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A Renormalizable SYK-Type Tensor Field Theory

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Abstract. In this paper we introduce a simple field theoretic version of the Carrozza–Tanasa–Klebanov–Tarnopolsky (CTKT) "uncolored" holographic tensor model. It gives a more familiar interpretation to the previously abstract modes of the SYK or CTKT models in terms of momenta. We choose for the tensor propagator the usual Fermionic propagator of condensed matter, with a spherical Fermi surface, but keep the CTKT interactions. Hence, our field theory can also be considered as an ordinary condensed matter model with a non-local and non-rotational invariant interaction. Using a multi-scale analysis, we prove that this field theory is just renormalizable to all orders of perturbation theory in the ultraviolet regime.

1. Introduction

Holography (and in particular the AdS/CFT correspondence) provides an effective definition of quantum gravity systems dual to certain conformal field theories. However, until recently the lack of simple solvable examples of this correspondence prevented to extract easily the gravitational content. A more serious shortcoming of AdS/CFT is that a second quantized version of quantum gravity should not be limited to an AdS geometry. It should give a meaning to some kind of functional integral over space–times, presumably pondered by an action of the Einstein–Hilbert (EH) type. This seems up to now intractable in the continuum.

Therefore, in parallel to string theory and AdS/CFT research, and largely independently from them, several formalisms have been developed in order to define a background-independent discretized version of the quantum gravity functional integral. They go under various names such as dynamical and causal triangulations [1], spin foams [2] and group field theory [3–8], which is their second quantized version [9,10], or random matrix and tensor models. The best success story in this direction is provided by random matrix models [11], for which the critical limit of 't Hooft topological expansion provides a universal random geometry [12] now *proven* equivalent to Liouville continuum gravity in dimension two [13].

The Feynman graphs of random matrix models are dual to two-dimensional triangulated surfaces. Random tensor models of higher ranks were therefore introduced to perform a similar sum but for higher-dimensional triangulated geometries [14–16]. They are indeed pondered by a discretized version of the EH action [17]. But their development was impaired by the lack of analytic tools.

Some years ago random tensors underwent a major upheaval. The theory was unlocked by the discovery of colored group field theory [18] and of the associated $1/N$ expansion [19–21] which has lead to a new universality theorem for random tensors [22,23]. It provided the missing hierarchy for the Feynman graphs of tensor models. The leading order was identified as the now famous melonic family [24]. Surprisingly this melonic family is *simpler* than the planar family that leads 't Hooft expansion at rank two. But it is essential to add that the tensor $1/N$ expansion itself (in its subleading orders) is *much more complicated* than the 't Hooft expansion. At rank d it organizes the huge geometric category of piecewise linear quasi-manifolds of dimension d. Several detailed reviews on this modern theory of random tensors are now available [22,23]. The corresponding revived approach to quantum gravity forms the "tensor track" [25,26].

AdS/CFT correspondence and tensor models were until recently unrelated. This is no longer the case. The Sachdev-Ye-Kitaev (SYK) model [27–29] provided two years ago a simple solvable example of an "almost" $\mathrm{AdS}_2/\mathrm{CFT}_1$ correspondence. It exhibits interesting properties such as maximal chaos [30] and approximate conformal invariance, explicitly broken through a kind of bilocal BCS mechanism. It is now clear that many details in the SYK model are not essential (Boson or Fermions, real or complex, particular rank, etc.). The only feature which is not optional is the presence of *at least one random tensor* which ensures that the large N limit is governed by the melonic family.

The link between SYK and tensor models was made even tighter in the Gurau–Witten (GW) [31,32] and Carrozza–Tanasa–Klebanov–Tarnopolsky (CTKT) models [34,35]. They open the new chapter of *holographic tensor models*. All this research enjoys currently tremendous activity, see [36–45] and more references therein. However, there is one category of random tensors still under the radar of the SYK and string community, namely *tensor field theories* (TFTs) [46–55]. TFTs distinguish themselves from tensor models by the presence of a non-trivial propagator. It allows to morph the $1/N$ limit into the physically more familiar picture of power counting, scales and a renormalization group analysis, opening the possibility to search numerically for non-trivial fixed points [56–58]. Until now in SYK and holographic tensor models, the modes are abstract and lack any spatial interpretation and the $N \to \infty$ limit is always performed at the beginning, keeping only the leading $1/N$ terms. Remark that subleading effects in $1/N$ depend on the detail of the model chosen [59,60]. In this way the $1/N$ limit cannot couple to the conformal limit. This seems to us somewhat unphysical.

In TFTs typical interactions still belong to the tensor theory space [61] but the propagator (i.e., the Gaussian measure covariance) is purposefully chosen to slightly break the tensor symmetry. This is quite natural if we consider the tensorial symmetry as a kind of abstract generalization of *locality* in field theory [62]. Propagators, as their name indicates, should *break* locality.

The main consequence of this slight breaking of the tensor symmetry is to allow for a separation of the tensor indices into (abstract, background-independent) ultraviolet and infrared degrees of freedom. Like in ordinary field theory most of the indices should have small covariances. They are identified with (abstract) ultraviolet degrees of freedom. They should be integrated to compute the effective theory for the few indices which form the infrared, effective degrees of freedom (not the other way around!). This picture seems also related to the general AdS/CFT philosophy in which the renormalization group, which flows between different conformal fixed points, precisely provides the extra bulk dimension of AdS [63].

At rank 2, TFTs reduce to non-commutative quantum field theory (NCQFT), which is an effective regime of string theory [64,65]. Mathematically it also corresponds to Kontsevich-type matrix models instead of ordinary matrix models [66,67]. In the Grosse–Wulkenhaar version, it can be renormalized [68,69] and the leading planar sector displays beautiful features such as asymptotic safety [70] and integrability [71], together with a completely unexpected restoration of Poincaré symmetry and of Osterwalder–Schrader positivity [72].

TFTs are the natural higher-rank generalizations of such NCQFTs. When equipped with additional gauge projectors, such TFTs coincide with tensor group field theory [73,74], whose divergencies and radiative corrections require regularization, hence non-trivial propagators, as argued in [75]. An important unexpected property of TFTs is their generic asymptotic freedom, at least for quartic melonic interactions [76–78].

For all these reasons we introduce in this paper a first example of a tensor field theory of the *SYK-type*.[1] The key is to choose an interesting propagator. Motivated by the condensed matter background of the SYK model, we choose the usual propagator of Fermions in 4 dimensions with a spherical Fermi surface (jellium model of non-relativistic many Fermions),[2] but we keep for interaction the two $O(N)^3$-invariant quartic tensor interactions of the Carrozza–Tanasa [34] model. Remark that the tensor theory space for rank three $O(N)$ invariants is not exactly the same as the $U(N)$ one, which has been more often considered in the literature. Remark that the complete graph interaction has been also used in the context of the large D-limit of matrix models [79] and recently

[1] We could also call it a holographic tensor field theory, but we prefer to wait until its holographic properties are better analyzed.

[2] Therefore, our model reminds of Horava–Lifschitz gravity or condensed matter physics, but beware that the abstract "space" of TFTs should not be necessarily identified with ordinary coordinates on a semi-classical effective background.

generalized to larger ranks in [80]. Note finally that group field theory has recently introduced tensor fields depending on a scalar field playing a role identical of "time" with the important feature that the interaction kernel in that theory is local in this field [81]. The introduction of such a field finds its interest in the search of an effective cosmological dynamics in group field theory. The model considered in [81] is, however, not the same as the one we study here simply because of the so-called gauge invariance imposed on the fields (that invariance is manifest with the presence of a $\star$-product) and the fact that the (Laplacian) propagator used in that theory is not that of a condensed matter theory.

In this paper we study the ultraviolet regime of this model. Our main result is to prove a "BPHZ-type" finiteness theorem at all orders through a multi-scale analysis in the spirit of [46,47,73,82]. We shall not discuss the non-perturbative stability here; see, however, [83–85] for the constructive tensor field program, entirely devoted to this issue.

The most interesting regime of the renormalization group in condensed matter physics is governed by the low temperature excitations close to the Fermi surface. If this "Fermi surface" infrared regime is also just renormalizable, this may prevent the formation of a conformal regime with anomalous dimension of the SYK type and saturation of the maximal chaos bound [30], as argued first in [33]. However, this requires a careful analysis in the style of [86–88] which is left for a future study.

Remark finally that our model is quite different from other types of tensor theories such as the Gross–Neveu tensor models studied in [89,90] in which the tensor invariance remains unbroken by the propagator.

2. The Model

2.1. Fields

Our goal is to extend into a tensor field theory the CTKT model [34,35], using the interactions of [34], the time dependence *à la* SYK of [35] and a new propagator which mixes time with additional *spatial* degrees of freedom. Since we want to use the Laplacian as our (non-relativistic) abstract spatial kinetic energy, and since it is a symmetric operator, we have first to double the number of fields. So we consider *a pair of Majorana tensor fields* which we write as $\{\chi(t, \vec{x}, \sigma)\}$ where σ is an abstract "spin" index taking two values, 1 or 2.[3] To stick for the moment as close as possible to the SYK and CTKT models, we keep the interaction *local in time*. But, and this is the defining feature of tensor field theory, our propagator is not local but has the ordinary form of a jellium condensed matter Fermionic propagator.

The coordinates $\vec{x}$ replace the three $O(N)^3$-symmetric tensor indices. They take value in a Cartesian product E^3. In this paper we choose either $E = \mathbb{R}$, hence $\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$, or a compactified version $E = U(1)$ and

[3] We could use the equivalent complex notation $\{\psi(t, \vec{x}), \bar{\psi}(t, \vec{x})\}$ but this would take us further away from the initial SYK formalism.

$\vec{x} = (\theta_1, \theta_2, \theta_3) \in U(1)^3$, the three-dimensional torus. Remember, however, not to identify this $\vec{x}$ variable with an ordinary direct space coordinate, as the CT interaction is neither rotation invariant nor local in terms of these variables.

The time variable is taken on the thermic circle $[-\frac{\beta}{2}, \frac{\beta}{2}]$. Since $\beta = \frac{1}{kT}$, this thermal circle becomes large at low temperature. We also introduce the dual momentum variables $(p_0, \vec{p})$. p_0, often called ω in condensed matter and SYK literature, is a Euclidean Matsubara frequency; hence, it takes values in a $\mathbb{Z}$ lattice of small mesh $\frac{2\pi}{\beta}$; if χ is Fermionic we should take antiperiodic conditions, which, since $p_0 = \frac{2\pi}{\beta}(n + \frac{1}{2})$, provide a natural infrared cutoff. This will not important in the subsequent analysis where p_0 is taken large compared to the lattice spacing.

Similarly, the momenta dual to the $\vec{x}$ variables will be denoted generically as $\vec{p}$. They take values in $\mathbb{R}^3$ or $\mathbb{Z}^3$ depending upon whether we choose $E = \mathbb{R}$ or $E = U(1)$, but this is again quite irrelevant for our analysis which considers a regime of the theory at large $\vec{p}$. We introduce the notations $p^2 = |\vec{p}|^2 = \sum_{i=1}^3 p_i^2$, and $\int \mathrm{d}^3p$ means either $\int_{\mathbb{R}^3} \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3$ in the non-compact case $E = \mathbb{R}$ or $\sum_{(p_1,p_2,p_3)\in\mathbb{Z}^3}$ in the compact case $E = U(1)$. The difference is not essential since in this paper we shall study the theory at large momenta only.

2.2. The Propagator

Using the Matsubara formalism [91] (up to an inessential difference of sign convention), the propagator in Fourier space $\hat{\underline{C}}$ of a condensed matter Fermionic field living on space $\mathbb{R}^3$ at finite temperature T is equal to:

$$\hat{\underline{C}}(p_0, \vec{p}) = \frac{1}{ip_0 + e(p)}, \qquad e(p) = \frac{p^2}{2m} - \mu, \tag{1}$$

where the vector $\vec{p}$ in (1) is three-dimensional, and the parameters m and μ correspond to the effective mass and to the chemical potential (which fixes the Fermi energy). To simplify we put for the moment $2m = \mu = 1$, so that $e(p) = p^2 - 1$. The corresponding direct space propagator at temperature T and position $(t, \vec{x})$ (where $\vec{x}$ is the three-dimensional spatial component) is

$$\underline{C}(t, \vec{x}) = \frac{T}{(2\pi)^3} \sum_{p_0} \int \mathrm{d}^3p \, e^{-ip_0 t + ip\cdot x} \, \hat{\underline{C}}(p_0, \vec{p}). \tag{2}$$

It is antiperiodic in the variable t with antiperiod $\frac{1}{T}$. This means that

$$\hat{\underline{C}}(p_0, \vec{p}) = \frac{1}{2} \int_{-\frac{1}{T}}^{\frac{1}{T}} \mathrm{d}t \int \mathrm{d}^3x \, e^{+ip_0 t - ip\cdot x} \, \underline{C}(t, \vec{x}) \tag{3}$$

is not zero only for discrete values (called the Matsubara frequencies):

$$p_0 = (2n+1)\pi T, \quad n \in \mathbb{Z}, \tag{4}$$

where we take $\hbar = k = 1$. Remark that only odd frequencies appear, because of antiperiodicity, hence $|p_0| \geq \pi T$ so that the temperature acts like an effective infrared cutoff.

The notation $\sum_{k_0}$ in (2) means really the discrete sum over the integer n in (4).[4] To simplify notations we write:

$$\int \mathrm{d}^4 p \;\equiv\; T \sum_{p_0} \int \mathrm{d}^3 p, \quad \int \mathrm{d}^4 x \;\equiv\; \frac{1}{2}\int_{-1/T}^{1/T} \mathrm{d}t \int \mathrm{d}^3 x. \tag{5}$$

$$\underline{\hat{C}}(p_0, \vec{p}) := \frac{-ip_0 + \ e(p)}{p_0^2 + e^2(p)} = \int_0^\infty \mathrm{d}\alpha\, (-ip_0 + \ e(p)) e^{-\alpha(p_0^2 + e^2(p))}. \tag{6}$$

To study the ultraviolet regime of the theory, we can consider only large values of p_0 and $e(p)$. In that regime we can write

$$\underline{\hat{C}}(p_0, \vec{p}) := \frac{-ip_0 + \ e(p)}{p_0^2 + e^2(p)} (1 - e^{-(p_0^2 + e^2(p))}) = \int_0^1 \mathrm{d}\alpha\, (-ip_0 + \ e(p)) e^{-\alpha(p_0^2 + e^2(p))}. \tag{7}$$

We then adopt then the following covariance for our free model with abstract spin is defined by the matrix covariance rules

$$\begin{aligned} \Big(\langle \chi_\sigma(p_0, \vec{p}) \chi_{\sigma'}(p_0', \vec{p}\,')\rangle\Big)_{\sigma\sigma'} &= \Big(C_{\sigma\sigma'}(p_0, \vec{p}) \delta(p_0 - p_0') \delta(\vec{p}, \vec{p}\,')\Big)_{\sigma\sigma'} \\ &= \left[\frac{-ip_0}{p_0^2 + e^2(p^2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e(p)}{p_0^2 + e^2(p^2)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] \delta(p_0 - p_0') \delta(\vec{p}, \vec{p}\,'), \end{aligned} \tag{8}$$

where $\chi_\sigma(p_0, \vec{p}) = \chi(p_0, \vec{p}, \sigma)$, and σ, σ' are the spin indices, and the matrices refer to these indices. Remark that these rules are globally antisymmetric, as they should be for Grassmann variables.

Denoting $d\mu_C(\chi)$ the corresponding Grassmann Gaussian measure [86–88], the free theory is defined with J_σ a Fermionic tensor source field (also with a two-valued spin index) and $J \cdot \chi = \sum_\sigma \int \mathrm{d}p_0 \mathrm{d}^3 p\, J_\sigma(p_0, \vec{p}) \chi_\sigma(p_0, \vec{p})$. C is the covariance of the Gaussian measure, or free propagator and $\int d\mu_C$ is the Gaussian integral of covariance C. We are interested in computing the partition function Z

$$Z(J) = \int \mathrm{d}\mu_C(\chi) e^{-S[\chi] + J \cdot \chi}, \tag{9}$$

and the generating function for cumulants of the theory

$$Z(J) = W(J) = \log Z(J). \tag{10}$$

2.3. The Tensor Interaction

We equip the free model with interactions inspired by those of Carrozza–Tanasa[5] [34]. The tetraedric part of that interaction was used also in [35].

