Ann. Henri Poincaré 17 (2016), 3473–3498

1424-0637/16/123473-26
published online April 29, 2016
DOI 10.1007/s00023-016-0493-6

Annales Henri Poincaré

On a Drift–Diffusion System for Semiconductor Devices

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Abstract. In this note, we study a fractional Poisson–Nernst–Planck equation modeling a semiconductor device. We prove several decay estimates for the Lebesgue and Sobolev norms in one, two and three dimensions. We also provide the first term of the asymptotic expansion as $t \to \infty$.

1. Introduction

We consider the drift–diffusion system given below:

$$\begin{cases} \partial_t u + (-\Delta)^{\frac{\alpha}{2}} u + \nabla \cdot (u \nabla \psi) = 0, \text{ for } (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ \partial_t v + (-\Delta)^{\frac{\beta}{2}} v - \nabla \cdot (v \nabla \psi) = 0, \text{ for } (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ \Delta \psi = u - v, \text{ for } (x,t) \in \mathbb{R}^d \times \mathbb{R}^+ \\ u(x,0) = u_0, v(x,0) = v_0, x \in \mathbb{R}^d \end{cases} \quad (1.1)$$

where u, v, and ψ are functions of space and time, the dimension $d \in \mathbb{Z}^+$ with $d \leq 3$, $0 < \alpha, \beta < 2$, and, if we denote the Fourier transform of the function ϕ by $\hat{\phi}$, then the fractional Laplacian is defined by

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} \phi} = |\xi|^\alpha \hat{\phi}.$$

The unknown functions $u(x,t)$ and $v(x,t)$ represent the density of electrons and positive holes in the semiconductor, respectively. Finally, the function ψ models the electromagnetic potential due to charges in a semiconductor. The fractional Laplacians are related to random trajectories, generalizing the concept of Brownian motion, which may contain jump discontinuities (the, so-called, α-stable Lévy processes). As an electron in a semiconductor may jump from a dopant into another, a nonlocal diffusion akin to the fractional Laplacian arises naturally.

1.1. Prior Results on (1.1)

Mock [29] first considered the drift–diffusion system (1.1) with $\alpha = \beta = 2$ on a bounded domain with the Neumann boundary condition (see also He et al. [20] and Liu and Wang [28]). A similar equation has been studied by Rodríguez and Ryzhik in a very different context [31]. Fang and Ito [19] proved the existence of a global weak solution in this bounded domain (see also the work by Bothe et al. [9] and Hineman and Ryham [21]). The asymptotic behavior of the solution in the case $\alpha = \beta = 2$ was studied by Jungel [22] and Biler and Dolbeault [3]. Kurokiba and Ogawa [26] and Kurokiba et al. [25] proved the existence and uniqueness of strong solutions to the Cauchy problem. Kawashima and Kobayashi [24] derived the optimal decay estimate by applying a weighted energy method and found an asymptotic result as $t \to \infty$. In the presence of an incompressible, viscous fluid, system (1.1) was studied by Schmuck [32], by Zhao et al. [17,41,42], by Bothe et al. [10]. Very recently, Kinderlehrer et al. provided a new approach to system (1.1) using that system (1.1) is a gradient flow driven by a $L \log L$-type free energy [23]. Each of these studies restricted their conclusions to $\alpha = \beta = 2$. The case of nonlinear diffusion has been considered by Zinsl [43].

When $v_0 \equiv 0$ (so the equation for v is dropped), the fractional case $0 < \alpha \leq 2$ of (1.1) has been studied by several authors. Yamamoto [34] obtained the asymptotic behavior in the local case $\alpha = 2$. Yamamoto [35] proceeded similarly, but derived the asymptotic expansion of the solution with the fractional Laplacian in the subcritical regime $1 < \alpha < 2$. Yamamoto et al. [36] showed the well-posedness and real analytic of the critical case corresponding to $\alpha = 1$. Sugiyama et al. [33] studied local and global existence and uniqueness of the system with the fractional Laplacian, focusing primarily on the supercritical and critical cases $0 < \alpha < 1$ and $\alpha = 1$, respectively. Yamamoto and Sugiyama [37,38] then derived lower bounds on the decay rates of a solution to the drift–diffusion system with the fractional Laplacian $0 < \alpha \leq 1$ and obtained the asymptotic behavior of the solution as $t \to \infty$. Similar systems arising in different contexts have been studied also by Li et al. [27], Escudero [18], Bournaveas and Calvez [11], Biler and Karch [4], Biler and Wu [8], Biler et al. [5] Biler and Woyczyński [6,7], Zhao [39,40], Ascasibar, Granero-Belinchón and Moreno [1] and Burczak and Granero-Belinchón [13,14].

The fractional case $1 < \alpha = \beta < 2$ of (1.1) with general v_0 has been studied by Ogawa and Yamamoto [30]. In particular, these authors proved the global existence and the asymptotic behavior of solutions. To the best of our knowledge, this is the only result concerning (1.1). Thus, by studying (1.1), this paper generalizes the current results in [30] in two different aspects:

1. it allows for diffusions with different strengths for u and v, i.e., α is not necessarily equal to β. The cases $\alpha \neq \beta$ and $\alpha = \beta$ present several differences at the level of the H^2 Sobolev norm and some *closeness* hypothesis needs to be imposed (see Theorem 2.4).
2. it allows for diffusions in the whole range $0 < \alpha, \beta < 2$. In particular, our work covers the supercritical and critical range $0 < \alpha, \beta \leq 1$.

1.2. Preliminaries

1.2.1. Singular Integral Operators. We write $\Lambda^\alpha = (-\Delta)^{\frac{\alpha}{2}}$, i.e.,

$$\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \hat{u}(\xi) \tag{1.2}$$

where $\hat{\cdot}$ denotes the usual Fourier transform. As a singular integral operator, the operator Λ^α possesses the kernel

$$\Lambda^\alpha u(x) = c_{\alpha,d} \text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(x-\eta) \mathrm{d}\eta}{|\eta|^{d+\alpha}}, \tag{1.3}$$

with

$$c_{\alpha,d} = \frac{4^s \Gamma(d/2+\alpha)}{\pi^{d/2} |\Gamma(-\alpha)|} > 0,$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t$$

is the Γ function.

1.2.2. Functional Spaces. We write $L^p(\mathbb{R}^d)$ for the usual Lebesgue spaces

$$L^p(\mathbb{R}^d) = \left\{ u \text{ measurable s.t. } \int_{\mathbb{R}^d} |u(x)|^p \mathrm{d}x < \infty \right\},$$

with norm

$$\|u\|_{L^p}^p = \int_{\mathbb{R}^d} |u(x)|^p \mathrm{d}x.$$

We write $H^s(\mathbb{R}^d)$ for the usual L^2-based Sobolev spaces:

$$H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d) \text{ s.t. } (1+|\xi|^s)\hat{u} \in L^2(\mathbb{R}^d) \right\},$$

with the norm

$$\|u\|_{H^s}^2 = \|u\|_{L^2}^2 + \|u\|_{\dot{H}^s}^2, \quad \|u\|_{\dot{H}^s} = \|\Lambda^s u\|_{L^2}.$$

1.3. Plan of the Paper

This note is organized as follows: in Sect. 2, we state our results. In Sect. 3, we prove the decay in the L^p spaces. In Sects. 4, 5 and 6, we prove the decay of the Sobolev norms. Then, in Sect. 7, we provide the first term in the asymptotic expansion. Finally, in Appendices A–C, we provide certain inequalities and estimates for fractional Laplacian that are used in the paper and may be interesting by themselves.

2. Main Results

Our first result concerns the global existence and decay of the solutions to (1.1):

Theorem 2.1. *Let $0 < \alpha, \beta < 2$, $d \in \mathbb{Z}^+$ with $d \leq 3$, be fixed constants and*

$$u_0(x), v_0(x) \in L^1(\mathbb{R}^d) \cap H^4(\mathbb{R}^d)$$

be the initial data. Then, there exists $(u(x,t), v(x,t))$ a global smooth solution to (1.1) satisfying

$$u \in L^\infty(0,T; L^1(\mathbb{R}^d) \cap H^4(\mathbb{R}^d)) \cap L^2(0,T; H^{4+\alpha/2}(\mathbb{R}^d)),$$
$$v \in L^\infty(0,T; L^1(\mathbb{R}^d) \cap H^4(\mathbb{R}^d)) \cap L^2(0,T; H^{4+\beta/2}(\mathbb{R}^d)),$$

for every $0 < T < \infty$. Furthermore, the functionals

$$\mathcal{F}_p[u(t), v(t)] := \|u(t)\|_{L^p}^p + \|v(t)\|_{L^p}^p,\ 1 \leq p < \infty,$$
$$\mathcal{F}_\infty[u(t), v(t)] := \|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty},$$

verify

$$\mathcal{F}_p[u(t), v(t)] \leq \mathcal{F}_p[u_0, v_0],\ 1 \leq p \leq \infty,$$

and there exist constants K and C_∞ such that

$$\mathcal{F}_\infty[u,v] \leq \frac{\mathcal{F}_\infty[u_0,v_0]}{(1+Kt)^{d/\max\{\alpha,\beta\}}},$$

$$\mathcal{F}_p[u,v] \leq (\|u_0\|_{L^1} + \|v_0\|_{L^1}) \frac{C_\infty^{p-1}}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}(p-1)}},$$

$$\|u(t)\|_{L^p} \leq \frac{\|u_0\|_{L^1}^{\frac{1}{p}} C_\infty^{1-\frac{1}{p}}}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}(1-\frac{1}{p})}},$$

$$\|v(t)\|_{L^p} \leq \frac{\|v_0\|_{L^1}^{\frac{1}{p}} C_\infty^{1-\frac{1}{p}}}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}(1-\frac{1}{p})}}.$$

Remark 2.2. In the case where the smooth initial data are

$$u_0(x), v_0(x) \in L^p(\mathbb{R}^d),\ 1 < p < \infty,$$

following the proof of Theorem 2.1, we have the pointwise estimates

$$\Lambda^\alpha u(x_t) \geq c(d,\alpha,p) \frac{u(x_t)^{1+\alpha p/d}}{\|u(t)\|_{L^p}^{\alpha p/d}} \geq c(d,\alpha,p) \frac{u(x_t)^{1+\alpha p/d}}{\mathcal{F}_p(u_0,v_0)^{\alpha/d}}$$
$$\Lambda^\beta v(y_t) \geq c(d,\beta,p) \frac{v(y_t)^{1+\beta p/d}}{\|v(t)\|_{L^p}^{\beta p/d}} \geq c(d,\beta,p) \frac{v(y_t)^{1+\beta p/d}}{\mathcal{F}_p(u_0,v_0)^{\beta/d}},$$

where x_t and y_t are such that

$$\|u(t)\|_{L^\infty} = u(x_t,t),\ \|v(t)\|_{L^\infty} = v(y_t,t).$$

Thus, instead of (3.2), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\infty[u,v] \leq -c(d,\alpha,p) \frac{u(x_t)^{1+\alpha p/d}}{\mathcal{F}_p(u_0,v_0)^{\alpha/d}} - c(d,\beta,p) \frac{v(y_t)^{1+\beta p/d}}{\mathcal{F}_p(u_0,v_0)^{\beta/d}}. \tag{2.1}$$

Our second result studies the behavior of Sobolev spaces H^s for $0 < s \leq 1$.

