



Small Cocycles, Fine Torus Fibrations, and a \mathbb{Z}^2 Subshift with Neither

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Abstract. Following an earlier similar conjecture of Kellendonk and Putnam, Giordano, Putnam, and Skau conjectured that all minimal, free \mathbb{Z}^d actions on Cantor sets admit “small cocycles.” These represent classes in H^1 that are mapped to small vectors in \mathbb{R}^d by the Ruelle–Sullivan (RS) map. We show that there exist \mathbb{Z}^2 actions where no such small cocycles exist, and where the image of H^1 under RS is \mathbb{Z}^2 . Our methods involve tiling spaces and shape deformations, and along the way we prove a relation between the image of RS and the set of “virtual eigenvalues,” i.e., elements of \mathbb{R}^d that become topological eigenvalues of the tiling flow after an arbitrarily small change in the shapes and sizes of the tiles.

1. Introduction and Statement of Results

In this paper we consider cohomological properties of minimal, free \mathbb{Z}^d actions on Cantor sets as studied in [11, 12, 15]. In particular, we consider the first group cohomology of a \mathbb{Z}^d action, and its image under the Ruelle–Sullivan map. Kellendonk and Putnam [20, Conj. 16] conjectured (under the additional assumption of unique ergodicity of the action) that the image of the Ruelle–Sullivan map is always dense in the dual space $(\mathbb{R}^d)^*$ to \mathbb{R}^d . Giordano et al. [13] conjectured (without the unique ergodicity assumption) that the image of this map is always dense in the dual space $(\mathbb{R}^d)^*$ to \mathbb{R}^d , and further that there always exist “small, positive cocycles.” Giordano et al. [13] showed that a number of interesting consequences would follow from these conjectures, including the existence of a free, minimal \mathbb{Z} action that is not orbit equivalent to any \mathbb{Z}^2 action.

In this paper we show that all these conjectures are in fact incorrect.

Theorem 1.1 (Theorem 8.3). *There exists a free, minimal, and uniquely ergodic \mathbb{Z}^2 subshift that does not admit any small cocycles, for which the image of*

H^1 under the Ruelle–Sullivan map is merely the natural \mathbb{Z}^2 that comes from constant cochains.

In fact, H^1 of this \mathbb{Z}^2 subshift is itself equal to \mathbb{Z}^2 (Theorem 8.4).

Before developing this counterexample, we consider the significance of the image of the Ruelle–Sullivan map. Here we work in the setting of \mathbb{R}^d actions on tiling spaces. \mathbb{Z}^d actions on Cantor sets, subshifts, and tilings with finite local complexity (FLC) are closely related. Every expansive \mathbb{Z}^d action can be realized as a subshift, the suspension of the \mathbb{Z}^d action on a subshift is a tiling space with FLC, and every tiling space with FLC is homeomorphic to the suspension of a subshift [23]. As a result, topological theorems about each of these categories can give important insights into the other two.

We then relate the first cohomology of a tiling space to spectral theory, and to realizations of that tiling space as a Cantor bundle over a torus.

Definition 1.2. If Ω is an aperiodic tiling space with FLC, and if $\lambda \in (\mathbb{R}^d)^*$, we say that λ is a **virtual eigenvalue** of the natural \mathbb{R}^d action if there exist arbitrarily small changes to the shapes and sizes of the tiles (aka arbitrarily small time changes) such that λ is a topological eigenvalue of the resulting \mathbb{R}^d actions.

Shape changes can also be used to make the translation dynamics topologically conjugate to the natural translation action on a Cantor bundle over a torus. (Henceforth, all Cantor bundles over tori will be assumed to carry this action.) Indeed, this is how the authors of [23] showed that all FLC tiling spaces are homeomorphic to suspensions of subshifts. In this paper we consider which tori can be bases of such bundles after arbitrarily small shape changes. For uniquely ergodic tiling spaces, the answer is especially simple:

Theorem 1.3. *Let Ω be a uniquely ergodic FLC tiling space whose natural \mathbb{R}^d action is minimal and free, and let $\lambda \in (\mathbb{R}^d)^*$. Then the following conditions are equivalent:*

1. λ is in the closure of the image of $H^1(\Omega)$ under the Ruelle–Sullivan map.
2. λ is a virtual eigenvalue.
3. There is an arbitrarily small shape change that transforms Ω into a Cantor bundle over a torus \mathbb{R}^d/L , such that λ is a period of the dual torus $(\mathbb{R}^d)^*/L^*$. (Here $L \subset \mathbb{R}^d$ is a lattice and $L^* \subset (\mathbb{R}^d)^*$ is the dual lattice.)

Similar results apply to linearly independent sets of virtual eigenvalues, and in particular to bases of eigenvalues.

Theorem 1.4. *Let Ω be a uniquely ergodic FLC tiling space whose natural \mathbb{R}^d action is minimal and free, and let $(\lambda_1, \dots, \lambda_d)$ be a basis for $(\mathbb{R}^d)^*$. Let $L' = \lambda_1\mathbb{Z} \oplus \dots \oplus \lambda_d\mathbb{Z}$ be the lattice spanned by the λ_i 's, dual to a lattice $L \subset \mathbb{R}^d$, and let $\mathbb{T} = \mathbb{R}^d/L$. Then the following are equivalent:*

1. All of the λ_i 's are virtual eigenvalues.
2. There exist arbitrarily small shape changes that convert Ω to a Cantor bundle over \mathbb{T} .

The following is then an immediate corollary:

Theorem 1.5. *Let Ω be a uniquely ergodic FLC tiling space with a minimal, free \mathbb{R}^d action. The image of $H^1(\Omega)$ is dense if and only if, for arbitrary length R , we can apply an arbitrarily small shape change to convert Ω into a Cantor bundle over $\mathbb{R}^d/(R\mathbb{Z})^d$.*

It is worth contrasting Theorem 1.3 with Theorem 3.9 of [13]. The latter theorem states that the existence of small positive cocycles implies that, for arbitrary length R , one can break up each tiling into locally defined regions such that each region contains a cube of side R . This in turn gives an easy orbit equivalence between the original \mathbb{Z}^d action on a Cantor set and a \mathbb{Z} action.

However, the converse is not true. The counterexample we construct to the conjectures of Giordano, Kellendonk, Putnam, and Skau does not admit small positive cocycles. However, it is built as a hierarchical tiling space, and so the tilings do admit partitions into arbitrarily large locally defined rectangular and square regions. By contrast to Theorem 3.9 of [13], Theorems 1.3, 1.4 and 1.5 are “if and only if” statements. Since our counterexample does not have elements of $H^1(\Omega)$ whose images under Ruelle–Sullivan are small, it does not admit large torus fibrations, and does not have any virtual eigenvalues beyond \mathbb{Z}^2 .

Definition 1.6. For a \mathbb{Z}^2 -subshift Ξ , let $\phi^{(n_1, n_2)}$ represent translation by $(n_1, n_2) \in \mathbb{Z}^2$. For each integer N , let N_+ denote the integers greater than N , and let N_- denote the integers less than N . We say that Ξ **admits a horizontal shear** if there exists an element u of the subshift space and integers N, N' such that, for every integer n , there is an element of Ξ that agrees with u on $\mathbb{Z} \times N_+$, and agrees with $\phi^{(n, 0)}u$ on $\mathbb{Z} \times N'_-$.

Vertical shears are defined similarly. Shears have a profound effect on the topological eigenvalues.

Theorem 1.7 (Theorem 7.1). *In a minimal \mathbb{Z}^2 subshift that admits shears in both coordinate directions, the spectrum of topological eigenvalues is precisely \mathbb{Z}^2 .*

We conjecture that this constraint on topological eigenvalues extends, upon perturbation, to a constraint on virtual eigenvalues.

Conjecture 1.8. *Let Ξ be a minimal, aperiodic and uniquely ergodic \mathbb{Z}^2 subshift, and let Ω be the suspension of Ξ . Let $\lambda = (\lambda_x, \lambda_y)$ be a virtual eigenvalue of Ω .*

1. *If Ξ admits a horizontal shear, then $\lambda_x \in \mathbb{Z}$.*
2. *If Ξ admits a vertical shear, then $\lambda_y \in \mathbb{Z}$.*
3. *If Ξ admits both a horizontal shear and a vertical shear, then $\lambda \in \mathbb{Z}^2$.*

Given Conjecture 1.8, proving Theorem 1.1 would reduce to exhibiting a \mathbb{Z}^2 subshift with shears in both directions. Such subshifts are already known. A particularly nice example, Natalie Frank’s non-Pisot Direct Product Variation (DPV) tiling, was developed in [6, 7] and further explored in [9, 10].

In fact, we are able to use the shear properties of the Frank DPV to prove that $RS(H^1(\Omega)) = \mathbb{Z}^2$. This proves Theorem 1.1 directly, without relying on Conjecture 1.8. Indeed, the methods of this proof generalize to a wide class of DPV tilings, providing evidence for Conjecture 1.8.

In a related vein, we consider bundle structures and the return dynamics induced by them, and how this depends on the (non)existence of small cocycles. These results are more technical, and we leave a precise statement of the theorems to Sects. 5 and 6.

