u')$ denote a variation of $g_{\mathcal{M}}$ with h the variation of g and u' the variation of u . We then set

$$\tilde{L}(h, u') = L(h, u') - \Delta_g \nu' \omega \otimes \omega,$$

where $\nu' = (\log u)' = \frac{u'}{u}$. Thus, the vertical component of \tilde{L} is given by $\frac{1}{2}s'(h) - \Delta_g \nu' + \delta_{g_{\mathcal{M}}}^* \delta_{g_{\mathcal{M}}} h(V, V)$, where $V = u^{-1}\partial_t$, (compare with the remark following (2.8)). We claim that \tilde{L} is a self-adjoint elliptic operator with respect to the boundary conditions

$$(\delta_{g_{\mathcal{M}}}\hat{h}, h^T, H'_h + N(\nu')) = (0, 0, 0) \quad \text{at } \partial\mathcal{M}. \tag{2.21}$$

(We note that $H'_h + N(\nu')$ is the variation of the mean curvature of $\partial\mathcal{M}$ in the direction \hat{h}). Namely, it is proved in [4, Lemma 3.2] that the operator L is self-adjoint and elliptic with respect to the closely related boundary conditions

$$(\delta_{g_{\mathcal{M}}}\hat{h}, [\hat{h}^T]_0, H'_h + N(\nu')) = (0, 0, 0) \quad \text{at } \partial\mathcal{M}, \tag{2.22}$$

where $[\hat{h}^T]_0$ denotes the trace-free part of \hat{h}^T (where \hat{h}^T is the restriction of \hat{h} to $\partial\mathcal{M}$, and the trace is taken with respect to the induced metric on $\partial\mathcal{M}$). Using exactly the same methods as in [4, Lemma 3.2] together with [4, Proposition 3.7], one easily sees that adding the term $-\Delta_g \nu' \omega \otimes \omega$ to L has the effect of changing the boundary conditions (2.22) to (2.21) while preserving ellipticity and self-adjointness.

Let $T_0 \subset T(\mathcal{S}_{\delta}^{m,\alpha}(M))$ be the subspace of tangent vectors (infinitesimal variations) satisfying the boundary conditions (2.21). By elliptic theory, the operator $\tilde{L}_0 := \tilde{L}|_{T_0} : T_0 \rightarrow T(\mathcal{S}_{\delta+2}^{m-2,\alpha}(M))$ is Fredholm, so has closed range, and

$$(\text{Im } \tilde{L}_0)^\perp = \text{Ker } \tilde{L}_0. \tag{2.23}$$

Let π_2 be the projection onto the second (i.e., vertical) factor of $T(\mathcal{S}_{\delta+2}^{m-2,\alpha}(M))$, as in (2.19). A simple calculation (cf. [4, (2.16)] for instance) shows that $2\delta_{g_M}^* \delta_{g_M} h(V, V) = \delta h(\nabla \log u)$. Hence,

$$(\pi_2 \circ \tilde{L}_0)(h, u') = \frac{1}{2}s'(h) - \Delta u' + \frac{1}{2}\delta h(\nabla \log u). \tag{2.24}$$

As noted in the beginning of the proof, the statement (2.17) does not depend on the choice of u . Hence, we may set $u = 1$, so that $(\pi_2 \circ \tilde{L}_0)(h, u') = \frac{1}{2}s'(h) - \Delta u'$.

Now the surjectivity of s' on $T\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ is equivalent to the statement that the restriction of \tilde{L}_0 to the horizontal subspace $(h, 0)$ (where $u' = 0$) of T_0 is surjective onto the vertical space $(0, f)$. By the self-adjoint property (2.23), this is equivalent to showing that if

$$\int_M \varphi s'(h) = 0, \tag{2.25}$$

for some $\varphi \in C_\delta^{m,\alpha}(M)$ and for all $(h, 0) \in T_0$, then $\varphi = 0$. To prove this, it follows from (2.25) that for all such h ,

$$0 = \int_M \varphi s'(h) = \int_M \langle S^* \varphi, h \rangle, \tag{2.26}$$

where, as in (2.10), S^* is the formal L^2 -adjoint of the linearization s' . (The condition $(h, 0) \in T_0$ implies the boundary terms in (2.26) vanish at ∂M and at infinity). In particular, (g, φ) solves

$$S^* \varphi = 0$$

on M so that (g, φ) is a static vacuum solution, (cf. the discussion following (2.11)).

Now the potential φ decays to zero at infinity at a rate at least $r^{-\delta}$. A simple asymptotic analysis of the static vacuum Eq. (2.10) then implies $\varphi = 0$ on M . Alternatively, this follows directly from Proposition 2.1 of [13] which shows that such potential functions, if not identically zero, are necessarily asymptotic to a nonzero constant or an affine function. This completes the proof that s' is surjective.

Finally, we prove that the kernel $\text{Ker } s'$ splits. Consider again the horizontal and vertical decomposition of the target space $\mathcal{S}_{\delta+2}^{m-2,\alpha}(M)$ as in (2.19). Define $S_1 = (\tilde{L}_0)^{-1}(*, 0)$ and $S_2 = (\tilde{L}_0)^{-1}(0, *)$. Clearly, S_1 and S_2 are closed subspaces of the domain T_0 of \tilde{L}_0 . It is easy to see that $S_1 + S_2$ is also a closed subspace, of finite codimension (the latter since the range of \tilde{L}_0 has finite codimension). Thus, $S_1 + S_2$ admits a closed complement, S_3 :

$$T_0 = (S_1 + S_2) \oplus S_3.$$

Now, the intersection $S_1 \cap S_2$ equals $\text{Ker } \tilde{L}_0$, which is finite dimensional since \tilde{L}_0 is Fredholm. Hence, $\text{Ker } \tilde{L}_0 \subset S_2$ has a closed complement S'_2 in S_2 . This gives a direct sum decomposition

$$T_0 = S_1 \oplus S'_2 \oplus S_3. \tag{2.27}$$

As in (2.24), $S_1 = \{\widehat{h} = (h, u') : \frac{1}{2}s'(h) - \Delta u' = 0\}$ when $u = 1$. Thus, $\text{Ker } s' \cap T_0$ is the horizontal subspace of S_1 , which splits by its vertical complement in S_1 . This proves that $\text{Ker } s' \cap T_0$ splits in T_0 . To complete the proof, let $\chi^{m-1,\alpha}(M)$ be the space of $C^{m-1,\alpha}$ vector fields on M and let $\chi_{\partial M}^{m-1,\alpha}(M)$ be the restriction of such vector fields to ∂M . Choose a smooth extension operator $E : \chi_{\partial M}^{m-1,\alpha}(M) \rightarrow \chi^{m-1,\alpha}(M)$ and a $C^{m+1,\alpha}$ smooth metric \widetilde{g} on M near (M, g) . Given h , consider the vector field Y_0 dual to $\delta h|_{\partial M}$ along ∂M and let $Y = E(Y_0)$. One may then uniquely solve the equation $\delta_g \delta_g^* X = Y$ with $X = 0$ on ∂M , and the decomposition $h = (h - \delta_g^* X) + \delta_g^* X$ gives a splitting of T_0 in the full tangent space $T(\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M))$. This proves that $\text{Ker } s'$ splits.

The implicit function theorem (regular value theorem) for Banach manifolds then implies the remaining part of the proposition. \square

Remark 2.5. Proposition 2.4 is closely related to the issue of linearization stability of solutions of the vacuum Einstein equations on \mathcal{M} and to the work of Fischer, Marsden and Moncrief [23]. While Proposition 2.4 is known to be false for compact manifolds M (i.e., there exist compact (M, g) for which Ds_g is not surjective), it is known to be true in the C^∞ setting for complete AF manifolds (in both cases without boundary).

Proposition 2.4 has the following useful corollary. Consider the map to Bartnik boundary data:

$$\pi_B : \mathcal{C}^{m,\alpha}(M) \rightarrow \text{Met}^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2), \quad \pi_B(g, u) = (\gamma, H), \quad (2.28)$$

where $\text{Met}^{m,\alpha}(S^2)$ is the space of $C^{m,\alpha}$ Riemannian metrics on S^2 with the $C^{m,\alpha}$ topology. Proposition 2.4 shows that $\mathcal{C}^{m,\alpha}(M)$ is a smooth Banach manifold; clearly, π_B is a smooth map of Banach manifolds.

Corollary 2.6. *The map π_B in (2.28) is a submersion, i.e., $D\pi_B$ is surjective, with splitting kernel. In particular, π_B is an open map.*

Proof. Proposition 2.4 implies that for any $(\gamma, H) \in \text{Im } \pi_B$, the map s is a submersion on $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$. Thus, for any given g (or, more precisely, (g, u)) in $\pi_B^{-1}(\gamma, H)$ and for any $f \in C_{\delta+2}^{m-2,\alpha}(M)$, there exists an h (or (h, u')) in $T_{(g,u)}\mathcal{S}^{m,\alpha}(M)$ satisfying $(h^T, H'_h) = (0, 0)$ at ∂M and such that $s'(h) = f$. Now given an arbitrary boundary variation (h^T, H'_h) , let $h_e \in T_{(g,u)}\mathcal{S}^{m,\alpha}(M)$ be an extension of (h^T, H'_h) to a variation of g on M of compact support. Then, $s'(h_e) = \varphi_e$, for some $\varphi_e \in C_{\delta+2}^{m-2,\alpha}(M)$. Let h_0 be a solution of $s'(h_0) = \varphi_e$ with zero boundary data. Then, $h := h_e - h_0$ satisfies $s'(h) = 0$ and h has the given boundary data (h^T, H'_h) . This proves that $D\pi_B$ is surjective.

Further, one has $\text{Ker } D\pi_B = T\mathcal{C}_{(\gamma,H)}^{m,\alpha}$ which was proved to split in Proposition 2.4. \square

More generally, for a given function $\sigma \in C_{\delta+2}^{m-2,\alpha}(M)$, let

$$\mathcal{C}_\sigma^{m,\alpha}(M) = \{g \in \text{Met}_\delta^{m,\alpha}(M) : s_g = \sigma\}. \quad (2.29)$$

One has a corresponding map π_B as in (2.28), and the same proof as above shows that π_B remains a submersion, for any σ .

Although the mass m_{ADM} has no critical points on $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$, it may have critical points on distinguished subsets of $\mathcal{S}_{(\gamma,H)}^{m,\alpha}(M)$ (constrained critical points). In view of Lemma 2.3, we focus in particular on the submanifold $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$. Clearly,

$$m_{\text{ADM}} : \mathcal{C}_{(\gamma,H)}^{m,\alpha} \rightarrow \mathbb{R}$$

is a smooth functional.

Proposition 2.7. *Critical points of the ADM mass m_{ADM} on $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$ are exactly metrics g that admit an AF function u such that (g, u) is an AF solution of the static vacuum Einstein equations on M with given boundary data (γ, H) .*

Note that we are not yet claiming that $u > 0$; this is addressed in Theorem 2.10 for Bartnik mass minimizers.

Proof. On $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$ (if nonempty), one has by (2.8)

$$\mathcal{H} = -16\pi m_{\text{ADM}} : \mathcal{C}_{(\gamma,H)}^{m,\alpha} \rightarrow \mathbb{R}. \tag{2.30}$$

The critical points of m_{ADM} on the constraint space $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$ are thus exactly the same as critical points of \mathcal{H} on $\mathcal{C}_{(\gamma,H)}^{m,\alpha}$, and in the following we work with \mathcal{H} . Note that the potential function u is irrelevant at this point (since m_{ADM} is independent of u); thus in the following and for the moment, we make a fixed (but arbitrary) choice of $u = u_0 > 0$, with $u_0 - 1 \in C_\delta^{m,\alpha}(M)$.

Let then (g, u_0) be a critical point of the constrained variational problem, i.e.,

$$d\mathcal{H}_{(g,u_0)}(h, u') = 0,$$

for all $(h, u') \in T_{(g,u_0)}\mathcal{C}_{(\gamma,H)}^{m,\alpha}$, i.e., $s'(h) = 0$ and $(h^T, H'_h) = (0, 0)$ at ∂M . A standard Lagrange multiplier theorem, discussed explicitly in a related context in [11, Theorem 6.3], shows that there is a distribution (bounded linear functional) λ on $C_{\delta+2}^{m-2,\alpha}(M)$, (the Lagrange multiplier), such that for all, i.e., unconstrained, variations $(h, u') \in T_{(g,u_0)}\mathcal{S}_{(\gamma,H)}^{m,\alpha}$,

$$d\mathcal{H}_{(g,u_0)}(h, u') = \lambda(s'(h)).$$

Since $s_g = 0$, one has by Proposition 2.2,

$$d\mathcal{H}_{(g,u_0)}(h, u') = \int_M \langle S^*u_0, h \rangle = \int_M u_0 s'(h),$$

for all variations $(h, u') \in T\mathcal{S}_{(\gamma,H)}^{m,\alpha}$ of compact support in $\text{int}(M)$. Combining these statements gives

$$\lambda(s'(h)) - \int_M u_0 s'(h) = 0, \tag{2.31}$$

for all such variations h . We claim that the distribution

$$T(w) := \int_M u_0 w - \lambda(w),$$

acting on functions $w \in C^{m-2,\alpha}(M)$ with compact support in $\text{int}(M)$ is a smooth solution of the static vacuum equations with respect to g on M , $C^{m,\alpha}$ up to ∂M . To see this, consider variations of the form $h = fg$, where f is a smooth function on M . Since $s'(fg) = -2\Delta f - sf = -2\Delta f$, we see from (2.31) that T is a weak (i.e., distribution) solution of $-2\Delta u = 0$ in $\text{int}(M)$. By elliptic regularity (i.e., the well-known Weyl Lemma and Schauder estimates), T is represented as a $C^{m,\alpha}$ function u , i.e.,

$$T(w) = \int_M uw$$

for all such w , where $\Delta u = 0$. Thus, for all $(h, u') \in T_{(g,u_0)}\mathcal{S}_{(\gamma,H)}^{m,\alpha}$ of compact support in $\text{int}(M)$,

$$0 = \int_M us'(h) = \int_M \langle S^*u, h \rangle,$$

where, as in (2.10), S^* is the formal L^2 -adjoint of the linearization s' . In particular, (g, u) solves $S^*u = 0$ in $\text{int}(M)$ so that (g, u) is a smooth static vacuum solution, (cf. the discussion following (2.11)). Since $\text{Ric}_g \in C^{m-2,\alpha}(M)$, integration of the static equations $u \text{ Ric} = D^2u$ shows that u is $C^{m,\alpha}$ up to ∂M .

