



Finite-Order Correlation Length for Four-Dimensional Weakly Self-Avoiding Walk and $|\varphi|^4$ Spins

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Abstract. We study the four-dimensional n -component $|\varphi|^4$ spin model for all integers $n \geq 1$ and the four-dimensional continuous-time weakly self-avoiding walk which corresponds exactly to the case $n = 0$ interpreted as a supersymmetric spin model. For these models, we analyse the correlation length of order p , and prove the existence of a logarithmic correction to mean-field scaling, with power $\frac{1}{2} \frac{n+2}{n+8}$, for all $n \geq 0$ and $p > 0$. The proof is based on an improvement of a rigorous renormalisation group method developed previously.

1. Introduction and Main Results

1.1. Introduction

Recently, using a rigorous renormalisation group method [5, 6, 10–13], the critical behaviour of the four-dimensional n -component $|\varphi|^4$ spin model [2, 18] and the four-dimensional continuous-time weakly self-avoiding walk [3, 4, 18] has been analysed. The latter model corresponds to the case $n = 0$ via an exact identity which represents the weakly self-avoiding walk as a supersymmetric field theory with quartic self-interaction. A typical result in this work is that for all $n \geq 0$, the susceptibility diverges as $\varepsilon^{-1} (\log \varepsilon^{-1})^{\frac{n+2}{n+8}}$, in the limit $\varepsilon \downarrow 0$ describing approach to the critical point. Related results have been obtained for the pressure, the specific heat, the critical two-point function, and other quantities. The existence of such logarithmic corrections to scaling for dimension 4 was predicted about 45 years ago in the physics literature [7, 17, 19]. For $n = 1$, the existence of logarithmic corrections was proven rigorously about 30 years ago in [15, 16].

A missing aspect in the analysis of critical scaling in [2–4, 18] is a determination of the divergence of correlation length scales, as the critical point is approached. A natural measure of length scale is the correlation length ξ defined as the reciprocal of the exponential decay rate of the two-point function. We do not study this correlation length (which was, however, studied in [16] for the case $n = 1$). Instead, we study ξ_p , the correlation length of order p , for all $p > 0$, and prove that its divergence takes the form $\varepsilon^{-\frac{1}{2}}(\log \varepsilon^{-1})^{\frac{1}{2} \frac{n+2}{n+8}}$. The independence of p in the exponents exemplifies the conventional wisdom that in critical phenomena, all naturally defined length scales should exhibit the same asymptotic behaviour. The correlation length ξ is predicted to diverge in the same manner, but our method would require further development to prove this.

1.2. Definitions of the Models

Before defining the models, we establish some notation. Let $L > 1$ be an integer (which we will need to fix large). Consider the sequence $\Lambda = \Lambda_N = \mathbb{Z}^d / (L^N \mathbb{Z}^d)$ of discrete d -dimensional tori of side lengths L^N , with $N \rightarrow \infty$ corresponding to the infinite volume limit $\Lambda_N \uparrow \mathbb{Z}^d$. Throughout the paper, we only consider $d = 4$, but we sometimes write d instead of 4 to emphasise the role of dimension. For any of the $2d$ unit vectors $e \in \mathbb{Z}^d$, we define the discrete gradient of a function $f : \Lambda_N \rightarrow \mathbb{R}$ by $\nabla^e f_x = f_{x+e} - f_x$ and the discrete Laplacian by $\Delta = -\frac{1}{2} \sum_{e \in \mathbb{Z}^d: |e|=1} \nabla^{-e} \nabla^e$. The gradient and Laplacian operators act componentwise on vector-valued functions. We also use the discrete Laplacian $\Delta_{\mathbb{Z}^d}$ on \mathbb{Z}^d and the continuous Laplacian $\Delta_{\mathbb{R}^d}$ on \mathbb{R}^d .

1.2.1. The $|\varphi|^4$ Model. Given $n \geq 1$, a spin field is a function $\varphi : \Lambda_N \rightarrow \mathbb{R}^n$. We write this function as $x \mapsto \varphi_x = (\varphi_x^1, \dots, \varphi_x^n)$.

On \mathbb{R}^n , we use the Euclidean inner product $v \cdot w = \sum_{i=1}^n v^i w^i$, the Euclidean norm $|v|^2 = v \cdot v$, and write $|v|^4 = (v \cdot v)^2$. Given $g > 0, \nu \in \mathbb{R}$, we define a function $U_{g,\nu,N}$ of the field by

$$U_{g,\nu,N}(\varphi) = \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{2} \varphi_x \cdot (-\Delta \varphi)_x \right). \tag{1.1}$$

Then, the expectation of a random variable $F : (\mathbb{R}^n)^{\Lambda_N} \rightarrow \mathbb{R}$ is defined by

$$\langle F \rangle_{g,\nu,N} = \frac{1}{Z_{g,\nu,N}} \int F(\varphi) e^{-U_{g,\nu,N}(\varphi)} d\varphi, \tag{1.2}$$

where $d\varphi$ is the Lebesgue measure on $(\mathbb{R}^n)^\Lambda$, and $Z_{g,\nu,N}$ is a normalisation constant (the partition function) chosen so that $\langle 1 \rangle_{g,\nu,N} = 1$. Given a lattice point x , we define the finite and infinite volume two-point functions (whenever the infinite volume limit exists):

$$G_{x,N}(g, \nu; n) = \frac{1}{n} \langle \varphi_0 \cdot \varphi_x \rangle_{g,\nu,N}, \quad G_x(g, \nu; n) = \lim_{N \rightarrow \infty} G_{x,N}(g, \nu; n). \tag{1.3}$$

In the above limit, we identify a point $x \in \mathbb{Z}^d$ with $x \in \Lambda_N$ for large N , by embedding the vertices of Λ_N as an approximately centred cube in \mathbb{Z}^d (say as

$[-\frac{1}{2}L^N + 1, \frac{1}{2}L^N]^d \cap \mathbb{Z}^d$ if L^N is even and as $[-\frac{1}{2}(L^N - 1), \frac{1}{2}(L^N - 1)]^d \cap \mathbb{Z}^d$ if L^N is odd).

1.2.2. Weakly Self-Avoiding Walk. Let X be the continuous-time simple random walk on the lattice \mathbb{Z}^d , with $d > 0$. In other words, X is the stochastic process with right-continuous sample paths that takes steps uniformly at random to one of the $2d$ nearest neighbours of the current position at the events of a rate- $2d$ Poisson process. Steps are independent of the Poisson process and of all other steps. Let E_0 denote the expectation for the process with $X(0) = 0 \in \mathbb{Z}^d$. The local time of X at x up to time T is the random variable $L_T(x) = \int_0^T \mathbb{1}_{X(t)=x} dt$, and the self-intersection local time up to time T is the random variable:

$$I(T) = \int_0^T \int_0^T \mathbb{1}_{X(t_1)=X(t_2)} dt_1 dt_2 = \sum_{x \in \mathbb{Z}^d} (L_T(x))^2. \tag{1.4}$$

Given $g > 0, \nu \in \mathbb{R}$, and $x \in \mathbb{Z}^d$, the continuous-time weakly self-avoiding walk two-point function is defined by the (possibly infinite) integral:

$$G_x(g, \nu; 0) = \int_0^\infty E_0 \left(e^{-gI(T)} \mathbb{1}_{X(T)=x} \right) e^{-\nu T} dT. \tag{1.5}$$

We write $G_{x,N}$ for the finite-volume analogue of (1.5) on the torus Λ_N .

1.2.3. Critical Point and Correlation Length of Order p . For both models, i.e., for all integers $n \geq 0$, the susceptibility is defined by

$$\chi(g, \nu; n) = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} G_{x,N}(g, \nu; n). \tag{1.6}$$

The limit exists for $n = 0$ [4], but the general case is incomplete for $n \geq 1$ due to a lack of correlation inequalities for $n > 2$ [14]. The existence of the limits (1.3) and (1.6) in the contexts we study is established in [2, 18] (assuming L is large).

We write $a \sim b$ to mean $\lim a/b = 1$. It is proved in [2, 4] that for $n \geq 0$ and small $g > 0$, there exists a critical value $\nu_c = \nu_c(g; n) < 0$, such that the susceptibility diverges according to the asymptotic formula:

$$\chi(g, \nu_c + \varepsilon; n) \sim A_{g,n} \varepsilon^{-1} (\log \varepsilon^{-1})^{\frac{n+2}{n+8}} \quad \text{as } \varepsilon \downarrow 0, \tag{1.7}$$

for some amplitude $A_{g,n} > 0$. In addition, in [4, 18], it is proved that

$$\lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} G_{x,N}(g, \nu_c + \varepsilon; n) \sim (1 + z_0^c(g; n)) (-\Delta_{\mathbb{Z}^d})_{0x}^{-1} \quad \text{as } |x| \rightarrow \infty, \tag{1.8}$$

for some function $1 + z_0^c(g; n) = 1 + O(g)$ (the field strength renormalisation).

When $g = 0, \nu_c(0; n) = 0$ for all $n \geq 0$. For the free two-point function (as opposed to interacting when $g > 0$), we use $m^2 \geq 0$ instead of $\nu \geq \nu_c = 0$, as we will extend (1.8) to approximate $G_{x,N}(g, \nu; n)$ by its free counterpart $(1 + z_0^c)G_{x,N}(0, m^2)$ with carefully chosen m^2 . The free two-point function is independent of $n \geq 0$, and its infinite-volume version is equal to the lattice Green function:

$$G_x(0, m^2) = (-\Delta_{\mathbb{Z}^d} + m^2)_{0x}^{-1}. \tag{1.9}$$

In probabilistic terms, $G_x(0, m^2)$ equals $\frac{1}{2d}$ times the expected number of visits to x of a simple random walk on \mathbb{Z}^4 with killing rate m^2 , started from 0. It is proved in [2, 4] that for $n \geq 0$ and for small $g > 0$, $\nu_c(g; n) = -ag + O(g^2)$ with $\mathbf{a} = (n + 2)(-\Delta_{\mathbb{Z}^4})_{00}^{-1} > 0$.

Given a unit vector $e \in \mathbb{Z}^d$, the correlation length ξ is defined by

$$\xi(g, \nu; n) = \limsup_{k \rightarrow \infty} \frac{k}{\log G_{ke}(g, \nu; n)}. \tag{1.10}$$

It provides a characteristic length scale for the model. We study a related quantity, the correlation length of order $p > 0$, defined in terms of the infinite-volume two-point function and susceptibility by

$$\xi_p(g, \nu; n) = \left[\frac{\sum_{x \in \mathbb{Z}^4} |x|^p G_x(g, \nu; n)}{\chi(g, \nu; n)} \right]^{\frac{1}{p}}. \tag{1.11}$$

It is predicted that ξ_p has the same asymptotic behaviour near $\nu = \nu_c$ as the correlation length ξ , for all $p > 0$.

For all quantities defined above, we often omit the argument n from the notation.

1.3. Main Result

Our main result is the following theorem. We define constants $c_p > 0$ by

$$c_p^p = \int_{\mathbb{R}^d} |x|^p (-\Delta_{\mathbb{R}^d} + 1)_{0x}^{-1} dx, \tag{1.12}$$

and set $\tilde{A}_{g,n} = A_{g,n}/(1 + z_0^c)$ for the constants $A_{g,n}$ and z_0^c in (1.7)–(1.8). (We expect that $A_{g,n}, z_0^c$ are independent of L , but this has not been proved.)