In Sect. 5 we consider the question of when tiling spaces (or subshifts) admit optimal “finest” torus fibrations. The results depend on whether we define “finest” in terms of the volume of the torus or the diameter of the fiber. In one case the answer depends on the structure of H^1 ; in the other case on the image of H^1 under the Ruelle–Sullivan map. In Sect. 6 we investigate the implications of the existence of small cocycles for return equivalence in tiling spaces. Two tiling spaces are return equivalent if given any two initial transversals (one in each space) there exist clopen subsets of these transversals so that the return dynamics of the translation action induced on these clopen subsets are conjugate. This study is especially well-suited to tiling spaces for which there exist arbitrarily small cocycles, for then we can find arbitrarily small clopen subsets of a transversal with induced \mathbb{Z}^d return actions, and the original space is homeomorphic to the suspension of these induced actions. In the presence of arbitrarily small cocycles, this allows us to show that return equivalence is the same as being homeomorphic for FLC tiling spaces as indicated in Theorem 6.1.

The organization of this paper is as follows. In Sect. 2 we review the definitions of group cohomology and the Ruelle–Sullivan map in the context of \mathbb{Z}^d actions on Cantor sets, and review some of the results of [13]. In Sect. 3 we review the connections between minimal \mathbb{Z}^d actions, subshifts, and FLC tiling spaces, and a formulation of tiling cohomology involving differential forms. In Sect. 4 we show how to implement small shape changes, and prove Theorems 1.3 and 1.4, leading to Theorem 1.5. Section 5 concerns the existence of “finest” torus fibrations, and Sect. 6 relates return equivalence to homeomorphisms. In Sect. 7 we investigate the role of shears and discuss Conjecture 1.8. Finally, in Sect. 8 we exhibit Frank’s DPV tiling and show that it has the necessary cohomological properties as a consequence of its shear properties. This then completes the proof of Theorem 1.1.

2. \mathbb{Z}^d Actions, Cochains, and Cohomology

Consider a \mathbb{Z}^d action on a Cantor set C . For $n \in \mathbb{Z}^d$, we denote the action of n on x by $\phi^n(x)$. A 1-cocycle with values in \mathbb{Z} is a continuous map $\theta : C \times \mathbb{Z}^d \rightarrow \mathbb{Z}$ such that, for all $n, m \in \mathbb{Z}^d$ and all $\chi \in C$,¹

$$\theta(\chi, n + m) = \theta(\chi, n) + \theta(\phi^n(\chi), m). \quad (1)$$

¹ We denote elements of a Cantor set C by Greek letters such as χ , elements of a subshift Ξ by Roman letters such as u , points in \mathbb{Z}^d by Roman letters such as n , tilings by capital Roman letters such as T , and points in \mathbb{R}^d by Roman letters such as x and y .

If $f : C \rightarrow \mathbb{Z}$ is a continuous function, then the **coboundary** of f is given by

$$\delta f(\chi, n) = f(\phi^n(\chi)) - f(\chi). \tag{2}$$

It's easy to check that all coboundaries are cocycles. The first group cohomology of C , denoted $H^1(C)$, is the quotient of the cocycles by the coboundaries.

Given an invariant measure on C , we can average a cocycle to get a function on \mathbb{Z}^d :

$$\bar{\theta}(n) = \int \theta(\chi, n) d\mu(\chi)$$

By the cocycle condition (1), this is a linear function of n , hence an element of $Hom(\mathbb{Z}^d, \mathbb{R}) = (\mathbb{R}^d)^*$. This averaging procedure is called the Ruelle–Sullivan (RS) map. It is easy to check that the RS map sends coboundaries to zero, and hence gives a linear map from cohomology classes to $(\mathbb{R}^d)^*$. (In fact, the RS map is a ring homomorphism from the full cohomology of the subshift to the exterior algebra of \mathbb{R}^d [20], but here we are only concerned with the image of H^1 .)

Example 1. If $\theta(\chi, n) = n_i$ (the i -th component of n) for all χ , then $\bar{\theta}(n) = n_i$, and $\bar{\theta}$ is the i -th basis vector in $(\mathbb{R}^d)^*$. This shows that the integer lattice within $(\mathbb{R}^d)^*$ is always in the image of the RS map.

In [13], Giordano et al. studied what they call “small, positive cocycles.” These are cocycles that are nonnegative for n in a quadrant (say, $n_i \geq 0$ for all i), and such that $\theta(\chi, n)$ is bounded above and below by constants plus small positive multiples of $|n|$ for n in the quadrant, with the constants and multiples being independent of χ . These cocycles are mapped by RS to small elements in a quadrant of $(\mathbb{R}^d)^*$. The authors discuss consequences of there existing arbitrarily small positive cocycles for minimal actions. Also, if true, the conjectures would allow one to construct a Bratteli diagram in which the dynamics are apparent, as one can do for \mathbb{Z} actions with the Bratteli–Vershik map. The main result of the present paper is to disprove these conjectures.

3. Subshifts, Tilings, and Pattern Equivariant Cohomology

Let \mathcal{A} be a set with n elements, called **letters**. The set \mathcal{A} is called the **alphabet**, and the space $\mathcal{A}^{\mathbb{Z}^d}$ is called the **full shift on n letters**. If $u \in \mathcal{A}^{\mathbb{Z}^d}$ (i.e., u is a map from \mathbb{Z}^d to \mathcal{A}), then the shift action is simply

$$(\phi^n u)(m) = u(m + n).$$

We give $\mathcal{A}^{\mathbb{Z}^d}$ the product topology. This is metrizable and is often described with a metric in which two functions u_1, u_2 are ϵ -close if they agree exactly on a ball of radius $1/\epsilon$ around the origin.

A **subshift** Ξ is a subset of $\mathcal{A}^{\mathbb{Z}^d}$ that is closed in the product topology and is invariant under the action of ϕ . A subshift is called **aperiodic** if $\phi^n u = u$ implies $n = 0$. That is, if the action of ϕ is free. A subshift is **minimal** if every orbit is dense. A minimal aperiodic subshift is homeomorphic to a Cantor set

since in this case the subshift has no isolated points, so aperiodic minimal subshifts are special cases of free minimal \mathbb{Z}^d actions on Cantor sets.

Conversely, every expansive \mathbb{Z}^d action on a Cantor set C can be identified with a subshift. To see this, partition C into finitely many clopen sets, each of which is smaller than the expansivity radius. Thus, for any distinct $\chi, \rho \in C$, there is an $n \in \mathbb{Z}^d$ such that $\phi^n(\chi)$ and $\phi^n(\rho)$ lie in different clopen sets. Let \mathcal{A} be the collection of clopen sets, and for each $\chi \in C$ define a function $u_\chi : \mathbb{Z}^d \rightarrow \mathcal{A}$ such that $u_\chi(n)$ is the clopen set containing $\phi^n(\chi)$. The image of this assignment is a subshift $\Xi \subset \mathcal{A}^{\mathbb{Z}^d}$, and gives an isomorphism between the given \mathbb{Z}^d action on C and the natural action of \mathbb{Z}^d on Ξ .

The **suspension** of a subshift Ξ is the set $\Xi \times \mathbb{R}^d / \sim$, where for each $u \in \Xi$, $n \in \mathbb{Z}^d$ and $v \in \mathbb{R}^d$,

$$(u, n + v) \sim (\phi^n(u), v).$$

There is a natural \mathbb{R}^d action by addition on the second factor. Such a suspension can be visualized as a tiling space, where the tiles are unit cubes labeled by the alphabet \mathcal{A} and meeting full-face to full-face. If $u \in \Xi$, then (u, v) is a tiling in which tiles with labels $u(n)$ occupy $n - v + [0, 1]^d$. (Acting by v on a tiling means translating the tiling by $-v$, or equivalently translating the origin, relative to the tiling, by $+v$).

However, there is no reason to restrict attention to tilings by cubes. We can begin with an arbitrary collection of labeled shapes, called prototiles and define tiles to be translates of prototiles. A **patch** of a tiling is a finite collection of tiles. A tiling is said to have **finite local complexity** (with respect to translation), or FLC, if for each radius R there are only finitely many possible patches of diameter less than R , up to translation. This is equivalent to there only being finitely many prototiles, with only finitely many ways that two tiles can meet. Suspensions of subshifts are clearly FLC tilings. Conversely, every FLC tiling space is homeomorphic to the suspension of a subshift [23]. Further, every FLC tiling space is topologically conjugate to a tiling space in which the tiles are polyhedra that meet full-face to full-face; we henceforth restrict attention to tilings of this form. For more details on this construction, and on topological properties of tiling spaces, see [22].

The metric on a tiling space Ω is similar to the metric on subshifts. Two tilings are ϵ -close if they agree on a ball of radius $1/\epsilon$ around the origin, up to a uniform translation by up to ϵ . This makes the \mathbb{R}^d action on Ω continuous. The canonical transversal Ξ of Ω is the set of tilings with a vertex at the origin.

Definition 3.1. Let T be a tiling. A function f on \mathbb{R}^d is **strongly pattern equivariant (with respect to T) with radius r** if, whenever $x, y \in \mathbb{R}^d$ and $T - x$ and $T - y$ agree on a ball of radius r around the origin, then $f(x) = f(y)$. In other words, the value of $f(x)$ depends only on the pattern of T on a ball of radius r around x . A function is **strongly pattern equivariant** if it is strongly pattern equivariant for some finite radius r . A function is **weakly pattern equivariant** if it is the uniform limit of strongly pattern equivariant functions. That is, for any $\epsilon > 0$ there is an r such that the value of $f(x)$ is determined,

up to ϵ , by the pattern of radius r around x . When we say a function is **pattern equivariant** (abbreviated PE), the pattern equivariance is strong unless stated otherwise.