[4] When $T \to 0$, k_0 becomes a continuous variable, the discrete sum becomes an integral $T \sum_{k_0} \to \frac{1}{2\pi} \int \mathrm{d}k_0$, and the corresponding propagator $C_0(k_0, \vec{k})$ becomes singular on the Fermi surface defined by $k_0 = 0$ and $|k| = 1$.

[5] The first term of this interaction with coupling λ_+ is also the one used by F. Ferrari for the large D limit of matrix models [79].

Consider the following interaction,

$$S_{\text{int}}(\chi) = \lambda_+ I_{\mathbf{b}_+}(\chi) + \lambda_m \sum_{c=1}^{3} I_{\mathbf{b}_c}(\chi) + V_2(\chi), \tag{11}$$

where the coupling constants λ_+ and λ_m (m standing for "melonic") are the bare coupling constants (which themselves decompose into renormalized constants plus counterterms) and where $I_{\mathbf{b}_+}$ and $I_{\mathbf{b}_c}$ are the quartic interaction terms fully expanded in $(p_0, \vec{p})$-space representation as

$$\begin{aligned} I_{\mathbf{b}_+} = & \sum_{\sigma=1,2} \int \left[\prod_{l=1}^{4} \mathrm{d}p_{0;l} \right] \mathrm{d}^3p \mathrm{d}^3p' \, \chi_\sigma(p_{0;1}, p_1, p_2, p_3)\chi_\sigma(p_{0;2}, p_1, p'_2, p'_3) \\ & \times \chi_\sigma(p_{0;3}, p'_1, p_2, p'_3)\chi_\sigma(p_{0;4}, p'_1, p'_2, p_3)\delta\left(\sum_{l=1}^{4} p_{0;l} \right), \\ \sum_{c=1}^{3} I_{\mathbf{b}_c}(\chi) = & \int \left[\prod_{l=1}^{4} \mathrm{d}p_{0;l} \right] \mathrm{d}^3p \mathrm{d}^3p' \, \chi_1(p_{0;1}, p_1, p_2, p_3)\chi_2(p_{0;2}, p'_1, p_2, p_3) \\ & \times \chi_1(p_{0;3}, p'_1, p'_2, p'_3)\chi_2(p_{0;4}, p_1, p'_2, p'_3)\delta\left(\sum_{l=1}^{4} p_{0;l} \right) + \text{sym}(1,2,3), \end{aligned} \tag{12}$$

where the integrals are understood as (5), and sym(1,2,3) replaces the sum over colors. The two types of interactions are associated with bubble diagrams $\mathbf{b}_+$ and $\mathbf{b}_c$ which represent orthogonal invariants as depicted in Fig. 1. Note that in this figure, only the melonic bubble $\mathbf{b}_1$ is drawn and the other bubbles with colors 2 and 3 can easily recovered.

Melonic interactions [22,24] belong to the family of dominant terms at large N, and we expect that they will be dominant in the ultraviolet regime. Remark also that the above interactions are local in the p_0-space but non-local in the $\vec{p}$-space. Attached to the local variables, a delta function $\delta(\sum_{l=1}^{4} p_{0;l})$ at each vertex manifests the conversation of momenta entering and exiting from the vertex. This is the usual standard of quantum field theory.

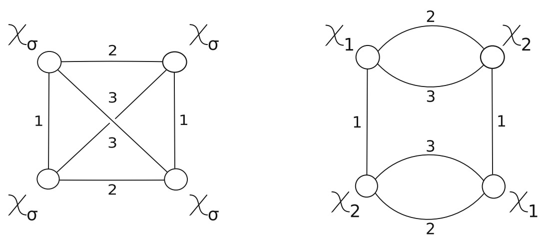

FIGURE 1. $O(N)$ invariants as interactions: on the left, tetraedric invariant associated with $\mathbf{b}_+$; on the right, le spheric melonic invariant $\mathbf{b}_1$

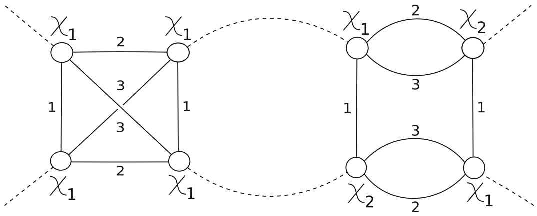

FIGURE 2. A Feynman graph in the theory

To be more specific, the most natural rule at this stage seems to keep the tetraedric interaction in λ_+ *diagonal in spin indices*, hence the sum over σ of an interaction with four χ_σ's, as if we had two independent Majorana fields. However, for the melonic interaction, since it is bipartite we feel the most natural interaction is to mix the spins hence to choose two spins and two anti spins cyclically along the melonic cycle, see Fig. 1 which shows the vertices associated with these interactions. These specific spin index choices for the interactions could be modified if that leads to more interesting infrared physics.

The remaining term in (11), V_2, gathers the two-point function mass and wave function counterterms:

$$\begin{aligned} V_2(\chi) = \Delta\mu(\lambda_m,\lambda_+) \sum_{\sigma<\sigma'} \int \mathrm{d}p_0 \mathrm{d}^3p \; \chi_\sigma(p_0,\vec{p})\chi_{\sigma'}(-p_0,\vec{p}) \\ + \Delta_{p_0}(\lambda_m,\lambda_+) \sum_{\sigma} \int \mathrm{d}p_0 \mathrm{d}^3p \; (ip_0)\chi_\sigma(p_0,\vec{p})\chi_\sigma(-p_0,\vec{p}) \\ + \Delta_{p^2}(\lambda_m,\lambda_+) \sum_{\sigma<\sigma'} \int \mathrm{d}p_0 \mathrm{d}^3p \; p^2\chi_\sigma(p_0,\vec{p})\chi_{\sigma'}(-p_0,\vec{p}). \end{aligned} \tag{13}$$

In this formula, as usual in perturbative renormalization, the mass counterterm $\Delta\mu(\lambda_m,\lambda_+)$ and wave function counterterms $\Delta_{p_0}(\lambda_m,\lambda_+)$ and $\Delta_{p^2}(\lambda_m,\lambda_+)$ are themselves perturbative series in the coupling constants.

A priori the counterterms could be power series in both couplings but we shall see below that only the melonic vertex is relevant in the ultraviolet regime. We nevertheless also included the tetraedric vertex because we feel it is the one which could be responsible for SYK physics in the infrared regime. Finally, a Feynman graph in this theory is formed with the gluing of vertices $\mathbf{b}_+$ and $\mathbf{b}_c$ with propagator lines that we draw as dashed lines in order to distinguish them from the internal structure of the vertices. See Fig. 2. As one quickly understands, a Feynman graph in this setting is a 4-regular edge (line) colored graph with half-lines. The propagator lines will be associated with the color 0.

2.4. Amplitudes

Expanding the theory in Feynman graphs, the amplitudes have to be arranged as Pfaffians of the antisymmetric matrix C [86] and have the general form

$$\left\langle \prod_{a=1}^{q} \psi(p_{0;a}, \vec{p}_a) \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}\mu_C(\psi) \left[\prod_{a=1}^{q} \psi(p_{0;a}, \vec{p}_a) \right] \Big[-S_{\text{int}}(\psi) \Big]^n = \sum_{\mathcal{G}} A_{\mathcal{G}}. \tag{14}$$

In the above expression, ψ stands either for χ_1 or for χ_2. As already emphasized, the spin index does not matter in the ultraviolet study, but can strongly affect the infrared regime.

Feynman amplitudes $A_{\mathcal{G}}$ will be our focus. For the moment and for simplicity, we neglect the presence of mass and wave function counterterms. They will be discussed in the following sections. Therefore, we consider a connected amputated graph $\mathcal{G}$ with vertex set $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_m$, with cardinal $V = |\mathcal{V}|$, where $\mathcal{V}_+$ is the set of tetraedric vertices with pattern $\mathbf{b}_+$ and $\mathcal{V}_m$ is the set of melonic vertices $\mathbf{b}_c$, and with line set $\mathcal{L}$, with cardinal $L = |\mathcal{L}|$. Note that $\mathcal{L}$ decomposes in two sorts of lines: $\mathcal{L}_1$ associated with the diagonal part of the covariance $C_{\sigma\sigma}$, and $\mathcal{L}_2$ associated with the off-diagonal entries. We denote N_{ext} the number of external fields also called external legs. Henceforth, the index σ will be mostly omitted in the notations but their presence is, however, indicated by the two sets $\mathcal{L}_i$, $i = 1, 2$.

The bare amplitude of a Feynman graph $\mathcal{G}$ is given by

$$\begin{aligned} A_{\mathcal{G}} = K_0 \left[\prod_{v \in \mathcal{V}} (-\lambda_v) \right] \int \left[\prod_{v \in \mathcal{V}} \mathrm{d}p_{0;v} \prod_s \mathrm{d}p_{v,s} \right] \\ \times \left[\prod_{\ell \in \mathcal{L}} C_\ell(\{p_{0;v(\ell)}, \vec{p}_{v(\ell),s}\}; \{p'_{0;v'(\ell)}, \vec{p}'_{v'(\ell),s}\}) \right] \\ \times \left[\prod_{v \in \mathcal{V}} \prod_s \delta(p_{v,s} - p'_{v,s'}) \right] \left[\prod_{v \in \mathcal{V}} \delta\left(\sum_{l=1}^{4} p_{0;l;v} \right) \right], \end{aligned} \tag{15}$$

where λ_v is a given coupling associated with $v \in \mathcal{V}$; $p_{0;\ell}$ and $p_{v(\ell),s}$ are the coordinates involved in the propagator labeled by a line index ℓ incident to its source and target vertices $v(\ell)$ and $v'(\ell)$; $p_{v,s}$ are the p coordinates of the vertex v and they possess a strand index s. The constant K_0 includes the Fermionic Pfaffian signs, the graph symmetry factor and a combinatorial constant. We will use the compact notation $K_0 \big[\prod_{v \in \mathcal{V}} (-\lambda_v) \big] = \kappa(\lambda)$.

Note that the propagator in the $\vec{p}$ coordinates is a product of Kroneckers δ and, similarly, the vertex kernels are also products of δ functions which convolute the different indices of the tensors. Integrating those δ produces conservation of the p coordinate index along a strand of the tensor graph. At the end of integration of all δ's in all propagators, one obtains a p_f coordinate per one-dimensional object f in the graph that we call *face*. Graphically a face is an alternating sequence of propagator lines with color 0 and colored lines c of vertices. A face is *closed* or *internal* if this sequence is a cycle in the colored graph; it is otherwise *open* or *external*. The set of closed faces is denoted $\mathcal{F}_{\text{int}}$, and its cardinality F_{int}; the set of open faces is denoted $\mathcal{F}_{\text{ext}}$, and its cardinality F_{ext}. We write $\mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}} = \mathcal{F}$ the set of all types of faces. Given a closed face (resp. open face) f, we denote p_f (resp. $p_f^{\rm ext}$) the momentum coordinate associated with f.

A face f is made of lines; hence, we write $\ell \in f$. We introduce an incidence matrix between line and faces which identifies if a line goes through a face or not:

$$\epsilon_{\ell f} = \begin{cases} 1, & \text{if } \ell \in f \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

We expand the amplitude (15) as follows:

$$\begin{aligned} A_{\mathcal{G}} = \kappa(\lambda) \int \left[\prod_{\ell \in \mathcal{L}} \mathrm{d}\alpha_\ell \right] \int \left[\prod_{v \in \mathcal{V}} \mathrm{d}p_{0;v} \right] \left[\prod_{f \in \mathcal{F}_{\rm int}} \mathrm{d}p_f \right] \\ \left[\left[\prod_{\ell \in \mathcal{L}_1} (-ip_{0;\ell}) \right] \left[\prod_{\ell \in \mathcal{L}_2} e\left(\sum_{f \in \mathcal{F}} \epsilon_{\ell f} p_f \right) \right] \right] \\ \times e^{-\alpha_\ell (p_{0;\ell}^2 + e^2(\sum_{f\in\mathcal{F}} \epsilon_{\ell f} p_f))} \right] \left[\prod_{v \in \mathcal{V}} \delta\left(\sum_{l=1}^{4} p_{0;l;v} \right) \right]. \end{aligned} \tag{17}$$

This amplitude must be regularized by a cutoff on momenta from which we will able to discuss the behavior of that amplitude. This is the task of the next section.

3. Multi-scale Analysis and Power Counting

We obtain, in this section, a power counting theorem for the amplitudes (17) in the ultraviolet regime using a multi-scale analysis of the Feynman amplitudes in the spirit of [82], adapted to the tensor context of non-local actions.

We begin with the slice decomposition of the propagator. This is a decomposition of the parametric integral obtained from (8) using a geometric progression with ratio $M > 0$. We write

$$\begin{aligned} \hat{C}_{\sigma\sigma'}(p_0, \vec{p}) &= \sum_{i=1}^{\infty} C_{\sigma\sigma';\, i}(p_0, \vec{p}), \\ C_{\sigma\sigma';\, i}(p_0, \vec{p}) &= \int_{M^{-2i}}^{M^{-2(i-1)}} \mathrm{d}\alpha \, (-ip_0 \delta_{\sigma\sigma'} + \varepsilon_{\sigma\sigma'} e(p)) e^{-\alpha(p_0^2 + e^2(p))} \end{aligned} \tag{18}$$

where we introduce the antisymmetric tensor $\varepsilon_{\sigma\sigma'}$, such that $\varepsilon_{12} = -1$ according to (8).

An ultraviolet cutoff is imposed in the space of indices i, such that $C^\rho = \sum_{i=1}^{\rho} C_i$ is the cut-offed propagator. The ultraviolet limit is obtained by taking $\rho \to \infty$. We omit to write the symbol ρ on each propagator, for simplicity. We expect our theory to be fully consistent in the $\rho \to \infty$ limit but this issue is postponed to a future study. In this paper we only establish perturbative renormalization at all orders.

In this ultraviolet regime the value of the chemical potential is unimportant and we have the bound

$$|C_{\sigma\sigma';\, i}(p_0, \vec{p})| \leq K M^{-i} \, e^{-M^{-i}(|p_0| + p^2)}, \tag{19}$$

for some inessential (M-dependent) constant K. To establish this rather trivial bound, one can use that $x = M^{-i}(|p_0| + p^2)$ is bounded by $\delta x^2 + \delta^{-1}$ for any $\delta > 0$; hence, $e^{-M^{-i}(|p_0|+p^2)}$ is bigger than $e^{-\delta M^{-2i}(|p_0|+p^2)^2}$, itself bigger than $e^{-2\delta M^{-2i}(p_0^2+p^4)}$. Then one can distinguish whether p^2 is smaller or greater than 3 to bound in the first case p^4 by 9 (and $|e(p)|$ by 2) and in the second case p^4 by $3e^2(p)$. Then choosing δ suitably small one concludes by trivial estimates on the integral in (18). Remark the anisotropy between p_0 and p and the fact that this bound does not depend on the σ indices. We therefore simplify our notations and omit to mention these in the remaining analysis.

The multi-scale analysis allows for an optimal amplitude bound. We consider a connected amputated Feynman graph $\mathcal{G}$ of the theory with vertex set $\mathcal{V} = \mathcal{V}_+ \cup \mathcal{V}_m$, $V = |\mathcal{V}|$ with propagator line set $\mathcal{L}$, $L = |\mathcal{L}|$. We work at this stage with amputated amplitudes, that are graphs with external vertices where test functions or external fields can be inserted. The number of those external fields also called external legs is N_{ext}.

