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Anderson Localization for Periodically Driven Systems

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Abstract. We study the persistence of localization for a strongly disordered tight-binding Anderson model on the lattice $\mathbb{Z}^d$, periodically driven on each site. Under two different sets of conditions on the driving, we show that Anderson localization survives if the driving frequency is higher than some threshold value. We discuss the implication of our results for recent development in condensed matter physics, we compare them with the predictions issuing from adiabatic theory, and we comment on the connection with Mott's law, derived within the linear response formalism.

1. Introduction

In this paper, we study the fate of Anderson localization in periodically driven systems. Let H_0 be the tight-binding Anderson Hamiltonian on the lattice $\mathbb{Z}^d$. At strong enough disorder, it is well known that all eigenstates of H_0 are exponentially localized (see [5,6,15] as well as [13] for more references). Let us then consider a periodic time-dependent Hamiltonian of the form

$$H(t) = H_0 + gH_1(t) \tag{1}$$

with $H_1(t) = H_1(t + T)$ for some period T, and with g some coupling constant. We assume that $H_1(t)$ acts everywhere locally: There exists R such that $|(x, H_1(t)y)| = 0$ for all $x, y \in \mathbb{Z}^d$, and all time t, as soon as $|x - y| > R$ (with the notation $(x, Ay) = (\delta_x, A\delta_y) = A(x, y)$ for an operator A).

The time evolution of an initial wave function $\psi(0)$ is governed by the time-dependent Schrödinger equation:

$$i\frac{\mathrm{d}\phi(t)}{\mathrm{d}t} = H(t)\phi(t).$$

The long time properties of the solutions of this equation are best understood through the Floquet eigenstates of $H(t)$ [19,32]. The question addressed in this paper can then be rephrased as follows: Under suitable regularity conditions

on the time dependence of $H_1(t)$, is there a range of values for g and T such that the structure of the eigenfunctions of H_0 is only weakly affected by the periodic potential $H_1(t)$, so that the Floquet eigenstates of $H(t)$ are themselves localized? We answer this question positively in Theorem 1 below, for two different regularity conditions on H_1, leading to different allowed values for g and T.

Localization and Floquet physics. The above question has already received some attention in the mathematical physics community. The connection with the discrete nonlinear Schrödinger equation (DNLS) constituted a first motivation, see [7,28]. In this context, the more general case of a quasi-periodic driving shows up naturally: In a first approximation, the non-linearity in the DNLS equation can be replaced by a quasi-periodic perturbation. On the other hand, in this perspective, it is natural to restrict oneself to spatially localized perturbations $((x, H_1(t)y)$ decays fast as x or y goes to infinity and not only as $|x - y|$ goes to infinity as we consider); indeed, stability results for the DNLS equation all deal with originally localized wave packets.

More recently, periodically driven Hamiltonian systems have been studied intensively in condensed matter theory. For two reasons at least:

First, from a theoretical perspective, driven systems constitute the first examples of dynamics out of equilibrium, lacking even energy conservation. The natural question that arises is whether the system will absorb energy until it reaches an infinite temperature state (i.e., a state with maximal entropy), as it would be the case for a chaotic system, or whether some extensive effectively conserved quantity emerges, forbidding energy absorption after some transient regime [1–4,9,10]. For non-interacting particles on a lattice, as we consider in this paper, this issue becomes trivial and fully independent of the issue of Anderson localization, once the driving frequency becomes higher than the bandwidth of individual particles, see [2,4]. Nevertheless, thanks to the Anderson localization phenomenon, our results guarantee the existence of an effective extensive conserved quantity for frequencies much below this trivial threshold, see Proposition 2 below.

Second, from a more practical point of view, driven systems furnish a way to engineer topological states of matter [25,27]. Though this possibility is not a priori related to the phenomenon of Anderson localization, it turns out that, for interacting many-body systems, localization makes it possible to "lift" phase transitions from the ground state to the full spectrum [20]. This observation is at the heart of very recent investigations of new phases of matter inside the many-body localized phase [22,29,30].

Hence, in view of the increasing role played by localized Floquet systems in modern condensed matter physics, it appeared useful to bring some firm mathematical foundations to the theory of Anderson localization in periodically driven systems, even though the need for mathematical rigor forces us to restrict the setup to non-interacting particles. Results in this direction already appeared in [18], where the localization for some random unitary operators is

established; this question is directly related to ours since the long time evolution of a periodically system is governed by the spectral properties of the unitary $U(T)$, where $U(t)$ solves $idU(t)/dt = H(t)U(t)$. However, for a Hamiltonian as in (1), we do not recover the particular form for U studied in [18].

Before stating our results, we now introduce two more specific aspects that deserved clarification and motivated the present article.

Adiabatic Theory.   Time-dependent Hamiltonian systems varying smoothly and slowly enough with time can be described through the use of adiabatic theory. Here, adapting the analysis from [3], we argue that localization emerges when level crossings in the system become typically non-adiabatic, and we determine the threshold frequency above which this happens.

Let us first remind the theory of the Landau–Zener effect for a time-dependent two-level Hamiltonian $G(t)$ [24,33]. To make the connection with our problem, let us assume that $G(t)$ is of the form $G(t) = PH(t)P$ where P projects on two eigenstates of H_0. Moreover, we assume that $G(t)$ varies smoothly on the scale of one period, i.e., we can write $G(t) = \tilde{G}(\nu t)$ for some smooth 2π-periodic function $\tilde{G}$ and $\nu = \frac{2\pi}{T}$. It is then convenient to move to the basis of the eigenstates of H_0, i.e., the basis where P is diagonal, and to decompose

$$G(t) = G_{\mathrm{dia}}(t) + G_{\mathrm{off}}(t),$$

as a sum of the diagonal and off-diagonal part. We notice that the time-dependent part of $G_{\mathrm{dia}}(t)$ is of order g. We set $(1, G_{\mathrm{off}}(t)2) =: g'$, where g' depends mainly on the distance between the two localization centers of the two states projected on by P and is typically much smaller than g. Finally, we assume that the two levels of PH_0P are close enough (g-close in fact) to each other so that the system undergoes an avoided crossing as time evolves: At some time, the levels of $G_{\mathrm{dia}}(t)$ cross, while $G_{\mathrm{off}}(t)$ induces level repulsion, leading to an avoided crossing for $G(t)$.

If the system is initially (i.e., before the crossing) prepared in an eigenstate of $G_{dia}(t)$, Landau–Zener theory tells us that, after the crossing, the state in which the systems end up depends on the value of

$$\frac{|(1, G_{\mathrm{off}}2)|^2}{v_{12}} \sim \frac{(g')^2}{g\nu}, \tag{2}$$

where v_{12} is the rate of change in the energy of $H_{dia}(t)$ at the crossing. At high frequency, when this value is much smaller than 1, the crossing is non-adiabatic and the system remains in the original state; at intermediate frequency, when this value is of order 1, the system ends up in a superposition of the eigenstates of $G_{\mathrm{dia}}(t)$; and finally at low frequency, when this value is much smaller than 1, the crossing is adiabatic and the system ends up in the other eigenstate of $G_{\mathrm{dia}}(t)$.

The above scenario, valid for a two level systems, may be seen as a caricature of the localization–delocalization transition: Non-adiabatic crossings do not entail hybridization of unperturbed eigenstates, while intermediate and adiabatic crossings, present at low enough frequency, allow the system to move

from one state to the other and constitute a possible mechanism for delocalization. Based on this picture, let us try to determine a critical value of ν above which localization survives. Let us fix g in (1) as well as W characterizing the strength of the disorder. Let us then pick a point $a \in \mathbb{Z}^d$. We first determine a minimal length L^* so that there is typically at least one crossing between the state centered around a and an other state with localization center in a ball of radius L^* around a. Since the probability of finding a crossing in a ball of radius L is of the order of $L^d \frac{g}{W}$, we find

$$L^* \sim \left(\frac{W}{g}\right)^{1/d}.$$

The effective coupling between a state centered around a and a state at a distance L of a, corresponding to g' in (2), is of the order of

$$g' \sim g\mathrm{e}^{-L/\xi},$$

where ξ is the localization length of H_0. Hence, from (2), we find that localization will survive if

$$\frac{g^2 \mathrm{e}^{-2L^*/\xi}}{g\nu} \ll 1 \quad \Leftrightarrow \quad \frac{\nu}{g} \gg \mathrm{e}^{-\frac{2}{\xi}(\frac{W}{g})^{1/d}}. \tag{3}$$

In Theorem 1 below, for a smooth driving (condition (C1)), we prove localization for ν larger than some threshold value comparable to what we obtain in (3). We notice that the Landau–Zener theory proceeds through nonperturbative arguments. Instead, our proof is based on the multi-scale analysis developed in [15], which is mainly a perturbative approach. It is thus somehow remarkable that the same upshot can be recovered in two a priori very different ways.

Finally, we notice that the approach through adiabatic theory outlined above is only expected to work for $H_1(t)$ depending smoothly on time. Unfortunately, both in theoretical and experimental physics works, it is a common protocol to just shift between two Hamiltonians periodically. This leads obviously to a non-smooth time dependence. As we wanted to cover this case as well, we also derived a result for H_1 being only in square integrable in time; see Theorem 1 below with the condition (C2). The lack of smoothness forced us to increase significantly the threshold on ν with respect to (3).

Mott's law. Mott's law asserts that the ac-conductivity of an Anderson insulator behaves as

$$\sigma(\nu) \ \sim \ \nu^2 \big(\log(1/\nu)\big)^{d+1} \quad \text{as } \nu \to 0$$

([26], see also [17] for the case of interacting electrons). An upper bound on $\sigma(\nu)$ was rigorously established in [23] (with $d+1$ replaced by $d+2$). The conductivity $\sigma(\nu)$ is derived within the linear response (LR) formalism; in our setup, this corresponds to fixing ν and taking the limit $g \to 0$ while observing the dynamics over a time of order ν/g^2. In such a regime, the hypotheses of Theorem 1 below are satisfied (we consider a monochromatic perturbation with frequency ν so that condition (C1) holds): The dynamics is localized for

g small enough once ν has been fixed.[1] It may thus come as a surprise that still $\sigma(\nu) > 0$.

This puzzling behavior was recently analyzed in detail for many-body systems in [16]. As it was pointed out to us by [12], the conductivity $\sigma(\nu)$ is computed for a system in equilibrium at zero or finite temperature. Moreover, as can be expected from its definition, for $g > 0$, LR should in general furnish only an accurate description of the dynamics for a transient regime in time of order ν/g^2. It is true though that, for "generic" or "ergodic" systems, it is reasonable to think that the predictions from LR remain valid for much longer timescales: While heating, the system remains approximately in equilibrium and LR can be applied iteratively until the infinite temperature state is reached. This is manifestly not true for localized systems as long as g is small enough compared to ν: The conductivity $\sigma(\nu) > 0$ represents mainly the Rabi oscillation of rare resonant spots ("cat states") in the Hamiltonian H_0, but these oscillations do not need to entail delocalization on the longer timescales described by the Floquet physics.

Organization of the paper. The precise definition of the model studied in this paper together with our results is presented in Sect. 2. The main steps of the proof of our main theorem are contained in Sect. 3, while some more technical intermediate results are shown in Sects. 4 to 6. The two corollaries are shown in Sect. 7. In several places, the proof of our results proceeds through a straightforward adaptation of delicate but well-known methods; as much as possible, we choose to describe in detail only the steps where some significant amount of new material was required.

2. Models and Results

2.1. The Models

We consider a lattice model on $\mathbb{Z}^d$ and we note $|x| = \sup_{i=1..d} |x_i|$. Our results could be of course extended to more general lattices. We are interested in the long time behavior of the Schrödinger equation:

$$i\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = H(t)\phi(t), \tag{4}$$

where the function $\phi(t)$ is defined on $L^2(\mathbb{Z}^d)$ for any t, and the Hamiltonian $H(t)$ is a periodic function with frequency $\nu = 2\pi/T$. The operator $H(t)$ is an idealized version of (1): We move to the basis where H_0 is diagonal and we replace it by an uncorrelated random potential V_ω, while we assume that $H_1(t)$ is still a nearest neighbor hopping (Anderson model):

$$H(t) = -g\Delta(t) + V_\omega. \tag{5}$$

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[1] Strictly speaking, our model does not coincide with that studied in, e.g., [23], as we do not explicitly include an electric field. However, it could be incorporated without affecting our conclusions.

