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Spectral Properties of Continuum Fibonacci Schrödinger Operators

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Abstract. We study continuum Schrödinger operators on the real line whose potentials are comprised of two compactly supported square-integrable functions concatenated according to an element of the Fibonacci substitution subshift over two letters. We show that the Hausdorff dimension of the spectrum tends to one in the small coupling and high-energy regimes, regardless of the shape of the potential pieces.

1. Introduction

1.1. Background

Quasicrystals were discovered in the early 1980s by Schechtman et al. [28] and have attracted a substantial amount of attention from researchers in mathematics and science. Broadly speaking, quasicrystals are solids characterized by the coexistence of two characteristics: *aperiodicity* (i.e., the absence of translation symmetries) and *long-range order*.

Thus, objects generated by or associated with strictly ergodic aperiodic subshifts over finite alphabets furnish concrete mathematical models of quasicrystals. Researchers in mathematics have studied such operators from various points of view including diffraction theory (see [1] and references therein) and spectral theory (see [4,5] and references therein). In particular, self-adjoint operators (and more recently, unitary operators) generated by such subshifts have been studied fairly heavily since the 1980s. Until somewhat recently, most of the effort was devoted to discrete Schrödinger operators arising in this manner. On the other hand, there have been several recent investigations concerning continuum Schrödinger operators [6,18,21], Jacobi matrices [14,22,31], and CMV matrices [7,8,10–13,23].

The (discrete) Fibonacci Hamiltonian is the central model of a one-dimensional quasicrystal in the discrete Schrödinger setting. This model was proposed in the physics literature by Kohmoto–Kadanoff–Tang [19] and Ostlund et al. [24]. Casdagli [3] and Sütő [26,27] wrote the seminal mathematics papers on this model, proving that it enjoys purely singular continuous spectrum supported on a Cantor set of zero Lebesgue measure. This spectral type is regarded as characteristic for this class of models. Our understanding of the discrete Fibonacci Hamiltonian is quite advanced; see [9] and references therein. However, our knowledge about the continuum Fibonacci Hamiltonian is substantially more rudimentary, owing to a proliferation of nontrivial obstructions, such as unboundedness of the operator, non-constancy of the Fricke–Vogt character on the spectrum, inability to explicitly compute transfer matrices (and hence the Fricke–Vogt character) in all but the simplest cases, inter alia.

Continuum versions of the Fibonacci Hamiltonian have been studied in [2,6,15–17,20,29,30]. The general theory for such continuum operators was established in the papers [6,21]. In the case of the Fibonacci subshift, [6] also established asymptotic behavior of the local Hausdorff dimension of the spectrum in the regimes of large energy and small potentials whenever the potential pieces are given by characteristic functions of intervals of length one. Their work relied on explicit formulae and calculations in an essential fashion and hence could not be immediately generalized to other potentials. Consequently, they posed the following question:

Question 6.8. *Is it true that [these asymptotics] hold regardless of the shape of the bump? That is, if we replace $f_a = \lambda \cdot \chi_{[0,1)}$ and $f_b = 0 \cdot \chi_{[0,1)}$ by general $f_a \in L^2(0, \ell_a)$ and $f_b \in L^2(0, \ell_b)$, do [these asymptotics] continue to hold as stated?*

In this paper, we answer this question in the affirmative in the regimes of small coupling and large energy. As of this moment, the regime of large coupling is out of reach. For the remainder of Sect. 1, we provide background on this class of operators and state our results at the end of the section.

1.2. Schrödinger Operators Associated with Subshifts Over Finite Alphabets

In this section, we will introduce the operators that we will study. First, we define the notion of concatenation of real-valued functions defined on intervals. Assume that for each $n \in \mathbb{Z}$, we have $\ell_n \in \mathbb{R}_+ \overset{\text{def}}{=} (0, \infty)$ and a real-valued function f_n defined on $[0, \ell_n)$. Moreover, assume that

$$\sum_{n \geq 0} \ell_n = \sum_{n < 0} \ell_n = \infty. \tag{1.1}$$

Condition (1.1) ensures that ensuing function will have domain $\mathbb{R}$. We define the *concatenation* of the sequence $\{f_n\}_{n \in \mathbb{Z}}$ as follows. Put