We complete the proof by arguing $u - 1 \in C_\delta^{m,\alpha}(M)$. Note

$$\lambda(w) = \int_M (u_0 - u)w,$$

for w compactly supported in $\text{int}(M)$, where we recall $u_0 \rightarrow 1$ at infinity. From [13, Proposition 2.1], the static vacuum potential u must approach a nonzero constant at infinity, be asymptotic to an affine function at infinity or else be identically zero. We will reach a contradiction (to λ being a bounded linear functional) in every case except $u \rightarrow 1$ at infinity. This together with $\Delta u = 0$ on M implies $u - 1 \in C_\delta^{m,\alpha}(M)$, and the proof will be complete.

Let $\chi_i : M \rightarrow [0, 1]$ be a sequence of smooth, nonnegative radially symmetric cutoff functions of compact support on M , with $\chi_i(r) = 1$ for $r \in [0, R_i]$, $|d^k \chi_i(r)| \leq c_k/r^k$ for $r \in [R_i, 2R_i]$ and $\chi_i(r) = 0$ for $r \geq 2R_i$ with $R_i \rightarrow \infty$.

First, suppose that either $u \equiv 0$, or u approaches a constant other than 1 at infinity. Then, $u_0 - u$ approaches a nonzero constant at infinity. Let $w \in C_{\delta+2}^{m-2,\alpha}(M)$ equal $r^{-2-\delta}$ outside of a compact set and vanish near ∂M . Let $w_i = \chi_i w$, a uniformly bounded sequence in $C_{\delta+2}^{m-2,\alpha}(M)$ each with compact support in $\text{int}(M)$. Since λ is a bounded linear functional, the sequence $\lambda(w_i)$ is uniformly bounded as well. However, a direct computation shows that $|\lambda(w_i)|$ diverges to infinity, a contradiction.

Last, consider the case in which u (and hence $u_0 - u$) limits to an affine function at infinity, i.e.,

$$(u_0 - u)(\vec{x}) = \vec{a} \cdot \vec{x} + O(|\vec{x}|^{1-\delta}),$$

where $\vec{a} \neq \vec{0}$ is constant (cf. [13]). Without loss of generality, assume $\vec{a} = (0, 0, 1)$, i.e., $(u_0 - u)(x, y, z) = z + O(|\vec{x}|^{1-\delta})$. A similar argument as that given above applies here with a different sequence of test functions. Let

$$w_i(\vec{x}) = |\vec{x}|^{-2-\delta} \chi_i(\vec{x} - (0, 0, 3R_i)),$$

a sequence of smooth functions of compact support in $\text{int}(M)$ vanishing near ∂M (for i large enough). It is straightforward to check that $\{w_i\}$ is uniformly bounded in $C_{\delta+2}^{m-2,\alpha}(M)$, but that $\lim_{i \rightarrow \infty} \lambda(w_i) = +\infty$. Again, this is a contradiction, since λ is a bounded linear functional. The proof is complete. \square

We note that the potential u of an AF static vacuum metric (M, g) is uniquely determined by g (up to multiplication by a scalar) if g is not flat, cf. [56, Proposition 10]. On the other hand, any affine function u is the potential of a flat static exterior solution (M, g_{Eucl}) . As discussed in Remark 2.11, it is not fully known in general whether, if (M, g) is an AF static metric, then the potential u must also be AF when a (nonminimal) boundary is present.

We now state and prove one of the main results of this section:

Theorem 2.8. *An AF metric $g \in \mathring{\mathcal{P}}^{m,\alpha}(M)$ on M realizing the Bartnik mass (2.5) of the boundary data (γ, H) is an AF static vacuum solution (g, u) (i.e., with $u \rightarrow 1$ at infinity), satisfying the boundary conditions (1.3).*

Proof. The Bartnik mass m_B is obtained by minimizing m_{ADM} subject to the no-horizon condition and constraints that $s \geq 0$, the boundary metric γ is fixed, and the mean curvature of ∂M is at most $H = H_{\partial\Omega}$ pointwise. If $g \in \mathring{\mathcal{P}}^{m,\alpha}(M)$ realizes m_B (say with mean curvature $H_{\partial M}$), then a neighborhood of g in $\mathcal{P}^{m,\alpha}(M)$ is in $\mathring{\mathcal{P}}^{m,\alpha}(M)$ by Lemma 2.1. Lemma 2.3 then implies that g must be scalar flat, $s_g = 0$. The result then follows from Proposition 2.7, since g is a critical point of m_{ADM} on $C_{(\gamma, H_{\partial M})}^{m,\alpha}$. \square

Remark 2.9. We recall briefly the reasoning that leads to the Bartnik boundary conditions (1.2) and (1.3). By combining the Gauss and Riccati equations on M at $\partial M = \partial\Omega$, one finds

$$N(H) = \frac{1}{2}(s_\gamma - s_g - |A|^2 - H^2). \tag{2.32}$$

Since the metric γ is fixed on $\partial\Omega$, the scalar curvature s_γ is fixed, while the last three terms in (2.32) are nonpositive, since $s_g \geq 0$. It follows that $N(H)$ is uniformly bounded above, but may a priori become arbitrarily negative in (weak) limits. Thus, in passing to a limit of a mass-minimizing sequence of extensions, one expects

$$H_{\partial M} \leq H_{\partial\Omega}, \tag{2.33}$$

so that the exterior mean curvature may drop from that given by the region Ω , as in (1.3). Note that the positive mass theorem still holds on such manifolds with corners, cf. [44, 53]. On the other hand, if A and s_g remain bounded on a minimizing sequence, then one has

$$H_{\partial M} = H_{\partial\Omega}. \tag{2.34}$$

Unfortunately, it is not clear how to give a topology on $\mathcal{P}^{m,\alpha}(M)$ to effectively implement such a structure in limits.

Conversely, given (say) smooth boundary data (γ, H) on S^2 that arise as boundary data for a smooth metric g_Ω of nonnegative scalar curvature on

$\Omega \simeq \bar{B}$, one expects that there is a smooth AF metric on \mathbb{R}^3 of nonnegative scalar curvature in which (Ω, g_Ω) isometrically embeds, (corresponding to the original definition (1.1)). This remains to be fully proved however.

Next, we prove that boundary conditions (2.34) are actually realized for a minimizer of the ADM mass, as defined in (1.3), provided one uses the definition (2.4) for $\mathring{P}^{m,\alpha}(M)$. This result was obtained by Miao for the case in which (γ, H) has strictly positive Gauss curvature (and $H > 0$) using a different technique; see [45, Proposition 3.4]. In addition, we prove that $u > 0$ everywhere on M .

Theorem 2.10. *Suppose that an AF metric g on M realizes the Bartnik mass of the boundary data (γ, H) in the sense of (2.5) (i.e., with boundary conditions (2.33)), where $H > 0$. Then, (2.34) holds, and the AF static vacuum potential u is strictly positive on M .*

Proof. Suppose (M, g) realizes the Bartnik mass (2.5) of (Ω, g_Ω) , so that as in (2.33), $H_{\partial M} \leq H_{\partial\Omega}$. We first show that (2.34) holds. If it fails, let $U \subset \partial M$ be the nonempty open set on which strict inequality holds:

$$H_{\partial M} < H_{\partial\Omega} \quad \text{on } U.$$

By Theorem 2.8, the metric g is static vacuum, and so in particular scalar flat. Let u be the corresponding AF static vacuum potential.

Consider the map π_B in (2.28). By Corollary 2.6, π_B is a submersion at $(g, 1)$ and so for any variation (h^T, H'_h) of the boundary data $(\gamma, H_{\partial M})$, there is a variation h of g such that $s'(h) = 0$. Choose

$$(h^T, H'_h) = (0, q), \tag{2.35}$$

where q is a smooth function on ∂M supported in U and such that

$$\int_{\partial M} uq > 0. \tag{2.36}$$

Clearly, there are many such choices of q , unless $u \equiv 0$ on U . However, if $u \equiv 0$ on U , then $H_{\partial M} \equiv 0$ on U , since the zero set of a static vacuum potential is totally geodesic. Consider two cases. First, if U is a proper subset of ∂M , then $H_{\partial M} \equiv 0$ on U contradicts $H_{\partial\Omega} > 0$, since $H_{\partial M} = H_{\partial\Omega}$ outside U . Second, if $U = \partial M$, then (M, g, u) is a Schwarzschild metric, by the black hole uniqueness theorem [17, Theorem 2]. In particular, $\gamma = \gamma_{2m}$ is a round metric. Since $H_{\partial\Omega} > 0$ and $H_{\partial M} = 0$, it is easy to see that (M, g) cannot be a minimal mass extension. (For example, one may take an equidistant round sphere $r > 2m$ close to the horizon $r = 2m$ of the Schwarzschild metric and rescale, decreasing the mass).

Now, let h be the corresponding variation of g with $s'(h) = 0$, satisfying (2.35). Since $s'(h) = 0$ and (g, u) is static vacuum, one has from (2.13), (2.30) and (2.35) that

$$-16\pi m'_{\text{ADM}}(h) = d\mathcal{H}_{(g,u)}(h, 0) = 2 \int_{\partial M} uq > 0, \tag{2.37}$$

so that $m'_{\text{ADM}}(h) < 0$. This gives, at the infinitesimal level, a mass-decreasing variation of $(g, 1)$ in $T_{(g,1)}\mathcal{C}^{m,\alpha}_{(\gamma,H_{\partial\Omega})}(M)$. Now consider the curve of boundary data $(\gamma, H_{\partial M} + tq)$ (for instance) with t small. Again by Corollary 2.6, this curve lifts (via π_B^{-1}) to a curve g_t in the slice or closed complement to $\text{Ker } s' = T_{(g,1)}\mathcal{C}^{m,\alpha}$ in $\mathcal{C}^{m,\alpha}$. It follows that $m_{\text{ADM}}(g_t) < m_{\text{ADM}}(g)$ for $t > 0$ small and since $\pi_B(g_t) = (\gamma, H + tq)$, $g_t \in \mathcal{C}^{m,\alpha}_{(\gamma,H_{\partial\Omega})}(M)$, again for t small. Since by Lemma 2.1, g_t has no horizons for t small, this contradicts the definition of Bartnik mass. Thus, (2.34) holds.

Next, suppose $u < 0$ somewhere on M . By the maximum principle, $u < 0$ at some point on ∂M . Let q be a smooth, nonpositive function on ∂M , supported in the set where $u < 0$, satisfying (2.36). A similar argument to that given above produces a mass-decreasing path of metrics in $\mathcal{C}^{m,\alpha}_{(\gamma,H_{\partial\Omega})}(M)$, again contradicting the definition of the Bartnik mass. Thus, $u \geq 0$ on M .

Finally, if $u(p) = 0$ for some $p \in M$, then $p \in \partial M$ by the maximum principle. At p , by the static vacuum equations, $0 = u \text{ Ric} = D^2u$ and hence $D^2u = 0$ at p . The restriction of D^2u to ∂M gives $(D^2)^T u + N(u)A = 0$, and taking then the trace over ∂M gives $\Delta_{\partial M}u + N(u)H = 0$. Since p is a minimum of u , $\Delta_{\partial M}u \geq 0$, while by the Hopf boundary point maximum principle, $N(u) > 0$. It follows then that $H(p) \leq 0$, a contradiction. This proves $u > 0$. □

Theorems 2.8 and 2.10 together imply Theorem 1.1 from Introduction.

Remark 2.11. Theorem 1.1 and its proof implement the original heuristic program suggested by Bartnik [10,11], based on the perspective initiated by Brill et al. [15]. It is also possible to prove part of Theorem 1.1 using different methods. We briefly summarize this alternate approach and some related results below.

To begin, Corvino [21] showed that metrics minimizing the Bartnik mass of a domain Ω are static vacuum outside $\bar{\Omega}$ by constructing suitable localized scalar curvature deformations. However, this result did not address the issues of the horizon conditions, nor the global behavior of the potential function u , and did not fully address the boundary conditions.

Using this method, an elementary argument in [21] shows that the boundary condition (1.3) is preserved; however, it is not clear whether the original (stronger) condition (1.2) is preserved. This issue has very recently been resolved in [22]. Note that the proof that a minimizer is static vacuum does require some stability condition, such as that in Lemma 2.1.

However, in such an approach, it is not clear whether the potential function u satisfies $u > 0$ or even $u \rightarrow 1$ at infinity M , i.e., (g, u) may not be globally static vacuum or asymptotically flat (AF) in the usual sense. Miao and Tam [47, Theorem 1.1] prove that a static potential is bounded, with a sign, outside a compact set, provided the metric is asymptotically Schwarzschild of nonzero mass. Unfortunately, this does not apply to the present setting, since it is not clear that a static vacuum metric is asymptotically Schwarzschild without knowing in advance its potential has a sign at infinity. Huang et al. [27,

Theorem 10] show that the static potential of a Bartnik mass minimizer approaches 1 at infinity (though this argument does not show $u > 0$ globally). Note that they use a different version of the Bartnik mass. We also refer the reader to some related results on static potentials by Galloway and Miao [24] (cf. Carlotto et al. [18, Corollary 1.9]), for example, in the boundaryless or minimal boundary case.

An immediate corollary of Theorem 2.10 is the strict monotonicity of the Bartnik mass, improving (2.6) when m_B is realized.

Corollary 2.12. *Suppose $0 < H \leq H'$ and $H < H'$ on some nonempty open set $U \subset \partial M$. If the data (γ, H) are realized by a mass-minimizing extension in $\mathring{\mathcal{P}}^{m,\alpha}(M)$, then*

$$m_B(\gamma, H') < m_B(\gamma, H). \tag{2.38}$$

3. Remarks on Conjecture I

In this section, we present several results that provide further positive evidence for the validity of Conjecture I, regarding the existence of AF extensions of nonnegative scalar curvature.