Introduce the multi-index $\boldsymbol{\mu} \in \mathbb{N}^L$ called (index) assignment which gives to each propagator line ℓ of the graph a scale $i_\ell \in [\![0, \rho]\!]$. Slicing all propagators, the initial amplitude becomes $A_{\mathcal{G}} = \sum_{\boldsymbol{\mu}} A_{\mathcal{G};\boldsymbol{\mu}}$ where $A_{\mathcal{G};\boldsymbol{\mu}}$ is called the multi-scale representation of the amplitude $A_{\mathcal{G}}$. After renormalizing the theory, the sum over $\boldsymbol{\mu}$ or overall possible assignments will be performed. We have at fixed index assignment $\boldsymbol{\mu}$:

$$A_{\mathcal{G};\boldsymbol{\mu}} = \kappa(\lambda) \int \left[\prod_{v\in\mathcal{V}} \mathrm{d}p_{0;v} \prod_{s} \mathrm{d}p_{v,s} \right] \left[\prod_{\ell\in\mathcal{L}} C_{i_\ell}(\{p_{0;v(\ell)}, \vec{p}_{v(\ell),s}\}; \{p'_{0;v'(\ell)}, \vec{p}'_{v'(\ell),s}\}) \right] \times \left[\prod_{v\in\mathcal{V}} \prod_{s} \delta(p_{v,s} - p'_{v,s'}) \right] \left[\prod_{v\in\mathcal{V}} \delta\left(\sum_{l} p_{0;l;v} \right) \right]. \tag{20}$$

Our goal is to find an optimal bound on $A_{\mathcal{G};\boldsymbol{\mu}}$ using, as much as possible, the decay of the lines. To do so, we introduce the so-called quasi-local subgraphs $\mathcal{G}^i$ of $\mathcal{G}$ as the subgraphs made of lines of $\mathcal{G}$ with index higher than i: $\forall \ell \in \mathcal{L}(\mathcal{G}^i) \cap \mathcal{L}, i_\ell \geq i$. $\mathcal{G}^i$ might have several connected components that we denote at fixed i, $\mathcal{G}^i_{(k)}$. Then $\{\mathcal{G}^i_{(k)}\}$ is the set of all quasi-local subgraphs of $\mathcal{G}$. Consider g a subgraph of $\mathcal{G}$, seeking a criterion for checking if g should coincide with some $\mathcal{G}^i_{(k)}$, we have the following: at fixed index assignment $\boldsymbol{\mu}$, define

$$i_g(\boldsymbol{\mu}) = \inf_{l \text{ internal line } \in g} i_l \qquad \text{and} \qquad e_g(\boldsymbol{\mu}) = \sup_{l \text{ external line } \in g} i_l, \tag{21}$$

then there exists (i, k), such that $g = \mathcal{G}^i_{(k)}$ if and only if $i_g(\boldsymbol{\mu}) > e_g(\boldsymbol{\mu})$, the so-called almost local condition. The value of i satisfies $i_g(\boldsymbol{\mu}) \geq i > e_g(\boldsymbol{\mu})$. An important property of the set of quasi-local graphs $\{\mathcal{G}^i_{(k)}\}$ is that it is partially ordered under inclusion, and, using this partial order, one forms an abstract tree namely the Gallavotti–Nicoló (GN) tree [93]. The rest of our program is to find an optimal bound for $A_{\mathcal{G};\boldsymbol{\mu}}$ in terms of the nodes of the GN tree, in other words, an optimal bound which expresses uniquely in terms of the $\{\mathcal{G}^i_{(k)}\}$.

At a fixed scale index i, we will need the following approximation of the sum

$$\sum_{p\in\mathbb{Z}} e^{-M^{-i}|p|^n} = cM^{\frac{i}{n}}(1+O(M^{-\frac{i}{n}})), \tag{22}$$

for $n>0$ and some positive constant $c=n^{-1}\Gamma(n^{-1})$ (see the detail of the calculations in Appendix A of [48]).

We are ready to perform the integration over internal variables of the $\{\mathcal{G}^i_{(k)}\}$ graphs. This can be organized in completely equivalent ways either in momentum or direct space, using respectively the bound (19), the important point being that it has to follow the GN tree structure. It means we sum inductively over the internal $(p_0,\vec{p})$ loop momenta of the $\{\mathcal{G}^i_{(k)}\}$ graphs, following the GN tree structure. At fixed $\boldsymbol{\mu}$, we integrate over delta's in the p-space and use (19) to obtain

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq K_1\left[\prod_{\ell\in\mathcal{L}} M^{-i_\ell}\right]\int\left[\prod_{v\in\mathcal{V}} \mathrm{d}p_{0;v}\right]\left[\prod_{\ell\in\mathcal{L}} e^{-M^{-i_\ell}|p_{0;\ell}|}\right]\left[\prod_{v\in\mathcal{V}}\left(\sum_l p_{0;l;v}\right)\right]$$
$$\times\int\left[\prod_{f\in\mathcal{F}_{\text{int}}} \mathrm{d}p_f\right]\left[\prod_{f\in\mathcal{F}_{\text{int}}} e^{-\left(\sum_{\ell\in f} M^{-i_\ell}\right)p_f^2}\right], \tag{23}$$

where $K_1 = K^L\kappa(\lambda)K_{\text{ext}}$, and K_{ext} is a bound over the product of external face amplitudes $e^{-(\sum_{\ell\in f} M^{-i_\ell})p^2_{\text{ext};f}}$ which can be easily achieved by bounding each factor by a constant. Note that the r.h.s bound factorizes along p_0-space and p-space. To find an optimal bound amplitude is therefore like combining a standard local QFT procedure and a non-local one.

The goal is to make the result of that summation/integration as low as possible. The integration over $p_{0;l;v}$ variables is standard in ordinary local QFT: we choose a vertex root and perform a momentum routine over the $p_{0;l;v}$. We can integrate over the set $\text{Cycle}_{\mathcal{G}}$ of independent cycles (loops in the underlying graph); along each cycle c choose the minimal index among the i_ℓ's: $i_c=\min_{\ell\in c} i_\ell$. In direct space, this is choosing a tree compatible with the GN tree, as explained in [82]. Concerning the non-local part, for each internal face f, we introduce the index $i_f=\min_{\ell\in f} i_\ell$ that will be important during the integration.

We are in position to find an optimal bound for any amplitude (23) as

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq K_1K_2\left[\prod_{\ell\in\mathcal{L}} M^{-i_\ell}\right]\left[\prod_{c\in\text{Cycle}_{\mathcal{G}}} M^{i_c}\right]\left[\prod_{f\in\mathcal{F}_{\text{int}}} M^{\frac{i_f}{2}}\right] \tag{24}$$

where K_2 is a constant. This result must be re-expressed in terms of the quasi-local subgraphs. Each of the factors have been already addressed in previous works. We have

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq K_3\left[\prod_{\ell\in\mathcal{L}}\prod_{i=1}^{i_\ell} M^{-1}\right]\left[\prod_{c\in\text{Cycle}_{\mathcal{G}}}\prod_{i=1}^{i_c} M\right]\left[\prod_{f\in\mathcal{F}_{\text{int}}}\prod_{i=1}^{i_f} M^{\frac{1}{2}}\right] \tag{25}$$

$$\leq K_3 \left[\prod_{\ell \in \mathcal{L}} \prod_{(i,k)/\ell \in \mathcal{L}(\mathcal{G}^i_{(k)})} M^{-1} \right] \left[\prod_{c \in \mathrm{Cycle}_{\mathcal{G}}} \prod_{(i,k)/\ell \in \mathcal{L}(\mathcal{G}^i_{(k)})} M \right] \left[\prod_{f \in \mathcal{F}_{\mathrm{int}}} \prod_{(i,k)/\ell \in \mathcal{L}(\mathcal{G}^{i\,f}_{(k)})} M^{\frac{1}{2}} \right]$$

with $K_3 = K_1 K_2$. The two first products are well-known (see [82]) and we simply rewrite them as:

$$\prod_{\ell \in \mathcal{L}} \prod_{(i,k)/\ell \in \mathcal{L}(\mathcal{G}^i_{(k)})} M^{-1} = \prod_{(i,k)} M^{-L(\mathcal{G}^i_{(k)})},$$

$$\prod_{c \in \mathrm{Cycle}_{\mathcal{G}}} \prod_{(i,k)/\ell \in \mathcal{L}(\mathcal{G}^{i\,c}_{(k)})} M = \prod_{(i,k)} M^{[L(\mathcal{G}^i_{(k)}) - (V(\mathcal{G}^i_{(k)}) - 1)]} \tag{26}$$

The last product decomposes into the following

$$\prod_{f \in \mathcal{F}_{\mathrm{int}}} \prod_{(i,k)/\ell \in \mathcal{L}(\mathcal{G}^{i\,f}_{(k)})} M^{\frac{1}{2}} = \prod_{f \in \mathcal{F}_{\mathrm{int}}} \prod_{(i,k)/\ell_f \in \mathcal{L}(\mathcal{G}^i_{(k)})} M^{\frac{1}{2}}$$

$$= \prod_{(i,k)} \prod_{f \in \mathcal{F}_{\mathrm{int}} \cap \mathcal{G}^i_{(k)}} M^{\frac{1}{2}} = \prod_{(i,k)} M^{\frac{1}{2} F_{\mathrm{int}}(\mathcal{G}^i_{(k)})} \tag{27}$$

Let us address now the vertices coming from the mass and wave functions couplings. The introduction of a mass coupling does not change the overall analysis. Adding wave function vertices changes the power counting by introducing an equal number of vertices V_{p_0} and V_{p^2}. The above analysis leading to (26) remains the same, the exponent therein becomes

$$L(\mathcal{G}^i_{(k)}) - (V(\mathcal{G}^i_{(k)}) + V_{p_0}(\mathcal{G}^i_{(k)}) + V_{p^2}(\mathcal{G}^i_{(k)}) - 1), \tag{28}$$

where $V(\mathcal{G}^i_{(k)})$ uniquely denotes the number of quartic vertices. Then the vertex weights $(-ip_0)$ and p^2 introduce back a factor

$$\prod_{(i,k)} M^{V_{p_0}(\mathcal{G}^i_{(k)}) + V_{p^2}(\mathcal{G}^i_{(k)})} \tag{29}$$

Collecting all contributions, the following statement holds:

Theorem 3.1 (Power counting). *Let $\mathcal{G}$ be a connected graph of the model* (12) *with Gaussian measure determined by the covariance* (8). *Considering $A_{\mathcal{G};\boldsymbol{\mu}}$ the amplitude associated with $\mathcal{G}$ at index assignment $\boldsymbol{\mu}$, there exists some large constant K such that*

$$|A_{\mathcal{G};\boldsymbol{\mu}}| \leq K^{V(\mathcal{G})} \prod_{(i,k) \in \mathbb{N}^2} M^{\omega_{\mathrm{deg}}(\mathcal{G}^i_{(k)})}, \tag{30}$$

where $\mathcal{G}^i_{(k)}$ are the quasi-local subgraphs and the divergence degree is given by

$$\omega_{\mathrm{deg}}(\mathcal{G}) = -(V(\mathcal{G}) - 1) + \frac{1}{2} F_{\mathrm{int}}(\mathcal{G}). \tag{31}$$

From the above theorem, we see that the just renormalizable model of this kind is of a different type than usual (ϕ^4_4) scalar field theory or of tensor field theory of the quartic type studied in [48].

4. Analysis of the Divergence Degree

We need to count the number of internal faces in a graph $\mathcal{G}$ *with external legs*. This requires to extend the notion of jackets into *pinched jackets* [46,47]. This is usually done in a bipartite (complex) framework but in our case we have real fields and the graphs are not bipartite so we shall use some new arguments.

Proposition 1. *Consider a connected rank* $d=3$ *graph* $\mathcal{G}$*, with boundary graph* $\partial\mathcal{G}$*. Let* C_∂ *be the number of connected components of* $\partial\mathcal{G}$*,* V_+ *the number of vertices of the kind* $\mathbf{b}_+$ *and* V_m *the number of vertices of the kind* $\mathbf{b}_c$ $V = V_+ + V_m$*.* $N_{\rm ext}$ *is the number of external legs of* $\mathcal{G}$*;*

$$F_{\rm int}(\mathcal{G}) = -(\omega(\mathcal{G}_{\rm color}) - g_{\partial\mathcal{G}}) + 3V_+ + 2V_m - N_{\rm ext} - (C_\partial - 1) + 2, \qquad (32)$$

where $\omega(\mathcal{G}_{\rm color}) = \sum_{\widetilde{J}} g_{\widetilde{J}}$ *is the sum of genera of the pinched jackets of* $\mathcal{G}_{\rm color}$ *the colored extension of* $\mathcal{G}$ *and* $g_{\partial\mathcal{G}}$ *is the genus of the boundary graph.*

Proof. Consider $\mathcal{G}$ a connected tensor graph and $\mathcal{G}_{\rm color}$ the colored extension of $\mathcal{G}$. We denote the number of vertices, $V_{\rm color}$, the number of lines, $L_{\rm color}$ of $\mathcal{G}_{\rm color}$. We recall our notations, V and L are, respectively, the same quantities for $\mathcal{G}$, while $F_{\rm int}$ is the number of internal faces of $\mathcal{G}$. For the boundary graph $\partial\mathcal{G}$, we denote V_∂, E_∂ and F_∂ the cardinality of the vertex set, edge set and face set. For a pinched jacket $\widetilde{J}$, we use $V_{\widetilde{J}}$ for the number of vertices, $E_{\widetilde{J}}$ for the number of edges and $F_{\widetilde{J}}$ for the number (necessarily closed) faces. Note that because $\mathcal{G}$ is connected, so is $\mathcal{G}_{\rm color}$ and any jacket within $\mathcal{G}_{\rm color}$ is also connected.

There are $d!/2 = 3$ jackets in $\mathcal{G}_{\rm color}$. Each $\mathbf{b}_c$ or $\mathbf{b}_+$ vertex of $\mathcal{G}$ decomposes in 4 vertices in $\mathcal{G}_{\rm color}$. Each of those vertices in $\mathcal{G}_{\rm color}$ decomposes again in 3, for each of the jackets. Each line of $\mathcal{G}$ splits in 3 to become an edge of a jacket. Furthermore, each vertex $\mathbf{b}_c$ or $\mathbf{b}_+$ in $\mathcal{G}$ is associated with 4 vertices in $\mathcal{G}_{\rm color}$ which gives 6 additional colored lines. This combinatorics gives:

$$\sum_J V_{\widetilde{J}} = 12(V_+ + V_m), \qquad \sum_J E_{\widetilde{J}} = 3L + 18(V_+ + V_m). \qquad (33)$$

The number of faces of the pinched jacket $\widetilde{J}$ decomposes in 3 terms:

$$F_{\widetilde{J}} = F_{{\rm int}\,;\,\widetilde{J};\,\mathcal{G}} + F_{{\rm int}\,;\,\widetilde{J};\,\mathcal{G}_{\rm color}} + F_{{\rm ext}\,;\widetilde{J}} \qquad (34)$$

where $F_{{\rm int}\,;\widetilde{J};\mathcal{G}}$ is the number faces of $\widetilde{J}$ which belong to $\mathcal{G}$ as well, $F_{{\rm int}\,;\widetilde{J};\mathcal{G}_{\rm color}}$ is the number of faces of $\widetilde{J}$ which belong to $\mathcal{G}_{\rm color}$ but do not belong to $\mathcal{G}$ and $F_{{\rm ext}\,;\widetilde{J}}$ are the faces of $\widetilde{J}$ which were external and are closed after pinching.