Theorem 2.3. *Let* $0 < \alpha, \beta < 2$, $d \in \mathbb{Z}^+$ *with* $d \leq 3$, *be fixed constants and*

$$u_0(x), v_0(x) \in L^1 \cap H^4$$

be the initial data. Then, there exists a constant C *such that the solution* $(u(x,t), v(x,t))$ *to* (1.1) *verifies*

$$\|u(t)\|_{\dot{H}^s} + \|v(t)\|_{\dot{H}^s} \leq \frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}\frac{1-s}{2}}}, \quad \forall t \geq 0, \quad 0 \leq s \leq 1.$$

Our third result regards the higher Sobolev norm H^s, $1 \leq s \leq 2$ and imposes restrictions on α and β:

Theorem 2.4. *Let* $0 < \alpha, \beta < 2$, $d \in \mathbb{Z}^+$, $d \leq 3$, *be fixed constants such that*

$$\frac{2d}{4 + 3\min\{\alpha,\beta\}} < 1, \tag{2.2}$$

$$\frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}}\frac{d}{4}\left(2 + \frac{2d}{4 + 3\max\{\alpha,\beta\} - 2d}\right) > 1, \tag{2.3}$$

$$\Gamma = \frac{d}{4 + 3\min\{\alpha,\beta\} - d}\left(2 + \frac{4|\alpha-\beta|}{4 + 3\min\{\alpha,\beta\}}\right) \leq 2, \tag{2.4}$$

and

$$u_0(x), v_0(x) \in L^1 \cap H^4$$

be the initial data. Furthermore, when $\Gamma < 2$, *we assume that*

$$\frac{4 + 3\max\{\alpha,\beta\}}{4 + 3\min\{\alpha,\beta\}} < 1 + \frac{d}{4 + 3\max\{\alpha,\beta\} - d}, \tag{2.5}$$

and

$$\frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}}\frac{d}{2} \geq 1. \tag{2.6}$$

Then, there exists a constant C *such that the solution* $(u(x,t), v(x,t))$ *to* (1.1) *verifies*

$$\|u(t)\|_{\dot{H}^s} + \|v(t)\|_{\dot{H}^s} \leq \frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}\frac{2-s}{4}}}, \quad \forall t \geq 0, \quad 0 \leq s \leq 2.$$

Notice that this result imposes restrictions on the difference $\alpha - \beta$. This result suggests that a big disparity in the strengths of the diffusive operators may lead to obstructions in higher Sobolev norms.

Our next theorem concerns the case of arbitrarily large Sobolev norms:

Theorem 2.5. *Let* $0 < \alpha, \beta < 2$, $d \in \mathbb{Z}^+$ *with* $d \leq 3$, *be fixed constants and*

$$u_0(x), v_0(x) \in L^1 \cap H^s, \quad s \geq 2,\ s \in \mathbb{R}$$

be the initial data. Assume that α, β *and* d *satisfy the same hypothesis as in Theorem* 2.4. *Then, there exists a constant* C *such that the solution* $(u(x,t), v(x,t))$ *to* (1.1) *verifies*

$$\|u(t)\|_{\dot{H}^r} + \|v(t)\|_{\dot{H}^r} \leq \frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}\frac{s-r}{2s}}}, \quad \forall t \geq 0,\ 0 \leq r \leq s.$$

Finally, we provide the first-order asymptotic estimate

Proposition 2.6. *Let* $0<\alpha,\beta<2$, $d\in\mathbb{Z}^+$ *with* $d\leq 3$, *be fixed constants and*

$$u_0(x),v_0(x)\in L^1\cap H^s,\quad s\geq 2,\, s\in\mathbb{R}$$

be the initial data. Then, there exists a constant C *such that the solution* $(u(x,t),v(x,t))$ *to* (1.1) *verifies*

$$\|u(t)-\mathrm{e}^{-t\Lambda^\alpha}u_0\|_{L^2}\leq\frac{C}{(1+t)^{\frac{d-1}{\max\{\alpha,\beta\}}-1}}$$
$$\|v(t)-\mathrm{e}^{-t\Lambda^\beta}v_0\|_{L^2}\leq\frac{C}{(1+t)^{\frac{d-1}{\max\{\alpha,\beta\}}-1}}$$

3. Proof of Theorem 2.1: Global Existence and L^p Decay Estimates

Step 1: Local existence The local existence and uniqueness follow from standard methods (see for instance [1]).
Step 2: Boundedness in L^p First notice that, given $u_0(x)\geq 0$ and $v_0(x)\geq 0$, we have that $u(t)\geq 0$ and $v(x,t)\geq 0$ $\forall t\geq 0$ (this can be shown with a contradiction argument and the use of pointwise methods [12]). Thus, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1[u,v]=0.$$

Furthermore, we have the stronger equalities

$$\|u(t)\|_{L^1}=\|u_0\|_{L^1},\ \|v(t)\|_{L^1}=\|v_0\|_{L^1}.$$

Consider now the case $1<p<\infty$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_p[u,v]&=\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^p}^p+\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{L^p}^p\\&=\int_{\mathbb{R}^d}pu(y,t)^{p-1}\partial_t u(y,t)\mathrm{d}y+\int_{\mathbb{R}^d}pv(y,t)^{p-1}\partial_t v(y,t)\mathrm{d}y\\&=\int_{\mathbb{R}^d}pu(y,t)^{p-1}[-\Lambda^\alpha u(y,t)-\nabla\cdot(u(y,t)\nabla\psi)]\mathrm{d}y\\&\quad+\int_{\mathbb{R}^d}pv(y,t)^{p-1}[-\Lambda^\beta v(y,t)+\nabla\cdot(v(y,t)\nabla\psi)]\mathrm{d}y\end{aligned}$$

The transport terms are

$$T_1=-\int_{\mathbb{R}^d}pu^{p-1}\nabla\cdot(u\nabla\psi)=\int_{\mathbb{R}^d}p(p-1)u^{p-1}\nabla u\cdot\nabla\psi=-\int_{\mathbb{R}^d}(p-1)u^p\Delta\psi,$$
$$T_2=\int_{\mathbb{R}^d}pv^{p-1}\nabla\cdot(v\nabla\psi)=-\int_{\mathbb{R}^d}p(p-1)v^{p-1}\nabla v\cdot\nabla\psi=\int_{\mathbb{R}^d}(p-1)v^p\Delta\psi.$$

Symmetrizing the diffusive terms, we get

$$\begin{aligned}
D_1 &= -\int_{\mathbb{R}^d} u(y,t)^{p-1}\Lambda^\alpha u(y,t)\mathrm{d}y \\
&= -p\int_{\mathbb{R}^d} u(y,t)^{p-1}\int_{\mathbb{R}^d}\frac{u(y,t)-u(\eta,t)}{|y-\eta|^{d+\alpha}}\mathrm{d}\eta\mathrm{d}y \\
&= -p\int_{\mathbb{R}^d} u(\eta,t)^{p-1}\int_{\mathbb{R}^d}\frac{u(\eta,t)-u(y,t)}{|\eta-y|^{d+\alpha}}\mathrm{d}\eta\mathrm{d}y \\
&= p\int_{\mathbb{R}^d} u(\eta,t)^{p-1}\int_{\mathbb{R}^d}\frac{u(y,t)-u(\eta,t)}{|\eta-y|^{d+\alpha}}\mathrm{d}\eta\mathrm{d}y \\
&= -\frac{p}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(u(y,t)^{p-1}-u(\eta,t)^{p-1}\right)\frac{u(y,t)-u(\eta,t)}{|y-\eta|^{d+\alpha}}\mathrm{d}\eta\mathrm{d}y \\
&\le 0
\end{aligned}$$

Following a similar procedure,

$$\begin{aligned}
D_2 &= -\int_{\mathbb{R}^d} v(y,t)^{p-1}\Lambda^\beta v(y,t)\mathrm{d}y \\
&= -\frac{p}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(v(y,t)^{p-1}-v(\eta,t)^{p-1}\right)\frac{v(y,t)-v(\eta,t)}{|y-\eta|^{d+\beta}}\mathrm{d}\eta\mathrm{d}y \\
&\le 0.
\end{aligned}$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_p[u,v] \le T_1+T_2 = -(p-1)\int_{\mathbb{R}^d}(u^p-v^p)(u-v)\mathrm{d}x \le 0,$$

and we conclude

$$\mathcal{F}_p[u,v] \le \mathcal{F}_p[u_0,v_0].$$

Step 3: Boundedness in L^∞ Due to the smoothness of $u(x,t)$ and $v(x,t)$ in space and time, we have that

$$M_u(t) := \|u(t)\|_{L^\infty} = u(x_t,t),\ M_v(t) := \|v(t)\|_{L^\infty} = v(y_t,t)$$

are Lipschitz. Thus, using Rademacher Theorem $M_u(t)$ and $M_v(t)$ are differentiable almost everywhere and (see [1,16])

$$\frac{\mathrm{d}}{\mathrm{d}t}M_u(t) = \partial_t u(x_t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}M_v(t) = \partial_t v(y_t).$$

Now, we show the $\mathcal{F}_\infty[u,v] = \|u(t)\|_{L^\infty} + \|v(t)\|_{L^\infty}$ is a Lyapunov functional:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\infty[u,v] \\
&\quad = M_u(t) + M_v(t) \\
&\quad = \partial_t u(x_t) + \partial_t v(y_t) \\
&\quad = -\Lambda^\alpha u(x_t) - \Lambda^\beta v(y_t) - u(x_t)\Delta\psi(x_t) + v(y_t)\Delta\psi(y_t)
\end{aligned}$$

$$= -\Lambda^\alpha u(x_t) - \Lambda^\beta v(y_t) - u(x_t)[u(x_t) - v(x_t)] + v(y_t)[u(y_t) - v(y_t)]$$
$$\leq -\Lambda^\alpha u(x_t) - \Lambda^\beta v(y_t) - u(x_t)^2 + 2u(x_t)v(y_t) - v(y_t)^2.$$