The main extension results to date are based on the quasi-spherical method introduced by Bartnik [8]. For example, using this method, it can be established that any boundary data (γ, H) with γ of positive Gauss curvature $K_\gamma > 0$ and $H > 0$ admit an extension in $\mathcal{P}(M)$ (see [8, 53, 54]). More recently, extension results have also been obtained by Lin [35] and Lin and Sormani [36] using a modified Ricci flow.

We write $(\gamma, H) \in \mathcal{P}^{m,\alpha}(M)$ if the boundary data (γ, H) on S^2 admit an admissible extension $g \in \mathcal{P}^{m,\alpha}(M)$, and similarly for $\mathring{\mathcal{P}}^{m,\alpha}(M)$.

We first note the following general result.

Proposition 3.1. *The spaces $\mathcal{P}^{m,\alpha}(M)$ and $\mathring{\mathcal{P}}^{m,\alpha}(M)$ are open in $Met^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2)$.*

Proof. This is an immediate consequence of Corollary 2.6, in the scalar-flat case. The general case follows as in (2.29). Lemma 2.1 then implies the statement for $\mathring{\mathcal{P}}^{m,\alpha}(M)$. □

For the discussion to follow, we will not address the horizon issue, which is more difficult to understand when dealing with more global problems.

We first prove a general result that the space $\mathcal{P}^{m,\alpha}(M)$ is invariant under pointwise increase of the mean curvature H , keeping the boundary metric γ fixed, (compare with the proof of Theorem 2.10). In fact, even a small decrease on H is allowed. The method of proof will also be used in the proof of Theorem 1.2 given in Sect. 4.

Proposition 3.2. *Suppose $(\gamma, H) \in \mathcal{P}^{m,\alpha}(M)$. There exists a $C^{m-1,\alpha}$ function $\mu > 0$ on S^2 , (depending on (γ, H)), such that for any $C^{m-1,\alpha}$ function H_0 on S^2 satisfying*

$$H_0 \geq H - \mu, \tag{3.1}$$

pointwise, one has $(\gamma, H_0) \in \mathcal{P}^{m,\alpha}(M)$.

Proof. Suppose $(\gamma, H) \in \mathcal{P}^{m,\alpha}(M)$, so that there is an AF extension g of (γ, H) with scalar curvature $s \geq 0$. For simplicity, we will assume (γ, H) and g are smooth. We first consider the case $H_0 \geq H$ and construct an AF extension of (γ, H_0) by a conformal deformation of g .

For a conformal metric $\tilde{g} = v^4g$ with $v > 0$, the scalar curvature s of g changes as

$$v^5\tilde{s} = -8\Delta v + sv =: f, \tag{3.2}$$

where Δ is the Laplacian operator on (M, g) . Since $s \geq 0$, $-8\Delta + s$ is a positive operator (for Dirichlet boundary data). Clearly, $\tilde{g} \in \mathcal{P}(M)$ requires $f \geq 0$.

For $r > 0$, let $B(r) = \{x \in M : \text{dist}(x, \partial M) \leq r\}$, so that $\partial B(r) = \partial M \cup S(r)$ is a regular hypersurface for r large. Given $f \geq 0$ of compact support on M , let v_r be the unique solution to (3.2) on $B(r)$ with Dirichlet boundary data $v_r = 1$ on $\partial M \cup S(r)$. By the maximum principle, $v_r > 0$ on $B(r)$. It is standard that letting $r \rightarrow \infty$, $v_r \rightarrow v$ with $v > 0$ on M , $v = 1$ at ∂M , and $v \rightarrow 1$ at infinity. In particular, \tilde{g} is a conformal AF metric on M , with induced boundary metric

$$\tilde{\gamma} = \gamma \quad \text{at} \quad \partial M,$$

and boundary mean curvature

$$\tilde{H} = v^{-2}H + 4v^{-2}N(\log v),$$

where N is the unit normal into M with respect to g . Hence,

$$\tilde{H} = H + 4N(v). \tag{3.3}$$

To obtain $\tilde{H} = H_0$, we will demonstrate how to choose $f \geq 0$ appropriately so that the solution v to (3.2) with boundary conditions of 1 on ∂M and at infinity satisfies $N(v) = \frac{1}{4}(H_0 - H)$.

Write (3.2) in the form

$$L(v) := \Delta v - \frac{1}{8}sv = -\frac{1}{8}f. \tag{3.4}$$

On the bounded domain $(B(r), g)$, L has a (negative) Green's function G , with $G(x, y) = 0$ for (say) $y \in \partial M \cup S(r)$. It is standard that we may take $r \rightarrow \infty$ to obtain a (negative) Green's function G of (M, g) with $O_2(1/|y|)$ decay at infinity for $G(x_0, y)$, for any fixed x_0 .

The Poisson kernel P of L is given by $P(x, y) = -N_x G(x, y)$, for $x \in \partial M$ and is positive for y in the interior of M . Green's formula gives for $x \in M$,

$$\begin{aligned} v(x) &= \frac{1}{8} \int_M G(x, y)(-f(y))dy + \int_{\partial M} P(y, x)v(y)dy \\ &+ \lim_{r \rightarrow \infty} \int_{S(r)} N_y G(x, y)v(y)dy. \end{aligned} \tag{3.5}$$

Here, dy represents the corresponding volume forms on M , ∂M , and $S(r)$, respectively, with respect to g .

Note that $v = 1$ solves (3.4) uniquely for the choice $f = s$; using (3.5) on the sequence v_i corresponding to $f_i = s\chi_i$, where $0 \leq \chi_i \leq 1$ are compactly supported functions on M that converge pointwise to 1 as $i \rightarrow \infty$, we may take a limit (using the decay $O(r^{-2-\delta})$ of s) to arrive at

$$1 = \frac{1}{8} \int_M G(x, y)(-s(y))dy + \int_{\partial M} P(y, x)1dy + \lim_{r \rightarrow \infty} \int_{S(r)} N_y G(x, y)1dy. \tag{3.6}$$

Since $v = 1$ on ∂M and $v \rightarrow 1$ at infinity, it follows that for general f of compact support, the solution v to (3.4) is given by

$$v(x) = 1 + \frac{1}{8} \int_M G(x, y)(s(y) - f(y))dy. \tag{3.7}$$

Thus, for $x \in \partial M$,

$$N(v)(x) = \frac{1}{8} \int_M P(x, y)(f(y) - s(y))dy. \tag{3.8}$$

It is standard that v has sufficient decay at infinity (e.g., $v(y) - 1 = O_m(|y|^{-\delta})$) to assure that \tilde{g} is asymptotically flat.

Now we claim that given any smooth function $\varphi \geq 0$ on ∂M , there is a C^k function $f \geq 0$ (for any $k > 0$) with compact support on M , such that

$$\varphi(x) = N(v)(x) = \frac{1}{8} \int_M P(x, y)(f(y) - s(y))dy. \tag{3.9}$$

This will complete the proof (in the case $H_0 \geq H$), by choosing $\varphi = \frac{1}{4}(H_0 - H) \geq 0$.

To prove the claim, note first that by the basic reproducing property of the Poisson kernel,

$$\int_{\partial M} P(x, y)\varphi(y)dy = \varphi(x). \tag{3.10}$$

Choose a constant $d_0 > 0$, smaller than the distance to the cut locus of the normal exponential map of ∂M into M , and let $\partial M_r = \{y \in M : \text{dist}(y, \partial M) = r\}$ for $0 \leq r \leq d_0$. Define a continuous linear operator $A_r : C^k(\partial M_r) \rightarrow C^k(\partial M_r)$ by

$$A_r(\chi)(x) = \chi_r(x) = \int_{\partial M_r} P_r(x, y)\chi(y)dy, \tag{3.11}$$

where $x \in \partial M_r$ and $P_r = P|_{\partial M_r}$. Using the identification of ∂M with ∂M_r via the normal exponential map, we may regard A_r as a map $C^k(\partial M) \rightarrow C^k(\partial M)$. It is well known (and easy to see) that

$$A_r \rightarrow Id, \quad \text{as } r \rightarrow 0,$$

as bounded linear operators on $C^k(\partial M)$. Since the space of invertible operators is open, we may shrink $d_0 > 0$ if necessary so that given $\varphi \in C^k(\partial M)$ there exists a unique $\varphi_r \in C^k(\partial M_r)$, $r \leq d_0$, satisfying

$$\int_{\partial M_r} P_r(x, y)\varphi_r(y)dy = \varphi(x). \tag{3.12}$$

Note that P_r is smooth in r for $r > 0$, and hence so is φ_r . The (higher order) normal derivatives $\partial_r^k P(x, y)$ govern (by convolution) the (higher order) normal derivatives of harmonic functions on M at ∂M ; it follows that φ_r is also smooth in r at $r = 0$.

Now, let $\rho(r)$ be a smooth function of $r \geq 0$ with $\rho(0) = 1$, $\rho(r) = 0$ for each $r \geq d_0$, and $\int_0^{d_0} \rho(r) dr = \frac{d_0}{2}$. Integrating over r and using the Gauss Lemma and Fubini theorem (or the coarea formula) gives

$$\int_M P(x, y)\rho(r(y))\varphi_r(y)dy = \varphi(x) \int_0^{d_0} \rho(r)dr = \frac{1}{2}d_0\varphi(x). \tag{3.13}$$

Thus, the C^k function given by $f(y) = s(y) + \frac{16}{d_0}\rho(r(y))\varphi_r(y)$ satisfies (3.9). It is clear that f is C^k smooth and extends smoothly by zero to M . This proves the claim (3.9). Note that f is not uniquely determined.

To complete the proof when $\mu > 0$, consider the given extension (M, g) of (γ, H) , and take the unique, smooth solution to

$$\begin{cases} \Delta u = 0 & \text{on } M \\ u = 1 & \text{on } \partial M \\ u \rightarrow \frac{1}{2} & \text{at } \infty. \end{cases}$$

Then, the conformal metric $\hat{g} = u^4g$ belongs to $\mathcal{P}^{m,\alpha}(M)$, induces the metric γ on its boundary and the induces mean curvature on ∂M given by

$$\hat{H} = H + 4N(u).$$

By the maximum principle, $N(u) < 0$ on ∂M . Thus, $(\gamma, H - \mu) \in \mathcal{P}^{m,\alpha}(M)$ for the choice $\mu = -4N(u) > 0$. The result now follows by applying the argument above to $(\gamma, H - \mu)$. □

It is useful to understand how the mass m_{ADM} changes under the deformations in Proposition 3.2. Thus, recall from (2.16) that if $\tilde{g} = v^4g$, then

$$\tilde{m} = m - \frac{1}{2\pi} \lim_{r \rightarrow \infty} \int_{S(r)} N(v)dA.$$

In the context of the proof of Proposition 3.2, suppose $f(y) \geq s(y) \geq 0$, so that, in particular, $\tilde{H} \geq H$. For simplicity, assume $f = s$ outside of a compact set. Then, (3.7) shows $v \geq 1$ and $v \rightarrow 1$ at infinity, and (3.4) shows v is subharmonic outside of a compact set. Then, similar to the proof of the last part of Lemma 2.3, we can enclose any coordinate sphere $S(r)$ with a smooth level set L of v , on which the outward normal derivative of v is nonpositive by the maximum principle. Applying the divergence theorem on the region between S_r and L , we see that that $\int_{S(r)} N(v)dA \leq 0$ for all r large, and thus

$$\tilde{m} \geq m.$$

Thus, roughly speaking, as one increases H , the mass m increases under conformal changes, when keeping the boundary metric fixed. On other hand, if

$0 \leq f(y) \leq s(y)$, so that $\tilde{H} \leq H$, then (3.7) shows $v \leq 1$ and $v \rightarrow 1$ at infinity, so that

$$\tilde{m} \leq m.$$

In particular, one can decrease the mass conformally if $s \geq 0$ is not identically zero; compare with Lemma 2.3.

When combined with existing results, Proposition 3.2 gives further partial evidence for the validity of Conjecture I.

Given a metric γ on S^2 , let $\lambda_1(-\Delta_\gamma + K_\gamma) > 0$ be the lowest eigenvalue of the operator $-\Delta_\gamma + K_\gamma$, where Δ_γ is the Laplacian with respect to γ and K_γ is the Gauss curvature.

Corollary 3.3. *One has $(\gamma, H) \in \mathcal{P}^{m,\alpha}(M)$ for all $H > 0$ and all γ such that $\lambda_1(-\Delta + K) > 0$.*

Proof. In Mantoulidis and Schoen [38] constructed extensions $g \in \mathcal{P}^{m,\alpha}(M)$ of $(\gamma, 0)$ for γ satisfying $\lambda_1(-\Delta + K) > 0$. The result then follows from Proposition 3.2. □

This generalizes (with a different proof) previous extension results of Bartnik [8] and Miao [45].

Corollary 3.4. *For any (γ, H) with $H > 0$, there is a $\lambda_0 > 0$ such that*

$$(\gamma, \lambda H) \in \mathcal{P}^{m,\alpha}(M),$$

for all $\lambda \geq \lambda_0$.

Proof. The proof is based on work in [2–4] on the moduli space $\mathcal{E}^{m,\alpha}$ of $C^{m,\alpha}$ AF static vacuum solutions (g, u) , $u > 0$, on $M = \mathbb{R}^3 \setminus B$. Namely, the map to Bartnik boundary data

$$\begin{aligned} \Pi_B : \mathcal{E}^{m,\alpha} &\rightarrow \text{Met}^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2), \\ \Pi_B(g, u) &= (\gamma, H), \end{aligned} \tag{3.14}$$

is a smooth Fredholm map, of Fredholm index 0. (This is discussed further in Sect. 5). Consider the map Π_B restricted to the space $\mathcal{E}_+^{m,\alpha}$ of static vacuum metrics with $H > 0$ at ∂M :

$$\Pi_B : \mathcal{E}_+^{m,\alpha} \rightarrow \text{Met}^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(M). \tag{3.15}$$

Consider also the action of scalars $\lambda \in \mathbb{R}^+$ on $C_+^{m-1,\alpha}(S^2)$ where $(\lambda, H) \rightarrow \lambda H$. Let $\mathcal{D}_+^{m-1,\alpha}(S^2)$ be the space of equivalence classes $[H] = [\lambda H]$. The space $\mathcal{D}_+^{m-1,\alpha}(S^2)$ is clearly a Banach manifold.