Theorem 1.1. *Let $d = 4$, $n \geq 0$ and $p > 0$. For L sufficiently large (depending on p, n), and for $g > 0$ sufficiently small (depending on p, n), as $\varepsilon \downarrow 0$*

$$\xi_p(g, \nu_c + \varepsilon; n) \sim c_p \tilde{A}_{g,n}^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}} (\log \varepsilon^{-1})^{\frac{1}{2} \frac{n+2}{n+8}}. \tag{1.13}$$

Some results related to Theorem 1.1 have been obtained previously. For $n = 1$, the $\varepsilon^{-\frac{1}{2}} (\log \varepsilon^{-1})^{\frac{1}{8}}$ behaviour on the right-hand side of (1.13) was proven in [16] for the correlation length ξ of (1.10), in the sense of upper and lower bounds with different constants. For the $n = 0$ model, the end-to-end distance of a hierarchical version of the continuous-time weakly self-avoiding walk, up to time T , was shown to have $T^{\frac{1}{2}} (\log T)^{\frac{1}{8}}$ behaviour [9].

The proof of Theorem 1.1 involves a modification of the renormalisation group strategy used in [2–4, 18] to analyse the susceptibility and the critical two-point function. That strategy is based on a multi-scale analysis using a finite-range decomposition of the covariance $(-\Delta + m^2)^{-1} = \sum_j C_j$. The new ingredient in our proof is to take better advantage of the decay of C_j when j exceeds the mass scale j_m given by $L^{j_m} \approx m^{-1}$. Using this decay, beyond the mass scale we obtain better control over the two-point function than what was obtained in [3, 18], sufficient to analyse ξ_p and to prove Theorem 1.1. It would be of interest to extend this, to seek the further improvements that would be needed to analyse the correlation length ξ . Our new treatment leads to the

simplification that at scales beyond j_m , the large-field regulator \tilde{G}_j used in [2–4, 18] becomes superfluous, and the fluctuation-field regulator G_j suffices.

1.4. The Non-Interacting Model

An elementary ingredient in the proof of Theorem 1.1 is the following result for the $g = 0$ case, which is independent of $n \geq 0$. For simplicity, we restrict attention to dimensions $d > 2$, as only $d = 4$ is used in this paper.

Proposition 1.2. *For all dimensions $d > 2$ and all $p > 0$, as $m^2 \downarrow 0$*

$$\sum_{x \in \mathbb{Z}^d} |x|^p G_x(0, m^2) = c_p^p m^{-(p+2)} (1 + O(m)), \tag{1.14}$$

with c_p given by (1.12). In particular, $\xi_p(0, \varepsilon) = c_p \varepsilon^{-1/2} (1 + O(\varepsilon^{1/2}))$ as $\varepsilon \downarrow 0$.

Proposition 1.2 is presumably well known, but since we have not found a proof in the literature, we provide a proof in the Appendix. Note that this $g = 0$ case does not exhibit a logarithmic correction.

2. Proof of Main Result

In this section, we state Proposition 2.1, an improvement on the results of [18] (this reference subsumes and extends the results of [3]), and show that Theorem 1.1 is a consequence of Proposition 2.1.

The main conclusions of [18] are based on a rigorous renormalisation group method. The method uses an approximation of the interacting model by the non-interacting one, encoded by an n -dependent map

$$(g, \varepsilon) \mapsto (m^2, g_0, \nu_0, z_0) \tag{2.1}$$

with domain $[0, \delta)^2$ (for some small $\delta > 0$). Properties of this map are discussed briefly in [18, Section 4.6] where further references to [4] and [2] are given. In particular, the map (2.1) identifies the values of m^2, z_0^c that lead to the approximate equality $G_{x,N}(g, \nu; n) \approx (1 + z_0^c) G_{x,N}(0, m^2)$ discussed in Sect. 1.2.3.

A key ingredient of the renormalisation group method is a flow of renormalised coupling constants. The flow of the important coupling constant g is well approximated by the sequence \bar{g} defined by

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \quad \bar{g}_0 = g_0, \tag{2.2}$$

where the coefficients $\beta_j = \beta_j(m^2) > 0$ are defined in [2, (3.19)]. The β_j obey $\beta_j(m^2) \approx \beta_j(0)$ for $j \leq j_m$ and $\beta_j(m^2) \approx 0$ for $j \geq j_m$, where

$$j_m = \lfloor \log_L m^{-1} \rfloor \tag{2.3}$$

is the mass scale. We are interested in small m with L fixed, so j_m is large and positive. It follows that \bar{g}_j decays like $1/j$ for $j \leq j_m$ and is approximately constant for $j > j_m$ (see [2, Section 3.2] for details).

We estimate sums over $x \in \mathbb{Z}^4$ by dividing \mathbb{Z}^4 into shells $S_1 = \{x : |x| < \frac{1}{2}L\}$ and, for $j \geq 2$, $S_j = \{x : \frac{1}{2}L^{j-1} \leq |x| < \frac{1}{2}L^j\}$. The number of points in

S_j is bounded by $O(L^{4j})$. We refer to the integer j as a scale. Given $x \in \mathbb{Z}^4$, we define the coalescence scale to be the unique scale j_x , such that

$$x \in S_{j_x+1}. \tag{2.4}$$

Equivalently, $j_x = \max\{0, \lfloor \log_L(2|x|) \rfloor\}$; this introduces a minor notational clash with the mass scale j_m defined in (2.3) that should not cause problems. It follows from [4, Proposition 6.1] that

$$\bar{g}_j = O((\log m^{-1})^{-1}) \text{ for } j \geq j_m, \quad \bar{g}_{j_x} = O((\log |x|)^{-1}) \text{ for } j_x \leq j_m. \tag{2.5}$$

In [18, Remark 6.5], a remainder R_x is identified, such that

$$\frac{1}{1+z_0} G_x(g, \nu; n) = (1 + O(\bar{g}_{j_x})) G_x(0, m^2) + R_x(m^2, g_0; n), \tag{2.6}$$

for any $n \geq 0$ with m^2, g_0 given in terms of (g, ν) by (2.1), with

$$|R_x| \leq O(\bar{g}_{j_x}) G_x(0, 0). \tag{2.7}$$

Thus, (2.6) compares the value of the interacting theory on the left-hand side, evaluated at (g, ν) , with the first term on the right-hand side. The first term on the right-hand side is the corresponding free quantity at renormalised parameter values $(0, m^2)$.

However, with (2.7), the exponential decay present in $G_x(0, m^2)$ when $m^2 > 0$ is overwhelmed by the remainder term which involves instead the massless free two-point function $G_x(0, 0)$, and control needed for the correlation length of order p gets lost. In the next proposition, we improve the estimate (2.7) of [18, Lemma 5.6] by providing a new factor $(m|x|)^{-2s}$ when $|x|$ is large compared to the mass. Roughly, $L^{j_x} \approx |x|$ and $L^{j_m} \approx m^{-1}$, so when the coalescence scale exceeds the mass scale, $m|x|$ becomes greater than 1. Thus, the factor $(m|x|)^{-2s}$ gives good decay when the coalescence scale exceeds the mass scale, and we are free to choose $s > 0$ to be as large as desired.

Proposition 2.1. *Let $d = 4$, $n \geq 0$, $\varepsilon \in (0, \delta)$ with δ sufficiently small, and $\nu = \nu_c + \varepsilon$. Let $x \in \mathbb{Z}^4$ with $x \neq 0$. Fix any $s \geq 0$. For L sufficiently large and for $g > 0$ sufficiently small (depending on s)*

$$|R_x| \leq \frac{O(\bar{g}_{j_x})}{|x|^2} \times \begin{cases} 1 & (m|x| \leq 1) \\ (m|x|)^{-2s} & (m|x| \geq 1), \end{cases} \tag{2.8}$$

with the constant depending on L and s .

The proof of Proposition 2.1 constitutes the main part of this paper and is given in Sects. 4–5.

The case $s = 0$ of Proposition 2.1 is already given by (2.7). This case is insufficient to prove Theorem 1.1, as the remainder term R_x is not summable over $x \in \mathbb{Z}^4$ when $s = 0$. The improvement to arbitrary $s > 0$ in (2.8) represents the key innovation in this paper. Note that, in particular, R_x is summable after multiplication by $|x|^p$, provided $2s > p + 2$.

Before proving Proposition 2.1, we prove Theorem 1.1 assuming Proposition 2.1. In the proof, we use the important relation that

$$m^2 \sim \tilde{A}_{g,n}^{-1} \varepsilon (\log \varepsilon^{-1})^{-\frac{n+2}{4}} \quad \text{as } \varepsilon \downarrow 0, \tag{2.9}$$

which is proved in [2, (4.35)] for $n \geq 1$ and [4, (4.63)] for $n = 0$. In particular, the dependence of m^2 on ε encompasses the logarithmic correction for the susceptibility, since

$$\chi(g, \nu; n) = \frac{1 + z_0}{m^2}, \tag{2.10}$$

according to [2, (4.24)] for $n \geq 1$ and [4, (4.34)] for $n = 0$.

Proof of Theorem 1.1. We multiply (2.6) by $|x|^p$, sum over $x \in \mathbb{Z}^4$, and use (2.10), to obtain

$$\xi_p^p(g, \nu) = \sum_{x \in \mathbb{Z}^4} |x|^p \frac{G_x(g, \nu)}{\chi(g, \nu)} = m^2 \sum_{x \in \mathbb{Z}^4} |x|^p \left(G_x(0, m^2) + r_x(g, m^2) \right), \tag{2.11}$$

with

$$r_x = O(\bar{g}_{j_x}) G_x(0, m^2) + R_x. \tag{2.12}$$

By Proposition 1.2, this gives (as $m^2 \downarrow 0$)

$$\xi_p^p(g, \nu) \sim c_p^p m^{-p} + m^2 \sum_{x \in \mathbb{Z}^4} |x|^p r_x(g, m^2). \tag{2.13}$$

By (2.9), it suffices to prove that the first term on the right-hand side of (2.13) is dominant.

For the term $O(\bar{g}_{j_x}) G_x(0, m^2)$ in (2.12), we apply (2.5) to obtain

$$\begin{aligned} & \sum_{x \in \mathbb{Z}^4} \bar{g}_{j_x} |x|^p G_x(0, m^2) \\ & \leq \sum_{x: 0 < j_x \leq j_m} \frac{c|x|^p}{\log|x|} G_x(0, m^2) + \frac{c}{\log m^{-1}} \sum_{x: j_x > j_m} |x|^p G_x(0, m^2). \end{aligned} \tag{2.14}$$

In the first term, we use $G_x(0, m^2) \leq G_x(0, 0) \leq O(|x|^{-2})$. The restriction $j_x \leq j_m$ ensures that $|x| \leq O(m^{-1})$. Therefore, the first term is bounded above by a multiple of $(m^{-1})^{d+p-2} (\log m^{-1})^{-1}$, which suffices. For the term with $j_x > j_m$, we extend the sum to $x \in \mathbb{Z}^4$ and apply Proposition 1.2 to obtain a bound of the same form as for the first term.

