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Long-Range Scattering for Discrete Schrödinger Operators

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Abstract. In this paper, we define time-independent modifiers to construct a long-range scattering theory for a class of difference operators on $\mathbb{Z}^d$, including the discrete Schrödinger operators on the square lattice. The modifiers are constructed by observing the corresponding Hamilton flow on $T^*\mathbb{T}^d$. We prove the existence and completeness of modified wave operators in terms of the above-mentioned time-independent modifiers.

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1. Introduction

We consider a class of generalized discrete Schrödinger operators H_0 and H on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, $d \geq 1$,

$$\begin{cases} H_0 u[x] = \displaystyle\sum_{y \in \mathbb{Z}^d} f[y] u[x-y], \\ Hu[x] = H_0 u[x] + V[x] u[x], \end{cases} \tag{1.1}$$

where $f \in \mathscr{S}(\mathbb{Z}^d) := \{u \in \ell^2(\mathbb{Z}^d) \mid u[x] = \mathcal{O}(\langle x \rangle^{-\infty})\}$, $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$, satisfies $f[-x] = \overline{f[x]}$, $x \in \mathbb{Z}^d$, and V is a real-valued bounded function on $\mathbb{Z}^d$. Then H_0 and H are bounded self-adjoint operators on $\mathcal{H}$.

We define the discrete Fourier transform F by

$$Fu(\xi) = (2\pi)^{-\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} e^{-ix \cdot \xi} u[x], \quad \xi \in \mathbb{T}^d = [-\pi, \pi)^d$$

for $u \in \ell^1(\mathbb{Z}^d)$. Then F is continuously extended to a unitary operator from $\mathcal{H}$ to $L^2(\mathbb{T}^d)$ and

$$H_0 u[x] = F^* \left(h_0(\cdot) F u(\cdot) \right) [x],$$

where

$$h_0(\xi) := \sum_{x\in\mathbb{Z}^d} e^{-ix\cdot\xi} f[x], \quad \xi \in \mathbb{T}^d. \tag{1.2}$$

The above condition on f implies h_0 is a real-valued smooth function on $\mathbb{T}^d$. We denote by $v(\xi)$ and $A(\xi)$ the generalized velocity and the Hessian of h_0, respectively:

$$v(\xi) = \nabla_\xi h_0(\xi),$$
$$A(\xi) = {}^t\nabla_\xi \nabla_\xi h_0(\xi) = (\partial_{\xi_j}\partial_{\xi_k} h_0(\xi))_{1\le j,k\le d}.$$

The set of threshold energies is denoted by $\mathcal{T}$,

$$\mathcal{T} = \left\{ h_0(\xi) \mid \xi \in \mathbb{T}^d, v(\xi) = 0 \right\}.$$

We note $\mathcal{T}$ has Lebesgue measure 0 by Sard's theorem. We first assume the condition below.

Assumption 1.1. The sets $\{\xi \in \mathbb{T}^d \mid v(\xi) = 0\}$ and $\{\xi \in \mathbb{T}^d \mid \det A(\xi) = 0\}$ have d-dimensional Lebesgue measure zero.

The above assumption implies the absence of point and singular continuous spectrum. The following assertion is a generalized version of Theorem 12.3.2 in [5].

Proposition 1.2. *Suppose that the set $\{\xi \in \mathbb{T}^d \mid v(\xi) = 0\}$ has d-dimensional Lebesgue measure zero. Then H_0 has purely absolutely continuous spectrum and $\sigma_{\mathrm{ac}}(H_0) = h_0(\mathbb{T}^d)$, where $\sigma_{\mathrm{ac}}(H_0)$ denotes the absolutely continuous spectrum of H_0.*

Proof. Fix a point $\xi_0 \in W := \{\xi \in \mathbb{T}^d \mid v(\xi) \neq 0\}$. Then it suffices to prove $C_c^\infty(U) \subset \mathcal{H}_{ac}(FH_0F^*)$ for some neighborhood $U \subset W$ of ξ_0; for any $f \in C_c^\infty(U)$,

$$\mathcal{B}(\sigma(H_0)) \to \mathbb{R},\ B \mapsto \int_{h_0^{-1}(B)\cap \operatorname{supp} f} |f(\xi)|^2 \mathrm{d}\xi$$

is an absolutely continuous Borel measure. The claim is proved by taking a local coordinate $U \ni x \mapsto (y(x), h_0(x)) \in \mathbb{R}^{d-1}\times\mathbb{R}$. □

If $V[x]$ decays at infinity, then V is a compact operator on $\mathcal{H}$ and hence $\sigma_{\mathrm{ess}}(H) = \sigma_{\mathrm{ess}}(H_0) = \sigma_{ac}(H_0) = h_0(\mathbb{T}^d)$, where $\sigma_{\mathrm{ess}}(H)$ and $\sigma_{\mathrm{ess}}(H_0)$ denotes the essential spectrum of H and H_0, respectively. We suppose a long-range condition on V.

Assumption 1.3. There exist $\tilde{V} \in C^\infty(\mathbb{R}^d;\mathbb{R})$ and $\varepsilon \in (0,1]$ such that $\tilde{V}|_{\mathbb{Z}^d} = V$ and

$$\left|\partial_x^\alpha \tilde{V}(x)\right| \le C_\alpha \langle x\rangle^{-|\alpha|-\varepsilon}, \quad x \in \mathbb{R}^d,\ \alpha \in \mathbb{Z}_+^d,$$

where $\mathbb{Z}_+ = \{0,1,2,\dots\}$.

Under Assumptions 1.1 and 1.3, the singular continuous spectrum of H is empty (see, e.g., [12]). In the following, we write V for $\tilde{V}$ without confusion.

Remark 1.4. Assumption 1.3 is equivalent to the following condition used in [11],

$$\left|\tilde{\partial}_x^\alpha V\,[x]\right| \leq C'_\alpha \langle x\rangle^{-|\alpha|-\varepsilon}, \quad x \in \mathbb{R}^d,\ \alpha \in \mathbb{Z}_+^d,$$

where $\tilde{\partial}_x^\alpha = \tilde{\partial}_{x_1}^{\alpha_1} \cdots \tilde{\partial}_{x_d}^{\alpha_d}$, and $\tilde{\partial}_{x_j} V\,[x] = V\,[x] - V\,[x-e_j]$ is the difference operator with respect to the jth variable. Here $\{e_j\}$ is the standard orthogonal basis of $\mathbb{R}^d$. See Lemma 2.1 in [11] for the detail.

In Sect. 2, we construct modified wave operators with time-independent modifiers, which are proposed by Isozaki and Kitada [7], so-called Isozaki–Kitada modifiers. Isozaki–Kitada modifiers are formally defined by

$$W_J^\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J e^{-itH_0}.$$

We construct J as an operator of the form

$$Ju\,[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi(x,\xi)-y\cdot\xi)} u[y] \mathrm{d}\xi, \tag{1.3}$$

where the phase function φ is a solution to the eikonal equation

$$h_0(\nabla_x \varphi(x,\xi)) + V(x) = h_0(\xi) \tag{1.4}$$

in the "outgoing" and "incoming" regions and considered in "Appendix A".

The next theorem is our main result.

Theorem 1.5. *Under* Assumptions 1.1 *and* 1.3, *there exists an operator J of the form* (1.3) *such that, for any $\Gamma \Subset h_0(\mathbb{T}^d)\backslash\mathfrak{T}$, the modified wave operators*

$$W_J^\pm(\Gamma) := \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} J e^{-itH_0} E_{H_0}(\Gamma) \tag{1.5}$$

exist, where E_{H_0} denotes the spectral measure of H_0. Furthermore, the following properties hold:

- (i) *Intertwining property: $HW_J^\pm(\Gamma) = W_J^\pm(\Gamma)H_0$.*
- (ii) *Partial isometries: $\|W_J^\pm(\Gamma)u\| = \|E_{H_0}(\Gamma)u\|$.*
- (iii) *Asymptotic completeness:* $\operatorname{Ran} W_J^\pm(\Gamma) = E_H(\Gamma)\mathcal{H}_{\mathrm{ac}}(H)$.

Examples 1.6. (i) In [11], a long-range scattering theory of the standard difference Laplacian $H_0u[x] = -\frac{1}{2}\sum_{|y-x|=1} u\,[y]\,,\ x \in \mathbb{Z}^d$ is considered. In this case, $h_0(\xi) = -\sum_{j=1}^d \cos\xi_j$ satisfies Assumption 1.1.

(ii) A model for two-dimensional triangle lattice is expressed by the operator $H_0u[x] = -\frac{1}{6}\sum_{j=1}^6 u[x+n_j],\ x \in \mathbb{Z}^2$, where $n_1 = (1,0)$, $n_2 = (-1,0)$, $n_3 = (0,1)$, $n_4 = (0,-1)$, $n_5 = (1,-1)$, $n_6 = (-1,1)$ (see, e.g., [2]). Since

$$h_0(\xi) = -\frac{1}{3}(\cos\xi_1 + \cos\xi_2 + \cos(\xi_1 - \xi_2))$$

in this case, Assumption 1.1 is satisfied.

Scattering theory for Schrödinger operators on $\mathbb{R}^d$ has been extensively studied [1,6,14,15]. If the perturbation is long range, i.e., $V(x) = O(\langle x\rangle^{-\varepsilon})$, $0 < \varepsilon \leq 1$, then the scattering theory needs a modification [6,7,15]. Discrete Schrödinger operator describes the state of electrons in solid matters with graph structure. Spectral properties of discrete Schrödinger operators have been studied in [2,4,8,11–13].

The main idea of the construction of modifiers is similar to [11]. We translate H into an operator on the flat torus $\mathbb{T}^d$ via discrete Fourier transform and consider the corresponding classical mechanics on $\mathbb{T}^d$. The proof is mainly based on [7]. We use the time-decaying method to construct the phase function φ in the definition of J, and then the stationary phase method and the Enss method to prove the existence and completeness of modified wave operators. The construction of φ is given in "Appendix A", which follows the argument of [9]. The main properties of φ are summarized in Proposition 2.1. In Sect. 2, we prepare some lemmas for the proof of Theorem 1.5. The Poisson summation formula is used to prove that pseudo-difference operators on $\mathbb{Z}^d$ are translated to pseudo-differential operators on $\mathbb{T}^d$ modulo smoothing operators (see the proof of Lemma 2.3 in "Appendix B"). This enables us to get over the difficulty derived from the discreteness of $\mathbb{Z}^d$. In Sect. 3, we prove Theorem 1.5.

2. Preliminaries

We first state a proposition on the Hamilton flow generated by $h(x,\xi) := h_0(\xi) + V(x)$, which is proved in "Appendix A". Here we note that h_0, v and A are extended periodically in ξ from $\mathbb{T}^d = [-\pi,\pi)^d$ to $\mathbb{R}^d$, and we identify integrations on $\mathbb{T}^d$ with those on $[-\pi,\pi)^d$. We also note that the following proposition concerns functions on $\mathbb{R}^d \times \left(\mathbb{R}^d \backslash v^{-1}(0)\right)$, not on $\mathbb{Z}^d \times \left(\mathbb{T}^d \backslash v^{-1}(0)\right)$.

We fix $\chi \in C^\infty(\mathbb{R}^d)$ such that

$$\chi(x) = \begin{cases} 0 & \text{if } |x| \le 1, \\ 1 & \text{if } |x| \ge 2, \end{cases} \tag{2.1}$$

and we define $\cos(x,y) := \frac{x\cdot y}{|x||y|}$ for $x, y \in \mathbb{R}^d \backslash \{0\}$. The following assertion is an analogue of Theorem 2.5 in [7].

Proposition 2.1. *There exists a real-valued function $\varphi \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \backslash v^{-1}(0)))$ satisfying the following properties: Set $a > 0$. Let $\varphi_a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ be defined by*

$$\varphi_a(x,\xi) = (\varphi(x,\xi) - x\cdot\xi)\chi\left(\frac{v(\xi)}{a}\right) + x\cdot\xi. \tag{2.2}$$

(1) *The function φ_a satisfies*

$$\varphi_a(x,\xi + 2\pi m) = \varphi_a(x,\xi) + 2\pi x\cdot m, \quad m \in \mathbb{Z}^d, \tag{2.3}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta \left[\varphi_a(x,\xi) - x\cdot\xi\right]\right| \le C_{\alpha\beta,a}\langle x\rangle^{1-\varepsilon-|\alpha|}, \tag{2.4}$$

$$\left|{}^t\nabla_x\nabla_\xi\varphi_a(x,\xi) - I\right| < \frac{1}{2} \tag{2.5}$$

for $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, where $|M| := \left(\sum_{j,k=1}^d |M_{jk}|^2\right)^{\frac{1}{2}}$ for a matrix M.

