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A Note on Harris' Ergodic Theorem, Controllability and Perturbations of Harmonic Networks

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Abstract. We show that elements of control theory, together with an application of Harris' ergodic theorem, provide an alternate method for showing exponential convergence to a unique stationary measure for certain classes of networks of quasi-harmonic classical oscillators coupled to heat baths. With the system of oscillators expressed in the form

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + F(X_t)\,\mathrm{d}t + B\,\mathrm{d}W_t$$

in $\mathbf{R}^d$, where A encodes the harmonic part of the force and $-F$ corresponds to the gradient of the anharmonic part of the potential, the hypotheses under which we obtain exponential mixing are the following: A is dissipative, the pair (A, B) satisfies the Kalman condition, F grows sufficiently slowly at infinity (depending on the dimension d), and the vector fields in the equation of motion satisfy the weak Hörmander condition in at least one point of the phase space.

1. Introduction

Thermally driven networks of oscillators play an important role in the investigation of various aspects of nonequilibrium statistical mechanics. On a mathematical level, a driven network of classical harmonic oscillators can be modelled as a d-dimensional process $(X_t)_{t\geq 0}$ described by a linear stochastic differential equation (SDE) of the form

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$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + B\,\mathrm{d}Z_t,$$

where the linear operators A and B satisfy certain structural conditions and where $(Z_t)_{t\geq 0}$ is a given n-dimensional stochastic process describing the noise due to thermal fluctuations. The integer $n \leq d$ is the number of degrees of freedom of the network that are coupled to heat baths. The noise is often taken to be a Wiener process, but other types of noise are physically interesting. A particularly important question regarding such systems and perturbations thereof is that of invariant measures.

In this work, we consider A and B satisfying the *Kalman condition*, a smooth globally Lipschitz perturbing vector field $x \mapsto F(x)$ that grows slower than $|x|^{1/2d}$ at infinity[1] and $(W_t)_{t\geq 0}$ a Wiener process, and show with arguments from control theory and an application of Hairer and Mattingly's version of Harris' ergodic theorem that the process described by the SDE

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + F(X_t)\,\mathrm{d}t + B\,\mathrm{d}W_t$$

admits a unique stationary measure when A is *dissipitaive* and a *weak Hörmander condition* on the vector fields in the SDE holds in at least one point x_0 of the phase space. Moreover, the convergence to this stationary measure then happens exponentially fast. The abstract mathematical setup and the result are made more precise in Sect. 2. The proof is provided in Sect. 3.

In Sect. 4, we introduce the mathematical description of perturbed networks of harmonic oscillators in this framework, both in the Langevin regime and in the so-called semi-Markovian regime, and for geometries that go beyond the one-dimensional chain. In this context, the matrix A encodes the friction, kinetic and harmonic terms (both the pinning and the interaction) while the perturbation F corresponds to minus the gradient of the anharmonic part of the potential.

In the case of a one-dimensional chain of oscillators connected to heat baths at both ends, results of this type have been established for a very general class of quasi-homogeneous potentials [4,7–9,15]. The recent paper [6] extends these results to more complicated networks. Roughly speaking, these results require that the pinning potential grows as $|q|^{k_1}$ at infinity, that the interaction potential grows as $|q|^{k_2}$ with $k_2 \geq k_1 \geq 2$, and that the interaction part of the potential has no flat piece or infinitely degenerate points. While our growth condition is considerably more restrictive than the ones found in these works, the form of local nondegeneracy that we require is weaker: we only need a weak Hörmander condition to hold at a single point. Moreover, our setup accommodates a wide variety of geometries and bounded many-body interaction terms (beyond pinning and two-body interactions).

Such results typically involve carefully studying smoothing properties of the associated Markov semigroup. The strategy here is different and instead relies on recent developments on the use of solid controllability in the study of

[1] The power $\frac{1}{2d}$ is generically not optimal. As we will see, d can be replaced by an integer d_* appearing in the formulation of the Kalman rank condition. In all cases $d_* \leq d$.

mixing properties of random dynamical systems [1,2,16,17]. The simplicity of the argument can in itself justify the presentation of such an application.

Another advantage is that our general strategy is not based on the Gaussian structure of Brownian motion and can thus be more easily adapted to different types of noise that are physically relevant. Similar arguments can be used to discuss the analogous problem with compound Poisson processes; this type of problem will be analysed in a subsequent work.

The proof can be summarized as follows. For a discrete-time Markov process, Harris' theorem states that the existence and uniqueness of an invariant measure, with exponentially fast convergence in the total variation metric, can be obtained from the existence of a suitable Lyapunov function and a minorization for the transition probabilities starting from any point in the interior of a suitable level set of that Lyapunov function. The precise statement we use is the one formulated in [11]; also see [12,14]. We then pass from discrete to continuous time.

The function $V(x) := \int_0^\infty |\mathrm{e}^{sA}x|^2 \,\mathrm{d}s$ is shown to be a suitable Lyapunov function using dissipativity of A, the behaviour of F at infinity, and basic Itô calculus. The details are given in Sect. 3.1.

In order to prove the lower bound on transitions, we use the Kalman condition on the pair (A, B) and again the estimate on the behaviour of F at infinity. These hypotheses yield that the point x_0 in which the weak Hörmander condition holds can be approached from $\{V \le R\}$ with a uniform lower bound on the probability. On the other hand, the weak Hörmander condition in x_0 implies solid controllability from x_0 and we can combine solid controllability and approachability to obtain the desired lower bound. The details are given in Sect. 3.2.

Different sufficient conditions for the hypotheses of the main theorem to hold are given in more concrete terms throughout Sects. 4 and 5. In the former, we give criteria for the dissipativity, Kalman and growth conditions in terms of more physical quantities for networks of oscillators based on [13]. In the latter, we give a perturbative condition for the weak Hörmander condition to hold.

2. Setup, Assumptions and Main Result

Notation. Throughout the paper, we use: $\|\cdot\|$ to denote the operator norm of linear maps; $\{e_i\}_{i=1}^n$ for the standard orthonormal basis of $\mathbf{R}^n$; $|\cdot|$ to denote the euclidean norm on $\mathbf{R}^d$ (arising from the standard inner product $\langle\,\cdot\,,\cdot\,\rangle$); $B(x,r)$ for the open ball of radius $r > 0$ centred at the point x in $\mathbf{R}^d$; $C_0^k([0,T];\mathbf{R}^n)$ to denote the space of k times continuously differentiable functions $\eta : [0,T] \to \mathbf{R}^n$ with $\eta(0) = 0$; $\operatorname{Prob}(\mathbf{R}^d)$ for the space of Borel probability measures on $\mathbf{R}^d$; $\mathcal{L}_G$ for the Lie derivative with respect to the vector field G; $\mathbf{1}_S$ to denote the indicator function of the set S. The natural numbers $\mathbf{N}$ start at 1. The underlying probability space is $(\Omega, \mathcal{F}, \mathbf{P})$ and we use the letter ω to denote elementary events there.

Let d and n be natural numbers with $n \leq d$ and let $\omega \mapsto (W_t(\omega))_{t\geq 0}$ be a Wiener process in $\mathbf{R}^n$. We are interested in the d-dimensional diffusion process $\omega \mapsto (X_t(x^{\text{in}}, \omega))_{t\geq 0}$ governed by the equation

$$X_t(x^{\text{in}}, \omega) = x^{\text{in}} + \int_0^t AX_s(x^{\text{in}}, \omega) + F(X_s(x^{\text{in}}, \omega))\, \mathrm{d}s + BW_t(\omega) \tag{1}$$

where $B : \mathbf{R}^n \to \mathbf{R}^d$ is a linear map, $A : \mathbf{R}^d \to \mathbf{R}^d$ is a linear map, F is a smooth globally Lipschitz vector field on $\mathbf{R}^d$, and $x^{\text{in}} \in \mathbf{R}^d$ is an initial condition. We often omit writing explicitly the dependence on x^{in} or ω and write the equation in differential notation. We assume the following dissipativity and controllability conditions on the linear maps A and B.

(D) the eigenvalues of the linear map A (considered over $\mathbf{C}^d$) each have strictly negative real part.

(K) the pair (A, B) satisfies the *Kalman condition*, meaning that the columns of B, AB, A^2B, A^3B and so forth span $\mathbf{R}^d$.

Then, by the Cayley–Hamilton theorem, there exists $d_* \leq d$ such that

$$\text{span}\{Be_i, ABe_i, A^2Be_i, \ldots, A^{d_*-1}Be_i : i = 1, \ldots, n\} = \mathbf{R}^d.$$

The Kalman condition is commonly used in the basic theory of controllability for linear systems (i.e. when $F \equiv 0$); it is then equivalent to several notions of controllability [5, §§1.2–1.3].

We further assume that the perturbing vector field F satisfies the following growth condition.

(G) there exists a constant $a \in [0, \frac{1}{2d_*})$ such that

$$\sup_{x\in\mathbf{R}^d} \frac{|F(x)|}{(1 + |x|)^a} < \infty. \tag{2}$$

Finally, we suppose the existence of a point x_0 where the *weak Hörmander condition* on the vector fields appearing in the stochastic equation (1) is satisfied.

(H) there exists a point $x_0 \in \mathbf{R}^d$ in which the family

$$\{V_0, \mathcal{L}_{V_2}V_1, \mathcal{L}_{V_3}\mathcal{L}_{V_2}V_1, \ldots : V_0 \in \mathcal{B} \text{ and } V_1, V_2, V_3, \ldots \in \mathcal{B} \cup \{A + F\}\}$$

of vector fields spans $T_{x_0}\mathbf{R}^d \cong \mathbf{R}^d$, where $\mathcal{B} = \{Be_1, \cdots, Be_n\}$.