Fix any $s > \frac{1}{2}(p + 2)$. For the term R_x of (2.12), we use Proposition 2.1 to see that

$$|R_x(g, m^2)| = O(\bar{g}_{j_x}) L^{-2j_x - 2s(j_x - j_m)_+}. \tag{2.15}$$

By (2.15) and (2.4)

$$\begin{aligned} \sum_{x \in \mathbb{Z}^4} |x|^p |R_x(g, m^2)| &= \sum_{j=1}^{\infty} \sum_{x \in S_j} |x|^p |R_x(g, m^2)| \\ &= \sum_{j=1}^{\infty} L^{4j+pj-2j-2s(j-j_m)_+} O(\bar{g}_j), \end{aligned} \tag{2.16}$$

with an L -dependent constant. By Lemma 2.2 (with $a = p + 2$ and $b = 1$), we obtain

$$m^2 \sum_{x \in \mathbb{Z}^4} |x|^p |R_x(g, m^2)| = O(m^{-p} (\log m^{-1})^{-1}). \quad (2.17)$$

The first term on the right-hand side of (2.13) therefore dominates, and the proof is complete. \square

The estimate used to obtain (2.17) is given by the following lemma, which is stated more generally for use in the proof of Proposition 1.2.

Lemma 2.2. *Let $L > 1$, $2s > a > 0$, $b \geq 0$, and let $\bar{g}_0 > 0$ be sufficiently small. Then*

$$\sum_{j=1}^{\infty} L^{aj-2s(j-j_m)+} \bar{g}_j^b = O(m^{-a} \bar{g}_{j_m}^b) = O(m^{-a} (\log m^{-1})^{-b}). \quad (2.18)$$

Proof. We divide the sum at the mass scale as

$$\sum_{j=1}^{\infty} L^{aj-2s(j-j_m)+} \bar{g}_j^b = \sum_{j=1}^{j_m} L^{aj} \bar{g}_j^b + \sum_{j=j_m+1}^{\infty} L^{aj-2s(j-j_m)} \bar{g}_j^b. \quad (2.19)$$

For the second sum on the right-hand side, we use $\bar{g}_j = O(\bar{g}_{j_m})$ for $j > j_m$ by (2.5), and obtain a bound consistent with the first equality of (2.18). For the first term, we use the crude bound $\bar{g}_i/\bar{g}_{i+1} = 1 + O(g_0)$ (by [6, Lemma 2.1]), and find

$$\sum_{j=1}^{j_m} L^{aj} \bar{g}_j^b \leq L^{aj_m} \bar{g}_{j_m}^b \sum_{j=1}^{j_m} ((1 + O(\bar{g}_0)) L^{-a})^{j_m-j} = O(L^{aj_m} \bar{g}_{j_m}^b), \quad (2.20)$$

for sufficiently small $\bar{g}_0 > 0$. This proves the first equality in (2.18). The second equality then follows, since $\bar{g}_{j_m} = O(\log m^{-1})$ by (2.5). \square

3. Renormalisation Group Method

We now provide a brief outline of the renormalisation group method developed in [5, 10–13], and identify the changes in the analysis of [18] needed to prove Proposition 2.1. We discuss some of the key points which have been explained in detail elsewhere, and make no attempt at completeness here. For notational simplicity, we consider the case $n \geq 1$ below; the case $n = 0$ is similar.

3.1. The Two-Point Function Via Gaussian Integration

The starting point of our study is the two-point function $G_{x,N}(g, \nu; n)$. The first step is a change of variables. Given $g > 0$, $\nu \in \mathbb{R}$, and given $m^2 > 0$ and $z_0 > -1$, let

$$g_0 = g(1 + z_0)^2, \quad \nu_0 = (1 + z_0)\nu - m^2. \tag{3.1}$$

Let $C = (-\Delta + m^2)^{-1}$ and let \mathbb{E}_C denote the expectation with respect to the Gaussian measure with covariance C . For $y \in \Lambda$, we define the monomials

$$\tau_y = \frac{1}{2}|\varphi_y|^2, \quad \tau_{\Delta,y} = \frac{1}{2}\varphi_y \cdot (-\Delta\varphi)_y. \tag{3.2}$$

Let $\mathbf{h} = n^{-1/2}(1, \dots, 1) \in \mathbb{R}^n$. With the two points $0, x \in \Lambda$ fixed, we introduce observable fields $\sigma_0, \sigma_x \in \mathbb{R}$, and define

$$U_0(\Lambda) = \sum_{y \in \Lambda} (g_0\tau_y^2 + \nu_0\tau_y + z_0\tau_{\Delta,y}) - \sigma_0(\varphi_0 \cdot \mathbf{h}) - \sigma_x(\varphi_x \cdot \mathbf{h}), \tag{3.3}$$

and

$$Z_0 = e^{-U_0(\Lambda)}. \tag{3.4}$$

It is then an elementary calculation (see [18, (6.3)]) to show that (for $n \geq 1$)

$$G_{x,N}(g, \nu; n) = (1 + z_0) \frac{\partial^2}{\partial \sigma_0 \partial \sigma_x} \Big|_0 \log \mathbb{E}_C Z_0. \tag{3.5}$$

The renormalisation group method provides a way to calculate the integral $\mathbb{E}_C Z_0$ and thereby compute (3.5). At this point, z_0 and m^2 are arbitrary, but careful choices of parameters will be required to make (3.5) useful, as in (2.1), and it is part of the method to determine this careful choice.

3.2. Progressive Integration

We evaluate the Gaussian integral $\mathbb{E}_C Z_0$ progressively, via the covariance decomposition

$$C = C_1 + \dots + C_{N-1} + C_{N,N} \tag{3.6}$$

constructed in [1] (see also [8]). For simplicity, we write $C_N = C_{N,N}$. For an integrable function F of the spin field φ , we let $\mathbb{E}_w \theta F$ be the convolution of F with the Gaussian measure of covariance w , i.e., $(\mathbb{E}_w \theta F)(\varphi) = \mathbb{E}_w F(\varphi + \zeta)$, where the expectation integrates the variable ζ . It is a property of Gaussian integration (see [10]) that

$$(\mathbb{E}_C \theta F)(\varphi) = (\mathbb{E}_{C_N} \theta \circ \mathbb{E}_{C_{N-1}} \theta \circ \dots \circ \mathbb{E}_{C_1} \theta F)(\varphi). \tag{3.7}$$

Let

$$Z_N = \mathbb{E}_C \theta Z_0 = \mathbb{E}_{C_N} \theta \circ \mathbb{E}_{C_{N-1}} \theta \circ \dots \circ \mathbb{E}_{C_1} \theta Z_0. \tag{3.8}$$

In particular

$$\mathbb{E}_C Z_0 = Z_N(0). \tag{3.9}$$

This allows us to evaluate the integral $\mathbb{E}_C Z_0$ by studying the dynamical system $Z_j \mapsto Z_{j+1}$ defined by

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j, \quad j < N. \tag{3.10}$$

For its analysis, we require a suitable space \mathcal{N} of functions of the spin and observable fields, on which the dynamical system acts. The space \mathcal{N} is discussed in detail in [18, Section 2.4.1]. The part of \mathcal{N} which does not involve the observable fields σ_0, σ_x is given by

$$\mathcal{N}^\varnothing = \mathcal{N}^\varnothing(\Lambda) = C^{\mathcal{P}\mathcal{N}}((\mathbb{R}^n)^\Lambda, \mathbb{R}). \tag{3.11}$$

The finite smoothness parameter $p_{\mathcal{N}}$ is discussed in Sect. 4.2 below, where it is explained that $p_{\mathcal{N}}$ must be chosen in a way that depends on the parameter p in Theorem 1.1. The part of \mathcal{N} involving the observable fields contains some subtleties that need not concern us here (see [18, Section 2.4.1] for details).

3.3. Local Field Polynomials

The dynamical system is analysed via a perturbative part which is tracked accurately to second order in g , together with a third-order non-perturbative part whose study forms the main part of our effort. For the perturbative part, we first introduce an appropriate space of local field polynomials.

For $y \in \Lambda$, we supplement (3.2) by defining

$$\tau_{\nabla\nabla,y} = \frac{1}{4} \sum_{e \in \mathbb{Z}^d: |e|=1} \nabla^e \varphi_y \cdot \nabla^e \varphi_y. \tag{3.12}$$

With $x \in \Lambda$ fixed, and given $g, \nu, z, y, u, \lambda_0, \lambda_x, q_0, q_x \in \mathbb{C}$, we extend (3.3) by defining the polynomial

$$\begin{aligned} V_y &= g\tau_y^2 + \nu\tau_y + z\tau_{\Delta,y} + y\tau_{\nabla\nabla,y} + u \\ &\quad - \mathbb{1}_{y=0}\lambda_0(\varphi_0 \cdot \mathbf{h})\sigma_0 - \mathbb{1}_{y=x}\lambda_x(\varphi_x \cdot \mathbf{h})\sigma_x \\ &\quad - \frac{1}{2}(\mathbb{1}_{y=0}q_0 + \mathbb{1}_{y=x}q_x)\sigma_0\sigma_x. \end{aligned} \tag{3.13}$$

Then, we define \mathcal{V} to be the space of functions $V = V_y$ of the form (3.13). Given $X \subset \Lambda$, we also define

$$\mathcal{V}(X) = \{V(X) = \sum_{y \in X} V_y : V \in \mathcal{V}\}. \tag{3.14}$$

We also make use of the subspaces $\mathcal{V}^{(1)} \subseteq \mathcal{V}$ consisting of polynomials with $y = 0$, as well as the subspace $\mathcal{V}^{(0)} \subseteq \mathcal{V}^{(1)}$ of polynomials with $u = y = q_0 = q_x = 0$. For $V \in \mathcal{V}$, we define maps $V \mapsto V^{(1)} \in \mathcal{V}^{(1)}$ and $V \mapsto V^{(0)} \in \mathcal{V}^{(0)}$. Both maps replace $z\tau_{\Delta} + y\tau_{\nabla\nabla}$ by $(z + y)\tau_{\Delta}$, and the latter additionally sets $u = q_0 = q_x = 0$.

3.4. Renormalisation Group Coordinates

For $j = 0, \dots, N$, we partition Λ into L^{N-j} disjoint scale- j blocks of side length L^j . A scale- j polymer is a union of scale- j blocks. The set of all scale- j blocks is denoted \mathcal{B}_j , and the set of all scale- j polymers is denoted \mathcal{P}_j . For $X \in \mathcal{P}_j$, we write $\mathcal{B}_j(X)$ for the set of scale- j blocks in X . For $F, G : \mathcal{P}_j \rightarrow \mathcal{N}$, we define the circle product $F \circ G : \mathcal{P}_j \rightarrow \mathcal{N}$ by

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}_j(X)} F(X \setminus Y)G(Y). \tag{3.15}$$

The evolution of Z_j can be tracked in the renormalisation group coordinates $\zeta_j \in \mathbb{R}, I_j, K_j : \mathcal{P}_j \rightarrow \mathcal{N}$, defined such that

$$Z_j = e^{\zeta_j}(I_j \circ K_j)(\Lambda), \quad \zeta_j = -u_j|\Lambda| + \frac{1}{2}(q_{0,j} + q_{x,j})\sigma_0\sigma_x. \tag{3.16}$$

The coordinate I_j tracks the evolution of the relevant and marginal directions. It is determined by a local polynomial $U \in \mathcal{V}^{(0)}$, and takes the form

$$I_j(X) = \prod_{B \in \mathcal{B}_j(X)} e^{-U(B)}(1 + W_j(B, U)), \quad X \in \mathcal{P}_j, \quad (3.17)$$

with W_j an explicit quadratic term in U (defined in [5, (3.21)]). The evolution of (ζ, U) to second order is called the perturbative flow and is given by the explicit map $V_{\text{pt}} : \mathcal{V} \rightarrow \mathcal{V}$ defined in [5, (3.23)]. In particular, it is shown in [18, Proposition 3.2] that the perturbative flow of q is given by

$$q_{\text{pt}} = q + \lambda_0 \lambda_x C_{j+1;0x}, \quad (3.18)$$

and that the perturbative flow of λ_0 and λ_x becomes the identity map once j exceeds the coalescence scale j_x .