A tiling T gives a CW decomposition of \mathbb{R}^d into d -cells (tiles), $(d-1)$ -cells (faces), and so on down to 1-cells (edges) and 0-cells (vertices). We can then speak of 0-cochains, 1-cochains, etc. A k -cochain with values in an Abelian group A is an assignment of an element of A to each oriented k -cell in T , and hence by linearity to each k -chain in T . As usual, the coboundary of a cochain α , applied to a chain c , is α applied to the boundary of c :

$$(\delta\alpha)(c) := \alpha(\partial c).$$

As with functions, we say a k -cochain is PE of radius r if its value on a k -cell depends only on the pattern of T on a ball of radius r around the center-of-mass of the cell. We say a k -cochain is (strongly) PE if it is PE with some finite radius r , and we say it is weakly PE if it is the uniform limit of strongly PE cochains. It is easy to check that the coboundary of a strongly PE cochain is strongly PE (albeit with a slightly larger radius), and that the coboundary of a weakly PE cochain is weakly PE.

Consider the cochain complex of (strongly) PE cochains. The cohomology of this complex is isomorphic to the Čech cohomology of the orbit closure of T with values in A [21]. In particular, if the \mathbb{R}^d action on Ω is minimal, then the cohomology of this complex is the same for all $T \in \Omega$, and is isomorphic to the Čech cohomology of Ω with values in A . We can then speak of the PE cohomology of Ω , by which we mean the cohomology of PE cochains on an arbitrary $T \in \Omega$.

We henceforth restrict our attention to *minimal* subshifts and minimal tiling spaces. That is, **all \mathbb{Z}^d actions on Cantor sets and translations actions on tiling spaces are assumed to be minimal unless explicitly noted otherwise.**

For \mathbb{Z}^d actions on Cantor sets, there is a natural correspondence between the first PE cohomology (with values in \mathbb{Z}) and the group-theoretic H^1 . If α is a PE cochain, then α extends by continuity to take values on k -cells of all tilings in Ω , since all tilings exhibit the same patterns. Thus, we may speak of the value of a 1-cochain α on an edge of a tiling. If $\delta\alpha = 0$, and if we associate a tiling by unit cubes to each function in a subshift, then we get a cocycle $\theta(u, n)$ by applying α to a path from 0 to n in the tiling associated with u . (We call this the **integral** of α along the path, even though we are merely summing rather than integrating.) Since $\delta\alpha = 0$, this result does not depend on the path taken. The cocycle condition (1) is just the statement that the integral from 0 to $n + m$ equals the integral from 0 to n plus the integral from n to $n + m$.

It is often convenient to define tiling cohomology via PE differential forms. Indeed, this is the setting in which pattern equivariance was first defined [18, 20]. A differential form is strongly PE if all of its coefficients are strongly PE. It is weakly PE if all of its coefficients, and the derivatives to all orders of those coefficients, are uniform limits of strongly PE functions. As long as the \mathbb{R}^d action on Ω is minimal, the cohomology of the de Rham complex of (strongly)

PE differential forms on an arbitrary tiling $T \in \Omega$ is isomorphic to the Čech cohomology of Ω with values in \mathbb{R} [18, 20, 21].

It sometimes happens that a real-valued strongly PE cochain (or differential form) α is the coboundary (or exterior derivative) of a weakly PE cochain (or form). In this case we say that the (strong) cohomology class of α is **asymptotically negligible** [5]. A theorem of [19], closely related to the classical Gottschalk–Hedlund theorem, says that the class of a closed PE 1-cochain (or form) α is asymptotically negligible if and only if the integral of α is bounded.

Recall that H^1 of a CW complex is always torsion-free, since the universal coefficients theorem relates the torsion in H^1 to the (nonexistent) torsion in H_0 . Tiling spaces are inverse limits of CW complexes, so there is never any torsion in H^1 of a tiling space. This implies that the first integer-valued cohomology of a (minimal) tiling space can be viewed as a subgroup of the real-valued cohomology, as represented either by real-valued PE cochains or by PE differential forms. What characterizes an integral class $[\alpha]$ is that there exists a radius r such that, for any two occurrences of a patch P containing a ball of radius r , the integral of α from the center of that ball in one occurrence of P to the corresponding point in the other occurrence is always an integer. (In the inverse limit construction of a tiling space Ω , such paths correspond precisely to closed loops in the CW complexes that approximate Ω .)

The Ruelle–Sullivan map is most easily defined using differential forms [20]. If $[\alpha]$ is an integral cohomology class, represented by the PE form α , then we average the value of $\alpha(0)$ over all tilings in the space, using an invariant measure. If the space is uniquely ergodic, then this is equivalent to picking one tiling T and averaging $\alpha(x)$ over larger and larger balls around the origin.

Definition 3.2. A closed PE 1-form α is **positive and ϵ -small** if

- At each point x in each tiling T , $|\alpha(x)| < \epsilon$,
- There is a cone $C \in (\mathbb{R}^d)^*$ such that $\alpha(x)$ applied to any vector in that cone is everywhere nonnegative, and
- There exists a vector v in that cone such that $\alpha(x)$ applied to v is everywhere positive.

We say that a tiling space **has small positive forms** if, for each $\epsilon > 0$, there exist positive ϵ -small forms. This is the natural analog, in the setting of differential forms, of the “small, positive cocycles” of [13].

If α is a closed, positive and ϵ -small form, then the Ruelle–Sullivan map sends $[\alpha]$ to an element of $(\mathbb{R}^d)^*$ of magnitude less than ϵ . In particular, if Ω has small positive forms, then the image of the Ruelle–Sullivan map is not discrete. Conversely, if Ω is uniquely ergodic and the Ruelle–Sullivan image of $[\alpha]$ is nonzero and has magnitude less than ϵ , then $[\alpha]$ can be represented by a positive and ϵ -small form [17].

4. Shape Changes and Virtual Eigenvalues

The shape of a tile is described by the vectors along all of the edges around the tile. The assignment of each edge to its corresponding vector is a (manifestly PE) closed vector-valued cochain and defines a class in $H^1(\Omega, \mathbb{R}^d)$. By varying this cochain in a strongly PE way, we can obtain a family of tiling spaces, all with the same combinatorics, but whose tiles have different shapes and sizes. We call this a “shape change” of the original tiling space. (See [22] for more details.) In [5], small shape changes were shown to be parametrized, up to local equivalence, by $H^1(\Omega, \mathbb{R}^d)$. In [19], all topological conjugacies between tiling spaces were shown to be a combination of shape changes and local equivalences (“mutual local derivability,” or MLD). In [17], building on [16], all homeomorphisms between uniquely ergodic tiling spaces were shown to be a combination of shape changes and local equivalences.

Shape changes can also be implemented with differential forms [17]. Let α be a closed PE vector-valued 1-form (i.e., an assignment of a square matrix to each point of a tiling) representing a class in $H^1(\Omega, \mathbb{R}^d)$. If $\alpha(x)$ is sufficiently close to a fixed invertible matrix M at each point x , we can apply a shape change in which the displacement from a vertex x to another vertex y becomes $\int_x^y \alpha$.

Now let P be a patch containing a ball whose radius is greater than the pattern equivariance radius of α . If x and y are corresponding vertices of different occurrences of P in a tiling, we say that $y - x$ is a **return vector** of P . If all return vectors are in \mathbb{Z}^d , then our tiling space is a fiber bundle over the torus $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$, where the map $\Omega \rightarrow \mathbb{T}$ just gives the coordinates of all occurrences of P , mod \mathbb{Z}^d (where we have chosen a particular vertex in P to represent the location of the patch). Note that translation in the tiling is equivalent to translation in the torus, so that this structure is a topological semi-conjugacy. Likewise, if all return vectors of P are in a lattice L , we get a bundle over $\mathbb{T}_L = \mathbb{R}^d/L$.

If we apply a shape change to Ω , generated by the vector-valued 1-form α , then the return vectors $y - x$ are replaced by integrals $\int_x^y \alpha$. If these are all in L , then the shape-changed tiling space Ω' is (topologically conjugate to) a bundle over \mathbb{T}_L .

The following theorem is a slightly stronger restatement of Theorem 1.4.

Theorem 4.1. *Suppose that Ω is a uniquely ergodic FLC tiling space with free, minimal \mathbb{R}^d action, and that $\lambda_1, \dots, \lambda_d$ are a basis for $(\mathbb{R}^d)^*$. Let L' be the lattice spanned by the λ_i 's, dual to a lattice $L \subset \mathbb{R}^d$. Then the following are equivalent:*

1. All of the λ_i 's are in the closure of $RS(H^1(\Omega))$.
2. For any $\epsilon > 0$, there is a shape change, implemented by a vector-valued 1-form that is pointwise ϵ -close to the identity, such that the resulting tiling space Ω' is topologically conjugate to a Cantor bundle over the torus \mathbb{R}^d/L .

3. For any $\epsilon > 0$, there is a shape change, implemented by a vector-valued 1-form that is pointwise ϵ -close to the identity, such that $\lambda_1, \dots, \lambda_d$ are all topological eigenvalues of Ω' .

Proof. We will show that (1) implies (2), that (2) implies (3), and that (3) implies (1).