It is proved in [3] that the induced quotient map

$$\begin{aligned} \tilde{\Pi}_B : \mathcal{E}_+^{m,\alpha} &\rightarrow \text{Met}^{m,\alpha}(S^2) \times \mathcal{D}_+^{m-1,\alpha}(S^2), \\ \tilde{\Pi}_B(g) &= (\gamma, [H]), \end{aligned}$$

is a smooth *surjective* Fredholm map of Fredholm index 1. Hence, for any given boundary data (γ, H) , $H > 0$, there exists $\lambda_0 = \lambda_0(\gamma, H)$ such that

$(\gamma, \lambda_0 H)$ are the Bartnik boundary data of a complete AF static vacuum solution (M, g, u) . Since $g \in \mathcal{P}^{m,\alpha}(M)$, the result then follows from Proposition 3.2. \square

Corollary 3.4 may be contrasted with the result in [30] that for given Bartnik boundary data (γ, H) , with $H > 0$, there is a largest value $\lambda^0 < \infty$ such that $(\gamma, \lambda H)$ has a $s \geq 0$ infilling for $\lambda < \lambda_0$, and no such infilling, for $\lambda > \lambda_0$. Here, an $s \geq 0$ infilling is a compact Riemannian 3-manifold with boundary inducing Bartnik boundary data (γ, H) that has nonnegative scalar curvature. That result required γ to have positive Gauss curvature K_γ , but a recent result of Mantoulidis and Miao ([37, Theorem 1.3]) implies that $\lambda^0 < \infty$ without assuming $K_\gamma > 0$.

4. Proof of Theorem 1.2

The main purpose of this section is to prove Theorem 1.2. Most of the section will be devoted to proving:

Theorem 4.1. *Let $F \in \mathcal{F}$ be as in the statement of Theorem 1.2. Then,*

$$m_B(\bar{B}, F^*(g_{\text{Eucl}})) = 0, \tag{4.1}$$

where m_B is the Bartnik mass defined by (1.4).

Remark 4.2. The result would be immediate if it were known that the Bartnik mass is continuous (or even lower semi-continuous) in the smooth topology on the space of metrics on \bar{B} of nonnegative scalar curvature, since $(\bar{B}, F^*(g_{\text{Eucl}}))$ can be smoothly approximated, up to isometry, by domains in \mathbb{R}^3 which have zero Bartnik mass. It is proved in [31, 33] that the ADM mass is lower semi-continuous in the pointed C^2 and C^0 topologies. However, this does not directly imply lower semicontinuity of the Bartnik mass: The main difficulty is that it is not known that “close” Bartnik data necessarily have “close” competitors for near-minimal mass extensions. Note that the recent work of McCormick [42] on the continuity of the Bartnik mass does not apply here—the regions we consider will not generally satisfy the required convexity condition in [42, Theorem 5.1].

Before proving Theorem 4.1, we first show how it is used to prove Theorem 1.2:

Proof of Theorem 1.2. Consider a pair $(\bar{B}, F^*(g_{\text{Eucl}}))$, where $F \in \mathcal{F}$. Let $F_0 : S^2 \rightarrow \mathbb{R}^3$ be $F|_{S^2}$, an immersion (but not embedding) of S^2 into \mathbb{R}^3 . Let $\gamma_0 = F_0^*g_{\text{Eucl}}$ and $H_0 : S^2 \rightarrow \mathbb{R}$ be the induced metric and mean curvature.

Now, suppose the Bartnik mass of $(\bar{B}, F^*(g_{\text{Eucl}}))$ is realized by an extension $(M, g) \in \check{\mathcal{P}}(M)$, so that (1.3) holds with $\Omega = \bar{B}$. By Theorem 4.1, the ADM mass of (M, g) vanishes.

Glue $(\bar{B}, F^*(g_{\text{Eucl}}))$ and (M, g) along their boundaries so as to satisfy (1.3) to produce a Riemannian manifold (N, h) , without boundary, that is asymptotically flat and smooth with nonnegative scalar curvature away from

the gluing hypersurface. The ADM mass of (N, h) also vanishes. By the rigidity case of the positive mass theorem “with corners” in dimension three [44, 53], (N, h) is isometric to $(\mathbb{R}^3, g_{\text{Eucl}})$ and $H_0 = H_{\partial M}$. In particular, there is an isometric embedding G of $(\bar{B}, F^*(g_{\text{Eucl}}))$ into $(\mathbb{R}^3, g_{\text{Eucl}})$. If we set $G_0 = G|_{S^2}$, then $G_0^*g_{\text{Eucl}} = F_0^*g_{\text{Eucl}} = \gamma_0$ and the mean curvature of the embedding G_0 is H_0 . Thus, we have two immersions F_0 and G_0 of S^2 into \mathbb{R}^3 both realizing the same induced metric and the same mean curvature. The contradiction will arise because F_0 is not an embedding but G_0 is, and we will show below that F_0 and G_0 are in fact congruent.

A pair of immersions F_1, F_2 of a surface into \mathbb{R}^3 with the same induced metric and mean curvature is called a Bonnet pair, and it is well known that there are no nontrivial Bonnet pairs of spherical topology. We recall the simple proof. Let A_i be the second fundamental form of F_i . Since $H_1 = H_2$ and $\gamma_1 = \gamma_2 \equiv \gamma$, the Gauss–Codazzi equations give

$$\delta_\gamma(A_1 - A_2) = 0,$$

where δ_γ is the divergence. Also $\text{tr}_\gamma(A_1 - A_2) = 0$. Thus, $A_1 - A_2$ is a holomorphic quadratic differential on S^2 . Since the only such is 0, one has $A_1 = A_2$. It then follows from the fundamental theorem for surfaces in \mathbb{R}^3 (rigidity) that the immersions F_1 and F_2 are congruent. \square

Corollary 4.3. *The space of compact regions (Ω, g_Ω) of nonnegative scalar curvature that admit a mass-minimizing extension in $\mathring{\mathcal{P}}(M)$ is not closed in the smooth topology.*

This corollary will make it hard to prove the existence of mass-minimizing extensions by studying limits of mass-minimizing sequences in general.

Remark 4.4. Recall a result of Huisken and Ilmanen [28] on the rigidity of the Bartnik mass: If $m_B(\Omega) = 0$, then Ω is locally flat. (Although note their proof applies to the “outward-minimizing” version of the Bartnik mass.) The proof of Theorem 1.2 above implies the converse of this result is false, for domains Ω for which Conjecture II holds. To see this, consider a locally flat domain $(\Omega, g_\Omega) = (\bar{B}, F^*(g_{\text{Eucl}}))$, where $F : \bar{B} \rightarrow \mathbb{R}^3$ is a smooth immersion which is not an embedding. If $m_B(\Omega) = 0$ and $m_B(\Omega)$ is realized by a minimum-mass extension (i.e., Conjecture II holds at Ω), then the proof of Theorem 1.2 gives a contradiction. Hence, either $m_B(\Omega) > 0$ or Conjecture II fails at Ω (or both).

Proof of Theorem 4.1. Let $F : \bar{B} \rightarrow \mathbb{R}^3$ be the immersion in \mathcal{F} . We detail the proof when the self-intersection set $Z \subset S^2 = \partial\bar{B}$ consists of two distinct points z, z' with $F(z) = F(z')$ but with F injective on $\bar{B} \setminus Z$. The proof in the general case of a finite number of double points is a straightforward modification of this case.

Let F_0 denote the restriction $F|_{S^2}$, an immersion (but not embedding) of S^2 into \mathbb{R}^3 , and let

$$(\gamma_0, H_0) = (F_0^*g_{\text{Eucl}}, H_{F_0})$$

be the induced metric and mean curvature of F_0 , defined on S^2 .

Let $\varepsilon > 0$. We will prove that $(\bar{B}, F^*(g_{\text{Eucl}}))$ admits an admissible extension in $\mathcal{P}(M)$ (for ε sufficiently small) whose ADM mass is $\leq C\varepsilon$ for a constant C depending only on F . For $\varepsilon > 0$ small enough, the extension will be in $\hat{\mathcal{P}}(M)$, i.e., it will contain no immersed minimal surfaces surrounding the boundary. The proof is rather long, consisting of five steps.

Step 1: Modification of g_{Eucl} to introduce positive scalar curvature. Let $\Omega \subset \mathbb{R}^3$ be the image of F , a compact set. Fix a neighborhood U of Ω , and let $R_0 = \frac{1}{\varepsilon}$ be chosen so that the interior of the ball $B(R_0)$ of radius R_0 contains the closure of U (decreasing ε if necessary).

To begin, we smoothly deform the Euclidean metric g_{Eucl} on \mathbb{R}^3 to produce a new Riemannian metric \hat{g} with the following properties: $\hat{g} = g_{\text{Eucl}}$ on U ; \hat{g} has nonnegative scalar \hat{s} that is strictly positive somewhere, zero on $\mathbb{R}^3 \setminus B(R_0)$, and $\int_{\mathbb{R}^3} \hat{s} dv \leq \varepsilon$; \hat{g} is asymptotically flat with ADM mass $\leq \varepsilon$. To be definite, we construct \hat{g} by applying a conformal factor w^4 to g_{Eucl} , where w is superharmonic, harmonic outside $B(R_0)$, identically 1 on U , and approaches a constant less than 1 at infinity. Additionally, we can choose w so that

$$1 - \varepsilon \leq w \leq 1 + \varepsilon, \quad 1 - \varepsilon \leq \frac{1}{w} \leq 1 + \varepsilon, \quad |\nabla w| \leq \varepsilon, \tag{4.2}$$

where $|\nabla w|$ is taken with respect to g_{Eucl} . In particular, there exists a closed ball K , contained in $B(R_0) \setminus U$, such that

$$\hat{s} \geq \alpha_0 > 0 \quad \text{on } K, \tag{4.3}$$

for some constant $\alpha_0 = \alpha_0(\varepsilon)$.

Step 2: Construction of family of metrics \bar{g}_t to obtain correct boundary metric. In this step, we will perturb the immersion F_0 to an embedding. This will of course alter the boundary data (γ_0, H_0) , so we will also perturb the metric \hat{g} so as to restore the original boundary metric γ_0 . This change will introduce a small amount of negative scalar curvature and will possibly violate (1.3); these issues will be addressed in Step 3.

To begin, let N_0 be the unit outward normal vector field along F_0 , viewed as a function on S^2 (taken with respect to \hat{g} , or equivalently, with respect to g_{Eucl}). Fix a number $\delta > 0$ sufficiently small so that $B_z(2\delta) \cap B_{z'}(2\delta) = \emptyset$; here, $B_z(r) \subset S^2$ is the open geodesic r -ball about z , with respect to the induced metric γ_0 . Let $q : S^2 \rightarrow \mathbb{R}^{\geq 0}$ be a smooth, nonnegative bump function that equals 1 on $B_z(\delta)$ and is zero outside $B_z(2\delta)$. Let $A \subset S^2$ be the open annular region:

$$A = \text{int}(B_z(2\delta) \setminus B_z(\delta)). \tag{4.4}$$

An example illustrating this setup is sketched in Fig. 1.

For $t \geq 0$, define a smooth family of maps $F_t : S^2 \rightarrow \mathbb{R}^3$ by

$$F_t(x) = F(x) - tq(x)N_0(z), \tag{4.5}$$

where $N_0(z)$ is treated as a constant vector field on \mathbb{R}^3 . For $0 < t < t_0$ sufficiently small, F_t is an embedding and $F_t(S^2)$ is contained inside U . The mapping $t \mapsto F_t(S^2)$ gives a local flow of surfaces in which the set $F_t(B_z(\delta))$

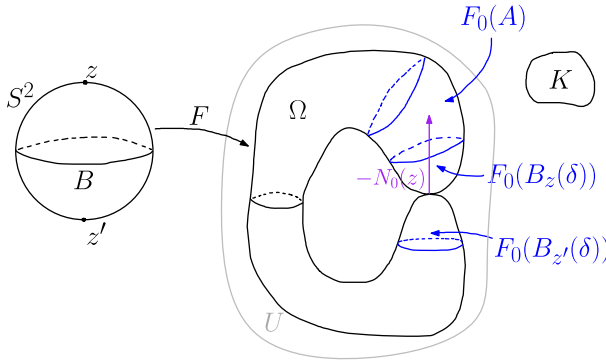


FIGURE 1. An illustration indicating the initial setup in the proof of Theorem 4.1

is translated in the $-N_0(z)$ direction at speed 1, and $F_t(S^2 \setminus B_z(2\delta))$ does not move. Thus, the only change to the geometry occurs in $F_t(A)$.

For $0 < t < t_0$, the smooth, embedded 2-sphere $F_t(S^2)$ bounds a smooth, compact region Ω_t in \mathbb{R}^3 that is diffeomorphic to a closed 3-ball. Let $M_t = \mathbb{R}^3 \setminus \text{int}(\Omega_t)$, a smooth manifold with compact boundary ∂M_t . Note that F_t is a diffeomorphism of S^2 onto ∂M_t .