Here $-\Delta(t)$ is Hermitian operator for any t such that $-\Delta(t)(x,y) = 0$ if $|x - y| > 1$ and

$$\| -\Delta(t)(x,y)\|_{L^2([0;T])} \leq 1 \tag{6}$$

for any x, y. We use the notation $-\Delta$ because in the usual time-independent Anderson model, $-\Delta(t)$ is the usual discrete Laplacian on $\ell(\mathbb{Z}^d)$

$$-\Delta\phi(x) = \frac{1}{2d} \sum_{|y-x|=1} \phi(y),$$

There exists a unitary operator $U(t)$, with $U(0) = Id$ such that $\phi(t) = U(t)\phi(0)$ and satisfying

$$i\frac{\mathrm{d}}{\mathrm{d}t}U(t) = H(t)U(t), \tag{7}$$

Existence and uniqueness of solution of (4) and (7) can be proved using a usual fixed point technique.

(RP) Potential regularity. We assume the following form for the random potential which are widely used in the literature:

$$V_\omega = \sum_{x \in \mathbb{Z}^d} v_x \delta_x \tag{8}$$

where v_x are i.i.d. random variables, with a bounded density ρ, such that $\|\rho\|_\infty < \infty$ defined on a bounded support $[-M; M]$. Furthermore, we will assume that the density ρ is piecewise $\mathcal{C}^1$.

The time-dependent term $-g\Delta(t)$ is considered to be a perturbation of order $g \ll 1$, usually referred to as the strong disorder regime. We treat this model in two particular cases.

(C1) Smooth driving. We suppose that $-\Delta(t)(x,y)$ is a monochromatic signal: For any x and y,

$$-\Delta(t)(x,y) = a_{x,y} + b_{x,y}\cos(\nu t) + b'_{x,y}\sin(\nu t) \tag{9}$$

with $a_{x,y} = a_{y,x}, b'_{x,y} = b'_{y,x}$ and $b_{x,y} = b_{y,x}$. In this regime, we are able to prove localization for frequencies ν up to a threshold comparable to the one given in (3). Moreover, we claim that the result can then be extended to a hopping $-\Delta(t)$ with Fourier coefficients that decay fast enough, but we focus on the case of single Fourier mode for simplicity.

(C2) L^2 driving. We only assume (6). In this case, a much larger threshold value for ν is needed, actually $\nu \geq 1$. We refer to [3] for the optimality of this condition.

Remark 1. Between these two extreme cases, one could obviously consider intermediate regularity cases, depending on the decay of the Fourier coefficients of $-\Delta(t)$. This should lead to other conditions on ν that are not investigated in this paper.

2.2. The Floquet Operator

We will work in the Fourier space instead of the time domain, and we denote by $\hat{x} = (x, k)$ a point of $\mathbb{Z}^d \times \mathbb{Z}$. Let us introduce the central object of our paper:

Definition 1. Let

$$\hat{H} = -g\hat{\Delta} + \hat{V}_\omega \tag{10}$$

be a Hamiltonian on $\mathbb{Z}^d \times \mathbb{Z}$, with

$$-\hat{\Delta}\hat{\psi}(x, k) = -\sum_{|y-x|\leq 1}\sum_{k'}\hat{\Delta}_{x,y}(k')\hat{\psi}(y, k - k') \tag{11}$$

where $\hat{\Delta}_{x,y}(k) = \frac{1}{T}\int_0^T \Delta_{x,y}(t)e^{-i\nu kt}dt$ and

$$\hat{V}_\omega = V_\omega + k\nu. \tag{12}$$

In the monochromatic case (C1), the Laplacian $-\hat{\Delta}$ is explicitly given by

$$-\hat{\Delta}\hat{\psi}(x, k) = \sum_{|y-x|\leq 1}\left[a_{x,y}\hat{\psi}(y, k) + \frac{b_{x,y} + ib'_{x,y}}{2}\hat{\psi}(y, k + 1)\right.$$

$$\left. + \frac{b_{x,y} - ib'_{x,y}}{2}\hat{\psi}(y, k - 1))\right]$$

We remark that it is a local operator, meaning it connects only sites $\hat{x}, \hat{y}$ such that $|\hat{x} - \hat{y}| = 1$ in the space Fourier graph $\mathbb{Z}^d \times \mathbb{Z}$. In the general L^2 case (C2), this is no longer true. Indeed, points $(x, k), (y, k')$ could be connected with $|k - k'|$ arbitrary large.

Let $H^1 = \{\hat{f} \in L^2(\mathbb{Z}^d \times \mathbb{Z}) : \sum_{(x,k)\in\mathbb{Z}^d\times\mathbb{Z}} |\hat{f}(x, k)|^2(1 + k^2) < +\infty\}$.

Proposition 1. *The operator $\hat{H}$ is well defined and self-adjoint on $L^2(\mathbb{Z}^d \times \mathbb{Z})$ with domain H^1.*

Proof. The multiplication operator by νk is a self-adjoint operator from $L^2(\mathbb{Z}^d \times \mathbb{Z})$ to itself with domain H^1. Let us show that $\hat{H}$ is also self-adjoint with domain H^1: By Theorem 4.3 in [21], it suffices to show that $H^1 \subset \text{dom}(\hat{\Delta})$ and that there exists a constant $a \in \mathbb{R}$ and $b < 1$ such that $\|g\hat{\Delta}\hat{f}\|_2^2 \leq a\|\hat{f}\|_2^2 + b\|\nu k\hat{f}\|_2^2$ for all $\hat{f} \in H^1$. There exists a constant $C < +\infty$ such that, for any $\hat{f} \in H^1$ and any $s \in]1/2, 1]$,

$$\|\hat{\Delta}\hat{f}\|_2^2 = \sum_{x,k}\left|\sum_{y:|y-x|\leq 1, k'}\hat{\Delta}_{x,y}(k - k')f(y, k')\right|^2$$

$$\leq (2d + 1)\sum_{x,k}\sum_{y:|y-x|\leq 1}\sum_{k'}\frac{|\hat{\Delta}_{x,y}(k - k')|^2}{(1 + (k')^2)^s}\sum_{k''}|f(y, k'')|^2(1 + (k'')^2)^s$$

$$\leq C\sum_{x,k}|f(x, k)|^2(1 + k^2)^s.$$

Therefore, $\hat{f} \in \text{dom}(\hat{\Delta})$. Let us now fix $1/2 < s < 1$. For any $0 < b' < 1$, there exists $a' \in \mathbb{R}$, such that $(1 + k^2)^s \leq a' + b'k^2$ for all $k \in \mathbb{Z}$. Therefore, taking b'

small enough, one gets $\|g\hat{\Delta}\hat{f}\|_2^2 \le a\|\hat{f}\|_2^2 + b\|\nu k\hat{f}\|_2^2$ for some $a \in \mathbb{R}$ and some $b < 1$. $\qquad\square$

The new Hamiltonian $\hat{H}$ gives the evolution of the "finite time Fourier series" of $\phi(t)$ defined as follows

$$\check{\phi}(x,k,t) = \frac{1}{T}\int_t^{t+T}\phi(x,u)e^{-i\nu ku}du. \tag{13}$$

We get formally a time-independent Schrödinger equation governed by the Hamiltonian $\hat{H}$:

Proposition 2.

$$i\partial_t\check{\phi}(x,k,t) = \hat{H}\check{\phi}(x,k,t) \tag{14}$$

Proof.

$$
\begin{aligned}
i\partial_t\check{\phi}(x,k,t) &= \frac{1}{T}\int_t^{t+T} i\partial_u\big[\phi(x,u)e^{-i\nu ku}\big]du \\
&= \frac{1}{T}\int_t^{t+T}\big(k\nu + H(u)\big)\phi(x,u)e^{-i\nu ku}du \\
&= \frac{1}{T}\int_t^{t+T}(k\nu + V_\omega)\phi(x,u)e^{-i\nu k} \\
&\quad + g\sum_{|y-x|\le 1}\sum_{k'}(-\hat{\Delta}_{x,y}(k'))\phi(y,u)e^{-i\nu(k-k')u}du \\
&= (V_\omega + k\nu)\check{\phi}(x,k,t) + g\sum_{|y-x|\le 1}\sum_{k'}(-\hat{\Delta}_{x,y}(k'))\check{\phi}(y,k-k',t) \\
&= \hat{H}\check{\phi}(x,k,t).
\end{aligned}
$$

$\qquad\square$

The time evolution of $\check{\phi}$ is deduced from the eigenvectors of $\hat{H}$:

$$\bar{\lambda}\hat{\psi} = \big(-g\hat{\Delta} + \hat{V}_\omega\big)\hat{\psi} \tag{15}$$

Looking for the eigenvectors of $\hat{H}$ is equivalent to the search of solution of the form $\phi(t) = e^{i\bar{\lambda}t}\psi(t)$ with ψ a T-periodic function (Floquet theory). Indeed, in the Fourier variables, (4) is equivalent to (15). In particular, as we will see, localization for $\hat{H}$ implies the absence of diffusion for ϕ.

Remark 2. Because $\psi(t)e^{i\bar{\lambda}t} = \psi(t)e^{-in\nu t}e^{i(n\nu+\bar{\lambda})t}$, if $(\psi,\bar{\lambda})$ is a solution then $(\psi(t)e^{-in\nu t}, n\nu + \bar{\lambda})$ is a solution as well for any $n \in \mathbb{Z}$. Hence, it is enough to consider the case $\bar{\lambda} \in [0;\nu]$.

2.3. Results

Our main theorem states Anderson localization for $\hat{H}$.

Theorem 1. *There exists $\epsilon > 0$ such that, if $g < \epsilon$, and if $\nu \ge e^{-g^{-\frac{1}{4p+8d}}}$ for some $p > 2d$ under the condition (C1), or if $\nu \ge 1$ under the condition (C2), then $\hat{H}$ exhibits localization: Its spectrum is pure point and its eigenvectors decay exponentially in space, $\mathbb{P}$ a.s.*

Remark 3. Under (C1), we will see that the eigenvectors are also deterministically exponentially localized along the frequency axis.

In the above theorem, the constant ϵ depends on M (the size of the support of the distribution of the disorder). However, for a uniform distribution on $[-M, M]$, one best replaces g by g/M and ν by ν/M everywhere; doing so, the value of ϵ becomes a dimensionless parameter that depends only on the dimension d. As a price to pay for mathematical rigor, we do not obtain good numerical estimates on ϵ. Nevertheless, it is worth pointing out that our estimate for ϵ is not worth than what is obtained for the most usual time-independent Anderson model in the tight-binding approximation, see, e.g., [13]. Indeed, in the time-independent case, the initialization of the multi-scale analysis requires to fix some large enough length scale $L_0 \sim 10$, and then take $\epsilon \sim L_0^{-8d}$. In our case, when the frequency ν goes to 0, the size of L_0 diverges. However, the constraint $g \leq [-\log(\nu)]^{-4p-8d}$ ensures precisely that no change in ϵ is needed.

In dimension $d = 1, 2$, we expect that the spectrum of $\hat{H}$ is actually exponentially localized at any value of g, ν. For the time-independent Anderson model, this is known to be the case in $d = 1$ (through the use of the transfer matrix formalism), and it is conjectured to be true in $d = 2$. The presence of a periodic dependence on time does not truly modify the dimensionality of the model: In the (x, k)-coordinates, the dimension becomes $d+1$ but the νk term forces eigenstates to be localized in a band of width $1/\nu$ in the k direction. Obviously, in $d > 2$, we expect mobility edges to appear at moderate disorder strengths.

The two following corollaries do not logically follow from Theorem 1, but rather from a refinement of its proof. The first one shows the absence of diffusion for solutions of (4) (dynamical localization):

Corollary 1. *There exist $\epsilon > 0$ and $q > 0$ (and one may take $q \to \infty$ as $\epsilon \to 0$) such that, if $g < \epsilon$ and $\nu \geq e^{-g^{-\frac{1}{4p+8d}}}$ for some $p > 2d$ under (C1), or $\nu \leq 1$ under (C2), then*

$$\mathbb{E}\left(\sup_{t>0} \sum_{x \in \mathbb{Z}^d} |x|^q |\phi(x,t)|^2\right) < \infty \qquad (16)$$

for any solution $\phi(x, t)$ of (4) with initial condition $\phi(x, 0)$ defined on a bounded support.