At scale $j = 0$, we are given U_0 as defined in (3.3) and we set $\zeta_0 = 0$. In particular, the initial values of u, q_0, q_x are zero, and the initial values of λ_0, λ_x are 1. By definition, $W_0 = 0$, and we have $I_0(X) = e^{-U_0(X)}$. We define $\mathbb{1}_\emptyset : \mathcal{P}_0 \rightarrow \mathcal{N}$ by

$$\mathbb{1}_\emptyset(X) = \begin{cases} 1 & X = \emptyset \\ 0 & \text{otherwise,} \end{cases} \quad (3.19)$$

and set $K_0 = \mathbb{1}_\emptyset$. With these choices, Z_0 of (3.4) takes the form (3.16), and we seek (ζ_j, U_j, K_j) , such that this continues to hold as the scale advances.

Equivalently, given (U_j, K_j) , we define $(\delta\zeta_{j+1}, U_{j+1}, K_{j+1})$, so that

$$\mathbb{E}_{j+1} \theta(I_j \circ K_j)(\Lambda) = e^{-\delta\zeta_{j+1}} (I_{j+1} \circ K_{j+1})(\Lambda), \quad (3.20)$$

where $\delta\zeta_{j+1} = \zeta_{j+1} - \zeta_j$. Moreover, we need K_j to contract as the scale advances, under an appropriate norm. The construction of (scale-dependent) maps V_+ and K_+ , such that (3.20) holds with

$$(\delta\zeta_{j+1}, U_{j+1}, K_{j+1}) = (V_+(U_j, K_j), K_+(U_j, K_j)) \quad (3.21)$$

is the main accomplishment of [13] and is summarised in Sect. 3.5, in a form adapted to our current setting.

3.5. The Main Theorem

We define a scale-dependent norm

$$\|V\|_{\mathcal{V}} = \max \left\{ |g|, L^{2j} |\nu|, |z|, |y|, L^{4j} |u|, \ell_j \ell_{\sigma,j} (|\lambda_0| \vee |\lambda_x|), \ell_{\sigma,j}^2 (|q_0| \vee |q_x|) \right\} \quad (3.22)$$

on $V \in \mathcal{V}$, which depends on parameters ℓ_j and $\ell_{\sigma,j}$. An innovation in this paper is that we define these parameters by

$$\ell_j = \ell_0 L^{-j-s(j-j_m)_+}, \quad \ell_{\sigma,j} = \ell_{j \wedge j_x}^{-1} 2^{(j-j_x)_+} \tilde{g}_j, \quad (3.23)$$

where the mass scale j_m is defined in (2.3), the coalescence scale j_x is defined in (2.4), and s is the parameter appearing in Proposition 2.1. The sequence $\tilde{g} = \tilde{g}(m^2, g_0)$ is defined in [4, (6.15)]; it is bounded above and below by constant multiples of the sequence \bar{g} defined in (2.2), by [4, Lemma 7.4]. We discuss the origin of the definition (3.23) in detail in Sect. 4.

The following theorem is a restatement of [18, Theorem 5.1] with three changes. The first, minor, change is the specialisation to the case $p = 1$ and $h = \mathfrak{h}$ (in the notation of [18]). The second change is the main accomplishment of this paper, namely, that the norms in the estimates (3.27) below use the new norm parameters (3.23). The third change is that we have omitted some technical details concerning the parameter m^2 to simplify this brief summary; these details are as in [18, Theorem 5.1]. In particular, m^2 must be chosen small in Theorem 3.1.

In [13], maps V_+, K_+ are defined which map a pair (U, K) at scale j to $(V_+(U, K), K_+(U, K))$ at scale $j + 1$, and which preserve the circle product $I \circ K$ under expectation as in (3.20). A norm has already been defined on the space \mathcal{V} in (3.22). We specify a domain \mathcal{D}_j for U in $\mathcal{V}^{(0)}$ in Sect. 5.2. We also require a norm on a space \mathcal{K} containing the non-perturbative coordinate K (see [18, Definition 4.5]), which is the \mathcal{W}_j norm of [13, (1.45)]. We denote the ball of radius r in the normed space \mathcal{W}_j by $B_{\mathcal{W}_j}(r)$. Given $\alpha > 0$, we define the domain

$$\mathbb{D}_j = \mathcal{D}_j \times B_{\mathcal{W}_j}(\alpha \tilde{\vartheta}_j \tilde{g}_j^3), \tag{3.24}$$

where \tilde{g}_j was discussed above, and $\tilde{\vartheta}_j$ is a sequence that, roughly, is equal to 1 below the mass scale and decays exponentially above the mass scale (discussed above [13, (1.65)]). Then, at scale j , the maps V_+, K_+ act on the domain \mathbb{D}_j and map into $\mathcal{V}_{j+1}^{(1)}, \mathcal{K}_{j+1}$, respectively. The deviation of the map V_+ from the perturbative map V_{pt} (mentioned above (3.18)) is denoted by R_+ , and is defined by

$$R_+(U, K) = V_+(U, K) - V_{\text{pt}}^{(1)}(U). \tag{3.25}$$

The following theorem is applied with $\alpha = 4M$ as a convenient choice.

Theorem 3.1. *Let $d = 4$ and let $n \geq 0$. Fix $s > 0$. Let $C_{\mathcal{D}}$ and L be sufficiently large. There exist $M > 0$ and $\delta > 0$, such that for $\tilde{g} \in (0, \delta)$, and with the domain \mathbb{D} defined using any $\alpha > M$, the maps*

$$R_+ : \mathbb{D} \rightarrow \mathcal{V}^{(1)}, \quad K_+ : \mathbb{D} \rightarrow \mathcal{W}_+ \tag{3.26}$$

define $(U, K) \mapsto (V_+, K_+)$ obeying (3.20), and satisfy the estimates

$$\|R_+\|_{\mathcal{V}} \leq M \tilde{\vartheta}_+ \tilde{g}_+^3, \quad \|K_+\|_{\mathcal{W}_+} \leq M \tilde{\vartheta}_+ \tilde{g}_+^3. \tag{3.27}$$

The proof of Theorem 3.1 is identical to the proof of [18, Theorem 5.1], via a version of [13, Theorems 1.10–1.11] that uses the norm parameters (3.23) with $s > 0$. The proof of the latter results with these new norm parameters amounts to checking that the proof of the $s = 0$ case contained in [12, 13] continues to hold with $s > 0$. A verification of this fact is carried out in Sect. 5.

Theorem 3.1 expresses a contractive property of the map K_+ , as it takes K in a ball whose radius involves $\alpha = 4M$ at scale j , to an image which lies in a ball whose radius involves the smaller number M at scale $j + 1$. The fact that K_+ is a contraction is used in [4, Proposition 8.1] (for $n = 0$) and [2, Theorem 3.6] (for $n \geq 1$) to prove that, for m^2 and g_0 sufficiently small, there exist critical initial conditions $\nu_0 = \nu_0^c(m^2, g_0)$ and $z_0 = z_0^c(m^2, g_0)$ such

that, for the case of no observables ($\sigma_0 = \sigma_x = 0$), iteration of the maps (V_+, K_+) defines a sequence $(V_j^{(0)}, K_j)$ which lies in the domain \mathbb{D}_j and obeys the estimates (3.27) for all $j = 1, \dots, N$. This construction of critical initial conditions uses the $s = 0$ version of (3.23).

The case with observable fields included is handled in [18]. Because we have increased $\ell_{\sigma,j}$ beyond the mass scale, the estimates on q_0, q_x given by the bound on R_+ in (3.27) are significantly improved compared to their versions with $s = 0$ in [18]. As is discussed in detail in [18, Section 5], $V_j^{(0)}$ remains in the domain \mathcal{D}_j for all j (also concerning $\lambda_{0,j}, \lambda_{x,j}$). Moreover, $\lambda_{0,j}, \lambda_{x,j}, q_{0,j}, q_{x,j}$, are independent of the volume parameter N and so can be extended to infinite sequences, and the following limits exist:

$$q_{u,\infty} = \lim_{j \rightarrow \infty} q_{u,j}, \quad u = 0, x. \tag{3.28}$$

3.6. Identity for the Two-Point Function

At scale N , the torus Λ is a single block, and (3.16) gives

$$Z_N = e^{\zeta_N} (I_N(\Lambda) + K_N(\Lambda)). \tag{3.29}$$

Evaluation at $\varphi = 0$ gives

$$Z_N(0) = e^{\zeta_N} (1 + K_N(\Lambda, 0)). \tag{3.30}$$

Thus, by (3.5)

$$\frac{1}{1 + z_0} G_{x,N}(g, \nu) = \frac{1}{2} (q_{0,N} + q_{x,N}) + \frac{D_{\sigma_a \sigma_b}^2 K_N}{1 + K_N} - \frac{(D_{\sigma_a} K_N)(D_{\sigma_b} K_N)}{(1 + K_N)^2} \tag{3.31}$$

(with derivatives taken at $\sigma_0 = \sigma_x = 0$ on the right-hand side). The bound [18, (6.13)] is only improved by the new choice of norm, to assert that, for $l = 0, 1, 2$, the l^{th} derivative of $K_N(\Lambda)$ with respect to the observable field is bounded above by a multiple of $2^{-l(N-j_x)} |x|^{-l} L^{-ls(N-j_x)} \vartheta_N \bar{g}_N^{3-l}$. In particular, the last two terms of (3.31) vanish as $N \rightarrow \infty$, and

$$\frac{1}{1 + z_0} G_x(g, \nu; n) = \frac{1}{2} (q_{0,\infty} + q_{x,\infty}). \tag{3.32}$$

It is then a consequence of (3.18) and (3.25) (as in the proof of [18, Lemma 5.6]) that

$$q_{u,\infty}(m^2) = \lambda_{0,j_x} \lambda_{x,j_x} G_x(0, m^2) + \sum_{i=j_x}^{\infty} R_i^{qu}, \quad u = 0, x, \tag{3.33}$$

where R_i^{qu} is the coefficient of $\mathbb{1}_{y=u} \sigma_0 \sigma_x$ (recall (3.13)) of $R_{+,i}$ (recall (3.25)). Moreover, as in [18, (5.30)] and [18, Corollary 6.4]

$$\lambda_{u,j_x} = 1 + O(\vartheta_{j_x} \bar{g}_{j_x}). \tag{3.34}$$

It follows that

$$\frac{1}{1 + z_0} G_x(g, \nu; n) = (1 + O(\bar{g}_{j_x})) G_x(0, m^2) + R_x, \tag{3.35}$$

with

$$R_x = \frac{1}{2} \sum_{i=j_x}^{\infty} (R_i^{q_0} + R_i^{q_x}). \tag{3.36}$$

This provides a more detailed statement of (2.6).