Remark 2.1. In the linear case (i.e. when $F \equiv 0$), a straightforward computation shows that the Kalman condition (K) implies the weak Hörmander condition (H). This suggests that the latter can be obtained from a perturbative argument in a point x_0 far from the origin if F can be neglected at infinity in a suitable sense; see Sect. 5.

It is convenient to study the properties of such a diffusion process through the corresponding controlled equation

$$\begin{cases} \dot{x}(t) = Ax(t) + F(x(t)) + B\dot{\eta}(t), \\ x(0) = x^{\text{in}}, \end{cases} \tag{3}$$

understood as

$$x(t) = x^{\text{in}} + \int_0^t Ax(s) + F(x(s))\,\mathrm{d}s + B(\eta(t) - \eta(0))$$

when η is a merely continuous function. We define, for $0 \le t \le T$,

$$\begin{aligned} S_t^F : \mathbf{R}^d \times C_0([0,T];\mathbf{R}^n) &\to \mathbf{R}^d \\ (x^{\text{in}}, \eta) &\mapsto x(t) \end{aligned}$$

giving the solution at time t of this problem. We refer to the second argument as the *control*. The function S_t^F is uniformly continuous in each argument. It is also Fréchet differentiable. We will make use of these regularity properties in Sect. 3.2.

Remark 2.2. The law for $\eta \in C_0([0,T];\mathbf{R}^n)$ corresponding to the Wiener process $W_t(\omega)$ restricted to the interval $[0,T]$ in (1), which we denote by ℓ, is *decomposable* in the following sense.

There exist a sequence $(F_N)_{N\in\mathbf{N}}$ of nested finite-dimensional subspaces and a sequence $(F'_N)_{N\in\mathbf{N}}$ of closed subspaces of the Banach space $C_0([0,T];\mathbf{R}^n)$ such that

- (i) the union $\bigcup_{N\in\mathbf{N}} F_N$ is dense in $C_0([0,T];\mathbf{R}^n)$;
- (ii) the space $C_0([0,T];\mathbf{R}^n)$ decomposes as the direct sum $F_N \oplus F'_N$ for each $N \in \mathbf{N}$, with corresponding (bounded) projections Π_N and Π'_N, and the measure ℓ decomposes as the product $\ell_N \otimes \ell'_N$ of its projected measures;
- (iii) the projected measure ℓ_N possesses a smooth positive density ρ_N with respect to the Lebesgue measure on the finite-dimensional space F_N.

The requirement of [17] that $\Pi_N\zeta \to \zeta$ in norm does not hold for all controls $\zeta \in C_0([0,T];\mathbf{R}^n)$. However, the convergence will hold true on nice enough subsets — which suffices for our endeavour. These decomposability properties play a central role in the arguments of [16,17] and are discussed here in "Appendix A".

We use $P_t^F(x^{\text{in}}, \cdot\,)$ to denote the distribution of the random variable $\omega \mapsto X_t(x^{\text{in}}, \omega)$ defined by (1). Then, P_t^F satisfies the Chapman–Kolmogorov equation:

$$P_T^F(x,\Gamma) = \int_{\mathbf{R}^d} P_{T-t}^F(y,\Gamma) P_t^F(x,\mathrm{d}y)$$

for all times $0 \le t \le T$, all $x \in \mathbf{R}^d$ and all Borel sets $\Gamma \subseteq \mathbf{R}^d$. We are interested in the large-time behaviour of P_t^F. Our main result is the following.

Theorem 2.3. *Suppose that the* SDE

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + F(X_t)\,\mathrm{d}t + B\,\mathrm{d}W_t$$

satisfies the conditions (D), (K), (G) *and* (H). *Then, it admits a unique invariant measure* $\mu^{\text{inv}} \in \text{Prob}(\mathbf{R}^d)$. *Moreover, the function* $V : \mathbf{R}^d \to [0,\infty)$ *defined by*

$$x \mapsto \int_0^\infty \langle \mathrm{e}^{sA}x, \mathrm{e}^{sA}x\rangle\,\mathrm{d}s$$

is integrable with respect to μ^{inv} *and there exist constants* $c, C > 0$ *such that*

$$\Big| \int_{\mathbf{R}^d} f(y) P_t^F(x^{\text{in}}, \mathrm{d}y) - \int_{\mathbf{R}^d} f(y) \mu^{\text{inv}}(\mathrm{d}y) \Big| \leq C(1 + V(x^{\text{in}})) \mathrm{e}^{-ct} \tag{4}$$

for all $x^{\text{in}} \in \mathbf{R}^d$*, all* $t \geq 0$ *and all measurable functions* f *with* $|f| \leq 1 + V$*.*

The proof of this theorem is developed throughout Sect. 3. The last key step there is an application of Hairer and Mattingly's version of Harris' ergodic theorem [11]. It requires two hypotheses: the existence of constants $\gamma \in (0, 1)$ and $K > 0$ such that

$$\Big| \int_{\mathbf{R}^d} V(y) P_t^F(x, \mathrm{d}y) \Big| \leq \gamma^t V(x) + K \tag{5}$$

for all $x \in \mathbf{R}^d$ and all $t \geq 0$, and the existence of a positive measure bounding from below the probability of reaching a set when starting from the interior of a suitable level set of V:

$$P_T^F(x, \cdot\,) \geq \nu_T \tag{6}$$

for all $x \in \mathbf{R}^d$ such that $V(x) \leq 1 + 2K(1 - \gamma)^{-1}$. The first one is dealt with in Sect. 3.1; the second one, in Sect. 3.2.

3. Proof of Theorem 2.3

3.1. Dissipativity and Lyapunov Stability

Condition (D) ensures that the integral defining $V : x \mapsto \int_0^\infty |\mathrm{e}^{sA} x|^2 \,\mathrm{d}s$ converges. To this function V is naturally associated a positive definite matrix M such that $V(x) = \langle x, Mx \rangle$. We wish to show that, under the conditions (D) and (G), this function satisfies the inequality (5) for some constants $\gamma \in (0, 1)$ and $K > 0$ that do not depend on x.

Lemma 3.1. *Under the conditions* (D) *and* (G)*, there exist constants* $K > 0$ *and* $\gamma \in (0, 1)$ *such that the function* V *satisfies*

$$\Big| \int_{\mathbf{R}^d} V(y) P_t^F(x, \mathrm{d}y) \Big| \leq \gamma^t V(x) + K$$

for all $x \in \mathbf{R}^d$ *and all* $t \geq 0$*.*

Proof. Fix an initial condition $X_0 \in \mathbf{R}^d$. First note that we have

$$\langle D_x V(x), Ax \rangle = 2 \langle x, MAx \rangle = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} \langle \mathrm{e}^{sA} x, \mathrm{e}^{sA} x \rangle \,\mathrm{d}s = -|x|^2.$$

On the other hand, by assumption (G), there exists $c_1 > 0$ such that $|F(x)| \leq \frac{1}{8\|M\|}|x| + c_1$ and thus there exists a constant $c_2 > 0$ depending on c_1 and $\|M\|$ such that

$$\langle D_x V(x), Ax + F(x) \rangle \leq -\tfrac{1}{2}|x^2| + c_2.$$

for all $x \in \mathbf{R}^d$.

By Itô's lemma applied to the smooth function V (with no explicit t-dependence),

$$\mathrm{d}V(X_t) = \langle DV(X_t), AX_t + F(X_t)\rangle \,\mathrm{d}t + 2\,\langle MX_t, B\,\mathrm{d}W_t\rangle + \operatorname{tr}(MBB^*)\,\mathrm{d}t$$

and thus

$$\mathbf{E}V(X_t) \le V(X_0) + \int_0^t \left(-\tfrac{1}{2}\mathbf{E}|X_s|^2 + c_2\right)\mathrm{d}s + \operatorname{tr}(MBB^*)t.$$

Since e^{sA} is nonsingular for any $s \in [0,1]$ by assumption (D), there exists $c_3 > 0$ depending on the eigenvalues of A such that

$$V(x) \ge \int_0^1 |\mathrm{e}^{sA}x|^2 \ge c_3|x|^2$$

for all $x \in \mathbf{R}^d$. Hence,

$$\mathbf{E}V(X_t) \le V(X_0) - \int_0^t \tfrac{1}{2c_3}\mathbf{E}V(X_s)\,\mathrm{d}s + (c_2 + \operatorname{tr}(MBB^*))t$$

By Grönwall's inequality, we conclude that there exists a constant $K > 0$ (independent of X_0) such that

$$\mathbf{E}V(X_t) \le \mathrm{e}^{-\frac{t}{2c_3}}V(X_0) + K.$$

□

3.2. Approachability and Solid Controllability

The goal of this section is to show the existence of a time $T > 0$ and a nontrivial measure ν_T on $\mathbf{R}^d$ such that the bound

$$P_T^F(x,\cdot\,) \ge \nu_T$$

holds for all $x \in \mathbf{R}^d$ such that $V(x) \le 1 + 2K(1-\gamma)^{-1}$, where γ and K are as in Lemma 3.1. This is done in two steps: we first control the probability of reaching neighbourhoods of x_0 where (H) holds, and then the probability of reaching an arbitrary set when starting from x' close enough to x_0.

Throughout this section, the controlled nonlinear system (3) is to be thought of as a perturbation of the controlled linear system

$$\begin{cases} \dot{z}(t) = Az(t) + B\dot{\eta}(t), \\ z(0) = x^{\mathrm{in}}. \end{cases} \tag{7}$$

For $\eta \in C_0([0,T];\mathbf{R}^n)$ and $0 \le t \le T$, $S_T(x^{\mathrm{in}},\eta)$ is defined as the solution at time t of the problem (7).

We set $R := 1 + 2K(1-\gamma)^{-1}$. We make extensive use of the compact set $\{x \in \mathbf{R}^d : V(x) \le R\}$, which we often write as $\{V \le R\}$ for short. We start by showing that the point x_0 in which the weak Hörmander condition (H) holds can be approximately reached with suitable control when starting from $\{V \le R\}$.[2] To do this, we need a technical lemma on a matrix often referred to as the *controllability Gramian*, which is used to construct relevant controls; see e.g. [5, §§1.2–1.3].

