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Infinite Volume Limit for Correlation Functions in the Dipole Gas

Tuan Minh Le

Abstract. We study a classical lattice dipole gas with low activity in dimension $d \geq 3$. We investigate long distance properties by a renormalization group analysis. We prove that various correlation functions have an infinite volume limit. We also get estimates on the decay of correlation functions.

1. Introduction

1.1. Overview

In this paper we study the classical dipole gas on a unit lattice $\mathbb{Z}^d$ with $d \geq 3$. Each dipole is described by its position coordinate $x \in \mathbb{Z}^d$ and a unit polarization vector (moment) $p \in \mathbb{S}^{d-1}$.

Let Λ_N be a box in $\mathbb{R}^d$

$$\Lambda_N = \left[\frac{-L^N}{2}, \frac{L^N}{2}\right]^d \tag{1}$$

where $L \geq 2^{d+3} + 1$ is a very large, odd integer. For $\Lambda_N \cap \mathbb{Z}^d$, the classical statistical mechanics of a gas of such dipoles with inverse temperature (for convenience) $\beta = 1$ and activity (fugacity) $z > 0$ is given by the grand canonical partition function

$$Z_N = \sum_{n\geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \left(\sum_{x_i \in \mathbb{Z}^d \cap \Lambda_N} \int_{\mathbb{S}^{d-1}} \mathrm{d}p_i \right) \exp\left(\frac{-1}{2} \sum_{1 \leq k,j \leq n} (p_k \cdot \partial)(p_j \cdot \partial) C(x_k, x_j) \right) \tag{2}$$

where $C(x, y)$ is the Coulomb potential on the unit lattice $\mathbb{Z}^d$. There are associated correlation functions describing correlations between various observables.

The main problems to study are infinite volume limit for the pressure defined by $p_N = |\Lambda_N|^{-1} \log Z_N$, infinite volume limit for correlation functions and decay of correlations. In fact the interaction between the dipoles in statistical ensemble at the long range is not summable; therefore, the classical approach of Mayer expansion would hardly work. To overcome this problem, we use a mathematical version of the physicists' renormalization group. For the small z, we show the existence of infinite volume limit for the pressure and various correlation functions, and get estimates on the decay of correlations. These results extend the results of other authors (see Sect. 1.3 for more details). The result on the infinite volume limit of correlations is new; and the result on the decay of correlations of the n-point function improves the results of most earlier authors who only consider the two point function.

1.2. Definition

Let $\{e_1, e_2, \dots, e_d\}$ be the standard basic for $\mathbb{Z}^d$. For $\varphi : \mathbb{Z}^d \to \mathbb{R}$ and $\mu \in \{1, 2, \dots, d\}$ we define $\partial_\mu \varphi$ as $\partial_\mu \varphi(x) = \varphi(x + e_\mu) - \varphi(x)$ and $[\varphi] = \frac{d-2}{2}$. Let $e_{-\mu} = -e_\mu$ with $\mu \in \{1, 2, \dots, d\}$. Then the definition of $\partial_\mu \varphi$ can be used to define the forward or backward lattice derivative along the unit vector e_μ with $\mu \in \{\pm 1, \pm 2, \dots, \pm d\}$. We have that ∂_μ and $\partial_{-\mu}$ are adjoint to each other and $-\Delta = 1/2 \sum_{\pm\mu=1}^{d} \partial_\mu^* \partial_\mu = 1/2 \sum_{\pm\mu=1}^{d} \partial_{-\mu} \partial_\mu$.[1]

As in [5], the potential energy between unit dipoles (x, p_1) and (y, p_2) is

$$(p_1 \cdot \partial)(p_2 \cdot \partial) C(x - y) \tag{3}$$

where $x, y \in \mathbb{Z}^d$ are positions, $p_1, p_2 \in \mathbb{S}^{d-1}$ are moments, $\partial = (\partial_1, \partial_2, \dots, \partial_d)$ and $C(x - y)$ is the Coulomb potential on the unit lattice $\mathbb{Z}^d$, which is the kernel of the inverse Laplacian

$$C(x, y) = (-\Delta)^{-1}(x, y) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \frac{e^{ip\cdot(x-y)}}{2\sum_{\mu=1}^{d}(1 - \cos p_\mu)} \mathrm{d}p. \tag{4}$$

And the potential energy of n dipoles at positions $x_1, x_2, \dots, x_n$, including self energy, has the form

$$\sum_{1 \le k,j \le n} (p_k \cdot \partial)(p_j \cdot \partial) C(x_k, x_j). \tag{5}$$

The grand canonical partition function Z_N can be equivalently expressed as a Euclidean field theory (due to Kac [13] and Siegert [14]) and is given by

$$\begin{aligned} {}_0Z_N \equiv Z_N &= \int \exp\left(zW(\Lambda_N, \phi)\right) \mathrm{d}\mu_C(\phi) \\ \text{where} \qquad W(\Lambda_N, \phi) &= \sum_{x \in \Lambda_N \cap \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} \mathrm{d}p \cos(p \cdot \partial\phi(x)) \end{aligned} \tag{6}$$

with

- $\mathrm{d}p$: the standard normalized rotation invariant measure on $\mathbb{S}^{d-1}$.
- The fields $\phi(x)$: a family of Gaussian random variables (on some abstract measure space) indexed by $x \in \mathbb{Z}^d$ with mean zero and covariance $C(x, y)$ which is a positive definite function as given above.

[1] We distinguish forward and backward derivatives to facilitate a symmetric decomposition of $V(\Lambda_N)$ (defined in (9)) into blocks.

- The measure μ_C: the underlying measure (see [6, Section 11.2] and [5, Appendix A] for more detail). We discuss about the equivalence of (2) and (6) in Appendix of this paper.

We define

$$\begin{aligned}\left\langle \prod_{k=1}^{n} \phi(x_k) \right\rangle &\equiv ({}_0Z_N)^{-1} \int \left(\prod_{k=1}^{n} \phi(x_k) \right) \exp\left(zW(\Lambda_N, \phi)\right) \mathrm{d}\mu_C(\phi),\\ \left\langle \prod_{k=1}^{n} \phi(x_k) \right\rangle^t &\equiv \left\langle \prod_{k=1}^{n} \phi(x_k) \right\rangle - \text{ (lower order terms).}\end{aligned} \tag{7}$$

For example, we have $\langle \phi(x_1)\phi(x_2)\rangle^t = \langle \phi(x_1)\phi(x_2)\rangle - \langle \phi(x_1)\rangle\langle \phi(x_2)\rangle$.

For investigating the truncated correlation functions, we consider a more general version of (6):

$${}_fZ_N = \int \exp\left(if(\phi) + zW(\Lambda_N, \phi)\right) \mathrm{d}\mu_C(\phi) \tag{8}$$

where $f(\phi)$ can be:

1. $f(\phi) = 0$ as in [5].
2. $f(\phi) = \sum_{k=1}^{m} t_k \partial_{\mu_k} \phi(x_k)$. We use this $f(\phi)$ to study the truncated correlation functions

$$\mathcal{G}^t(x_1, x_2, \ldots, x_m) \equiv \left\langle \prod_{k=1}^{m} \partial_{\mu_k} \phi(x_k) \right\rangle^t = i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log {}_fZ' \Big|_{t_1=0,\ldots t_m=0}$$

   which is nontrivial and previously investigated by Dimock and Hurd [7].
3. $f(\phi) = \sum_{k=1}^{m} t_k \exp(i\partial_{\mu_k}\phi(x_k))$. We use this $f(\phi)$ to study the dipole correlation

$$\left\langle \prod_{k=1}^{m} \exp\left(i\partial_{\mu_k}\phi(x_k)\right) \right\rangle^t = i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log {}_fZ' \Big|_{t_1=0,\ldots t_m=0}$$

4. Other general form which will be discussed at the end of this paper. The general form can be applied for truncated correlations of density of the dipoles which also has been studied by Brydges and Keller [2]. We think that this general form has more applications.

Here $x_k \in \mathbb{Z}^d$ are different points; $\mu_k \in \{\pm 1, \pm 2, \ldots, \pm d\}$ and $t_k \in \mathbb{C}$ small; $m \geq 2$. For $\{x_1, x_2, \ldots, x_m\} \subset \mathbb{Z}^d$, let

$$\text{diam}(x_1, x_2, \ldots, x_m) = \max_{1\leq i,j\leq m} \text{dist}(x_i, x_j)$$

where $\text{dist}(x_i, x_j)$ is the distance between x_i and x_j on lattice $\mathbb{Z}^d$.

1.3. Earlier Results

Actually Z_N is not expected to have a limit as $N \to \infty$. Using a method of correlation inequalities, Frohlich and Park [10, 1978], have shown the infinite volume limit for the pressure. After that, Frohlich and Spencer [11, 1981] also gave the result on the infinite volume limit for the dipole phase of the Coulomb gas by using correlation inequalities and multiscale analysis. In 1984, Gawedski and Kupiainen [12] used the block-spin renormalization group to show the long distance behavior of the correlation functions and the analyticity (in the fugacity z) of the pressure. In 1990, Brydges and Yau [4] introduced an improving and simpler renormalization group techniques for the continuum dipole model which they reproduced some results of [12]. Dimock and Hurd [7, 1992] has extended the Brydges–Yau's renormalization group analysis for the dipole gas and obtained the decay of some correlations of the 2-point and n-point functions. In 1994, Brydges and Keller [2] gave an "accurate" upper bound for correlation function of the density of dipoles at 2-point. In 2007, Brydges and Slade [1] developed a new renormalization group approach. In 2009, Dimock [5] has established an infinite volume limit for the pressure by using this new approach.

We follow particularly the new renormalization group approach developed by Brydges and Slade [1] and Dimock [5]. Generalizing Dimock's framework with an external field, we have reproduced some estimates on the correlation functions as in Dimock and Hurd [7], and extended results in Brydges and Keller [2] by giving the upper bound for correlations functions of the density of dipoles at n-point. Our main result is the existence of the infinite volume limit for correlations functions, which is new.

Besides the dipole gas papers mentioned above, we would like to cite some other papers on the Coulomb gas in $d = 2$ which has a dipole phase. There are the works of Dimock and Hurd [8], Falco [9] and Zhao [15].

1.4. The Main Result

For our RG approach we follow the analysis of Brydges and Slade [1]. Instead of (8), we use a different finite volume approximation. First, we add an extra term $(1-\varepsilon)V(\Lambda_N,\phi)$ where $0<\varepsilon$ is closed to 1 and

$$V(\Lambda_N,\phi)=\frac{1}{4}\sum_{x\in\Lambda_N\cap\mathbb{Z}^d}\sum_{\pm\mu=1}^{d}(\partial_\mu\phi(x))^2. \tag{9}$$

By replacing the covariance C by $\varepsilon^{-1}C$, this extra term will be partially compensated. Hence instead of (8) we will consider a new finite volume generating function

$${}_fZ_N={}_fZ'_N/Z''_N \tag{10}$$

where

$${}_fZ'_N=\int e^{if(\phi)}\exp\left(zW(\Lambda_N,\phi)-(1-\varepsilon)V(\Lambda_N,\phi)\right)\mathrm{d}\mu_{\varepsilon^{-1}C}(\phi) \tag{11}$$

and

$$Z''_N=\int\exp\left(-(1-\varepsilon)V(\Lambda_N,\phi)\right)\mathrm{d}\mu_{\varepsilon^{-1}C}(\phi). \tag{12}$$

We have

$$\begin{aligned}{}_fZ_N&={}_fZ'_N/Z''_N\\&=\int e^{if(\phi)}\exp\left(zW(\Lambda_N,\phi)\right)\left[(Z''_N)^{-1}e^{-(1-\varepsilon)V(\Lambda_N,\phi)}\mathrm{d}\mu_{\varepsilon^{-1}C}(\phi)\right].\end{aligned} \tag{13}$$

As N goes to infinity, $\exp(-(1-\varepsilon)V(\Lambda_N,\phi))$ formally becomes $\exp(1/2(1-\varepsilon)(\phi,-\Delta\phi))$, and $\mathrm{d}\mu_{\varepsilon^{-1}C}(\phi) = (1/2(\varepsilon)(\phi,-\Delta\phi))\mathrm{d}\phi$. Hence, when $N \to \infty$, the bracketed expression formally converges to $(\text{const.})(1/2(\phi,-\Delta\phi))\mathrm{d}\phi = (\text{const.})\mathrm{d}\mu_C(\phi)$. Formally this new ${}_fZ_N$ gives the same limit as (8). This result holds for any choice of ε. By definition (9), the extra term $(1-\varepsilon)V(\Lambda_N,\phi) = (1-\varepsilon)\frac{1}{4}\sum_{x\in\Lambda_N\cap\mathbb{Z}^d}\sum_{\pm\mu=1}^{d}(\partial_\mu\phi(x))^2$. Therefore, the choice of ε is a choice of how much $(\partial\phi)^2$ one is putting in the interaction and how much in the measure.

Similarly to the Theorem 1 in [5], our main theorems are:

Theorem 1. *For $|z|$ and $\max_k|t_k|$ sufficiently small, there is an $\varepsilon=\varepsilon(z)$ close to 1 so that ${}_fp_N = |\Lambda_N|^{-1}\log({}_fZ_N)$ has a limit as $N\to\infty$.*[2]

Using $f(\phi)=\sum_{k=1}^m t_k\partial_{\mu_k}\phi(x_k)$, we obtain the existence of infinite volume limit for correlation functions.

Theorem 2. *With L, A sufficiently large, the infinite volume limit of truncated correlation function $\lim_{N\to\infty}\langle\prod_{k=1}^m\partial_{\mu_k}\phi(x_k)\rangle^t$ exists.*

And we also can achieve some estimate for the correlation functions:

Theorem 3. *For any small $\iota>0$, with L, A sufficiently large (depending on ι), $\eta=\min\{d/2,2\}$, we have:*

$$\left|\left\langle\prod_{k=1}^{m}\partial_{\mu_k}\phi(x_k)\right\rangle^t\right| \le \frac{m!}{a^m}\ \mathit{diam}^{-\eta+\iota}(x_1,x_2,\ldots,x_m) \tag{14}$$

where a depends on ι, L, A.[3]

This bound is presumably not sharp since for example at $z=0$ the two point function goes like $\partial_\mu\partial_\nu C(x,y)\sim|x-y|^{-d}$. Also one expects tree decay rather than diameter decay. When $d=3$ or 4, the result in Theorem 3 looks like the result in [7], but here it is obtained with the new method.

Using $f(\phi)=\sum_{k=1}^m t_k\exp(i\partial_{\mu_k}\phi(x_k))$, we can obtain Theorems 4 and 5 which are similar to Theorems 2 and 3, just with different f.

Theorem 4. *With L, A sufficiently large, the infinite volume limit of the truncated correlation function $\lim_{N\to\infty}\langle\prod_{k=1}^m\exp(i\partial_{\mu_k}\phi(x_k))\rangle^t$ exists.*

Theorem 5. *For any small $\iota>0$, with L, A sufficiently large (depending on ι), let $\eta=\min\{d/2,2\}$ we have:*

$$\left|\left\langle\prod_{k=1}^{m}\exp\left(i\partial_{\mu_k}\phi(x_k)\right)\right\rangle^t\right| \le \frac{m!}{a^m}\ \mathit{diam}^{-\eta+\iota}(x_1,x_2,\ldots,x_m) \tag{15}$$

where a depends on ι, L, A.

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[2] In Theorem 1, $f(\phi)$ can be 0, $\sum_{k=1}^m t_k\partial_{\mu_k}\phi(x_k)$, or $\sum_{k=1}^m t_k\exp(i\partial_{\mu_k}\phi(x_k))$.

[3] The coefficient A is used in the definition of the norm of polymer activities in Sect. 2.4.

At the end of this paper (Sect. 6.3), we investigate a general form

$$f(\phi) = \sum_{k=1}^{m} t_k f_k(\phi)(x_k)$$

and obtain Theorems 6 and 7.

Theorem 6. *With L, A sufficiently large, the infinite volume limit of the truncated correlation function* $\lim_{N\to\infty}\langle\prod_{k=1}^{m} f_k(\phi)(x_k)\rangle^t$ *exists.*

Theorem 7. *For any small $\iota > 0$, with L, A sufficiently large (depending on ι), let $\eta = \min\{d/2, 2\}$ we have:*

$$\left|\left\langle\prod_{k=1}^{m} f_k(\phi)(x_k)\right\rangle^t\right| \leq \frac{m!}{a^m}\ diam^{-\eta+\iota}(x_1, x_2, \ldots, x_m) \tag{16}$$

where a depends on ι, L, A.

Applying Theorem 7 with a special f for density of dipoles

$$f = \sum_{k=1}^{m} t_k W_0(\{x_k\}),$$

where $W_0(\{x_k\}) = zW(1, \{x_k\})$ as in (64), we have obtained some estimates for truncated correlation functions of density of dipoles with ($m \geq 2$) points (Corollary 1) instead of only 2 points as Theorem 1.1.2 in [2].

Corollary 1. *For any small $\iota > 0$, with L, A sufficiently large (depending on ι), let $\eta = \min\{d/2, 2\}$ we have:*

$$\left|\left\langle\prod_{k=1}^{m} W_0(\{x_k\})\right\rangle^t\right| \leq \frac{m!}{a^m}\ diam^{-\eta+\iota}(x_1, \ldots, x_m) \tag{17}$$

where a depends on ι, L, A.

Then we apply Theorem 6 to establish the infinite volume limit for truncated correlation functions of density of dipoles (Corollary 2).

Corollary 2. *With L, A sufficiently large, the infinite volume limit of the truncated correlation function* $\lim_{N\to\infty}\langle\prod_{k=1}^{m} W_0(\{x_k\})\rangle^t$ *exists.*

For the proof of Theorem 1, we will show that, with a suitable choice of $\varepsilon = \varepsilon(z)$, the density $\exp(zW - (1-\varepsilon)V)$ likely goes to zero under the renormalization group flow and leaves a measure like $\mu_{\varepsilon(z)^{-1}C}$ to describe the long distance behavior of the system. Accordingly $\varepsilon(z)$ can be interpreted as a dielectric constant.