3.7. Proof of Proposition 2.1

By the first bound of (3.27) and the definition (3.22)

$$|R_{+,i}^{q_u}| \leq O(\ell_{\sigma,i}^{-2} \vartheta_i \bar{g}_i^3). \tag{3.37}$$

Using the old norm parameters $\ell_{\sigma}^{\text{old}}$, the sum R_x in (3.36) is bounded by the right-hand side of (2.7). With the new norm parameters, we instead get the result of Proposition 2.1.

Proof of Proposition 2.1 (assuming Theorem 3.1). We insert the definition of $\ell_{\sigma,j}$ from (3.23) into (3.37). We also use $\tilde{g}_j^{-2} = O(\bar{g}_j^{-2})$, $\vartheta_i \leq 1$, $\ell_0^2 \leq O(1)$, as well as $\bar{g}_j \leq O(\bar{g}_{j_x})$ for $j \geq j_x$. The definitions of the coalescence scale j_x and the mass scale j_m imply that $L^{-2j_x} \leq O(|x|^{-2})$ and $L^{-(j_x-j_m)_+} \leq O((m|x|)^{-1})$. All this leads to

$$\begin{aligned} \sum_{j=j_x}^{\infty} |R_j^{q_u}| &\leq L^{-2j_x - 2s(j_x-j_m)_+} \sum_{j=j_x}^{\infty} O(\bar{g}_j) 4^{-(j-j_x)} \\ &\leq |x|^{-2} (m|x|)^{-2s} O(\bar{g}_{j_x}). \end{aligned} \tag{3.38}$$

This gives the desired estimate (2.8). □

Thus, to prove Proposition 2.1, it suffices to show that Theorem 3.1 holds with the s -dependent choice (3.23), for arbitrary $s > 0$. Constants in estimates will depend on s , and since we used $s > \frac{1}{2}(p+2)$ in the proof of Theorem 1.1, such constants depend on p .

4. Improved Norm

The proof of Theorem 3.1 is based on the observation that it is possible to use the parameters (3.23) in the norm used in [12], instead of the $s = 0$ version used previously. In this section, we first state improved covariance estimates, thereby indicating why it is possible to improve the norm. This leads to a discussion of simplified norm pairs beyond the mass scale. A lemma concerning the fluctuation-field regulator indicates why the simplification is possible. In the following, we use the notation appropriate for the spin field $\varphi \in (\mathbb{R}^n)^\Lambda$ for $n \geq 1$; only notational modifications are needed for $n = 0$.

4.1. Covariance Bounds

The estimate in [18] which yields the $s = 0$ case of (2.8) uses the norms defined in [12]. One of these norms is the $\Phi_j(\ell_j)$ norm defined by

$$\|\varphi\|_{\Phi_j(\ell_j)} = \ell_j^{-1} \sup_{x \in \Lambda} \sup_{|\alpha|_1 \leq p_\Phi} L^{j|\alpha|_1} |\nabla^\alpha \varphi_x|, \tag{4.1}$$

which depends on the parameter ℓ_j , and on the maximal number of discrete derivatives p_Φ (fixed to be at least 4 in [12]). As in (3.23), we now define

$$\ell_j = \ell_0 L^{-j-s(j-j_m)_+}, \quad \ell_{\sigma,j} = \ell_{j \wedge j_x}^{-1} 2^{(j-j_x)_+} \tilde{g}_j. \tag{4.2}$$

The analysis of [12, 13] uses the norm parameters ℓ_j and $\ell_{\sigma,j}$ with $s = 0$. To distinguish these from our new choice (4.2) of ℓ_j and $\ell_{\sigma,j}$, we write

$$\ell_j^{\text{old}} = \ell_0 L^{-j}, \quad \ell_{\sigma,j}^{\text{old}} = (\ell_{j \wedge j_x}^{\text{old}})^{-1} 2^{(j-j_x)_+} \tilde{g}_j. \tag{4.3}$$

In the more general terminology and notation of [10, 12], we may regard a covariance C_j in the decomposition (3.6) as a test function depending on two arguments x, y , and with this identification its $\Phi_j(\ell_j)$ norm is

$$\|C_j\|_{\Phi_j(\ell_j)} = \ell_j^{-2} \sup_{x,y \in \Lambda} \sup_{|\alpha|_1 + |\beta|_1 \leq p_\Phi} L^{(|\alpha|_1 + |\beta|_1)j} |\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}|. \tag{4.4}$$

The purpose of the $\Phi_j(\ell_j)$ norm is to measure the size of typical fluctuation fields φ with covariance C_j . The parameter ℓ_j is chosen so that the norm of a typical field should be $O(1)$, independent of j .

The following lemma justifies our choice of ℓ_j in (4.2), by showing that the bound [12, (1.73)], proved there only for the $s = 0$ version ℓ_j^{old} of (4.3), remains true with the stronger choice of norm parameter ℓ_j that permits arbitrary $s \geq 0$. In its statement, the bounded sequence ϑ_j decays exponentially after the mass scale and may be taken to be equal to $2^{-(j-j_m)_+}$; its details are given in [12, Section 1.3.1] (where it is called χ_j rather than ϑ_j).

Lemma 4.1 (Extension of [12, (1.73)]). *Given $\mathfrak{c} \in (0, 1]$, ℓ_0 can be chosen large (depending on L, \mathfrak{c}, s), so that*

$$\|C_j\|_{\Phi_j(\ell_j)} \leq \min(\mathfrak{c}, \vartheta_j). \tag{4.5}$$

The proof of Lemma 4.1 uses an estimate from [5, Proposition 6.1(a)], which we repeat here as the following proposition.

Proposition 4.2 (Restatement of [5, Proposition 6.1(a)]). *Let $d > 2$, $L \geq 2$, $j \geq 1$, $\bar{m}^2 > 0$. For multi-indices α, β with ℓ^1 norms $|\alpha|_1, |\beta|_1$ at most some fixed value p , and for any k , and for $m^2 \in [0, \bar{m}^2]$*

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \leq c(1 + m^2 L^{2(j-1)})^{-k} L^{-(j-1)(d-2+|\alpha|_1+|\beta|_1)}, \tag{4.6}$$

where $c = c(p, k, \bar{m}^2)$ is independent of m^2, j, L . The same bound holds for $C_{N,N}$ if $m^2 L^{2(N-1)} \geq \varepsilon$ for some $\varepsilon > 0$, with c depending on ε but independent of N .

Proof of Lemma 4.1. For $d = 4$, insertion of (4.6) into (4.4) gives

$$\|C_j\|_{\Phi_j(\ell_j)} \leq c L^{p_\Phi} \ell_j^{-2} (1 + m^2 L^{2(j-1)})^{-k} L^{-2(j-1)}. \tag{4.7}$$

With $s = 0$ in (4.2), (4.7) gives $\|C_j\|_{\Phi_j(\ell_j)} \leq c_L \ell_0^{-2} (1 + m^2 L^{2(j-1)})^{-k}$ for an L -dependent constant c_L (whose value may now change from line to line). The estimate [12, (1.73)] is wasteful in that it does not make any use of the factor

$(1 + m^2 L^{2(j-1)})^{-k}$ in (4.7) beyond extraction of the factor ϑ_j . To improve this, we now allow arbitrary s , and fix the arbitrary parameter k to be $k = s + 1$ in (4.7), so that

$$(1 + m^2 L^{2j})^{-k} \leq c_L L^{-2(s+1)(j-j_m)_+}. \tag{4.8}$$

We insert (4.8) and the definition $\ell_j = \ell_0 L^{-j-s(j-j_m)_+}$ from (4.2) into (4.7), to conclude that there exists $c_0 = c_0(s, L)$, such that

$$\|C_j\|_{\Phi_j(\ell_j)} \leq c_0 \ell_0^{-2} L^{-2(j-j_m)_+}. \tag{4.9}$$

By definition of ϑ_j (see [12, Section 1.3.1]), $L^{-2(j-j_m)_+}$ is bounded by a multiple of ϑ_j . It thus suffices to choose ℓ_0 large enough that $\ell_0^2 \geq c_0 \mathfrak{c}^{-1}$. \square

4.2. New Choice of Norm Beyond the Mass Scale

As in [12, (1.36)], we use the localised version of (4.1), defined for subsets $X \subset \Lambda$ by

$$\|\varphi\|_{\Phi_j(X)} = \inf\{\|\varphi - f\|_{\Phi_j} : f \in \mathbb{C}^\Lambda \text{ such that } f_x = 0 \ \forall x \in X\}. \tag{4.10}$$

A small set is defined to be a connected polymer $X \in \mathcal{P}_j$ consisting of at most 2^d blocks (the specific number 2^d plays no direct role here), and $\mathcal{S}_j \subset \mathcal{P}_j$ denotes the set of small sets. The small set neighbourhood of $X \subset \Lambda$ is the enlargement of X defined by $X^\square = \bigcup_{Y \in \mathcal{S}_j: X \cap Y \neq \emptyset} Y$.

Given $X \subset \Lambda$ and $\varphi \in (\mathbb{R}^n)^\Lambda$, we recall from [12, (1.36)] that the fluctuation-field regulator G_j is defined by

$$G_j(X, \varphi) = \prod_{x \in X} \exp\left(|B_x|^{-1} \|\varphi\|_{\Phi_j(B_x^\square, \ell_j)}^2\right), \tag{4.11}$$

where $B_x \in \mathcal{B}_j$ is the unique block that contains x , and hence $|B_x| = L^{dj}$. The large-field regulator is defined in [12, (1.41)] by

$$\tilde{G}_j(X, \varphi) = \prod_{x \in X} \exp\left(\frac{1}{2} |B_x|^{-1} \|\varphi\|_{\tilde{\Phi}_j(B_x^\square, \ell_j)}^2\right). \tag{4.12}$$

The $\tilde{\Phi}_j$ norm appearing on the right-hand side of (4.12) is similar to the Φ_j norm, with the important difference that it is insensitive to shifts by linear test functions; see [12, (1.40)] for the precise definition. The two regulators serve as weights in the regulator norms of [12, Definition 1.1]. The regulator norms are defined, with $\gamma \in (0, 1]$ and for F in the space $\mathcal{N}(X^\square)$ of functionals of the field (see [10, (3.38)]), by

$$\|F\|_{G_j(\ell_j)} = \sup_{\varphi \in (\mathbb{R}^n)^\Lambda} \frac{\|F\|_{T_{\varphi, j}(\ell_j)}}{G_j(X, \varphi)}, \tag{4.13}$$

$$\|F\|_{\tilde{G}_j^\gamma(h_j)} = \sup_{\varphi \in (\mathbb{R}^n)^\Lambda} \frac{\|F\|_{T_{\varphi, j}(h_j)}}{\tilde{G}_j^\gamma(X, \varphi)}. \tag{4.14}$$

The parameter ℓ_j that appears in the regulators (4.11)–(4.12) and in the numerator of (4.13) was taken to be ℓ_j^{old} in [12], but now we use ℓ_j instead. As in [12], the parameter h_j and its observable counterpart $h_{\sigma,j}$ are given by

$$h_j = k_0 \tilde{g}_j^{-1/4} L^{-j}, \quad h_{\sigma,j} = (\ell_{j \wedge j_x}^{\text{old}})^{-1} 2^{(j-j_x)+} \tilde{g}_j^{1/4}. \quad (4.15)$$

In [12], estimates on $\|\cdot\|_{j+1}$ are given in terms of $\|\cdot\|_j$, where the pair $(\|\cdot\|_j, \|\cdot\|_{j+1})$ refers to either of the norm pairs

$$\|F\|_j = \|F\|_{G_j(\ell_j^{\text{old}})} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{T_{0,j+1}(\ell_{j+1}^{\text{old}})}, \quad (4.16)$$

or

$$\|F\|_j = \|F\|_{\tilde{G}_j(h_j)} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{\tilde{G}_{j+1}^\gamma(h_{j+1})}. \quad (4.17)$$

We will show that, above the mass scale j_m (see (2.3)), the results of [12] hold with both norm pairs in (4.16) and (4.17) replaced by the single new norm pair

$$\|F\|_j = \|F\|_{G_j(\ell_j)} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{G_{j+1}(\ell_{j+1})}, \quad (4.18)$$

with the improved ℓ_j of (4.2) with $s > 0$ fixed as large as desired.