Thus, using (1.3), we have that

$$-\Lambda^\alpha u(x_t) = -\int_{\mathbb{R}^d} \frac{u(x_t) - u(x_t - \eta)}{|\eta|^{d+\alpha}} \mathrm{d}\eta \leq 0,$$
$$-\Lambda^\beta v(y_t) = -\int_{\mathbb{R}^d} \frac{v(y_t) - v(y_t - \eta)}{|\eta|^{d+\beta}} \mathrm{d}\eta \leq 0,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\infty[u,v] \leq -\Lambda^\alpha u(x_t) - \Lambda^\beta v(y_t) - (u(x_t) - v(y_t))^2 \leq 0. \tag{3.1}$$

So

$$\mathcal{F}_\infty[u,v] \leq \mathcal{F}_\infty[u_0,v_0].$$

Step 4: Decay in L^∞ Furthermore, we have the following lower bounds (see Lemma A.1 and [2])

$$\Lambda^\alpha u(x_t) \geq \frac{c_{\alpha,d}\, u(x_t)\frac{\|u_0\|_{L^1}}{u(x_t)}}{\left(\left(\frac{2\|u_0\|_{L^1}}{u(x_t)}\right)^{1/d}\left(\frac{2}{\omega_d}\right)^{1/d}\right)^{d+\alpha}} \geq c(d,\alpha)\frac{u(x_t)^{1+\alpha/d}}{\|u_0\|_{L^1}^{\alpha/d}}$$
$$\Lambda^\beta v(y_t) \geq \frac{c_{\beta,d}\, v(y_t)\frac{\|v_0\|_{L^1}}{v(y_t)}}{\left(\left(\frac{2\|v_0\|_{L^1}}{v(y_t)}\right)^{1/d}\left(\frac{2}{\omega_d}\right)^{1/d}\right)^{d+\beta}} \geq c(d,\beta)\frac{v(y_t)^{1+\beta/d}}{\|v_0\|_{L^1}^{\beta/d}},$$

Thus, (3.1) can be sharpened and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\infty[u,v] \leq -c(d,\alpha)\frac{u(x_t)^{1+\alpha/d}}{\|u_0\|_{L^1}^{\alpha/d}} - c(d,\beta)\frac{v(y_t)^{1+\beta/d}}{\|v_0\|_{L^1}^{\beta/d}}. \tag{3.2}$$

Fix $\gamma > 0$. Then

$$(u(x_t) + v(y_t))^{1+\gamma}$$
$$\leq 2^{1+\gamma}\max\{u(x_t), v(y_t)\}^{1+\gamma}$$
$$\leq 2^{1+\gamma}\left(\max\{u(x_t), v(y_t)\}^{1+\gamma} + \min\{u(x_t), v(y_t)\}^{1-\gamma+(\alpha+\beta)/d}\right).$$

We define γ as:

$$\gamma := \begin{cases} \alpha/d & \text{if } \max\{u(x_t), v(y_t)\} = u(x_t) \\ \beta/d & \text{if } \max\{u(x_t), v(y_t)\} = v(y_t) \end{cases}. \tag{3.3}$$

With this definition of γ, we have

$$(u(x_t) + v(y_t))^{1+\gamma} \leq 2^{1+\gamma}\max\{u(x_t), v(y_t)\}^{1+\gamma}$$
$$\leq 2^{1+\max\{\alpha,\beta\}/d}\left(u(x_t)^{1+\alpha/d} + v(y_t)^{1+\beta/d}\right).$$

Let us denote

$$C_{min}(\alpha,\beta,d,u_0,v_0) := \min\left\{\frac{c(d,\alpha)}{\|u_0\|_{L^1}^{\alpha/d}}, \frac{c(d,\beta)}{\|v_0\|_{L^1}^{\beta/d}}\right\}, \tag{3.4}$$

then

$$\frac{C_{min}}{2^{1+\max\{\alpha,\beta\}/d}}\left(u(x_t)+v(y_t)\right)^{1+\gamma} \le c(d,\alpha)\frac{u(x_t)^{1+\alpha/d}}{\|u_0\|_{L^1}^{\alpha/d}} + c(d,\beta)\frac{v(y_t)^{1+\beta/d}}{\|v_0\|_{L^1}^{\beta/d}}.$$

We obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\infty[u,v] \le -\frac{C_{min}}{2^{1+\max\{\alpha,\beta\}/d}}\mathcal{F}_\infty[u,v]^{1+\gamma},$$

where γ is given by (3.3). We obtain the following rate of decay:

$$\mathcal{F}_\infty[u,v] \le \frac{\mathcal{F}_\infty[u_0,v_0]}{(1+Kt)^{1/\gamma}} \le \frac{\mathcal{F}_\infty[u_0,v_0]}{(1+Kt)^{d/\max\{\alpha,\beta\}}},$$

where

$$K = \min\left\{(\mathcal{F}_\infty[u_0,v_0])^{\alpha/d}, (\mathcal{F}_\infty[u_0,v_0])^{\alpha/d}\right\}\frac{\min\{\alpha,\beta\}}{d}\frac{C_{min}}{2^{1+\max\{\alpha,\beta\}/d}}.$$

As a consequence, we have

$$\|u(t)\|_{L^\infty}, \|v(t)\|_{L^\infty} \le \frac{C_\infty}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}}}.$$

Step 5: Decay in L^p Using interpolation and the conservation of mass, we obtain

$$\|u(t)\|_{L^p} \le \|u_0\|_{L^1}^{\frac{1}{p}}\frac{C_\infty^{1-\frac{1}{p}}}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}(1-\frac{1}{p})}},$$

$$\|v(t)\|_{L^p} \le \|v_0\|_{L^1}^{\frac{1}{p}}\frac{C_\infty^{1-\frac{1}{p}}}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}(1-\frac{1}{p})}},$$

$$\mathcal{F}_p[u,v] \le (\|u_0\|_{L^1} + \|v_0\|_{L^1})\frac{C_\infty^{p-1}}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}(p-1)}}.$$

Step 6: Global existence The global existence follows from the decay of $\|u\|_{L^\infty} + \|v\|_{L^\infty}$, energy estimates and a standard continuation argument (see [1]).

4. Proof of Theorem 2.3: Decay Estimates in Sobolev Spaces H^s, $0 < s < 1$

Step 1: Boundedness in H^1 $(d = 1)$

First, we deal with the one-dimensional case. We compute

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^1}^2 &= -\|u\|_{\dot{H}^{1+\alpha/2}}^2 - \int_{\mathbb{R}}\partial_x u\partial_x^2(u\partial_x\psi)\\ &\le -\|u\|_{\dot{H}^{1+\alpha/2}}^2 - \int_{\mathbb{R}}\partial_x u(\partial_x^2 u\partial_x\psi + u\partial_x(u-v) + 2\partial_x u(u-v)),\end{aligned}$$

and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\dot{H}^1} = -\|v\|^2_{\dot{H}^{1+\beta/2}} + \int_{\mathbb{R}} \partial_x v \partial_x^2 (v\partial_x\psi)$$
$$\leq -\|v\|^2_{\dot{H}^{1+\beta/2}} + \int_{\mathbb{R}} \partial_x v(\partial_x^2 v\partial_x\psi + v\partial_x(u-v) + 2\partial_x v(u-v)).$$

Adding them together and using Hölder inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\dot{H}^1} + \|v\|^2_{\dot{H}^1}\right) = -2\|u\|^2_{\dot{H}^{1+\alpha/2}} - 2\|v\|^2_{\dot{H}^{1+\beta/2}} + C(\|\partial_x u\|^2_{L^2} + \|\partial_x v\|^2_{L^2})(\|u\|_{L^\infty} + \|v\|_{L^\infty}).$$

Using the interpolation inequality

$$\|\Lambda^r f\|^2_{L^2} \leq \|f\|^2_{L^2} + \|f\|^2_{\dot{H}^{r+s}},\ \forall\, r,s \geq 0, \tag{4.1}$$

we conclude that, for $t \geq T^*$ and $T^* < \infty$ large enough (see Theorem 2.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\dot{H}^1} + \|v\|^2_{\dot{H}^1}\right) = -\|u\|^2_{\dot{H}^{1+\alpha/2}} - \|v\|^2_{\dot{H}^{1+\beta/2}} + C(\|u\|^2_{L^2} + \|v\|^2_{L^2})(\|u\|_{L^\infty} + \|v\|_{L^\infty}).$$

Recalling that

$$1 < \frac{2}{\max\{\alpha,\beta\}}$$

and using Theorem 2.1 to obtain that

$$(\|u\|^2_{L^2} + \|v\|^2_{L^2})(\|u\|_{L^\infty} + \|v\|_{L^\infty}) \leq \frac{C}{(1+t)^{\frac{2}{\max\{\alpha,\beta\}}}},$$

so

$$\int_{T^*}^t (\|u\|^2_{L^2} + \|v\|^2_{L^2})(\|u\|_{L^\infty} + \|v\|_{L^\infty})\mathrm{d}s \leq C,$$

we have that

$$\|u(t)\|^2_{\dot{H}^1} + \|v(t)\|^2_{\dot{H}^1} + \int_{T^*}^t \|u\|^2_{\dot{H}^{1+\alpha/2}} + \|v\|^2_{\dot{H}^{1+\beta/2}}\mathrm{d}s \leq C, \quad \forall\, t \geq T^*.$$

Standard energy estimates on the finite interval $[0,T^*]$ lead to

$$\|u(t)\|^2_{\dot{H}^1} + \|v(t)\|^2_{\dot{H}^1} \leq C, \quad \forall\, t \geq 0.$$

Step 2: Boundedness in H^1 ($d=2$, $d=3$)