Now we rewrite the generating function ${}_fZ_N$. First we scale $\phi \to \phi/\sqrt{\varepsilon}$ and then let $\sigma = \varepsilon^{-1} - 1$. Because ε is closed to 1, we have σ is near zero. We also have

$$\begin{aligned} {}_fZ'_N(z,\sigma) &= \int e^{if(\phi)} \exp\left(zW(\Lambda_N, \sqrt{1+\sigma}\phi) - \sigma V(\Lambda_N,\phi)\right) \mathrm{d}\mu_C(\phi), \\ Z''_N(\sigma) &= \int \exp\left(-\sigma V(\Lambda_N,\phi)\right) \mathrm{d}\mu_C(\phi), \\ {}_fZ_N(z,\sigma) &= {}_fZ'_N(z,\sigma)/Z''_N(\sigma). \end{aligned} \tag{18}$$

Then we need to show that with $|z|$ sufficiently small there is a (smooth) $\sigma = \sigma(z)$ near zero such that,

$$\begin{aligned} &|\Lambda_N|^{-1} \log {}_fZ_N(z,\sigma(z)) \\ &\quad = |\Lambda_N|^{-1} \log {}_fZ'_N(z,\sigma(z)) - |\Lambda_N|^{-1} \log Z''_N(\sigma(z)) \end{aligned} \tag{19}$$

has a limit when $N \to \infty$. And Theorem 1 is proved just by putting $\varepsilon(z) = (1+\sigma(z))^{-1}$ back. Dimock has proved that, for small real σ with $|\sigma| < 1$, we have $|\Lambda_N|^{-1} \log Z''_N(\sigma)$ converges as $N \to \infty$ [5, Theorem 2]. Hence we only need to investigate the first term in (19).

The paper is organized as follows:

- In Sect. 2, we give some general definitions on the lattice and their properties. We also give definitions about the norms we use together with their crucial properties and estimates. Then we define the basic Renormalization Group transformation as in [5].
- In Sect. 3, we accomplish the detailed analysis of the Renormalization Group transformation to isolate the leading terms. Then we simplify them for the next scale.
- In Sect. 4, we study the RG flow and find the stable manifold $\sigma = \sigma(z)$.
- In Sect. 5, we assemble the results and prove the infinite volume limit for $|\Lambda_N|^{-1} \log {}_fZ'_N$ exists.
- Finally in Sect. 6, by combining all the other estimates, we obtain some estimates for correlation functions and establish the infinite volume limit of correlation functions.

2. Preliminaries

In this section, we quote all notations and basic result from Dimock [5]. At the same time, we introduce some new notations which are useful for this paper.

2.1. Multiscale Decomposition

RG methods are based upon a multiscale decomposition of the basic lattice covariance C into a sequence of more controllable integrals and analyze the effects separately at each stage. Especially we choose a decomposition into finite range covariances which is developed by Brydges, Guadagni, and Mitter [3]. The decomposition of the lattice covariance C has the form

$$C(x-y) = \sum_{j=1}^{\infty} \Gamma_j(x-y) \tag{20}$$

such that

- $\Gamma_j(x)$ is defined on $\mathbb{Z}^d$, is positive semi-definite, and satisfies the finite range property: $\Gamma_j(x) = 0$ if $|x| \geq L^j/2$.
- There is a constant c_0 independent of L such that, for all j, x, we have

$$|\Gamma_j(x)| \leq c_0 L^{-2(j-1)[\phi]}. \tag{21}$$

  This implies that the series converges uniformly.
- There are constants c_α independent of L such that

$$|\partial^\alpha \Gamma_j(x)| \leq c_\alpha L^{-(j-1)(2[\phi]+|\alpha|)} \tag{22}$$

  where $\partial^\alpha = \prod_{\pm\mu=1}^d \partial_\mu^{\alpha_\mu}$ is a multiderivative and $|\alpha| = \sum_\mu |\alpha_\mu|$. Thus the differentiated series converges uniformly to $\partial^\alpha C$.
- [5, Lemma 2] There are some constants $C_{L,\alpha}$ such that

$$|\partial^\alpha C(x)| \leq C_{L,\alpha}(1+|x|)^{-2[\phi]-|\alpha|}. \tag{23}$$

For our RG analysis we need to break off pieces of covariance $C(x-y)$ one at a time. So we define

$$C_k(x-y) = \sum_{j=k+1}^{\infty} \Gamma_j(x-y). \tag{24}$$

Hence we have $C = C_0$ and

$$C_k(x-y) = C_{k+1}(x-y) + \Gamma_{k+1}(x-y). \tag{25}$$

2.2. Renormalization Group Transformation

The generating function (18) can be rewritten as

$$_fZ'_N(z,\sigma) = \int {}_f\mathcal{Z}_0^N(\phi)\mathrm{d}\mu_{C_0}(\phi) \tag{26}$$

with

$$_f\mathcal{Z}_0^N(\phi) = e^{if(\phi)} \exp\left(zW(\Lambda_N, \sqrt{1+\sigma}\phi) - \sigma V(\Lambda_N,\phi)\right). \tag{27}$$

We use the left subscript f as an extra notation for 3 cases at the same time:

- $f(\phi) = 0$ as in (Dimock [5]);
- $f(\phi) = \sum_{k=1}^m t_k \partial_{\mu_k}\phi(x_k))$;
- $f(\phi) = \sum_{k=1}^m t_k \exp(i\partial_{\mu_k}\phi(x_k))$.

Since $C_0 = C_1 + \Gamma_1$ we replace an integral over μ_{C_0} by an integral over μ_{Γ_1} and μ_{C_1}. So we have

$$_fZ'_N(z,\sigma) = \int {}_f\mathcal{Z}_0^N(\phi+\zeta)\mathrm{d}\mu_{\Gamma_1}(\zeta)\mathrm{d}\mu_{C_1}(\phi) = \int {}_f\mathcal{Z}_1^N(\phi)\mathrm{d}\mu_{C_1}(\phi). \tag{28}$$

We define a new density by the fluctuation integral

$$_f\mathcal{Z}_1^N(\phi) = (\mu_{\Gamma_1} * {}_f\mathcal{Z}_0^N)(\phi) \equiv \int {}_f\mathcal{Z}_0^N(\phi+\zeta)\mathrm{d}\mu_{\Gamma_1}(\zeta). \tag{29}$$

Because Γ_1, C_1 are only positive semi-definite, these are degenerate Gaussian measures.[4] By continuing this way, we will have the representation for $j = 0, 1, 2, \ldots$

$$ {}_fZ'_N(z,\sigma) = \int {}_f\mathcal{Z}_j^N(\phi)\mathrm{d}\mu_{C_j}(\phi) \tag{30}$$

here the density ${}_f\mathcal{Z}_j^N(\phi)$ is defined by

$$ {}_f\mathcal{Z}_{j+1}^N(\phi) = (\mu_{\Gamma_{j+1}} * {}_f\mathcal{Z}_j^N)(\phi) = \int {}_f\mathcal{Z}_j^N(\phi+\zeta)\mathrm{d}\mu_{\Gamma_{j+1}}(\zeta). \tag{31}$$

Our job is to investigate the growth of these densities when j go to ∞.

2.3. Local Expansion

We will rewrite each density ${}_f\mathcal{Z}_j^N(\phi)$ in a form which presents its locality properties known as a polymer representation. The localization becomes coarser when j gets bigger. First we will give some basic definitions on the unit lattice $\mathbb{Z}^d$.

2.3.1. Basic Definitions on the Lattice $\mathbb{Z}^d$. For $j = 0, 1, 2, \ldots$ we partition $\mathbb{Z}^d$ into *j-blocks* B. These blocks have side L^j and are translates of the center j-blocks

$$ B_j^0 = \{x \in \mathbb{Z}^d : |x| \leq 1/2(L^j - 1)\} \tag{32}$$

by points in the lattice $L^j\mathbb{Z}^d$. The set of all j-blocks in $\Lambda = \Lambda_N$ is denoted $\mathcal{B}_j(\Lambda_N)$, $\mathcal{B}_j(\Lambda)$ or just $\mathcal{B}_j$. A union of j-blocks X is called a *j-polymer*. Note that Λ is also a j-polymer for $0 \leq j \leq N$. The set of all j-polymers in $\Lambda = \Lambda_N$ is denoted $\mathcal{P}_j(\Lambda)$ or just $\mathcal{P}_j$. The set of all connected j-polymers is denoted by $\mathcal{P}_{j,c}$. For $X \in \mathcal{P}_j$, we use $\mathcal{C}(X)$ to denote the set of all connected components of polymer X and $\bar{X}$ to denote the smallest $Y \in \mathcal{P}_{j+1}$ such that $X \subset Y$.

For a j-polymer X, let $|X|_j$ be the number of j-blocks in X. We call j-polymer X *a small set* if it is connected and contains no more than 2^d j-blocks. The set of all small set j-polymers in Λ is denoted by $\mathcal{S}_j(\Lambda)$ or just $\mathcal{S}_j$. A j-block B has a small set neighborhood $B^* = \cup\{Y \in \mathcal{S}_j : Y \supset B\}$.

Note If B_1,B_2 are j-blocks and $B_2 \in B_1^*$ then, using the above definition, we have $B_1 \in B_2^*$. Similarly a j-polymer X has a small set neighborhood X^*.

For $l \geq 1$ and integer d, we define some constants $n_1(d), n_2(d), n_3(d,l)$ which are bounded and, for every $j \geq 0$, we have:

$$\begin{aligned} n_1(d) &\equiv \sum_{X\in\mathcal{S}_0, X\supset 0} 1/|X|_0 = \sum_{X\in\mathcal{S}_j, X\supset B_j^0} 1/|X|_j, \\ n_2(d) &\equiv \sum_{X\in\mathcal{S}_0, X\supset 0} 1 = \sum_{X\in\mathcal{S}_j, X\supset B_j^0} 1, \\ n_3(d,l) &\equiv \sum_{X\in\mathcal{S}_0, X\supset 0} \frac{l^{-|X|_0}}{|X|_0} = \sum_{X\in\mathcal{S}_j, X\supset B_j^0} \frac{l^{-|X|_j}}{|X|_j}, \\ n_3(d,l) &\leq n_3(d,1) = n_1(d) \leq n_2(d) \leq (2^d)!(2d)^{2^d}. \end{aligned} \tag{33}$$

[4] Dimock has discussed these in [5, Appendix A].

Furthermore, with a fixed d, we can get

$$0 \leq \lim_{l\to\infty} n_3(d,l) \leq \lim_{l\to\infty} \frac{n_1(d)}{l} = 0. \tag{34}$$

2.3.2. Local Expansion. Using the same approach as in [5], we rewrite the density ${}_f\mathcal{Z}_j^N(\phi)$ for $\phi : \mathbb{Z}^d \to \mathbb{R}$ in the general form

$${}_f\mathcal{Z} = ({}_fI \circ {}_fK)(\Lambda) \equiv \sum_{X\in\mathcal{P}_j(\Lambda)} {}_fI(\Lambda - X){}_fK(X). \tag{35}$$

Here ${}_fI(Y)$ is a background functional which is explicitly known and carries the main contribution to the density. The ${}_fK(X)$ is so called *a polymer activity*. It represents small corrections to the background.

In Sect. 5 we will show that the initial density ${}_fI_0$ has the factor property. We want to keep this factor property at all scales. Then we can use the analysis of Brydges' lecture [1]. Therefore, we assume ${}_fI(Y)$ always is in the form of

$${}_fI(Y) = \prod_{B\in\mathcal{B}_j : B\subset Y} {}_fI(B) \tag{36}$$

and ${}_fI(B,\phi)$ depends on ϕ only B^*, the small set neighborhood of B. Moreover we assume ${}_fK(X)$ factors over the connected components $\mathcal{C}(X)$ of X

$${}_fK(X) = \prod_{Y\subset\mathcal{C}(X)} {}_fK(Y) \tag{37}$$

and that ${}_fK(X,\phi)$ only depends on ϕ in X^*.

As in [5], the background functional ${}_fI(B)$ has a special form ${}_fI({}_fE,\sigma,B)$ $= \exp(-V({}_fE,\sigma,B))$ where[5]

$$V({}_fE,\sigma,B,\phi) = {}_fE(B) + \frac{1}{4}\sum_{x\in B}\sum_{\mu\nu} \sigma_{\mu\nu}(B)\partial_\mu\phi(x)\partial_\nu\phi(x) \tag{38}$$

for some functions ${}_fE, \sigma_{\mu\nu} : \mathcal{B}_j \to \mathbb{R}$. Indeed we usually can take $\sigma_{\mu\nu}(B) = \sigma\delta_{\mu\nu}$ for some constant σ. Then $V({}_fE,\sigma,B,\phi)$ becomes

$$V({}_fE,\sigma,B,\phi) = {}_fE(B) + \frac{\sigma}{4}\sum_{x\in B}\sum_{\mu}(\partial_\mu\phi(x))^2 \equiv {}_fE(B) + \sigma V(B). \tag{39}$$

Also in our model, when $f = 0$, we will have

$${}_0K(X,\phi) \ = \ {}_0K(X,-\phi), \qquad\qquad {}_0K(X,\phi) \ = \ {}_0K(X,\phi + c). \tag{40}$$

The later holds for any constant c which means that ${}_0K(X,\phi)$ only depends on derivatives $\partial\phi$.

2.4. About Norms and Their Properties

The norms provide a way of encoding the physicists' power counting arguments.

[5] Sums over μ are understood to range over $\mu = \pm 1, \ldots, \pm d$, unless otherwise specified.

2.4.1. Norms for ϕ. For a j-polymer X, let $\Phi_j(X)$ be the Banach space of functions $\phi : X \to \mathbb{R}$ modulo constants with the following norm

$$\|\phi\|_{\Phi_j(X)} = h_j^{-1} \max\left\{\|\nabla_j \phi\|_{X,\infty},\ \|\nabla_j^2 \phi\|_{X,\infty}\right\} \tag{41}$$

in that

$$\begin{aligned} \nabla_{j,\mu} &= L^j \partial_\mu, \qquad\qquad h_j = L^{-[\phi]j} h \text{ (for some } h \geq 1), \\ \|\nabla_j \phi\|_{X,\infty} &= \sup\left\{ |\nabla_{j,\mu}\phi(x)| \ \Big| x \in X, \mu \in \{\pm 1, \pm 2, \ldots, \pm d\}\right\}. \end{aligned} \tag{42}$$

If X is a $(j+1)$-polymer then X can also be considered as a j-polymer. And we have

$$\|\phi\|_{\Phi_j(X)} \leq L^{-[\phi]-1} \|\phi\|_{\Phi_{j+1}(X)} = L^{-d/2} \|\phi\|_{\Phi_{j+1}(X)} \leq L \|\phi\|_{\Phi_j(X)} \tag{43}$$

since $h_j^{-1} = L^{[\phi]j} h^{-1} = L^{[\phi](j+1)-[\phi]} h^{-1} = L^{-[\phi]} h_{j+1}^{-1}$ and $\nabla_j = L^{-1} \nabla_{j+1}$.

2.4.2. Norms for K. For $X \in \mathcal{P}_j$, we suppose that the polymer activities $K(X, \phi)$ only depends on ϕ in X^* (the smallset neighborhood of X) and is a $\mathcal{C}^3$ function on $\Phi_j(X^*)$. Then we define

- For $n = 0, 1, 2, 3$,

$$\|K_n(X, \phi)\|_j = \sup_{\substack{\|f_i\|_{\Phi_j(X^*)} \leq 1 \\ \forall i = 1 \ldots n}} \{\ |K_n(X, \phi; f_1, \ldots f_n)|\} \tag{44}$$

  where $K_n(X, \phi)$, the n^{th} derivative with respect to ϕ, is a multilinear functional on $f_i \in \Phi_j(X^*)$

$$K_n(X, \phi; f_1, \ldots, f_n) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n} K(X, \phi + t_1 f_1 + \cdots + t_n f_n)\Big|_{t_1 = \cdots = t_n = 0} \tag{45}$$

- A multiplicative norm $\|K(X, \phi)\|_j$

$$\|K(X, \phi)\|_j = \sum_{n=0}^{3} \frac{\|K_n(X, \phi)\|_j}{n!} \tag{46}$$

  satisfies

$$\|K(X, \phi) H(Y, \phi)\|_j \leq \|K(X, \phi)\|_j \|H(Y, \phi)\|_j \tag{47}$$

  where X, Y are disjoint polymers in $\mathcal{P}_j$.

-

$$\|K(X)\|_j = \sup_{\phi', \zeta} \|K(X, \phi' + \zeta)\|_j G_j(X, \phi', \zeta)^{-1} \tag{48}$$

  where $K(X, \phi)$ is considered as a function $K(X, \phi' + \zeta)$ of ϕ', ζ, and $G_j(X, \phi', \zeta) = G_j(X, \phi', 0) G_j(X, 0, \zeta)$ is the large field regulator and satisfies following conditions:
  - $G_j(X, \phi', \zeta)$ depends only on ϕ', ζ in X^*
  - $G_j(X, \phi', \zeta) \geq 1$ and $G_j(X, 0, 0) = 1$.

If we want to use the same norm but with K as a function of ϕ' only, we can define

$$\|K(X)\|'_j = \sup_{\phi',\zeta} \|K(X,\phi')\|_j G_j(X,\phi',\zeta)^{-1} = \sup_{\phi'} \|K(X,\phi')\|_j G_j(X,\phi',0)^{-1}. \quad (49)$$

There are two choices of large field regulators:

– The strong large field regulator

$$G_{s,j}(X,\phi',\zeta) = \prod_{B\in\mathcal{B}_j(X)} e^{\|\phi'\|^2_{\Phi_j(B^*)}+\|\zeta\|^2_{\Phi_j(B^*)}}. \quad (50)$$

We use $\|K(X)\|_{s,j}$ to denote the norm with this strong regulator.