The space \mathcal{N} containing the functionals F appearing above requires control on up to $p_{\mathcal{N}}$ derivatives of F with respect to the field φ , where $p_{\mathcal{N}}$ is a parameter of the T_φ norm. In the proof of Proposition 5.1, we must choose $p_{\mathcal{N}}$ to be large depending on p , to analyse the correlation length of order p . The renormalisation group analysis is predicated on fixed (but arbitrary) $p_{\mathcal{N}}$, so it can proceed with this modification. However, we do not prove that constants are uniform in $p_{\mathcal{N}}$, and in particular, we do not prove that the required smallness of g in Theorem 1.1 is uniform in the choice of $p_{\mathcal{N}}$. Thus, we do not have a result for all $p > 0$ for any fixed g .

The use of two norm pairs adds intricacy to [12,13]. The pair (4.16) is insufficient, on its own, because the scale- $(j + 1)$ norm is the T_0 semi-norm which controls only small fields, and an estimate in this norm does not imply an estimate for the G_{j+1} norm. The norm pair (4.17) is used to supplement the norm pair (4.16), and estimates in both the scale- $(j + 1)$ norms can be combined to provide an estimate for the G_{j+1} norm. This then sets the stage for the next renormalisation group step. Above the mass scale, the use of (4.18) now bypasses many issues. For example, for $j > j_m$, the \mathcal{W}_j norm of [13, (1.45)] is replaced simply by the $\mathcal{F}_j(G)$ norm, and there is no need for the \mathcal{Y}_j norm of [13, (2.12)] nor for [13, Lemma 2.4].

The need for both norm pairs (4.16)–(4.17) is discussed in [12, Section 1.2.1] and is related to the so-called large-field problem. Roughly speaking, the norm pair (4.17) is used to take advantage of the quartic term in the interaction to suppress the effects of large values of the fields. This approach relies on the fact that the interaction polynomial is dominated by the quartic term in the h norm, as expressed by [12, (1.91)], together with the lower bound [12, (1.90)] on the quartic term. However, above the mass scale, large fields are naturally suppressed by the rapid decay of the covariance. This idea is captured in Lemma 4.3, which replaces [12, Lemma 1.2] above the mass scale.

The regulators in its statement are defined by (4.11) with the s -dependent ℓ_j of (4.2).

Lemma 4.3 (Replacement for [12, Lemma 1.2]). *Let $X \subset \Lambda$ and assume that $s > 1$. For any $q > 0$, if L is sufficiently large depending on q , then for $j_m \leq j < N$*

$$G_j(X, \varphi)^q \leq G_{j+1}(X, \varphi). \tag{4.19}$$

Proof. By (4.11), it suffices to show that, for any scale- j block B_j and any scale- $(j + 1)$ block B_{j+1} containing B_j

$$q \|\varphi\|_{\Phi_j(B_j^\square, \ell_j)}^2 \leq L^{-4} \|\varphi\|_{\Phi_{j+1}(B_{j+1}^\square, \ell_{j+1})}^2. \tag{4.20}$$

In fact, since $\|\varphi\|_{\Phi_j(B_j^\square, \ell_j)} \leq \|\varphi\|_{\Phi_j(B_{j+1}^\square, \ell_j)}$ by definition, it suffices to prove the above bound with B_j replaced by B_{j+1} on the left-hand side. According to the definition of the norm in (4.10), to show this it suffices to prove that

$$q \|\varphi\|_{\Phi_j(\ell_j)}^2 \leq L^{-4} \|\varphi\|_{\Phi_{j+1}(\ell_{j+1})}^2 \tag{4.21}$$

(then we replace φ by $\varphi - f$ in the above and take the infimum).

By definition

$$\|\varphi\|_{\Phi_j(\ell_j)} \leq \ell_j^{-1} \ell_{j+1} \sup_{x \in \Lambda} \sup_{|\alpha| \leq p_\Phi} \ell_{j+1}^{-1} L^{(j+1)|\alpha|} |\nabla^\alpha \varphi_x|, \tag{4.22}$$

with the inequality due to replacement of $L^{j|\alpha|}$ on the left-hand side by $L^{(j+1)|\alpha|}$ on the right-hand side. Since $\ell_j^{-1} \ell_{j+1} = L^{-1-s}$ $j \geq j_m$

$$\|\varphi\|_{\Phi_j(\ell_j)} \leq L^{-1-s} \quad j \geq j_m \quad \|\varphi\|_{\Phi_{j+1}(\ell_{j+1})}. \tag{4.23}$$

Thus

$$q \|\varphi\|_{\Phi_j(\ell_j)}^2 \leq q L^{-4} L^{2-2s} \quad j \geq j_m \quad \|\varphi\|_{\Phi_{j+1}(\ell_{j+1})}^2, \tag{4.24}$$

and then (4.21) follows once L is large enough that $qL^{2-2s} \leq 1$. □

Remark 4.4. The elimination of the h norm after the mass scale is more than a convenience. It becomes a necessity when we improve the ℓ norm. Briefly, the reason is as follows. In the proof of [13, Lemma 2.4], the ratio ℓ_σ/h_σ must be bounded. For this, we would need to increase h_σ beyond the mass scale (since ℓ_σ has been increased). This forces a compensating decrease in h beyond j_m , to keep the product hh_σ bounded for stability (as in Sect. 5.2). However, if we do this, we lose the lower bound required on $\epsilon_{g\tau^2}$ required for stability in the h norm (see [12, (3.8)]).

5. Proof of Theorem 3.1

In this section, we show that Theorem 3.1 holds, thereby completing the proof of Proposition 2.1. The key steps in the proof of the $s = 0$ case of Theorem 3.1 are contained in [12, 13]. Our main objective in this section is to show that the results in [12, 13] continue to hold with the new norm parameters $\ell_j, \ell_{\sigma,j}$. To this end, we may and do use the fact that the estimates of [12] have already been established with the old norm parameters.

In the following, we indicate the changes in the analysis of [12, 13] that arise due to the new choice of norm parameters (4.2) beyond the mass scale, and due to the reduction from two norm pairs to one. This requires repeated reference to previous papers.

5.1. Norm Parameter Ratios

The analysis of [12] assumes that the norm parameters $\mathfrak{h}_j, \mathfrak{h}_{\sigma,j}$, for $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$, satisfy the estimates [12, (1.79)]; these assert that

$$\mathfrak{h}_j \geq \ell_j, \quad \frac{\mathfrak{h}_{j+1}}{\mathfrak{h}_j} \leq 2L^{-1}, \quad \frac{\mathfrak{h}_{\sigma,j+1}}{\mathfrak{h}_{\sigma,j}} \leq \text{const} \begin{cases} L & (j < j_x) \\ 1 & (j \geq j_x). \end{cases} \tag{5.1}$$

We do not change \mathfrak{h}_j or $\mathfrak{h}_{\sigma,j}$ for j below the mass scale, so there can be no difficulty until above the mass scale. Above the mass scale, the parameters $h_j, h_{\sigma,j}$ are eliminated, and requirements involving them become vacuous. Thus, for (5.1), we need only verify the second and third inequalities for the case $\mathfrak{h} = \ell$. By definition

$$\frac{\ell_{j+1}}{\ell_j} = L^{-(1+s \cdot \mathbb{1}_{j \geq j_m})}, \quad \frac{\ell_{\sigma,j+1}}{\ell_{\sigma,j}} = \frac{\tilde{g}_{j+1}}{\tilde{g}_j} \times \begin{cases} L^{1+s} & j \geq j_m & (j < j_x) \\ 2 & & (j \geq j_x). \end{cases} \tag{5.2}$$

According to [12, (1.77)], $\frac{1}{2}\tilde{g}_{j+1} \leq \tilde{g}_j \leq 2\tilde{g}_{j+1}$. Thus, the second estimate of (5.1) is satisfied (the ratio being improved when $j \geq j_m$), while the third is not when $s > 0$ and $j_m < j_x$. This potentially dangerous third estimate in (5.1) is used to prove the scale monotonicity lemma [12, Lemma 3.2], as well as the crucial contraction. We discuss [12, Lemma 3.2] next, and return to the crucial contraction in Sect. 5.4.

[12, Lemma 3.2]. There is actually no problem with the scale monotonicity lemma. Indeed, for the case $\alpha = ab$ of the proof of [12, Lemma 3.2], the hypothesis that $\pi_{0x}F = 0$ for $j < j_x$ ensures that this case only relies on the dangerous estimate for $j \geq j_x$, where the danger is absent in (5.2). For the cases $\alpha = a$ and $\alpha = b$ of the proof of [12, Lemma 3.2], what is important is the inequality $\ell_{\sigma,j+1}\ell_j \leq \text{const} \ell_{\sigma,j}\ell_j$, which continues to hold with (4.2) for all scales j , both above and below the mass scale. Therefore, [12, Lemma 3.2] continues to hold with the choice (4.2). In addition

$$\|F\|_{T_\varphi(\ell_j)} \leq \|F\|_{T_\varphi(\ell_j^{\text{old}})}. \tag{5.3}$$

This strengthened special case of the first inequality of [12, (3.6)] (strengthened due to the constant 1 on the right-hand side of (5.3) compared to the generic constant in [12, (3.6)]) can be seen from an examination of the proof of the $\alpha = a, b$ case of [12, Lemma 3.2], together with the observation that $\ell_{\sigma,j}\ell_j = \ell_{\sigma,j}^{\text{old}}\ell_j^{\text{old}}$ by definition.

5.2. Stability Domains

The stability domain \mathcal{D}_j is defined in [12, (1.83)]. We modify \mathcal{D}_j only for the coupling constant q , by replacing r_q in [12, (1.84)] by

$$L^{2j_x+2s(j_x-j_m)+}2^{2(j-j_x)}r_{q,j} = \begin{cases} 0 & j < j_x \\ C_{\mathcal{D}} & j \geq j_x. \end{cases} \tag{5.4}$$

[12, Proposition 1.5]. With (5.4), [12, Proposition 1.5] as it pertains to $\mathfrak{h} = \ell$ (omitting all reference to $\mathfrak{h} = h$) continues to hold beyond the mass scale by the same proof. In particular, with the smaller choice for the domain of q , [12, (3.14)] holds with the larger s -dependent $\ell_{\sigma,j}$.

Note that we do not need to change the domain of λ . This is because the bound [12, (3.13)] continues to hold with the new norm parameters. Indeed, while ℓ_j and $\ell_{\sigma,j}$ have been modified, their product $\ell_j\ell_{\sigma,j}$ has not changed below j_x , and improves above j_x . This guarantees that the T_0 semi-norm $\|\sigma\bar{\varphi}_a\|_{T_0} = \ell_{\sigma}\ell$ remains at least as good as it was with the old norm parameters, and therefore, there is no new stability requirement arising from this.