(2) *We set*

$$J_a u[x] := (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi_a(x,\xi)-y\cdot\xi)} u[y] \mathrm{d}\xi. \tag{2.6}$$

Then

$$(HJ_a - J_a H_0)u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi_a(x,\xi)-y\cdot\xi)} s_a(x,\xi) u[y] \,\mathrm{d}\xi, \tag{2.7}$$

where

$$\begin{aligned} s_a(x,\xi) &:= e^{-i\varphi_a(x,\xi)} H(e^{i\varphi_a(\cdot,\xi)})[x] - h_0(\xi) \\ &= \sum_{z\in\mathbb{Z}^d} f[z] e^{i(\varphi_a(x-z,\xi)-\varphi_a(x,\xi))} + V[x] - h_0(\xi) \end{aligned} \tag{2.8}$$

satisfies for $|x| \geq 1$ *and* $|v(\xi)| \geq a$

$$\left|\partial_\xi^\beta s_a(x,\xi)\right| \leq \begin{cases} C_{\beta,a}\langle x\rangle^{-1-\varepsilon}, & |\cos(x,v(\xi))| \geq \frac{1}{2}, \\ C_{\beta,a}\langle x\rangle^{-\varepsilon}, & |\cos(x,v(\xi))| \leq \frac{1}{2}. \end{cases} \tag{2.9}$$

We note that φ_a satisfies the eikonal equation (1.4) on $\{(x,\xi) \mid |x| \geq R_a, |v(\xi)| \geq a, |\cos(x,v(\xi))| \geq \frac{1}{2}\}$ and that the property is used for the proof of (2.9) in the $|\cos(x,v(\xi))| \geq \frac{1}{2}$ case (see Proposition A.9 and (A.51)).

In the rest of this section, we prepare some lemmas for the proof of properties (ii) and (iii). We choose $\gamma \in C_c^\infty(h_0(\mathbb{T}^d)\backslash\mathcal{T})$ and $\rho_\pm \in C^\infty([-1,1];[0,1])$ such that

$$\begin{aligned} &\rho_+(\sigma) + \rho_-(\sigma) = 1, \\ &\rho_+(\sigma) = 1, \quad \sigma \in \left[\frac{1}{4}, 1\right], \\ &\rho_-(\sigma) = 1, \quad \sigma \in \left[-1, -\frac{1}{4}\right]. \end{aligned}$$

Using γ and $\rho_\pm$, we define operators with cutoffs in the energy and the direction of x and $v(\xi)$. We set symbols $p_\pm$ and operators $P_\pm$, $\tilde{P}_\pm$ and $E_\pm(t)$ by

$$p_\pm(y,\xi) = \gamma(h_0(\xi))\chi(y)\rho_\pm(\cos(y,v(\xi))), \tag{2.10}$$

$$P_\pm u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x-y)\cdot\xi} p_\pm(y,\xi) u[y] \,\mathrm{d}\xi, \tag{2.11}$$

$$\tilde{P}_\pm u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x\cdot\xi-\varphi_a(y,\xi))} p_\pm(y,\xi) u[y] \,\mathrm{d}\xi, \tag{2.12}$$

$$E_\pm(t) = J_a e^{-itH_0} \tilde{P}_\pm, \quad t \in \mathbb{R}, \tag{2.13}$$

where J_a is defined by (2.6).

We consider properties of these operators. We use the stationary phase method as in the pseudo-differential operator calculus (see, e.g., [16]). The following two Lemmas correspond to Proposition 3.4 and Lemma 3.7 in [7], and the proofs are given in "Appendix B" (see also [3,7]).

Lemma 2.2. *J_a, $P_\pm$ and $\tilde{P}_\pm$ are bounded operators on $\mathcal{H}$.*

Lemma 2.3. *$\gamma(H_0) - P_+ - P_-$, $P_\pm^* - P_\pm$, $E_\pm(0) - P_\pm$, $J_a^* J_a - I$ and $J_a J_a^* - I$ are compact operators on $\mathcal{H}$.*

The next lemma, corresponding to Proposition 3.8 in [7], is an analogue of the intertwining property of wave operators.

Lemma 2.4. *For any $s \in \mathbb{R}$,*

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{itH_0} J_a^* E_\pm(t-s) = e^{isH_0}\tilde{P}_\pm. \tag{2.14}$$

Proof. The definition of $E_\pm(t)$ implies

$$\begin{aligned} e^{itH_0} J_a^* E_\pm(t-s) &= e^{itH_0} J_a^* J_a e^{-i(t-s)H_0}\tilde{P}_\pm \\ &= e^{itH_0}\left(J_a^* J_a - I\right) e^{-itH_0} e^{isH_0}\tilde{P}_\pm + e^{isH_0}\tilde{P}_\pm. \end{aligned}$$

Since $e^{-itH_0}u \to 0$ weakly as $t \to \pm\infty$ for any $u \in \mathcal{H} = \mathcal{H}_{\mathrm{ac}}(H_0)$, Lemma 2.3 implies that the first term converges strongly to 0 as $t \to \pm\infty$. □

Next we prove the norm convergence of $\lim_{t\to\pm\infty} e^{itH}E_\pm(t)$. If we set

$$G_\pm(t) := \left(\frac{\mathrm{d}}{i\mathrm{d}t} + H\right)E_\pm(t) = (HJ_a - J_a H_0)E_\pm(t),$$

then we have

$$e^{itH}E_\pm(t) - P_\pm = E_\pm(0) - P_\pm + i\int_0^t e^{i\tau H}G_\pm(\tau)\mathrm{d}\tau.$$

The following proposition is analogous to Theorem 3.5 in [7], and proves $G_\pm(t)$ is integrable in $\{\pm t \geq 0\}$, respectively.

Proposition 2.5. *$G_\pm(t)$ is norm continuous and compact for any $t \in \mathbb{R}$. Furthermore, $G_\pm(t)$ satisfies*

$$\|G_\pm(t)\| \leq C\langle t\rangle^{-1-\varepsilon}, \ \pm t \geq 0. \tag{2.15}$$

In particular, $e^{itH}E_\pm(t) - P_\pm$ converges to a compact operator with respect to the norm topology as $t \to \pm\infty$, respectively.

Proof. Let

$$\Phi(x, y, \xi; t) := \varphi_a(x, \xi) - th_0(\xi) - \varphi_a(y, \xi).$$

Then the definition (2.13) of $E_\pm(t)$ implies

$$\begin{aligned} G_\pm(t)u[x] &= (HJ_a - J_a H_0)e^{-itH_0}\tilde{P}_\pm u[x] \\ &= (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{i\Phi(x,y,\xi;t)} s_a(x,\xi)p_\pm(y,\xi)u[y]\mathrm{d}\xi. \end{aligned}$$

The norm continuity of $G_\pm(t)$ is obvious. Furthermore, (2.9) implies the compactness of $HJ_a - J_a H_0$ by the similar argument in the proof of Lemma 2.3, hence $G_\pm(t)$ is compact.

Let us prove (2.15). We consider the + case only. The other case is proved similarly. We use another decomposition $\rho^{\pm} \in C^{\infty}([-1,1];[0,1])$ which is different from $\rho_{\pm}$ in that

$$\rho^{+}(\sigma) + \rho^{-}(\sigma) = 1,$$
$$\rho^{+}(\sigma) = \begin{cases} 1, & \sigma \geq \frac{3}{4}, \\ 0, & \sigma \leq \frac{1}{2}. \end{cases}$$

We define

$$s_{-}(x,\xi) := s_a(x,\xi)\chi_{\{x\neq 0\}}\rho^{-}(\cos(x, v(\xi))),$$
$$s_{+}(x,\xi) := s_a(x,\xi) - s_{-}(x,\xi).$$

We then decompose G_+ as

$$\begin{aligned} G_+(t)u[x] &= (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{i\Phi(x,y,\xi;t)}(s_+p_+ + s_-p_+)(x,y,\xi)u[y]\mathrm{d}\xi \\ &=: (F_+(t) + F_-(t))u[x]. \end{aligned} \tag{2.16}$$

Now we claim that for any $t \geq 0$ and $\ell \geq 0$,

$$\|F_+(t)\| \leq C\langle at\rangle^{-1-\varepsilon}, \tag{2.17}$$
$$\|F_-(t)\| \leq C_\ell\langle at\rangle^{-\ell}. \tag{2.18}$$

If (2.17) and (2.18) hold, then (2.15) follows from (2.16).

For the proof of (2.17), we let

$$\phi(t;y,\xi) := th_0(\xi) + \varphi_a(y,\xi)$$

and set

$$L_1 := \langle\nabla_\xi\phi\rangle^{-2}(1 - \nabla_\xi\phi\cdot D_\xi).$$

Then (2.4) implies on the support of $s_+(x,\xi)p_+(y,\xi)$,

$$\langle\nabla_\xi\phi\rangle^{-1} \leq C\langle|y| + t|v(\xi)|\rangle^{-1}.$$

Thus, for any $\ell \in \mathbb{Z}_+$, we have

$$\begin{aligned} F_+(t)u[x] &= (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} L_1^{\ell}\left(e^{-i\phi(t;y,\xi)}\right)e^{i\varphi_a(x,\xi)}s_+(x,\xi)p_+(y,\xi)u[y]\mathrm{d}\xi \\ &= (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{-i\phi(t;y,\xi)}\left({}^tL_1\right)^{\ell}\left(e^{i\varphi_a(x,\xi)}s_+(x,\xi)p_+(y,\xi)\right)u[y]\mathrm{d}\xi \\ &= (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{i\Phi(t;y,\xi)}\left\{e^{-i\varphi_a(x,\xi)}\left({}^tL_1\right)^{\ell}\left(e^{i\varphi_a(x,\xi)}s_+p_+\right)\right\}u[y]\mathrm{d}\xi. \end{aligned}$$

The function in {} is a finite sum of functions of the form $s_j^{\ell}(x,\xi)p_j^{\ell}(y,\xi;t)$ such that

$$\begin{cases} \left|\partial_\xi^\beta s_j^\ell(x,\xi)\right| \leq C_\beta\langle x\rangle^{\ell-1-\varepsilon}, \\ \\ \left|\partial_\xi^\beta p_j^\ell(y,\xi;t)\right| \leq C_\beta\langle|y| + t|v(\xi)|\rangle^{-\ell}. \end{cases} \tag{2.19}$$

Indeed, (2.19) follows from (2.9) and (2.10). Letting

$$S_j^\ell u[x] := (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi_a(x,\xi)-y\cdot\xi)} s_j^\ell(x,\xi) u[y] \mathrm{d}\xi,$$

$$P_j^\ell(t) u[x] := (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x\cdot\xi-\varphi_a(y,\xi))} p_j^\ell(y,\xi;t) u[y] \mathrm{d}\xi,$$

we have

$$F_+(t) = \sum_j S_j^\ell e^{-itH_0} P_j^\ell(t).$$

Furthermore, we have by (2.19) and the argument in the proof of Lemma 2.2

$$\left\| \langle x\rangle^{1+\varepsilon-\ell} S_j^\ell \right\| < \infty,$$
$$\left\| P_j^\ell(t) \right\| \le C_\ell \langle at\rangle^{-\ell}.$$

Thus we obtain

$$\| \langle x\rangle^{1+\varepsilon-\ell} F_+(t) \| \le C'_\ell \langle at\rangle^{-\ell}$$

for any $\ell \in \mathbb{Z}_+$. Interpolation with respect to ℓ implies (2.17).

For the proof of (2.18), we note on the support of $s_-(x,\xi)p_+(y,\xi)$,

$$\langle \nabla_\xi \Phi\rangle^{-1} \le C\langle |x-y| + t|v(\xi)|\rangle^{-1}.$$

Letting

$$L_2 := \langle \nabla_\xi \Phi\rangle^{-2}(1 + \nabla_\xi\Phi\cdot D_\xi),$$

we have

$$\begin{aligned} F_-(t)u[x] &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i\Phi(x,y,\xi;t)} \left({}^tL_2\right)^\ell (s_-(x,\xi)p_+(y,\xi)) u[y]\mathrm{d}\xi \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi_a(x,\xi)-\varphi_a(y,\xi))} e^{-ith_0(\xi)} \left({}^tL_2\right)^\ell (s_-p_+) u[y]\mathrm{d}\xi \end{aligned}$$

for any $\ell \in \mathbb{Z}_+$. Since

$$q^\ell(x,y,\xi;t) := e^{-ith_0(\xi)} \left({}^tL_2\right)^\ell (s_-(x,\xi)p_+(y,\xi))$$

satisfies

$$\left| \partial_\xi^\beta q^\ell(x,y,\xi;t) \right| \le C_{\ell,\beta} \langle tv(\xi)\rangle^{|\beta|-\ell}$$

for any $\ell \in \mathbb{Z}_+$, we obtain (2.18) by the argument in the proof of Lemma 2.2. □

The next proposition claims that any particle in the energy Γ does not stay in any bounded domain in x.