The second one deals with the existence of a local effective Hamiltonian, i.e., an Hamiltonian H_{eff} such that

$$U(T) = e^{-iT H_{eff}}$$

and such that $H_{eff}(x, y)$ decays fast as $|x - y| \to \infty$. Under the conditions of Theorem 1, given $\bar{\lambda} \in [0, \nu[$ and a corresponding eigenfunction $\hat{\psi}_{\bar{\lambda}}$ of $\hat{H}$, and given $t \in \mathbb{R}$, let us denote by $P_{\psi_{\bar{\lambda}}(\cdot, t)}$ the projector

$$L^2(\mathbb{Z}^d) \to L^2(\mathbb{Z}^d), \quad f \mapsto \left(\psi_{\bar{\lambda}}(\cdot, 0), f\right) \psi_{\bar{\lambda}}(\cdot, t)$$

(since $\hat{\psi}_{\bar{\lambda}} \in H^1$ by Proposition 1, $\psi_{\bar{\lambda}}$ is continuous in time so that the above projector is well defined for all $t \in \mathbb{R}$). The representation

$$U(t) = \sum_{\bar{\lambda} \in [0,\nu[} e^{-i\bar{\lambda}t} P_{\psi_{\bar{\lambda}}(\cdot,t)}$$

holds. Hence, since the functions $\psi_{\bar{\lambda}}$ are T-periodic in time, we may set

$$H_{eff} = \sum_{\bar{\lambda} \in [0,\nu[} \bar{\lambda} P_{\psi_{\bar{\lambda}}(\cdot,0)}, \tag{17}$$

which defines an operator on $L^2(\mathbb{Z}^d)$. Under condition (C1), we have more:[2]

Corollary 2. *There exist $\epsilon > 0$ and $q > 0$ (and one may take $q \to \infty$ as $\epsilon \to 0$) such that, if $g < \epsilon$, $\nu \geq e^{-g^{-\frac{1}{4p+8d}}}$ for some $p > 2d$, and under condition (C1), then*

$$\mathbb{E}\big(|x-y|^q |H_{eff}(x,y)|\big) < \infty.$$

with H_{eff} as defined by (17).

3. Proof of Theorem 1

We will prove that the Hamiltonian $\hat{H}$ reveals localization by applying the classical tools of the multi-scale analysis (MSA). Thanks to the huge literature on MSA, it we will be enough for us to prove a probability estimate, usually referred to as Wegner estimate, and the initialization of the MSA to show the localization (as well as some extra technical results when dealing with the L^2 case, i.e., under assumption (C2)).

We start with the Wegner estimate. Below we call columns sets of the form $\Lambda_0 \times I \subset \mathbb{Z}^d \times \mathbb{Z}$, for some finite spatial box Λ_0 and some frequency interval I. Given $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$ and given $H \in L^2(\mathbb{Z}^d \times \mathbb{Z})$, we denote by $H_{|\Lambda}$ the operator acting on $L^2(\Lambda)$ such that $H_{|\Lambda}(\hat{x},\hat{y}) = H(\hat{x},\hat{y})$ for all $\hat{x}, \hat{y} \in \Lambda$.

Proposition 3 (Wegner estimate). *Let $\Lambda_0 \subset \mathbb{Z}^d$ be finite. Then*

1. *(The finite column case) For any $K \in \mathbb{N}, k_0 \in \mathbb{Z}$ so that $\Lambda_0 \times [k_0 - K; k_0 + K] \subset \mathbb{Z}^d \times \mathbb{Z}$, we have*

$$\forall E, \mathbb{P}(\exists \bar{\lambda} \text{ eigenvalue of } \hat{H}_{|\Lambda_0 \times [k_0-K;k_0+K]} : \bar{\lambda} \in [E-\epsilon, E+\epsilon])$$
$$\leq 2\pi\epsilon(2K+1)|\Lambda_0|\|\rho\|_\infty. \tag{18}$$

2. *(The infinite column case) There exists a constant C which depends only on $\|\rho\|_{L^\infty}$ and $\|\rho'\|_{L^\infty}$, such that for $\Lambda_0 \times \mathbb{Z} \subset \mathbb{Z}^d \times \mathbb{Z}$, we also have*

$$\mathbb{P}(\exists \bar{\lambda} \text{ eigenvalue of } \hat{H}_{|\Lambda_0 \times \mathbb{Z}} : \bar{\lambda} \in [E-\epsilon, E+\epsilon])$$
$$\leq 2\pi\sqrt{\epsilon}|\Lambda_0|\|\rho\|_\infty \max\left(1, \frac{M}{\nu}\right). \tag{19}$$

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[2] The result would be of little interest under condition (C2), since at high frequency, the existence of a local effective Hamiltonian follows from much more general considerations, see [4].

The proof of this proposition will be carried over in Sect. 4. Part 1 will be needed to establish Theorem 1 under the assumption (C1) and Part 2 under the assumption (C2). The crucial property that allows to show the second part of this proposition is contained in Remark 2: If $\hat{\psi}(x, k)$ is an eigenvector with eigenvalue $\bar{\lambda}$ of $\hat{H}_{|\Lambda_0 \times \mathbb{Z}}$, then $\hat{\psi}(x, k - k_0)$ is also an eigenvector with eigenvalue $\bar{\lambda} + \nu k_0$ for any $k_0 \in \mathbb{Z}$. Therefore, the eigenvalues are of the form $\{\bar{\lambda}_i : i = 1, \ldots, |\Lambda_0|\} + \nu \mathbb{Z}$, allowing to use $|\Lambda_0|$ in the RHS of (19) instead of the cardinal of the column which in this case is infinite.

The second ingredient in the MSA consists in proving the exponential decay of the resolvent $(\hat{H} - \lambda)^{-1}$ with high probability for a given $\lambda \in \mathbb{R}$. We will follow [13]. To initialize the MSA, we need to show that, given a point $\hat{x} \in \mathbb{Z}^d \times \mathbb{Z}$, there exists with high probability a finite domain around $\hat{x}$, called "good box," where the resolvent decay exponentially. From now on we fix some $\lambda \in [0, \nu]$. Indeed, it is enough to consider values of λ in this interval, because of the symmetry described in Remark 2.

For $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$, we will write

$$\partial^{in} \Lambda = \{\hat{x} \in \Lambda : \exists \hat{y} \notin \Lambda, \hat{\Delta}(\hat{x}, \hat{y}) \neq 0\} \tag{20}$$

$$\partial^{ext} \Lambda = \{\hat{x} \notin \Lambda : \exists \hat{y} \in \Lambda, \hat{\Delta}(\hat{x}, \hat{y}) \neq 0\} \tag{21}$$

3.1. Smooth Driving (C1)

Definition 2 (*Good box*). Under the assumption (C1), we say that $(x + [-L, L]^d) \times [k_0 - K, k_0 + K]$ is a μ-good box, for some $\mu > 0$, if, for any $(y, k) \in \partial^{in}\left(x + [-L, L]^d\right) \times [k_0 - K, k_0 + K]\right)$,

$$|((x, k_0), (\hat{H}_{|(x+[-L,L]^d) \times [k_0-K,k_0+K]} - \lambda)^{-1}(y, k))| \leq e^{-\mu|(x,k_1)-(y,k_2)|} \tag{22}$$

where $|(x, k_1) - (y, k_2)| = |k_2 - k_1| + \sum_{i=1}^{d} |x_i - y_i|$.

The difference between our model and the classical Anderson model is the absence of independence along the frequency axis. However, we have the following proposition.

Proposition 4. *If* $|k_0| > \frac{M + \sqrt{g}}{\nu} + K$ *then for any* $\Lambda_0 \subset \mathbb{Z}^d, \Lambda_0 \times [k_0 - K; k_0 + K]$ *is a* $-\ln(2(d+1)g)$ *good box.*

The proof of this proposition will appear as a simple case of the proof of Proposition 5 below (see Sect. 5 after the proof of Proposition 7). Thanks to this proposition, it is now enough then to study boxes close to the $k = 0$ axis. Once we restrict ourselves to such boxes, non-intersecting boxes are stochastically independent, and we can proceed with the usual MSA approach. So the idea of the proof is to show initialization of the MSA for boxes like $\Lambda_0 \times [-\frac{2(M+\sqrt{g})}{\nu}; \frac{2(M+\sqrt{g})}{\nu}]$.

Remark 4. For any $x \in \mathbb{Z}^d$, there exists k such that $|\hat{V}(x, k) - \lambda| \leq \nu$

Hence, there is no way avoiding a resonance of order ν for all x, and we cannot look for good boxes as free of any resonances. Nevertheless, we prove that good boxes appear with high probability when $g \ll 1$. Let $p > d$.

Proposition 5 (Initialization of the MSA under the assumption (C1)). *Assume that (C1) holds. For any $\mu > 0, L^* \in \mathbb{N}$, there exist $\epsilon > 0$ and $L \geq L^*$ such that for any $g < \epsilon$, such that if $\nu > \exp(-\frac{1}{g^{\frac{1}{8d+4p}}})$ then*

$$\mathbb{P}(B_L(x) \text{ is a } \mu\text{-goog box}) > 1 - \frac{1}{L^{2p}} \qquad (23)$$

where $B_L(x) = x + [-L; L]^d \times [-\frac{M}{\nu}; \frac{M}{\nu}]$.

The proof of this proposition will be carried over in Sect. 5. For the usual Anderson model, Theorem 1 would follow from (see Theorem 8.3 in [13]):

1. MSA initialization (Theorem 11.1 in [13]),
2. Wegner estimate (Theorem 5.23 in [13]),
3. Independence of these two properties for two distinct boxes (obvious in the usual model).

As already said, the only peculiarity of our model under assumption (C1) is the special form of the potential. In our case, it will thus be enough to prove

1. MSA initialization: Proposition 5,
2. Wegner estimate: Eq. (18) in Proposition 3,
3. Independence: Proposition 4.

Theorem 1 is then obtained as Theorem 8.3 in [13].

3.2. L^2 Driving (C2)

A new problem appears here: For which distance on $\mathbb{Z}^d \times \mathbb{Z}$ should we prove the exponential decay? In the smooth case, $\hat{\Delta}$ was a local operator, so the usual distance on works fine. But because $g(k' - k)$ is nonzero for $k - k'$ large if the driving is only in $L^2([0, T])$, the operator $\hat{\Delta}$ connects now points $(\hat{x}, \hat{y})$ that are not close to each other in $\mathbb{Z}^d \times \mathbb{Z}$ and there is no exponential decay along the frequency k. In order to prove exponential decay on $\mathbb{Z}^d$, we introduce a new decay function on $\mathbb{Z}^d \times \mathbb{Z}$, which can actually easily be used in the "random walk expansion" that appears in the MSA.

Definition 3. Let $G : (\mathbb{Z}^d \times \mathbb{Z})^2 \to \mathbb{R}$ such that for all any $\hat{x}_0 \in \mathbb{Z}^d \times \mathbb{Z}, G(\hat{x}_0, .) \in L^1(\mathbb{Z}^d \times \mathbb{Z})$ with $\|G(\hat{x}_0, .)\|_{L^1} < 1/2$. We define the decay function d_G by

$$d_G(\hat{x}, \hat{y}) = -\ln \left(\sum_{\mathcal{C}(\hat{x} \to \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) \qquad (24)$$

for any $\hat{x}, \hat{y} \in \mathbb{Z}^d \times \mathbb{Z}$ if $\hat{x} \neq \hat{y}$, and 0 otherwise, where $\mathcal{C}(\hat{x} \to \hat{y})$ is the set of all finite sequences of the type $(\hat{x} = \hat{z}_0, \hat{z}_1, \hat{z}_2, \dots, \hat{z}_k = \hat{y})$ (or "paths" from $\hat{x}$ to $\hat{y}$).