An internal face of a jacket contributing to $F_{{\rm int}\,;\,\widetilde{J};\,\mathcal{G}} + F_{{\rm int}\,;\,\widetilde{J};\,\mathcal{G}_{\rm color}}$ is shared exactly by another jacket; a jacket face contributing to $F_{{\rm ext}\,;\widetilde{J}}$ must be tracked at the level of the boundary graph $\partial\mathcal{G}$. We have, by summing over jackets:

$$\sum_J F_{\widetilde{J}} = 2F_{\rm int} + 2F_{\rm color;int} + \sum_J F_{{\rm ext}\,;\widetilde{J}} \qquad (35)$$

The quantity $F_{\rm color;int}$ is the number of additional internal faces brought by the colored expansion at the level of each vertex of $\mathcal{G}$. Each vertex of the type

$\mathbf{b}_c$ brings 4 of such closed faces; meanwhile, a vertex of the cross type $\mathbf{b}_+$ brings 3 of those. This computes explicitly as

$$F_{\text{color;int}} = 3V_+ + 4V_m. \tag{36}$$

The last piece of (35) is now treated. Consider the boundary graph $\partial\mathcal{G}$ which is a 3-regular ribbon graph.

$$V_\partial = N_{\text{ext}}\,, \qquad E_\partial = F_{\text{ext}}\,, \qquad 3V_\partial = 2E_\partial. \tag{37}$$

The boundary graph might have several connected components; hence, writing its Euler characteristic, we have

$$2C_\partial - 2g_{\partial\mathcal{G}} = V_\partial - E_\partial + F_\partial. \tag{38}$$

Each face of $\partial\mathcal{G}$ can be uniquely mapped to a face of a unique pinched jacket which closes after pinching. Hence,

$$\begin{aligned}\sum_J F_{\text{ext}\,;\widetilde{J}} = F_\partial &= 2C_\partial - 2g_{\partial\mathcal{G}} - (V_\partial - E_\partial)\\ &= 2C_\partial - 2g_{\partial\mathcal{G}} - \Big(1 - \frac{3}{2}\Big)N_{\text{ext}}\\ &= 2C_\partial - 2g_{\partial\mathcal{G}} + \frac{1}{2}N_{\text{ext}}\,. \end{aligned} \tag{39}$$

We are then in position to find an expression of the number of internal faces of $\mathcal{G}$. Combining the relations (33), (35), (36) and (39), we get:

$$\begin{aligned} F_{\text{int}} &= \frac{1}{2}\Bigg[\sum_J F_{\widetilde{J}} - 2F_{\text{color;int}} - \sum_J F_{\text{ext}\,;\widetilde{J}}\Bigg]\\ &= \frac{1}{2}\Bigg[\sum_J [2 - 2g_{\widetilde{J}} - (V_{\widetilde{J}} - E_{\widetilde{J}})] - 2F_{\text{color;int}} - \sum_J F_{\text{ext}\,;\widetilde{J}}\Bigg]\\ &= \frac{1}{2}\Bigg[2\cdot 3 - 2\omega(\mathcal{G}_{\text{color}}) + 3L + 18(V_+ + V_m) - 12(V_+ + V_m)\\ &\qquad - 2\Big[3V_+ + 4V_m\Big] - \Big[2C_\partial - 2g_{\partial\mathcal{G}} + \frac{1}{2}N_{\text{ext}}\Big]\Bigg]\\ &= -(\omega(\mathcal{G}_{\text{color}}) - g_{\partial\mathcal{G}}) + 3V_+ + 2V_m - N_{\text{ext}} - (C_\partial - 1) + 2, \end{aligned} \tag{40}$$

where we used the sum of the Euler characteristics of connected pinched jacket $2 - 2g_{\widetilde{J}} = V_{\widetilde{J}} - E_{\widetilde{J}} - F_{\widetilde{J}}$, define $\omega(\mathcal{G}_{\text{color}}) = \sum_J g_{\widetilde{J}}$ as the degree of the graph, and use the relation $4(V_+ + V_m) = 2L + N_{\text{ext}}$. □

Proposition 2 (Divergence degree). *In the above notations,*

$$\omega_{\deg}(\mathcal{G}) = -\frac{1}{2}[\omega(\mathcal{G}_{\text{color}}) - V_+ - g_{\partial\mathcal{G}} + (C_\partial - 1)] - \frac{1}{2}(N_{\text{ext}} - 4). \tag{41}$$

Proof. We insert $F_{\rm int}(\mathcal{G})$ of Proposition 1 in (31) of Theorem 3.1 and do some algebra to obtain:

$$\begin{aligned}\omega_{\rm deg}(\mathcal{G}) &= -(V(\mathcal{G})-1)+\frac{1}{2}[-(\omega(\mathcal{G}_{\rm color})-g_{\partial\mathcal{G}})+3V_{+}+2V_{m}-N_{\rm ext}-(C_{\partial}-1)+2]\\ &= -\frac{1}{2}[\omega(\mathcal{G}_{\rm color})-V_{+}-g_{\partial\mathcal{G}}+(C_{\partial}-1)]-\frac{1}{2}(N_{\rm ext}-4).\end{aligned}$$

which is (41). □

Lemma 7 in [92] with $D=d=3$ states that for vacuum graphs

$$\omega(\mathcal{G}_{\rm color}) \geq 3\left[\sum_{\mathbf{b}_c}\omega(\mathbf{b}_c)+\sum_{\mathbf{b}_+}\omega(\mathbf{b}_+)\right]. \tag{42}$$

Using $\omega(\mathbf{b}_c)=0$ and $\omega(\mathbf{b}_+)=\frac{1}{2}$ we find

$$\omega(\mathcal{G}_{\rm color}) \geq \frac{3}{2}V_{+}. \tag{43}$$

The quantity

$$\mathrm{ind}_0(\mathcal{G}) = \omega(\mathcal{G}_{\rm color})-\frac{3}{2}V_{+} \tag{44}$$

is called the index of the colored tensor graph $\mathcal{G}$ [80]. For vacuum graphs it coincides with the degree used in [34]. Deleting lines in a vacuum graph can only decrease the genus hence even for graphs with external legs we have

$$\omega(\mathcal{G}_{\rm color}) \geq \frac{3}{2}V_{+} \quad \Rightarrow \quad \mathrm{ind}_0(\mathcal{G})\geq 0. \tag{45}$$

In terms of this index

$$\omega_{\rm deg}(\mathcal{G}) = -\frac{1}{2}[\mathrm{ind}_0(\mathcal{G})+\frac{1}{2}V_{+}-g_{\partial\mathcal{G}}+(C_{\partial}-1)]-\frac{1}{2}(N_{\rm ext}-4). \tag{46}$$

From this point the renormalizability of the model could be addressed.

5. Renormalizability

We now prove that the divergence degree is strictly negative for operators with 6 or more external legs (also called convergent or irrelevant).

Lemma 5.1 (Bound on convergent graphs with $N_{\rm ext}\geq 6$). *For $\mathcal{G}$ any graph with $N_{\rm ext}\geq 6$,*

$$\omega_{\rm deg}(\mathcal{G}) \;\leq\; -\frac{1}{12}N_{\rm ext}. \tag{47}$$

Proof. Remark first that we need to prove the theorem only in the case $(C_{\partial}-1)=0$ since considering disconnected boundaries makes $\omega_{\rm deg}$ smaller. In this case

$$\omega_{\rm deg}(\mathcal{G}) = -\frac{1}{2}\left[\mathrm{ind}_0(\mathcal{G})+\frac{1}{2}V_{+}-g_{\partial\mathcal{G}}\right]-\frac{1}{2}(N_{\rm ext}-4). \tag{48}$$

Since $\mathrm{ind}_0(\mathcal{G})\geq 0$ (positivity of the index) and $V_{+}\geq 0$, we have

$$\omega_{\rm deg}(\mathcal{G}) \leq \frac{1}{2}[g_{\partial\mathcal{G}}-(N_{\rm ext}-4)]. \tag{49}$$

It is easy to check that

$$g_{\partial\mathcal{G}} \leq \frac{N_{\text{ext}}}{4} - \frac{1}{2} \tag{50}$$

since a three-colored graph, like $\partial\mathcal{G}$ is, has at least 3 faces. The above relation is derived using (37). Therefore,

$$\omega_{\text{deg}}(\mathcal{G}) \leq \frac{1}{2}\left[\frac{N_{\text{ext}}}{4} - \frac{1}{2} - (N_{\text{ext}} - 4)\right] = -\frac{3N_{\text{ext}}}{8} + \frac{7}{4}, \tag{51}$$

and the latter expression can be bounded by $-N_{\text{ext}}/12$ whenever $N_{\text{ext}} \geq 6$. □

It remains to treat the case of graphs with $N_{\text{ext}} \leq 4$.

Four-Point Subgraphs Let us set $N_{\text{ext}} = 4$, then by (41) the divergence degree for these graphs is

$$\omega_{\text{deg}}(\mathcal{G}) = -\frac{1}{2}\left[\text{ind}_0(\mathcal{G}) + \frac{1}{2}V_+ - g_{\partial\mathcal{G}} + (C_\partial - 1)\right] \tag{52}$$

and we want to check that $\omega_{\text{deg}}(\mathcal{G}) \leq 0$ so that we have at most logarithmic divergence for four-point functions. Having four external legs, a graph can have three types of possible boundaries:

- A disconnected boundary, hence $\partial\mathcal{G}$, is made of two quadratic melons. In that case $C_\partial = 2$ and $g_{\partial\mathcal{G}} = 0$ so that

$$\omega_{\text{deg}}(\mathcal{G}) = -\frac{1}{2}\left[\text{ind}_0(\mathcal{G}) + \frac{1}{2}V_+ + 1\right] \leq -\frac{1}{2}[\text{ind}_0(\mathcal{G}) + 1] \leq -\frac{1}{2}. \tag{53}$$

  This case does not require renormalization.
- A connected boundary with $\partial\mathcal{G}$ of the quartic melonic type $\mathbf{b}_c$, for some color c. In that case $C_\partial = 1$, $g_{\partial\mathcal{G}} = 0$ so that

$$\omega_{\text{deg}}(\mathcal{G}) = -\frac{1}{2}\left[\text{ind}_0(\mathcal{G}) + \frac{1}{2}V_+\right] \leq 0 \tag{54}$$

  can be zero if $\text{ind}_0(\mathcal{G}) = 0 = V_+$. In particular, there is such a nontrivial graph at one loop, with $V_m = 2$. This case certainly requires renormalization treatment.
- A connected boundary with $\partial\mathcal{G}$ of the $\mathbf{b}_+$ type. In that case $C_\partial = 1$ and $g_{\partial\mathcal{G}} = \frac{1}{2}$, so that

$$\omega_{\text{deg}}(\mathcal{G}) = -\frac{1}{2}\left[\text{ind}_0(\mathcal{G}) + \frac{1}{2}(V_+ - 1)\right]. \tag{55}$$

The following subcases could be discussed:

– $V_+ > 1$, then directly $\omega_{\text{deg}}(\mathcal{G}) < 0$; hence, all this class define graphs with convergent amplitude.

– $V_+ = 0$, this case is impossible to occur since the boundary is non-orientable $g_{\partial\mathcal{G}} = \frac{1}{2}$, there must be some non-orientable vertices.

– $V_+ = 1$. This is the final and most delicate point. We obtain $\omega_{\rm deg}(\mathcal{G}) \leq 0$, as expected. Apparently the bound could saturate, namely $\omega_{\rm deg}(\mathcal{G}) = 0$, when $\mathrm{ind}_0(\mathcal{G}) = \omega(\mathcal{G}_{\rm color}) - \frac{3}{2} = 0$. But a more careful analysis shows that this is impossible. More precisely we shall prove

Lemma 5.2. *If $N_{\rm ext} = 4$, $V_+ = 1$ and $g_{\partial\mathcal{G}} = \frac{1}{2}$, then $F_{\rm int} \leq 2V_m - 1$, hence by* (40) *and* (41) $\omega_{\rm deg}(\mathcal{G}) \leq -\frac{1}{2}$.

Proof. Let us call $\mathcal{G}'$ the graph made from $\mathcal{G}$ by cutting out V_+. It has V_m vertices, all of melonic type. The case $V_m = 1$ is easy, as $F_{\rm int} = 1$ in that case. Then we can complete the proof that $F_{\rm int} \leq 2V_m - 1$ by induction. If the vertex V_+ is attached to 2 external lines, $\mathcal{G}'$ is made of melonic vertices and has 4 external legs; hence, its number of faces is maximal if $\mathcal{G}'$ is *fully melonic*, in which case it has $2(V_m - 1)$ internal faces (the melonic rate). Joining $\mathcal{G}'$ to V_+ creates at most one new internal face and we are done.

If the vertex V_+ is attached to 2 external lines, since $\mathcal{G}'$ has 6 external legs, it can have *at most* $2(V_m - 2)$ *internal faces (again the maximal melonic rate)*. Joining $\mathcal{G}'$ to V_+ creates at most three new internal faces and we are done again.

Finally, when the vertex V_+ is attached to no external lines, $\mathcal{G}'$ has 8 external legs, hence at most $2(V_m - 3)$ internal faces (again the melonic rate). Joining $\mathcal{G}'$ to V_+ can create at most six new internal faces; hence, we are not done yet. To gain the crucial last improvement of one face, we shall prove that in this case the boundary graph cannot be of the V_+ type. Indeed if $\mathcal{G}'$ has exactly $2(V_m - 3)$ internal faces, its boundary must be a melonic colored graph with eight vertices. But if this graph, when joined to V_+, creates 6 additional faces, it must be that its boundary was *disconnected* into at least two pieces with 4 colored vertices each (since all circuits of the four external legs of $\mathcal{G}'$ joined to V_+ have to be internal). Under that condition of disconnected boundary the maximal "melonic" number of internal faces is no longer $2(V_m - 3)$ but $2(V_m - 2)$ and we are done. □

Two-Point Subgraphs There is no longer any choice for the boundary, as $g_{\partial\mathcal{G}} = 0$ and $C_\partial = 1$ (there is only a single invariant with two vertices). The degree of divergence takes the form:

$$\omega_{\rm deg}(\mathcal{G}) = -\frac{1}{2}\left[\mathrm{ind}_0(\mathcal{G}) + \frac{1}{2}V_+\right] + 1 = -\frac{1}{2}[\omega(\mathcal{G}_{\rm color}) - V_+] + 1 \tag{56}$$

and is at most 1. As usual this means that we should perform mass and wave function subtractions. We have therefore $\omega_{\rm deg}(\mathcal{G}) \geq 0$ equivalent to $\omega(\mathcal{G}_{\rm color}) - V_+ \in \{0, 1, 2\}$.

To summarize we have proved

Theorem 5.1. • *If $N_{\rm ext} \geq 6$*

$$\omega_{\rm deg}(\mathcal{G}) \leq -\frac{N_{\rm ext}}{12}; \tag{57}$$

hence, these functions are convergent.

- *If* $N_{\text{ext}} = 4$

$$\omega_{\deg}(\mathcal{G}) \leq 0; \tag{58}$$

hence, four-point functions are at most log-divergent and renormalized by a single subtraction.

- *If* $N_{\text{ext}} = 2$

$$\omega_{\deg}(\mathcal{G}) \leq 1; \tag{59}$$

hence, two-point functions are at most quadratically divergent.

6. Renormalization

This section undertakes the renormalization of the divergent graphs of the model. We focus on the expansion of the amplitudes around their divergent and "local" part. The goal is to subtract the local part of quasi-local graphs and this improves power counting of the amplitudes.

There are two types of graphs which have divergences: four- and two-point diagrams. They will be treated separately.

We consider amplitudes with external legs. There are therefore two types of lines in a diagram, internal lines that we denote l and external lines denoted l_{ext}. An internal line l is associated with a high scale i_l of an internal momentum and a parameter $\alpha_l \in [M^{-2i_l}, M^{-2(i_l-1)}]$. An external line l_{ext} is associated with a lower scale $j_{l_{\text{ext}}} < i_l$ of an external momentum, and a parameter $\alpha_{l_{\text{ext}}} \in [M^{-2j_{l_{\text{ext}}}}, M^{-2(j_{l_{\text{ext}}}-1)}]$.

We have two types of momenta: *time* momenta p_0 and *space* momenta p. Their treatment in the following expansion is different and urge us to introduce more notations. For space momenta, p_f^{ext} is associated with an external face f and and p_f denotes an internal momenta associated with a closed face. External time momenta associated with external lines are denoted $p_{0;l_{\text{ext}}}$ and those associated internal lines are denoted by $p_{0;l}$. Note that since there is conservation of time momenta at the vertices, the $p_{0;l}$'s might be very well (linearly) depending on $p_{0;l_{\text{ext}}}$. After imposing the vertex constraints, it remains one internal momenta per independent cycle c in the graph.

Four-Point Amplitudes Consider a four-point function which is log-divergent. It is of the boundary type: $g_{\partial\mathcal{G}} = 0$. Pick a diagram amplitude coming from the expansion of the correlator:

$$\langle \chi_{1;\, p_{0;1}123}\, \chi_{2;\, p_{0;2}1'23}\, \chi_{1;\, p_{0;3}1'2'3'}\, \chi_{2;\, p_{0;4}12'3'} \rangle \tag{60}$$

where the notation $\chi_{\sigma;\, p_{0;i}123}$ stands for $\chi(p_{0;i}, p_1, p_2, p_3, \sigma)$. Note that this correlator has a boundary graph which is of the form of the melonic interaction with particular color 1 (this is the bubble $\mathbf{b}_1$). We will perform the expansion of an amplitude with this boundary data, to perform a similar analysis for other melonic boundary with color $c = 2, 3$ will be straightforward.

