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Abstract Theory of Pointwise Decay with Applications to Wave and Schrödinger Equations

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Abstract. We prove pointwise in time decay estimates via an abstract conjugate operator method. This is then applied to a large class of dispersive equations.

1. Introduction

In the study of dispersive equations, linear or nonlinear, one is faced with the need to quantitatively estimate the decay rate of the solution in various norms. The known estimates which play a central role in the theory of dispersive equations include local decay estimates, pointwise decay estimates in time, L^p decay estimates and Strichartz estimates. More intricate are microlocal estimates and propagation estimates. The pointwise decay estimates for Schrödinger operators were proven first in three dimensions and were obtained for short range potentials [23]:

$$\|\langle x\rangle^{-\sigma}\mathrm{e}^{-\mathrm{i}tH}P_c(H)\langle x\rangle^{-\sigma}\| = \mathcal{O}(t^{-3/2}),$$

where $\langle x\rangle^2 \equiv 1+|x|^2$, σ is large enough, and $P_c(H)$ stands for the projection on the continuous spectral part of H. Here $H \equiv -\Delta + V(x)$.

This was later extended by various authors, unified to arbitrary dimension and allowing resonances at thresholds in [25]. These estimates play an important role in proving the L^p decay estimates:

$$\|\mathrm{e}^{-\mathrm{i}tH}P_c(H)\psi\|_{L^\infty(\mathbb{R}^n)} \leq ct^{-n/2}\|\psi\|_{L^1(\mathbb{R}^n)}.$$

Such estimates were proven in some generality in [27] in three or more dimensions. The Kato–Jensen estimate above was used to control the low energy part of the solution. Moreover, it was remarked by Ginibre (unpublished), that the

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Kato–Jensen estimate, when combined with iterated Duhamel formula, can imply directly a slightly weaker L^p estimate:

$$\|e^{-itH} P_c(H)\psi\|_{L^\infty(\mathbb{R}^n)+L^2} \leq ct^{-n/2}\|\psi\|_{L^1(\mathbb{R}^n)\cap L^2}.$$

This was extended to N-body charge transfer Hamiltonians in [45,46].

Subsequent works have extended the L^p estimates to all dimensions, and general classes of potential perturbations. See e.g. [11,12,28,38,49,55] and many more. Common to all these results is the explicit use of the kernel of the (unperturbed) Hamiltonian. Therefore, such methods are difficult to implement on manifolds. In fact, on manifolds most results are of the local decay type and Strichartz estimates [3,4,10,39,52]. The pointwise decay estimates and the L^p estimates are not known or not optimal. In contrast, the abstract method we develop here and in a subsequent paper is not using resolvent estimates. Thus, it is applicable in cases where explicit or perturbative methods of constructing the resolvent are not suitable. See e.g. Sects. 7.4–7.6.

A completely independent method of getting pointwise estimates is based on positive commutator techniques. Mourre's abstract theory to prove decay estimates is based on the Mourre estimate:

$$E_I(H)\mathrm{i}[H,A]E_I(H) \geq \theta E_I(H)$$

for some $\theta > 0$, where $E_I(H)$ is the spectral projection on an interval I. Under regularity assumptions on the pair H, A similar to ours, Mourre proved that the following local decay estimate holds:

$$\int \|\langle A\rangle^{-\sigma} e^{-itH} E_I(H)\psi\|^2 dt \leq c\|\psi\|^2.$$

Mourre's method was then generalized in [24] and later in [7,8] to prove pointwise in time decay estimates. See the discussion and details at the end of Sect. 5. Later, in [14,22,51] a new, time-dependent method was developed to prove pointwise decay estimates, local decay and other propagation estimates, starting only from the Mourre estimate. The propagation estimates of [51], which also allowed some classes of time-dependent Hamiltonians, and optimized in [22] read:

$$\|F(A \leq \epsilon t)e^{-itH} E_I(H)\psi\| = \mathcal{O}(t^{-m})\|\langle A\rangle^{m+1}\psi\|,$$
$$\|F(A \geq bt)e^{-itH} E_I(H)\psi\| = \mathcal{O}(t^{-m})\|\langle A\rangle^{m+1}\psi\|,$$

for all m depending only on the regularity of the potential and the localization of the initial data. ϵ, b depend only on the interval I, and the constant θ. This method and results provide a powerful tool to spectral and scattering theory, including the N-body systems and Quantum Field theory. However, the positivity condition in the Mourre estimate breaks down at (finite) thresholds.