Let $P : \mathbb{Z}^d \times \mathbb{Z} \to \mathbb{R}$ be defined by

$$P((x, k)) = \begin{cases} 1/\sqrt{g} & \text{if } k\nu \in [-M - \sqrt{g}, M + \sqrt{g}], \\ \frac{1}{\nu(|k| - 1) - M} & \text{if } k\nu \notin [-M - \sqrt{g}, M + \sqrt{g}]. \end{cases}$$

We say that $\hat{x}$ is a resonant site if $|\hat{V}_\omega(\hat{x}) - \lambda| < \sqrt{g}$. We have defined the function $P(\hat{x})$ such that if there is no resonant site on $x \times \mathbb{Z}$, then $P(\hat{x}) > \frac{1}{|\hat{V}_\omega(\hat{x}) - \lambda|}$.

Definition 4. Under assumption (C2), we say that $C_L(x) = (x + [-L, L]^d) \times \mathbb{Z} \subset \mathbb{Z}^d \times \mathbb{Z}$ is a μ-good column if there exists a decay function $\tilde{d}_G$ such that

$$|(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})| \leq P(\hat{x})e^{-\tilde{d}_G(\hat{x},\hat{y})}$$

for all $\hat{y} \in \partial^{in}C_L(x)$, and such that

$$\sum_{\hat{y} \in \partial^{in}C_L(x)} e^{-\tilde{d}_G(\hat{x},\hat{y})} < e^{-\mu L}.$$

Proposition 6 (Initialization of the MSA under the assumption (C2)). *Assume that (C2) holds. For any $\mu > 0, L^* \in \mathbb{N}$, there exist $\epsilon > 0$ and $L \geq L^*$ such that for any $g < \epsilon$, such that if $\nu > 1$ then*

$$\mathbb{P}\big(C_L(x) \text{ is a } \mu\text{-good column }\big) > 1 - \frac{1}{L^{2p}}. \tag{25}$$

As in the smooth case (C1), Theorem 1 will follow from the Wegner estimate (Eq. (19) in Proposition 3) the initialization of the MSA (Proposition 6), and the stochastic independence of distinct columns (obvious here). But there is still one difference: The MSA has to be performed with infinite columns. This issue will be addressed in Sect. 6.4, where we explain the technical adaptations to perform in the proof in [13].

4. Wegner Estimate

In this section, we prove Proposition 3 (Wegner estimate). For (18) (finite column), we closely follow [31], while for (19) (infinite column), we follow [14] (see also [8]). Thanks to the resolvent formula, we have the Shur formula: For any P projector and $B = PBP$,

$$P(A + B)^{-1}P = ((PA^{-1}P)^{-1} + B)^{-1} \tag{26}$$

where the two last "$\cdot^{-1}$" in the right-hand side correspond to the inverse for operators restricted to $Im(P)$.

Proof of (18). We follow the proof from [13]. Let $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}, E \in \mathbb{R}$. Let $P_x, x \in \mathbb{Z}^d$ the projectors on the subspace $\{x\} \times [k_0 - K, k_0 + K]$ and $\Lambda_0 \subset \mathbb{Z}^d$ the projection of Λ on its first parameters.

$$\mathbb{P}\left(\exists \bar{\lambda} \text{ eigenvalue of } \hat{H}_{|\Lambda} : \bar{\lambda} \in [E - \epsilon, E + \epsilon]\right)$$

$$\leq \mathbb{E}\left(\text{Tr}(1_{[E-\epsilon,E+\epsilon]}(\hat{H}_{|\Lambda}))\right)$$

$$\leq \mathbb{E}\left(2\epsilon\Im(\text{Tr}(\hat{H}_{|\Lambda} - E - i\epsilon)^{-1})\right)$$

$$= \mathbb{E}\left[2\epsilon\Im\big(\sum_{x \in \Lambda_0} \text{Tr}\left(P_x(\hat{H}_{|\Lambda} - E - i\epsilon)^{-1}P_x)\right)\right]$$

$$= 2\epsilon\Im\left(\sum_{x\in\Lambda_0}\mathbb{E}\left[\mathrm{Tr}\left(((P_x(\hat{H}_{|\Lambda}-v_xP_x-E-i\epsilon)^{-1}P_x)^{-1}+v_xP_x)^{-1})\right]\right)$$

$$= 2\epsilon\sum_{x\in\Lambda_0}\mathbb{E}_{V_y:y\neq x}$$

$$\times\left[\int\Im\left(\mathrm{Tr}\left(((P_x(\hat{H}_{|\Lambda}-v_xP_x-E-i\epsilon)^{-1}P_x)^{-1}+v_xP_x)^{-1})\rho(x)\mathrm{d}v_x\right)\right]$$

$$= 2\epsilon\sum_{x\in\Lambda_0}\mathbb{E}_{V_y:y\neq x}$$

$$\times\left[\int\left(\sum_{\mu_i\in\sigma((P_x(\hat{H}_{|\Lambda}-v_xP_x-E-i\epsilon)^{-1}P_x)^{-1})}\Im\left((\mu_i+v_x)^{-1}\right)\rho(x)\mathrm{d}v_x\right)\right]$$

$$\leq 2\epsilon\sum_{x\in\Lambda_0}\mathbb{E}_{V_y:y\neq x}\left(\pi\|\rho\|_\infty(2K+1)\right)$$

$$\leq 2\pi\epsilon(2K+1)|\Lambda_0|\|\rho\|_\infty,$$

where, to get the last equality, we used that P_x acts as the identity on the subspace generated by P_x.   □

Proof of (19). Let Λ_0 be a finite subset of $\mathbb{Z}^d$. We make a change of variable for the potential $\alpha = \frac{1}{|\Lambda_0|}\sum_{x\in\Lambda_0}V_\omega(x)$. As in [14] (see also [8]), the conditional probability of α knowing $\tilde{V}(x) = V_\omega(x) - \alpha$ for all $x \in \Lambda_0$, admits a density $\xi_{\tilde{V}}(\alpha)$ and there exists a constant C such that, on a set U belonging to the sigma-algebra generated by $\tilde{V}(x)$ for all $x \in \Lambda_0$, and with probability larger that $1 - C\sqrt{\epsilon}$,

$$\|\xi_{\tilde{V}}\|_\infty \leq \frac{C}{\sqrt{\epsilon}}\left((2M)^{1/2}\|\rho\|_\infty + (2M)^{3/2}\|\rho'\|_\infty\right) \tag{27}$$

Because of the symmetry described in Remark 2, for any realization $(\tilde{V}, \alpha_0)$, there exist $\bar{\lambda}_1, .., \bar{\lambda}_{|\Lambda_0|} \in [0, \nu]$ such that $\sigma(\hat{H}_{\tilde{V},\alpha_0}) = \{\bar{\lambda}_1, \ldots, \bar{\lambda}_{|\Lambda_0|}\} + \nu\mathbb{Z}$. Now, keeping $\tilde{V}$ fixed and changing α, one gets $\sigma(\hat{H}_{\tilde{V},\alpha}) = \{\bar{\lambda}_1 + (\alpha - \alpha_0), \ldots, \bar{\lambda}_{|\Lambda_0|} + (\alpha - \alpha_0)\} + \nu\mathbb{Z}$. Then, for any $E \in \mathbb{R}$,

$$\mathbb{P}(d(\sigma(\hat{H}), E) < \epsilon)$$

$$\leq C\sqrt{\epsilon} + \mathbb{P}(\{d(\sigma(\hat{H}), E) < \epsilon\} \cap U)$$

$$\leq C\sqrt{\epsilon} + \mathbb{E}_{\tilde{V}}\left(\mathbb{1}_U \sum_{i=1}^{\Lambda_0}\sum_{k\in\mathbb{Z}}\int\mathbb{1}(|\bar{\lambda}_i + k\nu + (\alpha - \alpha_0) - E| < \epsilon)\xi_{\tilde{V}}(\alpha)d\alpha\right)$$

$$\leq C\sqrt{\epsilon} + 2\epsilon\frac{1}{\sqrt{\epsilon}}C((2M)^{1/2}\|\rho\|_\infty + (2M)^{3/2}\|\rho'\|_\infty)K_0$$

where K_0 is the maximum number of eigenvalue $\bar{\lambda}$ in $\sigma(\hat{H}_{\tilde{V},\alpha_0})$ such that there exists $\alpha \in [-M, M]$ such that $|\bar{\lambda} + \alpha - \alpha_0 - E| < \epsilon$ with nonzero probability. In particular, we have $K_0 \leq 2|\Lambda_0|\frac{M}{\nu} + 1$.   □

5. Smooth Driving (C1)

Proof of Proposition 5. Proposition 5 is deduced from Propositions 7 and 8 below.   □

The key tool for the MSA is the following formula:

$$(\hat{v}_0, (\hat{H} - \lambda)^{-1}\hat{z}) = \sum_{\hat{u}\in\partial^{in}\Lambda, \hat{v}\in\partial^{ext}\Lambda} (\hat{v}_0, (\hat{H}_{|\Lambda} - \lambda)^{-1}\hat{u})(\hat{u}, g\hat{\Delta}\hat{v})(\hat{v}, (\hat{H} - \lambda)^{-1}\hat{z})$$

(28)

for any $\hat{v}_0 \in \Lambda$ and $\hat{z} \notin \Lambda$, and $\Lambda \subset \mathbb{Z}^d \times \mathbb{Z}$ with $z \notin \Lambda$, which is a direct application of the well-known resolvent formula. We will repeat it as many times as we can, replacing v for v_0 and choosing correctly the new Λ. The next subsection deals with this question.

5.1. Resonant Sites, Security Box and Propagation Decay

Remind that $\hat{v} = (x, k) \in \mathbb{Z}^d \times \mathbb{Z}$ is a resonant site if $|\hat{V}_\omega(\hat{v}) - \lambda| = |v_x + \nu k - \lambda| < \sqrt{g}$. Obviously, for any x there exits a segment $K_x \subset \mathbb{Z}$ so that (x, k) is a resonant site for $k \in K_x$, where K_x is of the form $K_x = \mathbb{Z} \cap [k_0 - \sqrt{g}/\nu, k_0 + \sqrt{g}/\nu]$ for some k_0 that depends on $V_\omega(x)$ (see Fig. 1). Around each segment of resonant sites K_x, we define a security box $\Lambda_{K_x} = \{z \in \mathbb{Z}^d \times \mathbb{Z} : d(z, K_x) < N\}$, where N is an integer that will be defined later and d is the usual graph distance on $\mathbb{Z}^d \times \mathbb{Z}$.

We will say that a set of the form $\Lambda_0 \times I \subset \mathbb{Z}^d \times \mathbb{Z}$ is not strongly resonant if $d(\sigma(\hat{H}_{|\Lambda_0 \times I}), \lambda) > \nu^2 \alpha(g)$, where $\alpha(g)$ is a function which will be defined at the end of the proof of Proposition 8 below.

Proposition 7. *Let $L \in \mathbb{N}$. If no security boxes intersect, if no security box is strongly resonant, and if $(x + [-L, L]^d) \times \mathbb{Z}$ is not strongly resonant, then for*
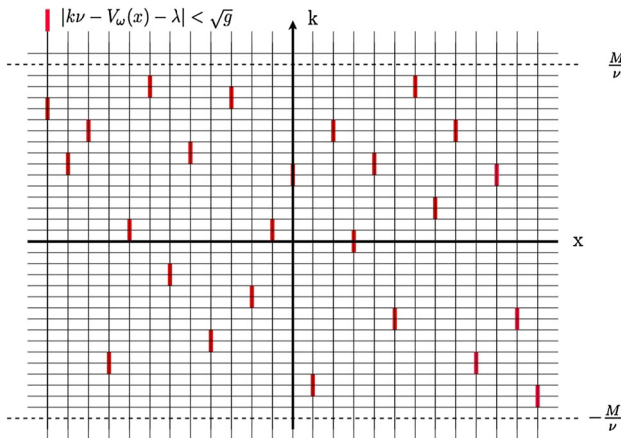

FIGURE 1. Resonant sites

any $y \in \partial^{in}(x + [-L, L]^d), k_1, k_2 \in \mathbb{Z}$,

$$\left((x, k_1), \left(\hat{H}_{|(x+[-L,L]^d) \times [k_0 - K, k_0 + K]} - \lambda\right)^{-1} (y, k_2)\right) \leq \frac{2(\sqrt{g}^{\frac{N}{2}})^{n_0}}{(\nu^2 \alpha(g))^2} \qquad (29)$$

where $n_0 = \lfloor \frac{d((x,k_1),(y,k_2))}{2N} \rfloor$.