– The weak large field regulator

$$G_j(X,\phi',\zeta) = \prod_{B\in\mathcal{B}_j(X)} \exp\left(c_1 h_j^{-2} L^{-dj}\|\nabla_j\phi'\|^2_{B,2} + c_2 h_j^{-2}\|\nabla_j^2\phi'\|^2_{B^*,\infty}\right) \times \exp\left(c_3 h_j^{-2} L^{-(d-1)j}\|\nabla_j\phi'\|^2_{\partial X,2}\right) \times \prod_{B\in\mathcal{B}_j(X)} \exp\left(c_4 h_j^{-2} \max_{0\le p\le 2}\|\nabla_j^p\zeta\|^2_{B^*,\infty}\right). \quad (51)$$

The norm with the weak regulator will be denoted just as $\|K(X)\|_j$.[6] Using the theorem 6.14 and (6.100) in [1], for proper choices of c_1, c_2, c_3, c_4, L sufficiently large, and h sufficiently large depending on L, we have that

$$G_{s,j}(X) \le G_{s,j}(X)^2 \le G_j(X) \quad (52)$$

which yields to

$$\|K(X)\|_j \le \|K(X)\|_{s,j}. \quad (53)$$

- For $A \ge 1$, we define

$$\|K\|_j = \sup_{X\in\mathcal{P}_{j,c}} \|K(X)\|_j A^{|X|_j}. \quad (54)$$

With this norm, polymer activities $K(X,\phi)$ ($X \in \mathcal{P}_{j,c}$) form a Banach space denoted as $\mathcal{K}_j(\Lambda_N)$.

2.4.3. Properties of Norms. Using the theorem 6.14 in [1], for proper choices of c_1, c_2, c_3, c_4, L sufficiently large, and h sufficiently large depending on L, we have the following properties:

1.
$$\|K(X)\|_j \le \prod_{Y\in\mathcal{C}(X)} \|K(Y)\|_j. \quad (55)$$

2.
$$\left\| \left(\prod_{B\subset X} F(B)\right) K(Y) \right\|_j \le \prod_{B\subset X} \|F(B)\|_{s,j} \|K(Y)\|_j \quad (56)$$

where X, Y are disjoint polymers (but possibly touching).

[6] Note that $h_j^{-2}L^{-dj}\|\nabla_j\phi'\|^2_{B,2} = h^{-2}\|\partial\phi'\|^2_{B,2}$ actually has no explicit dependence on j.

3.
$$\|K^{\#}(X)\|_j' \le 2^{|X|_j}\|K(X)\|_j \le (A/2)^{-|X|_j}\|K\|_j \tag{57}$$
with
$$K^{\#}(X,\phi) = \int K(X,\phi,\zeta)\mathrm{d}\mu_{\Gamma_{j+1}}(\zeta). \tag{58}$$

4. If U is a $(j+1)$-polymer, we can also consider U as a j-polymer. Hence,
$$\begin{aligned}\|K(U)\|_{j+1} &\le \|K(U)\|_j',\\ \|K(U)\|_{s,j+1} &\le \|K(U)\|_{s,j}'.\end{aligned} \tag{59}$$

2.4.4. Some Crucial Estimates. Now we consider potential $V(s,B,\phi)$ of the form
$$V(s,B,\phi) = \frac{1}{4}\sum_{x\in B}\sum_{\mu\nu} s_{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) \tag{60}$$
here the norms of functions $s_{\mu\nu}(x)$ are defined by
$$\|s\|_j = \sup_{B\in\mathcal{B}_j}|B|^{-1}\|s\|_{1,B} = \sup_{B\in\mathcal{B}_j} L^{-dj}\sum_{\mu\nu}\sum_{x\in B}|s_{\mu\nu}(x)|. \tag{61}$$

If $s_{\mu\nu}(x) = \sigma\delta_{\mu\nu}$ then $V(s,B) = \sigma V(B)$ as defined in (39) and the norm $\|s\|_j = 2d\ \sigma$.

Two following lemmas are borrowed from Section 3 in [5]:

Lemma 1 ([5], Lemma 3).

1. *For any functional $s_{\mu\nu}(x)$, we have*
$$\begin{aligned}\|V(s,B)\|_{s,j}' &\le h^2\|s\|_j,\\ \|V(s,B)\|_{s,j} &\le h^2\|s\|_j.\end{aligned} \tag{62}$$

2. *The function $\sigma \to \exp(-\sigma V(B))$ is complex analytic and if $h^2\sigma$ is sufficiently small, we have*
$$\begin{aligned}\left\|e^{-\sigma V(B)}\right\|_{s,j}' &\le 2,\\ \left\|e^{-\sigma V(B)}\right\|_{s,j} &\le 2.\end{aligned} \tag{63}$$

Let c be a constant such that the function $\sigma \to \exp(-\sigma V(B))$ is analytic in $|\sigma| \le ch^{-2}$ and satisfies $\|\exp(-\sigma V(B))\|_{s,j} \le 2$ on that domain.

To start the RG transformation, we also need some estimate on the initial interaction. When $j = 0$, $B \in \mathcal{B}_0$ is just a single site $x \in \mathbb{Z}^d$, so we consider
$$W(u,B,\phi) = \int_{\mathbb{S}^{d-1}} \mathrm{d}p\cos(p\cdot\partial\phi(x)u). \tag{64}$$

Lemma 2 ([5], Lemma 4).

1. *$W(u,B)$ is bounded by*
$$\|W(u,B)\|_{s,0} \le 2e^{\sqrt{d}hu}. \tag{65}$$

We also have that $W(u,B)$ is strongly continuously differentiable in u.

2. $e^{zW(u,B)}$ *is complex analytic in* z *and satisfies, for* $|z|$ *sufficiently small* $($*depending on* $d, h, u)$*, we have*

$$\left\| e^{zW(u,B)} \right\|_{s,0} \le 2. \tag{66}$$

And $e^{zW(u,B)}$ *is also strongly continuously differentiable in* u.

We also borrowed the following lemma [1] which is giving a treatment for large polymers.

Lemma 3 ([1], Lemma 6.18). *If* $U \in \mathcal{P}_{j+1}$ *and*

$$k(\alpha, U) = \sum_{\substack{X\in\mathcal{P}_{j,c}:\\ X\notin\mathcal{S}_j, \overline{X}=U}} (\alpha)^{-|X|_j}$$

then, for any $\lambda \in (0, 1]$, $\alpha \ge 1$, *we have*

$$\lim_{\alpha\to\infty} \sup_{U\in\mathcal{P}_{j+1}} k(\lambda\alpha, U)\alpha^{|U|_{j+1}} = 0.$$

The idea of lemma 6.18 in [1] is that, for large polymers X such that $\bar{X} = U$, the quantity $|X|_j$ must be much larger than $|U|_{j+1}$.

3. Analysis of the RG Transformation

Now we use the Brydges–Slade RG analysis and follow the framework of Dimock [5], but with an external field f.

3.1. Coordinates $({}_fI_j, {}_fK_j)$

Continuing to the Sect. 2.3.2 (Local Expansion), we suppose that we have ${}_f\mathcal{Z}(\phi) = ({}_fI \circ {}_fK)(\Lambda, \phi)$ with polymers on scale j. We rewrite it as

$${}_f\mathcal{Z}'(\phi') = (\mu_{\Gamma_{j+1}} * {}_f\mathcal{Z})(\phi') \equiv \int {}_f\mathcal{Z}(\phi' + \zeta)\mathrm{d}\mu_{\Gamma_{j+1}}(\zeta) \tag{67}$$

here we try to put it back to the form

$${}_f\mathcal{Z}'(\phi') = ({}_fI' \circ {}_fK')(\Lambda, \phi') \tag{68}$$

where the polymers are now on scale $(j + 1)$. Furthermore, supposed that we have chosen ${}_fI'$, we will find ${}_fK'$ so the identity holds. As explained before, our choice of ${}_fI'$ is to have the form

$${}_fI'(B', \phi') = \prod_{B\in\mathcal{B}_j, B\subset B'} {}_f\tilde{I}(B, \phi'), \qquad B' \in \mathcal{B}_{j+1}. \tag{69}$$

Now we define

$$\begin{aligned} \delta_fI(B, \phi', \zeta) &= {}_fI(B, \phi' + \zeta) - {}_f\tilde{I}(B, \phi'), \\ {}_fK \circ \delta_fI \equiv {}_f\tilde{K}(X, \phi', \zeta) &= \sum_{Y\subset X} {}_fK(Y, \phi' + \zeta)\delta_fI^{X-Y}(\phi', \zeta). \end{aligned} \tag{70}$$

For connected X we write ${}_f\tilde{K}(X,\phi',\zeta)$ in the form[7]

$$ {}_f\tilde{K}(X,\phi',\zeta) = \sum_{B\subset X} {}_fJ(B,X,\phi') + {}_f\check{K}(X,\phi',\zeta). \tag{71}$$

Given ${}_fK$ and ${}_fJ$ the Eq. (71) would give us a definition of ${}_f\check{K}(X)$ for X connected. And for any $X \in \mathcal{P}_j$, we define

$$ {}_f\check{K}(X,\phi',\zeta) = \prod_{Y\in\mathcal{C}(X)} {}_f\check{K}(Y,\phi',\zeta). \tag{72}$$

After using the finite range property and making some rearrangements as Proposition 5.1, Brydges [1], we have (68) holds with

$$ {}_fK'(U,\phi') = \sum_{X,\chi\to U} {}_fJ^{\chi}(\phi')\,{}_f\tilde{I}^{U-(X_\chi\cup X)}(\phi')\,{}_f\check{K}^{\#}(X,\phi'), \qquad U\in\mathcal{P}_{j+1} \tag{73}$$

where $\chi = (B_1, X_1, \dots B_n, X_n)$ and the condition $X,\chi \to U$ means that $X_1,\dots X_n, X$ be strictly disjoint and satisfy $\overline{(B_1^* \cup \cdots \cup B_n^* \cup X)} = U$. Moreover

$$ \begin{aligned} {}_fJ^{\chi}(\phi') &= \prod_{i=1}^{n} {}_fJ(B_l, X_l, \phi'),\\ {}_f\tilde{I}^{U-(X_\chi\cup X)}(\phi') &= \prod_{B\in U-(X_\chi\cup X)} {}_f\tilde{I}(B,\phi'), \end{aligned} \tag{74}$$

with $X_\chi = \cup_i X_i$. And ${}_f\check{K}^{\#}(X,\phi')$ is ${}_f\check{K}(X,\phi',\zeta)$ integrated over ζ as in (58).

At this point ${}_fK'$ is considered as a function of ${}_fI, {}_f\tilde{I}, {}_fJ, {}_fK$. It vanishes at the point $({}_fI, {}_f\tilde{I}, {}_fJ, {}_fK) = (1,1,0,0)$ since $\chi = \emptyset$ and $X=\emptyset$ iff $U=\emptyset$. We study its behavior in a neighborhood of this point. We use the norm in (54) for ${}_fK$ and define

$$ \begin{aligned} \|{}_fI\|_{s,j} &= \sup_{B\in\mathcal{B}_j} \|{}_fI(B)\|_{s,j},\\ \|{}_f\tilde{I}\|'_{s,j} &= \sup_{B\in\mathcal{B}_j} \|{}_f\tilde{I}(B)\|'_{s,j},\\ \|{}_fJ\|'_j &= \sup_{X\in\mathcal{S}_j, B\subset X} \|{}_fJ(X,B)\|'_j. \end{aligned} \tag{75}$$

Using the same argument as Theorem 3 in [5], we have the following result.

Theorem 8. *Let A be sufficiently large.*

1. *For $R>0$ there is an $r>0$ such that the following holds for all j. If $\max\{\|{}_0I-1\|_{s,j}, \|{}_fI-1\|_{s,j}\} < r$, $\max\{\|{}_0\tilde{I}-1\|'_{s,j}, \|{}_f\tilde{I}-1\|'_{s,j}\} < r$, $\max\{\|{}_0J\|'_j, \|{}_fJ\|'_j\} < r$ and $\max\{\|{}_0K\|_j, \|{}_fK\|_j\} < r$ then we have $\max\{\|{}_0K'\|_{j+1}, \|{}_fK'\|_{j+1}\} < R$. Furthermore ${}_fK'$ is a smooth function of ${}_fI, {}_f\tilde{I}, {}_fJ, {}_fK$ on this domain with derivatives bounded uniformly in j. The*

[7] As in Dimock [5], ${}_fJ(B,X)$ will be chosen to depend on ${}_fK$ and required ${}_fJ(B,X)=0$ unless $X\in\mathcal{S}_j, B\subset X$ and that ${}_fJ(B,X,\phi')$ depend on ϕ' only in B^*.

analyticity of ${}_fK'$ *in* $t_1, t_2, \ldots, t_m$ *still holds when we go from* j*-scale to* $(j+1)$*-scale.*

2. *If also*

$$\sum_{X \in \mathcal{S}_j : X \supset B} {}_fJ(B, X) = 0 \tag{76}$$

then the linearization of ${}_fK' = {}_fK'({}_fI, {}_f\tilde{I}, {}_fJ, {}_fK)$ *at* $({}_fI, {}_f\tilde{I}, {}_fJ, {}_fK) = (1, 1, 0, 0)$ *is*

$$\sum_{\substack{X \in \mathcal{P}_{j,c} \\ \overline{X} = U}} \Bigg({}_fK^{\#}(X) + ({}_fI^{\#}(X) - 1)1_{X \in \mathcal{B}_j} - ({}_f\tilde{I}(X) - 1)1_{X \in \mathcal{B}_j} - \sum_{B \subset X} {}_fJ(B, X) \Bigg) \tag{77}$$

where

$${}_fK^{\#}(X, \phi) = \int {}_fK(X, \phi + \zeta) \mathrm{d}\mu_{\Gamma_{j+1}}(\zeta) \tag{78}$$

and ${}_0J$ *actually is* ${}_fJ$ *at* $f = 0$.

3.2. Choosing J and Estimating $\mathcal{L}_1, \mathcal{L}_2$

3.2.1. Choosing J. For a smooth function $g(\phi)$ on $\phi \in \mathbb{R}^\Lambda$, let $T_2 g$ denote a second order Taylor expansion:

$$\begin{aligned} (T_2 g)(\phi) &= g(0) + g_1(0; \phi) + \frac{1}{2} g_2(0; \phi, \phi), \\ (T_0 g)(\phi) &= g(0). \end{aligned} \tag{79}$$

We also set

$$\delta_f K = {}_fK - {}_0K. \tag{80}$$

With ${}_fK^{\#}$ defined in (78), for $X \in \mathcal{S}_j$, $X \supset B$, $X \neq B$, we pick:

$$\begin{aligned} {}_fJ(B, X) &= \frac{1}{|X|_j} [T_2({}_0K^{\#}(X)) + T_0({}_fK^{\#}(X)) - T_0({}_0K^{\#}(X))] \\ &= \frac{1}{|X|_j} [T_2({}_0K^{\#}(X)) + T_0(\delta_f K^{\#}(X))] \end{aligned} \tag{81}$$

and choose ${}_fJ(B, B)$ so that (76) is satisfied. Otherwise, we let ${}_fJ(B, X) = 0$.[8] As in (39) we have picked

$${}_fI(B) = {}_fI({}_fE, \sigma, B) = \exp(-V({}_fE, \sigma, B)). \tag{82}$$

So we require ${}_f\tilde{I}$ to have the same form

$${}_f\tilde{I}(B) = {}_fI({}_f\tilde{E}, \tilde{\sigma}, B) = \exp(-V({}_f\tilde{E}, \tilde{\sigma}, B)) \tag{83}$$

with ${}_f\tilde{E}, \tilde{\sigma}$ which will be defined later. Because $\sum_{B \subset B'} V(B) = V(B')$, we have

$${}_fI'(B') = {}_fI({}_fE', \sigma', B') = \exp(-V({}_fE', \sigma', B)) \tag{84}$$

[8] When $f = 0$, the ${}_0J$ is exactly the same as the J in [5, section 4.2].

with

$$\begin{aligned} {}_f E'(B') &= \sum_{B \subset B'} {}_f\tilde{E}(B), \\ \sigma' &= \tilde{\sigma}. \end{aligned} \tag{85}$$

The map ${}_fK'$ becomes ${}_fK' = {}_fK'({}_f\tilde{E}, \tilde{\sigma}, {}_fE, \sigma, {}_fK, {}_0K)$. We use the standard norm on the energy

$$\|{}_fE\|_j = \sup_{B \in \mathcal{B}_j} |{}_fE(B)|. \tag{86}$$

And the Theorem 8 becomes:

Theorem 9. *Let A be sufficiently large.*

1. *For $R > 0$ there is an $r > 0$ such that the following holds for all j. If $\max\{\|{}_f\tilde{E}\|_j, \|{}_0\tilde{E}\|_j\}$, $\max\{\|{}_fE\|_j, \|{}_0E\|_j\}$, $\max\{\|{}_fK\|_j, \|{}_0K\|_j\}$, $|\tilde{\sigma}|$, $|\sigma| < r$ then $\max\{\|{}_fK'\|_{j+1}, \|{}_0K'\|_{j+1}\} < R$. Furthermore ${}_fK'$ is a smooth function of ${}_f\tilde{E}, \tilde{\sigma}, {}_fE, \sigma, {}_fK, {}_0K$ on this domain with derivatives bounded uniformly in j. The analyticity of ${}_fK'$ in $t_1, t_2, \ldots, t_m$ still holds when we go from j-scale to $(j+1)$-scale.*
2. *The linearization of ${}_fK'$ at the origin has the form*

$$\mathcal{L}_1({}_fK) + \mathcal{L}_2({}_fK) + \mathcal{L}_3({}_fE, \sigma, {}_f\tilde{E}, \tilde{\sigma}, {}_fK, {}_0K) \tag{87}$$

where

$$\begin{aligned} \mathcal{L}_1({}_fK)(U) &= \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X} = U}} {}_fK^{\#}(X) \\ &= \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X} = U}} {}_0K^{\#}(X) + \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X} = U}} \delta_f K^{\#}(X) \\ &= \mathcal{L}_1({}_0K)(U) + \mathcal{L}_1(\delta_f K)(U), \\ \mathcal{L}_2({}_fK)(U) &= \sum_{\substack{X \in \mathcal{S}_j \\ \overline{X} = U}} {}_fK^{\#}(X) - [T_2({}_0K^{\#}(X)) + T_0(\delta_f K^{\#}(X))] \\ &= \sum_{\substack{X \in \mathcal{S}_j \\ \overline{X} = U}} (I - T_2)({}_0K^{\#}(X)) + \sum_{\substack{X \in \mathcal{S}_j \\ \overline{X} = U}} (I - T_0)(\delta_f K^{\#}(X)) \\ &= \mathcal{L}_2({}_0K)(U) + \mathcal{L}_2(\delta_f K)(U), \\ \mathcal{L}_3(E, \sigma, \tilde{E}, \tilde{\sigma}, {}_fK, {}_0K)(U) &= \sum_{\bar{B} = U} \Big(V(\tilde{E}, \tilde{\sigma}, B) - V^{\#}(E, \sigma, B) \Big) \\ &\quad + \sum_{\bar{B} = U} \sum_{\substack{X \in \mathcal{S}_j \\ X \supset B}} \frac{1}{|X|_j} [T_2({}_0K^{\#}(X)) + T_0(\delta_f K^{\#}(X))]. \end{aligned} \tag{88}$$

Proof. The new map actually is the composition of the map ${}_fK' = {}_fK'({}_fI, {}_f\tilde{I}, {}_fJ, {}_fK)$ of Theorem 8 with the maps ${}_fI = {}_fI({}_fE, \sigma)$, ${}_f\tilde{I} = {}_fI({}_f\tilde{E}, \tilde{\sigma})$, ${}_fJ =$

${}_fJ({}_fK, {}_0K)$. Thus it suffices to establish uniform bounds and smoothness for the latter.