[2] This part of the argument actually holds for any $x_0 \in \mathbf{R}^d$, regardless of the Hörmander condition.

Lemma 3.2. *If A and B are such that the Kalman condition* (K) *is satisfied with d_*, then the symmetric positive definite matrix*

$$Q_T = \int_0^T \mathrm{e}^{tA} BB^* \mathrm{e}^{tA^*} \,\mathrm{d}t$$

has full rank and $\|Q_T^{-1}\| = O(T^{1-2d_*})$ as $T \to 0$.

Proof. Because Q_T is symmetric and by real-analyticity of the maps $(0,1) \ni T \mapsto \langle x, Q_T x\rangle \in \mathbf{R}_+$, it suffices to show that for each $x \in \mathbf{R}^d$ with $|x| = 1$, there exists $k \leq 2d_* - 1$ such that

$$\partial_T^k \langle x, Q_T x\rangle |_{T=0} \neq 0.$$

Suppose for contradiction that there exists such x with $|x| = 1$ and $0 = \partial_T^k \langle x, Q_T x\rangle$ for each $k \leq 2d_* - 1$. From the first derivative, we have

$$B^* x = 0.$$

From the third derivative, we have

$$\langle x, BB^*(A^*)^2 x\rangle + 2\langle x, ABB^*A^* x\rangle + \langle x, A^2BB^* x\rangle = 0,$$

but then, using again the consequence of the vanishing first derivative, we have

$$B^*A^* x = 0.$$

Inductively, from the $(2j+1)$th derivative, we have

$$B^*(A^*)^j x = 0,$$

for $j = 0, 1, \ldots d_* - 1$. We conclude that

$$x \in \bigcap_{j=0}^{d_*-1} \ker(B^*(A^*)^j) = \bigcap_{j=0}^{d_*-1} (\operatorname{ran}(A^j B))^\perp,$$

contradicting the Kalman condition. □

Proposition 3.3. *Fix $x_0 \in \mathbf{R}^d$. If the growth condition* (G) *and the Kalman condition* (K) *hold, then for any $x \in \mathbf{R}^d$, $\delta > 0$ and $T > 0$ there exists a control $\eta_{x,\delta,T} \in C_0^1([0,T]; \mathbf{R}^n)$ such that $S_T^F(x, \eta_{x,\delta,T}) \in B(x_0, \frac{1}{2}\delta)$.*

Proof. Let $x \in \mathbf{R}^d$ and $\delta > 0$ be arbitrary. Because the Kalman condition (K) holds, for any $T \in (0,1]$, the control

$$\zeta_{x,T}(t) := \int_0^t B^* \mathrm{e}^{(T-s)A^*} Q_T^{-1}(x_0 - \mathrm{e}^{-TA}x) \,\mathrm{d}s$$

is such that $S_T(x, \zeta_{x,T}) = x_0$; see e.g. [5, §1.2]. We immediately have the bound

$$|\dot{\zeta}_{x,T}(t)| \leq \|B\| \mathrm{e}^{T\|A\|} \|Q_T^{-1}\| (|x_0| + \mathrm{e}^{T\|A\|}|x|)$$

and the hypotheses yield through Lemma 3.2 the existence of a constant $C > 0$ depending on A and B such that

$$|\dot{\zeta}_{x,T}(t)| \leq C(|x| + |x_0|)T^{-m}$$

for all $T \in (0,1]$, where $m := 2d_* - 1$.

With $z_T(t) := S_t(x, \zeta_{x,T})$, $x_T(t) := S_t^F(x, \zeta_{x,T})$ and $y_T(t) := x_T(t) - z_T(t)$, we have

$$\begin{aligned} \dot{y}_T(t) &= A y_T(t) + F(x_T(t)), \\ y_T(0) &= 0. \end{aligned}$$

Then, for $t \in [0,T]$,

$$y_T(t) = \int_0^t \mathrm{e}^{(t-s)A} F(x_T(s)) \, \mathrm{d}s = \int_0^t \mathrm{e}^{(t-s)A} F(y_T(s) + z_T(s)) \, \mathrm{d}s.$$

By (G), there exists $C' > 0$ depending on A and F only such that

$$|y_T(t)| \le C' \int_0^t 1 + |y_T(s)|^a + |z_T(s)|^a \, \mathrm{d}s.$$

On the other hand,

$$\begin{aligned} |z_T(t)| &\le |\mathrm{e}^{tA} x| + \int_0^t |\mathrm{e}^{(t-s)A} B \dot{\zeta}_{x,T}(s)| \, \mathrm{d}s \\ &\le C'|x| + tC\mathrm{e}^{T\|A\|} \|B\| (|x| + |x_0|) T^{-m}. \end{aligned}$$

Combining these two inequalities, there exists a constant $C'' > 0$ such that

$$|y_T(t)| \le C'' \int_0^t |y_T(s)| \, \mathrm{d}s + tC''(1 + |x| + |x_0|)(1 + T^{a(1-m)})$$

Recall that $0 \le a < \frac{1}{2d_*}$ and $m = 2d_* + 1$. Hence,

$$a(1-m) + 1 > 0$$

and, by Grönwall's inequality, there exists $T_{x,\delta} \in (0,1]$ small enough, depending continuously on x and δ, such that $|S_T^F(x, \zeta_{x,T}) - x_0| = |y_T(T)| < \frac{1}{4}\delta$ for all $0 < T \le T_{x,\delta}$.

If $T \le T_{x,\delta}$, pick $\eta_{x,\delta,T} = \zeta_{x,T}$. If $T > T_{x,\delta}$, let

$$r_T := \sup_{0 \le t \le T} |S_t(x,0)| \quad \text{and} \quad s_T = \min\left\{ \tfrac{1}{2}T, \inf_{|y| \le r_{x,T}} T_{y,\delta} \right\}.$$

Then, $|S_{T-s_T}(x,0)| < r_T$ and by the above $\zeta_{S_{T-s_T}(x,0),s_T}$ is such that

$$S_{s_T}(S_{T-s_T}(x,0), \zeta_{S_{T-s_T}(x,0),s_T}) \in B(x_0, \tfrac{1}{4}\delta).$$

This corresponds to the control

$$\tilde{\eta}_{x,\delta,T}(t) := \mathbf{1}_{[T-s_T,T]}(t) \zeta_{S_{T-s_T}(x,0),s_T}(t - (T - s_T))$$

defined on $[0,T]$. A $C_0^1([0,T]; \mathbf{R}^n)$ regularization $\eta_{x,\delta,T}$ of $\tilde{\eta}_{x,\delta,T}$ will then satisfy $S_T(x, \eta_{x,\delta,T}) \in B(x_0, \frac{1}{2}\delta)$. □

Proposition 3.4. *Fix $x_0 \in \mathbf{R}^d$ and $\delta > 0$ and suppose that the conditions* (G) *and* (K) *hold. Then, the function*

$$(x,T) \mapsto P_T^F(x, B(x_0,\delta))$$

is positive and jointly lower semicontinuous.

Proof. For any $x \in \mathbf{R}^d$ and $T > 0$, there exists $\eta_{x,\delta,T} \in C_0^1([0,T];\mathbf{R}^n)$ such that $S_T^F(x, \eta_{x,\delta,T}) \in B(x_0, \frac{1}{2}\delta)$. By the Stroock–Varadhan support theorem,[3] the support of the distribution of paths $[0,T] \ni t \mapsto X_t(x,\omega)$ contains the closure of $\{[0,T] \ni t \mapsto S_t^F(x,\eta) : \eta \in C_0^1([0,T];\mathbf{R}^n)\}$ with respect to the supremum norm on $C_0([0,T];\mathbf{R}^d)$. In particular, $P_T^F(x, B(x_0,\delta)) > 0$.

For $\mathbf{P}$-almost every $\omega \in \Omega$, the path $t \mapsto W_t(\omega)$ is continuous. Since X_t satisfies the integral equation

$$X_t(x,\omega) = x + \int_0^t AX_s(x,\omega) + F(X_s(x,\omega))\,\mathrm{d}s + BW_t(\omega)$$

with $y \mapsto Ay + F(y)$ globally Lipschitz and $t \mapsto BW_t(\omega)$ continuous, a standard argument shows that the map $(x,T) \mapsto X_T(x,\omega)$ is jointly continuous. Therefore, the function

$$(x,T) \mapsto \mathbf{1}_{\{\omega' \in \Omega \,:\, X_T(x,\omega') \in B(x_0,\delta)\}}(\omega)$$

is jointly lower semicontinuous for $\mathbf{P}$-almost all $\omega \in \Omega$. Then, so is the map

$$(x,T) \mapsto \int_\Omega \mathbf{1}_{\{\omega' \in \Omega \,:\, X_T(x,\omega') \in B(x_0,\delta)\}}(\omega)\,\mathrm{d}\mathbf{P}(\omega)$$

by Fatou's lemma. □

Now that we have established that, starting from $\{V \le R\}$, any neighbourhood of x_0 can be suitably reached, we seek a minorization for transitions from points close to x_0 to arbitrary points of the space. In [17]'s study of SDEs on compact manifolds, the notions of decomposability and *solid controllability* are used to show that the weak Hörmander condition (H) in x_0 is sufficient to provide appropriate control of the transition probabilities from points x' close enough to x_0.

(sC) a system $S : \mathbf{R}^d \times E \to \mathbf{R}^d$, where E is a Banach space, is said to be *solidly controllable* from x_0, with compact $Q \Subset E$, if there is a ball G in $\mathbf{R}^d$ and a number $\epsilon > 0$ such that if a continuous map $\Phi : Q \to \mathbf{R}^d$ satisfies

$$\sup_{\zeta \in Q} |\Phi(\zeta) - S(x_0,\zeta)| \le \epsilon,$$

then $\Phi(Q) \supseteq G$.