Proposition 2.6. *For any $R > 0$ and $\ell \ge 0$,*

$$\| \chi_{\{|x|<R\}} E_\pm(s) \| \le C_{\ell,R} \langle s\rangle^{-\ell}, \quad \pm s \ge 0. \tag{2.20}$$

Proof. We prove (2.20) for the + case only. We first note

$$E_+(s)u[x] = (2\pi)^{-d}\int_{\mathbb{T}^d}\sum_{y\in\mathbb{Z}^d} e^{i\Phi(x,y,\xi;s)}p_+(y,\xi)u[y]\mathrm{d}\xi,$$

where $\Phi(x,y,\xi;t) = \varphi_a(x,\xi) - th_0(\xi) - \varphi_a(y,\xi)$. We observe that on the support of $p_+(y,\xi)$,

$$|sv(\xi) + \nabla_\xi\varphi_a(y,\xi)| \geq c(|y| + s|v(\xi)|)$$

for large s. Then, if $|x| \leq R$, we have for $s > 0$ large enough

$$|\nabla_\xi\Phi(x,y,\xi;s)| \geq c(|y| + s|v(\xi)|), \quad (y,\xi) \in \operatorname{supp} p_+.$$

Similarly to the proof of (2.18), we obtain (2.20). □

3. Proof of Theorem 1.5

3.1. Existence of Modified Wave Operators

We prove the existence of the limit (1.5) for the + case only. The other case is proved similarly. First we fix $\Gamma \Subset h_0(\mathbb{T}^d)\setminus\mathcal{T}$. We remark that, for any $u \in \mathcal{H}$ such that $Fu \in C^\infty(\mathbb{T}^d)$ and $\operatorname{supp} Fu \subset h_0^{-1}(\Gamma)$, we have

$$JE_{H_0}(\Gamma)u = J_a u \tag{3.1}$$

for some small enough $a > 0$. Then, to prove the existence of the limit (1.5), it suffices to show that

$$\begin{aligned}\int_0^\infty &\left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{itH}Je^{-itH_0}E_{H_0}(\Gamma)u\right)\right\|\mathrm{d}t\\ &= \int_0^\infty \left\|\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{itH}J_a e^{-itH_0}u\right)\right\|\mathrm{d}t\\ &= \int_0^\infty \|e^{itH}(HJ_a - J_aH_0)e^{-itH_0}u\|\mathrm{d}t\\ &= \int_0^\infty \|(HJ_a - J_aH_0)e^{-itH_0}u\|\mathrm{d}t\end{aligned} \tag{3.2}$$

is finite for such u. The last equality follows from the fact that e^{itH} is a unitary operator. Furthermore, by Assumption 1.1 and a partition of unity on $\mathbb{T}^d$, we may assume that $Fu \in C^\infty(\mathbb{T}^d)$ has a sufficiently small support in $\{\xi \in h_0^{-1}(\Gamma) \mid \det A(\xi) \neq 0\}$.

Let $w(t) := (HJ_a - J_aH_0)e^{-itH_0}u$. Then (2.7) implies

$$w(t)[x] = (2\pi)^{-\frac{d}{2}}\int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi) - th_0(\xi))}s_a(x,\xi)Fu(\xi)\mathrm{d}\xi.$$

Now we use the stationary phase method. The stationary point $\xi = \xi(x,t)$ is determined by

$$\frac{1}{t}\nabla_\xi\varphi_a(x,\xi) - v(\xi) = 0. \tag{3.3}$$

We define

$$D_t := \{x \in \mathbb{Z}^d \mid \exists \xi \in \operatorname{supp} Fu \text{ s.t. } (3.3) \text{ holds}\}.$$

By (2.4), there exists an open set $U \Subset \{\xi \in h_0^{-1}(\Gamma) \mid \det A(\xi) \neq 0\}$ such that $\operatorname{supp} Fu \Subset U$ and that for $t > 0$ large enough,

$$D_t \subset \left\{x \mid \frac{x}{t} \in v(U)\right\} =: D_t'.$$

On $(D_t')^c$, the non-stationary phase method implies

$$|w(t)[x]| \leq C_\ell \langle |x| + t\rangle^{-\ell}, \quad x \in \mathbb{Z}^d,\ t > 0$$

for any $\ell \geq 0$. Thus we learn for any $\ell \geq 0$

$$\|\chi_{(D_t')^c} w(t)\| \leq C_\ell' t^{-\ell}. \tag{3.4}$$

On D_t', the stationary phase method implies

$$w(t)[x] = t^{-\frac{d}{2}} A(t,x) s_a(x, \xi(x,t)) Fu(\xi(x,t)) + t^{-\frac{d}{2}-1} r(t,x),$$

where $A(t,x)$ is uniformly bounded in x and t with $x \in D_t'$, and

$$|r(t,x)| \leq C \sup_{|\beta| \leq d+3} \sup_{\xi \in \operatorname{supp} Fu} |\partial_\xi^\beta s_a(x,\xi)|.$$

Since $\cos(x, v(\xi)) \geq \frac{1}{2}$ for $x \in D_t'$ and $\xi \in \operatorname{supp} Fu$ if t is sufficiently large, we have by (2.9)

$$|s_a(x, \xi(x,t))| \leq C\langle x\rangle^{-1-\varepsilon},$$
$$|r(t,x)| \leq C\langle x\rangle^{-1-\varepsilon}.$$

We note $|x| \sim t$ on D_t' and the Lebesgue measure of D_t' is bounded by Ct^d. Thus we learn

$$\|\chi_{D_t'} w(t)\| \leq \left(\int_{D_t'} \left(Ct^{-\frac{d}{2}} \langle x\rangle^{-1-\varepsilon}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C' t^{-1-\varepsilon}. \tag{3.5}$$

Hence (3.4) and (3.5) imply

$$\|w(t)\| \leq \|\chi_{D_t'} w(t)\| + \|\chi_{(D_t')^c} w(t)\| \leq C'' t^{-1-\varepsilon},$$

which proves (3.2) is finite. □

3.2. Proof of the Properties (i), (ii) and (iii)

Proof of (i). The intertwining property is proved similarly to the short-range case (see, e.g., [14]). □

Proof of (ii). It suffices to show $\|W_J^\pm(\Gamma)u\| = \|u\|$ for $Fu \in C^\infty(\mathbb{T}^d)$ with $\operatorname{supp} Fu \subset h_0^{-1}(\Gamma)$. For such u, $Ju = J_a u$ holds for small $a > 0$. Thus letting $u_t = e^{-itH_0}u$, we learn

$$\left\|W_J^\pm(\Gamma)u\right\|^2 = \lim_{t\to\pm\infty} \|J_a u_t\|^2 = \lim_{t\to\pm\infty} ((J_a^* J_a - I)u_t, u_t) + \|u\|^2.$$

Using $\text{w-lim}_{t\to\pm\infty} u_t = 0$ and Lemma 2.3, we have $\lim_{t\to\pm\infty}(J_a^* J_a - I)u_t = 0$. This proves $W_J^\pm(\Gamma)$ are partial isometries. □

Proof of (iii). We prove the asymptotic completeness of $W_J^+(\Gamma)$ only. Since intertwining property implies $\operatorname{Ran} W_J^+(\Gamma) \subset E_H(\Gamma)\mathcal{H}_{\mathrm{ac}}(H)$, it suffices to prove $\operatorname{Ran} W_J^+(\Gamma) \supset E_H(\Gamma)\mathcal{H}_{\mathrm{ac}}(H)$.

We fix $v \in \mathcal{H}_{\mathrm{ac}}(H)$ and $\gamma \in C^\infty(\mathbb{R})$ so that $\gamma(H)v = v$ and $\operatorname{supp}\gamma \subset \Gamma$. We set $v_t := e^{-itH}v$ for simplicity. Then we show that $E_H(\Gamma)\mathcal{H}_{\mathrm{ac}}(H) \subset \operatorname{Ran} W_J^+(\Gamma)$ follows from

$$\lim_{s\to\infty}\limsup_{t\to\infty}\|v_s - e^{i(t-s)H}E_+(t-s)v_s\| = 0. \tag{3.6}$$

First, we observe

$$\begin{aligned}&\left\|e^{itH_0}J_a^* e^{-itH}v - e^{isH_0}\tilde{P}_+ v_s\right\|\\ &\quad\le \left\|e^{itH_0}J_a^*\left[v_t - E_+(t-s)v_s\right]\right\| + \left\|e^{itH_0}J_a^*E_+(t-s)v_s - e^{isH_0}\tilde{P}_+v_s\right\|.\end{aligned}$$

Lemma 2.4 implies the second term tends to 0 as $t \to \infty$. The first term is estimated by (3.6) since

$$\begin{aligned}&\left\|e^{itH_0}J_a^*\left[v_t - E_+(t-s)v_s\right]\right\|\\ &\quad\le \left\|e^{itH_0}J_a^*\right\|\,\|v_t - E_+(t-s)v_s\|\\ &\quad= \|J_a^*\|\left\|e^{i(t-s)H}(v_t - E_+(t-s)v_s)\right\|\\ &\quad= \|J_a^*\|\left\|v_s - e^{i(t-s)H}E_+(t-s)v_s\right\|.\end{aligned}$$

Thus we have

$$\lim_{s\to\infty}\limsup_{t\to\infty}\left\|e^{itH_0}J_a^*e^{-itH}v - e^{isH_0}\tilde{P}_+v_s\right\| = 0.$$

This implies $\left\{e^{itH_0}J_a^*e^{-itH}v\right\}_{t\ge 0}$ is a Cauchy sequence in $\mathcal{H}$, equivalently, there exists the limit

$$\lim_{t\to\infty} e^{itH_0}J_a^*e^{-itH}v =: \Omega^a v.$$

Hence we obtain for sufficiently small $a > 0$,

$$v = W_J^+(\Gamma)\Omega^a v \in \operatorname{Ran} W_J^+(\Gamma).$$

In the rest of the proof, we show (3.6). Since $v_s = \gamma(H)v_s$, we have

$$\begin{aligned}v_s - e^{i(t-s)H}E_+(t-s)v_s &= \gamma(H)v_s - e^{i(t-s)H}E_+(t-s)v_s\\ &= (\gamma(H) - \gamma(H_0))v_s\\ &\quad + (\gamma(H_0) - P_+ - P_-)v_s\\ &\quad + \left(P_+ - e^{i(t-s)H}E_+(t-s)\right)v_s + P_-v_s.\end{aligned} \tag{3.7}$$

We note $\text{w-lim}_{s\to\infty} v_s = 0$ and $\gamma(H) - \gamma(H_0)$ is compact by the compactness of $H - H_0 = V$. We also note $\gamma(H_0) - P_+ - P_-$ is compact by Lemma 2.3, and $P_+ - e^{i(t-s)H}E_+(t-s)$ converges to a compact operator independent of s as $t \to \infty$ by Proposition 2.5. Thus the terms on the RHS of (3.7) except the last one converge to 0.

To estimate the last term of (3.7), we observe

$$\begin{aligned}\|P_-v_s\|^2 &= (P_-^*P_-v_s, v_s)\\ &= ((P_-^* - P_-)P_-v_s, v_s)\\ &\quad + \left(\left(P_- - e^{-isH}E_-(-s)\right)P_-v_s, v_s\right)\\ &\quad + (P_-v_s, E_-(-s)^*v)\,. \end{aligned} \tag{3.8}$$

By the similar argument as above, we learn the first and second terms of (3.8) converge to 0 as $s \to \infty$. The third term of (3.8) is bounded by

$$\begin{aligned}&|(P_-v_s, E_-(-s)^*v)|\\ &\quad = |(P_-v_s, E_-(-s)^*(\chi_{\{|x|\geq R\}} + \chi_{\{|x|<R\}})v)|\\ &\quad \leq \|E_-(-s)P_-v_s\|\|\chi_{\{|x|\geq R\}}v\| + \|P_-v_s\|\|\chi_{\{|x|<R\}}E_-(-s)\|\|v\|\\ &\quad \leq C_v(\|\chi_{\{|x|\geq R\}}v\| + \|\chi_{\{|x|<R\}}E_-(-s)\|) \end{aligned} \tag{3.9}$$

for any $R > 0$. Using (2.20) and $\lim_{R\to\infty}\|\chi_{\{|x|\geq R\}}v\| = 0$, we learn that (3.9) converges to 0 as $s \to \infty$. Hence we obtain (3.6). □

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Appendix A. Classical Mechanics and the Construction of Phase Function

In this appendix, we use the following notations: For $\rho \in (0, 1)$, we define

$$\begin{aligned} h(x,\xi) &= h_0(\xi) + V(x),\\ V_\rho(t,x) &= V(x)\chi(\rho x)\chi\left(\frac{\langle \log\langle t\rangle\rangle x}{\langle t\rangle}\right),\\ h_\rho(t,x,\xi) &= h_0(\xi) + V_\rho(t,x),\\ \nabla_x^2 V_\rho(t,x) &= {}^t\nabla_x\nabla_x V_\rho(t,x), \end{aligned}$$

where $\chi \in C^\infty(\mathbb{R}^d)$ is a fixed function satisfying (2.1). Let ε be as in Assumption 1.3. We fix $\varepsilon_0, \varepsilon_1 > 0$ such that $\varepsilon_0 + \varepsilon_1 < \varepsilon$.