In the case $f = 0$, we have already have the proof in Theorem 4, Dimock [5]. So we only consider the case $f \neq 0$.

For ${}_fI = {}_fI({}_fE, \sigma)$, ${}_f\tilde{I} = {}_fI({}_f\tilde{E}, \tilde{\sigma})$ the proof is the same as the proof for I, I' in [5, Theorem 4]. We have:

$$
\begin{aligned}
{}_fJ(B, X) &= \frac{1}{|X|_j}[T_2({}_0K^{\#}(X)) + T_0({}_fK^{\#}(X)) - T_0({}_0K^{\#}(X))] \\
&= \frac{1}{|X|_j}[(T_2 - T_0)({}_0K^{\#}(X)) + T_0({}_fK^{\#}(X))].
\end{aligned} \tag{89}
$$

With the same argument as in [5, Theorem 4], we obtain:

$$
\begin{aligned}
\|(T_2 - T_0)({}_0K^{\#}(X))\|'_j &\leq \mathcal{O}(1)\|{}_0K^{\#}(X)\|'_j, \\
\|(T_0)({}_fK^{\#}(X))\|'_j &\leq \mathcal{O}(1)\|{}_fK^{\#}(X)\|'_j.
\end{aligned} \tag{90}
$$

By (57) these are bounded by $\mathcal{O}(1)[\|{}_fK\|_j + \|{}_0K\|_j]$. The same bound holds for $\|{}_fJ(B, B)\|'_j$. Therefore $\|{}_fJ\|'_j \leq \mathcal{O}(1)[\|{}_fK\|_j + \|{}_0K\|_j]$.

The linearization is just a computation. Indeed ${}_fJ(B, X)$ is designed so that

$$
\begin{aligned}
&\sum_{\substack{X \in \mathcal{S}_j \\ \overline{X} = U}} \left({}_fK^{\#}(X) - \sum_{B \subset X} {}_fJ(B, X) \right) \\
&= \sum_{\overline{B} = U} \sum_{\substack{X \in \mathcal{S}_j \\ X \supset B}} \frac{1}{|X|_j} \left[T_2({}_0K^{\#}(X)) + T_0(\delta_f K^{\#}(X)) \right] \\
&\quad + \sum_{\substack{X \in \mathcal{S}_j \\ \overline{X} = U}} {}_fK^{\#}(X) - \left[T_2({}_0K^{\#}(X)) + T_0(\delta_f K^{\#}(X)) \right]
\end{aligned} \tag{91}
$$

which accounts for the presence of these terms. Also the linearization of $({}_fI^{\#}(B) - 1)$ is $-V^{\#}({}_fE, \sigma, B)$, and so forth. This completes the proof. ☐

3.2.2. Estimating $\mathcal{L}_1, \mathcal{L}_2$—The First Two Linearization Parts. Next we make some estimates on the linearization's parts. First we estimate $\mathcal{L}_1$ which is the linearization on the large j-polymers.

Lemma 4. *Let A be sufficiently large depending on L. Then the operator $\mathcal{L}_1$ is a contraction with a norm which goes to zero as $A \to \infty$.*

Proof. We use the same proof as in Dimock [5, Lemma 5], but with updated notations. We estimate by using (57) and (59)

$$
\begin{aligned}
\|\mathcal{L}_1({}_0K)(U)\|_{j+1} \leq \|\mathcal{L}_1({}_0K)(U)\|'_j &\leq \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X} = U}} \|{}_0K^{\#}(X)\|'_j \\
&\leq \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X} = U}} (A/2)^{-|X|_j} \|{}_0K\|_j
\end{aligned} \tag{92}
$$

and

$$\|\mathcal{L}_1(\delta_f K)(U)\|_{j+1} \leq \|\mathcal{L}_1(\delta_f K)(U)\|'_j \leq \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X}=U}} \|\delta_f K^{\#}(X)\|'_j$$
$$\leq \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X}=U}} (A/2)^{-|X|_j} \|\delta_f K\|_j. \quad (93)$$

Multiplying by $A^{|U|_{j+1}}$ then taking the supremum over U, these yield

$$\|\mathcal{L}_1({}_0K)\|_{j+1} \leq \left[\sup_U A^{|U|_{j+1}} \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X}=U}} (A/2)^{-|X|_j}\right] \|{}_0K\|_j,$$
$$\|\mathcal{L}_1(\delta_f K)\|_{j+1} \leq \left[\sup_U A^{|U|_{j+1}} \sum_{\substack{X \in \mathcal{P}_{j,c}: \\ X \notin \mathcal{S}_j, \overline{X}=U}} (A/2)^{-|X|_j}\right] \|\delta_f K\|_j. \quad (94)$$

Using Lemma 3 (or [1, lemma 6.18]) with $\alpha = A$ and $\lambda = \frac{1}{2}$, the bracketed expression goes to zero as $A \to \infty$. We also have

$$\|\mathcal{L}_1({}_fK)\|_{j+1} \leq \|\mathcal{L}_1({}_0K)\|_{j+1} + \|\mathcal{L}_1(\delta_f K)\|_{j+1},$$
$$\|\delta_f K\|_j \leq \|{}_0K\|_j + \|{}_fK\|_j. \quad (95)$$

Hence, for A sufficiently large, $\|\mathcal{L}_1({}_fK)\|_{j+1}$ is arbitrarily small. □

Now we estimate and find an explicit upper bound for $\mathcal{L}_2$.

Lemma 5. *Let L be sufficiently large. Then the operator $\mathcal{L}_2$ is a contraction with a norm which goes to zero as $L \to \infty$.*

Proof. For $f = 0$, we have the Lemma 6, in [5].

For $f \neq 0$, we can write

$$\mathcal{L}_2({}_fK)(U) = \sum_{X \in \mathcal{S}_j, \overline{X}=U} \{(I - T_2){}_0K^{\#}(X) + (I - T_0)\delta_f K^{\#}(X)\}$$
$$= \mathcal{L}_2({}_0K)(U) + \mathcal{L}_2(\delta_f K)(U) \quad (96)$$

where

$$\mathcal{L}_2(\delta_f K)(U) = \sum_{X \in \mathcal{S}_j, \overline{X}=U} (I - T_0)\delta_f K^{\#}(X). \quad (97)$$

Using [1, (6.40)] as well as [5, Lemma 6] we get:

$$\|(I - T_2){}_0K^{\#}(X, \phi)\|_{j+1} \leq \left(1 + \|\phi\|_{\Phi_{j+1}(X^*)}\right)^3 \|{}_0K_3^{\#}(X, \phi)\|_{j+1}$$
$$\leq 4\left(1 + \|\phi\|^3_{\Phi_{j+1}(X^*)}\right) \|{}_0K_3^{\#}(X, \phi)\|_{j+1}, \quad (98)$$
$$\|(I - T_0)\delta_f K^{\#}(X, \phi)\|_{j+1} \leq \left(1 + \|\phi\|_{\Phi_{j+1}(X^*)}\right) \|\delta_f K_1^{\#}(X, \phi)\|_{j+1}.$$

Notice that $\delta_f K^{\#}(X, 0) = {}_fK^{\#}(X, 0) - {}_0K^{\#}(X, 0)$ only depend on ϕ in X^*. Moreover, ${}_0K$ and ${}_fK$ are different only on $\mathrm{supp}(f) = \{x_1, \ldots, x_m\}$. So, if $X^* \cap$

$\{x_1, x_2, \ldots, x_m\} = \emptyset$ then ${}_fK^{\#}(X,0) = {}_0K^{\#}(X,0)$ which means $\delta_f K^{\#}(X,0) = 0$. Therefore $\delta_f K^{\#}(X,0) = 0$ unless $X^* \cap \{x_1, x_2, \ldots, x_m\} \neq \emptyset$.

Using property (43), we have

$$\begin{aligned}\|{}_0K_3^{\#}(X,\phi)\|_{j+1} &\leq L^{-3d/2}\|{}_0K_3^{\#}(X,\phi)\|_j\\ &\leq 6(L^{-3d/2})\|{}_0K^{\#}(X,\phi)\|_j\\ &\leq 6(L^{-3d/2})\|{}_0K^{\#}(X)\|_j' G_j(X,\phi,0),\\ \|\delta_f K_1^{\#}(X,\phi)\|_{j+1} &\leq L^{-d/2}\|\delta_f K_1^{\#}(X,\phi)\|_j\\ &\leq L^{-d/2}\|\delta_f K^{\#}(X,\phi)\|_j\\ &\leq L^{-d/2}\left(\|\delta_f K^{\#}(X)\|_j'\right) G_j(X,\phi,0),\end{aligned} \tag{99}$$

and for $\phi = \phi' + \zeta$, using [1, (6.58)] we get:

$$\begin{aligned}\left(1 + \|\phi\|_{\Phi_{j+1}(X^*)}\right) G_j(X,\phi,0) &\leq \left(1 + \|\phi\|_{\Phi_{j+1}(X^*)}\right)^3 G_j(X,\phi,0)\\ &\leq 4\left(1 + \|\phi\|^3_{\Phi_{j+1}(X^*)}\right) G_j(X,\phi,0)\\ &\leq 4qG_{j+1}(\bar{X},\phi',\zeta)\end{aligned} \tag{100}$$

with q as in [1, (6.127)]. Combining all of them yields

$$\begin{aligned}\|(I-T_2){}_0K^{\#}(X,\phi)\|_{j+1} &\leq 24q(L^{-3d/2})\|{}_0K^{\#}(X)\|_j' G_{j+1}(\bar{X},\phi',\zeta),\\ \|(I-T_0)\delta_f K^{\#}(X,\phi)\|_{j+1} &\leq 4qL^{-d/2}\left(\|\delta_f K^{\#}(X)\|_j'\right) G_{j+1}(\bar{X},\phi',\zeta).\end{aligned} \tag{101}$$

By using (57), we obtain:

$$\begin{aligned}\|(I-T_2){}_0K^{\#}(X)\|_{j+1} &\leq 24q(L^{-3d/2})\|{}_0K^{\#}(X)\|_j'\\ &\leq 24q(L^{-3d/2})(A/2)^{-|X|_j}\|{}_0K\|_j,\\ \|(I-T_0)\delta_f K^{\#}(X)\|_{j+1} &\leq 4qL^{-d/2}\|\delta_f K^{\#}(X)\|_j'\\ &\leq 4qL^{-d/2}\left(\|\delta_f K\|_j\right) A^{-|X|_j}2^{|X|_j}.\end{aligned} \tag{102}$$

Therefore,

$$\begin{aligned}\|\mathcal{L}_2({}_fK)\|_{j+1} &\leq \|\mathcal{L}_2({}_0K)\|_{j+1} + \|\mathcal{L}_2(\delta_f K)\|_{j+1}\\ &\leq 24q(L^{-3d/2})\left[\sup_U A^{|U|_{j+1}} \sum_{X\in\mathcal{S}_j, \bar{X}=U} (A/2)^{-|X|_j}\right]\|{}_0K\|_j\\ &\quad + 4qL^{-d/2}\left(\|\delta_f K\|_j\right)\sup_U \sum_{\substack{X\in\mathcal{S}_j,\bar{X}=U\\ X^*\cap\{x_1,x_2,\ldots,x_m\}\neq\emptyset}} A^{|U|_{j+1}}A^{-|X|_j}2^{|X|_j}.\end{aligned} \tag{103}$$

The bracketed expression is less than $2^d 2^{2^d} n_2(d) L^d$ (using [1, (6.90)]), so we have

$$\|\mathcal{L}_2({}_0K)\|_{j+1} \leq 24q2^d2^{2^d}n_2(d)(L^{-d/2})\|{}_0K\|_j. \tag{104}$$

Because $|U|_{j+1} \le |X|_j \le 2^d$, we get:

$$\begin{aligned}
&4qL^{-d/2}\left(\|\delta_f K\|_j\right)\sup_U \sum_{\substack{X\in\mathcal{S}_j,\bar{X}=U\\ X^*\cap\{x_1,x_2,\ldots,x_m\}\neq\emptyset}} A^{|U|_{j+1}}A^{-|X|_j}2^{|X|_j}\\
&\quad\le 4qL^{-d/2}\left(\|\delta_f K\|_j\right)\sum_{\substack{X\in\mathcal{S}_j,\\ X^*\cap\{x_1,x_2,\ldots,x_m\}\neq\emptyset}} 2^{2^d}\\
&\quad\le 4qL^{-d/2}2^{2^d}\left(\|\delta_f K\|_j\right)\sum_{i=1}^{m}\sum_{\substack{X\in\mathcal{S}_j,\\ X^*\cap\{x_i\}\neq\emptyset}} 1\\
&\quad\le 4qmL^{-d/2}2^{2^d}\left(\|\delta_f K\|_j\right)n_2(d). \qquad (105)
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathcal{L}_2({}_fK)\|_{j+1} \le\ & 24q2^d2^{2^d}n_2(d)(L^{-d/2})\|{}_0K\|_j\\
&+L^{-d/2}\left(\|\delta_f K\|_j\right)n_2(d)2^{2^d}4qm. \qquad (106)
\end{aligned}$$

Moreover $\|\delta_f K\|_j \le \|{}_0K\|_j + \|{}_fK\|_j$. So we have the Lemma 5. □

3.3. Splitting $\mathcal{L}_3$

3.3.1. Splitting $\mathcal{L}_3$. Similarly in [5], we have a special treatment for the term $\mathcal{L}_3$. First we rewrite the final term in $\mathcal{L}_3$ which is

$$\begin{aligned}
&\sum_{\bar{B}=U}\sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}}\frac{1}{|X|_j}\left[T_2({}_0K^{\#}(X)) + T_0(\delta_f K^{\#}(X))\right]\\
&\quad=\sum_{\overline{B}=U}\sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}}\frac{1}{|X|_j}\left({}_0K^{\#}(X,0)+\frac{1}{2}{}_0K_2^{\#}(X,0;\phi,\phi)+{}_fK^{\#}(X,0)-{}_0K^{\#}(X,0)\right)\\
&\quad=\sum_{\overline{B}=U}\sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}}\frac{1}{|X|_j}\left({}_fK^{\#}(X,0)+\frac{1}{2}{}_0K_2^{\#}(X,0;\phi,\phi)\right). \qquad (107)
\end{aligned}$$

In ${}_0K_2^{\#}(X,0;\phi,\phi)$ we pick a point $z\in B$, then use the same argument as section 4.3 in [5] by replacing $\phi(x)$ with[9]

$$\phi(z)+\frac{1}{2}(x-z)\cdot\partial\phi(z) \;\equiv\; \phi(z)+\frac{1}{2}\sum_{\mu}(x_\mu - z_\mu)\partial_\mu\phi(z). \qquad (108)$$

[9] We need the factor 1/2 since the sum is over $\pm\mu = 1,\ldots,d$ and $x_{-\mu} = -x_\mu$.