In particular, this proposition implies that $(x + [-L, L]^d) \times [k_0 - K, k_0 + K]$ is a μ-good box with

$$\mu = -\left(\frac{\ln(g)}{4} - \frac{2\ln(\nu^2\alpha(g))}{L}\right).$$

Proof. For this proof, we work inside the space $L^2((x + [-L, L]^d) \times [k_0 - K, k_0 + K])$ and we write simply $\hat{H}$ instead of $\hat{H}_{|(x+[-L,L]^d) \times [k_0 - K, k_0 + K]}$.

Iterating (28), we obtain the usual random walk expansion for the resolvent (see, e.g., [13]): Given $\hat{x}, \hat{y} \in \mathbb{Z}^d \times \mathbb{Z}$, we get

$$(\hat{x}, (\hat{H} - \lambda)^{-1}\hat{y}) = \sum_{\hat{u}_i \in \partial_i^{in}\Lambda(\hat{v}_{i-1}), \hat{v}_i \in \partial^{ext}\Lambda(\hat{v}_{i-1})}$$
$$(\hat{x}, (\hat{H}_{|\Lambda(\hat{v}_0)} - \lambda)^{-1}, \hat{u}_1)(\hat{u}_1, g\hat{\Delta}\hat{v}_1)(\hat{v}_1, (\hat{H}_{|\Lambda(\hat{v}_1)} - \lambda)^{-1}$$
$$\hat{u}_2)(\hat{u}_2, g\hat{\Delta}\hat{v}_2) \ldots (\hat{v}_n, (\hat{H} - \lambda)^{-1}\hat{y}). \qquad (30)$$

In this writing, we need to specify when we stop iterating (28) and how $\Lambda(\hat{v}_{i-1})$ is defined. The following choice will guarantee the desired exponential decay:

1. If $|\hat{v} - \hat{y}| \leq N$, we stop iterating (28).
2. If $\hat{v}$ is not a resonant site, we choose $\Lambda(\hat{v}) = \{\hat{v}\}$. There are then at most $6d + 2$ points in $\partial^{ext}\Lambda(\hat{v})$.
3. If $\hat{v}$ is a resonance site, we choose $\Lambda(\hat{v}) = \Lambda_{K_x}$. There are at most $CdN^{d-1}(N + \sqrt{g}/\nu)$ points in $\partial^{ext}\Lambda(\hat{v})$ for some numerical constant $C > 0$.

See Fig. 2 for a typical path from $\hat{x}$ to $\hat{y}$.

From (30), we obtain

$$|(\hat{x}, (\hat{H} - \lambda)^{-1}\hat{y})| \leq \sum \left|(\hat{x}, (\hat{H}_{|\Lambda(\hat{v}_0)} - \lambda)^{-1}, \hat{u}_1)(\hat{u}_1, g\hat{\Delta}\hat{v}_1)\right|$$
$$\left|(\hat{v}_1, (\hat{H}_{|\Lambda(\hat{v}_1)} - \lambda)^{-1}\hat{u}_2)(\hat{u}_2, g\hat{\Delta}v_2)\right| \ldots \|(\hat{H} - \lambda)^{-1}\|. \qquad (31)$$

The factors in each term in this sum are bounded in two different ways, depending on whether they are resonant or not:

1. If $\hat{v}_i = (x, k)$ is not a resonant site, then $(\hat{H}_{|\Lambda} - \lambda) = (v_x + k\nu - \lambda)\delta_{(x,k)}$ so that

$$\left|\left((x, k), (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1}(x, k)\right)\left((x, k), g\hat{\Delta}(x', k')\right)\right|$$
$$\leq \frac{\left|\left((x, k), g\hat{\Delta}(x', k')\right)\right|}{\sqrt{g}} \leq \sqrt{g}. \qquad (32)$$
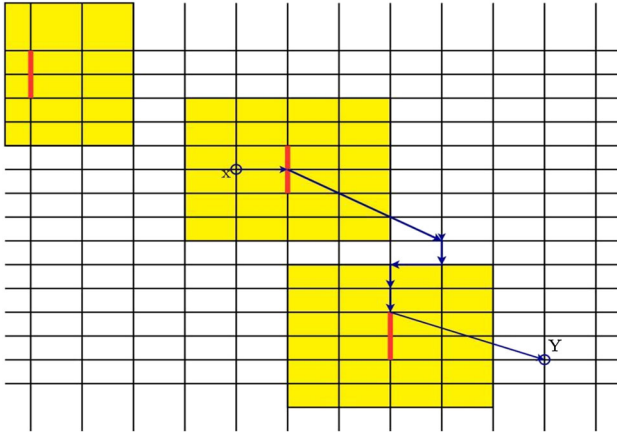
FIGURE 2. A typical path from $\hat{x}$ to $\hat{y}$. In *red* the resonant sites and in *yellow* the security *boxes* with $N = 2$ (color figure online)

2. If $\hat{v}_i = (x, k)$ belongs to K_x, then

$$\left|\left((x, k), (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1}(x', k')\right)\left((x', k'), g\hat{\Delta}(x'', k'')\right)\right|$$
$$\leq \frac{g}{d(\sigma(\hat{H}_{|\Lambda_{K_x}}), \lambda)}. \tag{33}$$

The sum in (31) will be small, if for every path joining $\hat{x}$ to $\hat{y}$, the number n of non-resonant sites is large enough to dominate the resonant terms (indexed by J), i.e.,

$$(2(d+1)\sqrt{g})^n \ll \prod_{j \in J} d\left(\sigma(\hat{H}_{|\Lambda_j}), \lambda\right) \tag{34}$$

We can now understand the reason why we have introduced the security boxes: Assuming that no security boxes intersect one another, then u_i is a resonant site implies that u_{i+1} is not resonant and its distance to any resonant sites is at least larger than N. From this we can deduce that for any path joining $\hat{x}$ to $\hat{y}$, every resonant term is followed by at least N non-resonant ones. Let $N \in \mathbb{N}$ such that

$$\frac{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2d+2)^{N+1}(\sqrt{g})^{\frac{N-1}{2}}}{\nu^2 \alpha(g)} < 1 \tag{35}$$

Then, if $\hat{u}_i$ is resonant, and assuming that, there is no strongly resonant security box, and no intersecting security boxes, we find that the following product of $N + 1$ consecutive factors can be bounded as

$$\left|(\hat{v}_i, (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1}\hat{u}_{i+1})(\hat{u}_{i+1}, g\hat{\Delta}v_{i+1})\right|$$
$$\dots \left|(\hat{v}_{i+N}, (\hat{H}_{|\Lambda(\hat{v}_{i+N})} - \lambda)^{-1}\hat{u}_{i+N+1})(\hat{u}_{i+N+1}, g\hat{\Delta}v_{i+N+1})\right|$$
$$\leq \frac{(\sqrt{g})^N}{d(\sigma(\hat{H}_{|\Lambda_{K_x}}), \lambda)} \leq \frac{(\sqrt{g})^{\frac{N+1}{2}}}{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2(d+1))^N}.$$

Hence, for a path connecting $\hat{x}$ to $\hat{y}$ in $l = k(N+1) + s$ steps ($s < N+1$), we obtain

$$\left| (\hat{x}, (\hat{H}_{|\Lambda(\hat{v}_i)} - \lambda)^{-1} \hat{u}_1)(\hat{u}_1, g\hat{\Delta} v_1) \right| \ldots \left| \left(\hat{v}_{i+l-1}, (\hat{H}_{|\Lambda(\hat{v}_{l-1})} - \lambda)^{-1} \hat{u}_l \right) (\hat{u}_l, g\hat{\Delta} v_l) \right|$$

$$\leq \left(\frac{(\sqrt{g})^{\frac{N+1}{2}}}{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2(d+1))^N} \right)^k \frac{(\sqrt{g})^{s-1}}{\nu^2 \alpha(g)}.$$

We can now conclude the proof. Indeed, any path connecting $\hat{x}$ to $\hat{y}$ contains at least $(d((x,k_1),(y,k_2)) - N)/2$ steps. Denoting by A_l the set of paths connecting $\hat{x}$ to $\hat{y}$ in l steps, we find

$$|(\hat{x}, (\hat{H} - \lambda)^{-1} \hat{y})| \leq \sum_{l=(d((x,k_1),(y,k_2))-N)/2}^{\infty} |A_l|$$

$$\left(\frac{(\sqrt{g})^{\frac{N+1}{2}}}{N^{d-1}((2N + \frac{\sqrt{g}}{\nu}))(2(d+1))^N} \right)^k \frac{(\sqrt{g})^{s-1}}{\nu^2 \alpha(g)} \frac{1}{\nu^2 \alpha(g)}$$

$$\leq \sum_{l=(d((x,k_1),(y,k_2))-N)/2}^{\infty} \sqrt{g}^{l/2} \frac{1}{(\nu^2 \alpha(g))^2}$$

$$\leq \frac{\left(\sqrt{g}^{\frac{N}{2}} \right)^{n_0}}{\left(1 - \sqrt{g} \right) (\nu^2 \alpha(g))^2}$$

$\square$

Proof of Proposition 4. For any $\hat{x} \in \Lambda_0 \times [k_0 - K; k_0 + K]$, $|\hat{V}(\hat{x}) - \lambda| \geq \sqrt{g}$. One can now do the random walk development as previously with no resonant term. $\square$

Proposition 8. *The probability of the event "there is no strongly resonant security box, and no intersecting security boxes" is smaller than $1/L^{2d}$ when g goes to 0 assuming $N = \mathcal{O}(\frac{\ln(\nu)}{\ln(g)}), L = m_1 N$, with m_1 a fixed large integer and $|\ln(\nu)| \leq g^{-\frac{1}{8d+4p}}$.*

Proof. To deal with the strongly resonant boxes, we use the Wegner-type estimate (18) with $\epsilon = \nu^2 \alpha(g)$:

$$\mathbb{P}(\Lambda_{K_x} \text{ is strongly resonant }) \leq \sum_{k_0=-M/\nu}^{M/\nu} \mathbb{P}(\Lambda_{K_x} \text{ is strongly resonant and}$$

$$\tag{36}$$

$$K_x = \mathbb{Z} \cap [k_0 - 1/(\nu\sqrt{g}), k_0 + 1/(\nu\sqrt{g})])$$

$$\leq \sum_{k_0=-M/\nu}^{M/\nu} \mathbb{P}(\Lambda_{\mathbb{Z} \cap [k_0 - 1/(\sqrt{g}\nu), k_0 + 1/(\sqrt{g}\nu)]}$$

$$\text{is strongly resonant)}$$

$$\leq \frac{2M}{\nu} 2\pi \nu^2 \alpha(g) \left(N^d \left(\frac{2\sqrt{g}}{\nu} + 2N \right) \right) \|\rho\|_\infty$$
$$\leq 4M \left(N^d \left(\frac{2\sqrt{g}}{\nu} + 2N \right) \right) \nu \alpha(g) \qquad (37)$$

We deal now with the probability of non-intersecting security boxes: For any $x, y \in [-L, L]^d$, $\Lambda_{K_x} \cap \Lambda_{K_y} = \emptyset$. This will be satisfied if there is no $|k| \leq 2N$ such that $|v_x - v_y + k\nu| \leq \sqrt{g}$. If $\nu \leq \sqrt{g}$, the probability P of intersecting security boxes is bounded by:

$$P \leq \frac{(2L)^d \big((2L)^d - 1 \big)}{2} \mathbb{P} \big(|v_x - v_y| < 2(N\nu + \sqrt{g}) \big)$$
$$\leq 2(2L)^d \big((2L)^d - 1 \big) (N\nu + \sqrt{g}) \|\rho\|_\infty \qquad (38)$$

and in any case (when $\nu > \sqrt{g}$) by

$$P \leq 2(2L)^d \big((2L)^d - 1 \big) (N+1) \sqrt{g} \|\rho\|_\infty \qquad (39)$$

From (38) (or (39)) and Proposition 7 we conclude the proof of our theorem. We need:

$$\begin{cases} 4M \left(N^d \left(\frac{2\sqrt{g}}{\nu} + 2N \right) \right) \nu \alpha(g) \leq \frac{1}{2L^{2p}} \\ 2(2L)^d \big((2L)^d - 1 \big) \left(N\nu + \sqrt{g} \right) \|\rho\|_\infty \leq \frac{1}{2L^{2p}} \\ - \left(\frac{\ln(g)}{4} - \frac{2\ln(\nu^2 \alpha(g))}{L} \right) > \mu \end{cases} \qquad (40)$$

or (when $\nu > \sqrt{g}$)

$$\begin{cases} 4M \left(N^d \left(\frac{2\sqrt{g}}{\nu} + 2N \right) \right) \nu \alpha(g) \leq \frac{1}{2L^{2p}} \\ 2(2L)^d \big((2L)^d - 1 \big) (N+1) \sqrt{g} \|\rho\|_\infty \leq \frac{1}{2L^{2p}} \\ - \left(\frac{\ln(g)}{4} - \frac{2\ln(\nu^2 \alpha(g))}{L} \right) > \mu \end{cases} \qquad (41)$$

and (35). We set $\alpha(g) = 1$ in case of $\nu < \sqrt{g}$ and $\alpha(g) = g$ in case of $\nu > \sqrt{g}$.