We start by noting that a diagram issued from (60) has four external propagator lines with momenta $p_{0;a}$, $a = 1, 2, 3, 4$, that we associate with external lines l_{ext} (depending of course on a) such that $p_{0;a} = p_{0;l_{\text{ext}}}$. An illustration of a four-point graph with external momenta is given in Fig. 3

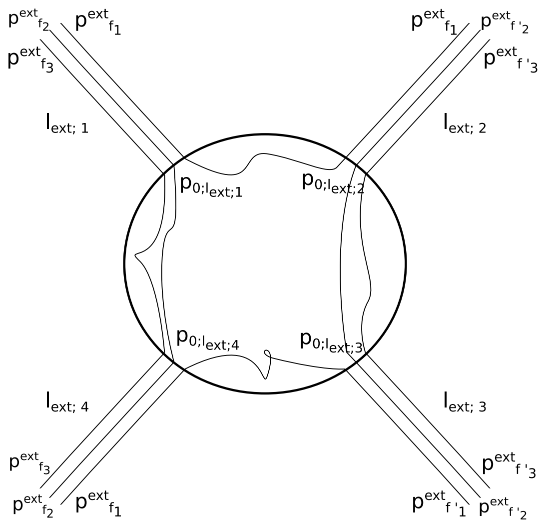

FIGURE 3. Four-point graph with external lines $l_{\text{ext};a}$, external momenta $p_{0,l_{\text{ext};a}}$ and external momenta p_f^{ext} associated with external faces

A graph amplitude of the model is of the form

$$\begin{aligned}
&A_{\mathcal{G};4}(\{p_{0;l_{\text{ext}}}\};\{p_f^{\text{ext}}\})\\
&\quad= \kappa(\lambda)\int\Bigg[\prod_{\ell\in\mathcal{L}} \mathrm{d}\alpha_\ell\Bigg]\int\Bigg[\prod_{v\in\mathcal{V}}\mathrm{d}p_{0;v}\Bigg]\Bigg[\prod_{f\in\mathcal{F}_{\text{int}}}\mathrm{d}p_f\Bigg]\Bigg[\Bigg[\prod_{\ell\in\mathcal{L}_1}(-ip_{0;\ell}]\Bigg[\prod_{\ell\in\mathcal{L}_2} e\Bigg(\sum_{f\in\mathcal{F}}\epsilon_{\ell f}p_f\Bigg)\Bigg]\Bigg]\\
&\qquad\times e^{-\sum_{\ell\in\mathcal{L}}\alpha_\ell\, e^2(\sum_{f\in\mathcal{F}}\epsilon_{\ell f}p_f)}\; e^{-\sum_{\ell\in\mathcal{L}}\alpha_\ell\, p_{0;\ell}^2}\Bigg[\prod_{v\in\mathcal{V}}\delta\Bigg(\sum_{l=1}^{4}p_{0;l;v}\Bigg)\Bigg]\\
&\quad= \kappa(\lambda)\int\Bigg[\prod_{\ell\in\mathcal{L}} \mathrm{d}\alpha_\ell e^{\alpha_\ell}\Bigg]\int\Bigg[\prod_{v\in\mathcal{V}}\mathrm{d}p_{0;v}\Bigg]\Bigg[\prod_{f\in\mathcal{F}_{\text{int}}}\mathrm{d}p_f\Bigg]\Bigg[\Bigg[\prod_{\ell\in\mathcal{L}_1}(-ip_{0;\ell}]\Bigg[\prod_{\ell\in\mathcal{L}_2} e\Bigg(\sum_{f\in\mathcal{F}}\epsilon_{\ell f}p_f\Bigg)\Bigg]\Bigg]\\
&\qquad\times\Bigg[\prod_{f\in\mathcal{F}_{\text{ext}}} e^{-(\sum_{\ell\in f}\alpha_\ell)[(p_f^{\text{ext}})^4-2(p_f^{\text{ext}})^2]}\Bigg]\Bigg[\prod_{\substack{f,f'\in\mathcal{F}_{\text{ext}}\\ f\neq f'}} e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f^{\text{ext}})^2(p_{f'}^{\text{ext}})^2}\Bigg]\\
&\qquad\times\Bigg[\prod_{f\in\mathcal{F}_{\text{int}}} e^{-(\sum_{\ell\in f}\alpha_\ell)[p_f^4-2p_f^2]}\Bigg]\Bigg[\prod_{\substack{f,f'\in\mathcal{F}_{\text{int}}\\ f\neq f'}} e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f)^2(p_{f'})^2}\Bigg]\\
&\qquad\times\Bigg[\prod_{f\in\mathcal{F}_{\text{ext}},f'\in\mathcal{F}_{\text{int}}} e^{-2(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f^{\text{ext}})^2(p_{f'})^2}\Bigg]\times e^{-\sum_{\ell\in\mathcal{L}}\alpha_\ell\, p_{0;\ell}^2}\Bigg[\prod_{v\in\mathcal{V}}\delta\Bigg(\sum_{l=1}^{4}p_{0;l;v}\Bigg)\Bigg].
\end{aligned}\tag{61}$$

Consider the decomposition of the set $\mathcal{L}$ of lines in internal lines $\mathcal{L}_{\text{int}}$ and external lines $\mathcal{L}_{\text{ext}}$. The treatment of the momenta $p_{0;\ell}$ resorts from a usual technique: first, lines must be oriented in an arbitrary way (but, at the end, the procedure is independent of the orientation); at each vertex v, if a line l is oriented toward v, the sign of the momentum $p_{0;l}$ associated with l in the δ-function is chosen positive, and negative otherwise; second we must fix a tree

$\mathcal{T}$ of internal lines and do a momentum routine along the lines of that tree. Using the δ-functions of the vertices, and expanding the squares produces a sign before the Schwinger parameter α, that we denote $\underline{\alpha} = \pm\alpha$. The following development and the conclusion of our analysis do not actually depend on the signs and we will keep a general notation $\underline{\alpha}$ without a concern about these signs. Note that for a given cycle $c \in \mathrm{Cycle}_{\mathcal{G}}$ of the graph that corresponds to a given high momentum $p_{0;c}$, there is a subset $\mathcal{T}_c \subset \mathcal{T}$ of lines. There is a line $l_c \in \mathcal{L}_{\mathrm{int}}$ such that the set of lines $\{l_c\} \cup \mathcal{T}_c = \mathcal{L}_c$ forms the cycle c. With each external momenta $p_{0;l_{\mathrm{ext}}}$, there is a path $\mathcal{T}_{l_{\mathrm{ext}}} \subset \mathcal{T}$ of internal lines l such that after the integration of the δ-functions, $p_{0;l}$ becomes a function of $p_{0;l_{\mathrm{ext}}}$. We then introduce another matrix, $|\mathcal{L}_{\mathrm{int}}| \times (|\mathrm{Cycle}_{\mathcal{G}}| + |\mathcal{L}_{\mathrm{ext}}|)$, which decomposes in to diagonal blocks:

$$\varepsilon_{lc} = \begin{cases} 1 \text{ if } \ l \in \mathcal{L}_c \\ 0 \text{ otherwise} \end{cases} \qquad \varepsilon_{l l_{\mathrm{ext}}} = \begin{cases} 1 \text{ if } \ l \in \mathcal{T}_{l_{\mathrm{ext}}} \\ 0 \text{ otherwise} \end{cases} \tag{62}$$

Then, we have the following expansion:

$$\begin{aligned}
&\left[\prod_{\ell\in\mathcal{L}} e^{-\alpha_\ell p_{0;\ell}^2}\right]\left[\prod_{v\in\mathcal{V}} \delta\left(\sum_{l=1}^{4} p_{0;l;v}\right)\right] = \left[\prod_{v\in\mathcal{V}} \delta\left(\sum_{l=1}^{4} p_{0;l;v}\right)\right] \\
&\quad\times\left[\prod_{c\in\mathrm{Cycle}_{\mathcal{G}}} e^{-(\sum_{l\in\mathcal{L}_c}\alpha_l)p_{0;c}^2}\right]\left[\prod_{\substack{c,c'\in\mathrm{Cycle}_{\mathcal{G}}\\ c\neq c'}} e^{-(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{c'}}\underline{\alpha}_l)p_{0;c}p_{0;c'}}\right] \\
&\quad\times\left[\prod_{l_{\mathrm{ext}}\in\mathcal{L}_{\mathrm{ext}}} e^{-(\alpha_{l_{\mathrm{ext}}}+\sum_{l\in\mathcal{T}_{l_{\mathrm{ext}}}}\alpha_l)p_{0;l_{\mathrm{ext}}}^2}\right]\left[\prod_{\substack{l_{\mathrm{ext}}\in\mathcal{L}_{\mathrm{ext}}\\ c\in\mathrm{Cycle}_{\mathcal{G}}}} e^{-2(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\mathrm{ext}}}}\underline{\alpha}_l)p_{0;c}p_{0;l_{\mathrm{ext}}}}\right] \\
&\quad\times\left[\prod_{\substack{l_{\mathrm{ext}},l'_{\mathrm{ext}}\in\mathcal{L}_{\mathrm{ext}}\\ l_{\mathrm{ext}}\neq l'_{\mathrm{ext}}}} e^{-(\sum_{l\in\mathcal{T}_{l_{\mathrm{ext}}}\cap\mathcal{T}_{l'_{\mathrm{ext}}}}\underline{\alpha}_l)p_{0;l_{\mathrm{ext}}}p_{0;l'_{\mathrm{ext}}}}\right],
\end{aligned} \tag{63}$$

where we used the δ-functions to perform the relevant substitutions.

We perform the following expansion for each factor associated with external momenta:

$$\begin{aligned}
&e^{-(\alpha_{l_{\mathrm{ext}}}+\sum_{l\in\mathcal{T}_{l_{\mathrm{ext}}}}\alpha_l)p_{0;l_{\mathrm{ext}}}^2} = e^{-\alpha_{l_{\mathrm{ext}}}p_{0;l_{\mathrm{ext}}}^2}\left[1 - Q^1_{l_{\mathrm{ext}}}\right] \\
&Q^1_{l_{\mathrm{ext}}} = \left(\sum_{l\in\mathcal{T}_{l_{\mathrm{ext}}}}\alpha_l\right)p_{0;l_{\mathrm{ext}}}^2\int_0^1 \mathrm{d}s\, e^{-s(\sum_{l\in\mathcal{T}_{l_{\mathrm{ext}}}}\alpha_l)p_{0;l_{\mathrm{ext}}}^2}, \\
&e^{-2(\sum_{c\in\mathrm{Cycle}_{\mathcal{G}}}(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\mathrm{ext}}}}\underline{\alpha}_l)p_{0;c})p_{0;l_{\mathrm{ext}}}} = 1 - Q^2_{l_{\mathrm{ext}}} \\
&Q^2_{l_{\mathrm{ext}}} = 2\left[\sum_{c\in\mathrm{Cycle}_{\mathcal{G}}}\left(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\mathrm{ext}}}}\underline{\alpha}_l\right)p_{0;c}\right]p_{0;l_{\mathrm{ext}}} \\
&\qquad\times\int_0^1 \mathrm{d}s\, e^{-2s(\sum_{c\in\mathrm{Cycle}_{\mathcal{G}}}(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\mathrm{ext}}}}\underline{\alpha}_l)p_{0;c})p_{0;l_{\mathrm{ext}}}}, \\
&e^{-(\sum_{l\in\mathcal{T}_{l_{\mathrm{ext}}}\cap\mathcal{T}_{l'_{\mathrm{ext}}}}\underline{\alpha}_l)p_{0;l_{\mathrm{ext}}}p_{0;l'_{\mathrm{ext}}}} = 1 - Q^3_{l_{\mathrm{ext}},l'_{\mathrm{ext}}}
\end{aligned}$$

$$Q^3_{l_{\rm ext},l'_{\rm ext}} = \left(\sum_{l \in \mathcal{T}_{l_{\rm ext}} \cap \mathcal{T}_{l'_{\rm ext}}} \underline{\alpha}_l \right) p_{0;l_{\rm ext}} p_{0;l'_{\rm ext}} \times \int_0^1 \mathrm{d}s\, e^{-s(\sum_{l \in \mathcal{T}_{l_{\rm ext}} \cap \mathcal{T}_{l'_{\rm ext}}} \underline{\alpha}_l) p_{0;l_{\rm ext}} p_{0;l'_{\rm ext}}}. \tag{64}$$

Focusing on the momentum associated with space coordinates, for the momenta $p_f^{\rm ext}$ associated with an external face f, we use the following decomposition and expansion:

$$e^{-(\sum_{\ell \in f} \alpha_\ell)[(p_f^{\rm ext})^4 - 2(p_f^{\rm ext})^2]} = e^{-(\alpha_{l_{\rm ext}} + \alpha_{l'_{\rm ext}})[(p_f^{\rm ext})^4 - 2(p_f^{\rm ext})^2]}(1 - Q^1_{{\rm ext}\,;f})$$
$$Q^1_{{\rm ext}\,;f} = \left(\sum_{l \in f} \alpha_l \right)[(p_f^{\rm ext})^4 - 2(p_f^{\rm ext})^2] \int_0^1 \mathrm{d}s\, e^{-s(\sum_{l \in f} \alpha_l)(p_f^{\rm ext})^2},$$
$$e^{-(\sum_{\ell \in f, \ell \in f'} \alpha_\ell)(p_f^{\rm ext})^2 (p_{f'}^{\rm ext})^2} = e^{-(\alpha_{l_{\rm ext}} + \alpha_{l'_{\rm ext}})(p_f^{\rm ext})^2 (p_{f'}^{\rm ext})^2}(1 - Q^2_{{\rm ext}\,;f})$$
$$Q^2_{{\rm ext}\,;f,f'} = \left(\sum_{l \in f, l \in f'} \alpha_l \right)(p_f^{\rm ext})^2 (p_{f'}^{\rm ext})^2 \int_0^1 \mathrm{d}s\, e^{-s(\sum_{l \in f, l \in f'} \alpha_l)(p_f^{\rm ext})^2 (p_{f'}^{\rm ext})^2},$$
$$e^{-(\sum_{\ell \in f, \ell \in f'} \alpha_\ell)(p_f^{\rm ext})^2 (p_{f'})^2} = 1 - Q^3_{{\rm ext}\,;f,f'}$$
$$Q^3_{{\rm ext}\,;f,f'} = \left(\sum_{l \in f, l \in f'} \alpha_l \right)(p_f^{\rm ext})^2 (p_{f'})^2 \int_0^1 \mathrm{d}s\, e^{-s(\sum_{l \in f, l \in f'} \alpha_l)(p_f^{\rm ext})^2 (p_{f'})^2}, \tag{65}$$

where, in the last expansion, we use the fact that there is no external lines which could belong to $f' \in \mathcal{F}_{\rm int}$.