Assume now that $d=2$ or $d=3$. Testing the equation for u against $\Lambda^2 u$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot{H}^1} = -\|u\|^2_{\dot{H}^{1+\alpha/2}} - \int_{\mathbb{R}^d} \Lambda u\Lambda(\nabla u\cdot\nabla\psi)\mathrm{d}x - \int_{\mathbb{R}^d} \Lambda^2 u u(u-v)\mathrm{d}x$$
$$= -\|u\|^2_{\dot{H}^{1+\alpha/2}} - \int_{\mathbb{R}^d} \Lambda u[\Lambda,\nabla\psi]\cdot\nabla u\mathrm{d}x - \int_{\mathbb{R}^d} \Lambda u\nabla\psi\cdot\nabla\Lambda u\mathrm{d}x$$
$$- \int_{\mathbb{R}^d} \Lambda u\Lambda(u(u-v))\mathrm{d}x$$

$$= -\|u\|_{\dot{H}^{1+\alpha/2}}^2 - \int_{\mathbb{R}^d} \Lambda u[\Lambda, \nabla\psi]\cdot\nabla u \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} |\Lambda u|^2(u-v)\mathrm{d}x$$
$$- \int_{\mathbb{R}^d} \Lambda u \Lambda(u(u-v))\mathrm{d}x.$$

In the same way

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{\dot{H}^1}^2 = -\|v\|_{\dot{H}^{1+\beta/2}}^2 + \int_{\mathbb{R}^d} \Lambda v \Lambda(\nabla v\cdot\nabla\psi)\mathrm{d}x + \int_{\mathbb{R}^d} \Lambda^2 v v(u-v)\mathrm{d}x$$
$$= -\|v\|_{\dot{H}^{1+\beta/2}}^2 + \int_{\mathbb{R}^d} \Lambda v[\Lambda, \nabla\psi]\cdot\nabla v \mathrm{d}x + \int_{\mathbb{R}^d} \Lambda v \nabla\psi\cdot\nabla\Lambda v \mathrm{d}x$$
$$+ \int_{\mathbb{R}^d} \Lambda v \Lambda(v(u-v))\mathrm{d}x$$
$$= -\|v\|_{\dot{H}^{1+\beta/2}}^2 + \int_{\mathbb{R}^d} \Lambda v[\Lambda, \nabla\psi]\cdot\nabla v \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\Lambda v|^2(u-v)\mathrm{d}x$$
$$+ \int_{\mathbb{R}^d} \Lambda v \Lambda(v(u-v))\mathrm{d}x.$$

Recalling the Sobolev embedding

$$\|f\|_{L^{\frac{2d}{d-s}}} \leq C\|\Lambda^{s/2} f\|_{L^2}, \tag{4.2}$$

and Theorem 2.1 (using $d \geq 2$, $\max\{\alpha,\beta\} < 2$), we have a time $T^* < \infty$ such that, for $t \geq T^*$,

$$\left|\int_{\mathbb{R}^d} |\Lambda u|^2(u-v)\mathrm{d}x\right| \leq \|\Lambda u\|_{L^{\frac{2d}{d-\alpha}}}^2 \|u-v\|_{L^{d/\alpha}}$$
$$\leq C\|u\|_{\dot{H}^{1+\frac{\alpha}{2}}}^2 \|u-v\|_{L^{d/\alpha}}$$
$$\leq \frac{1}{8}\|u\|_{\dot{H}^{1+\frac{\alpha}{2}}}^2, \tag{4.3}$$

$$\left|\int_{\mathbb{R}^d} |\Lambda v|^2(u-v)\mathrm{d}x\right| \leq \|\Lambda v\|_{L^{\frac{2d}{d-\beta}}}^2 \|u-v\|_{L^{d/\beta}}$$
$$\leq C\|v\|_{\dot{H}^{1+\frac{\beta}{2}}}^2 \|u-v\|_{L^{d/\beta}}$$
$$\leq \frac{1}{8}\|v\|_{\dot{H}^{1+\frac{\beta}{2}}}^2. \tag{4.4}$$

Using the fractional Leibniz rule

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|\Lambda^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}),$$

with

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

we have

$$\left|\int_{\mathbb{R}^d} \Lambda u \Lambda(u(u-v))\mathrm{d}x\right|$$
$$\leq C\|\Lambda u\|_{L^{\frac{2d}{d-\alpha}}}(\|\Lambda u\|_{L^2}\|u-v\|_{L^{\frac{2d}{\alpha}}} + \|u\|_{L^{\frac{2d}{\alpha}}}\|\Lambda(u-v)\|_{L^2})$$
$$\leq C\|u\|_{\dot{H}^{1+\alpha/2}}(\|\Lambda u\|_{L^2}\|u-v\|_{L^{\frac{2d}{\alpha}}} + \|u\|_{L^{\frac{2d}{\alpha}}}\|\Lambda(u-v)\|_{L^2}), \tag{4.5}$$

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} \Lambda v\Lambda(v(u-v))\mathrm{d}x\right| \\
&\quad\leq C\|\Lambda v\|_{L^{\frac{2d}{d-\beta}}}(\|\Lambda v\|_{L^2}\|u-v\|_{L^{\frac{2d}{\beta}}}+\|v\|_{L^{\frac{2d}{\beta}}}\|\Lambda(u-v)\|_{L^2}) \\
&\quad\leq C\|v\|_{\dot{H}^{1+\beta/2}}(\|\Lambda v\|_{L^2}\|u-v\|_{L^{\frac{2d}{\beta}}}+\|v\|_{L^{\frac{2d}{\beta}}}\|\Lambda(u-v)\|_{L^2}). \qquad (4.6)
\end{aligned}$$

Recalling the inequalities

$$\|\partial_{x_i}\partial_{x_j}f\|_{L^p}\leq C\|\Delta f\|_{L^p}, \quad \forall\, 1<p<\infty, \qquad (4.7)$$

$$\|\partial_{x_i}f\|_{L^p}\leq C\|\Lambda f\|_{L^p}, \quad \forall\, 1<p<\infty, \qquad (4.8)$$

and Lemma B.2, we have

$$\|[\Lambda,\nabla\psi]\nabla u\|_{L^{\frac{2d}{d+\alpha}}}\leq C\|\Delta\psi\|_{L^{\frac{2d}{\alpha}}}\|\Lambda u\|_{L^2}, \qquad (4.9)$$

$$\|[\Lambda,\nabla\psi]\nabla v\|_{L^{\frac{2d}{d+\beta}}}\leq C\|\Delta\psi\|_{L^{\frac{2d}{\beta}}}\|\Lambda v\|_{L^2}. \qquad (4.10)$$

Thus, due to (4.9) and (4.9), we have that

$$\begin{aligned}
\left|\int_{\mathbb{R}^d} \Lambda u[\Lambda,\nabla\psi]\cdot\nabla u\mathrm{d}x\right| &\leq C\|\Lambda u\|_{L^{\frac{2d}{d-\alpha}}}\|u-v\|_{L^{\frac{2d}{\alpha}}}\|\Lambda u\|_{L^2} \\
&\leq C\|u\|_{\dot{H}^{1+\alpha/2}}\|u-v\|_{L^{\frac{2d}{\alpha}}}\|\Lambda u\|_{L^2}, \qquad (4.11) \\
\left|\int_{\mathbb{R}^d} \Lambda u[\Lambda,\nabla\psi]\cdot\nabla v\mathrm{d}x\right| &\leq C\|\Lambda v\|_{L^{\frac{2d}{d-\beta}}}\|u-v\|_{L^{\frac{2d}{\beta}}}\|\Lambda v\|_{L^2} \\
&\leq C\|v\|_{\dot{H}^{1+\beta/2}}\|u-v\|_{L^{\frac{2d}{\beta}}}\|\Lambda v\|_{L^2}. \qquad (4.12)
\end{aligned}$$

Collecting the terms (4.3), (4.4), (4.5), (4.6), (4.11) and (4.12) and using Young's inequality, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^1}^2+\|v\|_{\dot{H}^1}^2\right)&\leq-\|u\|_{\dot{H}^{1+\alpha/2}}^2-\|v\|_{\dot{H}^{1+\beta/2}}^2+C\Big(\|\Lambda u\|_{L^2}^2\|u-v\|_{L^{\frac{2d}{\alpha}}}^2 \\
&\quad+\|u\|_{L^{\frac{2d}{\alpha}}}^2\|\Lambda(u-v)\|_{L^2}^2+\|\Lambda v\|_{L^2}^2\|u-v\|_{L^{\frac{2d}{\beta}}}^2 \\
&\quad+\|v\|_{L^{\frac{2d}{\beta}}}^2\|\Lambda(u-v)\|_{L^2}^2\Big).
\end{aligned}$$

Using the interpolation inequality (4.1), we conclude that, for $t\geq T^*$ and $T^*<\infty$ large enough (Theorem 2.1),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^1}^2+\|v\|_{\dot{H}^1}^2\right)&\leq C\Big(\|u\|_{L^2}^2\|u-v\|_{L^{\frac{2d}{\alpha}}}^2+\|u\|_{L^{\frac{2d}{\alpha}}}^2\|u-v\|_{L^2}^2 \\
&\quad+\|v\|_{L^2}^2\|u-v\|_{L^{\frac{2d}{\beta}}}^2+\|v\|_{L^{\frac{2d}{\beta}}}^2\|u-v\|_{L^2}^2\Big) \\
&\quad-\frac{1}{2}\left(\|u\|_{\dot{H}^{1+\alpha/2}}^2+\|v\|_{\dot{H}^{1+\beta/2}}^2\right).
\end{aligned}$$

Another application of Theorem 2.1 leads to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^1}^2+\|v\|_{\dot{H}^1}^2\right)+\frac{1}{2}\left(\|u\|_{\dot{H}^{1+\alpha/2}}^2+\|v\|_{\dot{H}^{1+\beta/2}}^2\right) \\
&\quad\leq C\left(\|v\|_{L^2}^2+\|u\|_{L^2}^2\right)\Big(\|v\|_{L^{\frac{2d}{\alpha}}}^2+\|u\|_{L^{\frac{2d}{\alpha}}}^2+\|u\|_{L^{\frac{2d}{\beta}}}^2+\|v\|_{L^{\frac{2d}{\beta}}}^2\Big)
\end{aligned}$$

$$\leq \frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}}}\left(\frac{1}{(1+t)^{\frac{2d(1-\frac{\alpha}{2d})}{\max\{\alpha,\beta\}}}}+\frac{1}{(1+t)^{\frac{2d(1-\frac{\beta}{2d})}{\max\{\alpha,\beta\}}}}\right)$$
$$\leq C\left(\frac{1}{(1+t)^{\frac{3d-\max\{\alpha,\beta\}}{\max\{\alpha,\beta\}}}}+\frac{1}{(1+t)^{\frac{3d-\beta}{\max\{\alpha,\beta\}}}}\right)$$
$$\leq \frac{C}{(1+t)^{\frac{3d-\max\{\alpha,\beta\}}{\max\{\alpha,\beta\}}}}.$$