The construction of time-decaying potential is same as Isozaki and Kitada [7] and is first used by Kitada and Yajima [10]. One of the merits of this construction is that V_ρ decays with respect to time t almost same as position x. The next lemma follows from Assumption 1.3 with elementary computations.

Lemma A.1. *For any $t \in \mathbb{R}$, $x \in \mathbb{R}^d$ and multi-index α,*

$$|\partial_x^\alpha V_\rho(t,x)| \le C_\alpha \min\{\rho^{\varepsilon_0}\langle t\rangle^{-|\alpha|-\varepsilon_1}, \langle x\rangle^{-|\alpha|-\varepsilon}\}, \tag{A.1}$$

where C_α's are independent of x, t and ρ.

Let $(q,p)(t,s) = (q,p)(t,s;x,\xi)$ be the solution to the canonical equation associated to the Hamiltonian h_ρ:

$$\begin{cases} \partial_t q(t,s) = \nabla_\xi h_\rho(t, p(t,s), q(t,s)), \\ \partial_t p(t,s) = -\nabla_x h_\rho(t, p(t,s), q(t,s)), \\ (q,p)(s,s) = (x,\xi). \end{cases}$$

This can be rewritten in the integral form:

$$q(t,s) = x + \int_s^t v(p(\tau,s))\mathrm{d}\tau, \tag{A.2}$$

$$p(t,s) = \xi - \int_s^t \nabla_x V_\rho(\tau, q(\tau,s))\mathrm{d}\tau. \tag{A.3}$$

Before proving Proposition 2.1, let us describe the outline of this section. First, we see in Proposition A.2 that $q(t,s) \sim x + (t-s)v(\xi)$ and $p(t,s) \sim \xi$ for sufficiently small $\rho > 0$. Then we construct a solution $\phi(t;x,\xi)$ of the Hamilton–Jacobi equation (A.30) by the method of characteristics. Also estimates for $y(s,t;x,\xi)$ and $\eta(t,s;x,\xi)$, characterized by (A.21) and (A.22), respectively, are given in Proposition A.3. Using the above ϕ, we define functions $\phi_\pm(x,\xi)$ by (A.33), and we confirm that $\phi_\pm$ satisfies the eikonal equation (1.4) and the estimate (2.4) in outgoing and incoming region, respectively. Finally, we construct a function $\varphi(x,\xi)$ such that Proposition 2.1 holds with $\phi_\pm$ and phase-space cutoffs.

Now, we start with estimates for classical orbits $(q,p)(t,s;x,\xi)$. The following proposition is the corresponding result of Proposition 2.1 in [7].

Proposition A.2. *For $\rho > 0$ small enough, there exist $C_\ell > 0$ ($\ell \in \mathbb{Z}_+$) such that, for any $x, \xi \in \mathbb{R}^d$, $0 \le \pm s \le \pm t$ and multi-indices α and β,*

$$|p(s,t;x,\xi) - \xi| \le C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}, \tag{A.4}$$

$$|p(t,s;x,\xi) - \xi| \le C_0\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}, \tag{A.5}$$

$$|\partial_x^\alpha [\nabla_x q(s,t;x,\xi) - I]| \le C_{|\alpha|}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}, \tag{A.6}$$

$$|\partial_x^\alpha \nabla_x p(s,t;x,\xi)| \le C_{|\alpha|}\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}, \tag{A.7}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta [\nabla_x q(t,s;x,\xi) - I]\right| \le C_{|\alpha|+|\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}|t-s|, \tag{A.8}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta \nabla_x p(t,s;x,\xi)\right| \le C_{|\alpha|+|\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}, \tag{A.9}$$

$$\left|\partial_\xi^\beta [\nabla_\xi q(t,s;x,\xi) - (t-s)A(\xi)]\right| \le C_{|\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}|t-s|, \tag{A.10}$$

$$\left|\partial_\xi^\beta [\nabla_\xi p(t,s;x,\xi) - I]\right| \le C_{|\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}, \tag{A.11}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta \left[q(t,s;x,\xi) - x - (t-s)v(p(t,s;x,\xi))\right]\right| \\ \leq C_{|\alpha|+|\beta|}\rho^{\varepsilon_0} \min\{|t-s|\langle s\rangle^{-\varepsilon_1}, \langle t\rangle^{1-\varepsilon_1}\}. \tag{A.12}$$

Here, $|x| = \left(\sum_{j=1}^d |x_j|^2\right)^{\frac{1}{2}}$ *for a vector* x *and* $|M| = \left(\sum_{j,k=1}^d |M_{jk}|^2\right)^{\frac{1}{2}}$ *for a matrix* M*.*

Proof. We prove in the $0 \leq s \leq t$ case. The other case is proved similarly. The proof is decomposed into 5 steps.

Step 1: Proof of (A.4) and (A.5). The inequalities (A.4) and (A.5) are shown by (A.1) and

$$p(t,t') - \xi = -\int_{t'}^t \nabla_x V_\rho(\tau, q(\tau,t'))\mathrm{d}\tau, \quad t,t' \in \mathbb{R}.$$

Step 2: Proof of (A.6) and (A.7). We use the induction with respect to $|\alpha|$. First we prove (A.6) and (A.7) for $\alpha = 0$. Differentiating (A.2) and (A.3) in x, we have

$$\begin{cases} \nabla_x q(s,t) = I + \int_t^s A(p(\tau,t))\nabla_x p(\tau,t)\mathrm{d}\tau, \\ \nabla_x p(s,t) = -\int_t^s \nabla_x^2 V_\rho(\tau, q(\tau,t))\nabla_x q(\tau,t)\mathrm{d}\tau. \end{cases}$$

Letting

$$Q_0(s) := \nabla_x q(s,t) - I,$$
$$P_0(s) := \nabla_x p(s,t),$$

we observe

$$\begin{cases} Q_0(s) = \int_t^s A(p(\tau,t))P_0(\tau)\mathrm{d}\tau, \\ P_0(s) = -\int_t^s \nabla_x^2 V_\rho(\tau, q(\tau,t))Q_0(\tau)\mathrm{d}\tau - \int_t^s \nabla_x^2 V_\rho(\tau, q(\tau,t))\mathrm{d}\tau. \end{cases} \tag{A.13}$$

Thus combining the two equations in (A.13), we learn

$$P_0(s) = B_t(P_0(\cdot))(s) + R_0(s),$$

where

$$B_t(P(\cdot))(s) := -\int_t^s \nabla_x^2 V_\rho(\tau, q(\tau,t))\left[\int_t^\tau A(p(\sigma,t))P(\sigma)\mathrm{d}\sigma\right]\mathrm{d}\tau,$$
$$R_0(s) := -\int_t^s \nabla_x^2 V_\rho(\tau, q(\tau,t))\mathrm{d}\tau.$$

Let $\|M(\cdot)\|_0 := \sup_{0\le s\le t}\langle s\rangle^{1+\varepsilon_1}|M(s)|$ for $M \in C([0,t]\,;M_d(\mathbb{R}))$. Then (A.1) implies

$$\begin{aligned}|B_t(P(\cdot))(s)| &\le \int_s^t C_2\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\int_\tau^t |P(\sigma)|\mathrm{d}\sigma\mathrm{d}\tau\\ &\le C_2\rho^{\varepsilon_0}\|P\|_0\int_s^\infty\langle\tau\rangle^{-2-\varepsilon_1}\int_\tau^\infty\langle\sigma\rangle^{-1-\varepsilon_1}\mathrm{d}\sigma\mathrm{d}\tau\\ &\le C_2C'\rho^{\varepsilon_0}\langle s\rangle^{-1-2\varepsilon_1}\|P\|_0,\\ |R_0(s)| &\le \int_s^t C_2\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\mathrm{d}\tau \le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}.\end{aligned}$$

If $\rho \le (2C_2C')^{-\frac{1}{\varepsilon_0}}$, the operator norm $\|B_t\|_0$ of B_t with respect to $\|\cdot\|_0$ is bounded by $\frac{1}{2}$. Hence we obtain

$$\|P_0(\cdot)\|_0 = \|(1-B_t)^{-1}(R_0(\cdot))\|_0 \le \frac{1}{1-\|B_t\|_0}\|R_0(\cdot)\|_0 \le 2C\rho^{\varepsilon_0}, \tag{A.14}$$

which proves (A.7) for $\alpha = 0$. The inequality (A.6) for $\alpha = 0$ follows directly from (A.13) and (A.14).

Next we confirm the induction is valid. We fix $\alpha \in \mathbb{Z}_+^d\backslash\{0\}$ and assume that (A.6) and (A.7) hold for α' with $|\alpha'| < |\alpha|$. Differentiating (A.13), we have

$$\begin{cases}\partial_x^\alpha Q_0(s) = \int_t^s A(p(\tau,t))\partial_x^\alpha P_0(\tau)\mathrm{d}\tau + R_{0,1}(s),\\ \partial_x^\alpha P_0(s) = -\int_t^s \nabla_x^2 V_\rho(\tau,q(\tau,t))\partial_x^\alpha Q_0(\tau)\mathrm{d}\tau\\ \qquad\qquad +R_{0,21}(s) + R_{0,22}(s),\end{cases} \tag{A.15}$$

where

$$\begin{aligned}R_{0,1}(s) &:= \sum_{0\lneq\alpha'\le\alpha}\binom{\alpha}{\alpha'}\int_t^s \partial_x^{\alpha'}\left[A(p(\tau,t))\right]\partial_x^{\alpha-\alpha'}P_0(\tau)\mathrm{d}\tau,\\ R_{0,21}(s) &:= -\sum_{0\lneq\alpha'\le\alpha}\binom{\alpha}{\alpha'}\int_t^s \partial_x^{\alpha'}\left[\nabla_x^2 V_\rho(\tau,q(\tau,t))\right]\partial_x^{\alpha-\alpha'}Q_0(\tau)\mathrm{d}\tau,\\ R_{0,22}(s) &:= -\int_t^s \partial_x^\alpha\left[\nabla_x^2 V_\rho(\tau,q(\tau,t))\right]\mathrm{d}\tau,\end{aligned}$$

and $\binom{\alpha}{\alpha'} := \prod_{j=1}^d \frac{\alpha_j!}{\alpha_j'!(\alpha_j-\alpha_j')!}$. By (A.1) and assumptions of the induction, we have

$$\begin{aligned}|R_{0,1}(s)| &\le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1},\\ |R_{0,21}(s)| &\le \int_s^t C\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\cdot C\rho^{\varepsilon_0}\langle\tau\rangle^{-\varepsilon_1}\mathrm{d}\tau \le C\rho^{\varepsilon_0}\langle s\rangle^{-1-2\varepsilon_1},\\ |R_{0,22}(s)| &\le \int_s^t C\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\mathrm{d}\tau \le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}.\end{aligned}$$

The similar argument as for $\alpha = 0$ implies $\|\partial_x^\alpha P_0(\cdot)\|_0 \le C_\alpha\rho^{\varepsilon_0}$ and (A.6).