It remains the following factor to study, for $l \in \mathcal{L}_{\rm int} \cap \mathcal{L}_1$,

$$(-ip_{0;l}) \left[\prod_{v \in \mathcal{V}} \delta \left(\sum_{l=1}^4 p_{0;l;v} \right) \right]$$
$$= \left[-i \sum_{c \in {\rm Cycle}_\mathcal{G}} \varepsilon_{lc} p_{0;c} - i \sum_{l_{\rm ext}} \varepsilon_{l l_{\rm ext}} p_{0;l_{\rm ext}} \right] \left[\prod_{v \in \mathcal{V}} \delta \left(\sum_{l=1}^4 p_{0;l;v} \right) \right]$$
$$= [-i \sum_{c \in {\rm Cycle}_\mathcal{G}} \varepsilon_{lc} p_{0;c}] \left[1 + Q_l^{4;1} \right] \left[\prod_{v \in \mathcal{V}} \delta \left(\sum_{l=1}^4 p_{0;l;v} \right) \right],$$
$$Q_l^{4;1} = \frac{\sum_{l_{\rm ext}} \varepsilon_{l l_{\rm ext}} p_{0;l_{\rm ext}}}{\sum_{c \in {\rm Cycle}_\mathcal{G}} \varepsilon_{lc} p_{0;c}}. \tag{66}$$

Then for elements $l \in \mathcal{L}_{\rm int} \cap \mathcal{L}_2$, we write

$$e \left(\sum_{f \in \mathcal{F}} \epsilon_{lf} p_f \right) \left[\prod_{v \in \mathcal{V}} \delta \left(\sum_{l=1}^4 p_{0;l;v} \right) \right]$$
$$= \left[e \left(\sum_{f \in \mathcal{F}_{\rm int}} \epsilon_{lf} p_f + \sum_{f \in \mathcal{F}_{\rm ext}} \epsilon_{lf} p_f^{\rm ext} \right) \right] \left[\prod_{v \in \mathcal{V}} \delta \left(\sum_{l=1}^4 p_{0;l;v} \right) \right]$$

$$= e\left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{lf} p_f\right)\left[1+Q_l^{4;2}\right]\left[\prod_{v\in\mathcal{V}}\delta\left(\sum_{l=1}^{4} p_{0;l;v}\right)\right],$$
$$Q_l^{4;2} = \frac{\sum_{f\in\mathcal{F}_{\rm ext}} \epsilon_{lf}(p_f^{\rm ext})^2}{e\left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{lf} p_f\right)}. \tag{67}$$

We are in position to provide the local expansion of (61). Plugging (64), (65), (66) and (67), in the four-point amplitude (61) we find:

$$A_{\mathcal{G};4}(\{p_{0;l_{\rm ext}}\};\{p_f^{\rm ext}\}) = \kappa(\lambda)\delta\left(\sum_{l_{\rm ext}} p_{0;l_{\rm ext}}\right)\int\left[\prod_{\ell\in\mathcal{L}} {\rm d}\alpha_\ell e^{\alpha_\ell}\right]$$
$$\times\int\left[\prod_{c\in{\rm Cycle}_{\mathcal{G}}} {\rm d}p_{0;c}\right]\left[\prod_{f\in\mathcal{F}_{\rm int}} {\rm d}p_f\right]$$
$$\times\left[\prod_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_1}\left[i\sum_{c\in{\rm Cycle}_{\mathcal{G}}} \varepsilon_{lc} p_{0;c}\right][1+Q_l^{4;1}]\right]$$
$$\times\left[\prod_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{lf} p_f\right)[1+Q_l^{4;2}]\right]$$
$$\times\left[\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}\cap\mathcal{L}_1}(-ip_{0;l_{\rm ext}})\right]\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\rm ext}} \epsilon_{l_{\rm ext}f} p_f^{\rm ext}\right)\right]\right]$$
$$\times\left[\prod_{f\in\mathcal{F}_{\rm ext}} e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})[(p_f^{\rm ext})^4-2(p_f^{\rm ext})^2]}(1-Q^1_{{\rm ext};f})\right]$$
$$\times\left[\prod_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2}(1-Q^2_{{\rm ext};f,f'})\right]$$
$$\times\left[\prod_{f\in\mathcal{F}_{\rm ext},f'\in\mathcal{F}_{\rm int}}[1-Q^3_{{\rm ext};f,f'}]\right]\left[\prod_{f\in\mathcal{F}_{\rm int}} e^{-(\sum_{\ell\in f}\alpha_\ell)[p_f^4-2p_f^2]}\right]$$
$$\times\left[\prod_{\substack{f,f'\in\mathcal{F}_{\rm int}\\ f\neq f'}} e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f)^2(p_{f'})^2}\right]$$
$$\times\left[\prod_{c\in{\rm Cycle}_{\mathcal{G}}} e^{-(\sum_{l\in\mathcal{L}_c}\alpha_l)p_{0;c}^2}\right]\left[\prod_{\substack{c,c'\in{\rm Cycle}_{\mathcal{G}}\\ c\neq c'}} e^{-(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{c'}}\underline{\alpha}_l)p_{0;c}p_{0;c'}}\right]$$
$$\times\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} e^{-\alpha_{l_{\rm ext}}p^2_{0;l_{\rm ext}}}[1-Q^1_{l_{\rm ext}}]\right]\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}}[1-Q^2_{l_{\rm ext}}]\right]$$

$$\times \left[\prod_{\substack{l_{\rm ext}, l'_{\rm ext} \in \mathcal{L}_{\rm ext} \\ l_{\rm ext} \neq l'_{\rm ext}}} [1 - Q^3_{l_{\rm ext}, l'_{\rm ext}}] \right] \tag{68}$$

and this recasts as

$$\begin{aligned} A_{\mathcal{G};4}(\{p_{0;l_{\rm ext}}\};\{p_f^{\rm ext}\}) &= \kappa(\lambda)\delta\left(\sum_{l_{\rm ext}} p_{0;l_{\rm ext}}\right) \int \left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} {\rm d}\alpha_{l_{\rm ext}}\, e^{\alpha_{l_{\rm ext}}}\right] \\ &\times \left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}\cap\mathcal{L}_1} (-ip_{0;l_{\rm ext}})\right]\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\rm ext}} \epsilon_{l_{\rm ext}\, f} p_f^{\rm ext}\right)\right] \\ &\times \left[\prod_{f\in\mathcal{F}_{\rm ext}} e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})[(p_f^{\rm ext})^4 - 2(p_f^{\rm ext})^2]}\right] \\ &\times \left[\prod_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2}\right]\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} e^{-\alpha_{l_{\rm ext}}\, p^2_{0;l_{\rm ext}}}\right] \\ &\times \int\left[\prod_{l\in\mathcal{L}_{\rm int}} {\rm d}\alpha_l e^{\alpha_l}\right]\int\left[\prod_{c\in{\rm Cycle}_{\mathcal{G}}} {\rm d}p_{0;c}\right]\left[\prod_{f\in\mathcal{F}_{\rm int}} {\rm d}p_f\right] \\ &\times \left[\prod_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_1} (-i\sum_{c\in{\rm Cycle}_{\mathcal{G}}} \varepsilon_{lc} p_{0;c})\right]\left[\prod_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{lf} p_f\right)\right] \\ &\times \left[\prod_{f\in\mathcal{F}_{\rm int}} e^{-(\sum_{\ell\in f}\alpha_\ell)[p_f^4 - 2p_f^2]}\right]\left[\prod_{\substack{f,f'\in\mathcal{F}_{\rm int}\\ f\neq f'}} e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f)^2(p_{f'})^2}\right] \\ &\times \left[\prod_{c\in{\rm Cycle}_{\mathcal{G}}} e^{-(\sum_{l\in\mathcal{L}_c}\alpha_l)p^2_{0;c}}\right]\left[\prod_{\substack{c,c'\in{\rm Cycle}_{\mathcal{G}}\\ c\neq c'}} e^{-(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{c'}} \underline{\alpha}_l)p_{0;c}p_{0;c'}}\right] \\ &\times \Bigg\{1 + \sum_{\sigma=1,2}\sum_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_\sigma} Q_l^{4;\sigma} - \sum_{f\in\mathcal{F}_{\rm ext}} Q^1_{{\rm ext};f} - \sum_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} Q^2_{{\rm ext};f,f'} \\ &- \sum_{f\in\mathcal{F}_{\rm ext}, f'\in\mathcal{F}_{\rm int}} Q^3_{{\rm ext};f,f'} - \sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^1_{l_{\rm ext}} - \sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^2_{l_{\rm ext}} \\ &- \sum_{\substack{l_{\rm ext},l'_{\rm ext}\in\mathcal{L}_{\rm ext}\\ l_{\rm ext}\neq l'_{\rm ext}}} Q^3_{l_{\rm ext},l'_{\rm ext}} + \sum Q\cdot Q + \ldots\Bigg\}. \end{aligned} \tag{69}$$

where the last expression $\sum Q \cdot Q + \ldots$ stands for higher-order products of the remainders Q.

The zeroth order in that expansion is of the form

$$A_{\mathcal{G};4}(\{p_{0;l_{\rm ext}}\};\{p_f^{\rm ext}\};0) = \kappa(\lambda)\delta\left(\sum_{l_{\rm ext}} p_{0;l_{\rm ext}}\right)\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}\cap\mathcal{L}_1}(-ip_{0;l_{\rm ext}})\right]$$
$$\times\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\rm ext}}\epsilon_{l_{\rm ext}\,f}\,p_f^{\rm ext}\right)\right] \tag{70}$$
$$\times\int\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} {\rm d}\alpha_{l_{\rm ext}}\,e^{\alpha_{l_{\rm ext}}}\right]\left[\prod_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} e^{-\alpha_{l_{\rm ext}}\,p_{0;l_{\rm ext}}^2}\right] \tag{71}$$
$$\times\left[\prod_{f\in\mathcal{F}_{\rm ext}} e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})[(p_f^{\rm ext})^4-2(p_f^{\rm ext})^2]}\right]$$
$$\times\left[\prod_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2}\right] \tag{72}$$
$$\times\int\left[\prod_{l\in\mathcal{L}_{\rm int}} {\rm d}\alpha_l e^{\alpha_l}\right]\int\left[\prod_{c\in{\rm Cycle}_{\mathcal{G}}} {\rm d}p_{0;c}\right]$$
$$\times\left[\prod_{f\in\mathcal{F}_{\rm int}} {\rm d}p_f\right]\left[\prod_{l\in\mathcal{L}_{\rm int}}\left[-i\sum_{c\in{\rm Cycle}_{\mathcal{G}}}\varepsilon_{lc}p_{0;c}+e\left(\sum_{f\in\mathcal{F}_{\rm int}}\epsilon_{lf}p_f\right)\right]\right]$$
$$\times\left[\prod_{f\in\mathcal{F}_{\rm int}} e^{-(\sum_{\ell\in f}\alpha_\ell)[p_f^4-2p_f^2]}\right]\left[\prod_{\substack{f,f'\in\mathcal{F}_{\rm int}\\ f\neq f'}} e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f)^2(p_{f'})^2}\right]$$
$$\times\left[\prod_{c\in{\rm Cycle}_{\mathcal{G}}} e^{-(\sum_{l\in\mathcal{L}_c}\alpha_l)p_{0;c}^2}\right]\left[\prod_{\substack{c,c'\in{\rm Cycle}_{\mathcal{G}}\\ c\neq c'}} e^{-(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{c'}}\underline{\alpha}_l)p_{0;c}p_{0;c'}}\right]. \tag{73}$$

By a small combinatorics and essentially variable renaming, the expressions (70), (71) and (72) can be combined to give 4 propagators glued together to form a vertex with pattern given by (60) and the three last lines are integrals over internal momenta and will give a log-divergent contribution. This term will therefore renormalize λ_m associated with the melonic vertex of the form $\mathbf{b}_1$.

We now address the Q remainder terms and recall that for an internal line l we have $p_{0;l}\sim M^{i_l}\sim\alpha_l^{-\frac{1}{2}}$, for an external line $l_{\rm ext}$, $p_{0;l_{\rm ext}}\sim M^{j_{l_{\rm ext}}}\sim\alpha_{l_{\rm ext}}^{-\frac{1}{2}}$. A momentum p_f associated with a closed or external face f is of the order $p_f\sim M^{-i_f/2}$, $i_f=\min_{\ell\in f} i_\ell$. Note that if f is external, then necessarily i_f is nothing but one of the index $j_{l_{\rm ext}}$ of one of the two external sliced propagators $l_{\rm ext}$.

Keeping in mind $i(\mathcal{G}^i_{(k)}) = \min_{l\in\mathcal{L}_{\rm int}(\mathcal{G}^i_{(k)})} i_l > e(\mathcal{G}^i_{(k)}) = \sup_{l_{\rm ext}\in\mathcal{L}_{\rm ext}(\mathcal{G}^i_{(k)})} j_{l_{\rm ext}}$, the following bounds are valid on a single $\mathcal{G}^i_{(k)}$ graph:

$$\left|\sum_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_1} Q^{4;1}_l\right| = \sum_{l\in\mathcal{L}_{\rm int}} \frac{\left|\sum_{l_{\rm ext}} \varepsilon_{l l_{\rm ext}} p_{0;l_{\rm ext}}\right|}{\left|\sum_{c\in {\rm Cycle}_{\mathcal{G}}} \varepsilon_{lc} p_{0;c}\right|}$$

$$\leq \frac{c_1 M^{e(\mathcal{G}^i_{(k)})}}{c_2 M^{i(\mathcal{G}^i_{(k)})}} \leq C_{4;1} M^{-(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2} Q^{4;2}_l\right| = \sum_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2} \frac{\left|\sum_{f\in\mathcal{F}_{\rm ext}} \epsilon_{lf}(p^{\rm ext}_f)^2\right|}{\left|e\left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{lf} p_f\right)\right|}$$

$$\leq \frac{c_1' M^{e(\mathcal{G}^i_{(k)})}}{c_2' M^{i(\mathcal{G}^i_{(k)})}} \leq C_{4;2} M^{-(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^1_{l_{\rm ext}}\right| = \left|\sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} \left(\sum_{l\in\mathcal{T}_{l_{\rm ext}}} \alpha_l\right) p^2_{0;l_{\rm ext}} \int_0^1 {\rm d}s\, e^{-(\sum_{l\in\mathcal{T}_{l_{\rm ext}}} \alpha_l) p^2_{0;l_{\rm ext}}}\right|$$

$$\leq C_1 M^{-2(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^2_{l_{\rm ext}}\right| = 2\left|\left[\sum_{c\in{\rm Cycle}_{\mathcal{G}}} \left(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}} \underline{\alpha}_l\right) p_{0;c}\right] p_{0;l_{\rm ext}}\right.$$

$$\left.\times \int_0^1 {\rm d}s\, e^{-2(\sum_{c\in{\rm Cycle}_{\mathcal{G}}}(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}} \underline{\alpha}_l) p_{0;c}) p_{0;l_{\rm ext}}}\right|$$

$$\leq C_2 M^{-(2i(\mathcal{G}^i_{(k)})-i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))} \leq C_2 M^{-(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{\substack{l_{\rm ext}, l'_{\rm ext}\in\mathcal{L}_{\rm ext}\\ l_{\rm ext}\neq l'_{\rm ext}}} Q^3_{l_{\rm ext}, l'_{\rm ext}}\right|$$

$$= \left|\sum_{\substack{l_{\rm ext}, l'_{\rm ext}\in\mathcal{L}_{\rm ext}\\ l_{\rm ext}\neq l'_{\rm ext}}} \left(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}} \underline{\alpha}_l\right) p_{0;l_{\rm ext}} p_{0;l'_{\rm ext}} \int_0^1 {\rm d}s\, e^{-(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}} \underline{\alpha}_l) p_{0;l_{\rm ext}} p_{0;l'_{\rm ext}}}\right|$$

$$\leq C_3 M^{-2(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{f\in\mathcal{F}_{\rm ext}} Q^1_{{\rm ext};f}\right| = \left|\sum_{f\in\mathcal{F}_{\rm ext}} \left(\sum_{l\in f} \alpha_l\right) [(p^{\rm ext}_f)^4 - 2(p^{\rm ext}_f)^2] \int_0^1 {\rm d}s\, e^{-s(\sum_{l\in f}\alpha_l)(p^{\rm ext}_f)^2}\right|$$

$$\leq C_1' M^{-2(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} Q^2_{{\rm ext};f,f'}\right|$$

$$= \left|\sum_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} \left(\sum_{l\in f, l\in f'} \alpha_l\right) (p^{\rm ext}_f)^2 (p^{\rm ext}_{f'})^2 \int_0^1 {\rm d}s\, e^{-s(\sum_{l\in f, l\in f'}\alpha_l)(p^{\rm ext}_f)^2 (p^{\rm ext}_{f'})^2}\right|$$

$$\leq C_2' M^{-2(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))},$$

$$\left|\sum_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} Q^3_{{\rm ext}\,;f,f'}\right| = \left|\left(\sum_{l\in f, l\in f'}\alpha_l\right)(p_f^{\rm ext})^2(p_{f'})^2\int_0^1 {\rm d}s\, e^{-s(\sum_{l\in f, l\in f'}\alpha_l)(p_f^{\rm ext})^2(p_{f'})^2}\right|$$
$$\leq C'_3 M^{-(2i(\mathcal{G}^i_{(k)})+i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))} \leq C'_3 M^{-(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))}, \tag{74}$$

where C_i, c_i, c'_i and C'_i are constants depending on the graph. Using these bounds, we have the following bound on the first-order corrections:

$$|A_{\mathcal{G};4}(\{p_{0;l_{\rm ext}}\};\{p_f^{\rm ext}\};1)| \leq C \prod_{(i,k)\in\mathbb{N}^2} M^{\omega_{\rm d}(\mathcal{G}^i_{(k)})} M^{-(i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)}))}, \tag{75}$$

where C is another constant. Hence, since $i(\mathcal{G}^i_{(k)}) - e(\mathcal{G}^i_{(k)}) > 0$, this bound shows that the remainder will bring enough decay to ensure the convergence during the sum over scale attributions. In the same vein, higher-order products of $Q^{(\cdot)}$'s will be even more convergent. Finally, after changing the pattern of external momenta in the four-point correlator in a way to produce other type of melonic interactions of the form $\mathbf{b}_c$ of any color $c = 0, 1, 2, 3$, we can perform an analysis entirely parallel to the above and show that the zeroth-order term will renormalize λ_m and remainders will be again convergent.