(1) \Rightarrow (2): Let M_0 be a matrix whose rows are the λ_i 's. We can find integral classes $[\alpha_1], \dots, [\alpha_d] \in H^1$ such that $M_0^{-1}M$ is $(\epsilon/2)$ -close to the identity matrix, where M is the matrix whose i -th row is the image of $[\alpha_i]$ under the Ruelle–Sullivan map. For convenience, package the d scalar-valued cohomology classes $[\alpha_1], \dots, [\alpha_d]$ into a single vector-valued class $[\alpha]$, represented by a vector-valued differential form α .

Since Ω is uniquely ergodic, the pointwise ergodic theorem implies that M is the spatial average of α over any tiling T in Ω , and that convergence to this average is uniform. Thus, if ρ_r is a bump function of total integral 1 and large support (say, achieving a constant maximum value on a ball of radius $r \gg 1$ and vanishing outside a ball of radius $r + 1$), and if we pick r large enough, the convolution of α with r will be nearly pointwise constant, and in particular $M_0^{-1}(\rho_r * \alpha)(x)$ will be $\epsilon/2$ close to $M_0^{-1}M$, and so ϵ -close to the identity. Let $\tilde{\alpha} = \rho_r * \alpha$. Since ρ_r has compact support, and since α is PE, $\tilde{\alpha}$ is also PE, albeit with a radius $r + 1$ greater than the pattern equivariance radius of α . Furthermore, $[\tilde{\alpha}] = [\alpha]$. (For more details of this construction, with precise estimates on the convergence, see [17].)

Since $[\alpha]$ is an integral class, the integral of $\tilde{\alpha}$ from one occurrence of a (sufficiently large) patch P to another occurrence must give an integer. If we then define a function $f(x) = \int_0^x \tilde{\alpha}$, then all occurrences of P will have the same value of $f \pmod{\mathbb{Z}^d}$. Likewise, if we define a function $g(x) = \int_0^x M_0^{-1}\tilde{\alpha}$, then all occurrences of P will have the same value of $g \pmod{L}$. Thus, the shape change implemented by $M_0^{-1}\tilde{\alpha}$ maps Ω to a Cantor bundle over \mathbb{R}^d/L .

(2) \Rightarrow (3): Since L and L' are dual lattices, all elements of L' are topological eigenvalues of any bundle over \mathbb{R}^d/L .

(3) \Rightarrow (1): If λ_i is a topological eigenvalue of Ω' , with corresponding eigenfunction ψ , then $\frac{-i d\psi}{2\pi\psi}$ is a constant 1-form (equaling λ_i) on any tiling $T' \in \Omega$ and represents an integral cohomology class. This form pulls back to a nearly-constant (but strongly PE) 1-form on a tiling $T \in \Omega$, representing the same integral class. The spatial average of this 1-form is then ϵ -close to λ_i . Since we can pick ϵ to be arbitrarily small, λ_i must be in the closure of the image of the Ruelle–Sullivan map. □

To prove Theorem 1.3, we simply apply Theorem 4.1 to a basis of $(\mathbb{R}^d)^*$ consisting of λ and all but one of the standard basis vectors for $(\mathbb{R}^d)^*$.

Theorems 1.3, 1.4, 1.5, and 4.1 all assume unique ergodicity. However, the unique ergodicity was only needed to produce collections of closed PE 1-forms with desired properties. The following is the analog of Theorem 4.1 without the assumption of unique ergodicity.

Theorem 4.2. *Suppose that Ω is an FLC tiling space with a free, minimal \mathbb{R}^d action, that M_0 is an invertible matrix, that α is a closed, PE, vector-valued differential form such that $\|M_0\alpha(x) - I\| < 1/4$ everywhere, and that α represents an integer cohomology class. Let L be a lattice in \mathbb{R}^d that is the integer span of the columns of M_0 . Then there is a shape change, induced by $M_0\alpha$, to a new tiling space Ω' that is a Cantor bundle over a torus \mathbb{R}^d/L .*

Proof. Let $f(x) = \int_0^x \alpha$ and let $g(x) = \int_0^x M_0\alpha$. The condition $\|M_0\alpha(x) - I\| < 1/4$ is sufficient to guarantee that $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bijective. Since M_0 is invertible, this also implies that f is bijective. As in the proof of Theorem 4.1, different occurrences of a patch P of size greater than the PE radius of α must have values of f that differ by elements of \mathbb{Z}^d , and so must have values of g that differ by elements of L . Doing a shape change that replaces x by $g(x)$ thus results in a Cantor bundle over \mathbb{R}^d/L . \square

If the determinant of α is pointwise small, then the determinant of M_0 must be large, and our torus has large volume. The less fluctuation there is in α , the closer we can get $M_0\alpha$ to be to the identity (everywhere), and the closer the shapes and sizes of the tiles in Ω' will be to the shapes and sizes in Ω .

5. Finest Bundle

In [23] the question is raised whether there is a “finest” possible bundle structure of a tiling space. There are a number of ways in which one might consider one bundle finer than another, and here we shall give, using two natural notions of measuring fineness, conditions allowing one to determine in many settings when a particular bundle structure on a tiling space admits a finest bundle structure.

In the first notion, we consider the bundle $\Omega \xrightarrow{p'} \mathbb{T}^d$ **finer** than the bundle $\Omega \xrightarrow{p} \mathbb{T}^d$ if there exists a commutative diagram

$$\begin{array}{ccc}
 \Omega & & \\
 p \downarrow & \searrow p' & \\
 \mathbb{T}^d & \xleftarrow{\pi} & \mathbb{T}^d
 \end{array} \tag{3}$$

in which the mapping π is a covering map, and we denote this relation as $p \preceq p'$. This is a natural notion in the bundle category since the bundles $\mathbb{T}^d \rightarrow \mathbb{T}^d$ are covering maps. If the degree of π is greater than one, we consider p' to be **strictly finer** than p and denote this by $p \prec p'$. We call an abelian group G **infinitely generated but of rational rank d** if G is not finitely generated but $G \otimes \mathbb{Q}$ is isomorphic to \mathbb{Q}^d .

Theorem 5.1. *Let Ω be a minimal FLC tiling space of dimension d . Suppose further that $H^1(\Omega, \mathbb{Z})$ contains no subgroup that is infinitely generated but of rational rank d . Then given any bundle projection $\Omega \xrightarrow{p} \mathbb{T}^d$, there is a bundle*

projection $\Omega \xrightarrow{p'} \mathbb{T}^d$ satisfying: $p \preceq p'$ and there is no bundle projection p'' with $p' \prec p''$. In general there is no such maximal element.

Proof. Assume that we have an infinite sequence $p_n : \Omega \rightarrow \mathbb{T}^d$ of bundle projections satisfying $p_n \prec p_{n+1}$ for all n . Consider then the following commutative diagram.

$$\begin{array}{ccccc}
 & & \Omega & & \\
 & \swarrow p_1 & \downarrow p_2 & \searrow p_n & \\
 \mathbb{T}^d & \xleftarrow{\pi_1} & \mathbb{T}^d \dots & \xleftarrow{\pi_n} & \mathbb{T}^d \dots
 \end{array} \tag{4}$$

If we consider the tiling space Ω to have the translational \mathbb{R}^d action given by the suspension construction based on the bundle projection p_1 (which sometimes is and sometimes is not conjugate to the original action, see [5]), then each p_n semi-conjugates the translational action on Ω with the natural translation action of \mathbb{R}^d on \mathbb{T}^d , and the induced map to the inverse limit $\Omega \xrightarrow{p} \lim_{\leftarrow} \{\mathbb{T}^d, \pi_n\}$ semi-conjugates the translation action on Ω with the \mathbb{R}^d action on $\lim_{\leftarrow} \{\mathbb{T}^d, \pi_n\}$ by the commutativity of the diagram. Since the action on $\lim_{\leftarrow} \{\mathbb{T}^d, \pi_n\}$ is equicontinuous, p induces an injection $p^* : H^1(\lim_{\leftarrow} \{\mathbb{T}^d, \pi_n\}, \mathbb{Z}) \rightarrow H^1(\Omega, \mathbb{Z})$, see, e.g., [2]. As $H^1(\lim_{\leftarrow} \{\mathbb{T}^d, \pi_n\}, \mathbb{Z})$ is infinitely generated but of rational rank d , $H^1(\Omega, \mathbb{Z})$ contains a subgroup that is infinitely generated but of rational rank d . \square

If $H^1(\Omega, \mathbb{Z})$ contains a subgroup that is infinitely generated but of rational rank d , there is in general no such maximal element. For example, in substitution tiling spaces arising from substitutions of constant length, or in the chair tilings of the plane, we will have exactly such a sequence of increasingly finer projections.

The theorem does show the existence of a finest bundle projection, for example, for the Penrose tiling space or any Euclidean cut and project tiling space.

There is a second notion of finer for which the notions we have developed here have direct implications. We say that the bundle $\Omega \xrightarrow{p'} \mathbb{T}^d$ is *fiber finer* than the bundle $\Omega \xrightarrow{p} \mathbb{T}^d$ if $p'^{-1}(\mathbf{0}) \subsetneq p^{-1}(\mathbf{0})$. The following is then a corollary to Theorem 1.5.