1. $N = n_1 \frac{ln(\nu)}{\sqrt{g}}$ with $n_1 > 7$.
2. $L = m_1 N$ with m_1 a large enough integer.

We have then $-(\frac{ln(g)}{4} - \frac{ln(\nu^2 \alpha(g))}{L}) > |\ln(g)|(\frac{1}{4} - \frac{1}{m_1})$. Then assume $|\ln(\nu)| \leq g^{-\frac{1}{8d+4p}}$. So we get $L^{4d+2p} \sqrt{g} = O(g^{1/4})$. Finally, the three conditions of (40) are satisfied in the limit $g \to 0$ and this is the end of the proof of 5. $\qquad \square$

6. L^2 Driving (C2)

We now consider the case of an L^2 driving. In this setup, we will work on infinite columns $C_L(x) = (x + [-L, L]^d) \times \mathbb{Z}$, so that distinct columns are independent with respect to the disorder. Instead, one should be careful in the random walk expansion since infinite sums appear. That this is not a problem comes from the decay of the Green function at the large frequencies:

6.1. Decay of the Green Function Along the Frequency Axes

Proposition 9. *Let $\hat{\phi}$ be an eigenfunction of $\hat{H}$ with eigenvalue $\bar{\lambda}$. Then*

$$\sum_{x,k} ||k\nu - \lambda|\hat{\phi}(x,k)|^2 \le (g+M)^2. \tag{42}$$

In particular,

$$|\hat{\phi}(x,k)| \le \frac{1+M+g}{1+|k\nu - \bar{\lambda}|} \tag{43}$$

for any x.

Proof. We use the time representation of $\hat{\phi}$. Recall that $\phi(t) = e^{i\lambda t}\psi(t)$ with ψ solution of (4). Since the evolution is unitary, for all $t \in [0,T]$, $\|\phi(t)\| = \|\psi(t)\| = \|\psi(0)\| = \|\phi(0)\|$. So

$$\sum_{x,k} ||k\nu - \lambda|\hat{\phi}(x,k)|^2 = \frac{1}{T} \int_0^T \|(i\partial_t - \bar{\lambda})\phi(t)\|^2 \mathrm{d}t$$

$$= \frac{1}{T} \int_0^T \|(-g\Delta(t) + V)\phi(t)\|^2 \mathrm{d}t$$

$$\le \frac{1}{T} \int_0^T \|(g\Delta(t) + V)\|^2 \mathrm{d}t$$

$$\le g^2 + 2M\frac{1}{T} \int_0^T \|(g\Delta(t)\|\mathrm{d}t + M^2$$

$$\le (g+M)^2,$$

and we deduce that $(1 + (|k\nu - \lambda|))\hat{\phi}(x,k)$ is square integrable. $\square$

From this we can deduce an estimate for the resolvent:

Proposition 10. *There exists a constant C depending only on ν so that we have*

$$|(\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})| \le \frac{(2L+1)^{d/2}(2+M)P(\hat{x})}{1+|k_z - k_y|} \left(\sup_i \frac{1}{|\lambda - \bar{\lambda}_i|} + C \right)$$

for any $\hat{z} = (z, k_z), \hat{y} = (y, k_y) \in C_L(x)$, where $\bar{\lambda}_i$ are the eigenvalue of $\hat{H}_{|C_L(x)}$.

Proof. We decompose $\hat{H}_{|C_L(x)}$ into its eigenvectors and we apply Cauchy–Schwarz. The eigenvalues of $\hat{H}_{|C_L(x)}$ are all of the form $\bar{\lambda}_i + k\nu$, where we can assume that $\bar{\lambda}_i$ are such that $|\bar{\lambda}_i + k\nu - \lambda| \ge \nu/2$ if $k \ne 0$. Then

$$(\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})$$

$$= \sum_{i=1}^{|\Lambda|} \sum_{k\in\mathbb{Z}} \frac{1}{\bar{\lambda}_i + \nu k - \lambda} \phi_{\bar{\lambda}_i + \nu k}(\hat{z}) \phi_{\bar{\lambda}_i + \nu k}(\hat{y})$$

$$\le \left(\sum_{i=1}^{|\Lambda|} \sum_{k\in\mathbb{Z}} (1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2 |\phi_{\bar{\lambda}_i + \nu k}(\hat{z})|^2 \right)^{1/2}.$$

$$\left(\sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} |\phi_{\bar{\lambda}_i + \nu k}(\hat{y})|^2 \right)^{1/2}$$

$$= \left(\sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} (1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2 |\phi_{\bar{\lambda}_i}(z, k_z - k)|^2 \right)^{1/2} .$$

$$\left(\sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} |\phi_{\bar{\lambda}_i + \nu k}(\hat{y})|^2 \right)^{1/2}$$

We use now (42) to control the first factor, and (43) to get an estimate on $|\phi_{\bar{\lambda}_i + \nu k}(\hat{y})|$ in the second one:

$$(\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y}) \le (1 + M + g)^2$$

$$\times \left(\sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_y)|)^2} \right)^{1/2}$$

$$= (1 + M + g)^2 \left(\sum_{i=1}^{|\Lambda|} \frac{1}{|\bar{\lambda}_i - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu k_y|)^2} \right.$$

$$\left. + \sum_{i=1}^{|\Lambda|} \sum_{k \in \mathbb{Z}^*} \frac{1}{|\bar{\lambda}_i + \nu k - \lambda|^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_z)|)^2} \frac{1}{(1 + |\bar{\lambda}_i + \nu(k - k_y)|)^2} \right)^{1/2}$$

$$\le |\Lambda|^{1/2}(1 + M + g)^2 \left(\sup_i \frac{1}{|\lambda - \bar{\lambda}_i|} + C \right) P(\hat{z}) \frac{1}{(1 + |k_z - k_y|)},$$

where the last inequality comes from the estimate of the integral

$$\int dk \frac{1}{1 + k^2} \frac{1}{1 + (k - k_z)^2} \frac{1}{1 + (k - k_y)^2} \sim \frac{1}{(1 + |k_z|)^2} \frac{1}{(1 + |k_z - k_y|)^2}.$$

$\square$

Definition 5. We say that $C_L(x)$ is not strongly resonant if

$$\inf_{\bar{\lambda}_i \in \sigma(\hat{H}_{|C_L(x)})} \{|\bar{\lambda}_i - \lambda|\} > e^{-\sqrt{L}}. \tag{44}$$

In particular, if $C_L(x)$ is not strongly resonant, we have

$$|(\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})| \le \frac{CL^{d/2}P(\hat{z})}{1 + |k_z - k_y|} e^{\sqrt{L}}$$

where C is a constant.

6.2. The Decay Function

If Anderson localization is most of the time studied over $\mathbb{Z}^d$, the problem could be raised on any set of point X. It is indeed easy to define a random potential $V(x), x \in X$ and a "Laplacian" $\Delta(x_1, x_2)$ without assuming a particular geometry of the system. But to recover the decay, one should then first define a decay function, and Δ is the only object that we can use to construct such a decay function. We first give a general definition.

Definition 6. Let $G : X \times X \to \mathbb{R}_+$, for any $\hat{x}, \hat{y} \in X$,

$$d_G(\hat{x}, \hat{y}) = -\ln \left(\sum_{\mathcal{C}(\hat{x} \to \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) \tag{45}$$

if $\hat{x} \neq \hat{y}$, and 0 otherwise, where $\mathcal{C}(\hat{x} \to \hat{y})$ is the set of all paths $\hat{x} = \hat{z}_0, \hat{z}_1, \hat{z}_2, ..., \hat{z}_k = \hat{y}$ from $\hat{x}$ to $\hat{y}$.

Proposition 11. *If for any $z \in X, \sum_{z_1} |G(z, z_1)| < 1/2$, then d_G is positive and satisfies the triangle inequality.*

Proof. We first check that d_G is positive. Let $\hat{x}, \hat{y}$

$$\sum_{\mathcal{C}(\hat{x} \to \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})|$$

$$\leq \sum_{\hat{y}'} \sum_{\mathcal{C}(\hat{x} \to \hat{y}')} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})|$$

$$\leq \sum_{n>0} \prod_{i=0}^{n} \left(\max_{\hat{z}_i} \sum_{\hat{z}_{i+1} \in X} |G(\hat{z}_i, \hat{z}_{i+1})| \right)$$

$$= \sum_{n>0} \left(\max_{\hat{x}} \sum_{\hat{y} \in X} |G(\hat{x}, \hat{y})| \right)^n$$

$$= \frac{\left(\max_{\hat{x}} \sum_{\hat{y} \in X} |G(\hat{x}, \hat{y})| \right)}{1 - \left(\max_{\hat{x}} \sum_{\hat{y} \in X} |G(\hat{x}, \hat{y})| \right)}$$

$$< 1.$$

We now check the triangle inequality. Let $\hat{z}$ be another point in X.

$$d_G(\hat{x}, \hat{y}) + d_G(\hat{y}, \hat{z})$$

$$= -\ln \left(\sum_{\mathcal{C}(\hat{x} \to \hat{y})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right) - \ln \left(\sum_{\mathcal{C}(\hat{y} \to \hat{z})} \prod_j |G(\hat{z}_j, \hat{z}_{j+1})| \right)$$

$$= -\ln \left(\sum_{\mathcal{C}(\hat{x} \to \hat{y})} \sum_{\mathcal{C}(\hat{y} \to \hat{z})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \prod_j |G(\hat{z}_j, \hat{z}_{j+1})| \right)$$

$$\geq -\ln \left(\sum_{\mathcal{C}(\hat{x} \to \hat{z})} \prod_i |G(\hat{z}_i, \hat{z}_{i+1})| \right)$$

$$= d_G(\hat{x}, \hat{z}).$$

$\square$

6.3. Initialization of the Multi-scale

Proof of Proposition 6. Proposition 6 follows from Propositions 13 and 14 below. $\qquad\square$

Definition 7. We will use d_G with $X = \mathbb{Z}^d \times \mathbb{Z}$ and

$$G(\hat{x}, \hat{y}) = g|\hat{\Delta}(\hat{x}, \hat{y})P(\hat{y})| \tag{46}$$

Remark that we also have $\Delta(\hat{z}, .)P(.) \in L^1$ because $\Delta(\hat{z}, .) \in L^2$ and $P(.) \in L^2$. We will write $\|G\|_{\ell^1 max} = \sup_x \sum_y G(x, y)$. This quantity goes to zero as $g \to 0$. The decay function is related to usual distance on $\mathbb{Z}^d$ through the following proposition:

Proposition 12. *For any $\hat{x} = (x, k_x)$,*

$$\sum_{z:|x-z|=L} \sum_k e^{-d_G((x,k_x),(z,k))} \le e^{L\ln((\|G\|_{\ell^1 max}) - \ln(1 - \|G\|_{\ell^1 max})} \tag{47}$$

in particular $\hat{z} = (z, k_z), |x - z| > L$.