Lemma 4.5. *There exist a domain $V \subset\subset U \subset \mathbb{R}^3$ and a smooth family of Riemannian metrics $\{\bar{g}_t\}_{0 \leq t < t_0}$ on \mathbb{R}^3 such that:*

$$\bar{g}_0 = \hat{g} \text{ on } \mathbb{R}^3; \quad \bar{g}_t = \hat{g} \text{ outside of } V, \quad \text{for } t \in [0, t_0),$$

and, on S^2 ,

$$F_t^* \bar{g}_t = F_0^* g_{\text{Eucl}} = \gamma_0, \quad \text{for } t \in [0, t_0).$$

Moreover, V satisfies

$$V \cap F_t(S^2) = F_t(\text{int}(B_z(\frac{9}{4}\delta) \setminus B_z(\frac{3}{4}\delta))). \tag{4.6}$$

Proof. The following construction takes place within U , so we regard g_{Eucl} and \hat{g} as equal in the remainder of this proof. Fix $t \in (0, t_0)$, and let γ_t be the metric on the embedded surface $F_t(S^2) \subset \mathbb{R}^3$ induced by g_{Eucl} . Let $r(x) = \text{dist}_{g_{\text{Eucl}}}(x, \partial M_t)$ be the Euclidean distance of $x \in \mathbb{R}^3$ to $\partial M_t = F_t(S^2)$. Recall that $\gamma_t = \gamma_0$ outside $F_t(B_z(2\delta))$ (where here we are identifying $\gamma_0 = F_0^* g_{\text{Eucl}}$ with the induced metric on $F_0(S^2)$). Let Σ_t^r be the image of $F_t(A)$ under the time r Euclidean exponential map normal to $F_t(A)$ (the r -equidistant surface to $F_t(A)$); here, $r \in (-r_0, r_0)$ and r_0 is chosen small enough so that $\Sigma_t^r \subset M_t \cap U$ for $r \geq 0$ and Σ_t^r is smooth for all $r \in (-r_0, r_0)$. Note that r_0 may be chosen independent of t ; it depends only on δ and the surface $F_0(S^2)$. (See the left side of Fig. 2).

Let O_t be the union of the surfaces Σ_t^r for $|r| < r_0$, an open set. In O_t , by the Gauss lemma for the normal exponential map,

$$g_{\text{Eucl}} = dr^2 + \gamma_t^r,$$

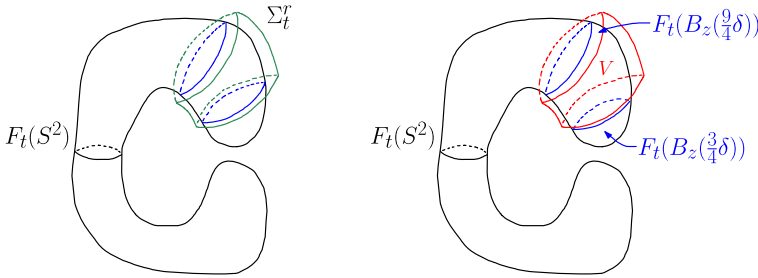


FIGURE 2. This illustration shows part of the setup of the proof of Lemma 4.5

where γ_t^r is the metric induced on Σ_t^r by g_{Eucl} . Note $\gamma_t^0 = \gamma_t$ on Σ_t^0 . Define a new metric on O_t by

$$\bar{g}_t = dr^2 + \bar{\gamma}_t^r,$$

where $\bar{\gamma}_t^r$ is a smooth metric on Σ_t^r , varying smoothly in r , such that $\bar{\gamma}_t^0 = \gamma_0|_{\Sigma_t^0}$ and $\bar{\gamma}_t^r = \gamma_t^r$, for $|r| \geq r_0/2$. (If $t = 0$, define $\bar{g}_0 = g_{\text{Eucl}}$ on O_0 .) Note that $\bar{\gamma}_t^0$, a metric on $F_t(A)$, extends smoothly to γ_t on $F_t(S^2)$, since $\gamma_t = \gamma_0$ outside $F_t(A)$.

To complete the construction, let V be an open set contained in U and containing $\bigcup_{0 < t < t_0} O_t$ and satisfying (4.6). (See the right side of Fig. 2).

Extend \bar{g}_t smoothly to V (and smoothly in $t \in [0, t_0]$ as well, which can be arranged in the above construction) so that \bar{g}_t induces the metric γ_0 on $F_t(S^2)$ and \bar{g}_t agrees with \hat{g} near ∂V . Then, \bar{g}_t extends to a smooth family of metrics on \mathbb{R}^3 , with $\bar{g}_t = \hat{g}$ outside of V . \square

Note that $\bar{g}_t = \hat{g} = g_{\text{Eucl}}$ on a neighborhood of $F_0(z) = F_0(z')$. Since the family \bar{g}_t is smooth in t and $\bar{g}_0 = \hat{g}$ has nonnegative scalar curvature, the scalar curvature \bar{s}_t of \bar{g}_t is nonnegative outside V and converges uniformly to zero inside V as $t \rightarrow 0$. We also note, for later reference, that \bar{V} does not contain $F_0(z)$.

To summarize at this point, we have the asymptotically flat Riemannian manifold (M_t, \bar{g}_t) , for each $t \in (0, t_0)$, for which the induced metric on the boundary (when pulled back to S^2 via F_t) equals the original boundary metric γ_0 . The mean curvature \bar{H}_t of ∂M_t (viewed as a function on S^2) converges uniformly to the original H_0 as $t \rightarrow 0$. Moreover,

$$\bar{H}_t = H_0 \quad \text{on} \quad S^2 \setminus A' \tag{4.7}$$

for all $t \in (0, t_0)$, where $A' = F_t^{-1}(V \cap F_t(S^2)) = \text{int}(B_z(\frac{9}{4}\delta) \setminus B_z(\frac{3}{4}\delta))$ is a slightly enlarged annulus.

The space (M_t, \bar{g}_t) is almost, but not quite, an admissible extension of $(\bar{B}, F^*(g_{\text{Eucl}}))$, for two reasons: First, the mean curvatures of the boundaries do not agree (although they are close, for t small) and more generally do not necessarily satisfy (1.3) (i.e., $\bar{H}_t \leq H_0$ may fail to hold at some points); second,

the scalar curvature \bar{s}_t is not nonnegative (although it is nearly so, for t small.). We address these two problems in the next step.

Step 3: Conformal deformation to correct scalar curvature and boundary mean curvature. Next, we perform a conformal deformation on (M_t, \bar{g}_t) . For each $t \in (0, t_0)$, consider the linear elliptic problem

$$\begin{cases} \bar{L}_t v_t = 0 & \text{in } M_t \\ v_t = 1 & \text{on } \partial M_t \\ v_t \rightarrow 1 & \text{at infinity,} \end{cases} \tag{4.8}$$

where $\bar{L}_t := \bar{\Delta}_t - \frac{1}{8}\bar{s}_t$ and $\bar{\Delta}_t$ is the Laplacian for \bar{g}_t . Assuming for the moment a smooth, positive solution v_t exists, define the conformal metric

$$\tilde{g}_t = v_t^4 \bar{g}_t \quad \text{on } M_t.$$

Then, the induced metric on the boundary stays the same:

$$F_t^* \tilde{g}_t = F_t^* \bar{g}_t = F_0^* g_{\text{Eucl}} = \gamma_0,$$

by Lemma 4.5 and the boundary condition on v_t . Also, the mean curvature \bar{H}_t of ∂M_t with respect to \tilde{g}_t is given by

$$\tilde{H}_t = \bar{H}_t + 4\bar{N}_t(v_t), \tag{4.9}$$

where \bar{N}_t is the unit boundary normal on (M_t, \bar{g}_t) (viewed as a function on ∂M_t , pointing into M_t). In this step, we will prove that a solution $v_t > 0$ to (4.8) exists and that, moreover,

$$\tilde{H}_t < H_0, \tag{4.10}$$

for t sufficiently small.

One small difficulty is that in contrast to the setting of Proposition 3.2, since \bar{s}_t may be negative at some points, $-\bar{L}_t$ may not automatically be a positive operator (with Dirichlet boundary conditions), so that the Eq. (4.8) may not a priori always be uniquely solvable. Similarly, the associated Green’s function and Poisson kernel may not be uniquely defined or have appropriate signs. On the other hand, these properties are relatively simple to prove. Note first that $\bar{L}_t : C_{\delta}^{k+2, \alpha}(M_t) \rightarrow C_{\delta+2}^{k, \alpha}(M_t)$ (where we recall the weighted Hölder space notation from Sect. 2) is formally $L^2(M_t, \bar{g}_t)$ -self-adjoint with respect to zero Dirichlet boundary conditions on ∂M_t . In the statement below, let $M_0 \subset \mathbb{R}^3$ be the closure of the complement of Ω . Note that $M_0 \setminus F_0(z)$ is a smooth manifold with (noncompact) boundary.

Before stating Lemma 4.6, we need a precise definition of a family of functions on M_t converging to a function on M_0 . First, if $\{f_t\}_{0 < t < t_0}$ are in $C_{\delta+2}^{k, \alpha}(M_t)$ and $f_0 \in C_{\delta+2}^{k, \alpha}(M_0)$, we define $f_t \rightarrow f_0$ in $C_{\delta+2}^{k, \alpha}(M_0)$ as $t \rightarrow 0$ if f_t and f_0 admit $C_{\delta+2}^{k, \alpha}$ extensions (say \tilde{f}_t and \tilde{f}_0 , respectively) to a domain $M_* \subset \mathbb{R}^3$ strictly containing M_0 and all M_t (for t sufficiently small) in its interior and $\tilde{f}_t \rightarrow \tilde{f}_0$ in $C_{\delta+2}^{k, \alpha}(M_*)$. Second, we define $f_t \rightarrow f_0$ in $C_{loc, \delta+2}^{k, \alpha}(M_0 \setminus F_0(z))$ if (i) given any compact set $K \subset M_0 \setminus F_0(z)$, there exists a smooth family of

embeddings $\Phi_t : K \rightarrow M_t$, Φ_0 being the identity, such that $f_t \circ \Phi_t$ converges to f in $C^{k,\alpha}(K)$, and (ii) $f_t|_{\mathbb{R}^3 \setminus U} \rightarrow f|_{\mathbb{R}^3 \setminus U}$ in $C_{\delta+2}^{k,\alpha}(\mathbb{R}^3 \setminus U)$.

Lemma 4.6. *For $t_0 > 0$ sufficiently small, $-\bar{L}_t$ is a positive operator for $0 < t < t_0$, with respect to Dirichlet boundary conditions. Hence, given $f_t \in C_{\delta+2}^{k,\alpha}(M_t)$, there is a unique solution $\omega_t \in C_{\delta}^{k+2,\alpha}(M_t)$ to $\bar{L}_t \omega_t = f_t$ with $\omega_t = 0$ on ∂M_t . Additionally, given $f_0 \in C_{\delta+2}^{k,\alpha}(M_0)$, there is a unique solution $\omega_0 \in C_{\delta}^{k+2,\alpha}(M_0)$ to $\bar{L}_0 \omega_0 = f_0$ with $\omega_0 = 0$ on ∂M_0 . Moreover, if $f_t \rightarrow f_0$ in $C_{\delta+2}^{k,\alpha}(M_0)$ as $t \rightarrow 0$, then the solutions ω_t converge to ω_0 in $C_{loc,\delta}^{k+2,\alpha}(M_0 \setminus F_0(z))$ as $t \rightarrow 0$. Finally, the Green’s function $\bar{G}_t(x, y)$ and Poisson kernel $\bar{P}_t(x, y)$ for \bar{L}_t exist and satisfy*

$$\bar{G}_t(x, y) \leq 0, \quad \bar{P}_t(x, y) \geq 0,$$

with strict inequality for y in the interior of M_t .

Proof. Note that M_0 is not a manifold with boundary (as $\partial M_0 = F_0(S^2)$ is not embedded), but it does satisfy the Poincaré “exterior cone condition” (cf. [25, p. 29, 203–205]) and is therefore a regular domain for the Dirichlet problem for the operator

$$\bar{L}_0 = \widehat{L} := \widehat{\Delta} - \frac{1}{8}\widehat{s},$$

where we recall that $\widehat{s} \geq 0$. Clearly, $-\bar{L}_0$ is a positive operator with respect to Dirichlet boundary conditions, $\omega = 0$ on ∂M_0 and $\omega \rightarrow 0$ at infinity. In particular, for the bottom of the L^2 spectrum, one has

$$\lambda_0 = \inf \frac{\int_{M_0} -f \bar{L}_0 f \overline{d}v_0}{\int_{M_0} f^2 \overline{d}v_0} > 0,$$

where the inf is taken over nonzero smooth functions f of compact support in M_0 . It is standard that there exists a positive Green’s function \bar{G}_0 for \bar{L}_0 on M_0 , and moreover the Dirichlet problem $\bar{L}_0 \omega_0 = f_0$ is uniquely solvable.

As $t \searrow 0$, the boundaries ∂M_t converge to ∂M_0 , smoothly away from $F_0(z)$. Similarly, the operators \bar{L}_t converge smoothly to the operator \bar{L}_0 away from $F_0(z)$ (and are equal outside a compact set). Moreover, the lowest eigenvalue λ_0^t of $-\bar{L}_t$ varies continuously with t as $t \rightarrow 0$, (cf. [5] for instance for a much more general result than this special situation), and hence $\lambda_0^t > 0$, for t sufficiently small. For convenience, we include a direct proof that $\lambda_0^t > 0$ in the next paragraph.

For $t > 0$, let

$$\lambda_0^t = \inf \frac{\int_{M_t} -f \bar{L}_t f \overline{d}v_t}{\int_{M_t} f^2 \overline{d}v_t}$$

(where the infimum is taken over nonzero smooth functions f of compact support in M_t), be the bottom eigenvalue of $-\bar{L}_t$. We claim that $\lambda_0^t > 0$ for t sufficiently small. Since the minimum value of $\bar{s}_t - \widehat{s}$ on M_t converges to 0 as $t \rightarrow 0$,

$$\begin{aligned} \lambda_0^t &= \inf \frac{\int_{M_t} \left(|\nabla f|_{\bar{g}_t}^2 + \frac{1}{8} \widehat{s} f^2 \right) \overline{dv}_t + \frac{1}{8} \int_{M_t} (\bar{s}_t - \widehat{s}) f^2 \overline{dv}_t}{\int_{M_t} f^2 \overline{dv}_t} \\ &= \inf \frac{\int_{M_t} \left(|\nabla f|_{\bar{g}_t}^2 + \frac{1}{8} \widehat{s} f^2 \right) \overline{dv}_t}{\int_{M_t} f^2 \overline{dv}_t} - O(t) \\ &\geq c_1 \inf \frac{\int_{M_t} \left(|\nabla f|_{\bar{g}}^2 + c_2 \widehat{s} f^2 \right) \widehat{dv}}{\int_{M_t} f^2 \widehat{dv}} - O(t) \end{aligned}$$

for some constants $c_1, c_2 > 0$. This is because $\{\bar{g}_t\}_{0 < t < t_0}$ and \widehat{g} are uniformly equivalent by uniform positive constants as $t \rightarrow 0$. Now take a manifold with boundary $M_* \subset \mathbb{R}^3$ that contains all M_t , and consider (M_*, \widehat{g}) . We have

$$\lambda_0^t \geq c_1 \inf \frac{\int_{M_*} \left(|\nabla f|_{\bar{g}}^2 + c_2 \widehat{s} f^2 \right) \widehat{dv}}{\int_{M_*} f^2 \widehat{dv}} - O(t),$$

where this infimum is taken over all nonzero smooth functions f of compact support in M_* . It is clear this infimum is strictly positive, so that $\lambda_0^t > 0$ for t sufficiently small.