Most of the ideas for the next three results are present in different parts of [17]; also see [16]. We retrieve the key steps and repiece them in a way that is suitable for our endeavour.

Lemma 3.5. *If there exists a closed ball $D \Subset \mathbf{R}^d$ and a continuous function $f : D \to E$ such that $S(x_0, f(x)) = x$ for all $x \in D$, then S satisfies the solid controllability condition* (sC) *from x_0, with $Q = f(D)$.*

[3] In the case of an additive noise, the Stroock–Varadhan support theorem can be given a direct proof by continuity arguments even if the vector field is unbounded, as long as the solutions are defined globally in time.

Proof. Take $\epsilon < \frac{1}{4}\operatorname{diam}(D)$ and set $G := \{x \in D : d(x, \partial D) \geq \epsilon\}$. Let Φ be a continuous map on $f(D)$ such that

$$\sup_{\zeta \in f(D)} |\Phi(\zeta) - S(x_0, \zeta)| \leq \epsilon.$$

Then, for any $x' \in G$, the continuous function $\Psi_{x'}$ defined on D by

$$\Psi_{x'}(x) = x' - \Phi(f(x)) + x$$

maps D to itself. Indeed,

$$\begin{aligned} |x' - \Psi_{x'}(x)| &= |x' - (x' - \Phi(f(x)) + x)| \\ &= |\Phi(f(x)) - S(x_0, f(x))| \leq \sup_{\zeta \in f(D)} |\Phi(\zeta) - S(x_0, \zeta)| \leq \epsilon. \end{aligned}$$

Hence, by the Brouwer fixed point theorem, there exists $x \in D$ such that $x = \Psi_{x'}(x)$, i.e. such that $x' = \Phi(f(x))$. We conclude $G \subseteq \Phi(f(D))$. □

We will use this for S_1^F defined in Sect. 2. In this case, the Banach space E of controls is $C_0([0,1]; \mathbf{R}^n)$ equipped with the supremum norm.

Proposition 3.6. *If the weak Hörmander condition* (H) *is satisfied in x_0, then S_1^F is solidly controllable from x_0, with a set Q consisting of functions that are all Lipschitz with a common Lipschitz constant κ.*

Proof. By the previous lemma, to show solid controllability, it suffices to provide a ball $D \Subset \mathbf{R}^d$ and a continuous function $f : D \to C_0([0,1]; \mathbf{R}^n)$ such that $S_1^F(x_0, f(x_*)) = x_*$ for all $x_* \in D$.

As part of Theorem 2.1 in [17, §2.2], it is shown in a similar setting that the Hörmander condition implies the existence of a ball $D \Subset \mathbf{R}^d$ and a continuous function $\tilde{f} : D \to L^2([0,1]; \mathbf{R}^n)$ such that the solution of

$$\begin{cases} \dot{x} = Ax + F(x) + B\,\tilde{f}(x_*) \\ x(0) = x_0 \end{cases}$$

satisfies $x(1) = x_*$. Moreover, $\kappa := \sup_{x_* \in D} \|\tilde{f}(x_*)\|_{C_0} < \infty$. The construction of D and $\tilde{f}$ uses local arguments and can be directly translated to our setup.

The idea behind the proof is the following. Consider the following extended problem for $y(t) = (x(t), s(t))$ in $\mathbf{R}^d \times \mathbf{R}$:

$$\begin{cases} \dot{y} = (Ax + F(x), 1) + (B\xi, 0) \\ y(0) = (x_0, 0) \end{cases}, \tag{8}$$

where the control ξ is taken in $L^2([0,1]; \mathbf{R}^n)$. The Hörmander condition implies that the Lie algebra generated by the family $\{\tilde{V}_\eta(x,s) = (Ax + F(x), 1) + (B\eta, 0) : \eta \in \mathbf{R}^n\}$ of vector fields has full rank at the point $(x_0, 0)$. Hence, one can show using ideas from the proof of Krener's theorem that there exists a choice of small intervals $(a_l, b_l) \subset [0,1]$ and vectors $\eta_l \in \mathbf{R}^n$ for $l = 0, 1, \ldots, d$ such that the parallelepiped

$$\tilde{\Pi} = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbf{R}^{d+1} : \alpha_l \in (a_l, b_l)\}$$

embeds into $\mathbf{R}^d \times \mathbf{R}$ via the map

$$\begin{aligned}\phi : \tilde{\Pi} &\to \mathbf{R}^d \times \mathbf{R}\\ \alpha &\mapsto \left(\mathrm{e}^{\alpha_d \tilde{V}_{\eta_d}} \circ \cdots \circ \mathrm{e}^{\alpha_0 \tilde{V}_{\eta_0}}\right)(x_0, 0).\end{aligned}$$

In other words, ϕ takes α to the solution y at time $T_\alpha := \alpha_0 + \alpha_1 + \cdots + \alpha_d$ of the extended problem (8) with the control

$$\xi_\alpha(t) = \mathbf{1}_{[0,\alpha_0)}(t)\eta_0 + \sum_{l=1}^{d} \mathbf{1}_{[\alpha_0+\cdots+\alpha_{l-1},\alpha_0+\cdots+\alpha_{l-1}+\alpha_l)}(t)\eta_l. \tag{9}$$

Fixing an $\hat{\alpha} \in \tilde{\Pi}$ with corresponding $T_{\hat{\alpha}} \in (0, 1]$, one finds that the solutions at time $T_{\hat{\alpha}}$ of the problem

$$\begin{cases}\dot{x} = Ax + F(x) + B\xi_\alpha\\ x(0) = x_0\end{cases}$$

provide a diffeomorphism between a neighbourhood of $\hat{\alpha}$ in $\{\alpha \in \tilde{\Pi} : T_\alpha = T_{\hat{\alpha}}\}$ and an open set $O \subset \mathbf{R}^d$. Inverting this diffeomorphism, one finds a function that associates with each point $x_* \in O$ a control $\xi_{\alpha(x_*)} \in L^2([0, T_{\hat{\alpha}}]; \mathbf{R}^n)$ of the form (9). By construction,

$$S^F_{T_{\hat{\alpha}}}\left(x_0, \int_0^{\cdot} \xi_{\alpha(x_*)}(s)\,\mathrm{d}s\right) = x_*$$

for all $x_* \in O$. A standard argument then allows to find a closed ball $D \Subset \mathbf{R}^d$ and a continuous function $\tilde{f} : D \to L^2([0, 1]; \mathbf{R}^n)$ such that

$$S^F_1\left(x_0, \int_0^{\cdot} (\tilde{f}(x_*))(s)\,\mathrm{d}s\right) = x_*$$

for all $x_* \in D$. The supremum κ is bounded by the sum of the $|\eta_l|$ used in the construction of the embedding ϕ.

Let $f : D \to C_0([0, 1]; \mathbf{R}^n)$ be defined by $f(x_*) := \int_0^{\cdot} (\tilde{f}(x_*))(s)\,\mathrm{d}s$. Then,

$$\begin{aligned}\|f(x_*) - f(x_{**})\|_{C_0} &= \sup_{t\in[0,1]} \left| \int_0^t (\tilde{f}(x_*))(s)\,\mathrm{d}s - \int_0^t (\tilde{f}(x_{**}))(s)\,\mathrm{d}s \right|\\ &\leq \|\tilde{f}(x_*) - \tilde{f}(x_{**})\|_{L^2}\end{aligned}$$

so that f is continuous. We conclude that S^F_1 is solidly controllable from x_0, with $Q = f(D)$. The constant κ is a common Lipschitz constant for all functions in Q. □

Proposition 3.7. *If the weak Hörmander condition* (H) *is satisfied in x_0, then there exist $\delta_0 > 0$ and a nonzero Borel measure $\tilde{\nu}$ on $\mathbf{R}^d$ such that*

$$P^F_1(x', \cdot\,) \geq \tilde{\nu}$$

for all $x' \in B(x_0, \delta_0)$.

Proof. By the previous proposition, we have solid controllability of the system S_1^F from the point x_0, with a set Q consisting of Lipschitz functions. Then, the strategy of [17, §1.2] (also see [16, §2.1]) yields the desired measure. We outline the argument for completeness and to emphasize that we do not need the full strength of the decomposability assumption made there.

Let Π_N be as in Remark 2.2 and Appendix A. Because all controls in Q have a common Lipschitz constant κ, we have

$$\lim_{N\to\infty} \sup_{\zeta\in Q} \|\zeta - \Pi_N\zeta\|_{C_0} = 0$$

by Lemma A.1. Then, because $S_1^F(x_0,\cdot\,) : C_0([0,1];\mathbf{R}^n) \to \mathbf{R}^d$ is uniformly continuous, there exists $N \in \mathbf{N}$ large enough that

$$\sup_{\zeta\in Q} |S_1^F(x_0,\Pi_N\zeta) - S_1^F(x_0,\zeta)| < \epsilon,$$

for the ϵ in (sC). Taking $\Phi = S_1^F(x_0,\Pi_N\,\cdot\,)$ there, $\Phi(Q)$ contains a ball (which has positive measure).

By Sard's theorem, there exists a point $\zeta_0 \in Q$ in which $D\Phi$ has full rank. Because $\Phi\circ\Pi_N = \Phi$, this property still holds true if we restrict Φ to $F_N = \operatorname{ran}\Pi_N$. There then exists a d-dimensional subspace $F_N^1 \subseteq F_N$ such that $D\Phi|_{\zeta_0}(F_N^1) = \mathbf{R}^d$. Let F_N^2 be such that $F_N^1 \oplus F_N^2 = F_N$. We will write $\zeta \in F_N$ as (ζ^1,ζ^2) according to this decomposition. More generally, we will write a generic element of C_0 as (ζ^1,ζ^2,ζ') with $\zeta' \in F_N'$. The Jacobian of the map $S_1^F(x_0,(\,\cdot\,,\zeta_0^2,0)) : F_N^1 \to \mathbf{R}^d$ at the point ζ_0^1 is a linear isomorphism between F_N^1 and $\mathbf{R}^d$.