$$s_n \overset{\text{def}}{=} \begin{cases} \sum_{j=0}^{n-1} \ell_j & n \geq 1 \\ 0 & n = 0 \\ -\sum_{j=n}^{-1} \ell_j & n \leq -1, \end{cases} \tag{1.2}$$

denote $I_n = [s_n, s_{n+1})$, and define

$$f(x) = f_n(x - s_n), \quad \text{for each } x \in I_n. \tag{1.3}$$

We will denote this concatenation

$$f = \left(\cdots \mid f_{-2} \mid f_{-1} \mid \boxed{f_0} \mid f_1 \mid f_2 \mid \cdots \right). \tag{1.4}$$

We use a box to indicate the position of the origin. One can also concatenate finite families of functions. Given $m \le 0 \le n$, and $\{f_j, \ell_j\}_{j=m}^n$ as above, we define $\left(f_m \mid \cdots \mid \boxed{f_0} \mid \cdots \mid f_n \right)$ on $[s_m, s_{n+1})$ via (1.2) and (1.3).

Let $\mathcal{A}$ be a finite set, called the *alphabet*. Equip $\mathcal{A}$ with the discrete topology and endow $\mathcal{A}^{\mathbb{Z}}$ with the corresponding product topology. The *left shift*

$$[T\omega](n) \stackrel{\text{def}}{=} \omega(n+1), \quad \omega \in \mathcal{A}^{\mathbb{Z}}, \ n \in \mathbb{Z},$$

defines a homeomorphism from $\mathcal{A}^{\mathbb{Z}}$ to itself. A subset $\Omega \subseteq \mathcal{A}^{\mathbb{Z}}$ is called T-*invariant* if $T^{-1}(\Omega) = \Omega$. Any compact T-invariant subset of $\mathcal{A}^{\mathbb{Z}}$ is called a *subshift*.

We can use the concatenation construction above to associate potentials (and hence Schrödinger operators) to elements of subshifts as follows. For each $\alpha \in \mathcal{A}$, we pick $\ell_\alpha > 0$ and a real-valued function $f_\alpha \in L^2[0, \ell_\alpha]$. Then, for any $\omega \in \mathcal{A}^{\mathbb{Z}}$, we define the action of the continuum Schrödinger operator H_ω in $L^2(\mathbb{R})$ by

$$H_\omega = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\omega, \tag{1.5}$$

where the potential V_ω is given by

$$V_\omega = V_{\omega, \{f_\alpha\}} \stackrel{\text{def}}{=} \left(\cdots f_{\omega_{-2}} \mid f_{\omega_{-1}} \mid \boxed{f_{\omega_0}} \mid f_{\omega_1} \mid f_{\omega_2} \cdots \right). \tag{1.6}$$

These potentials belong to $L^2_{\mathrm{loc,unif}}(\mathbb{R})$, and hence, each H_ω defines a self-adjoint operator on a dense subspace $L^2(\mathbb{R})$ in a canonical fashion.

1.3. The Fibonacci Subshift

In this paper, we study a special case of the foregoing construction, namely potentials generated by elements of the Fibonacci subshift. In this case, the alphabet contains two symbols, $\mathcal{A} \stackrel{\text{def}}{=} \{\mathsf{a}, \mathsf{b}\}$. The Fibonacci substitution is the map

$$S(\mathsf{a}) = \mathsf{ab}, \ S(\mathsf{b}) = \mathsf{a}.$$

This map extends by concatenation to $\mathcal{A}^*$, the free monoid over $\mathcal{A}$ (i.e., the set of finite words over $\mathcal{A}$), as well as to $\mathcal{A}^{\mathbb{N}}$, the collection of (one-sided) infinite words over $\mathcal{A}$. There exists a unique element

$$u = \mathsf{abaababa} \ldots \in \mathcal{A}^{\mathbb{N}}$$

with the property that $u = S(u)$. It is straightforward to verify that for $n \in \mathbb{N}$, $S^n(\mathsf{a})$ is a prefix of $S^{n+1}(\mathsf{a})$. Thus, one obtains u as the limit (in the product topology on $\mathcal{A}^{\mathbb{N}}$) of the sequence of finite words $\{S^n(\mathsf{a})\}_{n \in \mathbb{N}}$. With this setup,

the Fibonacci subshift is defined to be the collection of two-sided infinite words with the same local factor structure as u, that is,

$$\Omega \overset{\text{def}}{=} \{\omega \in \mathcal{A}^{\mathbb{Z}} : \text{every finite subword of } \omega \text{ is also a subword of } u\}.$$