Step 3: Proof of (A.10) and (A.11). We use the induction with respect to $|\beta|$. First we consider the $\beta = 0$ case. Similarly to Step 2, we have

$$\begin{cases} \nabla_\xi q(t,s) = \int_s^t A(p(\tau,s))\nabla_\xi p(\tau,s)\mathrm{d}\tau, \\ \nabla_\xi p(t,s) = I - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))\nabla_\xi q(\tau,s)\mathrm{d}\tau, \end{cases}$$

equivalently,

$$\begin{cases} Q'(t) = \int_s^t A(p(\tau,s))P'(\tau)\mathrm{d}\tau - \int_s^t (A(p(\tau,s)) - A(\xi))\mathrm{d}\tau, \\ P'(t) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))Q'(\tau)\mathrm{d}\tau \\ \qquad\qquad - \int_s^t (\tau - s)\nabla_x^2 V_\rho(\tau, q(\tau,s))A(\xi)\mathrm{d}\tau, \end{cases} \tag{A.16}$$

where

$$Q'(t) := \nabla_\xi q(t,s) - (t-s)A(\xi),$$
$$P'(t) := \nabla_\xi p(t,s) - I.$$

By (A.16), we have

$$P'(t) = B_s(P'(\cdot))(t) + R'(t),$$

where

$$R'(t) := -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s)) \int_s^\tau A(p(\sigma,s))\mathrm{d}\sigma\mathrm{d}\tau.$$

Letting $\|M(\cdot)\|_1 := \sup_{t\geq s} |M(t)|$ for $M \in C([s,\infty); M_d(\mathbb{R}))$, we have

$$\begin{aligned} |B_s(P(\cdot))(t)| &\leq \int_s^t C_2\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\int_s^\tau |P(\sigma)|\mathrm{d}\sigma\mathrm{d}\tau \\ &\leq C_2\rho^{\varepsilon_0}\|P\|_1 \int_s^t \langle\tau\rangle^{-2-\varepsilon_1}(\tau - s)\mathrm{d}\tau \\ &\leq C_2C'\rho^{\varepsilon_1}\langle s\rangle^{-\varepsilon_1}\|P\|_1, \\ |R'(t)| &\leq \int_s^t C\rho^{\varepsilon_1}\langle\tau\rangle^{-2-\varepsilon_1}(\tau - s)\mathrm{d}\tau \leq C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}. \end{aligned}$$

Thus, if $\rho \leq (2C_2C')^{-\varepsilon_0}$, we obtain

$$\|P'(\cdot)\|_1 = \|(1 - B_s)^{-1}R'(\cdot)\|_1 \leq \frac{1}{1 - \|B_s\|_1}\|R'(\cdot)\|_1 \leq 2C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}. \tag{A.17}$$

This proves (A.11) for $\beta = 0$. The inequality (A.10) for $\beta = 0$ follows from (A.5), (A.16) and (A.17).

Next we prove the induction works. Differentiating (A.16), we have

$$\begin{cases} \partial_\xi^\beta Q'(t) = \int_s^t A(p(\tau,s))\partial_\xi^\beta P'(\tau)\mathrm{d}\tau + R'_{11}(t) + R'_{12}(t), \\ \partial_\xi^\beta P'(t) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))\partial_\xi^\beta Q'(\tau)\mathrm{d}\tau + R'_{21}(t) + R'_{22}(t), \end{cases} \tag{A.18}$$

where

$$R'_{11}(t) := \sum_{0 \lneq \beta' \leq \beta} \binom{\beta}{\beta'} \int_s^t \partial_\xi^{\beta'} [A(p(\tau,s))] \partial_\xi^{\beta-\beta'} P'(\tau)\mathrm{d}\tau,$$

$$R'_{12}(t) := \int_s^t \partial_\xi^{\beta} [A(p(\tau,s)) - A(\xi)] \mathrm{d}\tau,$$

$$R'_{21}(t) := -\sum_{0 \lneq \beta' \leq \beta} \binom{\beta}{\beta'} \int_s^t \partial_\xi^{\beta'} \left[\nabla_x^2 V_\rho(\tau, q(\tau,s))\right] \partial_\xi^{\beta-\beta'} Q'(\tau)\mathrm{d}\tau,$$

$$R'_{22}(t) := -\int_s^t (\tau - s)\partial_\xi^{\beta} \left[\nabla_x^2 V_\rho(\tau, q(\tau,s))A(\xi)\right] \mathrm{d}\tau.$$

Thus we have

$$\partial_\xi^\beta P'(t) = B_s(\partial_\xi^\beta P'(\cdot))(t) - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))(R'_{11}(\tau) + R'_{12}(\tau))\mathrm{d}\tau + R'_{21}(t) + R'_{22}(t).$$

If (A.10) and (A.11) are true for β' with $|\beta'| < |\beta|$, we learn

$$|R'_{11}(t)| \leq C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}|t-s|,$$

$$|R'_{12}(t)| \leq C \sup_{|\beta'|\leq|\beta|} \int_s^t \left|\partial_\xi^{\beta'}[p(\tau,s) - \xi]\right| \mathrm{d}\tau \leq C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}|t-s|,$$

$$|R'_{21}(t)| \leq \int_s^t C\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1} \cdot C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}|\tau - s|\mathrm{d}\tau \leq C\rho^{2\varepsilon_0}\langle s\rangle^{-2\varepsilon_1},$$

$$|R'_{22}(t)| \leq \int_s^t C\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}|\tau - s|\mathrm{d}\tau \leq C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}.$$

Using the similar argument as for $\beta = 0$, we obtain (A.10) and (A.11) for any β.

Step 4: Proof of (A.8) and (A.9). We use the induction with respect to $|\alpha| + |\beta|$. In the $\alpha = \beta = 0$ case, differentiation in x implies

$$\begin{cases} \nabla_x q(t,s) = I + \int_s^t A(p(\tau,s))\nabla_x p(\tau,s)\mathrm{d}\tau, \\ \nabla_x p(t,s) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))\nabla_x q(\tau,s)\mathrm{d}\tau. \end{cases}$$

Letting

$$Q(t) := \nabla_x q(t,s) - I,$$
$$P(t) := \nabla_x p(t,s),$$

we observe

$$\begin{cases} Q(t) = \int_s^t A(p(\tau,s))P(\tau)\mathrm{d}\tau, \\ P(t) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))Q(\tau)\mathrm{d}\tau - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))\mathrm{d}\tau. \end{cases} \tag{A.19}$$

This implies

$$P(t) = B_s(P(\cdot))(t) + R(t),$$

where

$$R(t) := -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau, s))\mathrm{d}\tau.$$

Since

$$|R(t)| \le \int_s^t C_2 \rho^{\varepsilon_0} \langle\tau\rangle^{-2-\varepsilon_1}\mathrm{d}\tau \le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1},$$

we have

$$\|P(\cdot)\|_1 = \|(1-B_s)^{-1}R\|_1 \le 2C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1},$$

which proves (A.9) for $\alpha = \beta = 0$. The inequality (A.8) follows from (A.9) and (A.19).

We prove the induction with respect to $|\alpha| + |\beta|$ works. By (A.19), we have

$$\begin{cases} \partial_x^\alpha \partial_\xi^\beta Q(t) = \int_s^t A(p(\tau,s))\partial_x^\alpha\partial_\xi^\beta P(\tau)\mathrm{d}\tau + R_1(t), \\ \partial_x^\alpha \partial_\xi^\beta P(t) = -\int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))\partial_x^\alpha \partial_\xi^\beta Q(\tau)\mathrm{d}\tau \\ \qquad\qquad +R_{21}(t) + R_{22}(t), \end{cases} \tag{A.20}$$

where

$$R_1(t) := \sum_{\substack{\alpha'\le\alpha,\beta'\le\beta,\\ |\alpha'+\beta'|\ge 1}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \int_s^t \partial_x^{\alpha'}\partial_\xi^{\beta'}\left[A(p(\tau,s))\right]\partial_x^{\alpha-\alpha'}\partial_\xi^{\beta-\beta'}P(\tau)\mathrm{d}\tau,$$

$$R_{21}(t)$$

$$:= -\sum_{\substack{\alpha'\le\alpha,\beta'\le\beta,\\ |\alpha'+\beta'|\ge 1}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \int_s^t \partial_x^{\alpha'}\partial_\xi^{\beta'}\left[\nabla_x^2 V_\rho(\tau, q(\tau,s))\right]\partial_x^{\alpha-\alpha'}\partial_\xi^{\alpha-\beta'}Q(\tau)\mathrm{d}\tau,$$

$$R_{22}(t) := -\int_s^t \partial_x^\alpha\partial_\xi^\beta\left[\nabla_x^2 V_\rho(\tau, q(\tau,s))\right]\mathrm{d}\tau.$$

Thus we learn

$$\partial_x^\alpha\partial_\xi^\beta P(t) = B_s(\partial_x^\alpha\partial_\xi^\beta P(\cdot))(t) - \int_s^t \nabla_x^2 V_\rho(\tau, q(\tau,s))R_1(\tau)\mathrm{d}\tau + R_{21}(t) + R_{22}(t).$$

By (A.10), (A.11) and assumptions of the induction, we have

$$|R_1(t)| \le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}|t-s|,$$

$$|R_{21}(t)| \le \int_s^t C\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\cdot C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}|\tau - s|\mathrm{d}\tau \le C\rho^{2\varepsilon_0}\langle s\rangle^{-1-2\varepsilon_1},$$

$$|R_{22}(t)| \le \int_s^t C\rho^{\varepsilon_0}\langle\tau\rangle^{-2-\varepsilon_1}\mathrm{d}\tau \le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}.$$

Similarly to the argument for $\alpha = \beta = 0$, we obtain (A.8) and (A.9) for any α and β.

Step 5: Proof of (A.12). By (A.2) and (A.3), we have

$$q(t,s;x,\xi) = x + \int_s^t v(p(\tau,s))\mathrm{d}\tau \\ = x + \int_s^t v\left(p(t,s) + \int_\tau^t \nabla_x V_\rho(\sigma, q(\sigma,s))\mathrm{d}\sigma\right)\mathrm{d}\tau.$$

Thus

$$q(t,s;x,\xi) - x - (t-s)v(p(t,s)) \\ = \int_s^t \left[v\left(p(t,s) + \int_\tau^t \nabla_x V_\rho(\sigma, q(\sigma,s))\mathrm{d}\sigma\right) - v(p(t,s))\right]\mathrm{d}\tau.$$

This equality and (A.8)–(A.11) imply (A.12). □

Similarly to Proposition 2.2 in [7], we observe that, if ρ is small enough, the maps

$$y \mapsto q(s,t;y,\xi), \\ \eta \mapsto p(t,s;x,\eta)$$

have the corresponding inverses.

Proposition A.3. *Fix $\rho > 0$ so that $C_0\rho^{\varepsilon_0} < \frac{1}{2}$ holds, where C_0 is the constant in* Proposition A.2. *Then, for $x, \xi \in \mathbb{R}^d$ and $0 \le \pm s \le \pm t$, there exist $y(s,t) = y(s,t;x,\xi) \in \mathbb{R}^d$ and $\eta(t,s) = \eta(t,s;x,\xi) \in \mathbb{R}^d$ such that*

$$\begin{cases} q(s,t;y(s,t;x,\xi),\xi) = x, & \text{(A.21)} \\ p(t,s;x,\eta(t,s;x,\xi)) = \xi, & \text{(A.22)} \end{cases}$$

and

$$\begin{cases} q(t,s;x,\eta(t,s;x,\xi)) = y(s,t;x,\xi), & \text{(A.23)} \\ p(s,t;y(s,t;x,\xi),\xi) = \eta(t,s;x,\xi). & \text{(A.24)} \end{cases}$$

Furthermore, for any $x, \xi \in \mathbb{R}^d$, $0 \le \pm s \le \pm t$ and multi-indices α and β,

$$|\partial_x^\alpha [\nabla_x y(s,t;x,\xi) - I]| \le C'_\alpha \rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}, \tag{A.25}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta \nabla_x \eta(t,s;x,\xi)\right| \le C'_{\alpha\beta}\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1}, \tag{A.26}$$

$$\left|\partial_\xi^\beta [\eta(t,s;x,\xi) - \xi]\right| \le C'_\beta \rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}, \tag{A.27}$$

$$\left|\partial_\xi^\beta [y(s,t;x,\xi) - x - (t-s)v(\xi)]\right| \\ \le C'_\beta \rho^{\varepsilon_0}\min\{|t-s|\langle s\rangle^{-\varepsilon_1}, \langle t\rangle^{1-\varepsilon_1}\}. \tag{A.28}$$

Proof. Step 1. By $|\nabla_x q(s,t;x,\xi) - I| < \frac{1}{2}$, $|\nabla_\xi p(t,s;x,\xi) - I| < \frac{1}{2}$ and Schwartz's global inversion theorem ([6], Proposition A.7.1), we have the existence and uniqueness of $y(s,t;x,\xi)$ and $\eta(t,s;x,\xi)$ satisfying (A.21) and (A.22). The equalities (A.23) and (A.24) are shown by (A.21) and (A.22).