By the implicit function theorem, there exist neighbourhoods V^1 of ζ_0^1, V^2 of ζ_0^2, V' of 0, W of x_0, U of $S_1^F(x_0,(\zeta_0^1,\zeta_0^2,0))$; and a continuously differentiable function $g : W\times U\times V^2\times V' \to V^1$ such that, for points in the appropriate open sets, $S_1^F(x',(\zeta^1,\zeta^2,\zeta')) = x_*$ is equivalent to $\zeta^1 = g(x',x_*,\zeta^2,\zeta')$.

Recall that ℓ equals the product measure $\ell_N\times\ell_N'$ with ℓ_N possessing a continuous and positive density ρ_N on F_N. Let $\chi : \mathbf{R}^d\times C_0 \to [0,1]$ be continuous, supported in $W\times V^1\times V^2\times V'$, and equal to 1 at $(x_0,\zeta_0^1,\zeta_0^2,0)$. Then, for any Borel set $\Gamma\subseteq\mathbf{R}^d$,

$$\begin{aligned} &P_1^F(x',\Gamma) \\ &\quad\geq \iiint_{S_1^F(x',\cdot\,)^{-1}(\Gamma)} \chi(x',\zeta^1,\zeta^2,\zeta')\rho_N(\zeta^1,\zeta^2)\,\mathrm{d}\zeta^1\,\mathrm{d}\zeta^2\ell_N'(\mathrm{d}\zeta') \\ &\quad= \iint_{V^2\times V'}\int_\Gamma \frac{\chi(x',g(x',x_*,\zeta^2,\zeta'),\zeta^2,\zeta')\rho_N(g(x',x_*,\zeta^2,\zeta'),\zeta^2)}{\det[DS_1^F(x',(\,\cdot\,,\zeta^2,\zeta'))|_{g(x',x_*,\zeta^2,\zeta')}]}\,\mathrm{d}x_* \\ &\qquad \mathrm{d}\zeta^2\ell_N'(\mathrm{d}\zeta') \end{aligned}$$

for all $x'\in W$.

By continuity, there exist numbers $\delta_0 > 0$ and $\alpha > 0$ such that

$$P_1^F(x',\Gamma) \geq \alpha\operatorname{vol}(\Gamma\cap B(S_1^F(x_0,\zeta_0),\delta_0))$$

for all $x'\in B(x_0,\delta_0)$ and all Borel sets $\Gamma\subseteq\mathbf{R}^d$. □

Then, by the Chapman–Kolmogorov equation,

$$\begin{aligned}P_{T+1}^F(x,\Gamma) &\geq \int_{x'\in B(x_0,\delta_0)} P_T^F(x,\mathrm{d}x')P_1^F(x',\Gamma)\\ &\geq \int_{x'\in B(x_0,\delta_0)} P_T^F(x,\mathrm{d}x')\tilde{\nu}(\Gamma) = P_T^F(x,B(x_0,\delta_0))\tilde{\nu}(\Gamma)\end{aligned}$$

for any Borel set $\Gamma \subseteq \mathbf{R}^d$ and any $T > 0$. We conclude that for any $T > 1$ the nontrivial measure

$$\nu_T := \Big(\inf_{x\in\{V\leq R\}} P_{T-1}^F(x,B(x_0,\delta_0))\Big)\tilde{\nu}$$

is such that

$$P_T^F(x,\cdot\,) \geq \nu_T$$

for all $x \in \mathbf{R}^d$ such that $V(x) \leq R$. The infimum in the definition of ν_T is positive by Proposition 3.4.

3.3. Application of Harris' Ergodic Theorem

Recall that, by Lemma 3.1, the conditions (D) and (G) ensure the existence of constants $K > 0$ and $\gamma \in (0,1)$ such that the function V satisfies

$$\Big|\int_{\mathbf{R}^d} V(y)P_t^F(x,\mathrm{d}y)\Big| \leq \gamma^t V(x) + K \tag{10}$$

for all $x \in \mathbf{R}^d$ and all $t > 0$. Using the conditions (G) and (K), we also showed in Proposition 3.4 that, for any $\delta > 0$, $(x,T) \mapsto P_T^F(x,B(x_0,\delta))$ is positive and jointly lower semicontinuous. Then, we concluded from this, hypothesis (H) and the arguments of [17] that, for any $T > 1$, there is a nontrivial measure ν_T such that

$$P_T^F(x,\cdot\,) \geq \nu_T \tag{11}$$

for all $x \in \mathbf{R}^d$ such that $V(x) \leq R$.

The existence of a function V satisfying the condition (10) and a nontrivial measure ν_T satisfying (11) are precisely the hypotheses we need to apply Harris' theorem.

Indeed, considering the T-skeleton of our diffusion process[4] for $T = 2$, Theorem 1.2 in [11] yields constants $c, C > 0$ and a stationary measure $\mu^{\text{inv}} \in \operatorname{Prob}(\mathbf{R}^d)$ against which V is integrable and such that

$$\sup_{|f|\leq 1+V}\Big|\int_{\mathbf{R}^d} f(y)[P_{2m}^F(x,\mathrm{d}y) - \mu^{\text{inv}}(\mathrm{d}y)]\Big| \leq C\mathrm{e}^{-c(2m+2)}\big(1+V(x)\big) \tag{12}$$

for all $x \in \mathbf{R}^d$ and all $m \in \mathbf{N}\cup\{0\}$.

The measure μ^{inv} is the unique stationary probability measure for the 2-skeleton, but it could a priori depend on our choice of T-skeleton. However, we can show that this measure is actually stationary, not only for the 2-skeleton, but also for the continuous-time process.

[4] By T-skeleton of a (continuous time) stochastic process, we mean the restriction to times in the countable set $T\mathbf{N}$.

Note that with $f = \mathbf{1}_\Gamma$ the characteristic function of any Borel set $\Gamma \subseteq \mathbf{R}^d$, integrating (12) in the variable x yields that

$$\Big| \int_{\mathbf{R}^d} P^F_{2m}(x,\Gamma)\lambda(\mathrm{d}x) - \mu^{\mathrm{inv}}(\Gamma)\Big| \leq C\mathrm{e}^{-c(2m+2)}\Big(1 + \int_{\mathbf{R}^d} V(x)\lambda(\mathrm{d}x)\Big) \quad (13)$$

for any measure $\lambda \in \mathrm{Prob}(\mathbf{R}^d)$.

Putting λ defined by $\lambda(\Gamma) = \int P^F_s(x,\Gamma)\mu^{\mathrm{inv}}(\mathrm{d}x)$ in (13) for some $s \geq 0$, we have by the Chapman–Kolmogrov equation that

$$\begin{aligned}\Big| \int_{\mathbf{R}^d} P^F_{2m+s}(x,\Gamma)\mu^{\mathrm{inv}}(\mathrm{d}x) - \mu^{\mathrm{inv}}(\Gamma)\Big| \\ \leq C\mathrm{e}^{-c(2m+2)}\Big(1 + \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} V(y)P^F_s(x,\mathrm{d}y)\mu^{\mathrm{inv}}(\mathrm{d}x)\Big).\end{aligned}$$

Using (10),

$$\begin{aligned}\Big| \int_{\mathbf{R}^d} P^F_{2m+s}(x,\Gamma)\mu^{\mathrm{inv}}(\mathrm{d}x) - \mu^{\mathrm{inv}}(\Gamma)\Big| \\ \leq C\mathrm{e}^{-c(2m+2)}\Big(1 + K + \int_{\mathbf{R}^d} V(x)\mu^{\mathrm{inv}}(\mathrm{d}x)\Big).\end{aligned}$$

But the left-hand side does not depend on $m \in \mathbf{N}$ because μ^{inv} is invariant for the 2-skeleton. We therefore have $\int P^F_s(x,\cdot\,)\mu^{\mathrm{inv}}(\mathrm{d}x) = \mu^{\mathrm{inv}}$ for all $s \geq 0$, i.e. that μ^{inv} is stationary for the original continuous-time process.

Now, for any $|f| \leq 1 + V$, $s \in [0,2)$ and $m \in \mathbf{N} \cup \{0\}$,

$$\begin{aligned}&\Big| \int_{\mathbf{R}^d} f(y)[P^F_{2m+s}(x,\mathrm{d}y) - \mu^{\mathrm{inv}}(\mathrm{d}y)]\Big| \\ &\quad = \Big| \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} f(y)P^F_{2m}(x,\mathrm{d}z)P^F_s(z,\mathrm{d}y) - f(y)P^F_s(z,\mathrm{d}y)\mu^{\mathrm{inv}}(\mathrm{d}z)\Big| \\ &\quad = \Big| \int_{\mathbf{R}^d}\Big(\int_{\mathbf{R}^d} f(y)P^F_s(z,\mathrm{d}y)\Big)[P^F_{2m}(x,\mathrm{d}z) - \mu^{\mathrm{inv}}(\mathrm{d}z)]\Big|.\end{aligned}$$

Since $|f| \leq 1 + V$, we have by (10) that

$$\Big| \int_{\mathbf{R}^d} f(y)P^F_s(z,\mathrm{d}y)\Big| \leq \int_{\mathbf{R}^d} (1 + V(y))P^F_s(z,\mathrm{d}y) \leq (K+1)\big(1 + V(z)\big).$$

Therefore, we may apply (12) with f replaced by $\frac{1}{K+1}\int f(y)P^F_s(\,\cdot\,,\mathrm{d}y)$ to get

$$\Big| \int_{\mathbf{R}^d} f(y)[P^F_{2m+s}(x,\mathrm{d}y) - \mu^{\mathrm{inv}}(\mathrm{d}y)]\Big| \leq (K+1)C\mathrm{e}^{-c(2m+2)}\big(1 + V(x)\big).$$

Because any time $t > 0$ can be written as $2m + s$ with $s \in [0,2)$, this is — up to a relabelling of the constants — the assertion of Theorem 2.3.