$$d_G(\hat{x}, \hat{z}) \ge L(-\ln(\|G\|_{\ell^1 max})) + \ln(1 - \|G\|_{\ell^1 max}) \tag{48}$$

Proof. Because no path of length smaller than L connects $\hat{x}$ with the boundary of $\{(z, k) : |x - z| > L\}$,

$$\sum_{\mathcal{C}(\hat{x} \to \hat{z})} \prod_i |G(\hat{z}_i \hat{z}_{i+1})| \le \sum_{n > L} \|G\|_{\ell^1 max}^n \le \frac{\|G\|_{\ell^1 max}^L}{1 - \|G\|_{\ell^1 max}}. \tag{49}$$

So

$$d_G(\hat{x}, \hat{z}) \ge -L\ln((\|G\|_{\ell^1 max}) + \ln(1 - \|G\|_{\ell^1 max}).$$

$\qquad\square$

Proposition 13. *If there is no resonant site at all in $C_L(x)$, and if $\hat{H}_{|C_L(x)}$ has no eigenvalue $\bar{\lambda}_i$ with $|\bar{\lambda}_i - \lambda| \le \sqrt{g}$, then $C_L(x)$ is a $(\mu', \tilde{d}_G)$ good column*

Proof. We use here again the resolvent formula:

$$(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y}) = \sum_{\hat{z}} \frac{g\hat{\Delta}(\hat{x}, \hat{z})}{\hat{V}(\hat{x}) - \lambda}(\hat{z}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y}).$$

Applying it several times yields the usual random walk expansion:

$$(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y}) = \sum_{\hat{z}, \hat{z}_1, \hat{z}_2, \ldots, \hat{z}_n} \frac{g\hat{\Delta}(\hat{x}, \hat{z}_1)}{\hat{V}(\hat{x}) - \lambda} \frac{g\hat{\Delta}(\hat{z}_1, \hat{z}_2)}{\hat{V}(\hat{z}_1) - \lambda}$$

$$\cdots \frac{g\hat{\Delta}(\hat{z}_{n-1}, \hat{z}_n)}{\hat{V}(\hat{z}_{n-1}) - \lambda}(\hat{z}_n, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})$$

Because there is no resonant site, $\frac{1}{\hat{V}(\hat{z}) - \lambda} \le P(\hat{z})$ for any $\hat{z} \in C_L(x)$. So

$$|(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})|$$

$$= P(\hat{x}) \sum_{\hat{z}, \hat{z}_1, \hat{z}_2, \ldots, \hat{z}_n} |g\hat{\Delta}(\hat{x}, \hat{z}_1)P(\hat{z}_1)g\hat{\Delta}(\hat{z}_1, \hat{z}_2)\ldots P(\hat{z}_{n-1})$$

$$\times g\hat{\Delta}(\hat{z}_n, \hat{z}_{n-1})(\hat{z}_n, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})|$$

$$\leq CP(\hat{x}) \sum_{\hat{z}, \hat{z}_1, \hat{z}_2, \ldots, \hat{z}_n} |g\hat{\Delta}(\hat{x}, \hat{z}_1)P(\hat{z}_1)g\hat{\Delta}(\hat{z}_1, \hat{z}_2)\ldots P(\hat{z}_{n-1})$$

$$\times g\hat{\Delta}(\hat{z}_n, \hat{z}_{n-1})P(\hat{z}_n)|\frac{L^{d/2}}{\sqrt{g}}$$

$$\leq CL^{d/2}\frac{P(\hat{x})}{\sqrt{g}} \sum_{\mathcal{C}(x \to y)} \prod_i g|\hat{\Delta}(\hat{z}_i, \hat{z}_{i+1})|P(\hat{z}_{i+1})$$

where the first inequality is obtained through Proposition 10 and the hypothesis on the eigenvalues $\bar{\lambda}_i$. So one has

$$|(\hat{x}, (\hat{H}_{|C_L(x)} - \lambda)^{-1}\hat{y})| \leq CL^{d/2}\frac{P(\hat{x})}{\sqrt{g}}e^{-\tilde{d}_G(\hat{x}, \hat{y})}$$

$\square$

Proposition 14. *The probability of the event "there is no resonant site at all in $C_L(x)$, and $\hat{H}_{|C_L(x)}$ has no eigenvalue λ_i with $|\lambda_i - \lambda| \leq \sqrt{g}$" goes to 0 with $g \to 0$.*

Proof. First,

$$\mathbb{P}(\text{there is no resonant site in } C_L(x)) \leq ||\rho||_\infty \frac{2M}{\nu}(2L+1)^d\sqrt{2g}. \quad (50)$$

Next, thanks to Wegner estimate,

$$\mathbb{P}(C_L(x) \text{ is not strongly resonant }) \leq ||\rho||_\infty \frac{2M}{\nu}(2L+1)^d\sqrt{2g}. \quad (51)$$

This gives the proposition for $g \to 0$. $\square$

6.4. Technical Results for the Iteration of the MSA

We have proved that for a fixed $L, C_L(x)$ is a good column with high probability. MSA induces that the property is valid for all L_k with $L_{k+1} = L_k^\alpha, L_0 = L$, but some adaptations with wrt. [13] are needed, due to the long range hopping along the frequency axis. It turns out that only Theorems 10.14 and 10.20 need to be re-investigated. Here we prove Proposition 16 below that will play the role of Theorem 10.14 in [13] (the equivalent of Theorem 10.20 in [13] can then be obtained without any new idea).

Thanks to the estimates on Green function obtained in Sect. 6.1, we obtain

Proposition 15.

$$\sup_{\hat{x}, y} \sum_{k_y} \sum_{\hat{z}} \frac{1}{1 + |k_x - k_y|}|\Delta(\hat{y}, \hat{z})P(\hat{z})| < \infty \quad (52)$$

In particular, $G(\hat{x}, .) = \sum_{k_y} \frac{1}{1+|k_x-k_y|}|\Delta(\hat{y}, .)P(.)|$ is in L^1 uniformly in x.

Proof. We have $\sqrt{|\Delta(\hat{y},.)|} \in L^4$, with a norm that can be bounded uniformly in $\hat{y}$, $\frac{1}{1+|.|} \in L^{\frac{4}{3}}$ and $P(.) \in L^{\frac{4}{3}}$.

$$
\sup_{\hat{x},y} \sum_{k_y} \sum_{\hat{z}} \frac{1}{1+|k_x-k_y|}|\Delta(\hat{y},\hat{z})P(\hat{z})|
$$

$$
\leq \Big(\sup_{\hat{x},y,\hat{z}} \sum_{k_y} \frac{1}{1+|k_x-k_y|}\sqrt{|\Delta(\hat{y},\hat{z})|} \Big)\Big(\sup_{\hat{y}} \sum_{\hat{z}} \sqrt{|\Delta(\hat{y},\hat{z})|}P(z) \Big)
$$

$$
\leq \big(\|\frac{1}{1+|.|}\|_{L^{\frac{4}{3}}}\|\sqrt{|\Delta(\hat{y},.)|}\|_{L^4}\big)\big(\|\sqrt{|\Delta(\hat{y},.)|}\|_{L^4}\|P(.)\|_{L^{\frac{4}{3}}}\big)
$$

$$
< \infty
$$

$\square$

Proposition 16. *If there is no two distinct small-scale columns $C_{L_k}(y) \subset C_{L_{k+1}}(x)$ which are not μ-good, and there is no columns $C_{2L_k}(y') \subset C_{L_{k+1}}(x)$ that are strongly resonant and $C_{L_{k+1}}(x)$ is not strongly resonant, then $C_{L_{k+1}}(x)$ is μ' good with $\mu' > \mu - \frac{3L_k}{L_{k+1}}$.*

Proof. Let d_G be the decay function used for the small-scale good boxes. In the case of C_{L_k} is a bad column, we use the resolvent development twice

$$
|(\hat{x},(\hat{H}_{|C_{L_{k+1}}(x)}-\lambda)^{-1}\hat{y})|
$$

$$
\leq \sum_{\substack{\hat{z}_1\in\partial^{in}C_{L_{2k}}(x),\\ \hat{z}_2\in\partial^{ext}C_{L_{2k}}(x)}} |(\hat{x},(\hat{H}_{|C_{2L_k}(x)}-\lambda)^{-1}\hat{z}_1)g\hat{\Delta}(\hat{z}_1,\hat{z}_2)
$$

$$
(\hat{z}_2,(\hat{H}_{|C_{L_{k+1}}(x)}-\lambda)^{-1}\hat{y})|
$$

$$
\leq \sum_{\substack{\hat{z}_1\in\partial^{in}C_{L_{2k}}(x)\\ \hat{z}_2\in\partial^{ext}C_{L_{2k}}(x)}} \sum_{\substack{\hat{z}_3\in\partial^{in}C_{L_k}(z_2)\\ \hat{z}_4\in\partial^{ext}C_{L_k}(z_2)}} |(\hat{x},(\hat{H}_{|C_{2L_k}(x)}-\lambda)^{-1}\hat{z}_1)g\hat{\Delta}(\hat{z}_1,\hat{z}_2)
$$

$$
(\hat{z}_2,(\hat{H}_{|C_{L_k}(x)}-\lambda)^{-1}\hat{z}_3)g\hat{\Delta}(\hat{z}_3,\hat{z}_4)(\hat{z}_4,(\hat{H}_{|C_{L_{k+1}}(x)}-\lambda)^{-1}\hat{y})|
$$

$$
\leq P(\hat{x}) \sum_{\substack{\hat{z}_1\in\partial^{in}C_{L_{2k}}(x)\\ \hat{z}_2\in\partial^{ext}C_{L_{2k}}(x)}} \sum_{\substack{\hat{z}_3\in\partial^{in}C_{L_k}(z_2)\\ \hat{z}_4\in\partial^{ext}C_{L_k}(z_2)}} e^{\sqrt{L_k}}\frac{C(2L_k)^{d/2}}{1+|k_{\hat{x}}-k_{\hat{z}_1}|}|g\hat{\Delta}(\hat{z}_1,\hat{z}_2)|
$$

$$
P(\hat{z}_2)e^{-d_G(\hat{z}_2,\hat{z}_3)}g|\hat{\Delta}(\hat{z}_3,\hat{z}_4)(\hat{z}_4,(\hat{H}_{|C_{L_k}(x)}-\lambda)^{-1}\hat{y})|
$$

So let us define G' as follows:

$$
G'(\hat{x},\hat{y}) = e^{-d_G(\hat{x},\hat{y})}
$$

if $C_{L_k}(x)$ is a μ good box and $\hat{y} \in \partial^{ext}C_{L_k}(x)$, and

$$G'(\hat{x}, \hat{y}) = \sum_{\substack{\hat{z}_1 \in \partial^{in} C_{L_{2k}}(x) \\ \hat{z}_2 \in \partial^{ext} C_{L_{2k}}(x) \\ \hat{z}_3 \in \partial^{in} C_{L_k}(z_2)}} e^{\sqrt{L_k}} \frac{C(2L_k)^{d/2}}{1 + |k_{\hat{x}} - k_{\hat{z}_1}|}$$

$$\times |g\hat{\Delta}(\hat{z}_1, \hat{z}_2)| P(\hat{z}_2) e^{-d_G(\hat{z}_2, \hat{z}_3)} |g\hat{\Delta}(\hat{z}_3, \hat{y})| P(\hat{y})$$

if $C_L(x)$ is a bad box.

Thanks to Proposition 15, there is a constant C independent of L_k such that for the second case: $\|G'\|_{L^1} \leq C' e^{2\sqrt{L_k}} e^{-\mu L_k}$. We can then recover the usual tools, using that $e^{-\mu L_k}$ dominate the other terms for L_k large. In particular because for any path from x to $\partial C_L(x)$ there is at least $(\frac{L_{k+1}}{L_k} - 3) \mu$ good boxes. So, with the same argument as in the proof of Proposition 12,

$$\sum_{\hat{y} \in \partial^{in} C_L(x)} e^{-d_{G'}(\hat{x}, \hat{y})} \leq e^{-\mu(L_{k+1} - 3L_k) - \ln(1 - \|G'\|_{\ell^1 max})}$$

$$\square$$

7. Proof of the Corollaries

As said, Corollaries 1 and 2 do not follow logically from Theorem 1; instead, one should go through the MSA once again and refine several estimates. This work has been carried over in [11], and one indicates here only the main steps as well as the few needed extra adaptations.