Given $\ell_{\mathsf{a}}, \ell_{\mathsf{b}} \in \mathbb{R}_+$ and real-valued functions $f_{\mathsf{a}} \in L^2[0, \ell_{\mathsf{a}})$, $f_{\mathsf{b}} \in L^2[0, \ell_{\mathsf{b}})$, we consider the family of continuum Schrödinger operators $\{H_\omega\}_{\omega \in \Omega}$ defined by (1.5) and (1.6). Since (Ω, T) is a minimal dynamical system, one can verify that there is a uniform closed set $\Sigma = \Sigma(f_{\mathsf{a}}, f_{\mathsf{b}}) \subset \mathbb{R}$ with the property that

$$\sigma(H_\omega) = \Sigma \quad \text{for every } \omega \in \Omega.$$

Of course, one can choose f_{a} and f_{b} in such a way that every V_ω is a periodic potential (notice that as soon as V_{ω_0} is periodic for a single $\omega_0 \in \Omega$, then every V_ω is periodic by minimality). The main result of [6] is that this is the only possible obstruction to Cantor spectrum. Thus, we adopt the following assumption throughout the paper:

Assumption A. *The potential pieces f_{a} and f_{b} are chosen so that V_ω is aperiodic for one $\omega \in \Omega$ (hence for every $\omega \in \Omega$ by minimality).*

Theorem 1.1 (Damanik et al. [6]). *Let Ω denote the Fibonacci subshift over $\mathcal{A} = \{\mathsf{a}, \mathsf{b}\}$. If f_{a} and f_{b} satisfy Assumption A, then Σ is a Cantor set of zero Lebesgue measure.*

Remark 1.2. In [6], the authors also assumed a condition on $\{f_\alpha : \alpha \in \mathcal{A}\}$ that they called *irreducibility*. This condition is defined so that the potentials satisfy the *simple finite decomposition property* (SFDP) from [18]. However, since our alphabet only has two letters, SFDP follows from aperiodicity and [18, Proposition 3.5].

Spectral properties of the family $\{H_\omega\}_{\omega \in \Omega}$ are encoded by dynamical characteristics of an associated polynomial map $T : \mathbb{R}^3 \to \mathbb{R}^3$. Concretely, every energy $E \in \mathbb{R}$ corresponds to a point $\gamma(E) \in \mathbb{R}^3$ via a (model-dependent) map, γ, called the *curve of initial conditions*. An energy E belongs to Σ if and only if $\gamma(E)$ has a bounded forward orbit under the action of T. We will describe this correspondence (and precisely define γ and T) in Sect. 2.

Throughout the remainder of the paper, we assume that $\ell_{\mathsf{a}}, \ell_{\mathsf{b}} > 0$, $f_{\mathsf{a}} \in L^2[0, \ell_{\mathsf{a}})$, and $f_{\mathsf{b}} \in L^2[0, \ell_{\mathsf{b}})$ are given so that Assumption A holds. It is quite helpful to introduce a third pair $(\ell_{\mathsf{ab}}, f_{\mathsf{ab}})$, defined by

$$\ell_{\mathsf{ab}} = \ell_{\mathsf{a}} + \ell_{\mathsf{b}}, \quad f_{\mathsf{ab}}(x) = \begin{cases} f_{\mathsf{a}}(x) & 0 \le x < \ell_{\mathsf{a}} \\ f_{\mathsf{b}}(x - \ell_{\mathsf{a}}) & \ell_{\mathsf{a}} \le x < \ell_{\mathsf{ab}}. \end{cases}$$

Equivalently, using the notation from (1.4), we could write $f_{\mathsf{ab}} = (\,\boxed{f_{\mathsf{a}}}\,|f_{\mathsf{b}})$. We denote by $\widetilde{\mathcal{A}} = \{\mathsf{a}, \mathsf{b}, \mathsf{ab}\}$ the enlarged "alphabet."