If we also average over $z \in B$, (108) becomes

$$\begin{aligned}
&\sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} \left({}_fK^{\#}(X,0) + \frac{1}{2}\frac{1}{|B|}\sum_{z\in B} {}_0K_2^{\#}(X,0;\phi,\phi) \right) \\
&\quad = \sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} \left({}_fK^{\#}(X,0) \right) \\
&\qquad + \sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} \left(\frac{1}{8|B|} \sum_{z\in B}\sum_{\mu\nu} {}_0K_2^{\#}(X,0;x_\mu,x_\nu)\partial_\mu\phi(z)\partial_\nu\phi(z) \right) \\
&\qquad + \sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j:\\ X\supset B}} \frac{1}{|X|_j} \sum_{z\in B} \frac{1}{|B|} \left(\frac{1}{2} {}_0K_2^{\#}(X,0;\phi,\phi) \right) \\
&\qquad - \sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} \left(\frac{1}{8|B|} \sum_{z\in B}\sum_{\mu\nu} {}_0K_2^{\#}(X,0;x_\mu,x_\nu)\partial_\mu\phi(z)\partial_\nu\phi(z) \right) \\
&\quad = \sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} \left({}_fK^{\#}(X,0) + \frac{1}{8|B|} \sum_{z\in B}\sum_{\mu\nu} {}_0K_2^{\#}(X,0;x_\mu,x_\nu)\partial_\mu\phi(z)\partial_\nu\phi(z) \right) \\
&\qquad + \mathcal{L}_3'({}_fK)(U)
\end{aligned} \tag{109}$$

here $\mathcal{L}_3'({}_fK)(U) = \mathcal{L}_3'({}_0K)(U)$ is so called the error, namely

$$\begin{aligned}
\mathcal{L}_3'({}_0K)(U) = \sum_{\overline{B}=U} \sum_{\substack{X\in\mathcal{S}_j:\\ X\supset B}} \frac{1}{|X|_j} \sum_{z\in B} \frac{1}{|B|} \bigg(&\frac{1}{2} {}_0K_2^{\#}(X,0;\phi,\phi) \\
&- \frac{1}{8} {}_0K_2^{\#}(X,0;x\cdot\partial\phi(z), x\cdot\partial\phi(z)) \bigg)
\end{aligned} \tag{110}$$

and we can say $\mathcal{L}_3'(\delta_fK)(U) = 0$. Next we define

$$\begin{aligned}
{}_f\beta(B) = {}_f\beta({}_fK,B) &= -\sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} {}_fK^{\#}(X,0), \\
\alpha_{\mu\nu}(B) = \alpha_{\mu\nu}({}_fK,B) &= \alpha_{\mu\nu}({}_0K,B) \\
&= -\frac{1}{2}\frac{1}{|B|} \sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j} {}_0K_2^{\#}(X,0;x_\mu,x_\nu).
\end{aligned} \tag{111}$$

Note that $\alpha_{\mu\nu}$ is symmetric and satisfies $\alpha_{-\mu\nu} = -\alpha_{\mu\nu}$. We also let $\alpha_{\mu\nu}$ stand for the function $\alpha_{\mu\nu}(x)$ which takes the constant value $\alpha_{\mu\nu}(B)$ for $x \in B$. Now we write (109) as

$$
\begin{aligned}
&\sum_{\overline{B}=U}\sum_{\substack{X\in\mathcal{S}_j\\X\supset B}}\frac{1}{|X|_j}\left({}_fK^{\#}(X,0)+\frac{1}{8|B|}\sum_{z\in B}\sum_{\mu\nu}{}_0K_2^{\#}(X,0;x_\mu,x_\nu)\partial_\mu\phi(z)\partial_\nu\phi(z)\right)\\
&\quad+\mathcal{L}_3'({}_0K)(U)\\
&=\sum_{\overline{B}=U}\left(\frac{1}{4}\sum_{z\in B}\sum_{\mu\nu}\frac{1}{2}\frac{1}{|B|}\sum_{\substack{X\in\mathcal{S}_j\\X\supset B}}\frac{1}{|X|_j}{}_0K_2^{\#}(X,0;x_\mu,x_\nu)(B)\partial_\mu\phi(z)\partial_\nu\phi(z)\right)\\
&\quad+\sum_{\overline{B}=U}\sum_{\substack{X\in\mathcal{S}_j\\X\supset B}}\frac{1}{|X|_j}{}_fK^{\#}(X,0)+\mathcal{L}_3'({}_0K)(U)\\
&=-\sum_{\overline{B}=U}\left({}_f\beta(B)+\frac{1}{4}\sum_{z\in B}\sum_{\mu\nu}\alpha_{\mu\nu}(B)\partial_\mu\phi(z)\partial_\nu\phi(z)\right)+\mathcal{L}_3'({}_0K)(U)\\
&=-\left(\sum_{\overline{B}=U}V({}_f\beta,\alpha,B,\phi)\right)+\mathcal{L}_3'({}_0K)(U) \qquad (112)
\end{aligned}
$$

where $V({}_f\beta,\alpha,B,\phi)$ defined in (38). Combining all of the above, we get:

$$
\begin{aligned}
&\mathcal{L}_3({}_fE,\sigma,{}_f\tilde{E},\tilde{\sigma},{}_fK,{}_0K)(U)\\
&\quad=\sum_{\overline{B}=U}\Big(V({}_f\tilde{E},\tilde{\sigma},B)-V^{\#}({}_fE,\sigma,B)-V({}_f\beta,\alpha,B)\Big)+\mathcal{L}_3'({}_0K)(U). \qquad (113)
\end{aligned}
$$

3.3.2. Estimating α, ${}_f\beta$ and $\mathcal{L}_3'$. First we find some explicit upper bounds for α and ${}_f\beta$.

Lemma 6 (Estimates ${}_f\beta$ and α).

$$
\begin{aligned}
\|{}_f\beta\|_j &\equiv \sup_{B\in\mathcal{B}_j}|{}_f\beta(B)|\le 2n_2(d)A^{-1}\|{}_fK\|_j,\\
\|\alpha\|_j &\equiv \sup_{B\in\mathcal{B}_j}\sum_{\mu\nu}|\alpha_{\mu\nu}(B)|\le 4(2d)^2n_2(d)h^{-2}A^{-1}\|{}_0K\|_j.
\end{aligned} \qquad (114)
$$

Remark. The norm $\|\alpha\|_j$ agrees with the norm $\|s\|_j$ in (61) if $s_{\mu\nu}(x)=\alpha_{\mu\nu}(B)$ for $x\in B$.

Proof. Using (49) and (57), with A very large, we have:

$$
\begin{aligned}
|{}_fK^{\#}(X,0)| &\le \|{}_fK^{\#}(X)\|_j'\le (A/2)^{-1}\|{}_fK\|_j,\\
\|{}_0K_2^{\#}(X,0)\|_j &\le 2\|{}_0K^{\#}(X)\|_j'\le 4A^{-1}\|{}_0K\|_j.
\end{aligned} \qquad (115)
$$

From (33), the number of small sets containing a block B is $n_2(d)$ which is bounded and depends only on d, we have:

$$\begin{aligned}|{}_f\beta(B)| &\le \sum_{X\in\mathcal{S}_j, X\supset B} |{}_fK^{\#}(X,0)| \\ &\le \sum_{X\in\mathcal{S}_j, X\supset B} 2A^{-1}\|{}_fK\|_j \\ &\le 2n_2(d)A^{-1}\|{}_fK\|_j. \end{aligned} \tag{116}$$

We also have $\|x_\mu\|_{\Phi_j(X^*)} = h^{-1}L^{dj/2}$ and $|B| = L^{dj}$. By using the norm definition (44), we get

$$\begin{aligned}|B|^{-1}|{}_0K_2^{\#}(X,0;x_\mu,x_\nu)| &\le (h^{-1}L^{dj/2})^2L^{-dj}\|{}_0K_2^{\#}(X,0)\|_j \\ &= h^{-2}\|{}_0K_2^{\#}(X,0)\|_j \\ &\le 4h^{-2}A^{-1}\|{}_0K\|_j \end{aligned} \tag{117}$$

then

$$\begin{aligned}\sum_{\mu\nu}|\alpha_{\mu\nu}(B)| &\le \sum_{\mu\nu}\sum_{X\in\mathcal{S}_j, X\supset B} |B|^{-1}|{}_0K_2^{\#}(X,0;x_\mu,x_\nu)| \\ &\le \sum_{\mu\nu}\sum_{X\in\mathcal{S}_j, X\supset B} 4h^{-2}A^{-1}\|{}_0K\|_j \\ &\le \sum_{\mu\nu} n_2(d)4h^{-2}A^{-1}\|{}_0K\|_j \\ &\le (2d)^2n_2(d)4h^{-2}A^{-1}\|{}_0K\|_j. \end{aligned} \tag{118}$$

□

Now we give some estimate for $\mathcal{L}_3'$.

Lemma 7. *Let L be sufficiently large. Then the operator $\mathcal{L}_3'$ is a contraction with arbitrarily small norm*

$$\|\mathcal{L}_3'({}_0K)\|_{j+1} \le 72d^2 2^{2d} n_1(d)(L^{-2})\|{}_0K\|_j. \tag{119}$$

Proof. Based on the proof of Lemma 8 in [5], we make some modifications and obtain a better upper bound with some explicit coefficient. We have

$$\mathcal{L}_3'({}_0K)(U) = \sum_{\bar{B}=U}\sum_{\substack{X\in\mathcal{S}_j\\ X\supset B}} \frac{1}{|X|_j}\sum_{z\in B}\frac{1}{|B|}\frac{1}{2}{}_0K_2^{\#}\left(X,0;\phi-\frac{1}{2}x\cdot\partial\phi(z),\phi+\frac{1}{2}x\cdot\partial\phi(z)\right). \tag{120}$$

Using [5, (152)–(154)] we get:

$$\begin{aligned}\left\|\phi-\frac{1}{2}x\cdot\partial\phi(z)\right\|_{\Phi_j(X^*)} &\le 3d2^d(L^{-d/2-1})\|\phi\|_{\Phi_{j+1}(X^*)}, \\ \left\|\phi+\frac{1}{2}x\cdot\partial\phi(z)\right\|_{\Phi_j(X^*)} &\le 3d2^d(L^{-d/2-1})\|\phi\|_{\Phi_{j+1}(X^*)}. \end{aligned} \tag{121}$$

Now we estimate

$$ {}_0H_X(U,\phi) = {}_0K_2^{\#}\left(X,0;\phi - \frac{1}{2}x\cdot\partial\phi(z), \phi + \frac{1}{2}x\cdot\partial\phi(z)\right). \tag{122} $$

Using the same argument as (156)–(157) in [5], we obtain:

$$ \|{}_0H_X(U,\phi)\|_{j+1} \le 18d^2 2^{2d}(L^{-d-2})\|K_2^{\#}(X,0)\|_j(1+\|\phi\|^2_{\Phi_{j+1}(U^*)}). \tag{123} $$

But for $\phi = \phi' + \zeta$,

$$ (1+\|\phi\|^2_{\Phi_{j+1}(U^*)}) \le G_{s,j+1}(U,\phi,0) \le G^2_{s,j+1}(U,\phi',\zeta) \le G_{j+1}(U,\phi',\zeta). \tag{124} $$

Also using (115) we can get:

$$ \|H_X(U)\|_{j+1} \le 72d^2 2^{2d}(L^{-d-2})A^{-1}\|K\|_j, \tag{125} $$

which yields to

$$ \begin{aligned} \|\mathcal{L}_3'K(U)\|_{j+1} &\le n_1(d)\sum_{\bar{B}=U}\|H_X(U)\|_{j+1} \\ &\le n_1(d)L^d 72d^2 2^{2d}(L^{-d-2})A^{-1}\|K\|_j \\ &\le 72d^2 2^{2d} n_1(d)(L^{-2})A^{-1}\|K\|_j. \end{aligned} \tag{126} $$

Since $\mathcal{L}_3'K(U)$ is zero unless $|U|_{j+1} = 1$ this gives

$$ \|\mathcal{L}_3'K\|_{j+1} \le 72d^2 2^{2d} n_1(d)(L^{-2})\|K\|_j. \tag{127} $$

□

3.4. Identifying Invariant Parts and Estimating the Others

Now we investigate the 1st term of (113). We notice that $\alpha_{\mu\nu}(B) = \alpha_{\mu\nu}({}_fK,B) = \alpha_{\mu\nu}({}_0K,B)$ is independent from $f(\phi)$ and ${}_0E(B)$, ${}_0K(X,\phi)$ actually is the same as $E(B), K(X,\phi)$ in [5, lemma 9]. Therefore we have the same result as [5, lemma 9]

Lemma 8 ([5], Lemma 9). *Suppose ${}_0E(B), {}_0K(X,\phi)$ are invariant under lattice symmetries away from the boundary of Λ_N and ${}_0\tilde{E}(B)$ is invariant for B^* away from the boundary. Then*

1. *${}_0E'(B'), {}_0K'(U,\phi)$ are invariant for B', U away from the boundary.*
2. *If B^* is away from the boundary then ${}_0\beta(B), \alpha_{\mu\nu}(B)$ are independent of B and $\alpha_{\mu\nu}(B) = \hat{\alpha}_{\mu\nu}(B)$ defined for all B by*

$$ \hat{\alpha}_{\mu\nu}(B) = \frac{\alpha}{2}\,(\delta_{\mu\nu} - \delta_{\mu,-\nu}) \tag{128} $$

where α is a constant.

For all $B \in \mathcal{B}_j$ we define

$$ \alpha'_{\mu\nu}(B) = \alpha\;\delta_{\mu\nu} \tag{129} $$

and write, for any $U \in \mathcal{B}_{j+1}$

$$ \sum_{\overline{B}=U} V({}_f\beta,\alpha,B) = \sum_{\overline{B}=U} V({}_f\beta,\alpha',B) - \mathcal{L}_4({}_fK)(U) - \Delta({}_fK)(U) \tag{130} $$

with

$$\begin{aligned}\mathcal{L}_4({}_fK)(U) &= \mathcal{L}_4({}_0K)(U) = \sum_{\overline{B}=U} V(0,\alpha'-\hat{\alpha},B) = V(0,\alpha'-\hat{\alpha},U)\\ \Delta({}_fK)(U) &= \Delta({}_0K)(U) = \sum_{\overline{B}=U} V(0,\hat{\alpha}-\alpha,B) = V(0,\tilde{\alpha},U)\end{aligned} \tag{131}$$

where $\tilde{\alpha}_{\mu\nu}(x) = \hat{\alpha}_{\mu\nu}(B)-\alpha_{\mu\nu}(B)$ if $x \in B$. Then we can write that $\mathcal{L}_4(\delta_f K)(U) = 0$ and $\Delta(\delta_f K)(U) = 0$. By the above definition $\Delta({}_0K)(U)$ vanishes unless U touches the boundary. Now (113) becomes

$$\begin{aligned}&\mathcal{L}_3({}_fE,\sigma,{}_f\tilde{E},\tilde{\sigma},{}_fK,{}_0K)(U)\\ &\quad= \sum_{\overline{B}=U}\Big(V({}_f\tilde{E},\tilde{\sigma},B) - V^{\#}({}_fE,\sigma,B) - V({}_f\beta,\alpha',B)\Big)\\ &\qquad+\mathcal{L}_3'({}_0K)(U) + \mathcal{L}_4({}_0K)(U) + \Delta({}_0K)(U).\end{aligned} \tag{132}$$

Remark. Because $\mathcal{L}_4({}_fK) = \mathcal{L}_4({}_0K)$ and $\Delta({}_fK) = \Delta({}_0K)$ are independent from f, we will have the same results as Lemma 10 and Lemma 11 in Dimock [5]. Moreover, by using Lemma 6 above together with some calculating, we can obtain some explicit upper bounds for $\mathcal{L}_4({}_0K)$ and $\Delta({}_0K)$.

Lemma 9. *Let L be sufficiently large. Then the operator $\mathcal{L}_4$ is a contraction with*

$$\|\mathcal{L}_4({}_0K)\|_{j+1} \leq 4(2d)^3 n_2(d) L^{-(j+1)} \|{}_0K\|_j. \tag{133}$$

Lemma 10. *Let L be sufficiently large. Then the operator Δ is a contraction with*

$$\|\Delta({}_0K)\| \leq 4(2d)^5 2^d n_2(d) L^{-1} \|{}_0K\|_j. \tag{134}$$

3.5. Simplifying for the Next Scale

We now pick ${}_f\tilde{E}(B),\tilde{\sigma}$ so the V terms in (132) cancel. We have:

$$\begin{aligned}V^{\#}({}_fE,\sigma,B,\phi) &= {}_fE(B)+\int\frac{\sigma}{4}\sum_{x\in B}\sum_{\mu}(\partial_\mu\phi(x)+\partial_\mu\zeta(x))^2\,\mathrm{d}\mu_{\Gamma_{j+1}}(\zeta)\\ &= {}_fE(B) + \frac{\sigma}{4}\sum_{x\in B}\sum_{\mu}\partial_\mu\phi(x)^2\int \mathrm{d}\mu_{\Gamma_{j+1}}(\zeta)\\ &\quad+\frac{\sigma}{2}\sum_{x\in B}\sum_{\mu}\partial_\mu\phi(x)\int\partial_\mu\zeta(x)\mathrm{d}\mu_{\Gamma_{j+1}}(\zeta)\\ &\quad+\frac{\sigma}{4}\sum_{x\in B}\sum_{\mu}\int\partial_\mu\zeta(x)^2\mathrm{d}\mu_{\Gamma_{j+1}}(\zeta)\\ &= {}_fE(B)+\frac{\sigma}{4}\sum_{x\in B}\sum_{\mu}\partial_\mu\phi(x)^2+0+\frac{\sigma}{4}\sum_{x\in B}\sum_{\mu}(\partial_\mu\Gamma_{j+1}\partial_\mu^*)(x,x)\\ &\equiv V({}_fE,\sigma,B,\phi) + \frac{\sigma}{4}\sum_{\mu} Tr(1_B\partial_\mu\Gamma_{j+1}\partial_\mu^*)\end{aligned} \tag{135}$$

because

$$\begin{aligned} \int \partial_\mu \zeta(x) \mathrm{d}\mu_{\Gamma_{j+1}}(\zeta) &= 0, \\ \int \partial_\mu \zeta(x)^2 \mathrm{d}\mu_{\Gamma_{j+1}}(\zeta) &= \int (\zeta, \partial_\mu^* \delta_x)(\zeta, \partial_\mu^* \delta_x) \mathrm{d}\mu_{\Gamma_{j+1}}(\zeta) \\ &= (\partial_\mu^* \delta_x, \Gamma_{j+1} \partial_\mu^* \delta_x) \\ &= (\delta_x, \partial_\mu \Gamma_{j+1} \partial_\mu^* \delta_x) \\ &= (\partial_\mu \Gamma_{j+1} \partial_\mu^*)(x,x). \end{aligned} \tag{136}$$

If we choose ${}_f\tilde{E} = {}_f\tilde{E}({}_fE, \sigma, {}_fK)$

$${}_f\tilde{E}(B) = {}_fE(B) + \frac{\sigma}{4} \sum_\mu Tr(1_B \partial_\mu \Gamma_{j+1} \partial_\mu^*) + {}_f\beta({}_fK, B) \tag{137}$$

then the constant terms of (135) will be canceled. The second order terms of (135) would be vanish if we define $\tilde{\sigma} = \tilde{\sigma}(\sigma, {}_fK) = \tilde{\sigma}(\sigma, {}_0K)$ by

$$\tilde{\sigma} = \sigma + \alpha({}_fK) = \sigma + \alpha({}_0K). \tag{138}$$

Here we are canceling the constant term exactly for all B, but for the quadratic term we only cancel the invariant version away from the boundary.