Using Theorem 2.1 with $\max\{\alpha,\beta\}<2$, we obtain the inequality

$$1<\frac{2}{\max\{\alpha,\beta\}}\leq\frac{d}{\max\{\alpha,\beta\}}, \tag{4.13}$$

thus, we have that

$$2<\frac{3d}{\max\{\alpha,\beta\}}-1. \tag{4.14}$$

Integrating in time, we obtain

$$\|u(t)\|^2_{\dot H^1}+\|v(t)\|^2_{\dot H^1}+\frac{1}{2}\int_{T^*}^t\|u\|^2_{\dot H^{1+\alpha/2}}+\|v\|^2_{\dot H^{1+\beta/2}}\mathrm{d}s\leq C,\quad \forall\, t\geq T^*,$$

Taking then the maximum of the norms on the finite interval $[0,T^*]$, we obtain

$$\|u(t)\|^2_{\dot H^1}+\|v(t)\|^2_{\dot H^1}\leq C,\,\forall\, t\geq 0, \tag{4.15}$$

Step 3: Decay in H^s

Sobolev interpolation

$$\|f\|_{\dot H^s}\leq C\|f\|^{\frac{r-s}{r}}_{L^2}\|f\|^{\frac{s}{r}}_{\dot H^r}, \tag{4.16}$$

(with $r=1$) gives us the following decay in the intermediate spaces $\dot H^s$ for every $0\leq s<1$

$$\|u(t)\|_{\dot H^s}+\|v(t)\|_{\dot H^s}\leq\frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}\frac{1-s}{2}}},\quad \forall\, t\geq 0, \tag{4.17}$$

5. Proof of Theorem 2.4: Decay Estimates in Sobolev Spaces H^s, $0\leq s<2$

Step 1: Boundedness in H^2 Testing against $(-\Delta)^2u$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot H^2}&=-\|u\|^2_{\dot H^{2+\alpha/2}}-\int_{\mathbb{R}^d}\Delta u\Delta(\nabla u\cdot\nabla\psi)\mathrm{d}x-\int_{\mathbb{R}^d}\Delta u\Delta(u(u-v))\mathrm{d}x\\&=-\|u\|^2_{\dot H^{2+\alpha/2}}-\int_{\mathbb{R}^d}\Delta u[\Delta,\nabla\psi]\cdot\nabla u\mathrm{d}x-\int_{\mathbb{R}^d}\Delta u\nabla\psi\cdot\nabla\Delta u\mathrm{d}x\\&\quad-\int_{\mathbb{R}^d}\Delta u\Delta(u(u-v))\mathrm{d}x\\&=-\|u\|^2_{\dot H^{2+\alpha/2}}-\int_{\mathbb{R}^d}\Delta u[\Delta,\nabla\psi]\cdot\nabla u\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d}|\Delta u|^2(u-v)\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^d} \Delta u \left(\Delta u(u-v) + u\Delta(u-v) + 2\nabla u \cdot \nabla(u-v)\right) \mathrm{d}x \\
&= -\|u\|_{\dot{H}^{2+\alpha/2}}^2 - \int_{\mathbb{R}^d} \Delta u[\Delta, \nabla\psi] \cdot \nabla u \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^d} |\Delta u|^2 (u-v)\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \Delta u \left(u\Delta(u-v) + 2\nabla u \cdot \nabla(u-v)\right) \mathrm{d}x \\
&\leq -\|u\|_{\dot{H}^{2+\alpha/2}}^2 - \int_{\mathbb{R}^d} \Delta u[\Delta, \nabla\psi] \cdot \nabla u \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} |\Delta u|^2 v\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \Delta u \left(u\Delta v + 2\nabla u \cdot \nabla(u-v)\right) \mathrm{d}x,
\end{aligned}
$$

In the same way

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{\dot{H}^2}^2 &= -\|v\|_{\dot{H}^{2+\beta/2}}^2 + \int_{\mathbb{R}^d} \Delta v \Delta(\nabla v \cdot \nabla\psi)\mathrm{d}x + \int_{\mathbb{R}^d} \Delta v\Delta(v(u-v))\mathrm{d}x \\
&\leq -\|v\|_{\dot{H}^{2+\beta/2}}^2 + \int_{\mathbb{R}^d} \Delta v[\Delta, \nabla\psi] \cdot \nabla v \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^d} |\Delta v|^2 u\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \Delta v \left(v\Delta u + 2\nabla v \cdot \nabla(u-v)\right) \mathrm{d}x.
\end{aligned}
$$

We collect these estimates and use Hölder inequality to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^2}^2 + \|v\|_{\dot{H}^2}^2\right) \leq & -\|u\|_{\dot{H}^{2+\alpha/2}}^2 + \|\Delta u\|_{L^{\frac{2d}{d-\alpha}}} \|[\Delta, \nabla\psi] \cdot \nabla u\|_{L^{\frac{2d}{d+\alpha}}} \\
& + \frac{1}{2}\|\Delta u\|_{L^{\frac{2d}{d-\alpha}}}^2 \|v\|_{L^{\frac{d}{\alpha}}} + \frac{1}{2}\|\Delta v\|_{L^{\frac{2d}{d-\beta}}}^2 \|u\|_{L^{\frac{d}{\beta}}} \\
& + 2\|\Delta u\|_{L^{\frac{2d}{d-\alpha}}} \|\nabla u\|_{L^{\frac{4d}{d+\alpha}}} \|\nabla(u-v)\|_{L^{\frac{4d}{d+\alpha}}} \\
& + 2\|\Delta v\|_{L^{\frac{2d}{d-\beta}}} \|\nabla v\|_{L^{\frac{4d}{d+\beta}}} \|\nabla(u-v)\|_{L^{\frac{4d}{d+\beta}}} \\
& - \|v\|_{\dot{H}^{2+\beta/2}}^2 + \|\Delta v\|_{L^{\frac{2d}{d-\beta}}} \|[\Delta, \nabla\psi] \cdot \nabla v\|_{L^{\frac{2d}{d+\beta}}} \\
& + \|\Delta u\|_{L^{\frac{2d}{d-\alpha}}} \|\Delta v\|_{L^{\frac{2d}{d-\beta}}} \|u+v\|_{L^{\frac{2d}{\alpha+\beta}}}.
\end{aligned}
$$

Due the Sobolev embedding (4.2), we have that, for $t \geq T^*$ and $T^* < \infty$ large enough, the previous inequality simplifies to

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^2}^2 + \|v\|_{\dot{H}^2}^2\right) \leq & -2\|u\|_{\dot{H}^{2+\alpha/2}}^2 + C\|u\|_{\dot{H}^{2+\alpha/2}} \|[\Delta, \nabla\psi] \cdot \nabla u\|_{L^{\frac{2d}{d+\alpha}}} \\
& + C\|u\|_{\dot{H}^{2+\alpha/2}} \|\nabla u\|_{L^{\frac{4d}{d+\alpha}}} \|\nabla(u-v)\|_{L^{\frac{4d}{d+\alpha}}} \\
& + C\|v\|_{\dot{H}^{2+\beta/2}} \|\nabla v\|_{L^{\frac{4d}{d+\beta}}} \|\nabla(u-v)\|_{L^{\frac{4d}{d+\beta}}} \\
& - 2\|v\|_{\dot{H}^{2+\beta/2}}^2 + C\|v\|_{\dot{H}^{2+\beta/2}} \|[\Delta, \nabla\psi] \cdot \nabla v\|_{L^{\frac{2d}{d+\beta}}}.
\end{aligned}
$$

Lemma B.2 together with (4.7) and (4.8) gives us the following estimates:

$$
\begin{aligned}
&\|[\Delta, \nabla\psi]\nabla u\|_{L^{\frac{2d}{d+\alpha}}} \\
&\quad \leq C\left(\|u-v\|_{L^{\frac{2d}{\alpha}}} \|\Delta u\|_{L^2} + \|\nabla(u-v)\|_{L^{\frac{4d}{d+\alpha}}} \|\nabla u\|_{L^{\frac{4d}{d+\alpha}}}\right), \qquad (5.1)
\end{aligned}
$$

$$\|[\Delta, \nabla\psi]\nabla v\|_{L^{\frac{2d}{d+\beta}}} \leq C\left(\|u-v\|_{L^{\frac{2d}{\beta}}}\|\Delta v\|_{L^2} + \|\nabla(u-v)\|_{L^{\frac{4d}{d+\beta}}}\|\nabla v\|_{L^{\frac{4d}{d+\beta}}}\right). \tag{5.2}$$

Consequently, due to the interpolation inequality (4.1) with $r = 2$, we can further simplify and get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{\dot{H}^2}^2 + \|v\|_{\dot{H}^2}^2\right) \leq & -\|u\|_{\dot{H}^{2+\alpha/2}}^2 - \|v\|_{\dot{H}^{2+\beta/2}}^2 \\ & + C\|u-v\|_{L^{\frac{2d}{\beta}}}^2\|v\|_{L^2}^2 + C\|u-v\|_{L^{\frac{2d}{\alpha}}}^2\|u\|_{L^2}^2 \\ & + C\|u\|_{\dot{H}^{2+\alpha/2}}\|\nabla u\|_{L^{\frac{4d}{d+\alpha}}}\|\nabla(u-v)\|_{L^{\frac{4d}{d+\alpha}}} \\ & + C\|v\|_{\dot{H}^{2+\beta/2}}\|\nabla v\|_{L^{\frac{4d}{d+\beta}}}\|\nabla(u-v)\|_{L^{\frac{4d}{d+\beta}}}.\end{aligned}$$