Our main results show that the local Hausdorff dimension of the spectrum tends to one in the high-energy and small coupling regimes, regardless of the

shapes of f_{a} and f_{b}; the local Hausdorff dimension of a set S at a point $x \in S$ is defined by

$$\dim_{\mathrm{H}}^{\mathrm{loc}}(S; x) = \lim_{\varepsilon \downarrow 0} \dim_{\mathrm{H}}(S \cap (x - \varepsilon, x + \varepsilon)),$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. This answers [6, Question 6.8] in the affirmative in those two asymptotic regimes in full generality.

Theorem 1.3. *The local Hausdorff dimension of Σ tends to 1 in the high-energy region. That is,*

$$\lim_{K \to \infty} \inf_{E \in \Sigma \cap [K, \infty)} \dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma; E) = 1.$$

More specifically, there exists a constant $c = c(f_{\mathsf{a}}, f_{\mathsf{b}}) > 0$ with the property that

$$\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma; E) \geq \max\left(1 - cE^{-1/4}, 0\right) \qquad (1.7)$$

for all $E \in \Sigma \cap \mathbb{R}_+$.

Theorem 1.4. *With notation as above, let $\Sigma_\lambda = \Sigma(\lambda f_{\mathsf{a}}, \lambda f_{\mathsf{b}})$ for $\lambda \in \mathbb{R}$. We have*

$$\lim_{\lambda \downarrow 0} \inf_{E \in \Sigma_\lambda \cap \mathbb{R}_+} \dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma_\lambda; E) = 1.$$

The rest of the paper is laid out as follows. In Sect. 2, we describe some background information, including the trace map formalism for the operator family $\{H_\omega\}_{\omega \in \Omega}$. We prove Theorem 1.3 in Sect. 3, and we prove Theorem 1.4 in Sect. 4.

2. Trace Map, Fricke–Vogt Invariant, and Local Hausdorff Dimension of the Spectrum

The spectrum (and many spectral characteristics) of the continuum Fibonacci model can be encoded in terms of an associated polynomial diffeomorphism of $\mathbb{R}^3$, called the *trace map*. We will make this correspondence explicit, following [6]. First, we consider the differential equation

$$- y''(x) + f_\alpha(x) y(x) = E y(x), \quad \alpha \in \widetilde{\mathcal{A}}, \ E \in \mathbb{R}, \ x \in [0, \ell_\alpha]. \qquad (2.1)$$

Denote the solution of (2.1) obeying $y(0) = 0$, $y'(0) = 1$ (respectively, $y(0) = 1$, $y'(0) = 0$) by $\psi_{\alpha,\mathrm{D}}(\cdot, E)$ (respectively, $\psi_{\alpha,\mathrm{N}}(\cdot, E)$). The associated transfer matrices are then given by

$$M_\alpha(E) = \begin{bmatrix} \psi_{\alpha,\mathrm{N}}(\ell_\alpha, E) & \psi_{\alpha,\mathrm{D}}(\ell_\alpha, E) \\ \psi'_{\alpha,\mathrm{N}}(\ell_\alpha, E) & \psi'_{\alpha,\mathrm{D}}(\ell_\alpha, E) \end{bmatrix}, \quad \alpha \in \widetilde{\mathcal{A}}, \ E \in \mathbb{R},$$

and

$$x_\alpha(E) = \frac{1}{2} \mathrm{tr}(M_\alpha(E)) = \frac{1}{2}\left(\psi_{\alpha,\mathrm{N}}(\ell_\alpha, E) + \psi'_{\alpha,\mathrm{D}}(\ell_\alpha, E)\right), \quad \alpha \in \widetilde{\mathcal{A}}, \ E \in \mathbb{R}.$$

It is straightforward to verify that

$$M_{\mathsf{ab}}(E) = M_{\mathsf{b}}(E) M_{\mathsf{a}}(E).$$

The map $\gamma(E) \overset{\text{def}}{=} (x_{\mathsf{ab}}(E), x_{\mathsf{a}}(E), x_{\mathsf{b}}(E))$ is known as the *curve of initial conditions*. Then, the *trace map* is defined by

$$T(x, y, z) \overset{\text{def}}{=} (2xy - z, x, y), \quad x, y, z \in \mathbb{R}.$$