By composing ${}_fK' = {}_fK'({}_f\tilde{E}, \tilde{\sigma}, {}_fE, \sigma, {}_fK, {}_0K)$ in Theorem 9 with newly defined ${}_f\tilde{E} = {}_f\tilde{E}({}_fE, \sigma, {}_fK)$ and $\tilde{\sigma} = \tilde{\sigma}(\sigma, {}_fK) = \tilde{\sigma}(\sigma, {}_0K)$ we obtain a new map ${}_fK' = {}_fK'({}_fE, \sigma, {}_fK, {}_0K)$. We also have new quantities ${}_fE'({}_fE, \sigma, {}_fK)$ defined by ${}_fE'(B') = \sum_{B \subset B'} {}_f\tilde{E}(B)$ and $\sigma' = \sigma'(\sigma, {}_fK) = \sigma'(\sigma, {}_0K)$ defined by $\sigma' = \tilde{\sigma} = \sigma + \alpha({}_fK) = \sigma + \alpha({}_0K)$ as normal. These quantities satisfy (67)

$$\mu_{\Gamma_{j+1}} * \left({}_fI({}_fE, \sigma) \circ {}_fK\right)(\Lambda) = \left({}_fI'({}_fE', \sigma') \circ {}_fK'\right)(\Lambda). \tag{139}$$

Here we still assume that L is sufficiently large, and that A is sufficiently large depending on L.

Theorem 10. 1. *For $R > 0$ there is an $r > 0$ such that the following holds for all j. If $\|{}_fE\|_j$, $|\sigma|$, $\max\{\|{}_fK\|_j, \|{}_0K\|_j\} < r$ then $\max\{\|{}_fK'\|_{j+1}, \|{}_0K'\|_{j+1}\}$, $\|{}_fE'\|_{j+1}$, $|\sigma'| < R$. Furthermore ${}_fE', {}_fK', \sigma'$ are smooth functions of ${}_fE, \sigma, {}_fK, {}_0K$ on this domain with derivatives bounded uniformly in j. The analyticity of ${}_fK'$ in $t_1, t_2, \ldots, t_m$ still holds when we go from j-scale to $(j+1)$-scale.*

2. *The linearization of ${}_fK' = {}_fK'({}_fE, \sigma, {}_fK, {}_0K)$ at the origin is the contraction $\mathcal{L}({}_fK)$ where*

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3' + \mathcal{L}_4 + \Delta. \tag{140}$$

Proof. For the first part, by combining with Theorem 9, it suffices to show that the linear maps ${}_f\tilde{E}$ and $\tilde{\sigma}$ have norms bounded uniformly in j. Using the estimate $|\alpha({}_fK)| = |\alpha({}_0K)| \le 4(2d)^2 n_2(d) h^{-2} A^{-1} \|{}_0K\|_j$ from Lemma 6, we have $\tilde{\sigma}$ is bounded. From Lemma 6 we also have the bound on $\|{}_f\beta({}_fK)\|_j \le 2n_2(d)A^{-1}\|{}_fK\|_j$. For $B \in \mathcal{B}_j$, the estimate (22) gives us

$$\left| \frac{\sigma}{4} \sum_\mu Tr(1_B(\partial_\mu \Gamma_{j+1} \partial_\mu^*) \right| \le dc_{1,1}|\sigma| \sum_{x \in B} L^{-dj} \le dc_{1,1}|\sigma| \tag{141}$$

where $c_{1,1}$ as in (22). Combining with (137) we have that ${}_f\tilde{E} = {}_f\tilde{E}({}_fE, \sigma, {}_fK)$ satisfies

$$\|{}_f\tilde{E}\|_j \leq \|{}_fE\|_j + \mathcal{C}(|\sigma| + A^{-1}\|{}_fK\|_j) \tag{142}$$

where $\mathcal{C} = \max\{dc_{1,1}, 2n_2(d)\}$.

The second part follows since the linearization of the new function ${}_fK'$ is the linearization of the old function ${}_fK'$ in Theorem 9 composed with ${}_f\tilde{E} = {}_f\tilde{E}({}_fE, \sigma, {}_fK), \tilde{\sigma} = \tilde{\sigma}(\sigma, {}_fK) = \tilde{\sigma}(\sigma, {}_0K)$. (All of them vanish at zero.) The cancelation gives us only with $\mathcal{L}({}_fK)$. □

3.6. Forming RG Flow

It is easier for us if we can extract the energy from the other variables. Assume that we start with $E(B) = 0$ in (139)

$$\mu_{\Gamma_{j+1}} * ({}_fI(0,\sigma) \circ {}_fK)(\Lambda_N) = \left({}_fI'({}_fE', \sigma') \circ {}_fK'\right)(\Lambda) \tag{143}$$

where $\sigma' = \sigma'(\sigma, {}_fK) = \sigma'(\sigma, {}_0K)$, ${}_fK' = {}_fK'(0, \sigma, {}_fK)$ and ${}_fE' = {}_fE'(0, \sigma, {}_fK)$ as above. Then we remove the ${}_fE'$ by making an adjustment in ${}_fK'$

$$\begin{aligned}
&\mu_{\Gamma_{j+1}} * ({}_fI(0,\sigma) \circ {}_fK)(\Lambda_N) = \left({}_fI'({}_fE', \sigma') \circ {}_fK'\right)(\Lambda) \\
&\quad = \sum_{U \in \mathcal{P}_{j+1}} \left({}_fI'({}_fE', \sigma')(\Lambda - U)\right)\left({}_fK'(0, \sigma, {}_fK, U)\right) \\
&\quad = \sum_{U \in \mathcal{P}_{j+1}} \left(\prod_{B' \in \mathcal{B}_{j+1}(\Lambda - U)} {}_fI'({}_fE', \sigma')(B')\right)\left({}_fK'(0, \sigma, {}_fK, U)\right) \\
&\quad = \sum_{U \in \mathcal{P}_{j+1}} \left(\prod_{B' \in \mathcal{B}_{j+1}(\Lambda - U)} \exp({}_fE'(B'))[{}_fI'(0, \sigma')(B')]\right)\left({}_fK'(0, \sigma, {}_fK, U)\right) \\
&\quad = \sum_{U \in \mathcal{P}_{j+1}} \left(\exp\left(\sum_{B' \in \mathcal{B}_{j+1}(\Lambda - U)} {}_fE'(B')\right) {}_fI'(0, \sigma')(\Lambda - U)\right)\left({}_fK'(0, \sigma, {}_fK, U)\right) \\
&\quad = \exp\left(\sum_{B' \in \mathcal{B}_{j+1}(\Lambda_N)} {}_fE^+(B')\right)\left({}_fI'(0, \sigma^+) \circ {}_fK^+\right)(\Lambda_N)
\end{aligned} \tag{144}$$

where ${}_fE^+(\sigma, {}_fK, B'), \sigma^+(\sigma, {}_fK), {}_fK^+(\sigma, {}_fK, U)$ are defined as following $(U \in \mathcal{P}_{j+1}, B' \in \mathcal{B}_{j+1})$

$$\begin{aligned}
{}_fE^+(\sigma, {}_fK, B') &\equiv {}_fE'(0, \sigma, {}_fK, B') = \sum_{B \subset B'} {}_f\tilde{E}(0, \sigma, {}_fK, B), \\
\sigma^+(\sigma, {}_fK) &\equiv \sigma'(\sigma, {}_fK) = \sigma'(\sigma, {}_0K) = \sigma + \alpha({}_0K), \\
{}_fK^+(\sigma, {}_fK, U) &\equiv \exp\left(-\sum_{B' \in \mathcal{B}_{j+1}(U)} {}_fE^+(B')\right) {}_fK'(0, \sigma, {}_fK, U).
\end{aligned} \tag{145}$$

The dynamical variables are now $\sigma^+(\sigma, {}_fK)$ and ${}_fK^+(\sigma, {}_fK)$. The extracted energy ${}_fE^+(\sigma, K)$ is controlled by the other variables. Because everything vanishes at the origin the linearization of ${}_fK^+(\sigma, {}_fK)$ is still $\mathcal{L}({}_fK)$. The bound (142) on ${}_f\tilde{E}$ would give us an upper bound on ${}_fE^+$ and our Theorem 10 becomes:

Theorem 11. 1. *For $R > 0$ there is an $r > 0$ such that the following holds for all j. If $|\sigma|, \max\{\|{}_fK\|_j, \|{}_0K\|_j\} < r$ then $|\sigma^+|, \max\{\|{}_fK^+\|_{j+1}, \|{}_0K^+\|_{j+1}\} < R$. Furthermore $\sigma^+, {}_fK^+$ are smooth functions of $\sigma, {}_fK$ on this domain with derivatives bounded uniformly in j. The analyticity of ${}_fK^+$ in $t_1, \ldots, t_m$ still holds when we go from j-scale to $(j+1)$-scale.*

2. *The extracted energies satisfy*

$$\|{}_fE^+(\sigma, {}_fK)\|_{j+1} \leq \mathcal{C}(L^d)\left(|\sigma| + A^{-1}\|{}_fK\|_j\right). \tag{146}$$

3. *The linearization of K^+ at the origin is the contraction $\mathcal{L}$.*

4. The Stable Manifold

Up to now, we have not specialized to the dipole gas, but take a general initial point $\sigma_0, {}_fK_0$ corresponding to an integral $\int ({}_fI(0, \sigma_0) \circ {}_fK_0)(\Lambda_N) \mathrm{d}\mu_{C_0}$. We assume ${}_0K_0(X, \phi)$ has the lattice symmetries and satisfies the conditions (40). We also assume $|\sigma_0|, \max\{\|{}_fK\|_0, \|{}_0K\|_0\} < r$ where r is small enough so the theorem 11 holds, say with $R = 1$, then we can take the first step. We apply the transformation (144) for $j = 0, 1, 2, \ldots$ and continue as far as we can. Then we get a sequence $\sigma_j, {}_fK_j^N(X)$ by $\sigma_{j+1} = \sigma^+(\sigma_j, {}_fK_j^N)$ and ${}_fK_{j+1}^N = {}_fK^+(\sigma_j, {}_fK_j^N)$ with extracted energies ${}_fE_{j+1}^N = {}_fE^+(\sigma_j, {}_fK_j^N)$. Then we have, for any l, with ${}_fI_j(\sigma_j) = {}_fI_j(0, \sigma_j)$

$$\begin{aligned} &\int \left({}_fI_0(\sigma_0) \circ {}_fK_0\right)(\Lambda_N) \mathrm{d}\mu_{C_0} \\ &\quad = \exp\left(\sum_{j=1}^{l} \sum_{B \in \mathcal{B}_j(\Lambda_N)} {}_fE_j^N(B)\right) \int \left({}_fI_l(\sigma_l) \circ {}_fK_l^N\right)(\Lambda_N) \mathrm{d}\mu_{C_l}. \end{aligned} \tag{147}$$

The quantities ${}_0K_j^N(X)$ and ${}_0E_j^N(B)$ are independent of N and have the lattice symmetries if X, B are away from the boundary $\partial\Lambda_N$ in the sense that they have no boundary blocks. These properties are true initially and are preserved by the iteration. In these cases we denote these quantities by just ${}_0K_j(X)$ and ${}_0E_j(B)$

With our construction α defined in (111), (128) only depends on ${}_0K_j$. By splitting K^+ into a linear and a higher order piece the sequence $\sigma_j, {}_fK_j^N(X)$ is generated by the RG transformation

$$\begin{aligned} \sigma_{j+1} &= \sigma_j + \alpha(K_j), \\ {}_0K_{j+1}^N &= \mathcal{L}({}_0K_j^N) + {}_0g(\sigma_j, {}_0K_j^N), \\ \delta_fK_{j+1}^N &= (\mathcal{L}_1 + \mathcal{L}_2)(\delta_fK_j^N) + {}_fg(\sigma_j, {}_fK_j^N, {}_0K_j^N) - {}_0g(\sigma_j, {}_0K_j^N). \end{aligned} \tag{148}$$

This is regarded as a mapping from the Banach space $\mathbb{R} \times (\mathcal{K}_j(\Lambda_N) \times \mathcal{K}_j(\Lambda_N))$ to the Banach space $\mathbb{R} \times (\mathcal{K}_j(\Lambda_N) \times \mathcal{K}_j(\Lambda_N))$. The second equation of (148) defines ${}_0g$ which is smooth with derivatives bounded uniformly in j and satisfies ${}_0g(0,0) = 0$, $D({}_0g)(0,0) = 0$. The last equation of (148) defines ${}_fg$ which is also smooth with derivatives bounded uniformly in j and satisfies ${}_fg(0,0) = 0$, $D({}_fg)(0,0) = 0$.

Now we consider the first two equations in (148). Around the origin there are a neutral direction σ_j and a contracting direction K_j (since $\mathcal{L}$ is a contraction.). Hence we expect there is a stable manifold. We quote a version of the stable manifold theorem due to Brydges [1], as applied in Theorem 7 in Dimock [5]

Theorem 12 ([5], Theorem 7). *Let L be sufficiently large, A sufficiently large (depending on L), and r sufficiently small (depending on L, A). Then there is $0 < \tau < r$ and a smooth real-valued function $\sigma_0 = h({}_0K_0)$, $h(0) = 0$, mapping $\|{}_0K_0\|_0 < \tau$ into $|\sigma_0| < r$ such that with these start values the sequence $\sigma_j, {}_0K_j^N$ is defined for all $0 \le j \le N$ and*

$$|\sigma_j| \le r2^{-j}, \qquad \left\|{}_0K_j^N\right\|_j \le r2^{-j}. \tag{149}$$

Furthermore the extracted energies satisfy

$$\|{}_0E_{j+1}^N\|_{j+1} \le 2\mathcal{C}(L^d)r2^{-j}. \tag{150}$$

Remark. Using the Lemma 11 below, given $r > 0$, we can always choose z, σ_0 and $\max_k |t_k|$ sufficiently small then $\max\{\|{}_fK\|_0, \|{}_0K\|_0\} \le r$. Now we claim that $\|{}_fK_j^N\|_j$ has the same bound as the $\|{}_0K_j^N\|_j$ in the last theorem.

Supposed that at $j = k$, we have: $\|{}_fK_j^N\|_j \le r2^{-k}$. As in the proof of Theorem 7 in Dimock [5], we can say that $\mathcal{L}$ and $(\mathcal{L}_1 + \mathcal{L}_2)$ is a contraction with norm less than 1/8 and ${}_fg(\sigma_j, {}_fK_j^N, {}_0K_j)$ is second order. Hence there are some constant H such that: $\|{}_fg(\sigma_j, {}_fK_j^N, {}_0K_j^N)\| \le H(|\sigma_j|^2 + \|{}_0K_j^N\|_j^2 + \|{}_fK_j^N\|_j^2)$ with $|\sigma_j|, \|{}_0K_j^N\|_j, \|{}_fK_j^N\|_j$ small. Then we have:

$$\begin{aligned}
\left\|{}_fK_{j+1}^N\right\|_{j+1} &\le \frac{1}{8}\left(\left\|{}_0K_j^N\right\|_j + \left\|\delta_f K_j^N\right\|_j\right) \\
&\quad + H\left(|\sigma_j|^2 + \left\|{}_0K_j^N\right\|_j^2 + \left\|{}_fK_j^N\right\|_j^2\right) \\
&\le \frac{1}{8}\left(2\left\|{}_0K_j^N\right\|_j + \left\|{}_fK_j^N\right\|_j\right) + 3H\left(r2^{-j}\right)^2 \\
&\le \frac{1}{8}\left(3r2^{-j}\right) + 3H\left(r2^{-j}\right)^2 \\
&\le r2^{-j-1}
\end{aligned} \tag{151}$$

for r sufficiently small.

The bound for $\|{}_fE^N\|_{j+1}$ comes from the bound on $\sigma_j, \|{}_fK_j^N\|_j$, (146) and $A > 1$.

Combining with the last theorem, for all $0 \leq j \leq N$ we can have:

$$|\sigma_j| \leq r2^{-j}, \qquad \left\| {}_fK_j^N \right\|_j \leq r2^{-j}, \tag{152}$$

and the extracted energies satisfy

$$\left\| {}_fE_{j+1}^N \right\|_{j+1} \leq 2\mathcal{C}(L^d) r2^{-j}. \tag{153}$$

5. The Dipole Gas

5.1. The Initial Density

Now we consider the generating function:

$${}_fZ_N(z,\sigma) = \int e^{if(\phi)} \exp\left(zW(\Lambda_N, \sqrt{1+\sigma}\phi) - \sigma V(\Lambda_N,\phi)\right) \mathrm{d}\mu_{C_0}(\phi). \tag{154}$$

When $f = 0$, it becomes

$${}_0Z_N(z,\sigma) = \int \exp\left(zW(\Lambda_N, \sqrt{1+\sigma}\phi) - \sigma V(\Lambda_N,\phi)\right) \mathrm{d}\mu_{C_0}(\phi). \tag{155}$$

For $B \in \mathcal{B}_0$, we define: $W_0(B) = zW(\sqrt{1+\sigma_0}, B)$ as in (64) and $V_0(B) = \sigma_0 V(B)$ as in (39). Then we follow with a Mayer expansion to put the density in the form we want

$$\begin{aligned} {}_f\mathcal{Z}_0^N &= \prod_{B\subset\Lambda_N} e^{if(\phi)+W_0(B)-V_0(B)} \\ &= \prod_{B\subset\Lambda_N} \left(e^{-V_0(B)} + \left(e^{if(\phi)+W_0(B)} - 1\right) e^{-V_0(B)}\right) \\ &= \sum_{X\subset\Lambda_N} {}_fI_0(\sigma_0, \Lambda_N - X)\, {}_fK_0(X) \\ &= \left({}_fI_0(\sigma_0) \circ {}_fK_0\right)(\Lambda_N) \end{aligned} \tag{156}$$

where $I_0(\sigma_0, B) = e^{-V_0(B)}$ and ${}_fK_0(X) = {}_fK_0(z,\sigma_0,X)$ is given by

- ${}_fK_0(X) = \prod_{B\subset X}(e^{if(\phi)|_B + W_0(B)} - 1)e^{-V_0(B)}$ when $f(\phi) = \sum_{k=1}^m t_k \exp(i\partial_{\mu_k}\phi(x_k))$, $if(\phi)|_B = t_k \exp(i\partial_{\mu_k}\phi(x_k))$ if $B = \{x_k\}$ for some k, otherwise $if(\phi)|_B = 0$.
- ${}_fK_0(X) = \prod_{B\subset X}(e^{if(\phi)|_B + W_0(B)} - 1)e^{-V_0(B)}$ when $f(\phi) = \sum_{k=1}^m t_k\partial_{\mu_k}\phi(x_k)$, $if(\phi)|_B = t_k\partial_{\mu_k}\phi(x_k)$ if $B = \{x_k\}$ for some k, otherwise $if(\phi)|_B = 0$.
- ${}_fK_0(X) = \prod_{B\subset X}(e^{W_0(B)} - 1)e^{-V_0(B)}$ when $f(\phi) = 0$.