Theorem 5.2. *Let Ω be a uniquely ergodic FLC tiling space with a minimal \mathbb{R}^d action. If the image of $H^1(\Omega)$ is dense under the RS map, then Ω admits a sequence of bundle projections p_n with p_{n+1} fiber finer than p_n for all n .*

Proof. Consider the sequence of bundle projections p_n corresponding to $\mathbb{R}^d / (n!\mathbb{Z})^d$ as in Theorem 1.5. \square

But by the above, the projections p_n will not generally be related by the relation \prec . One could also consider this result in terms of induced actions. A tiling space with a given bundle structure $\Omega \xrightarrow{p} \mathbb{T}^d$ is also the suspension of a \mathbb{Z}^d action ϕ given by the global holonomy to a fiber F of the projection as discussed in Sect. 3. When the conditions of the theorem are met, for any

given clopen set K of F it follows that one can find an induced \mathbb{Z}^d action $\phi_{K'}$ on some proper clopen $K' \subset K$. This induced action is the analog of the first return map for a flow.

6. Return Equivalence in Tiling Spaces

In [4], the authors developed the notion of return equivalence for matchbox manifolds, a class of foliated spaces including FLC tiling spaces which are generally characterized by having totally disconnected transversals. Two matchbox manifolds $\mathcal{M}_1, \mathcal{M}_2$ are **return equivalent** if given any clopen subsets K_i of transversals of \mathcal{M}_i , there exist clopen subsets $V_i \subset K_i$ such that the induced holonomy pseudogroups on the V_i are isomorphic. Roughly, a pseudogroup on X is a non-empty collection of homeomorphisms between open subsets of X that is closed under inverses, compositions, restrictions, and unions. An isomorphism between a pseudogroup Ψ_X on X and a pseudogroup Ψ_Y on Y is given by a homeomorphism $h : X \rightarrow Y$ such that for each $g \in \Psi_X$, $h \circ g \circ h^{-1} \in \Psi_Y$ and conversely, and this correspondence respects composition. In the case of an FLC tiling space Ω , the induced holonomy pseudogroup on a transversal \mathcal{T} is easier to conceive than is usual since it is generated by all the return maps of the \mathbb{R}^d translation action on Ω to \mathcal{T} . (We refer the reader to [4] for detailed definitions of these concepts in the general case.) In the case of the suspension of a \mathbb{Z}^d action ϕ on a Cantor set C as in Sect. 3, C can be regarded as a transversal, and the induced holonomy pseudogroup on C is the pseudogroup on C generated by homeomorphisms generating ϕ .

In [4], it is shown that if $\mathcal{M}_1, \mathcal{M}_2$ are homeomorphic **minimal** matchbox manifolds (meaning that each leaf is dense), then \mathcal{M}_1 and \mathcal{M}_2 are return equivalent. To see why this should be true in the context of minimal, aperiodic FLC tiling spaces, first consider that a homeomorphism $h : \Omega_1 \rightarrow \Omega_2$ maps local transversals (which we regard as subspaces of the domain of a chart) of Ω_1 to those of Ω_2 and conversely, and similar facts hold for atlases of charts. Thus, it suffices to show that given any two clopen subsets U_i of local transversals \mathcal{T}_i of a single space Ω there are clopen subsets $V_i \subset U_i$ with isomorphic induced holonomy pseudogroups. By aperiodicity and minimality of the translation action $\Phi : \mathbb{R}^d \times \Omega \rightarrow \Omega$, there is a topological disk $D \subset \mathbb{R}^d$ such that for some clopen sets $V_i \subset U_i$ we have for $B := \Phi(D \times V_1)$ that $B \cap V_i = V_i$ and, for each $v \in V_1$, $\Phi(D \times \{v\})$ intersects V_2 in a single point. This yields for each point of V_1 a uniquely paired element of D which Φ maps to a uniquely determined element of V_2 . This correspondence between the elements of V_1 and V_2 is the homeomorphism that yields the isomorphism between the respective holonomy pseudogroups. Similar considerations show that in the case of the suspension of a \mathbb{Z}^d action ϕ on a Cantor set C , any sufficiently small transversal has holonomy pseudogroup isomorphic to the holonomy pseudogroup of a clopen subset of C .

It is also shown in [4] that there are large classes of matchbox manifolds for which return equivalence implies homeomorphism. A key tool in most of

the proofs is the basic fact that if $\mathcal{M}_1, \mathcal{M}_2$ are both bundles over the same closed manifold B with conjugate global holonomy actions on the fiber, then \mathcal{M}_1 and \mathcal{M}_2 are homeomorphic, see, e.g., [3, Theorem 2.1.7]. It is however also determined in [4] that there are return equivalent minimal matchbox manifolds that both have the structure of the total space of a principal bundle over a surface of genus 2 which are return equivalent but not homeomorphic.

The question of whether “return equivalent” implies “homeomorphic” for tiling spaces involves both topological and ergodic assumptions:

Theorem 6.1. *Let Ω_1 be a uniquely ergodic FLC tiling space with a free, minimal \mathbb{R}^d action for which the image of $H^1(\Omega)$ is dense. Then an aperiodic minimal FLC tiling space Ω_2 is return equivalent to Ω_1 if and only if Ω_1 and Ω_2 are homeomorphic.*

Proof. We regard the Ω_i as suspensions of minimal, aperiodic \mathbb{Z}^d actions ϕ_i on Cantor sets C_i . Assuming that the Ω_i are return equivalent, we must show that they are homeomorphic. Let \mathcal{T}_i be given transversals of the Ω_i . As discussed above, we may consider without loss of generality the \mathcal{T}_i to be clopen subsets of C_i with holonomy actions that induced by ϕ_i . By return equivalence, there exist clopen subsets $K_i \subset \mathcal{T}_i \subset C_i$ and a homeomorphism $h : K_1 \rightarrow K_2$ that conjugates the pseudogroups on the K_i induced by ϕ_i . By our hypothesis on Ω_1 and Theorem 1.5 we then know that K_1 has a clopen subset V_1 such that ϕ_1 induces a \mathbb{Z}^d action $\phi_1|_{V_1}$ of V_1 and Ω_1 is the suspension of this restricted action. The induced pseudogroup action on V_1 is then the pseudogroup generated by $\phi_1|_{V_1}$ and h conjugates this \mathbb{Z}^d action with a \mathbb{Z}^d action $\phi_2|_{V_2}$ on $V_2 := h(V_1)$ which in turn is induced by the holonomy action of ϕ_2 on V_2 . Then Ω_2 is homeomorphic to the suspension of $\phi_2|_{V_2}$ and so Ω_1 and Ω_2 are homeomorphic as they are both suspensions over the torus of conjugate actions on a Cantor set. □

The key ingredient in this proof is the existence of arbitrarily small, positive forms. We suspect that a similar result holds under much weaker conditions, but the proof would necessarily involve different ideas due to the lack of bundles over arbitrarily large tori without a small form condition.

7. The Role of Shears

In preparation for a discussion of Conjecture 1.8, we show that the topological eigenvalues of a subshift with shears are restricted.

Theorem 7.1. *Let Ξ be a minimal, and uniquely ergodic \mathbb{Z}^2 subshift, and let Ω be the suspension of Ξ . Let $\lambda = (\lambda_x, \lambda_y)$ be a topological eigenvalue of Ω .*

1. *If Ξ admits a horizontal shear, then $\lambda_x \in \mathbb{Z}$.*
2. *If Ξ admits a vertical shear, then $\lambda_y \in \mathbb{Z}$.*
3. *If Ξ admits both a horizontal shear and a vertical shear, then $\lambda \in \mathbb{Z}^2$.*

Proof. Suppose that λ is a topological eigenvalue with eigenfunction ψ , normalized so that $|\psi| = 1$ everywhere. Since ψ is a continuous function on a

compact space Ω , it is uniformly continuous. So for each $\epsilon > 0$ there exists a $\delta > 0$ such that if two tilings T, T' are δ -close, then $|\psi(T) - \psi(T')| < \epsilon$.

Now suppose that Ξ admits a horizontal shear with function u and corresponding tiling T_1 . Let T_2 be a tiling in Ω that agrees with T_1 on $\mathbb{R} \times [N, \infty)$ and agrees with $T_1 - (1, 0)$ on $\mathbb{R} \times (-\infty, N']$. Pick $\epsilon > 0$ and a constant $K > \delta^{-1} + \max(|N|, |N'|)$.

Since $T_1 - (0, K)$ and $T_2 - (0, K)$ agree on $\mathbb{R} \times [N - K, \infty)$, they are δ -close, and so $\psi(T_1 - (0, K))$ and $\psi(T_2 - (0, K))$ are ϵ -close. By the eigenvalue property, $\psi(T_2 - (0, -K)) = \exp(-4\pi i \lambda_y K) \psi(T_2 - (0, K))$. Since $T_2 - (0, -K)$ and $T_1 - (1, -K)$ agree on $\mathbb{R} \times (\infty, K + N']$, they are δ -close, so $\psi(T_2 - (0, -K))$ and $\psi(T_1 - (1, -K))$ are ϵ -close. Finally, $\psi(T_1 - (1, K))$ equals $\exp(4\pi i \lambda_y K) \psi(T_1 - (1, -K))$ by the eigenvalue property. That is,

$$\begin{aligned} \psi(T_1 - (1, K)) &= \exp(4\pi i \lambda_y K) \psi(T_1 - (1, -K)) \\ &\approx \exp(4\pi i \lambda_y K) \psi(T_2 - (0, -K)) \\ &= \psi(T_2 - (0, K)) \\ &\approx \psi(T_1 - (0, K)). \end{aligned}$$

However, $\psi(T_1 - (1, K)) = \exp(2\pi i \lambda_x) \psi(T_1 - (0, K))$, so $\exp(2\pi i \lambda_x)$ must be 2ϵ -close to 1. Since ϵ was arbitrary, $\exp(2\pi i \lambda_x)$ must be exactly 1, and λ_x must be an integer.