This map is known to have a first integral given by the so-called *Fricke–Vogt invariant*, defined by

$$I(x, y, z) \overset{\text{def}}{=} x^2 + y^2 + z^2 - 2xyz - 1, \quad x, y, z \in \mathbb{R}.$$

More precisely, the trace map T preserves I (in the sense that $I \circ T = I$), and hence, T preserves the level surfaces of I:

$$S_V \overset{\text{def}}{=} \{(x, y, z) \in \mathbb{R}^3 : I(x, y, z) = V\}.$$

Consequently, every point of the form $T^n(x_{\mathsf{ab}}(E), x_{\mathsf{a}}(E), x_{\mathsf{b}}(E))$ with $n \in \mathbb{Z}_+$ lies on the surface $S_{I(\gamma(E))}$. For the sake of convenience, we put

$$I(E) \overset{\text{def}}{=} I(\gamma(E)) = I(x_{\mathsf{ab}}(E), x_{\mathsf{a}}(E), x_{\mathsf{b}}(E)),$$

with a minor abuse of notation.

The surfaces S_V experience a transition at $V = 0$. When $V < 0$, S_V has one compact connected component which is homeomorphic to the 2-sphere $\mathbb{S}^2$, and four unbounded connected components, each of which is homeomorphic to the open unit disk. When $V = 0$, each of the four unbounded components meet the compact component, forming four conical singularities. As soon as $V > 0$, the singularities resolve; then, S_V is smooth, connected, and homeomorphic to $\mathbb{S}^2$ with four points removed.

The trace map is important in the study of operators of this type as its dynamical spectrum encodes the operator-theoretic spectrum of H, where the dynamical spectrum is defined by

$$B \overset{\text{def}}{=} \{E \in \mathbb{R} : \{T^n(\gamma(E)) : n \in \mathbb{Z}_+\} \text{ is bounded}\}.$$

Proposition 2.1 (Damanik et al. [6]). *We have $\Sigma = B$.*

There are several substantial differences between the continuum setting and the discrete setting that we should point out. First, in the discrete case, the Fricke–Vogt invariant is constant (viewed as a function of E). However, the invariant may enjoy nontrivial dependence on E in the continuum setting, which is demonstrated by examples in [6]. This dependence is related to new phenomena that emerge in the continuum setting and make its study worthwhile.

Second, the Fricke–Vogt invariant is always nonnegative in the discrete setting, but one cannot a priori preclude negativity of I in the continuum setting. However, it is proved in [6] that any energies for which $I(E) < 0$ must lie in the resolvent set of the corresponding continuum Fibonacci Hamiltonian.

Proposition 2.2 (Damanik et al. [6]). *For every $E \in \Sigma$, one has $I(E) \geq 0$*

In order to study the fractal dimension of the spectrum, we will use the following theorem from [6], which relates local fractal characteristics near an energy E in the spectrum to the value of the invariant at E.

Theorem 2.3 (Damanik et al. [6]). *There exists a continuous map $D : [0, \infty) \to$ $(0, 1]$ that is real-analytic on $\mathbb{R}_+$ with the following properties:*

(i) $\dim_{\mathrm{H}}^{\mathrm{loc}}(\Sigma; E) = D(I(E))$ *for each $E \in \Sigma$.*

(ii) *We have $D(0) = 1$ and $1 - D(I) \sim \sqrt{I}$ as $I \downarrow 0$.*

(iii) *As $I \to \infty$, we have*

$$\lim_{I \to \infty} D(I) \cdot \log I = 2 \log(1 + \sqrt{2})$$

Thus, to study the local fractal dimensions of the spectrum, it suffices to understand the invariant I.