Step 2: Proof of (A.25). Differentiation of (A.21) in x implies

$$\nabla_x q(s,t;y(s,t),\xi)\nabla_x y(s,t) = I. \tag{A.29}$$

We have by (A.6)

$$\begin{aligned}|\nabla_x y(s,t)-I| &= |(\nabla_x q(s,t;y(s,t),\xi))^{-1}-I|\\ &\le C|\nabla_x q(s,t;y(s,t),\xi)-I|\\ &\le C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}.\end{aligned}$$

Differentiating (A.29), we have for $\alpha \neq 0$

$$\begin{aligned}&\nabla_x q(s,t;y(s,t),\xi)\partial_x^\alpha \nabla_x y(s,t)\\ &\quad = -\sum_{0\lneq\alpha'\le\alpha}\binom{\alpha}{\alpha'}\partial_x^{\alpha'}\left[\nabla_x q(s,t;y(s,t),\xi)\right]\partial_x^{\alpha-\alpha'}\nabla_x y(s,t).\end{aligned}$$

Using (A.6) and the induction with respect to $|\alpha|$, we observe that the RHS of the above equality is bounded by $C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}$. Thus we have $|\partial_x^\alpha\nabla_x y(s,t)| \le C'_\alpha\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}$.

Step 3: Proof of (A.27). By (A.24), we observe for $\beta = 0$

$$\begin{aligned}|\eta(t,s)-\xi| &= |p(s,t;y(s,t),\xi)-\xi|\\ &= \left|\int_s^t \nabla_x V_\rho(\tau,q(\tau,t;y(s,t),\xi))\mathrm{d}\tau\right|\\ &\le C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}.\end{aligned}$$

In the case of $|\beta| = 1$, we have by differentiation of (A.22) in ξ

$$\nabla_\xi p(t,s;x,\eta(t,s))\nabla_\xi\eta(t,s) = I.$$

Similarly to Step 2, we obtain by (A.11)

$$\begin{aligned}|\nabla_\xi\eta(t,s)-I| &\le C|\nabla_\xi p(t,s;x,\eta(t,s))-I|\\ &\le C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}.\end{aligned}$$

In the other cases, we learn by (A.22)

$$\begin{aligned}&\nabla_\xi p(t,s;x,\eta(t,s))\partial_\xi^\beta\nabla_\xi\eta(t,s)\\ &\quad = -\sum_{0\lneq\beta'\le\beta}\binom{\beta}{\beta'}\partial_\xi^{\beta'}[\nabla_\xi p(t,s;x,\eta(t,s))]\partial_\xi^{\beta-\beta'}\nabla_\xi\eta(t,s), \quad \beta\neq 0.\end{aligned}$$

The induction with respect to $|\beta|$ and (A.11) imply each term in the RHS is bounded by $C\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1}$. Thus (A.27) holds for any β.

Step 4: Proof of (A.26). Differentiating (A.22) in x, we have

$$\nabla_x p(t,s;x,\eta(t,s)) + \nabla_\xi p(t,s;x,\eta(t,s))\nabla_x\eta(t,s) = 0.$$

This equality and (A.9) imply

$$\begin{aligned}|\nabla_x\eta(t,s)| &= |(\nabla_\xi p(t,s;x,\eta(t,s)))^{-1}\nabla_x p(t,s;x,\eta(t,s))|\\ &\le C|\nabla_x p(t,s;x,\eta(t,s))|\\ &\le C\rho^{\varepsilon_0}\langle s\rangle^{-1-\varepsilon_1},\end{aligned}$$

which proves (A.26) for $\alpha = \beta = 0$. If $\alpha + \beta \neq 0$, we have

$$\begin{aligned}
&\nabla_\xi p(t,s;x,\eta(t,s))\partial_x^\alpha \partial_\xi^\beta \nabla_x \eta(t,s) \\
&\quad = -\partial_x^\alpha \partial_\xi^\beta \left[\nabla_x p(t,s;x,\eta(t,s))\right] \\
&\qquad - \sum_{\substack{\alpha' \le \alpha, \beta' \le \beta, \\ |\alpha'+\beta'| \ge 1}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'} \partial_x^{\alpha'} \partial_\xi^{\beta'} [\nabla_\xi p(t,s;x,\eta(t,s))] \partial_x^{\alpha-\alpha'} \partial_\xi^{\beta-\beta'} \nabla_x \eta(t,s).
\end{aligned}$$

Thus (A.26) is proved by (A.27), (A.9), (A.11) and the induction with respect to $|\alpha| + |\beta|$.

Step 5: Proof of (A.28). Similarly to the proof of (A.12) in Proposition A.2, we have

$$\begin{aligned}
&y(s,t) - x - (t-s)v(\xi) \\
&\quad = q(t,s;x,\eta(t,s)) - x - (t-s)v(p(t,s;x,\eta(t,s))) \\
&\quad = \int_s^t \left[v\left(\xi + \int_\tau^t \nabla_x V_\rho(\sigma, q(\sigma,s;x,\eta(t,s)))\mathrm{d}\sigma \right) - v(\xi) \right] \mathrm{d}\tau.
\end{aligned}$$

Using this equality, (A.10) and (A.27), we obtain (A.28). □

We define

$$\phi(t;x,\xi) := u(t;x,\eta(t,0;x,\xi)),$$

where

$$u(t;x,\eta) := x\cdot\eta + \int_0^t \{h_\rho - x\cdot\nabla_x h_\rho\}(\tau, q(\tau,0;x,\eta), p(\tau,0;x,\eta))\mathrm{d}\tau.$$

Then a direct calculus implies that ϕ satisfies the Hamilton–Jacobi equation

$$\begin{cases} \partial_t \phi(t;x,\xi) = h_\rho(t, \nabla_\xi \phi(t;x,\xi), \xi), \\ \phi(0;x,\xi) = x\cdot\xi, \end{cases} \tag{A.30}$$

and the relation between ϕ and the functions y and η in Proposition A.3:

$$\begin{cases} \nabla_x \phi(t;x,\xi) = \eta(t,0;x,\xi), \\ \nabla_\xi \phi(t;x,\xi) = y(0,t;x,\xi). \end{cases} \tag{A.31}$$

Remark A.4. The relation (A.31) and Proposition A.3 imply the estimate

$$|\partial_x^\alpha \partial_\xi^\beta \left[\nabla_x y(s,t;x,\xi) - I\right]| \le C'_{|\alpha|+|\beta|}\rho^{\varepsilon_0}\langle s\rangle^{-\varepsilon_1} \tag{A.32}$$

holds for $|\beta| \ge 1$. Hence (A.25) is extended to (A.32) for any α and β.

Now, we construct outgoing and incoming solutions of the eikonal equation (1.4).

Lemma A.5. *The limits*

$$\phi_\pm(x,\xi) := \lim_{t\to\pm\infty} (\phi(t;x,\xi) - \phi(t;0,\xi)) \tag{A.33}$$

exist, are smooth in $\mathbb{R}^{2d}$ *and*

$$\phi_\pm(x,\xi+2\pi m) = \phi_\pm(x,\xi) + 2\pi x\cdot m, \quad x,\xi\in\mathbb{R}^d,\ m\in\mathbb{Z}^d. \tag{A.34}$$

Proof. We define

$$R(t,x,\xi) := \phi(t;x,\xi) - \phi(t;0,\xi).$$

Then we have

$$\begin{aligned}\nabla_x R(t,x,\xi) &= \eta(t,0;x,\xi) = p(0,t;y(0,t;x,\xi),\xi)\\ &= \xi + \int_0^t (\nabla_x V_\rho)(\tau, q(\tau,t;y(0,t;x,\xi),\xi))\mathrm{d}\tau\\ &= \xi + \int_0^t (\nabla_x V_\rho)(\tau, q(\tau,0;x,\eta(t,0;x,\xi)))\mathrm{d}\tau.\end{aligned}$$

Since

$$\left|\partial_x^\alpha \partial_\xi^\beta [(\nabla_x V_\rho)(\tau, q(\tau,0;x,\eta(t,0;x,\xi)))]\right| \le C_{\alpha\beta}\langle\tau\rangle^{-1-\varepsilon_1},$$

$\nabla_x R(t,x,\xi)$ converges to a smooth function uniformly in $(x,\xi)\in\mathbb{R}^{2d}$. Thus

$$\partial_\xi^\beta R(t,x,\xi) = x\cdot\int_0^1 \nabla_x\partial_\xi^\beta R(t,\theta x,\xi)\mathrm{d}\theta \tag{A.35}$$

converges locally uniformly in $\mathbb{R}^{2d}$. This implies the smoothness of $\phi_\pm$.

It is easy to see (A.34) if we remark

$$\begin{aligned}\eta(t,0;x,\xi+2\pi m) &= \eta(t,0;x,\xi) + 2\pi m,\\ q(t,0;x,\xi+2\pi m) &= q(t,0;x,\xi)\end{aligned}$$

for $x,\xi\in\mathbb{R}^d$, $t\in\mathbb{R}$ and $m\in\mathbb{Z}^d$. □

Next we consider properties of $\phi_\pm$ in the "outgoing" and "incoming" regions. We prepare improved estimates of Proposition A.2 for an orbit which is outgoing or incoming.

Lemma A.6. *Let* $(q,p)(t) = (q,p)(t,0;x,\xi)$ *be an orbit satisfying* (A.2) *and* (A.3). *Suppose*

$$|q(\tau)| \ge b|\tau| + d, \quad \pm\tau \ge 0$$

for some $b>0$ *and* $d\ge 0$. *Then there exist* $l_{\alpha\beta}, l_\beta \ge 2$ *such that for* $\pm t\ge 0$ *and* α, $\beta\in\mathbb{N}_{\ge 0}^d$,

$$|p(t)-\xi| \le Cb^{-1}\langle d\rangle^{-\varepsilon}, \tag{A.36}$$

$$\left|\partial_x^\alpha\partial_\xi^\beta\left[\nabla_x q(t) - I\right]\right| \le C_{\alpha\beta} b^{-l_{\alpha\beta}}\langle d\rangle^{-1-|\alpha|-\varepsilon}|t|, \tag{A.37}$$

$$\left|\partial_x^\alpha\partial_\xi^\beta \nabla_x p(t)\right| \le C_{\alpha\beta} b^{-l_{\alpha\beta}}\langle d\rangle^{-1-|\alpha|-\varepsilon}, \tag{A.38}$$

$$\left|\partial_\xi^\beta\left[\nabla_\xi q(t) - tA(\xi)\right]\right| \le C_\beta b^{-l_\beta}\langle d\rangle^{-\varepsilon}|t|, \tag{A.39}$$

$$\left|\partial_\xi^\beta\left[\nabla_\xi p(t) - I\right]\right| \le C_\beta b^{-l_\beta}\langle d\rangle^{-\varepsilon}. \tag{A.40}$$

Proof. We calculate similarly to Proposition A.2, whereas we use the following estimate instead:

$$|\partial_x^\alpha V_\rho(t,q(t))| \le C_\alpha \langle q(t)\rangle^{-|\alpha|-\varepsilon} \le C_\alpha \langle b|t|+d\rangle^{-|\alpha|-\varepsilon}.$$

□

The next lemma gives improved estimates of Proposition A.3 for outgoing or incoming orbits.

Lemma A.7. *Let $b,d \ge 0$, $b \ne 0$ and $x,\ \xi \in \mathbb{R}^d$ satisfy*

$$|q(\tau,0;x,\eta(t,0;x,\xi))| \ge b|\tau| + d, \quad 0 \le \pm\tau \le \pm t$$

for any $\pm t \ge 0$. Then there exist $l'_{\alpha\beta}, l'_\beta \ge 2$ such that, for $\pm t \ge 0$,

$$\left|\partial_x^\alpha \partial_\xi^\beta \left[\nabla_x \eta(t,0;x,\xi)\right]\right| \le C_{\alpha\beta} b^{-l'_{\alpha\beta}} \langle d\rangle^{-1-|\alpha|-\varepsilon}, \tag{A.41}$$

$$\left|\partial_\xi^\beta \left[\eta(t,0;x,\xi) - \xi\right]\right| \le C_\beta b^{-l'_\beta} \langle d\rangle^{-\varepsilon}. \tag{A.42}$$

Proof. The proofs are similar to those of (A.26) and (A.27) if we use

$$|\partial_x^\alpha V_\rho(\tau, q(\tau,0;x,\eta(t,0;x,\xi)))| \le C_\alpha \langle b|\tau|+d\rangle^{-|\alpha|-\varepsilon}, \quad 0 \le \pm\tau \le \pm t.$$

□

Using the above two lemmas, we have the estimate of $\phi_\pm(x,\xi) - x\cdot\xi$ on the outgoing and incoming region, respectively. See Proposition 2.4 in [7] for the case of Schrödinger operators.