This proves statement (1). Statement (2) is similar, and statement (3) follows from statements (1) and (2). □

It is tempting to try to prove Conjecture 1.8 by modifying the proof of Theorem 7.1 to take into account the effects of the small shape change. Unfortunately, we have not succeeded in this approach, insofar as small shape changes, integrated over large distances, can result in non-negligible displacements.

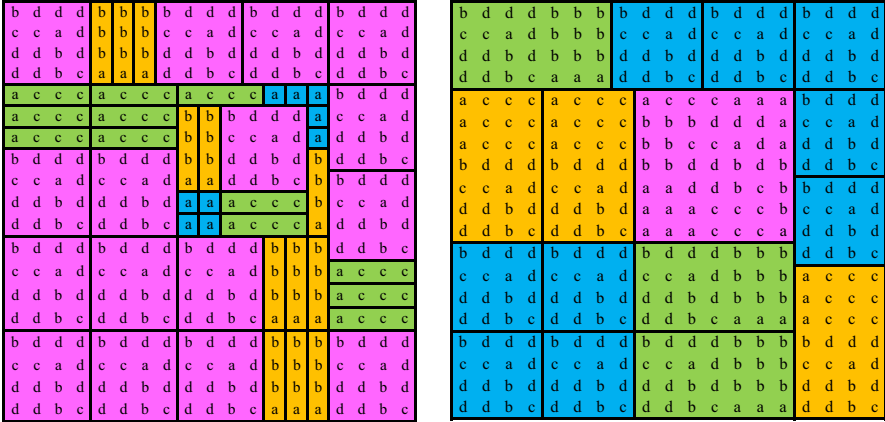
Instead, we study specific examples where we can track the effects of the shears directly. In [8], the authors study tiling spaces with continuous shears (and with infinite local complexity) and show that the shears serve to kill off parts of H^1 . In the following section, we obtain qualitatively similar results for tilings with FLC.

8. The Counterexample

Natalie Frank’s Direct Product Variation (DPV) tiling (see [6, 7] for basic properties and [8, 10] for further development) comes from a fusion rule [9], which can be considered as a generalization of a substitution, yielding a hierarchical structure. In the particular example Ω_F we consider, there are four tiles, a, b, c , and d , which we take to be unit squares in \mathbb{R}^2 . All tilings in Ω_F consist of translates of these four tiles. For the fusion rule used to construct Ω_F , supertiles $P_n(a, b, c, \text{ or } d)$ are defined recursively. The 0-supertiles are just the tiles themselves. For $n \geq 0$, we combine n -supertiles into $n + 1$ -supertiles $P_{n+1}(a, b, c, d)$ by the following combinatorial rule:

$$P_{n+1}(a) = \begin{bmatrix} P_n(b) & P_n(d) & P_n(d) & P_n(d) \\ P_n(c) & P_n(c) & P_n(a) & P_n(d) \\ P_n(d) & P_n(d) & P_n(b) & P_n(d) \\ P_n(d) & P_n(d) & P_n(b) & P_n(c) \end{bmatrix}, \quad P_{n+1}(c) = \begin{bmatrix} P_n(b) \\ P_n(b) \\ P_n(b) \\ P_n(a) \end{bmatrix}, \\
 P_{n+1}(b) = [P_n(a) \quad P_n(c) \quad P_n(c) \quad P_n(c)], \quad P_{n+1}(d) = [P_n(a)]. \quad (5)$$

The 1 and 2-supertiles within $P_3(a)$



Notice that the fusion rule (5) admits an involution in which we swap the vertical and horizontal directions, while swapping the labels b and c at the same time. Thus, many of the facts established for the horizontal direction apply equally to the vertical direction.

By the primitivity of the fusion rule, the tilings in Ω_F consist of those tilings T for which every patch of T is contained in $P_n(a)$ for some $n \geq 0$, where we can replace a with any of the other 3 symbols. Using the standard equivalence between the occurrence of patches with bounded gaps and the minimality of the translation action, this allows one to see the minimality of the translation action directly. Any patch P of a tiling in Ω_F occurs in $P_n(a)$ for some n by construction. Now every $(n + 1)$ -supertile contains at least one copy of $P_n(a)$, and the $(n + 1)$ -supertiles appear with bounded gaps because they have bounded diameters.² Similar arguments as used for showing the freeness of the translation action for substitution tiling spaces can be applied to show that the translation action on Ω_F is free, but this can also be directly inferred directly from the existence of the shears established in Theorem 8.1. That the translation action on Ω_F is uniquely ergodic follows from the general result [9, Cor. 3.10] for fusion tilings.

To see how the shears come about, consider the way that the supertiles are situated within supertiles of larger order. For $n > 0$, the different n -supertiles have different sizes, but the fusion is set up so that things always fit together.

² We are using the fact that every tiling $T \in \Omega_F$ admits a decomposition into $(n + 1)$ -supertiles. Although the uniqueness of such a decomposition is sometimes subtle, the existence follows from the axioms of hierarchical tilings [9].

In fact, $P_n(a)$ and $P_n(d)$ are always squares, with the a -supertile being roughly $(1 + \sqrt{13})/2$ times wider than the d supertile. $P_n(b)$ and $P_n(c)$ are wide and tall rectangles, respectively, with aspect ratios approaching $1 : (1 + \sqrt{13})/2$ as $n \rightarrow \infty$.

However, as seen in the above figure, the n -supertiles within an $(n + 1)$ -supertile do *not* typically meet full-edge to full-edge. They meet with a number of possible offsets. The $(n - 1)$ -supertiles on the boundaries of these n -supertiles meet with a greater number of possible offsets. The $(n - 2)$ -supertiles on the boundaries of *these* $(n - 1)$ -supertiles meet with even more possible offsets.

Theorem 8.1. *The Frank DPV tiling Ω_F admits shears in both directions.*

Proof. Most of this proof is already well known, in particular the fact that it is possible to apply infinitely many shears to a legal tiling and obtain another legal tiling [7]. What is new is proving that shears by *arbitrary* integer distances are possible. We will show this for shears in the horizontal direction. The vertical direction is similar by the symmetry in the fusion rule.

The horizontal shears come about from a mismatch in the one-dimensional dynamics just above and below a horizontal boundary between high-order supertiles. On the north side of the boundary, we have a one-dimensional substitution σ_n that is obtained by looking at the bottom row of each supertile. That is,

$$\sigma_n(a) = ddbc, \quad \sigma_n(b) = accc, \quad \sigma_n(c) = a, \quad \sigma_n(d) = a.$$

Likewise, on the south side of the boundary, we have a substitution σ_s derived from the top row of the two-dimensional supertiles:

$$\sigma_s(a) = bddd, \quad \sigma_s(b) = accc, \quad \sigma_s(c) = b, \quad \sigma_s(d) = a.$$

On both sides of this “fault line,” the n -supertiles come in two widths, namely the entries of $(1, 1)M^n$, where $M = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$. These widths are relatively prime. To see this, note that the matrix M has determinant -3 , and so is invertible over all \mathbb{Z}_p with p a prime other than 3. Since the widths of the basic tiles are not zero (mod p), we cannot have both widths of the n -supertiles divisible by p . Meanwhile, it is easy to check that $(1, 1)M^n = (1, 1) \pmod{3}$, and hence that no supertiles have lengths divisible by 3. Thus, the widths of the n -supertiles share no common factors.

Comparing the one-dimensional substitution dynamics above and below the line, we find regions where there are more wide supertiles above and narrow supertiles below, or vice versa. This difference in population above and below the line (sometimes called the *discrepancy*) is known to be unbounded [7]. This means that the wider n -supertiles above and below the fault line are offset by arbitrary multiples of the narrower width (mod the wider width). Since the two widths are relatively prime, arbitrary offsets between wide n -supertiles are possible. Since the substitution σ_n is primitive, this implies that arbitrary offsets between *any* supertile above, and *any* supertile below, are possible.

Now consider a tiling T where the upper half plane consists of an infinite-order supertile (meaning the union of supertiles of higher and higher order),

and the lower half plane consists of another infinite-order supertile. That it is possible to form such a tiling follows from an application of the Extension Theorem [14, 3.8] where we use patches formed by two supertiles of increasing order which share a boundary along the x -axis. Translating the lower half plane in T sideways by an arbitrary integer yields another legal tiling $T' \in \Omega_F$, since all the patterns of T' near the x -axis consist of northern supertiles meeting southern supertiles with arbitrary offsets, and these in turn are already found elsewhere in the tiling, from which it follows that all patterns in the resulting tiling are legal. \square

Since the tiling just above the fault line is governed by σ_n , it is useful to establish the topology of the one-dimensional tiling space Ω_{σ_n} generated by the substitution σ_n .

Lemma 8.2. $H^1(\Omega_{\sigma_n}; \mathbb{R}) = \mathbb{R}^4$, and the group $H^1_{an}(\Omega_{\sigma_n}; \mathbb{R})$ of asymptotically negligible classes is trivial.