Note that, when $f = 0$, ${}_0K_0$ actually is the K_0 in [5, lemma 12]. We also can prove the same result for ${}_fK_0$.

Lemma 11. *Given $1 > r > 0$, there are some sufficiently small $a(r), b(r)$ and $c(r)$ such that if $\max_k |t_k| \leq a(r)$, $|z| \leq b(r)$ and $|\sigma_0| \leq c(r)$ then $\|{}_fK_0(z,\sigma_0)\|_0 \leq r$. Furthermore ${}_fK_0$ is a smooth function of (z,σ_0), and analytic in t_k for all $k = 1,\ldots,m$.*

Proof. We consider these cases:

(i) When $f = 0$, using [5, lemma 12], we have some $b_0(r), c_0(r)$ such that $\|{}_0K_0(z,\sigma_0)\|_0 \le r$ if $|z| \le b_0(r)$ and $|\sigma_0| \le c_0(r)$.

(ii) In the case $f(\phi) = \sum_{k=1}^m t_k \partial_{\mu_k}\phi(x_k)$, using [5, (95)], for $\phi = \phi' + \zeta$, we have:

$$\begin{aligned}
\|(e^{if(\phi)|_B + W_0(B)} - 1)\|_0 &= \|(e^{if(\phi)|_B + zW(\sqrt{1+\sigma_0},B)} - 1)\|_0 \\
&\le \sum_{n=1}^{\infty} \frac{1}{n!} \|zW(\sqrt{1+\sigma_0},B) + if(\phi)|_B\|_0^n \\
&\le \sum_{n=1}^{\infty} \frac{1}{n!} \left(\|zW(\sqrt{1+\sigma_0},B)\|_0 + \|if(\phi)|_B\|_0\right)^n \\
&\le \sum_{n=1}^{\infty} \frac{1}{n!} \left(2|z|e^{h\sqrt{d(1+\sigma_0)}} + \max_k |t_k| h^{-1} \|\phi\|_{\Phi_0(B^*)}\right)^n. \qquad (157)
\end{aligned}$$

We can assume that $\max_k |t_k| h^{-1} \le 1$. Applying lemma 1, we obtain $\|e^{-V_0(B)}\|_{s,0} \le 2$. Then

$$\begin{aligned}
\|{}_fK_0(B)\|_0 &= \sup_{\phi',\zeta} \left\|{}_fK_0(B,\phi'+\zeta)\right\|_0 G_0(X,\phi',\zeta)^{-1} \\
&\le \sup_{\phi',\zeta} \left\|(e^{if(\phi)|_B + W_0(B)} - 1)\right\|_0 \left\|e^{-V_0(B)}\right\|_0 G_{s,0}(X,\phi',\zeta)^{-2} \\
&\le \left\|e^{-V_0(B)}\right\|_{s,0} \sup_{\phi',\zeta} \left\|(e^{if(\phi)|_B + W_0(B)} - 1)\right\|_0 G_{s,0}(X,\phi',\zeta)^{-1} \\
&\le 2\sup_{\phi',\zeta} \left(\exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}} + \max_k |t_k| h^{-1} \|\phi'+\zeta\|_{\Phi_0(B^*)}\right) - 1\right) G_{s,0}(X,\phi',\zeta)^{-1} \\
&\le 2\sup_{\phi',\zeta} \left(\exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}\right) - 1\right) \exp\left(\max_k |t_k| h^{-1} \|\phi'+\zeta\|_{\Phi_0(B^*)}\right) G_{s,0}(X,\phi',\zeta)^{-1} \\
&\quad + 2\sup_{\phi',\zeta} \left(\exp\left(\max_k |t_k| h^{-1} \|\phi'+\zeta\|_{\Phi_0(B^*)}\right) - 1\right) G_{s,0}(X,\phi',\zeta)^{-1} \\
&\le 2\sup_{\phi',\zeta} \left(\exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}\right) - 1\right) \exp\left(\|\phi'+\zeta\|_{\Phi_0(B^*)}\right) e^{-\|\phi'\|^2_{\Phi_0(B^*)} - \|\zeta\|^2_{\Phi_0(B^*)}} \\
&\quad + 2\sup_{\phi',\zeta} \left(\exp\left(\max_k |t_k| h^{-1} \|\phi'+\zeta\|_{\Phi_0(B^*)}\right) - 1\right) e^{-\|\phi'\|^2_{\Phi_0(B^*)} - \|\zeta\|^2_{\Phi_0(B^*)}}. \qquad (158)
\end{aligned}$$

Because $\exp(\|\phi'+\zeta\|_{\Phi_0(B^*)}) \exp(-\|\phi'\|^2_{\Phi_0(B^*)} - \|\zeta\|^2_{\Phi_0(B^*)})$ is bounded and

$$\lim_{z,\sigma_0 \to 0} \left(\exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}\right) - 1\right) = 0, \qquad (159)$$

there exist some sufficiently small $b_1(r), c_1(r) > 0$ such that we have

$$\begin{aligned}
2\sup_{\phi',\zeta} &\left(\exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}\right) - 1\right) e^{\|\phi'+\zeta\|_{\Phi_0(B^*)} - \|\phi'\|^2_{\Phi_0(B^*)} - \|\zeta\|^2_{\Phi_0(B^*)}} \\
&\le \frac{r}{4A} \qquad (160)
\end{aligned}$$

for all $|z| \le b_1(r)$ and $|\sigma_0| \le c_1(r)$.

For other part, we have:

$$2\sup_{\phi',\zeta}\left(\exp\left(\max_k |t_k|h^{-1}\|\phi'+\zeta\|_{\Phi_0(B^*)}\right)-1\right)e^{-\|\phi'\|^2_{\Phi_0(B^*)}-\|\zeta\|^2_{\Phi_0(B^*)}}$$
$$\leq 2\sup_{\phi',\zeta}\left(\exp\left(\|\phi'\|_{\Phi_0(B^*)}+\|\zeta\|_{\Phi_0(B^*)}\right)-1\right)e^{-\|\phi'\|^2_{\Phi_0(B^*)}-\|\zeta\|^2_{\Phi_0(B^*)}}. \quad (161)$$

We can also find some sufficiently large H such that: if $\|\phi'\|_{\Phi_0(B^*)}+\|\zeta\|_{\Phi_0(B^*)}\geq H$ then

$$2\left(\exp\left(\|\phi'\|_{\Phi_0(B^*)}+\|\zeta\|_{\Phi_0(B^*)}\right)-1\right)e^{-\|\phi'\|^2_{\Phi_0(B^*)}-\|\zeta\|^2_{\Phi_0(B^*)}}\leq\frac{r}{4A}. \quad (162)$$

For $\|\phi'\|_{\Phi_0(B^*)}+\|\zeta\|_{\Phi_0(B^*)}\leq H$, we have

$$\|\phi'+\zeta\|_{\Phi_0(B^*)}\leq\|\phi'\|_{\Phi_0(B^*)}+\|\zeta\|_{\Phi_0(B^*)}\leq H.$$

So with $\max_k |t_k|\leq a_1(r)$ sufficiently small and $\|\phi'\|_{\Phi_0(B^*)}+\|\zeta\|_{\Phi_0(B^*)}\leq H$,

$$2\left(\exp\left(\max_k |t_k|h^{-1}\|\phi'+\zeta\|_{\Phi_0(B^*)}\right)-1\right)e^{-\|\phi'\|^2_{\Phi_0(B^*)}-\|\zeta\|^2_{\Phi_0(B^*)}}\leq\frac{r}{4A}. \quad (163)$$

In summary we can always choose sufficiently small $a(r), b(r), c(r)$ such that if $\max_k |t_k|\leq a_1(r)$, $|z|\leq b_1(r)$, and $|\sigma_0|\leq c_1(r)$ then

$$\|{}_fK_0(B)\|_0\leq 2\frac{r}{4A}=\frac{r}{2A}, \qquad \forall B\in\mathcal{B}_0. \quad (164)$$

For those $a_1(r), b_1(r), c_1(r)$,$\max_k |t_k|\leq a_1(r)$, $|z|\leq b_1(r)$, and $|\sigma_0|\leq c_1(r)$, we have

$$\begin{aligned}\|{}_fK_0\|_0 &= \sup_{X\in\mathcal{P}_{0,c}}\|{}_fK_0(X)\|_0 A^{|X|_0}\\ &\leq \sup_{X\in\mathcal{P}_{0,c}}\left(\prod_{B\subset X}\|{}_fK_0(B)\|_0\right)A^{|X|_0}\\ &\leq \sup_{X\in\mathcal{P}_{0,c}}\left(\frac{r}{2A}\right)^{|X|_0}A^{|X|_0}\leq\frac{r}{2}<r. \end{aligned} \quad (165)$$

(iii) In the last case, $f(\phi)=\sum_{k=1}^m t_k\exp(i\partial_{\mu_k}\phi(x_k))$, we have:

$$\begin{aligned}\|(e^{if(\phi)|_B+W_0(B)}-1)\|_0 &\leq \sum_{n=1}^{\infty}\frac{1}{n!}\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}+\max_k|t_k|\right)^n\\ &= \exp\left(2|z|e^{h\sqrt{d(1+\sigma_0)}}+\max_k|t_k|\right)-1. \end{aligned} \quad (166)$$

Using the same argument as above, we can choose some sufficiently small $a_2(r)$, $b_2(r)$, $c_2(r)$ such that $\|{}_fK_0(z,\sigma_0)\|_0\leq r$ when $\max_k |t_k|\leq a_2(r)$, $|z|\leq b_2(r)$ and $|\sigma_0|\leq c_2(r)$.

Now we just simply pick $a(r)=\max\{a_1(r),a_2(r)\}$, $b(r)=\max\{b_0(r),b_1(r),b_2(r)\}$ and $c(r)=\max\{c_0(r),c_1(r),c_2(r)\}$.

The smoothness follows similarly from Lemma 12 (Dimock [5]).[10] □

Remark. We have ${}_fK_0$ is analytic. For each step when we jump from j-scale to $(j+1)$-scale, the analyticity of ${}_fK$ still holds for the next scale.

Noticing that ${}_0K_0$ is just the K_0 in Section 6 (Dimock [5]), we need the following lemma to apply Theorem 12.

Lemma 12 ([5, Lemma 13]). *The equation $\sigma = h({}_0K_0(z,\sigma))$ defines a smooth implicit function $\sigma = \sigma(z)$ near the origin which satisfies $\sigma(0) = 0$.*

Taking $|z|$ sufficiently small and choosing $\sigma_0 = \sigma(z)$, we can apply theorem 12. For $0 \leq l \leq N$, we have

$$ {}_fZ_N = \exp\left(\sum_{j=1}^{l} \sum_{B\in\mathcal{B}_j(\Lambda_N)} {}_fE_j^N(B)\right) \int ({}_fI_l(\sigma_l) \circ {}_fK_l^N)(\Lambda_N) \mathrm{d}\mu_{C_l} \tag{168} $$

where $|\sigma_j| \leq r2^{-j}$, $\|{}_fK_j^N\|_j \leq r2^{-j}$ and $\|{}_fE_{j+1}^N\|_{j+1} \leq \mathcal{O}(L^d)r2^{-j}$.

5.2. Completing the Proof of Theorem 1

Theorem 13. *For $|z|$ and $\max_k |t_k|$ sufficiently small the following limit exists:*

$$ \lim_{N\to\infty} |\Lambda_N|^{-1} \log {}_fZ'_N(z,\sigma(z)). \tag{169} $$

Proof. With updated index, the proof can go exactly the same as the proof of [5, Theorem 8]. We take $l = N$ in (168). At this scale there is only one block $\Lambda_N \in \mathcal{B}_N(\Lambda_N)$ and so we have

$$ \begin{aligned} |\Lambda_N|^{-1}\log {}_fZ'_N(z,\sigma(z)) &= |\Lambda_N|^{-1} \sum_{j=1}^{N} \sum_{B\in\mathcal{B}_j(\Lambda_N)} {}_fE_j^N(B) \\ &+ |\Lambda_N|^{-1}\log\left(\int \left[{}_fI_N(\sigma_N,\Lambda_N) + {}_fK_N^N(\Lambda_N)\right] \mathrm{d}\mu_{C_N}\right). \end{aligned} \tag{170} $$

The second term has the form

$$ |\Lambda_N|^{-1} \log\left(1 + \int {}_fF_N \mathrm{d}\mu_{C_N}\right) \tag{171} $$

where

$$ {}_fF_N(\Lambda_N) = {}_fF_N = {}_fI_N(\sigma_N,\Lambda_N) - 1 + {}_fK_N^N(\Lambda_N). \tag{172} $$

By (126) in [5] and norm definition (54), we have

$$ \begin{aligned} \|{}_fI_N(\sigma_N,\Lambda_N) - 1\|_N &\leq 4c^{-1}h^2|\sigma_N| \leq 4c^{-1}h^2 r2^{-N} \\ \|{}_fK_N^N(\Lambda_N)\|_N &\leq A^{-1}\|{}_fK_N^N\|_N \leq A^{-1}r2^{-N} \end{aligned} \tag{173} $$

so that $\|{}_fF_N(\Lambda_N)\|_N \leq (4c^{-1}h^2 + A^{-1})r(2^{-N})$ which is $\mathcal{O}(2^{-N})$ as $N \to \infty$.

[10] Instead of using the usual estimates, such as $(1 + \|\phi\|^2_{\Phi_j(B^*)}) \leq \exp(\|\phi\|^2_{\Phi_j(B^*)}) = G_{s,j}(B,\phi,0)$, we can use

$$ \left(1 + \|\phi\|^2_{\Phi_j(B^*)}\right) = k\left(\frac{1}{k} + \frac{1}{k}\|\phi\|^2_{\Phi_j(B^*)}\right) \leq k\exp\left(\frac{1}{k}\|\phi\|^2_{\Phi_j(B^*)}\right) = kG^{\frac{1}{k}}_{s,j}(B,\phi,0) \tag{167} $$

for any positive integer k, and so forth.

In [5, Lemma 14], Dimock has proved that for h sufficiently large

$$\int G_N(\Lambda_N, 0, \zeta) \mathrm{d}\mu_{C_N}(\zeta) \leq 2. \tag{174}$$

Then we estimate

$$\begin{aligned}\left|\int {}_fF_N(\Lambda_N)\mathrm{d}\mu_{C_N}\right| &\leq \left\|{}_fF_N(\Lambda_N)\right\|_N \int G_N(\Lambda_N, 0, \zeta)\mathrm{d}\mu_{C_N}(\zeta)\\ &\leq 2\|F(\Lambda_N)\|_N\\ &\leq 2\left(4c^{-1}h^2 + A^{-1}\right) r(2^{-N}).\end{aligned} \tag{175}$$

Hence the expression (171) is $\mathcal{O}(2^{-N})|\Lambda_N|^{-1}$ and goes to zero very quickly as $N \to \infty$

The rest of the proof came as in the proof of Theorem 8 in [5]. □

6. Correlation Functions: Estimates and Infinite Volume Limit

Note: We always can assume that $L \gg 2^{d+3} + 1$

6.1. In the Case: $f(\phi) = \sum_{k=1}^m t_k \partial_{\mu_k} \phi(x_k)$

For $x_k \in \mathbb{Z}^d$ are different points; $\mu_k \in \{\pm 1, \ldots, \pm d\}$ and t_k complex and $|t_k| \leq a = a(r)$ for $\forall k : 1, 2, \ldots, m$.

6.1.1. Proof of Theorem 3. Using (168) with $l = N$, for the truncated correlation functions, we have:

$$\begin{aligned}\mathcal{G}^t(x_1, x_2, \ldots, x_m) &\equiv \left\langle \prod_{k=1}^m \partial_{\mu_k}\phi(x_k)\right\rangle^t \equiv i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log {}_fZ'\Big|_{t_1=0,\ldots t_m=0}\\ &= i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m}\left(\sum_{j=1}^N \sum_{B\in\mathcal{B}_j(\Lambda_N)} {}_fE_j^N(B)\right)\Bigg|_{t_1=0,\ldots t_m=0}\\ &\quad + i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log \int ({}_fI_N(\sigma_N) \circ {}_fK_N^N)(\Lambda_N)\mathrm{d}\mu_{C_N}\Big|_{t_1=0,\ldots t_m=0}\\ &= i^m \sum_{j=1}^N \sum_{B\in\mathcal{B}_j(\Lambda_N)} \frac{\partial^m}{\partial t_1 \ldots \partial t_m} {}_fE_j^N(B)\Big|_{t_1=0,\ldots t_m=0}\\ &\quad + i^m \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log\left(1 + \int ({}_fI_N(\sigma_N) - 1 + {}_fK_N^N)(\Lambda_N)\mathrm{d}\mu_{C_N}\right)\Bigg|_{t_1=0,\ldots t_m=0}\end{aligned} \tag{176}$$

Now we consider the quantity:

$$\begin{aligned}
{}_fF_N &\equiv \sum_{j=1}^{N} \sum_{B\in\mathcal{B}_j(\Lambda_N)} \frac{\partial^m}{\partial t_1\dots\partial t_m} {}_fE_j^N(B)\Big|_{t_1=0,\dots,t_m=0}\\
&= \sum_{j=0}^{N-1} \sum_{B\in\mathcal{B}_j(\Lambda_N)} \frac{\partial^m}{\partial t_1\dots\partial t_m} {}_f\beta({}_fK_j^N, B)\Big|_{t_1=0,\dots,t_m=0}\\
&= \sum_{j=0}^{N-1} \sum_{B\in\mathcal{B}_j(\Lambda_N)} \frac{\partial^m}{\partial t_1\dots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N\#}(X,0)\Big|_{t_1=0,\dots,t_m=0}
\end{aligned} \tag{177}$$

by the definition of ${}_f\beta$ in (111).