Using the Sobolev embedding

$$\|\nabla f\|_{L^{\frac{4d}{d+s}}} \leq C\|f\|_{\dot{H}^{1+\frac{d-s}{4}}} \leq C\|\Lambda^{1-\frac{s}{4}}f\|_{\dot{H}^{\frac{d}{4}}},$$

and the interpolation inequality (4.16), we have that

$$\begin{aligned} I_1 &= \|u\|_{\dot{H}^{2+\alpha/2}}\|\nabla u\|_{L^{\frac{4d}{d+\alpha}}}\|\nabla(u-v)\|_{L^{\frac{4d}{d+\alpha}}} \\ &\leq C\|u\|_{\dot{H}^{2+\alpha/2}}\|\Lambda^{1-\frac{\alpha}{4}}u\|_{\dot{H}^{\frac{d}{4}}}\left(\|\Lambda^{1-\frac{\alpha}{4}}u\|_{\dot{H}^{\frac{d}{4}}} + \|\Lambda^{1-\frac{\alpha}{4}}v\|_{\dot{H}^{\frac{d}{4}}}\right) \\ &\leq C\|u\|_{\dot{H}^{2+\alpha/2}}^{1+\frac{d}{4+3\alpha}}\|\Lambda^{1-\frac{\alpha}{4}}u\|_{L^2}^{1-\frac{d}{4+3\alpha}}\left(\|\Lambda^{1-\frac{\alpha}{4}}u\|_{\dot{H}^{\frac{d}{4}}} + \|\Lambda^{1-\frac{\alpha}{4}}v\|_{\dot{H}^{\frac{d}{4}}}\right) \\ &\leq C\|u\|_{\dot{H}^{2+\alpha/2}}^{1+\frac{d}{4+3\alpha}}\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+3\alpha}} \\ &\quad\times\left(\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+3\alpha}}\|u\|_{\dot{H}^{2+\alpha/2}}^{\frac{d}{4+3\alpha}} + \|v\|_{\dot{H}^{2+\beta/2}}^{\frac{d}{4+\alpha+2\beta}}\|v\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+\alpha+2\beta}}\right), \end{aligned}$$

and

$$\begin{aligned} I_2 &= \|v\|_{\dot{H}^{2+\beta/2}}\|\nabla v\|_{L^{\frac{4d}{d+\beta}}}\|\nabla(u-v)\|_{L^{\frac{4d}{d+\beta}}} \\ &\leq C\|v\|_{\dot{H}^{2+\beta/2}}\|\Lambda^{1-\frac{\beta}{4}}v\|_{\dot{H}^{\frac{d}{4}}}\left(\|\Lambda^{1-\frac{\beta}{4}}u\|_{\dot{H}^{\frac{d}{4}}} + \|\Lambda^{1-\frac{\beta}{4}}v\|_{\dot{H}^{\frac{d}{4}}}\right) \\ &\leq C\|v\|_{\dot{H}^{2+\beta/2}}^{1+\frac{d}{4+3\beta}}\|v\|_{\dot{H}^{1-\frac{\beta}{4}}}^{1-\frac{d}{4+3\beta}} \\ &\quad\times\left(\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+\alpha+2\beta}}\|u\|_{\dot{H}^{2+\alpha/2}}^{\frac{d}{4+\alpha+2\beta}} + \|v\|_{\dot{H}^{2+\beta/2}}^{\frac{d}{4+3\beta}}\|v\|_{\dot{H}^{1-\frac{\beta}{4}}}^{1-\frac{d}{4+3\beta}}\right). \end{aligned}$$

We write

$$I_1 + I_2 = J_1 + J_2 + J_3 + J_4,$$

where

$$\begin{aligned} J_1 &= C\|u\|_{\dot{H}^{2+\alpha/2}}^{1+\frac{2d}{4+3\alpha}}\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{2-\frac{2d}{4+3\alpha}} \\ J_2 &= C\|v\|_{\dot{H}^{2+\beta/2}}^{1+\frac{2d}{4+3\beta}}\|v\|_{\dot{H}^{1-\frac{\beta}{4}}}^{2-\frac{2d}{4+3\beta}} \end{aligned}$$

$$J_3 = C\|u\|_{\dot{H}^{2+\alpha/2}}^{1+\frac{d}{4+3\alpha}}\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+3\alpha}}\|v\|_{\dot{H}^{2+\beta/2}}^{\frac{d}{4+\alpha+2\beta}}\|v\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+\alpha+2\beta}}$$
$$J_4 = C\|v\|_{\dot{H}^{2+\beta/2}}^{1+\frac{d}{4+3\beta}}\|v\|_{\dot{H}^{1-\frac{\beta}{4}}}^{1-\frac{d}{4+3\beta}}\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{1-\frac{d}{4+\alpha+2\beta}}\|u\|_{\dot{H}^{2+\alpha/2}}^{\frac{d}{4+\alpha+2\beta}}$$

Using hypothesis (2.2), so that

$$\frac{2d}{4+3\min\{\alpha,\beta\}}<1,$$

we can apply Young's inequality with

$$p=2-\frac{4d}{4+3\alpha+2d},\ q=2+\frac{4d}{4+3\alpha-2d}$$

and, recalling Theorem 2.3, we obtain that

$$\begin{aligned}J_1 &\leq \frac{1}{4}\|u\|_{\dot{H}^{2+\alpha/2}}^2 + C\|u\|_{\dot{H}^{1-\frac{\alpha}{4}}}^{(2-\frac{2d}{4+3\alpha})q}\\ &\leq \frac{1}{4}\|u\|_{\dot{H}^{2+\alpha/2}}^2 + \frac{C}{(1+t)^{\theta_1}},\end{aligned}$$

where

$$\theta_1 = \frac{d}{\max\{\alpha,\beta\}}\frac{\alpha}{8}\left(2-\frac{2d}{4+3\alpha}\right)q = \frac{d}{\max\{\alpha,\beta\}}\frac{\alpha}{4}\frac{8+6\alpha-2d}{4+3\alpha-2d}. \tag{5.3}$$

We need to have $\theta>1$. Then, in the case where $\beta=\max\{\alpha,\beta\}$ and $\alpha\ll 1$, the previous exponent may be arbitrarily small. However, in the case where (2.3) holds, we have that

$$\theta_1 \geq \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}}\frac{d}{4}\left(2+\frac{2d}{4+3\max\{\alpha,\beta\}-2d}\right)>1.$$

Applying Young's inequality now with

$$p=2-\frac{4d}{4+3\beta+2d},\ q=2+\frac{4d}{4+3\beta-2d},$$

we have that

$$\begin{aligned}J_2 &\leq \frac{1}{4}\|v\|_{\dot{H}^{2+\beta/2}}^2 + C\|v\|_{\dot{H}^{1-\frac{\beta}{4}}}^{(2-\frac{2d}{4+3\beta})q}\\ &\leq \frac{1}{4}\|v\|_{\dot{H}^{2+\alpha/2}}^2 + \frac{C}{(1+t)^{\theta_2}},\end{aligned}$$

where

$$\theta_2 = \frac{d}{\max\{\alpha,\beta\}}\frac{\beta}{8}\left(2-\frac{2d}{4+3\beta}\right)q = \frac{d}{\max\{\alpha,\beta\}}\frac{\beta}{4}\frac{8+6\beta-2d}{4+3\beta-2d}. \tag{5.4}$$

Thus, using hypothesis (2.3), we have that

$$\theta_2>1.$$

Using again Young's inequality with

$$p = 2 - \frac{2d}{4+3\alpha+d},\ q = 2 + \frac{2d}{4+3\alpha-d}$$
$$J_3 \leq \frac{1}{4}\|u\|^2_{\dot H^{2+\alpha/2}} + C\|u\|^{(1-\frac{d}{4+3\alpha})q}_{\dot H^{1-\frac{\alpha}{4}}}\|v\|^{\lambda}_{\dot H^{2+\beta/2}}\|v\|^{(1-\frac{d}{4+\alpha+2\beta})q}_{\dot H^{1-\frac{\alpha}{4}}}.$$

Due to hypothesis (2.4), the exponent is

$$\begin{aligned}\lambda &= \frac{d}{4+\alpha+2\beta}q\\ &= \frac{d}{4+3\alpha-d}\left(2+\frac{4(\alpha-\beta)}{4+\alpha+2\beta}\right)\\ &\leq \frac{d}{4+3\min\{\alpha,\beta\}-d}\left(2+\frac{4|\alpha-\beta|}{4+3\min\{\alpha,\beta\}}\right)\\ &\leq 2.\end{aligned}$$

Assume that $\lambda < 2$ (if $\Gamma = 2$, we can finish with J_3 straightforwardly by waiting for a large enough time and applying Theorem 2.3), thus we can apply Young's inequality again

$$P = \frac{2}{\lambda},\ Q = \frac{2}{2-\lambda}$$

and obtain

$$\begin{aligned}J_3 &\leq \frac{1}{4}\|u\|^2_{\dot H^{2+\alpha/2}} + \frac{1}{4}\|v\|^2_{\dot H^{2+\beta/2}} + C\|u\|^{(1-\frac{d}{4+3\alpha})qQ}_{\dot H^{1-\frac{\alpha}{4}}}\|v\|^{(1-\frac{d}{4+\alpha+2\beta})qQ}_{\dot H^{1-\frac{\alpha}{4}}}\\ &\leq \frac{1}{4}\|u\|^2_{\dot H^{2+\alpha/2}} + \frac{1}{4}\|v\|^2_{\dot H^{2+\beta/2}} + \frac{C}{(1+t)^{\theta_3}}.\end{aligned}$$

Notice that the condition

$$\left(1-\frac{d}{4+3\alpha}\right)qQ > 2$$

is implied by the stricter condition

$$\left(1-\frac{d}{4+3\alpha}\right)Q > 1,$$

or, equivalently,

$$\frac{2}{4+3\alpha} < \frac{q}{4+\alpha+2\beta} = \frac{2+\frac{2d}{4+3\alpha-d}}{4+\alpha+2\beta}.$$

A further computation shows that this latter condition is implied by hypothesis (2.5)

$$\frac{4+\alpha+2\beta}{4+3\alpha} \leq \frac{4+3\max\{\alpha,\beta\}}{4+3\min\{\alpha,\beta\}} < 1 + \frac{d}{4+3\max\{\alpha,\beta\}-d}.$$

Then, using Theorem 2.3, the integrability condition $\theta_3 > 1$ is implied by

$$\frac{d}{\max\{\alpha,\beta\}}\frac{\alpha}{2} \geq 1,$$

and hypothesis (2.6).