Proof. We compute $H^1(\Omega_{\sigma_n}, \mathbb{R})$ using the methods of Barge and Diamond [1]. Let $A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \end{pmatrix}$ be the substitution matrix of σ_n . Let X be the graph that describes all possible adjacencies between final letters of n -supertiles and beginning letters of the subsequent n -supertile. For $n \geq 3$, there are four possible adjacencies: $a.a$, $a.d$, $c.a$, and $c.d$, so X is isomorphic to a circle. According to [1], the cohomology of Ω_{σ_n} fits into the exact sequence

$$0 \rightarrow \tilde{H}^0(X; \mathbb{R}) \rightarrow \lim(\mathbb{R}^2, A^T) \rightarrow H^1(\Omega_{\sigma_n}; \mathbb{R}) \rightarrow H^1(X; \mathbb{R}) \rightarrow 0.$$

Since $\tilde{H}^0(X; \mathbb{R}) = 0$ and $H^1(X; \mathbb{R}) = \mathbb{R}$, and since A has rank 3, with eigenvalues $(1 \pm \sqrt{13})/2$, -1 and 0 , $H^1(\Omega_{\sigma_n}; \mathbb{R}) = \mathbb{R}^4$. The substitution homeomorphism maps Ω_{σ_n} to itself, and so induces a linear transformation on $H^1(\Omega_{\sigma_n}; \mathbb{R})$. The eigenvalues of this linear transformation are $(1 \pm \sqrt{13})/2$, -1 and 1 , with the 1 coming from the action of substitution on $H^1(X; \mathbb{R})$.

By a theorem of [5], H^1_{an} is the contracting subspace of H^1 under substitution. Since all eigenvalues are of magnitude 1 or greater, H^1_{an} is trivial. \square

Theorem 8.3. For the Frank DPV tiling, the image of $H^1(\Omega_F)$ under the Ruelle–Sullivan map is \mathbb{Z}^2 .

Proof. Suppose that α is a PE differential form representing an integral class in $H^1(\Omega)$, and that the Ruelle–Sullivan map sends $[\alpha]$ to $(\mu, \nu) \in \mathbb{R}^2$. The proof can be broken down into 5 steps.

1. Using the fact that Ω admits horizontal shears, we bound the fluctuations in $\int \alpha$ along different intervals of a fixed horizontal line in a fixed class of tilings.
2. We use properties of Ω_{σ_n} to control the values of α along this horizontal line. In particular, we show that $\alpha = \mu dx + d\beta$ along this line, where β is a strongly PE function.
3. Using the fact that $[\alpha]$ is an integral class, we show that for all sufficiently large values of n , μ times the length of any n -supertile must be an integer.

4. Since the lengths of the n -supertiles are relatively prime, μ must itself be an integer.
5. The same arguments, with the roles of x and y reversed, show that ν is an integer.

Step 1: Let r be the PE radius of α . Pick an integer $N > r$ and another integer n such that the height of the smallest n -supertile is greater than $N + r$. Consider tilings where there is a boundary between infinite-order supertiles on the x axis and where the vertices are located on \mathbb{Z}^2 . Let ℓ_+ and ℓ_- be the horizontal lines $y = N$ and $y = -N$. For any tiling in our class, we consider the integrals $\int_{(0,N)}^{(L,N)} \alpha$ along ℓ_+ and $\int_{(0,-N)}^{(L,-N)} \alpha$ along ℓ_- , where L is an arbitrary positive integer. Since α is closed, the difference between these integrals is the difference between the integrals along vertical paths from $(0, N)$ to $(0, -N)$ and from (L, N) to $(L, -N)$. However, these vertical integrals are of bounded length, independent of L and independent of the tiling in question. Since α is PE, and hence bounded, there is a constant K such that $\int_{(0,N)}^{(L,N)} \alpha$ is always within K of $\int_{(0,-N)}^{(L,-N)} \alpha$, regardless of the tiling in question or the length L .

Since $N > r$, the integrals along ℓ_+ and ℓ_- depend only on the tiling structure in the upper and lower half plane, respectively. Furthermore, since the tiling space admits shears, a given horizontal path along ℓ_+ can be paired with any horizontal path of the same length along ℓ_- . Thus, the integrals along all such paths of length L along ℓ_- must take values within $2K$ of one another, and likewise the integrals along all paths of length L in ℓ_+ . In particular, the integral of α along any path of length L in ℓ_+ must be within $2K$ of $L \cdot$ (average value of α on ℓ_+).

Step 2: Since α is closed, all horizontal lines in a given tiling T give the same average value $\lim_{L \rightarrow \infty} \frac{1}{L} \int_p^{p+(L,0)} \alpha$, and the unique ergodicity of the translation action implies that (the horizontal component of) $RS([\alpha])$ can be computed by averaging (the horizontal component of) $\alpha(x)$ over an arbitrary tiling T and is equal to this common linear average value. Thus, the average value of α on ℓ_+ is exactly μ . We can then write α , restricted to ℓ_+ , as $\mu dx + \alpha_0$, where α_0 has average zero. The results of the previous paragraph imply that the integral of α_0 along any path in ℓ_+ is bounded by $2K$.

Note that ℓ_+ is a row (N from the bottom) of a sequence of n -supertiles at the bottom of an infinite-order supertile. We associate to T a tiling $T_0 \in \Omega_{\sigma_n}$ given by the sequence of n -supertiles lying just above the x -axis. T_0 is a one-dimensional tiling whose tiles have the labels a, b, c , and d and the widths of the corresponding n -supertiles in Ω_F . On T_0 we define a 1-cochain $\tilde{\alpha}_0$ whose value on a tile (say, running from x_1 to x_2) is the integral of α_0 across the corresponding stretch of ℓ_+ (i.e., from (x_1, N) to (x_2, N)). Since ℓ_+ lies a distance r or greater from the top or bottom of these supertiles, this integral depends only on which of the four supertile types we are working and on the identities of the supertile's predecessors and successors to distance r . In particular, $\tilde{\alpha}_0$ is strongly PE.

Since the integral of α_0 along ℓ_+ is bounded, the integral of $\tilde{\alpha}_0$ is bounded, so $\tilde{\alpha}_0$ must represent an asymptotically negligible class in $H^1(\Omega_{\sigma_n})$. However, $H^1_{an}(\Omega_{\sigma_n})$ is trivial. Thus, $\tilde{\alpha}_0$ represents the zero class and is the derivative of a PE function $\tilde{\beta}$ on T_0 . That is, if P_0 is a patch of sufficient length in Ω_{σ_n} (specifically, greater than twice the PE radius of $\tilde{\beta}$), and if this patch occurs at two different places x_1, x_2 , then $\int_{(x_1, N)}^{(x_2, N)} \alpha_0 = \tilde{\beta}(x_2) - \tilde{\beta}(x_1) = 0$, and $\int_{(x_1, N)}^{(x_2, N)} \alpha = \mu(x_2 - x_1)$.

Step 3: Now we use the fact that α is an integral class and is the pullback of a class on an approximant that describes the tiling out to a distance equal to the pattern equivariance radius r . Thus, for any patch of size greater than r , the integral of α from one occurrence of the patch to another occurrence of the same patch must be an integer. In the setting of the previous step, $\mu(x_2 - x_1) \in \mathbb{Z}$.

Step 4: There exists a value of m such that P_0 appears in all m -supertiles of Ω_{σ_n} , and also in all supertiles of order greater than m . Since the word cc appears in the language of σ_n , one can find P_0 in corresponding locations of m -supertiles. That is, we can take $x_2 - x_1$ to be the length of an m -supertile of type c , or the length of an $(m + 1)$ -supertile of type c (which is the same as the length of an m -supertile of type a). Since a tile in Ω_{σ_n} is actually an n -supertile in Ω_F , these are the widths W_c and W_a of $(m + n)$ -supertiles of type c and a , respectively. The upshot is that μW_c and μW_a are integers.

We have already shown that W_c and W_a are relatively prime, so there are integers j and k such that $1 = jW_c + kW_a$. But then $\mu = j(\mu W_c) + k(\mu W_a)$ is an integer.

Step 5: The involution of the fusion rule (5) extends to an involution of Ω_F itself. Let $[\alpha']$ be the pullback of $[\alpha]$ by this involution. The Ruelle–Sullivan map sends $[\alpha']$ to (ν, μ) . The previous arguments, applied to $[\alpha']$, then show that $\nu \in \mathbb{Z}$. □

Theorem 8.3 says that the image of $H^1(\Omega_F)$ under the Ruelle–Sullivan map is \mathbb{Z}^2 . When it comes to the Kellendonk–Putnam and Giordano–Putnam–Skau conjectures, to virtual eigenvalues, and to possible bundle structures, that is the important result. However, it is possible to say more.

Theorem 8.4. $H^1(\Omega_F) = \mathbb{Z}^2$, and is generated by the classes of the constant forms dx and dy .

Proof. We must show that $[\alpha] = \mu[dx] + \nu[dy]$. That is, we must show that, for any two occurrences of any sufficiently large patch P , say at positions (x_1, y_1) and (x_2, y_2) ,

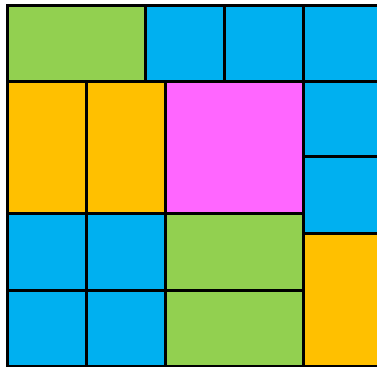
$$\int_{(x_1, y_1)}^{(x_2, y_2)} \alpha = \mu(x_2 - x_1) + \nu(y_2 - y_1). \tag{6}$$

By primitivity, we can restrict attention to the case that P is an n -supertile (for some sufficiently large value of n) of type d .