3. The High-Energy Region

Proof of Theorem 1.3. Clearly, it suffices to prove (1.7). In view of Theorem 2.3, it suffices to show that

$$I(E) = \mathcal{O}(E^{-1/2}) \tag{3.1}$$

as $E \to \infty$. Here and in what follows, we use $\| \cdot \|$ to denote an L^2 norm on an appropriate interval, i.e.,

$$\|f_\alpha\| = \left(\int_0^{\ell_\alpha} |f_\alpha(x)|^2 \, \mathrm{d}x \right)^{1/2}, \quad \alpha \in \widetilde{\mathcal{A}}.$$

Now, put

$$Q \stackrel{\text{def}}{=} \max\left(\|f_{\mathsf{ab}}\|, 1 \right), \quad C \stackrel{\text{def}}{=} Q \exp\left(Q \sqrt{\ell_{\mathsf{ab}}} \right).$$

For each energy $E \geq 0$, we denote

$$\kappa = \kappa(E) = \sqrt{E},$$

and we introduce functions

$$c_\kappa(x) \stackrel{\text{def}}{=} \cos(\kappa x), \quad s_\kappa(x) \stackrel{\text{def}}{=} \frac{\sin(\kappa x)}{\kappa}, \quad x \in \mathbb{R}.$$

Then, by [25, Theorem 1.3], one has the following estimate for $\alpha \in \widetilde{\mathcal{A}}$, $E \in \mathbb{C}$, $x \in [0, \ell_\alpha]$:

$$|\psi_{\alpha,\mathrm{N}}(x, E) - c_\kappa(x)| \leq \frac{1}{\sqrt{E}} \exp\left(\left| \mathrm{Im}\sqrt{E} \right| x + \|f_\alpha\| \sqrt{x} \right).$$

For $E > 0$, we get

$$|\psi_{\alpha,\mathrm{N}}(x, E) - c_\kappa(x)| \leq \frac{1}{\sqrt{E}} \exp\left(\|f_\alpha\| \sqrt{\ell_\alpha} \right) \leq C\kappa^{-1}.$$

Similarly, we obtain

$$\left| \psi'_{\alpha,\mathrm{D}}(x, E) - c_\kappa(x) \right| \leq C\kappa^{-1}$$

for every $\alpha \in \widetilde{\mathcal{A}}$, $E > 0$, and $x \in [0, \ell_\alpha]$. In particular, we have

$$x_\alpha(E) = c_\kappa(\ell_\alpha) + \mathcal{O}(\kappa^{-1})$$

for $\alpha \in \widetilde{\mathcal{A}}$ and $E > 0$ large. Then, the asymptotics above yield

$$I(E) = I(x_{\mathsf{ab}}(E), x_{\mathsf{a}}(E), x_{\mathsf{b}}(E))$$
$$= I(c_\kappa(\ell_{\mathsf{ab}}),\, c_\kappa(\ell_{\mathsf{a}}),\, c_\kappa(\ell_{\mathsf{b}})) + \mathcal{O}\left(\kappa^{-1}\right).$$

A straightforward calculation reveals that

$$I(c_\kappa(\ell_{\mathsf{ab}}),\, c_\kappa(\ell_{\mathsf{a}}),\, c_\kappa(\ell_{\mathsf{b}})) = 0,$$

so $I(E) = \mathcal{O}(E^{-1/2})$, which proves (3.1). □

4. The Small Coupling Regime

For the regime of small coupling, we fix f_{a} and f_{b} as before so that Assumption A holds, fix a Fibonacci-type potential V_ω, and ask about the behavior of

$$H_{\omega,\lambda} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \lambda V_\omega$$

as $\lambda \to 0$. We now view the relevant spectral data as functions of λ as well, and we write Σ_λ for the common spectrum of $H_{\omega,\lambda}$. Similarly, we view the Dirichlet and Neumann solutions as functions of λ, so we write $\psi_{\alpha,\mathrm{B}}(\cdot, E, \lambda)$ for the solution of

$$-y'' + \lambda f_\alpha y = Ey$$

obeying the appropriate boundary condition for $\mathrm{B} \in \{\mathrm{D}, \mathrm{N}\}$. Then, the transfer matrices and invariant are also functions of λ, and we denote them by $M_\alpha(E, \lambda)$ and $I(E, \lambda)$ to reflect this dependence.