4. Networks of Oscillators

We introduce the mathematical description of important physical systems that our main result covers, from the simplest to the most intricate. Based on [13], we also discuss the assumptions (K), (D) and (G) of our main result in this

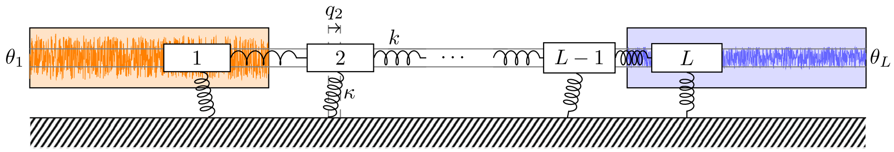

Figure 1. Depiction of the linear harmonic chain where the 1st and Lth oscillator are connected to heat baths at temperatures θ_1 and θ_L, respectively

context. Discussion of the weak Hörmander condition (H) is postponed to the next section.

4.1. The Linear Chain Coupled to Langevin Thermostats

Consider L unit masses, each labelled by an index in $\{1, 2, \ldots, L-1, L\}$ and whose position is restricted to a line. For $i = 1, 2, \ldots, L-1$, the ith mass is attached to the $(i+1)$th mass by a spring of spring constant $k \geq 0$. Each mass is also pinned by a spring of spring constant $\kappa > 0$. The position coordinate q_i of the ith mass is measured relative to a rest position q_i^{eq}; see Fig. 1. Perturbations of this system are described by Hamiltonians of the form

$$
\begin{aligned}
h &: \mathbf{R}^L \oplus \mathbf{R}^L \to \mathbf{R} \\
&(p,q) \mapsto \frac{1}{2}\sum_{i=1}^{L} p_i^2 + \frac{1}{2}\sum_{i=1}^{L} \kappa q_i^2 + \frac{1}{2}\sum_{i=1}^{L-1} k(q_{i+1}-q_i)^2 + U(q)
\end{aligned}
$$

where $U \in C^\infty(\mathbf{R}^L; \mathbf{R})$ is a perturbing potential.

Coupling the 1st and Lth oscillator to Langevin heat baths at positive temperatures θ_1 and θ_L with positive coupling constants γ_1 and γ_L yields the equations of motion

$$
\begin{aligned}
\mathrm{d}q_i &= p_i \,\mathrm{d}t, && 1 \leq i \leq L, \\
\mathrm{d}p_i &= -[\kappa q_i + k(q_i - q_{i-1}) - k(q_{i+1} - q_i) + \partial_i U(q)]\,\mathrm{d}t, && 1 < i < L, \\
\mathrm{d}p_1 &= -[\kappa q_1 - k(q_2 - q_1) + \partial_1 U(q)]\,\mathrm{d}t - \gamma_1 p_1 \,\mathrm{d}t + \sqrt{2\gamma_1\theta_1}\,\mathrm{d}W_{1,t}, \\
\mathrm{d}p_L &= -[\kappa q_L + k(q_L - q_{L-1}) + \partial_L U(q)]\,\mathrm{d}t - \gamma_L p_L \,\mathrm{d}t + \sqrt{2\gamma_L\theta_L}\,\mathrm{d}W_{L,t},
\end{aligned}
$$

where $(W_{1,t})_{t\geq 0}$ and $(W_{L,t})_{t\geq 0}$ are independent one-dimensional Wiener processes.

This system can be put into the form (1) with $d = 2L$ and $n = 2$ by setting

$$
X = \begin{pmatrix} p \\ \\ q \end{pmatrix}, \qquad A = \begin{pmatrix}
\begin{matrix} -\gamma_1 & 0 & & \\ 0 & 0 & & \\ 0 & 0 & & \\ & & \ddots & \\ & & 0 & 0 \\ & & 0 & 0 \\ & & 0 & -\gamma_L \end{matrix} & \begin{matrix} -k-\kappa & k & & \\ k & -2k-\kappa & & \\ 0 & k & & \\ & & \ddots & \\ & & k & 0 \\ & & -2k-\kappa & k \\ & & k & -k-\kappa \end{matrix} \\
\mathbf{1} & 0
\end{pmatrix},
$$

$$
B = \begin{pmatrix}
\begin{matrix} \sqrt{2\gamma_1\theta_1} & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & \sqrt{2\gamma_L\theta_L} \end{matrix} \\
0
\end{pmatrix}
$$

and $F(X) = F(p, q) = -\nabla_q U(q)$.

The Kalman condition (K) is met for the pair (A, B) (with $d_* \leq L$) as soon as $k > 0$ and the eigenvalues of A then have strictly negative real part (condition (D) holds) [13].

The growth condition (G) on the vector field F in the general setting is to be imposed on the gradient $\nabla_q U$ of the perturbing potential U for the chain of oscillators: we require that it is Lipschitz and that there exists $a \in [0, \frac{1}{2d_*})$ such that $|\nabla_q U(q)| = O(1+|q|)^a$ as $|q| \to \infty$. This potential is not restricted to one-body (pinning) or two-body interaction terms; it can for example include a sum of bounded three-body interaction terms.

4.2. More General Geometries in the Langevin Regime

Let I be a finite set and distinguish a nonempty subset $J \subset I$ of the sites, where the thermal noise will act. Fix a temperature $\theta_j > 0$ for the bath associated with each site $j \in J$. We can then generalize the above model to different geometries and different spring constants by considering

$$
X = \begin{pmatrix} p \\ \omega q \end{pmatrix}, \qquad A = \begin{pmatrix} -\frac{1}{2}\iota\iota^* & -\omega^* \\ \omega & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \iota \\ 0 \end{pmatrix} \vartheta^{1/2}, \tag{14}
$$

and $F(p, \omega q) = -\nabla_q U(q)$, where

$$
\omega : \mathbf{R}^I \to \mathbf{R}^I,
$$

is a nonsingular linear map[5] and where ϑ and ι are of the form

$$
\begin{aligned} \vartheta : \qquad & \mathbf{R}^J \to \mathbf{R}^J \\ & (u_j)_{j\in J} \mapsto (\theta_j u_j)_{j\in J}, \end{aligned}
$$

[5] We use the symbol ω for the linear map encoding the frequencies of the system in order to ease the comparison with other works to which we refer. Unfortunately, ω is also standard notation for elements of the underlying probability space. We trust that the meaning of the symbol is clear from the context.

and

$$\begin{aligned}\iota: \qquad \mathbf{R}^J &\to \mathbf{R}^I\\ (u_j)_{j\in J} &\mapsto (\sqrt{2\gamma_j}u_j)_{j\in J} \oplus 0_{I\setminus J}.\end{aligned}$$

Again, γ_j is the coupling constant for the jth oscillator of the boundary. More explicitly, the equations of motion then take the familiar form

$$\begin{aligned}\mathrm{d}q &= p\,\mathrm{d}t,\\ \mathrm{d}p &= -\omega^*\omega q\,\mathrm{d}t - \nabla_q U(q)\,\mathrm{d}t - \tfrac{1}{2}\iota\iota^* p\,\mathrm{d}t + \iota\vartheta^{1/2}\,\mathrm{d}W_t.\end{aligned}$$

Lemma 4.1 in [13] states that if the pair $(\omega^*\omega, \iota)$ satisfies the Kalman condition (K), then the pair (A, B) defined by (14) also satisfies the Kalman condition. By Theorem 5.1(2) there, it then immediately implies the dissipativity condition (D). In Sect. 4.1 there, the case of the triangular network is treated and explicit sufficient conditions for the Kalman condition are given in terms of the spring constants. Again, the growth condition (G) is to be imposed on the gradient $\nabla_q U$ of the perturbing potential U.

As mentioned in the introduction, the recent work of Cuneo, Eckmann, Hairer and Rey-Bellet [6] provides a result of existence, uniqueness and exponentially fast convergence in a similar setup. Their conditions C3–C5 on the behaviour of the potential at infinity are significantly less restrictive than our conditions (D) and (G), allowing for strong anharmonicity. However, their nondegeneracy condition C2 is needed in all points of the phase space while our Hörmander condition (H) is only needed in one point. Their controllability condition C1 on the topology of the graph plays a role similar to that of our Kalman condition (K).

4.3. Coupling Through Additional Degrees of Freedom

As pointed out e.g. in [13], models where the noise acts through auxiliary degrees of freedom enjoy the same structural properties, and are thus also suitable for our framework. We refer the reader to [9,10,18] for discussions of the physical interpretation and derivation of such models. Because of these auxiliary degrees of freedom, the model is sometimes said to be semi-Markovian.