Our previous arguments show that Eq. (6) holds whenever $y_2 = y_1$, when the path of integration is horizontal, and when all the supertiles that appear on the path are aligned on their bottom edges. Similar arguments, involving the topology of Ω_{σ_s} , apply when the path is horizontal and the supertiles are aligned along their top edges. Likewise, the equation applies when the path is vertical and successive supertiles are aligned on either their right or left edges. To complete the proof, we must show that there is a path from (x_1, y_1) to (x_2, y_2) in which successive supertiles are aligned on one side or another.

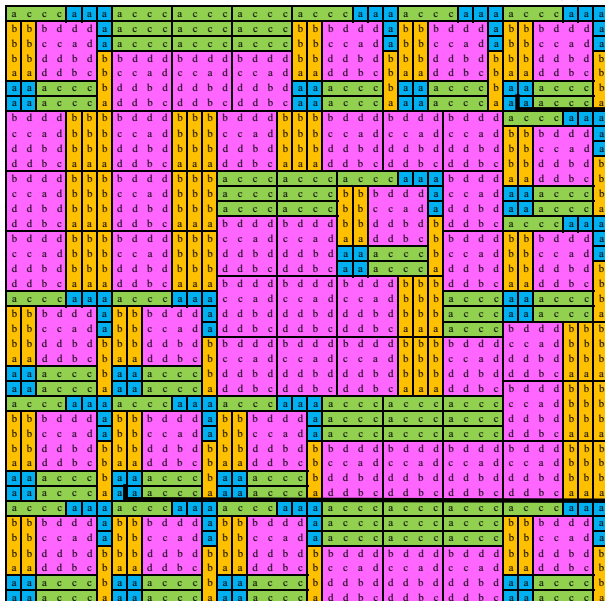
This follows from the structure of the fusion rule (5). All of the n -supertiles in an $(n + 1)$ supertile of type b, c or d are aligned. Within an $(n + 1)$ -supertile of type a , all of the n -supertiles on the top row are aligned, all of the supertiles on the right column are aligned, and the nine remaining supertiles are aligned. Since the upper right n -supertile is aligned with both the right column and the top row, and since the upper left supertile is aligned with both the top row and some of the elements of the lower left 3×3 block, it is possible to get from any n -supertile to any other within the $(n + 1)$ -supertile by following aligned edges.

The n -supertiles within $P_{n+1}(a)$



That $(n + 1)$ -supertile sits inside of an $(n + 2)$ -supertile. Once at the boundary of the $(n + 1)$ -supertile, one can get to any other $(n + 1)$ supertile in the same $(n + 2)$ -supertile by following edges of $(n + 1)$ -supertiles. By induction, if an n -supertile sits inside of an n' -supertile, for any $n' > n$, it is possible to go from the given n -supertile to the boundary of the n' -supertile by following aligned n -supertiles.

A path, in bold, along aligned 1-supertiles within $P_4(a)$



If the two occurrences of P lie in the same n' supertile of some order, it is thus possible to go from either one to a common boundary point, and hence to go from one to the other, only following aligned n -supertiles. On a generic tiling, this is always the case, in that the only infinite-order supertile is the entire plane. Since Eq. (6) applies for every pair of patches in a generic tiling, and since minimality implies that the pattern equivariant cohomology of every tiling is the same, Eq. (6) applies to all pairs of patches in all tilings. \square

It is worth noting three key features of Frank’s DPV tiling that make the proofs of Theorems 8.3 and 8.4 work. These features, or features that work just as well, are to be expected from a large class of FLC tilings of \mathbb{R}^2 with shears in both directions. In fact, there are no known examples of tilings with shears that do *not* have these properties, giving support for Conjecture 1.8.

1. In Frank’s DPV tiling, the one-dimensional tiling space along the lower boundary of supertiles had no asymptotically negligible (AN) classes. This was very convenient, but in cases where AN classes do exist, we can fall back on the following argument:

It is impossible for an integral class to be asymptotically negligible without being trivial, since asymptotically negligible means that integrals are determined locally up to ϵ , while integrality leaves no room for small uncertainty. Thus, it is impossible to have an integral class that is a rational multiple of length plus something asymptotically negligible.

Next, we must show that an integral class cannot be an *irrational* multiple of length plus an AN class. Equivalently, we must show that

we cannot have two integral classes whose images in the so-called mixed cohomology (that is, $H^1 \text{ mod AN}$) are irrational multiples of one another.

This depends on number-theoretic properties of the stretching factor, specifically on the size of the Galois group of its splitting field. We do not know whether our desired property is always true, but it is certainly true whenever the characteristic polynomial has more eigenvalues of magnitude 1 or bigger than of magnitude less than 1, and in particular is true whenever the stretching factor is a non-Pisot quadratic or cubic. Note that having a non-Pisot stretching factor was needed to get shears in the first place [7], and that all constructions to date of tilings with shears have involved quadratic or cubic stretching factors.

2. The lengths of the supertiles have no common factor. Again, this is a necessary condition for having shears. If the lengths of all n -supertiles had a common factor, then the offsets between adjacent n -supertiles would have to be multiples of that common factor.
3. It was possible to find a path from any n -supertile to any other along aligned n -supertiles. This is a general feature of DPV tilings, which are set up to be products of one-dimensional fusions, only with a block (in this case the lower left 3×3 block of $P_{n+1}(a)$) rotated or reflected. The block remains aligned, the rest of the supertile remains aligned, and it is possible to go from the block to the rest at a corner of the block.

In summary, the Frank DPV tiling is exceptionally simple to work with and allows us to prove Theorem 1.1 directly. Other tilings with shears, constructed from DPV's with non-Pisot stretching factors of degree 2 or 3, are likewise counterexamples to the conjectures of Giordano, Putnam, and Skau. The question of whether *all* subshifts with shears are counterexamples (c.f. Conjecture 1.8) remains open.

Example 2. Our main counterexample had shears in both directions and $H^1 = \mathbb{Z}^2$. The following example, taken from [8], is a tiling space with shears in only one direction. It admits small cocycles, but the image of the Ruelle–Sullivan map is not dense.

Consider a two-dimensional tiling with two tile types, a and b , both of which are unit squares. These form the basis of a fusion tiling with the rule

$$P_{n+1}(a) = \begin{bmatrix} P_n(a) & P_n(b) \\ P_n(b) & P_n(a) \end{bmatrix}, \quad P_{n+1}(b) = \begin{bmatrix} P_n(a) & P_n(a) & P_n(a) \\ P_n(a) & P_n(a) & P_n(a) \end{bmatrix}.$$

As with the previous example, there is a horizontal shear.

Restricting to a single row, the substitution $a \rightarrow ab$ (or ba), $b \rightarrow aaa$ has eigenvalues $(1 \pm \sqrt{13})/2$, both of which are bigger than 1. The second eigenvalue controls the discrepancy in how many m -supertiles of type a (versus b) appear on either side of a horizontal boundary between n -supertiles, where $n \gg m$. This discrepancy is unbounded, growing as $|(1 - \sqrt{13})/2|^n$, and the lengths of $P_m(a)$ and $P_m(b)$ are relatively prime, so m -supertiles meet with arbitrary offsets. Taking a limit as $m \rightarrow \infty$ proves that this tiling admits horizontal shears.

Also exactly as before, the horizontal part of a closed PE cochain α has to be an integer multiple of length, plus something exact. This is because the first cohomology of the one-dimensional tiling space with substitution $a \rightarrow ba$, $b \rightarrow aaa$ has no asymptotically negligible classes, insofar as the eigenvalues of $\begin{pmatrix} 1 & 3 \\ 1 & 0 \end{pmatrix}$ are both larger than 1. Thus, if (λ_x, λ_y) is in the image of the Ruelle–Sullivan map, λ_x must be an integer.

However, λ_y can be arbitrarily small. For each n , we can construct a class that essentially counts n -supertiles in the vertical direction. This class can be represented by a 1-cochain α that evaluates to 0 on all horizontal edges, to 1 on the vertical edges of the bottom row of each n -supertile, and to 0 on all other horizontal edges. Since there are no vertical shears, these bottom rows can be identified in a strongly PE manner. The Ruelle–Sullivan map sends this class to $(0, 2^{-n})$.

Since the image of RS contains small elements but is not dense, we can write Ω as a bundle over tori with large volume, but not over tori that are large in both directions.

The corresponding subshift does not admit small, positive cocycles with respect to the obvious coordinates, since $(0, 2^{-n})$ is not a positive vector on the first quadrant. However, if we rotate our axes by 45 degrees (i.e., restrict our \mathbb{Z}^2 action to combinations of $(1, 1)$ and $(1, -1)$), we obtain a \mathbb{Z}^2 action that does admit small, positive cocycles. This implies the existence of small equivalence relations. However, those relations were already manifest from the hierarchical structure of the tiling.

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Communicated by Jean Bellissard.

Received: February 15, 2017.

Accepted: March 6, 2017.