The choice (5.4) places a more stringent requirement on the domain than does the $s = 0$ version. To see that this requirement is actually met by the renormalisation group flow, we note a minor improvement to the proof of [13, Lemma 6.2(ii)], where the bound $|\delta q| \leq cL^{-2j}$ is used to show that $v(X)$ (defined there) satisfies

$$\|v(X)\| \leq cL^{-2j}(\ell_{\sigma,j}^{\text{old}})^2 \leq c'. \tag{5.5}$$

Here, the factor L^{-2j} arises as a bound on the covariance $C_{j+1;00}$ in the perturbative flow [12, (3.35)] of q , and it can, therefore, be improved to $L^{-2j-2s(j-j_m)+}$ by Lemma 4.1. Thus, also with $\ell^{\text{old}}, \ell_{\sigma}^{\text{old}}$ replaced by ℓ, ℓ_{σ} , the required bound $\|v(X)\| \leq c'$ remains valid.

5.3. Extension of Stability Analysis

In this and the next section, we verify that the results of [12, Section 2] remain valid with ℓ^{old} replaced by ℓ . In this section, we deal with the results whose proofs need only minor modification.

First, we note that the supporting results of [12, Section 4] hold with the new norms. Indeed, it is immediate from (5.3) that analogues of [12, Proposition 4.1] and [12, Lemmas 3.4, 4.11–4.12] hold with the new ℓ_j . Moreover, [12, Lemma 4.7] and [12, Proposition 4.0] hold for general values of the parameters \mathfrak{h}_j (which are implicit in the $T_{0,j}$ norm). We discuss [12, Proposition 4.9] in Sect. 5.4, and the remaining results of [12, Section 4] do not make use of norms.

[12, Proposition 2.1]. With $\mathfrak{h} = \ell$, [12, (2.1)] continues to hold with the same proof; in fact, the proof does not depend on the explicit choice of \mathfrak{h} . We do not need [12, (2.2)], as it is only applied with $\mathfrak{h} = h$.

[12, Proposition 2.2]. The only change to the proof is for the case $j_* = j+1$. To get [12, (2.9)], we proceed as previously in the case $\mathfrak{h} = h$ but applying Lemma 4.3 rather than [12, Lemma 1.2] following [12, (5.22)]. In the same way, we get [12, (2.10)] and the remaining parts of the proposition follow without changes to the proof.

[12, Proposition 2.3]. Again, the only required change in the proof is the use of Lemma 4.3 in the case $j_* = j + 1$, for which as previously we use Lemma 4.3 instead of [12, Lemma 1.2].

[12, Proposition 2.4]. No changes need to be made to the proof. In fact, it is necessary not to use the $\mathfrak{h} = \ell$ case of the estimate [12, (5.32)]. Instead, the $\mathfrak{h} = \ell^{\text{old}}$ case of this estimate should be used for g_Q . This is possible, since the renormalisation group map, and in particular, the coupling constants are independent of the choice of norm.

[12, Proposition 2.5]. Using (5.3), we see that the proof continues to hold above the mass scale. The only change to the proof is that in the application of [12, Proposition 2.2], j should be replaced by $j + 1$ in [12, (2.9)] with $j_* = j + 1$ (corresponding to the G_{j+1} norm). This yields [12, (6.6)] with a G_{j+1} norm on the left-hand side.

[12, Proposition 2.6]. A version of [12, Lemma 6.1] with the new ℓ continues to hold. This lemma makes use of $\tilde{\ell}$, which superficially depends on the choice of ℓ in its definition [12, (3.17)]. However, brief scrutiny of [12, (3.17)] reveals that the apparent dependence on ℓ actually cancels and there is in fact no dependence. Similarly, [12, Lemma 3.4] continues to hold without any changes to its proof. The proof of [12, Proposition 2.6] then applies without change.

[12, Proposition 2.7]. With the new choice of ℓ (and $\mathcal{G} = G$), [12, Lemma 7.1] continues to hold with no changes to its proof. Thus, by [12, (3.6)] and [12, Lemma 1.7]

$$\begin{aligned} & \|\mathbb{E}_{j+1} \delta I^X \theta F(Y)\|_{T_{\varphi, j+1}(\ell_{j+1})} \\ & \leq \|\mathbb{E}_{j+1} \delta I^X \theta F(Y)\|_{T_{\varphi, j}(\ell_j)} \\ & \leq \alpha_{\mathbb{E}}^{|X|_j + |Y|_j} (C_{\delta V} \bar{\epsilon})^{|X|_j} \|F(Y)\|_{G_j(\ell_j)} G_j(X \cup Y, \varphi)^5. \end{aligned} \tag{5.6}$$

By Lemma 4.3, $G_j(X \cup Y, \varphi)^5 \leq G_{j+1}(X \cup Y, \varphi)$. Now, we divide both sides by $G_{j+1}(X \cup Y, \varphi)$ and take the supremum over φ to complete the proof.

5.4. Extension of the Crucial Contraction

The proof of the ‘‘crucial contraction’’ [12, Proposition 2.8] makes use of the third estimate in (5.1), which is now violated above the mass scale due to our new choice of ℓ_j . On the other hand, the second estimate of (5.1) is improved by the new choice and compensates for the degraded third estimate, as we explain in this section.

Below the mass scale, we continue to use the crucial contraction as stated in [12, Proposition 2.8] in terms of two norm pairs. Next, we state a version of the crucial contraction for use above the mass scale using the new norm pair (4.18). The statement uses the notation of [12] (which we do not redefine here), with the exception that now we have replaced a by 0, b by x , and j_{ab} by j_x for consistency with our present notation. Throughout this section, we sometimes write the dimension as d for emphasis, although we only consider $d = 4$.

Proposition 5.1 (Improvement of [12, Proposition 2.8]). *Let $j_m \leq j < N$ and $V \in \mathcal{D}_j$. Let $X \in \mathcal{S}_j$ and $U = \bar{X}$. Let $F(X) \in \mathcal{N}(X^\square)$ be such that $\pi_\alpha F(X) = 0$ when $X(\alpha) = \emptyset$, and such that $\pi_{0x} F(X) = 0$ unless $j \geq j_x$. There is a constant C (independent of L) such that*

$$\|\tilde{I}_{\text{pt}}^{U \setminus X} \mathbb{E}_{C_{j+1}} \theta F(X)\|_{G_{j+1}(\ell_{j+1})} \leq C \left((L^{-d-1} + L^{-1} \mathbb{1}_{X \cap \{0,x\} \neq \emptyset}) \kappa_F + \kappa_{\text{Loc}F} \right), \tag{5.7}$$

with $\kappa_F = \|F(X)\|_{G_j(\ell_j)}$ and $\kappa_{\text{Loc}F} = \|\tilde{I}_{\text{pt}}^X \text{Loc}_X \tilde{I}_{\text{pt}}^{-X} F(X)\|_{G_j(\ell_j)}$.

An ingredient in the proof of Proposition 5.1 is [11, Lemma 3.6], which is the $s = 0$ version of the following lemma. For simplicity, we state only the conclusion of the lemma, and the notation and hypotheses are those in [11, Lemma 3.6], except now we use the s -dependent norm parameters $\mathfrak{h}_j = \ell_j$ of (4.2) (h_j is not needed above the mass scale, and the $s = 0$ case applies below the mass scale).

Lemma 5.2 (Improvement of [11, Lemma 3.6]). *With the same hypotheses and notation as in [11, Lemma 3.6]*

$$\|g\|_{\bar{\Phi}(X)} \leq \bar{C}_3 L^{-(1+s \ j \geq j_m) d'_+} \|g\|_{\bar{\Phi}'(X_+)}. \tag{5.8}$$

Proof. The proof of [11, Lemma 3.6] is based on the assumption $\ell_{j+1}/\ell_j \leq cL^{-1}$ (we take $[\varphi_i] = 1$; the parameters $\ell_{\sigma,j}$ are not used). For our new values of ℓ , the stronger assumption $\ell_{j+1}/\ell_j \leq L^{-1-s \ j \geq j_m}$ holds. The unique change to the proof occurs in the transition from [11, (3.42)] to [11, (3.43)], where the ratio ℓ_{j+1}/ℓ_j is used. With the new ratio, [11, (3.43)] becomes

$$\|r\|_{\Phi(X)} \leq \sup_{z \in \mathbf{X}_+} (cK \ell'^{-1})^z \sup_{|\beta|_\infty \leq p_\Phi} L^{-(p(z)+p(z)s \ j \geq j_m + |\beta|_1)} |\nabla_{R'}^\beta r_z|. \tag{5.9}$$

Here, $r = h - \text{Tay}_a h$, where h is an arbitrary test function and a is the largest point which is lexicographically no larger than any point in X . The test function h depends on sequences of points (x_1, \dots, x_p) , and $\text{Tay}_a h$ is a discrete version of Taylor’s approximation which approximates h by a discrete Taylor polynomial localised at point a in each argument (see [11] for details). By definition, for the empty sequence \emptyset , $(\text{Tay}_a h)_\emptyset = h_\emptyset$, and thus $r_\emptyset = 0$.

It follows that we can take $p(z) \geq 1$ in the supremum over $z \in \mathbf{X}_+$ in (5.9). Thus

$$\|r\|_{\Phi(X)} \leq L^{-s \ j \geq j_m} \sup_{z \in \mathbf{X}_+} (cK \ell'^{-1})^z \sup_{|\beta|_\infty \leq p_\Phi} L^{-(p(z)+|\beta|_1)} |\nabla_{R'}^\beta r_z|. \tag{5.10}$$

The quantity

$$\sup_{z \in \mathbf{X}_+} (cK \ell'^{-1})^z \sup_{|\beta|_\infty \leq p_\Phi} L^{-(p(z)+|\beta|_1)} |\nabla_{R'}^\beta r_z| \tag{5.11}$$

is identical to the right-hand side of [11, (3.43)] when $[\varphi_i] = 1$. In [11], it is shown that this quantity can be bounded by a constant times

$$L^{-d'_+} \|h\|_{\Phi'(X_+)}. \tag{5.12}$$

Thus

$$\|r\|_{\Phi(X)} \leq \bar{C}_3 L^{-s} \sum_{j \geq j_m} L^{-d'_+} \|h\|_{\Phi'(X_+)}. \tag{5.13}$$

With this improvement to [11, (3.43)] in the proof of [11, Lemma 3.6], the conclusion of [11, Lemma 3.6] is improved to (5.8). \square

Roughly speaking, the L -dependent factor in (5.8) implements the dimensional gain for irrelevant directions in a renormalisation group step when passing from one scale to the next. In other words, we may regard the dimension of the field as improving from 1 below the mass scale to $1 + s$ above the mass scale. The $s = 0$ version of Lemma 5.2 is adapted to the scaling at the critical point, where $m^2 = 0$. In the noncritical case $m^2 > 0$, the dimensional gain improves greatly for $j > j_m$, as apparent from (4.6), and is captured more accurately by the general- s version of (5.8).

As a consequence of the former improvement, we have the following two further improvements. From now on, we always assume $\mathfrak{h} = \ell$ and $j > j_m$, as this is the only case relevant for the improvement of [12, Proposition 2.8].