Thus, $-\bar{L}_t$ is a positive operator for t sufficiently small; it is then standard, cf. [34] for instance, that the Green’s function \bar{G}_t exists and is strictly negative in the interior of M_t and hence the Poisson kernel is strictly positive (since $\bar{P}_t(x, y) = -N_x \bar{G}_t(x, y)$ and $\bar{G}_t(x, y) = 0$ for $x \in \partial M_t$). The existence and uniqueness, along with the local $C^{k, \alpha}$ convergence of ω_t away from $F_0(z)$ and the weighted $C_\delta^{k+2, \alpha}(\mathbb{R}^3 \setminus U)$ convergence in the end, then follow from standard elliptic estimates. \square

It follows from Lemma 4.6 that (4.8) has a unique, smooth solution.

Remark 4.7. The main technical problem that arises in the discussion to follow is that for $x_t \rightarrow x = F_0(z)$, i.e., x_t converging to the singular point,

$$\bar{P}_t(x_t, \cdot) \rightarrow 0$$

(uniformly) on M_t . Thus, the Poisson kernel $\bar{P}_t(x_t, \cdot)$ degenerates at $F_0(z)$. Closely related to this is the fact that the Martin boundary of M_0 equals the Euclidean boundary away from the point $F_0(z)$ but at the cusp point $F_0(z)$ is much larger; there is a minimal positive harmonic function supported at $F_0(z)$ for each angle of approach to the singular point $F_0(z)$. This is discussed in Example 3 of [39].

Returning to the analysis of (4.8), it follows as in the discussion concerning (3.7) that

$$v_t(x) = 1 + \frac{1}{8} \int_{M_t} \bar{G}_t(x, y) \bar{s}_t(y) dy, \tag{4.11}$$

for $t \in (0, t_0)$, where dy denotes the volume form $\overline{dv}_t(y)$ of \overline{g}_t . As in (3.8), this gives for $x \in \partial M_t$,

$$\overline{N}_t(v_t)(x) = -\frac{1}{8} \int_{M_t} \overline{P}_t(x, y) \overline{s}_t(y) dy. \tag{4.12}$$

Let

$$C_t(\delta) = F_t(B_z(\delta) \cup B_{z'}(\delta)) \quad \text{and} \quad D_t(\delta) = F_t(S^2 \setminus (B_z(\delta) \cup B_{z'}(\delta))).$$

The geometry of ∂M_t is controlled in $D_t(\delta)$ but degenerates in $C_t(\delta)$ as $t \rightarrow 0$. Of course,

$$\partial M_t = C_t(\delta) \cup D_t(\delta).$$

Lemma 4.8. *If $\varepsilon > 0$ (from just before Step 1) is sufficiently small, then for $t_0 > 0$ sufficiently small, the solution v_t to (4.8) is positive and satisfies*

$$\overline{N}_t(v_t) < 0, \quad \text{on } \partial M_t, \quad \text{for } 0 < t < t_0. \tag{4.13}$$

Moreover, there exists $b > 0$, independent of $t \in (0, t_0)$, such that

$$\overline{N}_t(v_t)(x) < -b, \quad \text{for } x \in D_t(\frac{1}{2}\delta), \quad 0 < t < t_0. \tag{4.14}$$

Proof. By Lemma 4.6, as $t \searrow 0$, v_t converges in $C_{loc, \delta}^{k+2, \alpha}(M_0 \setminus F_0(z))$ to the limit solution v_0 to

$$\widehat{L}v_0 := \widehat{\Delta}v_0 - \frac{1}{8} \widehat{s}v_0 = 0, \tag{4.15}$$

on (M_0, \widehat{g}) with boundary conditions $v_0 = 1$ on ∂M_0 and $v_0 \rightarrow 1$ at infinity. By the maximum principle, since $\widehat{s} \geq 0$ (and is not identically zero), one has the following facts:

$$\begin{aligned} 0 < v_0 \leq 1 & \quad \text{on } M_0, \\ N_0(v_0) < 0 & \quad \text{on } F_0(S^2 \setminus \{z, z'\}). \end{aligned}$$

Here, the unit normal N_0 is viewed as a (well-defined) vector field on $F_0(S^2 \setminus \{z, z'\}) \subset \partial M_0$. In particular, $N_0(v_0) < -2b$ on $D_0(\frac{1}{2}\delta)$ for some constant $b > 0$. Since $v_t \rightarrow v_0$ locally in C^1 away from $F_0(z)$ (by Lemma 4.6), and $D_t(\frac{1}{2}\delta)$ converges smoothly to $D_0(\frac{1}{2}\delta)$ as $t \rightarrow 0$, we have

$$\overline{N}_t(v_t) < -b \quad \text{on } D_t(\frac{1}{2}\delta),$$

for t sufficiently small, which proves (4.14).

Next, we claim that v_t is uniformly controlled even in a neighborhood of $F_0(z)$; specifically, we establish (4.17) below. Let B' be a Euclidean ball centered at $F_0(z)$ of sufficiently small radius so that B' is disjoint from \overline{V} . Then, in the region $M_t \cap B'$, the metric \overline{g}_t is the flat Euclidean metric, so that \overline{L}_t is the Euclidean Laplacian Δ ; see Fig. 3. By the maximum principle applied to the harmonic function $1 - v_t$ on the domain $M_t \cap B'$, we have

$$\sup_{x \in M_t \cap B'} |1 - v_t(x)| = \sup_{x \in \partial(M_t \cap B')} |1 - v_t(x)|.$$

Since $v_t = 1$ on ∂M_t , this can be rewritten as

$$\sup_{x \in M_t \cap B'} |1 - v_t(x)| = \sup_{x \in M_t \cap \partial B'} |1 - v_t(x)|. \tag{4.16}$$

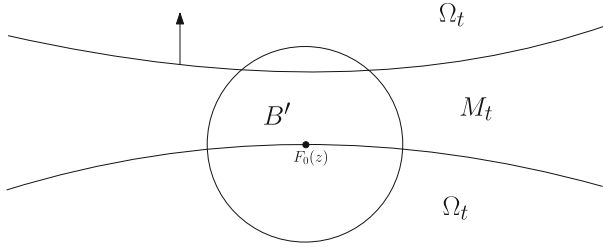


FIGURE 3. A sketch of the main sets involved in the proof of (4.17). B' is a fixed ball about $F_0(z)$. Ω_t is a connected set, but only its “caps” are shown here, the top one of which translates upward as t increases

By Lemma 4.6, $v_t \rightarrow v_0$ in $C_{loc,\delta+2}^{k,\alpha}(M_0 \setminus F_0(z))$, which implies (since $\partial B'$ is a fixed positive distance away from $F_0(z)$) that there exists a compact set K in M_0 containing a neighborhood of $M_0 \cap \partial B'$ in M_0 and a smooth family of embeddings $\Phi_t : K \rightarrow M_t$ such that $v_t \circ \Phi_t$ converges to v_0 in $C^{k,\alpha}(K)$, and that $\Phi_t(K)$ contains a neighborhood of $M_t \cap \partial B'$ (for t sufficiently small). In particular, this implies that $\sup_{x \in M_t \cap \partial B'} |v_0 \circ \Phi_t^{-1}(x) - v_t(x)| \rightarrow 0$ as $t \rightarrow 0$. From (4.16), we have

$$\begin{aligned} \sup_{x \in M_t \cap B'} |1 - v_t(x)| &\leq \sup_{x \in M_t \cap \partial B'} |1 - v_0 \circ \Phi_t^{-1}(x)| \\ &\quad + \sup_{x \in M_t \cap \partial B'} |v_0 \circ \Phi_t^{-1}(x) - v_t(x)|, \end{aligned}$$

and we just argued the second term to the right is $O(t)$. To address the first term on the right, since the metric \hat{g} is uniformly ε -close to the Euclidean metric, one has $|1 - v_0| = O(\varepsilon)$ on M_t . Thus, $\sup_{x \in M_t \cap \partial B'} |1 - v_0 \circ \Phi_t^{-1}(x)|$ is $O(\varepsilon) + O(t)$, so

$$\sup_{x \in M_t \cap B'} |1 - v_t(x)| \leq O(\varepsilon) + O(t), \tag{4.17}$$

as claimed.

Hence, if ε is sufficiently small, v_t is positive (and more generally $O(\varepsilon)$) near $F_0(z)$ for t sufficiently small, proving the claim. In particular, by this together with the appropriate convergence of v_t to v_0 as in Lemma 4.6, we have $v_t > 0$ on M_t , for all $t > 0$ sufficiently small.

Estimate (4.13) for x in $C_t(\frac{1}{2}\delta)$ for t is somewhat more subtle. Considering (4.12), note that $\bar{P}_t(x, y) > 0$ and $\bar{s}_t \geq \alpha_0 > 0$ in K from (4.3), while \bar{s}_t slightly negative of order t in V , for V as in Lemma 4.5. However, by Remark 4.7, the Poisson kernel $\bar{P}_t(x, y)$ degenerates at the singular point $x = F_0(z)$ as $t \rightarrow 0$: $\bar{P}_t(F_t(z), y) \rightarrow 0$ as $t \rightarrow 0$, uniformly in y . Thus, the relative behavior of \bar{P}_t in these two regions is not immediately clear.

We will use the following boundary Harnack estimate to obtain uniform control, as $t \rightarrow 0$, on the relative behavior of $\bar{P}_t(x, y)$ for y near to and away from the boundary ∂M_t .

Sub-Lemma 4.9. *There exists a constant $C > 0$ such that*

$$\sup_{y \in V \cap M_t} \bar{P}_t(x, y) \leq C \inf_{y' \in K} \bar{P}_t(x, y'), \tag{4.18}$$

for $x \in C_t(\frac{1}{2}\delta)$ and $t \in (0, t_0)$, where C is independent of such x and t .

The proof appears later.

Now it follows from (4.12) and the fact that any negative scalar curvature of \bar{g}_t lies within V together with the lower bound $\bar{s}_t \geq \alpha_0$ on K that for $x \in F_t(S^2)$,

$$\bar{N}_t(v_t)(x) \leq -\frac{1}{8} \int_{V \cap M_t} \bar{P}_t(x, y) \bar{s}_t(y) dy - \frac{1}{8} \int_K \bar{P}_t(x, y) \alpha_0 dy,$$

since $\bar{P}_t(x, y) \geq 0$. It follows then from (4.18) that for $x \in C_t(\frac{1}{2}\delta)$,

$$\begin{aligned} \bar{N}_t(v_t)(x) &\leq \frac{1}{8} \left(\sup_{y \in V \cap M_t} \bar{P}_t(x, y) \right) \left(\sup_{y \in V \cap M_t} |\bar{s}_t(y)| \right) \text{vol}_{\bar{g}_t}(V \cap M_t) \\ &\quad - \frac{1}{8} \alpha_0 \left(\inf_{y' \in K} \bar{P}_t(x, y') \right) \text{vol}_{\bar{g}}(K) \leq \frac{1}{8} \left(\inf_{y' \in K} \bar{P}_t(x, y') \right) \\ &\quad \times \left[C \left(\sup_{y \in V \cap M_t} |\bar{s}_t(y)| \right) \text{vol}_{\bar{g}_t}(V \cap M_t) - \alpha_0 \text{vol}_{\bar{g}}(K) \right]. \end{aligned}$$

Since $\sup_{y \in V \cap M_t} |\bar{s}_t(y)|$ converges to 0 (because $\hat{s} = 0$ on V) and $\text{vol}_{\bar{g}_t}(V \cap M_t)$ is bounded as $t \rightarrow 0$, the above is negative for t sufficiently small, independent of x . This completes the proof of Lemma 4.8. \square

Now, we explain why (4.10) holds. Recall from (4.7) that $\bar{H}_t = H_0$ on $C_t(\frac{3}{4}\delta)$. Thus, (4.13) and (4.9) show $\tilde{H}_t < H_0$ on $C_t(\frac{3}{4}\delta)$. Also, \bar{H}_t converges uniformly to H_0 , so (4.9) and (4.14) show (shrinking t_0 if necessary) that $\tilde{H}_t < H_0$ on $D_t(\frac{1}{2}\delta)$. This proves (4.10).

To conclude this step, we note that $\tilde{g}_t = v_t^4 \bar{g}_t$ is asymptotically flat: Since \bar{s}_t vanishes outside a compact set, v_t is \bar{g}_t -harmonic outside a compact set. Since $v_t \rightarrow 1$ at infinity, it is well known (and not hard to show) that $v_t^4 \bar{g}_t$ is asymptotically flat. Moreover, \tilde{g}_t has zero scalar curvature since $\bar{L}_t v_t = 0$. Thus, \tilde{g}_t is an admissible extension of $(\bar{B}, F^*(g_{\text{Eucl}}))$ in $\mathcal{P}(M)$, for t sufficiently small.