Two-Point Amplitudes There is a unique boundary graph for any two-point amplitude and it is such that $g_{\partial\mathcal{G}} = 0$. As discussed in Sect. 5, there are several types of two-point graphs which could diverge. Their general degree of divergence is of the form $\omega_{\rm deg}(\mathcal{G}) = 1 - p/2$, $p \in \{0, 1, 2\}$. We will focus on the maximal degree case, that is $p = 0, \omega_{\rm deg}(\mathcal{G}) = 1$, where the expansion needs to be pushed at second order. There other cases can be understood from this point.

We consider a perturbative amplitude issued from the expansion if the correlator

$$\langle \psi_{p_{0;1};123}\, \psi_{p_{0;1}123}\rangle, \tag{76}$$

where $\psi = \chi_\sigma$. We use the same notation as above for external line momenta, external face momenta. See Fig. 4.

The expression (61) remains true for any graph amplitude. We now expand the exponentials appearing therein:

$$e^{-(\alpha_{l_{\rm ext}}+\sum_{l\in\mathcal{T}_{l_{\rm ext}}}\alpha_l)p^2_{0;l_{\rm ext}}} = e^{-\alpha_{l_{\rm ext}}\,p^2_{0;l_{\rm ext}}}\left[1 - Q^1_{l_{\rm ext}} + Q^{1'}_{l_{\rm ext}}\right],$$

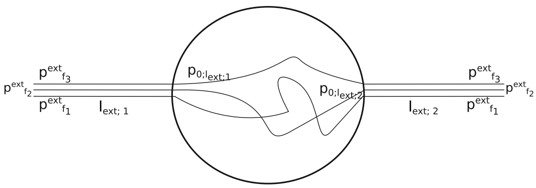

Figure 4. A two-point graph with external lines and the resulting external momenta

$$
\begin{aligned}
Q^1_{l_{\rm ext}} &= \left(\sum_{l\in\mathcal{T}_{l_{\rm ext}}}\alpha_l\right)p^2_{0;l_{\rm ext}}\,,\\
Q^{1'}_{l_{\rm ext}} &= \left[\left(\sum_{l\in\mathcal{T}_{l_{\rm ext}}}\alpha_l\right)p^2_{0;l_{\rm ext}}\right]^2\int_0^1 \mathrm{d}s\,(s-1)e^{-s(\sum_{l\in\mathcal{T}_{l_{\rm ext}}}\alpha_l)p^2_{0;l_{\rm ext}}}\,,\\
& e^{-2(\sum_{c\in{\rm Cycle}_{\mathcal{G}}}(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}}\underline{\alpha}_l)p_{0;c})p_{0;l_{\rm ext}}} = 1-Q^2_{l_{\rm ext}}+Q^{2'}_{l_{\rm ext}}\,,\\
Q^2_{l_{\rm ext}} &= 2\left[\sum_{c\in{\rm Cycle}_{\mathcal{G}}}\left(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}}\underline{\alpha}_l\right)p_{0;c}\right]p_{0;l_{\rm ext}}\,,\\
Q^{2'}_{l_{\rm ext}} &= \left[2\left[\sum_{c\in{\rm Cycle}_{\mathcal{G}}}\left(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}}\underline{\alpha}_l\right)p_{0;c}\right]p_{0;l_{\rm ext}}\right]^2\\
&\quad\times\int_0^1 \mathrm{d}s\,(1-s)e^{-2s(\sum_{c\in{\rm Cycle}_{\mathcal{G}}}(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}}\underline{\alpha}_l)p_{0;c})p_{0;l_{\rm ext}}}\,,\\
& e^{-(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}}\underline{\alpha}_l)p_{0;l_{\rm ext}}\,p_{0;l'_{\rm ext}}} = 1-Q^3_{l_{\rm ext},l'_{\rm ext}}+Q^{3'}_{l_{\rm ext},l'_{\rm ext}}\,,\\
Q^3_{l_{\rm ext},l'_{\rm ext}} &= \left(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}}\underline{\alpha}_l\right)p_{0;l_{\rm ext}}\,p_{0;l'_{\rm ext}}\,,\\
Q^{3'}_{l_{\rm ext},l'_{\rm ext}} &= \left[\left(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}}\underline{\alpha}_l\right)p_{0;l_{\rm ext}}\,p_{0;l'_{\rm ext}}\right]^2\\
&\quad\times\int_0^1 \mathrm{d}s\,(1-s)e^{-s(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}}\underline{\alpha}_l)p_{0;l_{\rm ext}}\,p_{0;l'_{\rm ext}}}\,. \qquad (77)
\end{aligned}
$$

Meanwhile, for momenta associated with faces, we have

$$
\begin{aligned}
& e^{-(\sum_{\ell\in f}\alpha_\ell)[(p_f^{\rm ext})^4-2(p_f^{\rm ext})^2]} = e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})[(p_f^{\rm ext})^4-2(p_f^{\rm ext})^2]}(1-Q^1_{{\rm ext}\,;f}+Q^{1'}_{{\rm ext}\,;f}),\\
& Q^1_{{\rm ext}\,;f} = \left(\sum_{l\in f}\alpha_l\right)[(p_f^{\rm ext})^4-2(p_f^{\rm ext})^2],\\
& Q^{1'}_{{\rm ext}\,;f} = \left[\left(\sum_{l\in f}\alpha_l\right)[(p_f^{\rm ext})^4-2(p_f^{\rm ext})^2]\right]^2\int_0^1 \mathrm{d}s\,(1-s)e^{-s(\sum_{l\in f}\alpha_l)(p_f^{\rm ext})^2},\\
& e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2} = e^{-(\alpha_{l_{\rm ext}}+\alpha_{l'_{\rm ext}})(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2}(1-Q^2_{{\rm ext}\,;f}+Q^{2'}_{{\rm ext}\,;f}),\\
& Q^2_{{\rm ext}\,;f,f'} = \left(\sum_{l\in f,l\in f'}\alpha_l\right)(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2, \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
& Q^{2'}_{{\rm ext}\,;f,f'} = \left[\left(\sum_{l\in f,l\in f'}\alpha_l\right)(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2\right]^2\int_0^1 \mathrm{d}s\,(1-s)e^{-s(\sum_{l\in f,l\in f'}\alpha_l)(p_f^{\rm ext})^2(p_{f'}^{\rm ext})^2},\\
& e^{-(\sum_{\ell\in f,\ell\in f'}\alpha_\ell)(p_f^{\rm ext})^2(p_{f'})^2} = 1-Q^3_{{\rm ext}\,;f,f'}+Q^{3'}_{{\rm ext}\,;f,f'},
\end{aligned}
$$

$$Q^3_{\text{ext};f,f'} = \left(\sum_{l\in f, l\in f'} \alpha_l\right)(p_f^{\text{ext}})^2 (p_{f'})^2,$$

$$Q^3_{\text{ext};f,f'} = \left[\left(\sum_{l\in f, l\in f'} \alpha_l\right)(p_f^{\text{ext}})^2 (p_{f'})^2\right]^2 \int_0^1 \mathrm{d}s\,(1-s) e^{-s(\sum_{l\in f,l\in f'}\alpha_l)(p_f^{\text{ext}})^2(p_{f'})^2}. \tag{79}$$

The last factor to expand becomes:

$$\left[\prod_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_1} (-ip_{0;l})\right]\left[\prod_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}} \epsilon_{lf} p_f\right)\right]\left[\prod_{v\in\mathcal{V}} \delta\left(\sum_{l=1}^4 p_{0;l;v}\right)\right]$$
$$= [Q^1 + Q^{2;1} + Q^{2;2} + Q^3]\left[\prod_{v\in\mathcal{V}} \delta\left(\sum_{l=1}^4 p_{0;l;v}\right)\right],$$
$$Q^1 = \left[\prod_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_1} (-i \sum_{c\in\text{Cycle}_{\mathcal{G}}} \varepsilon_{lc} p_{0;c})\right]$$
$$\times \left[\prod_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\text{int}}} \epsilon_{lf} p_f\right)\right],$$
$$Q^{2;1} = \sum_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_1} \left\{\left[-i\sum_{l_{\text{ext}}} \varepsilon_{ll_{\text{ext}}} p_{0;l_{\text{ext}}}\right]\left[\prod_{\substack{l'\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_1\\ l'\neq l}} (-i \sum_{c\in\text{Cycle}_{\mathcal{G}}} \varepsilon_{l'c} p_{0;c})\right]\right\}$$
$$\times \left[\prod_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\text{int}}} \epsilon_{l'f} p_f\right)\right],$$
$$Q^{2;2} = \sum_{l\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_2} \left\{\left[\sum_{f\in\mathcal{F}_{\text{ext}}} \epsilon_{lf} (p_f^{\text{ext}})^2\right]\left[\prod_{\substack{l'\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_2\\ l'\neq l}} e\left(\sum_{f\in\mathcal{F}_{\text{int}}} \epsilon_{l'f} p_f\right)\right]\right\}$$
$$\times \left[\prod_{l'\in\mathcal{L}_{\text{int}}\cap\mathcal{L}_1} (-i \sum_{c\in\text{Cycle}_{\mathcal{G}}} \varepsilon_{l'c} p_{0;c})\right], \tag{80}$$

and Q^3 is the sum of all remainder terms invoking all higher orders of the product

$$\left|\prod_{l\in A}\left[-i\sum_{l_{\text{ext}}} \varepsilon_{ll_{\text{ext}}} p_{0;l_{\text{ext}}}\right]\prod_{l\in B}\left[\sum_{f\in\mathcal{F}_{\text{ext}}} \epsilon_{lf}(p_f^{\text{ext}})^2\right]\right|,$$

for two subsets A and B of internal lines, $A, B \subset \mathcal{L}_{\text{int}}$, with cardinality $|A| \geq 2$ if $|B| = 0$, or $|B| > 1$ if $|A| = 0$, or $|A| + 2|B| \geq 3$, if $A > 0$ and $B > 0$.

We insert these expansions in the two-point amplitude and get:

$$A_{\mathcal{G};2}(\{p_{0;l_{\text{ext}}}\};\{p_f^{\text{ext}}\}) = \kappa(\lambda)\delta\left(\sum_{l_{\text{ext}}} p_{0;l_{\text{ext}}}\right)\int\left[\prod_{l_{\text{ext}}\in\mathcal{L}_{\text{ext}}} \mathrm{d}\alpha_{l_{\text{ext}}}\, e^{\alpha_{l_{\text{ext}}}}\right]$$
$$\times\left[\prod_{l_{\text{ext}}\in\mathcal{L}_{\text{ext}}\cap\mathcal{L}_1} (-ip_{0;l_{\text{ext}}})\right]\left[\prod_{l_{\text{ext}}\in\mathcal{L}_{\text{ext}}\cap\mathcal{L}_2} e\left(\sum_{f\in\mathcal{F}_{\text{ext}}} \epsilon_{l_{\text{ext}}f} p_f^{\text{ext}}\right)\right]$$

$$
\begin{aligned}
&\times \Bigg[\prod_{f \in \mathcal{F}_{\text{ext}}} e^{-(\alpha_{l_{\text{ext}}} + \alpha_{l'_{\text{ext}}})[(p_f^{\text{ext}})^4 - 2(p_f^{\text{ext}})^2]} \Bigg] \\
&\times \Bigg[\prod_{\substack{f,f' \in \mathcal{F}_{\text{ext}} \\ f \neq f'}} e^{-(\alpha_{l_{\text{ext}}} + \alpha_{l'_{\text{ext}}})(p_f^{\text{ext}})^2 (p_{f'}^{\text{ext}})^2} \Bigg] \Bigg[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} e^{-\alpha_{l_{\text{ext}}} p^2_{0;l_{\text{ext}}}} \Bigg] \\
&\times \Bigg[\prod_{f \in \mathcal{F}_{\text{int}}} e^{-(\sum_{\ell \in f} \alpha_\ell)[p_f^4 - 2p_f^2]} \Bigg] \Bigg[\prod_{\substack{f,f' \in \mathcal{F}_{\text{int}} \\ f \neq f'}} e^{-(\sum_{\ell \in f, \ell \in f'} \alpha_\ell)(p_f)^2 (p_{f'})^2} \Bigg] \\
&\times \Bigg[\prod_{c \in \text{Cycle}_{\mathcal{G}}} e^{-(\sum_{l \in \mathcal{L}_c} \alpha_l) p^2_{0;c}} \Bigg] \Bigg[\prod_{\substack{c,c' \in \text{Cycle}_{\mathcal{G}} \\ c \neq c'}} e^{-(\sum_{l \in \mathcal{T}_c \cap \mathcal{T}_{c'}} \underline{\alpha}_l) p_{0;c} p_{0;c'}} \Bigg] \\
&\times \Big(Q^1 + \sum_{\sigma=1,2} Q^{2;\sigma} + Q^3 \Big) \\
&\times \Bigg\{ 1 - \sum_{f \in \mathcal{F}_{\text{ext}}} (Q^1_{\text{ext}\,;f} + Q^{1'}_{\text{ext}\,;f}) - \sum_{\substack{f,f' \in \mathcal{F}_{\text{ext}} \\ f \neq f'}} (Q^2_{\text{ext}\,;f,f'} + Q^{2'}_{\text{ext}\,;f,f'}) \\
&- \sum_{f \in \mathcal{F}_{\text{ext}}, f' \in \mathcal{F}_{\text{int}}} (Q^3_{\text{ext}\,;f,f'} + Q^{3'}_{\text{ext}\,;f,f'}) \\
&- \sum_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} (Q^1_{l_{\text{ext}}} + Q^{1'}_{l_{\text{ext}}}) - \sum_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} (Q^2_{l_{\text{ext}}} + Q^{2'}_{l_{\text{ext}}}) \\
&- \sum_{\substack{l_{\text{ext}}, l'_{\text{ext}} \in \mathcal{L}_{\text{ext}} \\ l_{\text{ext}} \neq l'_{\text{ext}}}} (Q^3_{l_{\text{ext}}, l'_{\text{ext}}} + Q^{3'}_{l_{\text{ext}}, l'_{\text{ext}}}) \\
&+ \sum (Q+Q) \cdot (Q+Q) + \dots \Bigg\}. \qquad (81)
\end{aligned}
$$

where $\sum(Q+Q)\cdot(Q+Q)+\dots$ involves all types of higher-order products of the remainders $Q^{(\cdot)}_{\text{ext}\,;-}$ and $Q^{(\cdot)}_{l_{\text{ext}}}$, $Q^{(\cdot)}_{l_{\text{ext}}, l'_{\text{ext}}}$.