Proposition A.8.

$$\left|\partial_x^\alpha \partial_\xi^\beta [\phi_\pm(x,\xi) - x\cdot\xi]\right| \le C_{\alpha\beta} |v(\xi)|^{-l_{\alpha\beta}} \langle x\rangle^{1-|\alpha|-\varepsilon} \tag{A.43}$$

on $\{(x,\xi) \mid |x|^{\varepsilon_1}|v(\xi)|^{1-\varepsilon_1} \ge C_{\varepsilon_1}, \pm\cos(x,v(\xi)) \ge 0\}$, respectively.

Proof. On $\{(x,\xi) \mid x, v(\xi) \ne 0,\ \pm\cos(x,v(\xi)) \ge 0\}$, (A.4), (A.5) and (A.12) imply for $0 \le \pm\tau \le \pm t$,

$$\begin{aligned} |q(\tau,0;x,\eta(t,0;x,\xi))| &\ge |x + \tau v(p(\tau,0;x,\eta(t,0;x,\xi)))| - C_0\langle\tau\rangle^{1-\varepsilon_1} \\ &= |x + \tau v(p(\tau,t;y(0,t;x,\xi),\xi))| - C_0\langle\tau\rangle^{1-\varepsilon_1} \\ &\ge |x + \tau v(\xi)| - C\langle\tau\rangle^{1-\varepsilon_1} - C_0\langle\tau\rangle^{1-\varepsilon_1} \\ &\ge \frac{1}{\sqrt{2}}(|x| + |\tau v(\xi)|) - C\langle\tau\rangle^{1-\varepsilon_1}. \end{aligned}$$

If we remark

$$|x| + |\tau v(\xi)| \ge \left(\frac{1}{\varepsilon_1}|x|\right)^{\varepsilon_1} \left(\frac{1}{1-\varepsilon_1}|\tau v(\xi)|\right)^{1-\varepsilon_1} = \frac{|x|^{\varepsilon_1}|v(\xi)|^{1-\varepsilon_1}}{\varepsilon_1^{\varepsilon_1}(1-\varepsilon_1)^{1-\varepsilon_1}}|\tau|^{1-\varepsilon_1},$$

we learn for $|x|^{\varepsilon_1}|v(\xi)|^{1-\varepsilon_1} \ge C_{\varepsilon_1}$

$$|q(\tau,0;x,\eta(t,0;x,\xi))| \ge \frac{1}{2}(|x| + |\tau v(\xi)|), \quad 0 \le \pm\tau \le \pm t. \tag{A.44}$$

Hence the proposition is proved by (A.44), (A.31), (A.33), (A.35) and Lemma A.7. □

The following proposition says $\phi_\pm$ is a solution to the eikonal equation (1.4).

Proposition A.9. *For any $a > 0$, there exists $R_a > 1$ such that $\phi_\pm$ satisfies the eikonal equation*

$$h(x, \nabla_x \phi_\pm(x,\xi)) = h_0(\xi) \tag{A.45}$$

on the outgoing (or incoming) region

$$\{(x,\xi) \mid |x| \geq R_a,\ |v(\xi)| \geq a, \pm\cos(x, v(\xi)) \geq 0\},$$

respectively.

Proof. By (A.31) and (A.33), we have

$$\nabla_x \phi_\pm(x,\xi) = \lim_{t\to\pm\infty} \eta(t,0;x,\xi) = \lim_{t\to\pm\infty} p(0,t;y(0,t;x,\xi),\xi).$$

If $|x| \geq 2\rho^{-1}$, then we have by the definition of V_ρ

$$h(x, \nabla_x \phi_\pm(x,\xi)) = \lim_{t\to\pm\infty} h_\rho(0,x,p(0,t;y(0,t;x,\xi),\xi)). \tag{A.46}$$

Now we claim

$$\begin{aligned} E(\tau) &:= h_\rho(\tau, q(\tau,t;y(0,t;x,\xi),\xi), p(\tau,t;y(0,t;x,\xi),\xi)) \\ &= h_\rho(\tau, q(\tau,0;x,\eta(t,0;x,\xi)), p(\tau,0;x,\eta(t,0;x,\xi))) \end{aligned}$$

is a constant for $0 \leq \pm\tau \leq \pm t$. A direct calculus implies

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}\tau}(\tau) &= \partial_t h_\rho(\tau, q(\tau,0;x,\eta(t,0;x,\xi)), p(\tau,0;x,\eta(t,0;x,\xi))) \\ &= \partial_t V_\rho(\tau, q(\tau,0;x,\eta(t,0;x,\xi))). \end{aligned}$$

We note (A.44) holds on $\{(x,\xi) \mid |x| \geq R_a,\ |v(\xi)| \geq a,\ \pm\cos(x,v(\xi)) \geq 0\}$ for R_a large enough, and hence

$$\begin{aligned} |q(\tau,0;x,\eta(t,0;x,\xi))| &\geq \frac{1}{2}(R_a + a|\tau|) \\ &\geq 2\max\left\{\rho^{-1}, \frac{\langle\tau\rangle}{\langle\log\langle\tau\rangle\rangle}\right\}, \quad 0 \leq \pm\tau \leq \pm t. \end{aligned}$$

We also note $\partial_t V_\rho(t,x) = 0$ if $|x| \geq 2\max\{\rho^{-1}, \frac{\langle t\rangle}{\langle\log\langle t\rangle\rangle}\}$. Thus we have $\frac{\mathrm{d}E}{\mathrm{d}\tau}(\tau) = 0$ if $0 \leq \pm\tau \leq \pm t$, in particular,

$$\begin{aligned} h_\rho(0,x,p(0,t;y(0,t;x,\xi),\xi)) &= E(0) = E(t) \\ &= h_\rho(t, y(0,t;x,\xi),\xi). \end{aligned} \tag{A.47}$$

Hence, (A.46) and (A.47) imply

$$h(x, \nabla_x \phi_\pm(x,\xi)) = \lim_{t\to\pm\infty} h_\rho(t, y(0,t;x,\xi),\xi) = h_0(\xi).$$

□

Proof of Proposition 2.1. Let $\varphi \in C^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus v^{-1}(0)))$ be defined by

$$\varphi(x,\xi) = (\phi_+(x,\xi) - x\cdot\xi)\chi_+(x,\xi) + (\phi_-(x,\xi) - x\cdot\xi)\chi_-(x,\xi) + x\cdot\xi, \tag{A.48}$$

where

$$\chi_\pm(x,\xi) = \chi\left(\mu|v(\xi)|^\ell x\right)\psi_\pm(\cos(x, v(\xi))) \tag{A.49}$$

and $\psi_\pm \in C^\infty([-1,1];[0,1])$ satisfy

$$\psi_\pm(\sigma) = \begin{cases} 1, & \pm\sigma \geq \frac{1}{2}, \\ 0, & \pm\sigma \leq 0. \end{cases}$$

If μ and ℓ are fixed so that μ is sufficiently small and that ℓ is sufficiently large, then φ satisfies (2.3), (2.4) and (2.5).

Finally we prove (2.9). Let s_a be defined by (2.8). We decompose s_a by

$$s_a(x,\xi) = s_a^1(x,\xi) + s_a^2(x,\xi), \tag{A.50}$$

where

$$s_a^1(x,\xi) = \sum_{z\in\mathbb{Z}^d} f[z]\, e^{i(\varphi_a(x-z,\xi)-\varphi_a(x,\xi))} - h_0(\nabla_x\varphi_a(x,\xi)),$$
$$s_a^2(x,\xi) = h(x, \nabla_x\varphi_a(x,\xi)) - h_0(\xi).$$

For s_a^2, (A.45) and Assumption 1.3 imply for $|x| \geq R_a$ and β,

$$\partial_\xi^\beta s_a^2(x,\xi) = \begin{cases} 0, & |\cos(x, v(\xi))| \geq \frac{1}{2}, \\ \mathcal{O}(\langle x\rangle^{-\varepsilon}), & |\cos(x, v(\xi))| \leq \frac{1}{2}. \end{cases} \tag{A.51}$$

For s_a^1, we have

$$\begin{aligned} s_a^1(x,\xi) &= \sum_{z\in\mathbb{Z}^d} f[z]\left(e^{i(\varphi_a(x-z,\xi)-\varphi_a(x,\xi))} - e^{-iz\cdot\nabla_x\varphi_a(x,\xi)}\right) \\ &= \sum_{z\in\mathbb{Z}^d} f[z]\, e^{-iz\cdot\nabla_x\varphi_a(x,\xi)}\left(e^{i\Phi_a(x,\xi,z)} - 1\right), \end{aligned}$$

where

$$\begin{aligned} \Phi_a(x,\xi,z) &= \varphi_a(x-z,\xi) - \varphi_a(x,\xi) + z\cdot\nabla_x\varphi_a(x,\xi) \\ &= z\cdot\left(\int_0^1 \theta_1 \int_0^1 \nabla_x^2\varphi_a(x-\theta_1\theta_2 z,\xi)\mathrm{d}\theta_2\mathrm{d}\theta_1\right) z. \end{aligned}$$

By (2.4), we observe

$$\left|\partial_\xi^\beta[e^{-iz\cdot\nabla_x\varphi_a(x,\xi)}]\right| \leq C_\beta\langle z\rangle^{|\beta|}$$

and

$$\begin{aligned} \left|\partial_\xi^\beta\Phi_a(x,\xi,z)\right| &\leq C_\beta|z|^2\int_0^1\theta_1\int_0^1\langle x-\theta_1\theta_2 z\rangle^{-1-\varepsilon}\mathrm{d}\theta_2\mathrm{d}\theta_1 \\ &\leq C_\beta\langle x\rangle^{-1-\varepsilon}\langle z\rangle^{3+\varepsilon}. \end{aligned}$$

Thus we obtain

$$\left|\partial_\xi^\beta s_a^1(x,\xi)\right| \le C_\beta \langle x\rangle^{-1-\varepsilon}. \tag{A.52}$$

Hence (2.9) is proved by (A.50), (A.51) and (A.52). □

Appendix B. Proofs of Lemmas 2.2 and 2.3

B.1. Proof of Lemma 2.2

First we remark that $J_a, P_\pm, \tilde{P}_\pm$ and their formal adjoint operators

$$J_a^* u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x\cdot\xi-\varphi_a(y,\xi))} u[y] \mathrm{d}\xi,$$

$$P_\pm^* u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x-y)\cdot\xi} p_\pm(x,\xi) u\left[y\right] \mathrm{d}\xi,$$

$$\tilde{P}_\pm^* u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi_a(x,\xi)-y\cdot\xi)} p_\pm(x,\xi) u\left[y\right] \mathrm{d}\xi$$

map from $\mathscr{S}(\mathbb{Z}^d)$ to itself.

Letting $L := \langle x-y\rangle^{-2}(1+(x-y)\cdot D_\xi)$, $D_\xi := \frac{1}{i}\nabla_\xi$, we easily see $L\left(e^{i(x-y)\cdot\xi}\right) = e^{i(x-y)\cdot\xi}$. Thus we have

$$\begin{aligned} P_\pm u\left[x\right] &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} L^k \left(e^{i(x-y)\cdot\xi}\right) p_\pm(y,\xi) u\left[y\right] \mathrm{d}\xi \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x-y)\cdot\xi} (L^*)^k \left(p_\pm(y,\xi)\right) u\left[y\right] \mathrm{d}\xi \end{aligned}$$

for any $k \in \mathbb{N}_{\ge 0}$. We define $|p_\pm| := \sup_{|\beta|\le d+1} \sup_{(x,\xi)\in\mathbb{Z}^d\times\mathbb{T}^d} |\partial_\xi^\beta p_\pm(x,\xi)|$. Then we learn that, setting $k = d+1$,

$$|P_\pm u\left[x\right]| \le C|p_\pm| \sum_{y\in\mathbb{Z}^d} \langle x-y\rangle^{-d-1} |u[x]|.$$

This and Young's inequality imply $\|P_\pm u\| \le C|p_\pm|\|u\|$, where $\|u\| := \left(\sum_{x\in\mathbb{Z}^d} |u[x]|^2\right)^{\frac{1}{2}}$. Hence $P_\pm$ are bounded.