Step 4: Control of ADM mass of \tilde{g}_t . By the conformal deformation formula (2.16), and the fact that $\hat{g} = \bar{g}_0 = \bar{g}_t$ outside a compact set,

$$\begin{aligned} m_{\text{ADM}}(\tilde{g}_t) &= m_{\text{ADM}}(\bar{g}_t) - \frac{1}{2\pi} \lim_{r \rightarrow \infty} \int_{S_r} \bar{N}_t(v_t) \bar{d}\bar{A}_t \\ &= m_{\text{ADM}}(\hat{g}) - \frac{1}{2\pi} \int_{S_{R_1}} \bar{N}_t(v_t) \bar{d}\bar{A}_t \leq \varepsilon - \frac{1}{2\pi} \int_{S_{R_1}} \bar{N}_0(v_t) \bar{d}\bar{A}_0 \\ &\leq \varepsilon + \frac{1}{2\pi} \int_{S_{R_1}} |\bar{\nabla}_0(v_t)| \bar{d}\bar{A}_0, \end{aligned} \tag{4.19}$$

by the divergence theorem (since v_t is \bar{g}_t -harmonic outside U for all t) and since $\bar{g}_t = \bar{g}_0$ on S_{R_1} . Here, $R_1 \geq R_0 + 1$, where R_0 is the value chosen in Step 1, i.e., S_{R_0} encloses $U \supset \partial M_t$. Increasing R_1 if necessary, we arrange that

$$\frac{|S_{R_1}|_{\bar{g}_0}}{4\pi(R_1)^2} \leq 2, \tag{4.20}$$

by asymptotic flatness.

By Lemma 4.6, the convergence of v_t to v_0 is sufficient to guarantee that, by (4.19),

$$m_{\text{ADM}}(\tilde{g}_t) \leq \varepsilon + \frac{2}{2\pi} \int_{S_{R_1}} |\bar{\nabla}_0(v_0)|_{\bar{d}A_0} \tag{4.21}$$

for t sufficiently small. Using the Green’s function \bar{G}_0 to represent v_0 , we have, as in (4.11):

$$v_0(x) = 1 + \frac{1}{8} \int_{M_0} \bar{G}_0(x, y) \bar{s}_0(y) dy = 1 + \frac{1}{8} \int_{B(R_0)} \bar{G}_0(x, y) \bar{s}_0(y) dy,$$

since \bar{s}_0 vanishes outside $B(R_0)$. By the standard decay of the Green’s function, there exists constant C_1 depending only on the initial immersion F such that

$$|\bar{\nabla}_0 \bar{G}_0(x, y)|_{\bar{g}_0} \leq \frac{C_1}{|x|^2}$$

for $|x| \geq R_1$ and $y \in B(R_0)$. From the ε -bound on the L^1 norm of the scalar curvature of $\bar{g}_0 = \hat{g}$ from Step 1, this gives

$$|\bar{\nabla}_0 v_0(x)|_{\bar{g}_0} \leq \frac{C_1 \varepsilon}{8|x|^2},$$

for $|x| \geq R_1$. Combining this with (4.21) and using (4.20) imply that

$$\begin{aligned} m_{\text{ADM}}(\tilde{g}_t) &\leq \varepsilon + \frac{2}{2\pi} \frac{C_1 \varepsilon}{8(R_1)^2} |S_{R_1}|_{\bar{g}_0} \\ &\leq (1 + C_1) \varepsilon. \end{aligned}$$

Step 5: Absence of Horizons. In this final step, we argue that if $\varepsilon > 0$ was chosen small enough to begin with in Step 1, then (M_t, \tilde{g}_t) will not contain any immersed minimal surfaces that surround ∂M_t , for t sufficiently small.

Recall from Step 1 that $R_0 = \frac{1}{\varepsilon} > 0$ was chosen so that $B(R_0)$ contains U , and \hat{g} was constructed to be conformally flat with zero scalar curvature outside $B(R_0)$. In particular, \bar{g}_t and \tilde{g}_t also have zero scalar curvature and are conformally flat outside $B(R_0)$. It follows that $\tilde{g}_t = u_t^4 g_{\text{Eucl}}$ in $\mathbb{R}^3 \setminus B(R_0)$, where $u_t > 0$ is g_{Eucl} -harmonic for each t (specifically, $u_t = wv_t$).

In particular, using Lemma 4.6 and the decay of w , one has the Euclidean estimates

$$|u_t(x) - 1 + a| \leq \frac{c_0}{|x|} \quad \text{and} \quad |\nabla u_t(x)|_{g_{\text{Eucl}}} \leq \frac{c_1}{|x|^2}, \tag{4.22}$$

for $x \in \mathbb{R}^3 \setminus B(R_0)$, where c_0 and c_1 depend only on F , and $1 - a$ is the constant that w (and hence u_t) approaches at infinity. By (4.2), $0 < a \leq \varepsilon$.

Suppose Σ is a compact, immersed minimal surface in (M_t, \tilde{g}_t) that surrounds ∂M_t . The mean curvatures \tilde{H}_t and H_{Eucl} of Σ with respect to \tilde{g}_t and g_{Eucl} are related by

$$0 = \tilde{H}_t = (1 + f_1)H_{\text{Eucl}} + f_2$$

for smooth functions f_1 and f_2 on Σ , where $|f_1|$ and $|f_2|$ are bounded above by the C^1 norm of $\tilde{g}_t - g_{\text{Eucl}}$. (This can be seen from the first variation of area formula, for instance). Fix a Euclidean ball B'' centered at $F_0(z)$ that does not contain Σ . In particular, by (4.2), the fact that \tilde{g}_t converges smoothly to \hat{g} as $t \rightarrow 0$ and that ν_t converges smoothly to ν_0 outside B'' , we have, for t sufficiently small,

$$|H_{\text{Eucl}}| \leq C_2\varepsilon \quad \text{on} \quad \Sigma \setminus B'', \tag{4.23}$$

for a constant C_2 depending only on F and B'' .

Let $r = |x|$ be the Euclidean distance function from $F_0(z)$, and let $R_1 = \max_{\Sigma} r$, achieved at a point $p_0 \in \Sigma \setminus B''$. By a standard comparison of mean curvature of Σ ,

$$H_{\text{Eucl}}(p_0) \geq H_{\text{Eucl}}(S_{R_1}) = \frac{2}{R_1}. \tag{4.24}$$

Thus, $\frac{1}{R_1} \leq \frac{C_2\varepsilon}{2}$. If $\varepsilon > 0$ is sufficiently small, then $R_1 > R_0$. In particular, $p_0 \in \mathbb{R}^3 \setminus B(R_0)$, so that $\tilde{g}_t = u_t^4 g_{\text{Eucl}}$ in a neighborhood of p_0 . Thus

$$0 = \tilde{H}_t = u_t^{-2}H_{\text{Eucl}} + 4u_t^{-3}N_{\text{Eucl}}(u_t),$$

at p_0 . Combining this with (4.22),

$$|H_{\text{Eucl}}(p_0)| \leq 4 \frac{|\nabla u_t|}{u_t}(p_0) \leq \frac{4c_1}{(R_1)^2} \frac{1}{1 - \varepsilon - \frac{c_0}{R_1}} \tag{4.25}$$

Estimates (4.24) and (4.25) give a contradiction if ε is sufficiently small, since ε controls $\frac{1}{R_1}$.

Thus, if ε is chosen to be sufficiently small in Step 1, then \tilde{g}_t is an admissible extension of $(\bar{B}, F^*(g_{\text{Eucl}}))$ in $\hat{\mathcal{P}}(M)$ for t sufficiently small. The proof of Theorem 4.1 is now complete, except for the proof of Sub-Lemma 4.9, to which we now return. \square

Proof of Sub-Lemma 4.9. We will use the boundary Harnack principle for the elliptic operator \bar{L}_t , cf. [12, Theorem 1.1], for instance. Recall that the open set V , from Lemma 4.5, satisfies $F_0(z) \notin \bar{V}$, $V \cap F_t(S^2) = F_t(A') = F_t(\text{int}(B_z(\frac{9}{4}\delta) \setminus B_z(\frac{3}{4}\delta)))$ for each t and that K is a set disjoint from U on which $\hat{s} \geq \alpha_0 > 0$.

Let $O_2 \supset O_1 \supset (V \cup K)$ be connected, bounded open sets in \mathbb{R}^3 chosen so that $\bar{O}_2 \cap C_t(\frac{5}{8}\delta) = \emptyset$ for all $t \in (0, t_0)$ and that $\bar{O}_1 \subset O_2$. In particular, \bar{O}_2 does not include $F_0(z)$. See Fig. 4.

Then, for any $x \in C_t(\frac{1}{2}\delta)$, the function $y \mapsto \bar{P}_t(x, y)$ is smooth and bounded on $O_2 \cap M_t$ and vanishes on $\partial M_t \cap O_2$.

Let ν_t be the unique solution to $\bar{L}_t\nu_t = 0$ in M_t with boundary conditions 0 on ∂M_t and 1 at infinity. By Lemma 4.6, ν_t converges smoothly, away from $F_0(z)$, as $t \rightarrow 0$, to a function ν_0 on M_0 satisfying $\hat{L}\nu_0 = 0$, cf. also the

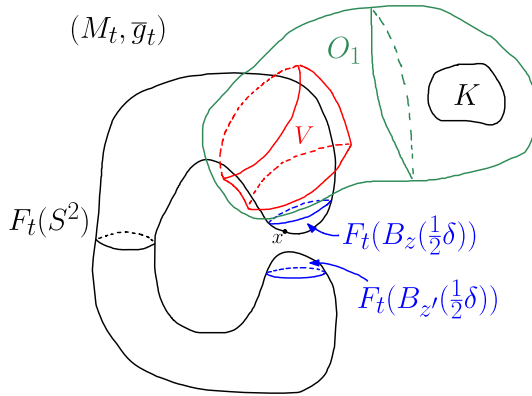


FIGURE 4. Main sets used in the proof of Sub-Lemma 4.9 are shown above. O_2 is not pictured but can be viewed as a slight enlarging of O_1

discussion around (4.17). By the maximum principle, $0 < \nu_0 < 1$ in the interior of M_0 . The convergence in the region O_2 (which again is a finite distance from $F_0(z)$) is sufficient to guarantee that, shrinking t_0 if necessary,

- $0 < \nu_t \leq 2$ in $O_2 \cap \text{int}(M_t)$,
- $\nu_t(y) \geq \beta$ for $y \in K$, where $\beta > 0$ is some constant independent of t .

By Lemma 4.6 again, for $x \in C_t(\frac{1}{2}\delta)$ and $y \in O_2$, the Poisson kernel $\bar{P}_t(x, y)$ satisfies $\bar{P}_t(x, \cdot) = 0$ on $O_2 \cap \partial M_t$ and $\bar{P}_t(x, \cdot) > 0$ in $O_2 \cap \text{int}(M_t)$. Since \bar{L}_t is elliptic and $\nu_t(y)$ and $y \mapsto \bar{P}_t(x, y)$ are both \bar{L}_t -harmonic on $O_2 \cap M_t$, the boundary Harnack principle (cf. [12, Theroem 1.1]) implies that: For all $y, y' \in O_1 \cap \text{int}(M_t)$ and all $x \in C_t(\frac{1}{2}\delta)$, one has

$$\frac{\bar{P}_t(x, y)}{\nu_t(y)} \leq c_t \frac{\bar{P}_t(x, y')}{\nu_t(y')} \tag{4.26}$$

for some constant $c_t > 0$ (depending on t), but independent of x . However, since \bar{L}_t and (M_t, \bar{g}_t) converge smoothly in the region O_2 (away from $F_0(z)$) as $t \rightarrow 0$, we may take the constant c_t independent of $t \in (0, t_0)$; call it C_0 .

Thus, using (4.26) and the relations on ν_t above,

$$\begin{aligned} \sup_{y \in V \cap M_t} \bar{P}_t(x, y) &= \sup_{y \in V \cap \text{int}(M_t)} \bar{P}_t(x, y) \leq \sup_{y \in O_1 \cap \text{int}(M_t)} \bar{P}_t(x, y) \\ &\leq \sup_{y \in O_1 \cap \text{int}(M_t)} \frac{2\bar{P}_t(x, y)}{\nu_t(y)} \leq 2C_0 \inf_{y' \in O_1 \cap \text{int}(M_t)} \frac{\bar{P}_t(x, y')}{\nu_t(y')} \\ &\leq 2C_0 \inf_{y' \in K} \frac{\bar{P}_t(x, y')}{\nu_t(y')} \leq \frac{2C_0}{\beta} \inf_{y' \in K} \bar{P}_t(x, y'). \end{aligned}$$

This proves the result with $C = \frac{2C_0}{\beta}$.

The proof of Theorem 4.1 is now complete. □

Remark 4.10. In the proof of Theorem 4.1, we constructed admissible extensions of $(\bar{B}, F^*(g_{\text{Eucl}}))$ that obeyed the boundary conditions (1.3). However, it may be possible with some further work to achieve equality of the mean curvatures, i.e., (1.2), in the construction by following an argument similar to the proof of Proposition 3.2. Specifically, one may replace (4.8) with $\bar{L}_t v_t = f_t$, where the functions $f_t \geq 0$ are chosen to be supported near ∂M_t and so that the normal derivatives $\bar{N}_t(v_t)$ satisfy (4.9) with $\bar{H}_t = H_0$. Note that f_t may blow up near the singular point $F_0(z)$, which complicates the analysis; we do not pursue this further here.

To conclude this section, we note that it is not difficult to see that the proof of Theorem 1.2 generalizes to a larger class of immersions \mathcal{F} at the boundary of the space of embeddings than the particular class \mathcal{F} used in Theorem 1.2. We will not pursue this in any further detail here. Instead, we make the following more general:

Conjecture 4.11. *Conjecture II is false for any locally flat 3-ball. That is, if F is any smooth immersion of a 3-ball \bar{B} in \mathbb{R}^3 that is not an embedding, then $(\bar{B}, F^*(g_{\text{Eucl}}))$ admits no admissible extension realizing its Bartnik mass.*

5. Remarks on Conjecture III

In this section, we discuss several aspects of Conjecture III, related to the analysis in the previous section on Conjecture II.