Let I and J be finite sets as above and consider $X = (r, p, \omega q) \in \mathbf{R}^J \oplus \mathbf{R}^I \oplus \mathbf{R}^I$ for some nonsingular linear map $\omega : \mathbf{R}^I \to \mathbf{R}^I$. In addition, let $\Lambda : \mathbf{R}^J \to \mathbf{R}^I$ be a linear injection and let $\iota : \mathbf{R}^J \to \mathbf{R}^J$ and $\vartheta : \mathbf{R}^J \to \mathbf{R}^J$ be linear bijections. We set

$$A = \begin{pmatrix} -\frac{1}{2}\iota\iota^* & -\Lambda^* & 0\\ \Lambda & 0 & -\omega^*\\ 0 & \omega & 0\end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix}\iota\\ 0\\ 0\end{pmatrix}\vartheta^{1/2}; \tag{15}$$

the important structural constraints are

$$\vartheta > 0, \qquad\qquad B^*B > 0, \tag{16}$$

$$\ker(A - A^*) \cap \ker B^* = \{0\}, \qquad\qquad A + A^* = -B\vartheta^{-1}B^*. \tag{17}$$

The perturbation F is taken to be of the form

$$F : X = (r, p, \omega q) \mapsto -\nabla_q U(q)$$

for some smooth potential $U : \mathbf{R}^I \to \mathbf{R}$ encoding the anharmonic part of both the interaction and the pinning potential. More explicitly, the equations of motion then read

$$\begin{aligned} \mathrm{d}q &= p\,\mathrm{d}t, \\ \mathrm{d}p &= -\omega^*\omega q\,\mathrm{d}t - \nabla_q U(q)\,\mathrm{d}t + \Lambda r(t)\,\mathrm{d}t, \\ \mathrm{d}r &= -\tfrac{1}{2}\iota^*\iota r\,\mathrm{d}t - \Lambda^* p\,\mathrm{d}t - \iota\vartheta^{1/2}\,\mathrm{d}W_t. \end{aligned}$$

Proposition 4.1. *If the pair* $(\omega^*\omega, \Lambda)$ *satisfies the Kalman condition, then the pair* (A, B) *also satisfies the Kalman condition* (K)*.*

Proof. Let $(\hat{r}, \hat{p}, \omega\hat{q})$ be a target for the system in time $T > 0$. If $(\omega^*\omega, \Lambda)$ satisfies the Kalman condition, then there exists $\eta_1 \in C_0^1([0, T]; \mathbf{R}^n)$ such that the solution $(p_1(t), q_1(t))$ of

$$\begin{aligned} \dot{p}_1 &= -\omega^*\omega q_1 + \Lambda\dot{\eta}_1, & p_1(0) &= 0, \\ \dot{q}_1 &= p_1, & q_1(0) &= 0, \end{aligned}$$

satisfies $(p_1(T), q_1(T)) = (\hat{p}, \hat{q})$. Note that $(\frac{t}{T}\hat{r}, p_1(t), q_1(t))$ is then a solution of the system

$$\begin{aligned} \dot{r}_2 &= -\tfrac{1}{2}\iota\iota^*(r_2 + \dot{\eta}_2) - \Lambda^* p_2 + \iota\vartheta^{1/2}\dot{\zeta}_2, & r_2(0) &= 0, \\ \dot{p}_2 &= -\omega^*\omega q_2 + \Lambda(r_2 + \dot{\eta}_2), & p_2(0) &= 0, \\ \dot{q}_2 &= p_2 & q_2(0) &= 0, \end{aligned}$$

for the choices of control

$$\begin{aligned} \eta_2(t) &= \eta_1(t) - \int_0^t \tfrac{s}{T}\hat{r}\,\mathrm{d}s, \\ \zeta_2(t) &= (\iota\vartheta^{1/2})^{-1}\int_0^t \tfrac{1}{T}\hat{r} + \tfrac{1}{2}\iota\iota^*\dot{\eta}_1(s) + \Lambda^* p_1(s)\,\mathrm{d}s, \end{aligned}$$

hitting the prescribed target at time $t = T$.

Finally, note that with $\tilde{\eta}$ a smooth approximation of $\dot{\eta}_2$ that is 0 at times $t = 0$ and $t = T$, $(r_2(t) + \tilde{\eta}(t), p_2(t), q_2(t))$ is an approximate solution of

$$\begin{aligned} \dot{r} &= -\tfrac{1}{2}\iota\iota^* r - \Lambda^* p + \iota\vartheta^{1/2}\dot{\zeta}, & r(0) &= 0, \\ \dot{p} &= -\omega^*\omega q + \Lambda r, & p(0) &= 0, \\ \dot{q} &= p, & q(0) &= 0, \end{aligned}$$

for the choice of control

$$\zeta(t) = \zeta_2(t) + (\iota\vartheta^{1/2})^{-1}\tilde{\eta}(t).$$

Therefore, the original system is approximately controllable from 0. Because the system is linear, we conclude that the pair (A, B) satisfies the Kalman condition. □

Then, Theorem 5.1(2) of [13] states that, in this setup, the Kalman condition (K) implies that all the eigenvalues of A have strictly negative real part, i.e. condition (D).

In particular, for A and B arising from a pair $(\omega^*\omega, \Lambda)$ satisfying the Kalman condition (K), as long as $|\nabla_q U(q)| = O(1+|q|)^a$ as $|q| \to \infty$, and as long as there exists a point where the weak Hörmander condition holds, the field $q \mapsto \omega^*\omega q + \nabla_q U(q)$ is allowed to be degenerate in nonnegligible regions of the position space. This is to be compared the nondegeneracy hypothesis H2) in [4,9,15] and C2 in [6] that are needed everywhere.

5. The Weak Hörmander Condition

As a starting point, we note that under the assumption (K), the condition (H) is automatically satisfied for any F with compact support or any F whose derivatives up to order $d-1$ vanish at a point. Also note that a standard perturbative argument shows that if the conditions (D), (K) and (G) are satisfied, then there exists $\lambda_0 > 0$ such that the system

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t + \lambda F(X_t)\,\mathrm{d}t + B\,\mathrm{d}W_t$$

admits a unique invariant measure satisfying (4) as soon as $0 < \lambda < \lambda_0$.

A more subtle perturbative argument is presented in Proposition 5.1. We then give an example of a physically motivated potential to which this proposition applies in the context of networks of oscillators.

In view of the definition of the weak Hörmander condition, we are interested in the part of the tangent space spanned by Lie derivatives. The Lie derivatives, $\mathcal{L}_G b$, $\mathcal{L}_G^2 b$, …, $\mathcal{L}_G^{d_*-1} b$ with $G : x \mapsto Ax + F(x)$ and b a constant vector field will play a particularly important role. A direct computation shows

$$\begin{aligned}
\mathcal{L}_G b &= -DG[b],\\
\mathcal{L}_G^2 b &= +DG^2[b] - D^2G[b, G],\\
\mathcal{L}_G^3 b &= -DG^3[b] + 2DG[D^2G[b, G]]\\
&\quad + D^2G[DG[b], G] - D^3G[b, G, G] - D^2G[b, DG[G]],
\end{aligned}$$

and so forth. Here, the point of the space at which the vectors fields are taken is implicit and we use

$$D^jG[\cdot\,, \cdot, \ldots, \cdot\,] : \underbrace{\mathbf{R}^d \times \mathbf{R}^d \times \cdots \times \mathbf{R}^d}_{j \text{ times}} \to \mathbf{R}^d$$

for the jth Fréchet derivative of the map $G : \mathbf{R}^d \to \mathbf{R}^d$ at this point. The above pattern generalizes in the following way.

Claim. The difference between $\mathcal{L}_G^k b$ and $(-1)^k DG^k[b]$ is a linear combination over $\mathbf{Z}$ of compositions of Fréchet derivatives of G with b. In each term, b appears once, G appears N_0 times, DG appears N_1 times, …, D^kG appears N_k times, with $N_1 \neq k$ and

$$\sum_{j=0}^{k} N_j = \sum_{j=0}^{k} jN_j = k. \tag{18}$$

Proof. We proceed by induction on k. For $k = 1$ we have

$$\mathcal{L}_G b = -DG[b],$$

which satisfies the claim. Assume now that the result holds for some $k \in \mathbf{N}$ so that $\mathcal{L}_G^k b - (-1)^k DG^k[b]$ is a sum of terms satisfying (18). Since

$$\mathcal{L}_G^{k+1} b = -DG[\mathcal{L}_G^k b] + D(\mathcal{L}_G^k b)[G],$$

the first term yields $-(-1)^k DG[DG^k[b]]$ and terms with the same form as those of $\mathcal{L}_G^k b$, but with the changes $k \mapsto k+1$ (adding $N_{k+1} = 0$) and $N_1 \mapsto N_1 + 1$. It indeed satisfies the right condition on the N's if $\mathcal{L}_G^k b$ does. As for the second term, by the product rule, each term in $\mathcal{L}_G^k b$ yields a sum of terms undergoing $N_0 \mapsto N_0 + 1$ and $N_j \mapsto N_j - 1$ and $N_{j+1} \mapsto N_{j+1} + 1$ for one and only one $j \in \{1, \dots, k\}$. □

Proposition 5.1. *Suppose that the pair* (A, B) *satisfies the Kalman condition* (K) *and that there exists a sequence* $(y^{(n)})_{n\in\mathbf{N}}$ *in* $\mathbf{R}^d$ *that is bounded away from 0 and such that*

$$\lim_{n\to\infty} |y^{(n)}|^{k-1} \|D^k F(y^{(n)})\| = 0$$

for each $k = 1, 2, \dots, d_* - 1$. *Then, there exists a point* $x_0 \in \mathbf{R}^d$ *where the weak Hörmander condition* (H) *is satisfied.*

Proof. Let G denote $y \mapsto Ay + F(y)$ and let b stand for a column of B. By our previous claim, we have the bound

$$\begin{aligned}
&|(\mathcal{L}_G^k b)(y) - (-1)^k (DG(y))^k[b]| \\
&\quad\le \sum_{N\in\mathcal{A}} |C_N||b||G(y)|^{N_0} \|DG(y)\|^{N_1} \|D^2G(y)\|^{N_2} \cdots \|D^kG(y)\|^{N_k} \\
&\quad\le \sum_{N\in\mathcal{A}} |C_N||b|(\|A\||y| + \tfrac{1}{8}\|M\|^{-1}|y| + c_1)^{N_0} \\
&\qquad \|DG(y)\|^{N_1} \|D^2G(y)\|^{N_2} \cdots \|D^kG(y)\|^{N_k}
\end{aligned}$$

where $\mathcal{A} := \{N = (N_0, N_1, \dots, N_k) \in (\mathbf{N} \cup \{0\})^k$ satisfying (18) and $N_1 \neq k\}$ and C_N is a combinatorial factor in $\mathbf{Z}$.