Next we prove $\tilde{P}_\pm$ are bounded. A direct calculus implies

$$\begin{aligned} \tilde{P}_\pm^* \tilde{P}_\pm u[x] &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(\varphi_a(x,\xi)-\varphi_a(y,\xi))} p_\pm(x,\xi) p_\pm(y,\xi) u\left[y\right] \mathrm{d}\xi \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y\in\mathbb{Z}^d} e^{i(x-y)\cdot\eta(\xi;x,y)} p_\pm(x,\xi) p_\pm(y,\xi) u\left[y\right] \mathrm{d}\xi, \end{aligned}$$

where η in the last equality is defined by

$$\eta(\xi;x,y) := \int_0^1 \nabla_x \varphi_a(y+\theta(x-y),\xi) \mathrm{d}\theta. \tag{B.1}$$

Then (2.5) implies $\eta(\cdot; x, y) : \mathbb{T}^d \to \mathbb{T}^d$ has its inverse map $\xi(\cdot; x, y)$. Changing the variable ξ to η, we have

$$\tilde{P}_\pm^* \tilde{P}_\pm u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\eta} r(x, y, \eta) u\,[y]\,\mathrm{d}\eta,$$

where

$$r(x, y, \eta) = p_\pm(x, \xi(\eta; x, y)) p_\pm(y, \xi(\eta; x, y)) \left| \det\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)\right|.$$

Since (2.4) implies

$$\left|\partial_\eta^\beta \left[\det\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right) - 1\right]\right| \le C_\beta \langle x\rangle^{-\varepsilon}, \tag{B.2}$$

the similar argument for $P_\pm$ proves the boundedness of $\tilde{P}_\pm^* \tilde{P}_\pm$. Thus, for $u \in \mathscr{S}(\mathbb{Z}^d)$, we obtain

$$\|\tilde{P}_\pm u\|^2 = |(\tilde{P}_\pm^* \tilde{P}_\pm u, u)| \le \|\tilde{P}_\pm^* \tilde{P}_\pm\| \|u\|^2,$$

which implies $\tilde{P}_\pm$ are bounded. The boundedness of J_a is proved similarly. □

B.2. Proof of Lemma 2.3

Since

$$\gamma(H_0) - P_+ - P_- = \gamma(H_0)(1 - \chi),$$

the compactness of the support of $1 - \chi$ implies $P_+ + P_- - \gamma(H_0)$ is a finite rank operator, in particular, a compact operator.

We show $P_\pm^* - P_\pm$ are compact. We observe

$$\begin{aligned}
&(P_\pm^* - P_\pm)u[x] \\
&\quad = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\xi} (p_\pm(x, \xi) - p_\pm(y, \xi)) u\,[y]\,\mathrm{d}\xi \\
&\quad = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\xi} (x - y) \cdot \int_0^1 \nabla_x p_\pm(y + \theta(x - y), \xi) \mathrm{d}\theta\; u[y] \mathrm{d}\xi \\
&\quad = (2\pi)^{-d} i \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\xi} \int_0^1 \nabla_\xi \cdot \nabla_x p_\pm(y + \theta(x - y), \xi) \mathrm{d}\theta\; u[y] \mathrm{d}\xi,
\end{aligned}$$

where the last equality follows from integral by parts in ξ. Since

$$\begin{aligned}
\left|\int_0^1 \partial_\xi^\beta [\nabla_\xi \cdot \nabla_x p_\pm(y + \theta(x - y), \xi)] \mathrm{d}\theta\right| &\le C_\beta \int_0^1 \langle y + \theta(x - y)\rangle^{-1} \mathrm{d}\theta \\
&\le C'_\beta \langle x\rangle^{-1},
\end{aligned}$$

similar argument in Lemma 2.2 proves $\langle x\rangle(P_\pm^* - P_\pm)$ are bounded. By the compactness of $\langle x\rangle^{-1}$ as an operator on $\mathcal{H}$, $P_\pm^* - P_\pm = \langle x\rangle^{-1} \cdot \langle x\rangle(P_\pm^* - P_\pm)$ are compact.

We next prove the compactness of $E_{\pm}(0) - P_{\pm}$. Using (B.1), we have

$$\begin{aligned} E_{\pm}(0)u[x] &= J_a \tilde{P}_{\pm} u[x] \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(\varphi_a(x,\xi) - \varphi_a(y,\xi))} p_{\pm}(y,\xi) u\,[y]\,\mathrm{d}\xi \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\eta} p_{\pm}(y,\xi(\eta)) \left|\det\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)\right| u\,[y]\,\mathrm{d}\eta. \end{aligned}$$

Thus

$$(E_{\pm}(0) - P_{\pm})u[x] = (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\eta} r(x,y,\eta) u\,[y]\,\mathrm{d}\eta,$$

where

$$r(x,y,\eta) = p_{\pm}(y,\xi(\eta)) \left|\det\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)\right| - p_{\pm}(y,\eta).$$

By (B.2), we have $|\partial_\eta^\beta [r(x,y,\eta)]| \le C_\beta \langle x \rangle^{-\varepsilon}$, and hence $\langle x \rangle^{\varepsilon}(E_{\pm}(0) - P_{\pm})$ are bounded. This proves $E_{\pm}(0) - P_{\pm}$ are compact.

The compactness of $J_a J_a^* - I$ is proved similarly to that of $E_{\pm}(0) - P_{\pm}$, since

$$\begin{aligned} (J_a J_a^* - I)u[x] &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(\varphi_a(x,\xi) - \varphi_a(y,\xi))} u\,[y]\,\mathrm{d}\xi - u[x] \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \sum_{y \in \mathbb{Z}^d} e^{i(x-y)\cdot\eta} \left(\left|\det\left(\frac{\mathrm{d}\xi}{\mathrm{d}\eta}\right)\right| - 1\right) u\,[y]\,\mathrm{d}\eta. \end{aligned}$$

Finally, we prove $J_a^* J_a - I$ is compact. Now we mimic the proof of Lemma 7.1 in [12]. For $f \in L^2(\mathbb{T}^d)$, we denote

$$\begin{aligned} L_a f(\xi) &= F J_a^* J_a F^* f(\xi) \\ &= (2\pi)^{-d} \sum_{x \in \mathbb{Z}^d} \int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi) - \varphi_a(x,\eta))} f(\eta) \mathrm{d}\eta, \\ \tilde{L}_a f(\xi) &= (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi) - \varphi_a(x,\eta))} f(\eta) \mathrm{d}\eta \mathrm{d}x. \end{aligned}$$

First we show that, for any $\psi \in C^\infty(\mathbb{T}^d)$ with sufficiently small support,

$$K_{a,\psi} := \psi \circ (L_a - \tilde{L}_a)$$

is a compact operator on $L^2(\mathbb{T}^d)$. We define $\Pi : L^1(\mathbb{R}^d) \to L^1(\mathbb{T}^d)$ by

$$\Pi f(\xi) := \sum_{m \in \mathbb{Z}^d} f(\xi + 2\pi m).$$

Then (2.3) implies

$$\begin{aligned}\Pi\tilde{L}_a f(\xi) &= (2\pi)^{-d}\sum_{m\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi+2\pi m)-\varphi_a(x,\eta))}f(\eta)\mathrm{d}\eta\mathrm{d}x\\ &= (2\pi)^{-d}\sum_{m\in\mathbb{Z}^d}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi)+2\pi x\cdot m-\varphi_a(x,\eta))}f(\eta)\mathrm{d}\eta\mathrm{d}x.\end{aligned}$$

Using Poisson's summation formula

$$\sum_{m\in\mathbb{Z}^d} e^{2\pi i x\cdot m} = \sum_{m\in\mathbb{Z}^d}\delta_{x-m} \tag{B.3}$$

in the sense of distribution, we have

$$\Pi\tilde{L}_a f(\xi) = (2\pi)^{-d}\sum_{x\in\mathbb{Z}^d}\int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi)-\varphi_a(x,\eta))}f(\eta)\mathrm{d}\eta = L_a f(\xi).$$

Thus we learn

$$\begin{aligned}K_{a,\psi}f(\xi) &= \psi\circ(\Pi\tilde{L}_a-\tilde{L}_a)f(\xi)\\ &= \sum_{m\in\mathbb{Z}^d\setminus\{0\}}\psi(\xi)\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{i(\varphi_a(x,\xi+2\pi m)-\varphi_a(x,\eta))}f(\eta)\mathrm{d}\eta\mathrm{d}x\\ &= \int_{\mathbb{T}^d} k_{a,\psi}(\xi,\eta)f(\eta)\mathrm{d}\eta,\end{aligned}$$

where the integral kernel

$$k_{a,\psi}(\xi,\eta) = \sum_{m\in\mathbb{Z}^d\setminus\{0\}}\psi(\xi)\int_{\mathbb{R}^d} e^{i(\varphi_a(x,\xi+2\pi m)-\varphi_a(x,\eta))}\mathrm{d}x$$

is smooth. This implies the compactness of $K_{a,\psi}$.

In order to show the compactness of $\psi\circ(\tilde{L}_a - I)$, we note

$$\tilde{L}_a f(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{i\int_0^1\nabla_\xi\varphi_a(x,\eta+\theta(\xi-\eta))\mathrm{d}\theta\cdot(\xi-\eta)}f(\eta)\mathrm{d}\eta\mathrm{d}x.$$

Letting

$$y(x;\xi,\eta) := \int_0^1\nabla_\xi\varphi_a(x,\eta+\theta(\xi-\eta))\mathrm{d}\theta,$$

we observe $y(\cdot;\xi,\eta)$ has its inverse map by (2.5). Thus we have

$$\tilde{L}_a f(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{T}^d} e^{iy\cdot(\xi-\eta)}\left|\det\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)\right| f(\eta)\mathrm{d}\eta\mathrm{d}y.$$

This equality and

$$\left|\partial_y^\alpha\partial_\xi^\beta\partial_\eta^\gamma\left[\det\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)-1\right]\right| \le C_{\alpha\beta\gamma}\langle y\rangle^{-|\alpha|-\varepsilon}$$

imply the compactness of $\psi\circ(\tilde{L}_a - I)$.

Hence, with the help of a partition of unity $\{\psi_j\}_{j=1}^J$ on $\mathbb{T}^d$, we observe

$$J_a^* J_a - I = F^*(L_a - I)F = F^* \sum_{j=1}^{J} \left(K_{a,\psi_j} + \psi_j \circ (\tilde{L}_a - I) \right) F$$

is compact. □

References

[1] Amrein, W., Boutet de Monvel, A., Georgescu, V.: C_0-groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians. Progress in Mathematics, vol. 135. Birkhäuser, Basel (1996)

[2] Ando, K., Isozaki, H., Morioka, H.: Spectral properties of Schrödinger operators on perturbed lattices. Ann. Henri Poincaré **17**, 2103–2171 (2016)

[3] Asada, K., Fujiwara, D.: On some oscillatory integral transformations in $L^2(\mathbb{R}^n)$. Jpn. J. Math. (N.S.) **4**(2), 299–361 (1978)

[4] Boutet de Monvel, A., Sahbani, J.: On the spectral properties of discrete Schrödinger operators: (the multi-dimensional case). Rev. Math. Phys. **11**, 1061–1078 (1999)

[5] de Oliveira, C.R.: Intermediate Spectral Theory and Quantum Dynamics. Birkhäuser, Basel (2009)

[6] Dereziński, J., Gérard, C.: Scattering Theory of Classical and Quantum N-Particle Systems. Springer, Berlin (1997)

[7] Isozaki, H., Kitada, H.: Modified wave operators with time-independent modifiers. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **32**(1), 77–104 (1985)

[8] Isozaki, H., Korotyaev, I.: Inverse problems, trace formulae for discrete Schorödinger operators. Ann. Henri Poincaré **13**, 751–788 (2012)

[9] Kitada, H.: Scattering theory for Schrödinger equations with time-dependent potentials of long-range type. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **29**, 353–369 (1982)

[10] Kitada, H., Yajima, K.: A scattering theory for time-dependent long-range potentials. Duke Math. J. **49**, 341–376 (1982)

[11] Nakamura, S.: Modified wave operators for discrete Schrödinger operators with long-range perturbations. J. Math. Phys. **55**, 112101 (2014)

[12] Nakamura, S.: Microlocal properties of scattering matrices. Commun. Partial Differ. Equ. **41**, 894–912 (2016)

[13] Parra, D., Richard, S.: Spectral and scattering theory for Schrödinger operators on perturbed topological crystals. Rev. Math. Phys. **30**, 1850009-1–1850009-39 (2018)

[14] Reed, M., Simon, B.: The Methods of Modern Mathematical Physics, Volume III, Scattering Theory. Academic Press, London (1979)

[15] Yafaev, D.R.: Mathematical Scattering Theory. Analytic Theory. Mathematical Surveys and Monographs. American Mathematical Society, Providence (2010)

[16] Zworski, M.: Semiclassical Analysis. Graduate Studies in Mathematics, vol. 138. American Mathematical Society, Providence (2012)

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