To begin, (as noted briefly in Introduction), it is proved in [2, 4] that the moduli space $\mathcal{E}^{m,\alpha}$ of $C^{m,\alpha}$ AF static vacuum solutions (g, u) , $u > 0$, on $M = \mathbb{R}^3 \setminus B$ is a smooth Banach manifold. The moduli space $\mathcal{E}^{m,\alpha}$ is the space of all such static vacuum metrics (g, u) which are $C^{m,\alpha}$ smooth up to ∂M , modulo the action of the $C^{m+1,\alpha}$ diffeomorphisms $\text{Diff}_1^{m+1,\alpha}(M)$ of M equal to the identity on ∂M and asymptotic to the identity at infinity. Moreover, the map to Bartnik boundary data

$$\begin{aligned} \Pi_B : \mathcal{E}^{m,\alpha} &\rightarrow \text{Met}^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2) := \mathcal{B}, \\ \Pi_B(g, u) &= (\gamma, H), \end{aligned} \tag{5.1}$$

is a smooth Fredholm map, of Fredholm index 0, i.e., $\dim \text{Ker } D\Pi_B = \dim \text{Coker } D\Pi_B$, at any (g, u) .

The ADM mass of (M, g) is given by a simple Komar integral

$$m_{\text{ADM}}(g) = \frac{1}{4\pi} \int_{\partial M} N(u) dA_\gamma, \tag{5.2}$$

and clearly the mass

$$m_{\text{ADM}} : \mathcal{E}^{m,\alpha} \rightarrow \mathbb{R}, \tag{5.3}$$

is a smooth function on $\mathcal{E}^{m,\alpha}$.

Conjecture III is the statement that the map Π_B is a bijection when $H > 0$. As in (3.15), let $\mathcal{E}_+^{m,\alpha}$ be the open Banach submanifold of static vacuum metrics with $H > 0$ at ∂M . The map

$$\begin{aligned} \Pi_B : \mathcal{E}_+^{m,\alpha} &\rightarrow \text{Met}^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2), \\ \Pi_B(g) &= (\gamma, H), \end{aligned} \tag{5.4}$$

is clearly also a smooth Fredholm map, of Fredholm index 0.

The question of whether Π_B in (5.4) is a bijection is a PDE issue (global existence and uniqueness for an elliptic boundary value problem) which is now disconnected from the extension issue in Conjecture I. The interior behavior in $B = \mathbb{R}^3 \setminus M$ no longer plays any role (besides assigning boundary data). In particular, the mass function m_{ADM} in (5.3) may well have negative values on $\mathcal{E}_+^{m,\alpha}$. Put another way, it is not at all clear (at least in general) how to restrict the boundary data (γ, H) to the smaller space \mathcal{B}_+ of such data which have nonnegative scalar curvature in-fillings in order to obtain meaningful information about the restricted map $\Pi_B|_{\mathcal{D}_+}$, where $\mathcal{D}_+ = \Pi_B^{-1}(\mathcal{B}_+)$.

It is proved in [4] that Π_B in (5.4) is not a homeomorphism. In fact, Π_B is not proper, and if the inverse map is defined, it is not continuous. The reasons for this are more or less the same as the behavior discussed in Theorem 1.2, namely the passage from embedded spheres to immersed spheres, and it is worth discussing this in more detail.

Let $\text{Imm}^{m+1,\alpha} := \text{Imm}^{m+1,\alpha}(S^2, \mathbb{R}^3)$ be the space of $C^{m+1,\alpha}$ immersions $F : S^2 \rightarrow \mathbb{R}^3$. This is a smooth Banach manifold (an open submanifold of the full mapping space $C^{m+1,\alpha}(S^2, \mathbb{R}^3)$). Similarly, the space $\text{Emb}^{m+1,\alpha} := \text{Emb}^{m+1,\alpha}(S^2, \mathbb{R}^3)$ of $C^{m+1,\alpha}$ embeddings is an open submanifold of $\text{Imm}^{m+1,\alpha}$. Of course, embeddings $F \in \text{Emb}^{m+1,\alpha}$ give static vacuum solutions $(M, g_{\text{Eucl}}, 1)$ where M is the unbounded component of $\mathbb{R}^3 \setminus \text{Im } F$; thus,

$$\text{Emb}^{m+1,\alpha} \subset \mathcal{E}^{m+1,\alpha} \tag{5.5}$$

upon an appropriate identification. Immersions that are not embeddings no longer give such flat static vacuum solutions. It is then natural to consider the behavior of the inclusion (5.5) at the (point-set theoretic) boundary of $\text{Emb}^{m+1,\alpha}$ within $\text{Imm}^{m+1,\alpha}$; denote this space as $\partial \text{Emb}^{m+1,\alpha}$.

In the following, we will identify immersions into \mathbb{R}^3 that differ by a rigid motion of \mathbb{R}^3 . Rigid motions, i.e., the isometry group of \mathbb{R}^3 , act freely on $\text{Imm}^{m+1,\alpha}$ by post-composition. Let $\mathcal{I}mm^{m+1,\alpha}$ be the resulting smooth quotient space.

For $F \in \mathcal{I}mm^{m+1,\alpha}$, the induced metric $\gamma = F^*(g_{\text{Eucl}})$ is a $C^{m,\alpha}$ metric on S^2 while the mean curvature $H = H_F$ is in $C^{m-1,\alpha}(S^2)$. Note that the data (γ, H) are well defined for $F \in \mathcal{I}mm^{m+1,\alpha}$. Thus, the map Π_B in (5.1) or (5.4) defined initially on $\text{Emb}^{m+1,\alpha}$ extends to a smooth map on the larger space $\mathcal{I}mm^{m+1,\alpha}$; to avoid confusion, we denote this extended map as $\Pi_B^{\mathcal{I}}$.

Lemma 5.1. *The map*

$$\Pi_B^{\mathcal{I}} : \mathcal{I}mm^{m+1,\alpha} \rightarrow \mathcal{B}, \quad \Pi_B^{\mathcal{I}}(F) = (\gamma, H) = (F^*(g_{\text{Eucl}}), H_F),$$

is a smooth proper embedding of Banach manifolds.

Proof. The map $\Pi_B^{\mathcal{I}}$ is injective by the proof of Theorem 1.2, i.e., the nonexistence of (nontrivial) Bonnet pairs. The proof that $D\Pi_B^{\mathcal{I}}$ is injective is

essentially the same. Thus, the Gauss–Codazzi (constraint) equations for the immersion F are $\delta_\gamma(A - H\gamma) = 0$, where δ_γ is the divergence. Linearizing gives $\delta'_{\gamma'}(A - H\gamma) + \delta_\gamma(A' - H'\gamma - H\gamma') = 0$, where $\delta'_{\gamma'}$ is the variation of the divergence. If $(\gamma', H') = (0, 0)$, this becomes $\delta_\gamma A' = 0$. Since $\text{tr}A' = (\text{tr}A)' + \langle A, \gamma' \rangle = 0$, it follows as before that A' is a holomorphic quadratic differential on S^2 and hence $A' = 0$. Thus, the full Cauchy data (γ', A') of the immersion vanish. It follows by (infinitesimal) rigidity of surfaces that F' is an infinitesimal rigid motion, so $F' = 0$ in $T\mathcal{I}mm^{m+1,\alpha}$.

Next, we show that Π_B^T is proper. Suppose F_i satisfy $\Pi_B^T(F_i) = (\gamma_i, H_i) \rightarrow (\gamma, H)$ in \mathcal{B} . Then, one has uniform control on δA and $\text{tr}A$. It is well known that (δ, tr) form an elliptic system for symmetric bilinear forms on S^2 . Since the system has trivial kernel on $\mathcal{I}mm^{m+1,\alpha}$, elliptic regularity gives uniform control on $\{A_i\}$ in $C^{m-1,\alpha}$. It is then standard that this gives uniform control on $\{F_i\}$ in $\mathcal{I}mm^{m+1,\alpha}$. Thus, a (sub)sequence of $\{F_i\}$ converges to a limit F , which proves that Π_B^T is proper. \square

Let

$$\mathcal{M} = \text{Im}(\Pi_B^T) \subset \mathcal{B},$$

a properly embedded Banach submanifold representing the Bartnik boundary data of immersions $F : S^2 \rightarrow \mathbb{R}^3$. Let

$$\mathcal{M}^{emb} \subset \mathcal{M}$$

be the open submanifold of embedded Bartnik boundary data, i.e., $\mathcal{M}^{emb} = \text{Im}(\Pi_B^T(\text{Emb}^{m+1,\alpha}))$. Thus

$$\mathcal{M}^{emb} \subset \text{Im}(\Pi_B).$$

However, it is not at all clear whether the full space of immersed boundary data $\mathcal{M} \subset \text{Im}(\Pi_B)$.

As discussed in [4], the map Π_B is not proper when restricted to $\text{Emb}^{m+1,\alpha} \subset \mathcal{E}^{m,\alpha}$. Namely, take any sequence of embeddings F_i converging to an immersion F that is not an embedding. The Bartnik boundary data (γ_i, H_i) of F_i converge to the boundary data (γ, H) of F . However, the sequence of static vacuum solutions $(M, g_i, 1)$ determined by F_i does not converge to a limit in $\mathcal{E}^{m,\alpha}$ (since $\partial(\text{Emb}^{m+1,\alpha})$ is not contained in $\mathcal{E}^{m,\alpha}$).

In analogy to Conjecture 4.11, we make:

Conjecture 5.2. *The Bartnik boundary data (γ, H) , with $H > 0$, of any locally flat 3-ball (that is not an embedded ball in \mathbb{R}^3) do not have a static vacuum extension, i.e., such $(\gamma, H) \in \mathcal{M} \setminus \mathcal{M}^{emb}$ are not in the image of Π_B .*

One may similarly conjecture that Conjecture 5.2 above holds more generally for immersions $F : S^2 \rightarrow M$ into any static vacuum solution (M, g, u) in place of flat \mathbb{R}^3 , where $F(S^2)$ is a surface surrounding ∂M .

We also conjecture there is a second region where Conjecture III breaks down. Recall that the black hole uniqueness theorem [17, 29], together with [46], states that the only static vacuum extension of the boundary data $(\gamma, 0)$

is given by the Schwarzschild metric with $\gamma = \gamma_{2m}$ a round metric of radius $2m$.

Conjecture 5.3. *For any $\gamma \in \text{Met}^{m,\alpha}(S^2)$ of nonconstant Gauss curvature, there is a neighborhood $\mathcal{U}_\gamma \subset C_+^{m-1,\alpha}(S^2)$ with $0 \in \bar{\mathcal{U}}_\gamma$, such that for $H \in \mathcal{U}_\gamma$, the boundary data (γ, H) do not bound a static vacuum metric (M, g, u) . In particular, Π_B is not surjective near the Schwarzschild metric.*

Partial evidence for this conjecture is given by the main compactness theorem (Theorem 1.2) in [4]. Namely, if (M, g_i, u_i) is a sequence of static vacuum solutions, $u_i > 0$, with boundary data $(\gamma_i, H_i) \rightarrow (\gamma, H)$ in \mathcal{B} , with $H_i > 0$ and $H \geq 0$, and if ∂M is strictly outer-minimizing in (M, g_i, u_i) for all i , then a subsequence of (M, g_i, u_i) converges in $\mathcal{E}^{m,\alpha}$ to a limit (M, g, u) realizing the data (γ, H) . Setting $H = 0$, one has a contradiction to the black hole uniqueness theorem if $\gamma \neq \gamma_{2m}$ for some $m > 0$. Hence, such outer-minimizing solutions cannot exist for i sufficiently large.

It remains an open question as to whether such static vacuum extensions exist with ∂M not outer-minimizing. If such a sequence exists, either the curvature of g_i must blow up near ∂M or the distance to the cut locus of the normal exponential map must tend to zero (or both), as $i \rightarrow \infty$.

Remark 5.4. The compactness result above suggests modifying the Bartnik mass m_B to \tilde{m}_B by allowing for only outer-minimizing extensions, as suggested by Bray [14]. Note this rules out the constructions above in the proof of Theorem 1.2 and the discussion above on Conjecture III, which are certainly *not* outer-minimizing extensions. One may also restrict the map Π_B in (5.4) to the space $\tilde{\mathcal{E}}^{m,\alpha}$ of static vacuum solutions for which ∂M is strictly outer-minimizing. Note that $\tilde{\mathcal{E}}^{m,\alpha}$ is an open domain in $\mathcal{E}_+^{m,\alpha}$. However, as discussed above, the restricted map Π_B on $\tilde{\mathcal{E}}^{m,\alpha}$ is not surjective (onto a product neighborhood of the Schwarzschild boundary data). Thus, Conjecture III also fails for the modified mass \tilde{m}_B .

Observe that boundary data (γ, H) near Schwarzschild data $(\gamma_{2m}, 0)$ do have outer-minimizing extensions in $\mathring{\mathcal{P}}^{m,\alpha}$. The discussion above (together with Theorem 1.1) strongly suggests that Conjecture II also fails for the modified mass \tilde{m}_B , i.e., there exist (γ, H) for which there is no mass-minimizing extension realizing \tilde{m}_B .

Although Theorem 1.2 shows that Conjecture II is false in general, (and similarly the discussion above indicates that Conjecture III is likely to be false in general) one would still like to find natural geometric conditions on the boundary data (γ, H) of the region Ω under which these conjectures could remain valid. In a simpler but related setting, the guiding light along these lines is the famous Weyl embedding theorem [48, 50] that a 2-sphere S^2 with metric γ of positive Gauss curvature embeds isometrically in \mathbb{R}^3 as the boundary of a convex body Ω . In particular, the normal exponential map \exp_N into the exterior $M = \mathbb{R}^3 \setminus \Omega$ has no cut or focal points.

As discussed in [3, 4] and seen here in Theorem 1.2, the presence of nearby cut or focal points of \exp_N is the primary difficulty in establishing Conjecture III and is of course also basic in establishing Conjecture II. Thus, it is natural to ask:

Question. Are there natural geometric conditions on (γ, H) such that any extension of (γ, H) in $\mathcal{P}^{m,\alpha}$ or any static vacuum extension in $\mathcal{E}_+^{m,\alpha}$ has a lower bound on the distance to the cut locus of \exp_N ?

Unfortunately, there is little evidence (if any) to suggest that the conditions $K_\gamma > 0$ and $H > 0$ are sufficient for this purpose, i.e., a simple, direct generalization of the Weyl embedding theorem has little support for its validity. On the other hand, it would of course be interesting to find any examples where $K_\gamma > 0$, $H > 0$ with the distance to the cut locus of \exp_N arbitrarily small.

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