[11, Proposition 1.19]. The improvement in Lemma 5.2 propagates to [11, Proposition 1.19], which now holds as stated except with $\gamma_{\alpha,\beta}$ improved to

$$\gamma_{\alpha,\beta} = \left(L^{-(d'_\alpha + s)} \sum_{j \geq j_m} + L^{-(A+1)} \right) \left(\frac{\ell_{\sigma,j+1}}{\ell_{\sigma,j}} \right)^{|\alpha \cup \beta|}. \tag{5.14}$$

The right-hand side can be estimated as follows. By (5.2)

$$\frac{\ell_{\sigma,j+1}}{\ell_{\sigma,j}} \leq 4 \begin{cases} L^{1+s} & j < j_x \\ 1 & j \geq j_x, \end{cases} \tag{5.15}$$

and hence

$$\gamma_{\alpha,\beta} \leq C'' \left(L^{-(d'_\alpha + s)} \sum_{j \geq j_m} + L^{-(A+1)} \right) \times \begin{cases} L^{(1+s) \sum_{j \geq j_m} |\alpha \cup \beta|} & j < j_x \\ 1 & j \geq j_x. \end{cases} \tag{5.16}$$

[12, Proposition 4.9]. As we explain next, using (5.14) and identical notation to that defined in and around [12, Proposition 4.9], the proposition holds as stated also for the improved norms, provided we take $A \geq 5 + s$. For this, what is required is to show that under the hypotheses of [12, Proposition 4.9], the $\gamma_{\alpha,\beta}$ that arise in its proof obey

$$\gamma_{\alpha,\beta} \leq C \begin{cases} L^{-5} & |\alpha \cup \beta| = 0 \\ L^{-1} & |\alpha \cup \beta| = 1, 2. \end{cases} \tag{5.17}$$

For $|\alpha \cup \beta| = 0$, the first term of (5.16) obeys the bound of (5.17), since $d'_\emptyset = d + 1$. For the remaining cases, $d'_\alpha = 2$ for $j < j_x$ and $d'_\alpha = 1$ for $j \geq j_x$. For $|\alpha \cup \beta| = 2$, the assumption that $F_1, F_2, F_1 F_2$ have no component in \mathcal{N}_{0x} unless $j \geq j_x$ means that we are in the case with no growth due the ratio $\ell_{\sigma,j+1}/\ell_{\sigma,j}$ in (5.16), and its first term again obeys the bound (5.17) with room to spare. Finally, when $|\alpha \cup \beta| = 1$, the first term of (5.16) also obeys the estimate (5.17), and again with room to spare. Concerning the second term

of (5.16), given our choice of A and the fact that we need only consider the growing factor in (5.16) for $|\alpha \cup \beta| = 1$, it suffices to observe that

$$L^{-(A+1)}L^{1+s} \sum_{j \geq jm} \leq L^{-5}. \tag{5.18}$$

This completes the proof of the improved version of [12, Proposition 4.9].

Proof of Proposition 5.1. We complete the proof of Proposition 5.1 by modifying the proof of [12, Proposition 2.8] above the mass scale. The estimate [12, (7.22)] follows from [12, Proposition 2.7] as an estimate in terms of the modified norm pair (4.18), for which [12, Proposition 2.7] was verified in Sect. 5.3. The bound [12, (7.25)] with improved γ is obtained by applying the improved version of [12, Proposition 4.9]. In the remainder of the proof of [12, Proposition 2.8], we specialise each occurrence of \mathcal{G} to the case $\mathcal{G} = G$ and we conclude by obtaining an analogue of [12, (7.31)] with \tilde{G} replaced by G by applying Lemma 4.3 rather than [12, Lemma 1.2].

An additional detail is that it is required that we choose the parameter defining the space \mathcal{N} to obey $p_{\mathcal{N}} > A$. Since we have changed A (depending on s), we must make a corresponding change to $p_{\mathcal{N}}$. This does not pose problems (beyond the previously discussed requirement that g needs to be chosen small depending on p), as this parameter may be fixed to be an arbitrary and sufficiently large integer (see [18, Section 7.1.3] where this point is addressed in a different context). Similarly, the value of A is immaterial and can be any fixed number in the proof of [12, Proposition 2.8]. \square

Acknowledgements

The work of GS, AT, BW was supported in part by NSERC of Canada. The authors are grateful to David Brydges for important conversations and advice, and to an anonymous referee for useful suggestions.

Appendix: Moments of the Free Green Function

We now prove Proposition 1.2, which we repeat as the following proposition.

Proposition A.1. *Let c_p be the constant defined by (1.12). For all dimensions $d > 2$ and all $p > 0$, as $m^2 \downarrow 0$,*

$$\sum_{x \in \mathbb{Z}^d} |x|^p G_x(0, m^2) = c_p^p m^{-(p+2)}(1 + O(m)). \tag{A.1}$$

In particular, $\xi_p(0, \varepsilon) = c_p \varepsilon^{-1/2}(1 + O(\varepsilon^{1/2}))$ as $\varepsilon \downarrow 0$.

The last sentence in the the proposition follows immediately from (A.1) and the fact that $\chi(0, m^2) = m^{-2}$, so it suffices to prove (A.1).

The case $p = 2$ of (A.1) can be obtained easily from the identity

$$\sum_{x \in \mathbb{Z}^d} |x|^2 G_x(0, m^2) = -\Delta_{\mathbb{R}^d} \hat{G}(0), \tag{A.2}$$

where \hat{G} is the Fourier transform of G . Higher even moments could in principle be computed by further differentiating \hat{G} . We adopt a different approach for general $p > 0$, based on the finite-range decomposition of $(-\Delta_{\mathbb{Z}^d} + m^2)^{-1}$ given in [1, 8]. This finite-range decomposition also provides the basis for the renormalisation group method. The finite-range decomposition is

$$G_x(0, m^2) = \sum_{j=1}^{\infty} C_{j;x}(m^2). \tag{A.3}$$

The finite-range property refers to the fact that $C_{j;x}(m^2) = 0$ if $|x| \geq \frac{1}{2}L^j$, where $L > 1$ is fixed arbitrarily. We review some properties of this decomposition, from [1, 5], before proving Proposition A.1. The positive definiteness of the finite-range decomposition is not needed here, and L need not be large.

The terms $C_{j;x}(m^2)$ are defined in [5, Section 6.1] by

$$C_{j;x}(m^2) = \begin{cases} \int_0^{\frac{1}{2}L} \phi_t^*(x; m^2) \frac{dt}{t} & (j = 1) \\ \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t^*(x; m^2) \frac{dt}{t} & (j \geq 2) \end{cases} \tag{A.4}$$

(in [5], the notation $C_{j;0,x}$ and $\phi_t^*(0, x; m^2)$ was used instead). Here, ϕ_t^* is a function of $x \in \mathbb{R}^d$ and $m^2 > 0$ given in [1, Example 1.1]. It satisfies the finite-range property that $\phi_t^*(x; m^2) = 0$ for $|x| > t$. It was also shown in [1] that there exists a function ϕ_t satisfying the same finite-range property but giving a decomposition of the continuum Green function:

$$(-\Delta_{\mathbb{R}^d} + m^2)_{0x}^{-1} = \int_0^{\infty} \phi_t(x; m^2) \frac{dt}{t}. \tag{A.5}$$

Moreover, by [1, (1.37)], for $|x| \leq t$

$$\phi_t^*(x; m^2) = \phi_t(x; m^2) + O(t^{-(d-1)}(1 + m^2t^2)^{-k}). \tag{A.6}$$

This allows us to approximate the discrete Green function by the continuum one, for which the moments are easily computed. We have set the constant c present in [1] equal to 1, which we can do by rescaling ϕ_t^* .

As t approaches 0, the error bound in (A.6) degenerates. However, to estimate (A.1), it suffices to restrict to $x \neq 0$. Then, since $x \in \mathbb{Z}^d$, the finite-range property permits replacement of the lower bound in the range of integration for $j = 1$ in (A.4) by $\frac{1}{2}$, and the contribution due to $j = 1$ can be estimated in the same way as the terms $j \geq 2$.

In addition, by [1, (1.34)], for any k , there is a constant C_k , such that

$$|D_x \phi_t(x; m^2)| \leq C_k t^{-(d-1)}(1 + m^2t^2)^{-k}. \tag{A.7}$$

We fix a choice of k which obeys $k > \frac{1}{2}(p + 1)$ and use only this choice. By [1, (1.38)], there exists a function $\bar{\phi}$, such that

$$\phi_t(x; m^2) = t^{-(d-2)} \bar{\phi}\left(\frac{x}{t}; m^2t^2\right). \tag{A.8}$$

Proof of Proposition 1.2. We begin by writing

$$\sum_{x \in \mathbb{Z}^d} |x|^p G_x(0, m^2) = \sum_{x \in \mathbb{Z}^d} |x|^p \sum_{j=1}^{\infty} C_{j;x}(m^2) = M(m^2) + E(m^2), \tag{A.9}$$

where the main and error terms are, respectively

$$M(m^2) = \sum_{x \in \mathbb{Z}^d} |x|^p \sum_{j=1}^{\infty} \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x; m^2) \frac{dt}{t}, \tag{A.10}$$

$$E(m^2) = \sum_{x \in \mathbb{Z}^d} |x|^p \sum_{j=1}^{\infty} \left(C_{j;x} - \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x; m^2) \frac{dt}{t} \right). \tag{A.11}$$

We first compute the main term M . By (A.8)

$$\phi_t(x; m^2) = m^{d-2} \phi_{mt}(mx; 1). \tag{A.12}$$

Therefore, by Riemann sum approximation

$$\sum_{x \in \mathbb{Z}^d} |x|^p \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x; m^2) \frac{dt}{t} \tag{A.13}$$

$$= m^{-(p+2)} m^d \sum_{x \in \mathbb{Z}^d} |mx|^p \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_{mt}(mx; 1) \frac{dt}{t} \tag{A.14}$$

$$= m^{-(p+2)} \int_{\mathbb{R}^d} |x|^p \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_{mt}(x; 1) \frac{dt}{t} + O(L^{(p+1)j} L^{-2k(j-j_m)_+}),$$

where the error estimate follows from (A.7) and (4.8). Summation over j gives

$$M(m^2) = c_p^p m^{-(p+2)} + O(m^{-(p+1)}), \tag{A.15}$$

where we used (A.5) for the first term, and we used $2k > p + 1$ and Lemma 2.2 for the second term.

For the error term, it follows from (A.4), (A.6), and the observation that the lower bound in the range of integration for the $j = 1$ term in (A.4) can be changed to $\frac{1}{2}$ that

$$C_{j;x} = \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x; m^2) \frac{dt}{t} + O(L^{-j(d-1)}(1 + m^2 L^{2j})^{-k}) \mathbb{1}_{|x| \leq L^j}. \tag{A.16}$$

Therefore, using (4.8) again we have

$$E(m^2) = \sum_{j=1}^{\infty} \sum_{|x| \leq L^j} |x|^p O(L^{-j(d-1)} L^{-2k(j-j_m)_+}) \tag{A.17}$$

$$= \sum_{j=1}^{\infty} O(L^{(p+1)j} L^{-2k(j-j_m)_+}). \tag{A.18}$$

With $2k > p + 1$ and Lemma 2.2, this gives $E(m^2) = O(m^{-(p+1)})$, and the proof is complete. \square

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Communicated by Abdelmalek Abdesselam.

Received: November 7, 2015.

Accepted: May 10, 2016.