By condition (18),

$$\begin{aligned}
&|y|^{N_0} \|DG(y)\|^{N_1} \|D^2G(y)\|^{N_2} \cdots \|D^kG(y)\|^{N_k} \\
&\quad = |y|^{\sum_{j'=2}^k (j'-1)N_{j'}} \prod_{j=1}^k \|D^jG(y)\|^{N_j} \\
&\quad = \|DG(y)\|^{N_1} \prod_{j=2}^k |y|^{(j-1)N_j} \|D^jG(y)\|^{N_j}.
\end{aligned}$$

Along the subsequence $(y^{(n)})_{n\in\mathbf{N}}$ in the hypothesis, for each $j \ge 2$,

$$\lim_{n\to\infty} |y^{(n)}|^{j-1} \|D^jG(y^{(n)})\| = \lim_{n\to\infty} |y^{(n)}|^{j-1} \|D^jF(y^{(n)})\| = 0.$$

In the case $j = 1$, we have

$$\limsup_{n\to\infty} \|DG(y^{(n)})\| \leq \|A\| + \limsup_{n\to\infty} \|DF(y^{(n)})\| = \|A\|.$$

Therefore,

$$\lim_{n\to\infty} |(\mathcal{L}_G^k b)(y^{(n)}) - (-1)^k (DG(y^{(n)}))^k[b]| = 0$$

for $k = 1, 2, \ldots, d_* - 1$, and for n large enough,

$$\operatorname{span}\{b, \mathcal{L}_G b, \ldots, \mathcal{L}_G^{d_*-1} b\}_{y=y^{(n)}} = \operatorname{span}\{b, DG(y)b, \ldots, (DG(y))^{d_*-1} b\}_{y=y^{(n)}}.$$

Finally note that

$$\lim_{n\to\infty} \|(DG(y^{(n)}))^k b - A^k b\| \leq \limsup_{n\to\infty} \sum_{j=1}^{k} \binom{k}{j} \|A\|^{k-j} \|DF(y^{(n)})\|^j = 0.$$

We conclude from the Kalman condition that for $N \in \mathbf{N}$ large enough

$$\operatorname{span}\{b, DG(y)b, \ldots, (DG(y))^{d_*-1} b : b \in \operatorname{ran} B\}_{y=y^{(N)}}$$

coincides with

$$\operatorname{ran}\{B, AB, \ldots, A^{d_*-1}B\} = \mathbf{R}^d.$$

The results holds with $x_0 = y^{(N)}$. □

Example 5.2. Consider that the masses in the models of Sect. 4, although restricted to a single spatial degree of freedom, live in three-dimensional space and each hold an electric charge of Gaussian density

$$\rho_i(\,\cdot\,) = \frac{Q}{(2\pi)^{3/2}\sigma^3} \exp\left(-\frac{|\,\cdot\, - (q_i + q_i^{\mathrm{eq}})|^2}{2\sigma^2}\right)$$

where σ is a parameter with dimension of length and Q is the electric charge of each mass. In view of Poisson's equation in $\mathbf{R}^3$, this gives rise to the term

$$U(q) = \sum_{i\in I} \sum_{\substack{i'\in I\\ i'\neq i}} \frac{Q^2}{4\pi\epsilon_0 |(q_i + q_i^{\mathrm{eq}}) - (q_{i'} + q_{i'}^{\mathrm{eq}})|} \frac{2}{\sqrt{\pi}} \int_0^{\frac{|(q_i + q_i^{\mathrm{eq}}) - (q_{i'} + q_{i'}^{\mathrm{eq}})|}{\sqrt{2}\sigma}} \mathrm{e}^{-s^2}\, \mathrm{d}s$$

in the Hamiltonian. This potential satisfies the condition of the previous proposition: take for example a sequence with $q_i^{(n)} = in\sigma$.

For the sake of matching exactly the setup of [4,9,15], consider that $I = \{1, \ldots, L\}$ and $J = \{1, L\}$ and that only nearest neighbours interact through the Coulomb force. Let us use the shorthand $\tilde{q}_i := q_i + q_i^{\mathrm{eq}}$. Then, the corresponding perturbing potential

$$U^{\mathrm{n.n.}}(q) = \sum_{i=1}^{L-1} \frac{Q^2}{4\pi\epsilon_0 |\tilde{q}_i - \tilde{q}_{i+1}|} \frac{2}{\sqrt{\pi}} \int_0^{\frac{|\tilde{q}_i - \tilde{q}_{i+1}|}{\sqrt{2}\sigma}} \mathrm{e}^{-s^2}\, \mathrm{d}s$$

also satisfies the hypotheses of our previous proposition. However, note that

$$\partial_{q_2}\partial_{q_3}U^{\text{n.n.}}(q)$$
$$= \frac{Q^2}{4\pi^{\frac{3}{2}}\epsilon_0\sigma^3}\left(- \frac{4\int_0^{\frac{|\tilde{q}_2-\tilde{q}_3|}{\sqrt{2}\sigma}} \mathrm{e}^{-s^2}\,\mathrm{d}s}{|\tilde{q}_2-\tilde{q}_3|^3/\sigma^3} + \frac{2\sqrt{2}e^{-\frac{|\tilde{q}_2-\tilde{q}_3|^2}{2\sigma^2}}}{|\tilde{q}_2-\tilde{q}_3|^2/\sigma^2} + \sqrt{2}\mathrm{e}^{-\frac{|\tilde{q}_2-\tilde{q}_3|^2}{2\sigma^2}}\right)$$

does not have a definite sign. Hence, for large values of $Q^2\sigma^{-3}$ (very concentrated charge distribution), the uniform condition H2) in [4,8,9,15] is not satisfied.

Appendix A. Decomposability properties

We devote this appendix to the decomposability properties of ℓ in Remark 2.2. We consider the case $T = 1$ and $n = 1$ but the argument can be easily adapted to the general case. Although we use results from the theory of Gaussian measures to show the decomposability properties, these properties are not specific to Gaussian processes and can be proved for other types of noises.

The Wiener process restricted to the interval $[0, 1]$ is a nondegenerate Gaussian measure on the Banach space $C_0([0, 1]; \mathbf{R})$. It has as its Cameron–Martin space the space $W_0^{1,2}([0, 1]; \mathbf{R})$ equipped with the inner product

$$\langle \eta, \zeta\rangle_{W_0^{1,2}} = \int_0^1 \dot{\eta}(s)\dot{\zeta}(s)\,\mathrm{d}s.$$

This Hilbert space has orthonormal basis $\{\psi_m\}_{m\in\mathbf{N}}$ where

$$\psi_m(t) = \int_0^t \phi_m(s)\,\mathrm{d}s$$

and where $\{\phi_m\}_{m\in\mathbf{N}}$ is a Fourier basis for $L^2([0, 1]; \mathbf{R})$. It is dense as a subspace of $C_0([0, 1]; \mathbf{R})$ equipped with the supremum norm.

Let $F_N := \operatorname{span}\{\psi_m : m \le N\}$ and let F_N' be the closure in $C_0([0, 1]; \mathbf{R})$ of the linear span of $\{\psi_m : m > N\}$. These sequences of subspaces satisfy (i) and provide a decomposition $F_N \oplus F_N'$: any $\eta \in C_0([0, 1]; \mathbf{R})$ can be written in a unique way as $\eta_N + \eta_N'$ with $\eta_N \in F_N$ and $\eta_N' \in F_N'$. To this decomposition are associated the projectors Π_N and Π_N'.

By the general theory of Gaussian measures (see e.g. [3, §3.5]), Brownian motion can be represented as the almost surely convergent sum

$$W_t(\omega) = \sum_{m\le N} \Xi_m(\omega)\psi_m(t) + \sum_{m>N} \Xi_m(\omega)\psi_m(t),$$

where $(\Xi_m)_{m\in\mathbf{N}}$ is a sequence of independent scalar standard normal random variables. The two sums are independent and provide the decomposition (ii) of ℓ as the product of the projected laws. Property (iii) clearly holds.

These abstract results from the theory of Gaussian measures do not provide strong convergence of Π_N to the identity operator on the Banach space $C_0([0, 1]; \mathbf{R})$ as $N \to \infty$ (or boundedness of the set of norms $\{\|\Pi_N\| : N \in \mathbf{N}\}$, which is used in [17]). However, we have the following weaker convergence result for regular enough sets of functions.

Lemma A.1. *If Q is a subset of $C_0([0,1];\mathbf{R})$ that is bounded in the norm induced by the inner product $\langle\,\cdot\,,\cdot\,\rangle_{W_0^{1,2}}$, then*

$$\lim_{N\to\infty}\sup_{\eta\in Q}\|\eta-\Pi_N\eta\|_{C_0}=0.$$

Proof. First note that by construction of the basis,

$$\sum_{m\in\mathbf{N}}\|\psi_m\|_{C_0}^2<\infty.$$

For $\eta\in W_0^{1,2}([0,1];\mathbf{R})$, the decomposition into the two subspaces can be made explicit:

$$\eta(t)=\sum_{m\le N}\psi_m(t)\int_0^1\phi_m(s)\dot\eta(s)\,\mathrm{d}s+\sum_{m>N}\psi_m(t)\int_0^1\phi_m(s)\dot\eta(s)\,\mathrm{d}s$$

and by the Cauchy–Schwarz inequality

$$\begin{aligned}\|\eta-\Pi_N\eta\|_{C_0}&\le\left(\sum_{m>N}\|\psi_m\|_{C_0}^2\right)^{\frac12}\left(\sum_{m>N}\left|\int_0^1\phi_m(s)\dot\eta(s)\,\mathrm{d}s\right|^2\right)^{\frac12}\\&\le\left(\sum_{m>N}\|\psi_m\|_{C_0}^2\right)^{\frac12}\|\eta\|_{W_0^{1,2}}.\end{aligned}$$

The convergence thus follows from the hypothesis $\sup_{\eta\in Q}\|\eta\|_{W_0^{1,2}}<\infty$. □

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