Let us start with Corollary 1.

Proposition 17. *There exists $p > 0$ (and one can take $p \to \infty$ as $\epsilon \to 0$) such that:*

$$\mathbb{E}\left(\sup_{t>0} \sum_{x \in \mathbb{Z}^d} \sum_k |x|^p |\check{\phi}(x, k, t)|^2\right) < \infty \tag{53}$$

for any $\check{\phi}(x, k, 0)$ defined on a bounded support.

Proof. Thanks to the MSA carried over in this paper, one can check that the results of [11] holds; in particular, the assumptions of Theorem 3.1 in [11] are satisfied. $\square$

In order to recover ϕ from $\check{\phi}$ we use the following proposition. Remind that, thanks to (6), we have $\|H(t)\|_{L^1[0;T]} \leq \sqrt{T} \|H(t)\|_{L^2[0;T]}$.

Proposition 18. *Let $\psi(t) \in L^2(\mathbb{Z}^d)$ satisfying $\|\psi(t)\|_{L^2} = 1$ for all $t \in \mathbb{R}$ be a solution of*

$$i\partial_t \psi(t) = A(t)\psi(t) \tag{54}$$

where for any t $A(t)$ is Hermitian, $C = \|A(.)\|_{L^1([0,T])} < \infty$ and $(x, A(t)y) = 0$ if $|x - y| > 1$. For any $t \in [0, T]$ and any $x_0 \in \mathbb{Z}^d$, we have

$$\sum_{|z-x_0|<R} |\psi(z,t)|^2 \geq |\psi(x_0,0)|^2 \left(1 - e^C \sum_{k \geq R} \frac{(2dC)^k}{k!}\right)$$
$$-e^C \sum_{k \geq R} \frac{(2dC)^k}{k!} |\psi(x_0)|. \tag{55}$$

Proof. Let us separate $\psi(0) = \mathbb{1}_{x=x_0}\psi(0) + \mathbb{1}_{x \neq x_0}\psi(0)$. Because the $A(t)$ is Hermitian, there exists $U(t)$ unitary such that

$$\psi(t) = U(t)\psi(0) = U(t)(\mathbb{1}_{x=x_0}\psi(0)) + U(t)(\mathbb{1}_{x \neq x_0}\psi(0)) \tag{56}$$

Calling $\psi_1 = U(t)(\mathbb{1}_{x=x_0}\psi(0))$, $\psi_2 = U(t)(\mathbb{1}_{x \neq x_0}\psi(0))$ we have $(\psi_1, \psi_2) = 0$ and $\|\psi_1\|^2 + \|\psi_2\|^2 = 1$. Because $\mathbb{1}_{|z-x_0|<R}$ is a projector,

$$\left(\psi_1 + \psi_2, \mathbb{1}_{|z-x_0|<R}(\psi_1 + \psi_2)\right)$$
$$= \left(\psi_1, \mathbb{1}_{|z-x_0|<R}\psi_1\right) + \left(\psi_2, \mathbb{1}_{|z-x_0|<R}\psi_2\right) + 2\left(\psi_1, \mathbb{1}_{|z-x_0|<R}\psi_2\right)$$
$$\geq \left(\psi_1, \mathbb{1}_{|z-x_0|<R}\psi_1\right) - 2\left|\left(\psi_2, \mathbb{1}_{|z-x_0|\geq R}\psi_1\right)\right|$$
$$\geq \|\psi_1\|^2 - \|\mathbb{1}_{|z-x_0|\geq R}\psi_1\|^2 - 2\left|\left(\psi_2, \mathbb{1}_{|z-x_0|\geq R}\psi_1\right)\right|$$
$$\geq \|\psi_1\|^2 - \|\mathbb{1}_{|z-x_0|\geq R}\psi_1\|^2 - 2\|\mathbb{1}_{|z-x_0|\geq R}\psi_1\|$$

We now proof that the locality of $A(t)$ implies that $\|\mathbb{1}_{|z-x_0|\geq R}\psi_1\|^2$ is small.

$$i\frac{d}{dt}\psi_1(y,t) = A(t)\psi_1(y,t) = \sum_{|y'-y|\leq 1} A_{y,y'}(t)\psi_1(y',t).$$

Hence,

$$\frac{d}{dt}|\psi_1(y,t)| \leq \sum_{|y'-y|\leq 1} |A_{y,y'}(t)||\psi_1(y',t)| \leq \|A(t)\| \sum_{|y'-y|\leq 1} |\psi_1(y',t)|$$

Let $a(y,t)$ be solution of the system

$$\begin{cases} \frac{d}{dt}a(y,t) = \|A(t)\| \sum_{|y'-y|\leq 1} a(y',t) \\ a(y,0) = |\psi_1(x_0,0)|\mathbb{1}_{y=x_0} \end{cases} \tag{57}$$

We have then for any (y,t)

$$|\psi_1(y,t)| \leq a(y,t) \tag{58}$$

We can evaluate a with the following remark: Let $X(t)$ be the classical Markovian random walk on $\mathbb{Z}$ of variable rate $\|A(t)\|$ and starting at point x_0. Its generator is

$$\frac{d}{dt}\mathbb{P}_{x_0}(X(t) = y) = \|A(t)\| \sum_{|y'-y|} (\mathbb{P}_{x_0}(X(t) = y') - \mathbb{P}_{x_0}(X(t) = y)) \tag{59}$$

and then we have

$$e^{-(2d+1)\int_0^t \|A(u)\|du} a(y,t) = a(x_0,0)\mathbb{P}_{x_0}(X(t) = y) \tag{60}$$

We can then deduce

$$\sum_{y \geq R} a(y,t) \leq a(x_0, 0) e^{(2d+1)\int_0^t \|A(u)\| \mathrm{d}u} \mathbb{P}(N_{2d\int_0^t \|A(u)\| \mathrm{d}u} \geq R) \tag{61}$$

where $N_{2d\int_0^t \|A(u)\| \mathrm{d}u}$ is the Poisson process of parameter $2d\int_0^t \|A(u)\| \mathrm{d}u$. So for any $t \leq T$

$$\sum_{y \geq R} a(y,t) \leq a(x_0, 0) e^C \sum_{k \geq R} \frac{(2dC)^k}{k!} \tag{62}$$

We can now conclude

$$\sum_{|z-x_0|<R} |\psi(z,t)|^2 = \left(\psi_1 + \psi_2, \mathbb{1}_{|z-x_0|<R}(\psi_1 + \psi_2)\right)$$

$$\geq \|\psi_1\|^2 - \|\mathbb{1}_{|z-x_0|\geq R}\psi_1\|^2 - \|\mathbb{1}_{|z-x_0|\geq R}\psi_1\|$$

$$\geq |\psi(x_0, 0)|^2 - |\psi(x_0, 0)|^2 \left(e^C \sum_{k \geq R} \frac{(2dC)^k}{k!}\right)^2$$

$$- |\psi(x_0, 0)| \left(e^C \sum_{k \geq R} \frac{(2dC)^k}{k!}\right)$$

$$\square$$

The above proposition and the dynamical localization of $\check{\phi}$ enable us to conclude:

Proposition 19. *For any $\epsilon > 0$, there exist some constants C_ϵ, D_ϵ such that*

$$C_\epsilon \sum_{x \in \mathbb{Z}^d} \sum_k |x|^p |\check{\phi}(x,k,t)|^2 + D_\epsilon \geq \sum_{x_0 \in \mathbb{Z}^d} |x_0|^{p-\epsilon} |\phi(x_0, t)|^2 \tag{63}$$

Proof. Let $\epsilon > 0$. Let $R : \mathbb{Z}^d \to \mathbb{R}, x \mapsto R(x_0)$ be a function satisfying the four following conditions:

$$\sum_{x_0 \in \mathbb{Z}^d} |x_0|^p \sum_{k \geq R(x_0)} \frac{(2dC)^k}{k!} < \infty, \tag{64}$$

for all $x_0 \in \mathbb{Z}^d$,

$$e^C \sum_{k \geq R(x_0)} \frac{(2dC)^k}{k!} < \frac{1}{2}, \tag{65}$$

for all $x, x_0 \in \mathbb{Z}^d$ such that $|x-x_0| < R(x_0)$, it holds that $|x-x_0| < (1+\epsilon)R(x)$, there exists a constant C_ϵ such that

$$\sum_{|x-x_0| \leq (1+\epsilon)R(x)} |x_0|^{p-\epsilon} \leq C_\epsilon |x|^p \tag{66}$$

for $|x_0| > 1$. For example, we can take R defined by $R(x) = ln^2(|x|)$ for $|x| > a$ and $R(x) = In^2(a)$ for a large enough.

$$\mathbb{E}\left(\sup_{t>0}\sum_{x\in\mathbb{Z}^d}\sum_k |x|^p|\check{\phi}(x,k,t)|^2\right)$$

$$=\mathbb{E}\left(\sup_{t>0}\sum_{x\in\mathbb{Z}^d}|x|^p\frac{1}{T}\int_t^{t+T}|\phi(x,u)|^2\mathrm{d}u\right)$$

$$\geq\frac{1}{C_\epsilon}\mathbb{E}\left(\sup_{t>0}\sum_{x\in\mathbb{Z}^d}\sum_{|x-x_0|\leq(1+\epsilon)R(x)}\frac{1}{T}\int_t^{t+T}|x_0|^{p-\epsilon}|\phi(x,u)|^2\mathrm{d}u\right)$$

$$\geq\frac{1}{C_\epsilon}\mathbb{E}\left(\sup_{t>0}\sum_{x_0\in\mathbb{Z}^d}|x_0|^{p-\epsilon}\frac{1}{T}\int_t^{t+T}\sum_{|x-x_0|\leq R(x_0)}|\phi(x,u)|^2\mathrm{d}u\right)$$

$$\geq\frac{1}{C_\epsilon}\mathbb{E}\left(\sup_{t>0}\sum_{x_0\in\mathbb{Z}^d}|x_0|^{p-\epsilon}\frac{1}{T}\int_t^{t+T}|\phi(x_0,t)|^2-|\phi(x_0,t)|^2\right.$$

$$\left.\times\left(e^C\sum_{k\geq R}\frac{(2dC)^k}{k!}\right)^2-|\phi(x_0,t)|\left(e^C\sum_{k\geq R}\frac{(2dC)^k}{k!}\right)\mathrm{d}u\right)$$

$$\geq\frac{1}{2C_\epsilon}\mathbb{E}\left(\sup_{t>0}\sum_{x_0\in\mathbb{Z}^d}|x_0|^{p-\epsilon}|\phi(x_0,t)|^2\right)$$

$$-e^C\frac{1}{C_\epsilon}\sum_{x_0\in\mathbb{Z}^d}|x_0|^p\sum_{k\geq R(x_0)}\frac{(2dC)^k}{k!}$$

So

$$\mathbb{E}\left(\sup_{t>0}\sum_{x_0\in\mathbb{Z}^d}|x_0|^{p-\epsilon}|\phi(x_0,t)|^2\right)<\infty \tag{67}$$

$\square$

Let us now come to Corollary 2:

Proof of Corollary 2. Since

$$\psi_{\overline{\lambda}}(\cdot,0)=\sum_{k\in\mathbb{Z}}\hat{\psi}_{\overline{\lambda}}(\cdot,k),$$

we can write

$$H_{eff}(x,y)=\sum_{(k,l)\in\mathbb{Z}^2}\sum_{\overline{\lambda}\in[0,\nu[}\overline{\lambda}\psi_{\overline{\lambda}}(x,k)\overline{\psi}_{\overline{\lambda}}(y,l)$$

$$=\sum_{k,l}\left((x,k),\eta(\hat{H})(y,k)\right)$$

with

$$\eta:\mathbb{R}\to\mathbb{R}, s\mapsto\eta(s)=1_{[0,\nu[}(s)s.$$

Again, thanks to the MSA shown in this paper, and the deterministic exponential decay along the frequency axis of the eigenfunctions under Assumption (C1), we can reuse the methods leading to Theorem 3.1 in [11], to get our result. □

Acknowledgements

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