Proof of Theorem 1.4. By Theorem 2.3, it suffices to show that $I(E, \lambda)$ goes to zero uniformly in $E > 0$ as $\lambda \to 0$. To that end, let $\varepsilon > 0$ be given. By the estimates in the proof of Theorem 1.3, we may choose a compact set $K \subset \mathbb{R}$ such that

$$|I(E, \lambda)| < \varepsilon$$

whenever $|\lambda| \leq 1$ and $E \in \mathbb{R}_+ \backslash K$. By [25, Theorem 1.5], we have that

$$\psi_{\alpha,\mathrm{N}}(\ell_\alpha, E, \lambda) \to c_\kappa(\ell_\alpha)$$

as $\lambda \to 0$, uniformly for $\alpha \in \widetilde{\mathcal{A}}$, $E \in K$. Thus, we may choose $\lambda_0 > 0$ so that

$$|\psi_{\alpha,\mathrm{N}}(\ell_\alpha, E, \lambda) - c_\kappa(\ell_\alpha)| < \varepsilon \qquad (4.1)$$

for all $\alpha \in \widetilde{\mathcal{A}}$ and $E \in K$ whenever $|\lambda| < \lambda_0$.

Differentiating the integral equation from [25, Theorem 1.1] , we get that

$$\psi_{\alpha,\mathrm{D}}'(\ell_\alpha, E, \lambda) = c_\kappa(\ell_\alpha) + \int_0^{\ell_\alpha} \cos(\kappa(\ell_\alpha - t)) \cdot \lambda f_\alpha(t) \cdot \psi_{\alpha,\mathrm{D}}(t, E, \lambda) \, \mathrm{d}t.$$

Again by [25, Theorem 5], $\psi_{\alpha,\mathrm{D}}(t, E, \lambda) \to s_\kappa(t)$ as $\lambda \to 0$, uniformly for $E \in K$ and $t \in [0, \ell_\alpha]$. Consequently (possibly after shrinking λ_0), we obtain

$$\left|\psi_{\alpha,\mathrm{D}}'(\ell_\alpha, E, \lambda) - c_\kappa(\ell_\alpha)\right| < \varepsilon \qquad (4.2)$$

for all $\alpha \in \widetilde{\mathcal{A}}$ and $E \in K$ whenever $|\lambda| < \lambda_0$. Therefore, by (4.1) and (4.2),

$$I(E, \lambda) = I(E, 0) + \mathcal{O}(\varepsilon)$$

for $E \in K$ and $|\lambda| < \lambda_0$, with a uniform implicit constant. Since $I(E, 0) = 0$, we have $I(E, \lambda) = \mathcal{O}(\varepsilon)$ for all $|\lambda| < \lambda_0$, as desired. $\square$

References

[1] Baake, M., Birkner, M., Grimm, U.: Non-periodic systems with continuous diffraction measures. In: Mathematics of Aperiodic Order. Progress in Mathematical Physics, vol. 309, pp. 1–32. Birkhäuser, Basel (2015)

[2] Baake, M., Joseph, D., Kramer, P.: Periodic clustering in the spectrum of quasiperiodic Kronig-Penney models. Phys. Lett. A **168**, 199–208 (1992)

[3] Casdagli, M.: Symbolic dynamics for the renormalization map of a quasi-periodic Schrödinger equation. Commun. Math. Phys. **107**, 295–318 (1986)

[4] Damanik, D.: Strictly ergodic subshifts and associated operators. In: Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday. Proceedings of Symposia in Pure Mathematics, Part 2, vol. 76, pp. 505–538. American Mathematical Society, Providence, RI (2007)

[5] Damanik, D., Embree, M., Gorodetski, A.: Spectral properties of Schrödinger operators arising in the study of quasicrystals. In: Mathematics of Aperiodic Order. Progress in Mathematical Physics, vol. 309, pp. 307–370. Birkhäuser, Basel (2015)

[6] Damanik, D., Fillman, J., Gorodetski, A.: Continuum Schrödinger operators associated with aperiodic subshifts. Ann. Henri Poincaré **15**, 1123–1144 (2014)

[7] Damanik, D., Fillman, J., Ong, D.C.: Spreading estimates for quantum walks on the integer lattice via power-law bounds on transfer matrices. J. Math. Pures Appl. **105**, 293–341 (2016)

[8] Damanik, D., Fillman, J., Vance, R.: Dynamics of unitary operators. J. Fractal Geom. **1**, 391–425 (2014)