The term J_4 is akin to J_3 and can be handled similarly. Then, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\dot{H}^2}+\|v\|^2_{\dot{H}^2}\right)\leq \frac{C}{(1+t)^{\Theta}},\ \forall\, t\geq T^*,$$

and $\Theta>1$. Thus,

$$\|u(t)\|^2_{\dot{H}^2}+\|v(t)\|^2_{\dot{H}^2}\leq C,\ \forall\, t\geq 0.$$

Step 2: Decay in H^s Using (4.16), we have

$$\|u(t)\|_{\dot{H}^s}+\|v(t)\|_{\dot{H}^s}\leq \frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}\frac{2-s}{4}}},\ \forall\, t\geq 0, \tag{5.5}$$

6. Proof of Theorem 2.5: Decay Estimates in Sobolev Spaces H^s, $s>2$

Let us fix $\delta=2-d/2$. Testing against $\Lambda^{2s}u$, we have the following estimate:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot{H}^s}&=-\|u\|^2_{\dot{H}^{s+\alpha/2}}-\int_{\mathbb{R}^d}\Lambda^s u\Lambda^s\nabla\cdot(u\nabla\psi)\mathrm{d}x\\&=-\|u\|^2_{\dot{H}^{s+\alpha/2}}-\int_{\mathbb{R}^d}\Lambda^s u[\Lambda^s\nabla,\nabla\psi]u\mathrm{d}x-\int_{\mathbb{R}^d}\Lambda^s u\Lambda^s\nabla u\cdot\nabla\psi\mathrm{d}x\\&=-\|u\|^2_{\dot{H}^{s+\alpha/2}}-\int_{\mathbb{R}^d}\Lambda^s u[\Lambda^s\nabla,\nabla\psi]u\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^d}|\Lambda^s u|^2(u-v)\mathrm{d}x.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\dot{H}^s}&=-\|v\|^2_{\dot{H}^{s+\beta/2}}+\int_{\mathbb{R}^d}\Lambda^s v\Lambda^s\nabla\cdot(v\nabla\psi)\mathrm{d}x\\&=-\|v\|^2_{\dot{H}^{s+\beta/2}}+\int_{\mathbb{R}^d}\Lambda^s v[\Lambda^s\nabla,\nabla\psi]v\mathrm{d}x+\int_{\mathbb{R}^d}\Lambda^s v\Lambda^s\nabla v\cdot\nabla\psi\mathrm{d}x\\&=-\|v\|^2_{\dot{H}^{s+\beta/2}}+\int_{\mathbb{R}^d}\Lambda^s v[\Lambda^s\nabla,\nabla\psi]v\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}|\Lambda^s v|^2(u-v)\mathrm{d}x\end{aligned}$$

Using Lemma B.1, we have that

$$\begin{aligned}\|[\Lambda^s\nabla,\nabla\psi]f\|_{L^2}&\leq C\left(\|\Lambda^s f\|_{L^2}\|\widehat{\Lambda\nabla\psi}\|_{L^1}+\|\Lambda^{s+1}\nabla\psi\|_{L^2}\|\widehat{f}\|_{L^1}\right)\\&\leq C\left(\|\Lambda^s f\|_{L^2}\|\widehat{\Delta\psi}\|_{L^1}+\|\Lambda^s\Delta\psi\|_{L^2}\|\widehat{f}\|_{L^1}\right)\\&\leq C\left(\|\Lambda^s f\|_{L^2}\left(\|\hat{u}\|_{L^1}+\|\hat{v}\|_{L^1}\right)\right.\\&\qquad\left.+\left(\|\Lambda^s u\|_{L^2}+\|\Lambda^s v\|_{L^2}\right)\|\widehat{f}\|_{L^1}\right).\end{aligned}$$

That means that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2_{\dot{H}^s}+\|v\|^2_{\dot{H}^s}\right)&\leq -2\|u\|^2_{\dot{H}^{s+\alpha/2}}-2\|v\|^2_{\dot{H}^{s+\beta/2}}\\&\quad+C\left(\|u\|^2_{\dot{H}^s}\left(\|\hat{u}\|_{L^1}+\|\hat{v}\|_{L^1}\right)\right.\\&\quad+\left(\|u\|^2_{\dot{H}^s}+\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s}\right)\|\widehat{u}\|_{L^1}\end{aligned}$$

$$+ \|v\|^2_{\dot{H}^s} \left(\|\hat{u}\|_{L^1} + \|\hat{v}\|_{L^1}\right)$$
$$+ \left(\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s} + \|v\|^2_{\dot{H}^s}\right)\|\widehat{v}\|_{L^1}\right),$$

where we have used the inequality

$$\|f\|_{L^\infty} \leq \|\hat{f}\|_{L^1}.$$

We obtain that

$$\frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\dot{H}^s} + \frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_{\dot{H}^s}\right) \leq -\|u\|^2_{\dot{H}^{s+\alpha/2}} - \|v\|^2_{\dot{H}^{s+\beta/2}}$$
$$+ C\left(\|v\|^2_{\dot{H}^s} + \|u\|^2_{\dot{H}^s}\right)\left(\|\hat{u}\|_{L^1} + \|\hat{v}\|_{L^1}\right). \quad (6.1)$$

Using Lemma C.1 (inequality (C.2)) and Theorem 2.4, we have that

$$\|\hat{u}\|_{L^1(\mathbb{R}^d)} + \|\hat{v}\|_{L^1(\mathbb{R}^d)} \leq C(1+t)^{-\frac{\delta}{2}\frac{d/2}{\max\{\alpha,\beta\}}}\left(\|u\|^{\frac{d/2}{2}}_{\dot{H}^2(\mathbb{R}^d)} + \|v\|^{\frac{d/2}{2}}_{\dot{H}^2(\mathbb{R}^d)}\right)$$
$$\leq C(1+t)^{-\frac{\delta}{2}\frac{d/2}{\max\{\alpha,\beta\}}}.$$

Thus, waiting for a large enough time T^* and using (4.1) and

$$\int_{T^*}^t \|u\|^2_{L^2} + \|v\|^2_{L^2} = C < \infty,$$

we conclude

$$\|u\|^2_{\dot{H}^s} + \|v\|^2_{\dot{H}^s} \leq C, \quad \forall\, t \geq T^*.$$

Considering the a priori estimates in the finite interval $[0, T^*]$, we conclude

$$\|u\|^2_{\dot{H}^s} + \|v\|^2_{\dot{H}^s} \leq C, \quad \forall\, t \geq 0. \quad (6.2)$$

Now, the decay follows from interpolation (4.16).

7. Proof of Proposition 2.6: Asymptotic Profile

Step 1: Decay of the potential ψ

This step is similar to the one in [38]. We have

$$\psi = \Delta^{-1}(u - v),$$

so, using Theorem 2.1, we have that

$$\nabla\psi = C_d \int_{\mathbb{R}^d} \frac{x_i - y_i}{|x-y|^d}(u(y) - v(y))\mathrm{d}y$$
$$= C_d\left(\int_{|y|\leq(1+t)^r} + \int_{|y|>(1+t)^r}\right)\frac{x_i - y_i}{|x-y|^d}(u(y) - v(y))\mathrm{d}y$$
$$\leq C\left(\|u\|_{L^\infty} + \|v\|_{L^\infty}\right)(1+t)^r + \left(\|u\|_{L^1} + \|v\|_{L^1}\right)(1+t)^{r(d-1)}$$
$$\leq C\,(1+t)^{r-\frac{d}{\max\{\alpha,\beta\}}} + C(1+t)^{-r(d-1)}.$$

We choose $r = \frac{1}{\max\{\alpha,\beta\}}$ and, thus, we obtain

$$\|\nabla\psi\|_{L^\infty} \leq \frac{C}{(1+t)^{\frac{d-1}{\max\{\alpha,\beta\}}}}. \quad (7.1)$$

Step 2: Mild solution Using Duhamel's principle, the mild solutions are given by

$$u(t) - \mathrm{e}^{-t\Lambda^\alpha}u_0 = -\int_0^t \mathrm{e}^{-(t-s)\Lambda^\alpha}\nabla\cdot(u(s)\nabla\psi(s))\mathrm{d}s$$
$$v(t) - \mathrm{e}^{-t\Lambda^\beta}v_0 = \int_0^t \mathrm{e}^{-(t-s)\Lambda^\beta}\nabla\cdot(v(s)\nabla\psi(s))\mathrm{d}s.$$

Step 3: Estimate on the difference Using the hypercontractive inequality

$$\|\mathrm{e}^{-t\Lambda^\alpha}h\|_{L^2} \leq Ct^{-\frac{d}{2\alpha}}\|h\|_{L^1}$$

we have that

$$\begin{aligned}\|u(t) - \mathrm{e}^{-t\Lambda^\alpha}u_0\|_{L^2} &= \left\|\int_0^t \mathrm{e}^{-(t-s)\Lambda^\alpha}\nabla\cdot(u(s)\nabla\psi(s))\mathrm{d}s\right\|_{L^2}\\ &\leq C\int_0^{t/2}\frac{f(s)}{(t-s)^{\frac{d}{2\alpha}}}\mathrm{d}s + C\int_{t/2}^t g(s)\mathrm{d}s,\end{aligned}$$

where the forcing is

$$\begin{aligned}f(s) &= \|u(s)(u(s)-v(s))\|_{L^1}\\ &\leq \frac{C}{(1+s)^{\frac{d}{\max\{\alpha,\beta\}}}}\end{aligned}$$

and

$$\begin{aligned}g(s) &= \|\nabla u(s)\cdot\nabla\psi(s)\|_{L^2} + \|u(s)(u(s)-v(s))\|_{L^2}\\ &\leq \frac{C}{(1+s)^{\frac{d-1}{\max\{\alpha,\beta\}}}} + \frac{C}{(1+s)^{\frac{3}{4}\frac{d}{\max\{\alpha,\beta\}}}},\end{aligned}$$

Thus, using

$$\begin{aligned}&\int_0^{t/2}\frac{C}{(t-s)^{\frac{d}{2\alpha}}(1+s)^{\frac{d}{\max\{\alpha,\beta\}}}}\mathrm{d}s\\ &\quad\leq \frac{C}{t^{\frac{d}{2\alpha}}}\int_0^{t/2}\frac{C}{(1+s)^{\frac{d}{\max\{\alpha,\beta\}}}}\mathrm{d}s \leq \frac{C}{t^{\frac{d}{2\alpha}}}\frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}-1}}\\ &\quad\times\|u(t)-\mathrm{e}^{-t\Lambda^\alpha}u_0\|_{L^2}\\ &\quad\leq \frac{C}{(1+t)^{\frac{d}{\max\{\alpha,\beta\}}-1}t^{\frac{d}{2\alpha}}} + \frac{C}{(1+t)^{\frac{d-1}{\max\{\alpha,\beta\}}-1}} + \frac{C}{(1+t)^{\frac{3}{4}\frac{d}{\max\{\alpha,\beta\}}-1}}\\ &\quad\leq \frac{C}{(1+t)^{\frac{d-1}{\max\{\alpha,\beta\}}-1}}\end{aligned}$$

In the same way,

$$\|v(t) - \mathrm{e}^{-t\Lambda^\beta}v_0\|_{L^2} \leq \frac{C}{(1+t)^{\frac{d-1}{\max\{\alpha,\beta\}}-1}}$$

Acknowledgments

The author is partially supported by the Grant MTM2014-59488-P from the former Ministerio de Economía y Competitividad (MINECO, Spain).

Appendix A. Inequalities for the Fractional Laplacian

In this appendix, we recall several inequalities involving the fractional Laplacian.