At zeroth order, we have the following amplitude

$$
\begin{aligned}
A_{\mathcal{G};2}(\{p_{0;l_{\text{ext}}}\}; \{p_f^{\text{ext}}\}; 0) &= \kappa(\lambda) \delta\Bigg(\sum_{l_{\text{ext}}} p_{0;l_{\text{ext}}} \Bigg) \int \Bigg[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} \mathrm{d}\alpha_{l_{\text{ext}}} \, e^{\alpha_{l_{\text{ext}}}} \Bigg] \\
&\times \Bigg[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}} \cap \mathcal{L}_1} (-ip_{0;l_{\text{ext}}}) \Bigg] \Bigg[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}} \cap \mathcal{L}_2} e\Bigg(\sum_{f \in \mathcal{F}_{\text{ext}}} \epsilon_{l_{\text{ext}}\, f} p_f^{\text{ext}} \Bigg) \Bigg] \\
&\times \Bigg[\prod_{f \in \mathcal{F}_{\text{ext}}} e^{-(\alpha_{l_{\text{ext}}} + \alpha_{l'_{\text{ext}}})[(p_f^{\text{ext}})^4 - 2(p_f^{\text{ext}})^2]} \Bigg] \qquad (82) \\
&\times \Bigg[\prod_{\substack{f,f' \in \mathcal{F}_{\text{ext}} \\ f \neq f'}} e^{-(\alpha_{l_{\text{ext}}} + \alpha_{l'_{\text{ext}}})(p_f^{\text{ext}})^2 (p_{f'}^{\text{ext}})^2} \Bigg]
\end{aligned}
$$

$$\times \left[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} e^{-\alpha_{l_{\text{ext}}} p^2_{0;l_{\text{ext}}}} \right]$$
$$\times \int \left[\prod_{l \in \mathcal{L}_{\text{int}}} \mathrm{d}\alpha_l e^{\alpha_l} \right] \int \left[\prod_{c \in \text{Cycle}_{\mathcal{G}}} \mathrm{d}p_{0;c} \right]$$
$$\times \left[\prod_{f \in \mathcal{F}_{\text{int}}} \mathrm{d}p_f \right] \left[\prod_{l \in \mathcal{L}_{\text{int}} \cap \mathcal{L}_1} \left(-i \sum_{c \in \text{Cycle}_{\mathcal{G}}} \varepsilon_{lc} p_{0;c} \right) \right] \left[\prod_{l \in \mathcal{L}_{\text{int}} \cap \mathcal{L}_2} e \left(\sum_{f \in \mathcal{F}_{\text{int}}} \epsilon_{lf} p_f \right) \right]$$
$$\times \left[\prod_{f \in \mathcal{F}_{\text{int}}} e^{-(\sum_{\ell \in f} \alpha_\ell)[p_f^4 - 2p_f^2]} \right] \left[\prod_{\substack{f,f' \in \mathcal{F}_{\text{int}} \\ f \neq f'}} e^{-(\sum_{\ell \in f, \ell \in f'} \alpha_\ell)(p_f)^2 (p_{f'})^2} \right]$$
$$\times \left[\prod_{c \in \text{Cycle}_{\mathcal{G}}} e^{-(\sum_{l \in \mathcal{L}_c} \alpha_l) p^2_{0;c}} \right] \left[\prod_{\substack{c,c' \in \text{Cycle}_{\mathcal{G}} \\ c \neq c'}} e^{-(\sum_{l \in \mathcal{T}_c \cap \mathcal{T}_{c'}} \underline{\alpha}_l) p_{0;c} p_{0;c'}} \right]. \tag{83}$$

Some change of variables allows us to show that the contribution of the external momenta can be recast as two propagators glued together and the factors from the integral over internal momenta which produces a linearly divergent term. This term renormalizes the mass (or the chemical potential, hence the Fermi radius in a condensed matter interpretation). Beware that this mass renormalization has a logarithmically divergent part corresponding to the constant part of the Q^1 term in (81).

We focus on the next order that we denote:

$$A_{\mathcal{G};2}(\{p_{0;l_{\text{ext}}}\}; \{p_f^{\text{ext}}\}; 1) = \kappa(\lambda) \delta \left(\sum_{l_{\text{ext}}} p_{0;l_{\text{ext}}} \right) \int \left[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} \mathrm{d}\alpha_{l_{\text{ext}}} e^{\alpha_{l_{\text{ext}}}} \right]$$
$$\times \left[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}} \cap \mathcal{L}_1} (-i p_{0;l_{\text{ext}}}) \right] \left[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}} \cap \mathcal{L}_2} e \left(\sum_{f \in \mathcal{F}_{\text{ext}}} \epsilon_{l_{\text{ext}} f} p_f^{\text{ext}} \right) \right]$$
$$\times \left[\prod_{f \in \mathcal{F}_{\text{ext}}} e^{-(\alpha_{l_{\text{ext}}} + \alpha_{l'_{\text{ext}}})[(p_f^{\text{ext}})^4 - 2(p_f^{\text{ext}})^2]} \right] \left[\prod_{\substack{f,f' \in \mathcal{F}_{\text{ext}} \\ f \neq f'}} e^{-(\alpha_{l_{\text{ext}}} + \alpha_{l'_{\text{ext}}})(p_f^{\text{ext}})^2 (p_{f'}^{\text{ext}})^2} \right]$$
$$\times \left[\prod_{l_{\text{ext}} \in \mathcal{L}_{\text{ext}}} e^{-\alpha_{l_{\text{ext}}} p^2_{0;l_{\text{ext}}}} \right]$$
$$\times \int \left[\prod_{l \in \mathcal{L}_{\text{int}}} \mathrm{d}\alpha_l e^{\alpha_l} \right] \int \left[\prod_{c \in \text{Cycle}_{\mathcal{G}}} \mathrm{d}p_{0;c} \right] \left[\prod_{f \in \mathcal{F}_{\text{int}}} \mathrm{d}p_f \right] \left[\prod_{f \in \mathcal{F}_{\text{int}}} e^{-(\sum_{\ell \in f} \alpha_\ell)[p_f^4 - 2p_f^2]} \right]$$
$$\times \left[\prod_{\substack{f,f' \in \mathcal{F}_{\text{int}} \\ f \neq f'}} e^{-(\sum_{\ell \in f, \ell \in f'} \alpha_\ell)(p_f)^2 (p_{f'})^2} \right] \tag{84}$$
$$\times \left[\prod_{c \in \text{Cycle}_{\mathcal{G}}} e^{-(\sum_{l \in \mathcal{L}_c} \alpha_l) p^2_{0;c}} \right] \left[\prod_{\substack{c,c' \in \text{Cycle}_{\mathcal{G}} \\ c \neq c'}} e^{-(\sum_{l \in \mathcal{T}_c \cap \mathcal{T}_{c'}} \underline{\alpha}_l) p_{0;c} p_{0;c'}} \right]$$

$$\times \left\{ \sum_{\sigma=1,2} Q^{2;\sigma} + Q^1 \left[- \sum_{f\in\mathcal{F}_{\rm ext},f'\in\mathcal{F}_{\rm int}} Q^3_{{\rm ext};f,f'} - \sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^2_{l_{\rm ext}} \right] \right\}. \tag{85}$$

We focus on the Q's terms and put them in the form

$$\begin{aligned}
Q^1 \sum_{f\in\mathcal{F}_{\rm ext},f'\in\mathcal{F}_{\rm int}} Q^3_{{\rm ext};f,f'} &= \sum_{f\in\mathcal{F}_{\rm ext}} (p_f^{\rm ext})^2 \left[\left(\sum_{f'\in\mathcal{F}_{\rm int}} (p_{f'})^2 \right) Q^1 \left(\sum_{l\in f, l\in f'} \alpha_l \right) \right] \\
Q^1 \sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^2_{l_{\rm ext}} &= \sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} p_{0;l_{\rm ext}} \left[2Q^1 \sum_{c\in{\rm Cycle}_{\mathcal{G}}} \left(\sum_{l\in\mathcal{T}_c\cap\mathcal{T}_{l_{\rm ext}}} \underline{\alpha}_l \right) p_{0;c} \right] \\
Q^{2;1} = -i \sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} & p_{0;l_{\rm ext}} \left\{ \sum_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_1} \varepsilon_{l l_{\rm ext}} \prod_{\substack{l'\in\mathcal{L}_{\rm int}\cap\mathcal{L}_1\\ l'\neq l}} \left[-i \sum_{c\in{\rm Cycle}_{\mathcal{G}}} \varepsilon_{l'c} p_{0;c} \right] \right\} \\
& \times \left[\prod_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2} e \left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{l'f} p_f \right) \right] \\
Q^{2;2} = \sum_{f\in\mathcal{F}_{\rm ext}} & (p_f^{\rm ext})^2 \left\{ \sum_{l\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2} \epsilon_{lf} \prod_{\substack{l'\in\mathcal{L}_{\rm int}\cap\mathcal{L}_2\\ l'\neq l}} \left[e \left(\sum_{f\in\mathcal{F}_{\rm int}} \epsilon_{l'f} p_f \right) \right] \right\} \\
& \times \left[\prod_{l'\in\mathcal{L}_{\rm int}\cap\mathcal{L}_1} \left(-i \sum_{c\in{\rm Cycle}_{\mathcal{G}}} \varepsilon_{l'c} p_{0;c} \right) \right].
\end{aligned} \tag{86}$$

At this point, one observes that the integral over all internal momenta of the above expressions could be brought as $\sum_{f\in\mathcal{F}_{\rm ext}} (p_f^{\rm ext})^2 \times {\rm coeff}(f)$ and $-ip_{0;l_{\rm ext}} \times {\rm coeff}'(l_{\rm ext})$, where ${\rm coeff}(f)$ and ${\rm coeff}'(l_{\rm ext})$ are constants depending on the graph. To be able to put those results as

$$\begin{aligned}
& -ip_{0;l_{\rm ext}} {\rm coeff}'(l_{\rm ext}) + \left[\sum_{f\in\mathcal{F}_{\rm ext}} (p_f^{\rm ext})^2 \right] \times {\rm coeff} \\
& \quad = -ip_{0;l_{\rm ext}} {\rm coeff}'(l_{\rm ext}) + \left[(p_{1;l_{\rm ext}})^2 + (p_{2;l_{\rm ext}})^2 + (p_{3;l_{\rm ext}})^2 \right] \times {\rm coeff},
\end{aligned} \tag{87}$$

which is of the form of the prefactor of the kinetic term and where coeff is another constant independent of f, we must gather all colored graphs which only differ through color permutation, and sum their contributions which must be all equal. Thus, this term (and the like by symmetrizing the graph) renormalizes the two wave functions Δ_{p_0} and Δ_{p^2}.

The last step is to prove the convergence of all remainder terms. We provide the following bounds of the remainders Q (under bounded integrals)

$$\begin{aligned}
\left| \sum_{f\in\mathcal{F}_{\rm ext}} Q^1_{{\rm ext};f} \right| &= \left| \sum_{f\in\mathcal{F}_{\rm ext}} [(p_f^{\rm ext})^4 - 2(p_f^{\rm ext})^2] \left[\left(\sum_{l\in f} \alpha_l \right) \right] \right| \leq k_1 M^{-2[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]}, \\
\left| \sum_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} Q^2_{{\rm ext};f,f'} \right| &= \left| \sum_{\substack{f,f'\in\mathcal{F}_{\rm ext}\\ f\neq f'}} (p_f^{\rm ext})^2 (p_{f'}^{\rm ext})^2 \left[\left(\sum_{l\in f, l\in f'} \alpha_l \right) \right] \right| \leq k_2 M^{-2[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},
\end{aligned}$$

$$\Big|\sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} Q^1_{l_{\rm ext}}\Big| = \Big|\sum_{l_{\rm ext}\in\mathcal{L}_{\rm ext}} (p_{0;l_{\rm ext}})^2\Big[\Big(\sum_{l\in\mathcal{T}_{l_{\rm ext}}}\alpha_l\Big)\Big]\Big| \le k_3 M^{-2[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},$$

$$\Big|\sum_{\substack{l_{\rm ext},l'_{\rm ext}\in\mathcal{L}_{\rm ext}\\ l_{\rm ext}\neq l'_{\rm ext}}} Q^3_{l_{\rm ext},l'_{\rm ext}}\Big|$$

$$= \Big|\sum_{\substack{l_{\rm ext},l'_{\rm ext}\in\mathcal{L}_{\rm ext}\\ l_{\rm ext}\neq l'_{\rm ext}}} p_{0;l_{\rm ext}}\,p_{0;l'_{\rm ext}}\Big[\Big(\sum_{l\in\mathcal{T}_{l_{\rm ext}}\cap\mathcal{T}_{l'_{\rm ext}}}\underline{\alpha}_l\Big)\Big]\Big| \le k_4 M^{-2[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]}, \tag{88}$$

where k's are constants. Using $i(\mathcal{G}^i_{(k)}) - e(\mathcal{G}^i_{(k)}) > 1$, and the fact that the integral over Q^1 brings the mass divergence $M^{\omega_{\rm deg}(\mathcal{G}^i_{(k)})=1}$ and that the integral over Q^2 brings lead to a log-divergent contribution, these bounds show that any term in the expansion involving one of the above expression as a factor has a strictly negative divergence degree. Further, we have

$$\begin{aligned}
&|Q^{1'}_{l_{\rm ext}}| \le k_5 M^{-4[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},\\
&|Q^{2'}_{l_{\rm ext}}| \le k_6 M^{-2[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},\\
&|Q^{3'}_{l_{\rm ext},l'_{\rm ext}}| \le k_7 M^{-4[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},\\
&|Q^{1'}_{{\rm ext};f}| \le k_7 M^{-4[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},\\
&|Q^{2'}_{{\rm ext};f,f'}| \le k_8 M^{-4[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]},\\
&|Q^{3}_{{\rm ext};f,f'}| \le k_9 M^{2[-2i(\mathcal{G}^i_{(k)})+i(\mathcal{G}^i_{(k)})+e(\mathcal{G}^i_{(k)})]} \le M^{-2[i(\mathcal{G}^i_{(k)})-e(\mathcal{G}^i_{(k)})]}.
\end{aligned} \tag{89}$$

Hence, any product of the above with Q^1 or $Q^{2;\sigma}$ will immediately lead a negative degree of divergence. In the same way, we can also show that Q^3 and the other higher-order products of Q's will contribute to convergent terms. After removing the divergences, all these contributions bring a sufficient decay to sum over the scale attributions and will lead to convergence. Thus, the model becomes renormalizable at all orders of perturbations.

7. Conclusion

We have proved the renormalizability of a tensor SYK model with a pair of Majorana tensor fields, in which time and tensor indices both govern a kind of renormalization group $(t_0,\vec{x}) \in [-\frac{\beta}{2},\frac{\beta}{2}] \times \mathbb{R}^3$ or $\in [-\frac{\beta}{2},\frac{\beta}{2}] \times U(1)^3$. Our model considers the orthogonal invariant (melonic and tetraedric) interactions introduced by Carrozza–Tanasa and uses the local-time interaction introduced by Klebanov–Tarnolposki which is common to all the SYK-type models. But it is endowed with a new notion of renormalization since it is based on the standard propagator of non-relativistic condensed matter. We achieved the proof of the perturbative ultraviolet renormalizability of the model through a multi-scale analysis and a power counting theorem which, interestingly, mixes the ordinary power counting of local field theory and the power counting of a non-local part coming from the tensorial convolution of the indices. A detailed study of the degree of divergence of an arbitrary graph proves that only the quartic melonic interactions renormalize as expected from the large N limit.

Having shown perturbative ultraviolet renormalizability, a next step is to compute the perturbative and non-perturbative flow equations for this model. Quartic melonic tensor field theory are generally UV asymptotically free [76]. A natural question is to check whether this remains true for the tensor SYK field theories introduced in this paper. Here the model is somehow different with two wave function couplings (Δ_{p_0} and Δ_{p^2}). The property of asymptotic safety or asymptotic freedom in the UV for tensor field theories mainly rests on the existence of a rapid growth of the coefficients of the wave function renormalization relatively to the quartic coupling. For our present situation, we foresee that at one loop, the tadpole gives no contribution to Δ_{p_0} but there will be still a contribution to Δ_{p^2}. All ingredients which trigger asymptotic freedom are therefore still present.

Of course the most interesting physics of this model lies in the infrared regime, which we intend to explore in a future study. We expect the tetraedric interaction to become more interesting in this regime. We may also have to consider variants of the model action (12), obtained by coloring differently the vertices with the two fields χ_1 and χ_2.

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