[9] Damanik, D., Gorodetski, A., Yessen, W.: The Fibonacci Hamiltonian. Invent. Math. **206**, 629–692 (2016)

[10] Damanik, D., Lenz, D.: Uniform Szegő cocycles over strictly ergodic subshifts. J. Approx. Theory **144**, 133–138 (2007)

[11] Damanik, D., Munger, P., Yessen, W.: Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients, I. The essential support of the measure. J. Approx. Theory **173**, 56–88 (2013)

[12] Damanik, D., Munger, P., Yessen, W.: Orthogonal polynomials on the unit circle with Fibonacci Verblunsky coefficients II. Applications. J. Stat. Phys. **153**, 339–362 (2013)

[13] Fillman, J.: Purely singular continuous spectrum for Sturmian CMV matrices via strengthened Gordon Lemmas. Proc. Am. Math. Soc. **145**, 225–239 (2017)

[14] Fillman, J., Takahashi, Y., Yessen, W.: Mixed spectral regimes for square Fibonacci Hamiltonians. J. Fractal Geom. **3**, 377–405 (2016)

[15] Ghosh, P.: The Kronig-Penney model on a generalized Fibonacci lattice. Phys. Lett. A **161**, 153–157 (1991)

[16] Holzer, M.: Three classes of one-dimensional, two-tile Penrose tilings and the Fibonacci Kronig-Penney model as a generic case. Phys. Rev. B **38**, 1709–1720 (1988)

[17] Kaminaga, M., Nakano, F.: Spectral properties of quasiperiodic Kronig-Penney model. Tsukuba J. Math. **26**, 205–228 (2002)

[18] Klassert, S., Lenz, D., Stollmann, P.: Delone measures of finite local complexity and applications to spectral theory of one-dimensional continuum models of quasicrystals. Discrete Contin. Dyn. Syst. **29**, 1553–1571 (2011)

[19] Kohmoto, M., Kadanoff, L.P., Tang, C.: Localization problem in one dimension: mapping and escape. Phys. Rev. Lett. **50**, 1870–1872 (1983)

[20] Kollar, J., Sütő, A.: The Kronig-Penney model on a Fibonacci lattice. Phys. Lett. A **117**, 203–209 (1986)

[21] Lenz, D., Seifert, C., Stollman, P.: Zero measure Cantor spectra for continuum one-dimensional quasicrystals. J. Differ. Equ. **256**, 1905–1926 (2014)

[22] Mei, M., Yessen, W.: Tridiagonal substitution Hamiltonians. Math. Model. Nat. Phenom. **9**, 204–238 (2014)

[23] Munger, P., Ong, D.: The Hölder continuity of spectral measures of an extended CMV matrix. J. Math. Phys. **55**(9), 093507 (2014)

[24] Ostlund, S., Pandit, R., Rand, D., Schnellnhuber, H.J., Siggia, E.D.: One-dimensional Schrödinger equation with an almost-periodic potential. Phys. Rev. Lett. **50**, 1873–1876 (1983)

[25] Pöschel, J., Trubowitz, E.: Inverse Spectral Theory. Academic Press, Cambridge (1987)

[26] Sütő, A.: The spectrum of a quasiperiodic Schrödinger operator. Commun. Math. Phys. **111**, 409–415 (1987)

[27] Sütő, A.: Singular continuous spectrum on a Cantor set of zero Lebesgue measure for the Fibonacci Hamiltonian. J. Stat. Phys. **56**, 525–531 (1989)

[28] Shechtman, D., Blech, I., Gratias, D., Cahn, J.W.: Metallic phase with long-range orientational order and no translational symmetry. Phys. Rev. Lett. **53**(20), 1951–1953 (1984)

[29] Thomas, U., Kramer, P.: The Fibonacci quasicrystal reconsidered: variety of energy spectra for continuous Schrödinger equations with simple potentials. Int. J. Modern Phys. B **3**, 1205–1235 (1989)

[30] Würtz, D., Soerensen, M., Schneider, T.: Quasiperiodic Kronig-Penney model on a Fibonacci superlattice. Helv. Phys. Acta **61**, 345–362 (1988)

[31] Yessen, W.: Spectral analysis of tridiagonal Fibonacci Hamiltonians. J. Spectr. Theory **3**, 101–128 (2013)

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