We notice that $\frac{\partial^m}{\partial t_1\dots\partial t_m} {}_fE_j(B)|_{t_1=0,\dots,t_m=0} = 0$ unless $B^* \supset \{x_1, x_2, \dots, x_m\}$. Hence,

$${}_fF_N = \sum_{j=0}^{N-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\dots,x_m\}}} \frac{\partial^m}{\partial t_1\dots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N\#}(X,0)\Big|_{t_1=0,\dots,t_m=0} \tag{178}$$

Note: Let $\eta = \min\{d/2, 2\}$. For any small $\iota > 0$, we can always find A, L sufficiently large such that:

$$\begin{aligned}
\|\left(\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3'+\mathcal{L}_4\right)({}_0K)\|_{j+1} &\le \frac{1}{4L^{\eta-\iota}}\|{}_0K\|_j\\
\|\left(\mathcal{L}_1+\mathcal{L}_2\right)(\delta_fK)\|_{j+1} \le \frac{1}{4L^{\eta-\iota}}\left(\|\delta_fK\|_j\right) &\le \frac{1}{4L^{\eta-\iota}}\left(\|{}_0K\|_j + \|{}_fK\|_j\right)
\end{aligned} \tag{179}$$

with $j \ge 1$ by using the explicit upper bounds in Lemmas 4, 5, 7, and 9.

Then we can replace $\mu = 1/2$ in Theorem 7 by $\mu = 1/M$ for $M = L^{\eta-\iota} \ge 2$. We still have $|\sigma_j| \le rM^{-j}$, $\|{}_fK_j^N\|_j \le rM^{-j}$ and $\|{}_fE_{j+1}^N\|_{j+1} \le \mathcal{O}(L^d)rM^{-j}$ with $\max_k |t_k| < a$ sufficiently small and $0 \le j \le N-1$. Because ${}_fK_j^{N\#}(X,0)$ is analytic, using Cauchy's bound and (57), we have:

$$\begin{aligned}
\left|\frac{\partial^m}{\partial t_1\dots\partial t_m} {}_fK_j^{N\#}(X,0)\Big|_{t_1=0,\dots,t_m=0}\right| &\le \frac{m!}{a^m}\left(\frac{A}{2}\right)^{-|X|_j} \|{}_fK_j^N\|_j\\
&\le \frac{m!}{a^m}\left(\frac{A}{2}\right)^{-|X|_j} rM^{-j}.
\end{aligned} \tag{180}$$

Then

$$\begin{aligned}
&\left|\frac{\partial^m}{\partial t_1\dots\partial_{t_m}} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N\#}(X,0)\Big|_{t_1=0,\dots,t_m=0}\right|\\
&\quad\le \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} \left|\frac{\partial^m}{\partial t_1\dots\partial t_m} {}_fK_j^{N\#}(X,0)\Big|_{t_1=0,\dots,t_m=0}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} \frac{m!}{a^m} \left(\frac{A}{2}\right)^{-|X|_j} rM^{-j} \\
&\leq n_3(d, \frac{A}{2}) \frac{m! rM^{-j}}{a^m}. \qquad (181)
\end{aligned}
$$

So

$$
|{}_fF_N| \leq \sum_{j=0}^{N-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} n_3\left(d, \frac{A}{2}\right) \frac{m! rM^{-j}}{a^m}. \qquad (182)
$$

By (171)–(175), we have:

$$
\begin{aligned}
&\left|\log\left(1 + \int ({}_fI_N(\sigma_N) - 1 + {}_fK_N^N)(\Lambda_N) \mathrm{d}\mu_{C_N}\right)\right| \\
&\quad \leq \log\left(1 + 2\|F(\Lambda_N)\|_N\right) \\
&\quad \leq \log\left(1 + 2[4c^{-1}h^2 + A^{-1}]r2^{-N}\right). \qquad (183)
\end{aligned}
$$

Using the Cauchy's bound as above, we obtain:

$$
\begin{aligned}
&\left|\frac{\partial^m}{\partial t_1\ldots\partial t_m} \log\left(1 + \int ({}_fI_N(\sigma_N) - 1 + {}_fK_N^N)(\Lambda_N) \mathrm{d}\mu_{C_N}\right)\Big|_{t_1=0,\ldots,t_m=0}\right| \\
&\quad \leq \frac{m!}{a^m} \log\left(1 + 2[4c^{-1}h^2 + A^{-1}]r2^{-N}\right). \qquad (184)
\end{aligned}
$$

So

$$
\lim_{N\to\infty} \left|\frac{\partial^m}{\partial t_1\ldots\partial t_m} \log\left(1 + \int ({}_fI_N(\sigma_N) - 1 + {}_fK_N^N)(\Lambda_N) \mathrm{d}\mu_{C_N}\right)\Big|_{t_1=0,\ldots,t_m=0}\right| = 0. \qquad (185)
$$

Now let j_0 be the smallest integer such that $\exists B \in \mathcal{B}_{j_0} : B^* \supset \{x_1, x_2, \ldots x_m\}$. Without loosing generality, we can assume that $|x_1 - x_2| =$ diam$(x_1, \ldots, x_m)$. For every $j \geq j_0$, let $B_j^1 \in \mathcal{B}_j$ be the unique j-block that contains $\{x_1\}$. For any $B \in \mathcal{B}_j, j \geq j_0$ with $B^* \supset \{x_1, x_2, \ldots, x_m\}$, B must be in ${B_j^1}^*$.

We have

$$
\begin{aligned}
|{}_fF_N| &\leq \sum_{j=0}^{N-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} n_3(d, \frac{A}{2}) \frac{m! rM^{-j}}{a^m} \\
&= \sum_{j=j_0}^{N-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} n_3(d, \frac{A}{2}) \frac{m! rM^{-j}}{a^m}. \qquad (186)
\end{aligned}
$$

Since $M \geq 2$, the last part of (186) is bounded by

$$\sum_{j=j_0}^{N-1} \sum_{\substack{B \in \mathcal{B}_j(\Lambda_N) \\ B^* \supset \{x_1, x_2, \ldots, x_m\}}} n_3\left(d, \frac{A}{2}\right) \frac{m! r M^{-j}}{a^m} \leq \sum_{j=j_0}^{N-1} \sum_{\substack{B \in \mathcal{B}_j(\Lambda_N) \\ B \in B_j^{1*}}} n_3\left(d, \frac{A}{2}\right) \frac{m! r M^{-j}}{a^m}$$
$$\leq \sum_{j=j_0}^{N-1} (2^d 2)^d n_3(d, \frac{A}{2}) \frac{m! r M^{-j}}{a^m}$$
$$\leq 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 2 \frac{m! r M^{-j_0}}{a^m}. \tag{187}$$

Therefore, we have:

$$|{}_f F_N| \leq 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 2 \frac{m! r M^{-j_0}}{a^m}. \tag{188}$$

By the definition of j_0, we have: $|x_1 - x_2| \leq d 2^{d+1} L^{j_0}$. Because $M = L^{\eta - \iota}$, we get

$$M^{-j_0} = L^{-j_0(\eta - \iota)} \leq (d 2^{d+1})^\eta |x_1 - x_2|^{-\eta + \iota}$$
$$= (d 2^{d+1})^\eta \operatorname{diam}^{-\eta + \iota}(x_1, \ldots, x_m). \tag{189}$$

Hence, we have:

$$|{}_f F_N| \leq 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 2 \frac{m! r}{a^m} \operatorname{diam}^{-\eta + \iota}(x_1, \ldots, x_m) \left(d^\eta 2^{\eta(d+1)}\right). \tag{190}$$

Using this with (185), we obtain:

$$\left| \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log {}_f Z' \Big|_{t_1 = 0, \ldots, t_m = 0} \right|$$
$$\leq 2^{d(d+1)} 4 n_3(d, \frac{A}{2}) \frac{m! r}{a^m} \operatorname{diam}^{-\eta + \iota}(x_1, \ldots, x_m) \left(d^\eta 2^{\eta(d+1)}\right). \tag{191}$$

Combining with (34), we get $n_3(d, \frac{A}{2}) 2^{d(d+1)} 4 r (d 2^{d+1})^\eta \leq 1$ with sufficiently large A. Therefore, with sufficiently large A, we have:

$$\left| \mathcal{G}^t(x_1, x_2, \ldots, x_m) \right| = \left| \left\langle \prod_{k=1}^m \partial_{\mu_k} \phi(x_k) \right\rangle^t \right|$$
$$= \left| \frac{\partial^m}{\partial t_1 \ldots \partial t_m} \log {}_f Z' \Big|_{t_1 = 0, \ldots, t_m = 0} \right|$$
$$\leq \frac{m!}{a^m} \operatorname{diam}^{-\eta + \iota}(x_1, \ldots, x_m). \tag{192}$$

We complete the proof of Theorem 3.

Remark. Actually for any $N-1 \geq q \geq j_0$, similarly to (187), we have

$$\left| \sum_{j=q}^{N-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \right|$$
$$\leq \sum_{j=q}^{N-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} n_3\left(d, \frac{A}{2}\right) \frac{m! r M^{-j}}{a^m}$$
$$\leq 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 2 \frac{m! r M^{-q}}{a^m}. \tag{193}$$

6.1.2. Proof of Theorem 2. Now we fix the set $\{x_1, x_2, \ldots, x_m\}$. Let j_1 be the smallest integer such that $B^0_{j_1} \supset \{x_1, x_2, \ldots, x_m\}$. Then j_1 is the smallest integer which is greater than $\log_L \max_i 2\|x_i\|_\infty$. We also have: $j_0 \leq j_1$.

Let q be any number such that $q \geq j_1 + 1 \geq j_0 + 1$. And let N_1, N_2 be any integers such that $N_2 \geq N_1 > q$. Using the definition of j_0, we have

$${}_fF_{N_1} = \sum_{j=j_0}^{q-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0}$$
$$+ \sum_{j=q}^{N_1-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \tag{194}$$

and

$${}_fF_{N_2} = \sum_{j=j_0}^{q-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0}$$
$$+ \sum_{j=q}^{N_2-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \tag{195}$$

We also notice that:

$$\sum_{j=j_0}^{q-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0}$$
$$= \sum_{j=j_0}^{q-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N)\\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_1\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \tag{196}$$

because for $0 \le j \le q-1$, ${}_fK_j^{N\#}(X,0)$ only depend on ϕ within X^* and $X^* \subset \Lambda_q$ which is the center q-block of $\Lambda_{N_1} \subset \Lambda_{N_2}$. Therefore,

$$\begin{aligned} &|{}_fF_{N_2} - {}_fF_{N_1}| \\ &\quad \le \left| \sum_{j=q}^{N_2-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N) \\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \right| \\ &\qquad + \left| \sum_{j=q}^{N_1-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N) \\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_1\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \right| \end{aligned} \tag{197}$$

Then using (193) with $\mu = 1/2$ instead of $\mu = 1/M = L^{-\eta+\iota}$, we obtain:

$$\begin{aligned} &\left| \sum_{j=q}^{N_2-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N) \\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_2\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \right| \\ &\quad \le 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 2\frac{m! r 2^{-q}}{a^m}, \\ &\left| \sum_{j=q}^{N_1-1} \sum_{\substack{B\in\mathcal{B}_j(\Lambda_N) \\ B^*\supset\{x_1,x_2,\ldots,x_m\}}} \frac{\partial^m}{\partial t_1\ldots\partial t_m} \sum_{X\in\mathcal{S}_j, X\supset B} \frac{1}{|X|_j} {}_fK_j^{N_1\#}(X,0)\Big|_{t_1=0,\ldots,t_m=0} \right| \\ &\quad \le 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 2\frac{m! r 2^{-q}}{a^m}. \end{aligned} \tag{198}$$

That means we have:

$$\left| {}_fF_{N_2} - {}_fF_{N_1} \right| \le 2^{d(d+1)} n_3\left(d, \frac{A}{2}\right) 4\frac{m! r 2^{-q}}{a^m} \to 0 \tag{199}$$

when $q \to \infty$.

Combining this with (176) and (185), we can conclude that $\lim_{N\to\infty}\langle\prod_{k=1}^m \partial_{\mu_k}\phi(x_k)\rangle^t$ exists.

Remark. We have N-uniformly boundedness on correlation functions and $\lim_{N\to\infty}\mathcal{G}^t(x_1,x_2,\ldots,x_m)$ exists. Therefore the bounds are held for infinite volume limit

6.2. When $f(\phi) = \sum_{k=1}^m t_k \exp(i\partial_{\mu_k}\phi(x_k))$

Using exactly the same argument as the above subsection, we obtain these following results:

Theorem 4. *With L, A sufficiently large, the infinite volume limit of the truncated correlation function $\lim_{N\to\infty}\langle\prod_{k=1}^m \exp(i\partial_{\mu_k}\phi(x_k))\rangle^t$ exists.*

Theorem 5. *For any small $\iota > 0$, with L, A sufficiently large (depending on ι), let $\eta = \min\{d/2, 2\}$ we have:*

$$\left| \left\langle \prod_{k=1}^{m} \exp\left(i\partial_{\mu_k}\phi(x_k)\right) \right\rangle^t \right| \leq \frac{m!}{a^m}\ diam^{-\eta+\iota}(x_1, x_2, \ldots, x_m)$$

where a depends on ι, L, A.

6.3. Other Cases

We can consider $f(\phi) = \sum_{k=1}^{m} t_k f_k(\phi)(x_k)$ with

- * $t_k \in \mathbb{C}$,
- * $x_k \in \mathbb{Z}^d$ are different points,
- * f_k is bounded in the sense that there are some $M_k, m_k \geq 0$ such that

$$\|f_k(\{x_k\}, \phi)\|_0 \leq M_k \|\phi\|_{\Phi_0} + m_k. \tag{200}$$

With the same argument as above cases, we have:

Theorem 6. *With L, A sufficiently large, the infinite volume limit of the truncated correlation function $\lim_{N\to\infty} \langle \prod_{k=1}^{m} f_k(\phi)(x_k)\rangle^t$ exists.*

Theorem 7. *For any small $\iota > 0$, with L, A sufficiently large (depending on ι), let $\eta = \min\{d/2, 2\}$ we have:*

$$\left| \left\langle \prod_{k=1}^{m} f_k(\phi)(x_k) \right\rangle^t \right| \leq \frac{m!}{a^m}\ diam^{-\eta+\iota}(x_1, x_2, \ldots, x_m)$$

where a depends on ι, L, A.

In the case $f = \sum_{k=1}^{m} t_k W_0(\{x_k\})$, with $W_0(\{x_k\}) = zW(1, \{x_k\})$ as in (64). Using the Lemma 2 (or the lemma 4 in [5]), these $W_0(\{x_k\})$ satisfy those above conditions. The $W_0(\{x_k\})$ are actually the density of the dipoles at x_k used in [2]. Applying theorems 7 and 6, we obtain these results:

Corollary 1. *For any small $\iota > 0$, with L, A sufficiently large (depending on ι), let $\eta = \min\{d/2, 2\}$ we have:*

$$\left| \left\langle \prod_{k=1}^{m} W_0(\{x_k\}) \right\rangle^t \right| \leq \frac{m!}{a^m}\ diam^{-\eta+\iota}(x_1, x_2, \ldots, x_m)$$

where a depends on ι, L, A.

This result somehow looks like the theorem (1.1.2) in [2]. However it gives estimates for truncated correlation functions of ($m \geq 2$) points instead of some estimate for only 2 points.

Corollary 2. *With L, A sufficiently large, the infinite volume limit of the truncated correlation function $\lim_{N\to\infty} \langle \prod_{k=1}^{m} W_0(\{x_k\})\rangle^t$ exists.*

Remark. We can consider the more general form $f(\phi) = \sum_{k=1}^{m} t_k f_k(\phi)$ with
* $t_k \in \mathbb{C}$,
* $A_k \equiv \mathrm{supp} f_k$ are pairwise disjoint and $|A_k| < \infty$,
* f_k is bounded in the sense that there are some $M_k, m_k \geq 0$ such that

$$\|f_k(A_k, \phi)\|_0 \leq M_k \|\phi\|_{\Phi_0} + m_k. \tag{201}$$

Then we still get similar results as in Theorems 7 and 6.

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Appendix A: Kac–Siegert Transformation

By expanding the exponential in (6) and carrying out the Gaussian integrals, we can rewrite ${}_0Z_N$ as

$$\begin{aligned}
{}_0Z_N &= \int \left(\sum_{n\geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \sum_{x_i \in \Lambda_N \cap \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} \mathrm{d}p_i \left(\frac{e^{ip_i\cdot\partial\phi(x_i)} + e^{-ip_i\cdot\partial\phi(x_i)}}{2} \right) \right) \mathrm{d}\mu_C(\phi) \\
&= \int \left(\sum_{n\geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \sum_{x_i \in \Lambda_N \cap \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} \mathrm{d}p_i e^{ip_i\cdot\partial\phi(x_i)} \right) \mathrm{d}\mu_C(\phi) \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \left(\sum_{x_i \in \Lambda_N \cap \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} \mathrm{d}p_i \right) \int e^{i\sum_{k=1}^{n} p_k\cdot\partial\phi(x_k)} \mathrm{d}\mu_C(\phi) \\
&= \sum_{n\geq 0} \frac{z^n}{n!} \prod_{i=1}^{n} \left(\sum_{x_i \in \Lambda_N \cap \mathbb{Z}^d} \int_{\mathbb{S}^{d-1}} \mathrm{d}p_i \right) \exp\left(\frac{-1}{2} \sum_{1\leq k,j\leq n} (p_k\cdot\partial)(p_j\cdot\partial)C(x_k,x_j) \right)
\end{aligned} \tag{202}$$

which is exactly the same as the grand canonical partition function (2).

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Tuan Minh Le
Department of Mathematics
SUNY at Buffalo
Buffalo, NY 14260, USA

and

Department of Mathematics and Applications
Saigon University
273 An Duong Vuong Street
Dist. 5, Ho Chi Minh City, Vietnam
e-mail: tuanle@buffalo.edu;
tuanle@sgu.edu.vn

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