Lemma A.1. *Let* $h \in \mathcal{S}(\mathbb{R}^d)$ *be a Schwartz function. We write* $h(x^*) := \max_x h(x)$, $h(x_*) := \min_x h(x)$ *and*

$$\|h\|_{L^p \cap L^\infty} := \max\{\|h\|_{L^p}, \|h\|_{L^\infty}\}.$$

Then

- *if* $h(x^*) > 0$,

$$\Lambda^\alpha h(x^*) \geq c(d,\alpha,p)\frac{h(x^*)^{1+\alpha p/d}}{\|h\|_{L^p}^{\alpha p/d}},$$

- *if* $h(x_*) < 0$,

$$\Lambda^\alpha h(x_*) \leq c(d,\alpha,p)\frac{h(x_*)|h(x_*)|^{\alpha p/d}}{\|h\|_{L^p}^{\alpha p/d}},$$

These bounds imply the norm

$$\|\mathrm{e}^{-\Lambda^\alpha t}\|_{L^p\cap L^\infty(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq \frac{C(d,\alpha,p)}{(t+1)^{d/(\alpha p)}}.$$

Proof. **Step 1** Let us assume that h takes both signs. Then, we have $h(x^*) = \max_x h(x) > 0$. We take $r > 0$ a positive number and define

$$\mathcal{U}_1 = \{\eta \in B(0,r) \ s.t. \ h(x^*) - h(x^* - \eta) > h(x^*)/2\},$$

and $\mathcal{U}_2 = B(0,r) - \mathcal{U}_1$. We have

$$\|h\|_{L^p}^p = \int_{\mathbb{R}^d} |h(x^*-\eta)|^p \mathrm{d}\eta \geq \int_{\mathcal{U}_2} |h(x^*-\eta)|^p \mathrm{d}\eta \geq \frac{|h(x^*)|^p}{2^p}|\mathcal{U}_2|,$$

so,

$$-\left(\frac{2\|h\|_{L^p}}{|h(x^*)|}\right)^p \leq -|\mathcal{U}_2|. \tag{A.1}$$

$$\begin{aligned}\Lambda^\alpha h(x^*) &= c_{\alpha,d}\mathrm{P.V.}\int_{\mathbb{R}^d}\frac{h(x^*)-h(x^*-\eta)}{|\eta|^{d+\alpha}}\mathrm{d}\eta \\ &\geq c_{\alpha,d}\mathrm{P.V.}\int_{\mathcal{U}_1}\frac{h(x^*)-h(x^*-\eta)}{|\eta|^{d+\alpha}}\mathrm{d}\eta \\ &\geq c_{\alpha,d}\frac{h(x^*)}{2r^{d+\alpha}}|\mathcal{U}_1|\end{aligned}$$

$$\geq c_{\alpha,d}\frac{h(x^*)}{2r^{d+\alpha}}\left(\omega_d r^d-|\mathcal{U}_2|\right)$$
$$\geq c_{\alpha,d}\frac{h(x^*)}{2r^{d+\alpha}}\left(\omega_d r^d-\left(\frac{2\|h\|_{L^p}}{h(x^*)}\right)^p\right),$$

where we have used

$$|B(0,r)|-|\mathcal{U}_2|=|\mathcal{U}_1|.$$

We take r such that

$$\omega_d r^d=2\left(\frac{2\|h\|_{L^p}}{h(x^*)}\right)^p,$$

thus

$$\Lambda^\alpha h(x^*)\geq c_{\alpha,d}\frac{h(x^*)2^p\left(\frac{\|h\|_{L^p}}{h(x^*)}\right)^p}{2\left(\left(\frac{2\|h\|_{L^p}}{h(x^*)}\right)^{p/d}\left(\frac{2}{\omega_d}\right)^{1/d}\right)^{d+\alpha}}=c(d,\alpha,p)\frac{h(x^*)^{1+\alpha p/d}}{\|h\|_{L^p}^{\alpha p/d}}.$$

Step 2 We have $h(x_*)=\min_x h(x)<0$. As before, we take $r>0$ a positive number and define

$$\mathcal{U}_1=\{\eta\in B(0,r)\ s.t.\ h(x^*)-h(x^*-\eta)<h(x^*)/2\},$$

and $\mathcal{U}_2=B(0,r)-\mathcal{U}_1$. In the same way, we obtain inequality (A.1). With the appropriate choice of r, we get

$$\Lambda^\alpha h(x_*)\leq c_{\alpha,d}\frac{h(x_*)}{2r^{d+\alpha}}\left(\omega_d r^d-\left(\frac{2\|h\|_{L^p}}{|h(x_*)|}\right)^p\right)$$
$$\leq c(d,\alpha,p)\frac{h(x_*)|h(x_*)|^{\alpha p/d}}{\|h\|_{L^p}^{\alpha p/d}}.$$

Step 3 Now, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{e}^{-\Lambda^\alpha t}h\|_{L^\infty}\leq -c(d,\alpha,p)\frac{\|\mathrm{e}^{-\Lambda^\alpha t}h\|_{L^\infty}^{1+\alpha p/d}}{\|h\|_{L^p}^{\alpha p/d}},$$

and, integrating,

$$\|\mathrm{e}^{-\Lambda^\alpha t}h\|_{L^\infty}\leq C(d,\alpha,p)\frac{\max\{\|h\|_{L^p},\|h\|_{L^\infty}\}}{(t+1)^{d/(\alpha p)}}.$$

□

Appendix B. Commutator Estimates

We prove now a commutator estimate akin to the one in [15]:

Lemma B.1. *Fix $s\geq 0$. Then, the following estimate holds true:*

$$\|[\Lambda^s\nabla,g]f\|_{L^2}\leq C\left(\|\Lambda^s f\|_{L^2}\|\widehat{\Lambda g}\|_{L^1}+\|\Lambda^{s+1}g\|_{L^2}\|\widehat{f}\|_{L^1}\right).$$

Proof. The proof is similar to the one in [15]. After taking the Fourier transform and using the inequality

$$|\chi|^s \leq 2^{s-1}\left(|\chi-\xi|^s + |\xi|^s\right),$$

we have

$$|\widehat{[\Lambda^s\nabla, g]f}(\chi)| \leq C\left(\int_{\mathbb{R}^d} |\chi-\xi|^s|\hat{f}(\chi-\xi)||\xi||\hat{g}(\xi)|\mathrm{d}\xi + \int_{\mathbb{R}^d} |\hat{f}(\chi-\xi)||\xi|^{1+s}|\hat{g}(\xi)|\mathrm{d}\xi\right).$$

Then, we conclude via Plancherel's Theorem and Young's inequality for convolutions. □

We also recall the classical Kato–Ponce commutator estimate

Lemma B.2. *Fix $s > 0$ and $1 < p < \infty$. Then, the following estimate holds true*

$$\|[\Lambda^s, g]f\|_{L^p} \leq C\left(\|\nabla g\|_{L^{p_1}}\|\Lambda^{s-1}f\|_{L^{p_2}} + \|\Lambda^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}\right),$$

for

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}, 1 < p_1, p_4 \leq \infty,\ 1 < p_2, p_3 < \infty.$$

Appendix C. Interpolation Inequalities for the Wiener's Algebra

In this appendix, we recall and prove several inequalities involving fractional Sobolev and the Wiener's algebra that may be interesting by themselves.

Lemma C.1. *Assume that*

$$u \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \cap \dot{H}^{d/2+\delta}(\mathbb{R}^d).$$

Then, the following inequalities hold:

$$\|\hat{u}\|_{L^1(\mathbb{R}^d)} \leq C\|u\|_{L^1(\mathbb{R}^d)}^{\frac{\delta}{d+\delta}}\|u\|_{\dot{H}^{d/2+\delta}(\mathbb{R}^d)}^{\frac{d}{d+\delta}}, \quad \forall\, \delta > 0, \tag{C.1}$$

$$\|\hat{u}\|_{L^1(\mathbb{R}^d)} \leq C\|u\|_{L^2(\mathbb{R}^d)}^{\frac{\delta}{d/2+\delta}}\|u\|_{\dot{H}^{d/2+\delta}(\mathbb{R}^d)}^{\frac{d/2}{d/2+\delta}}, \quad \forall\, \delta > 0. \tag{C.2}$$

Proof. We have

$$\|\hat{u}\|_{L^1(\mathbb{R}^d)} = \int_{|\xi|<R} |\hat{u}(\xi)|\mathrm{d}\xi + \int_{|\xi|>R} \frac{|\xi|^{d/2+\delta}}{|\xi|^{d/2+\delta}}|\hat{u}(\xi)|\mathrm{d}\xi$$
$$\leq \|u\|_{L^1(\mathbb{R}^d)}R^d\frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} + \|u\|_{\dot{H}^{d/2+\delta}(\mathbb{R}^d)}\sqrt{\int_{|\xi|>R}|\xi|^{-d-2\delta}\mathrm{d}\xi}$$

$$\leq \|u\|_{L^1(\mathbb{R}^d)} R^d \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} + \|u\|_{\dot{H}^{d/2+\delta}(\mathbb{R}^d)} \sqrt{\int_R^\infty r^{d-1-d-2\delta} dr}$$
$$\leq \|u\|_{L^1(\mathbb{R}^d)} R^d \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} + \|u\|_{\dot{H}^{d/2+\delta}(\mathbb{R}^d)} C_\delta R^{-\delta}$$

With the choice

$$R = \left(\frac{\|u\|_{\dot{H}^{d/2+\delta}}}{\|u\|_{L^1}}\right)^{\frac{1}{d+\delta}}$$

and we conclude the inequality (C.1). To prove the second inequality (C.2), we compute

$$\|\hat{u}\|_{L^1(\mathbb{R}^d)} = \int_{|\xi|<R} |\hat{u}(\xi)| \mathrm{d}\xi + \int_{|\xi|>R} \frac{|\xi|^{d/2+\delta}}{|\xi|^{d/2+\delta}} |\hat{u}(\xi)| \mathrm{d}\xi$$
$$\leq \|\hat{u}\|_{L^2(\mathbb{R}^d)} \sqrt{R^d \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)}} + \|u\|_{\dot{H}^{d/2+\delta}(\mathbb{R}^d)} C_\delta R^{-\delta}.$$

Now, we can take

$$R = \left(\frac{\|u\|_{\dot{H}^{d/2+\delta}}}{\|u\|_{L^1}}\right)^{\frac{1}{d/2+\delta}}$$

□

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Communicated by Nader Masmoudi.
Received: January 10, 2016.
Accepted: March 9, 2016.