Another way around this problem is the Morawetz type estimates. They apply at thresholds, but limited to nontrapping potentials. The extension to repulsive potentials and low dimension was established as well in some cases. Mourre's method was extended in many works to include thresholds [5–7,17,18,31–34,40–43,47,50,54]. They are based on requiring the Mourre estimate to hold with a lower bound given by some positive operator, which is not a multiple of the identity. However, these methods so far could not be versatile enough to include many common systems, mainly due to complicated assumptions or the use of abstract weighted spaces. They only imply local decay estimates of the type mentioned above.

In this work and forthcoming papers, we develop a new abstract theory to prove *pointwise* decay estimates in weighted spaces, starting only from a general commutator identity that should be satisfied by the Hamiltonian. We will show that for Schrödinger type equations generated by an abstract Hamiltonian H, as well as Klein Gordon and wave equations, pointwise decay estimates of the Kato–Jensen type hold using the following assumptions:

(a) The pair H, A with A self-adjoint, should satisfy regularity conditions similar to Mourre's method.
(b) A commutator identity of the type $\mathrm{i}[H, A] = \theta(H) + Q$ with $[Q, H] = 0$ and Q is H-bounded.
(c) The subspace $\mathcal{E}$ of vectors which satisfy local decay (as above) and are in the domain of A is known in some explicit sense, e.g. that it is all vectors in the domain of $\langle A\rangle^m$ and in some invariant subspace under the dynamics (generated by H).

These conditions differ in several aspects from the standard theories above. The commutator condition is restrictive, e.g. it is unstable under small perturbations of the Hamiltonian. However, we show in a subsequent work that local decay estimates are sufficient to absorb the effects of classes of perturbations; it is done by constructing a modified conjugate operator $\tilde{A}$ that satisfies the above conditions. So, in fact, the main condition is to identify, in some explicit way the subspace of vectors which satisfies local decay. It should be noted that positivity of commutators is not used. But, positivity would imply local decay, and better decay estimates in certain cases. To prove local decay estimates, either some resolvent bounds or some weakly positive commutators can be employed. In practice, this can be achieved using Morawetz type estimates or elementary perturbative resolvent estimates, relying only on the Fredholm alternative and compactness arguments.

We will then give several examples to show that such estimates follow effortlessly from the general theory. In followup papers we extend this method to include perturbations of H of the types described above.

2. Evanescent States

Let H be a self-adjoint operator on the Hilbert space $\mathcal{H}$ with spectral measure E. If $u \in \mathcal{H}$ let E_u be the measure $E_u(J) = \|E(J)u\|^2$ and $\psi_u : \mathbb{R} \to \mathbb{C}$ the function $\psi_u(t) = \langle u|\mathrm{e}\rangle^{\mathrm{i}tH}u = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}t\lambda} E_u(\mathrm{d}\lambda)$. We are interested in vectors u such that $\psi_u(t) \to 0$ as $t \to \infty$ and in the rapidity of this decay.

Note that $|\psi_u(t)|^2$ is a physically meaningful quantity if we think of H as the Hamiltonian of a system whose state space is $\mathcal{H}$. Indeed, if u, v are vectors of norm one then $|\langle v|\mathrm{e}\rangle^{\mathrm{i}tH}u|^2$ is the probability of finding the system in the state v at moment t if the initial state is u, hence $|\psi_u(t)|^2$ *is the probability that at moment t the system be in the same state u as at moment $t=0$.*

Remark 2.1. In this paper, we are interested in the decay properties of the functions ψ_u for u in the absolute continuity subspace $\mathcal{H}_H^{\mathrm{ac}}$ of $\mathcal{H}$ relatively to H. We shall see that $\psi_u \in L^2(\mathbb{R})$ for u in a dense subspace of $\mathcal{H}_H^{\mathrm{ac}}$ but in rather simple cases it may happen that $\psi_u \in L^1(\mathbb{R})$ only for $u=0$. Formally speaking, the physically interesting quantity $|\psi_u(t)|^2$ generically decays more rapidly than $\langle t\rangle^{-1}$ but not as rapidly as $\langle t\rangle^{-2}$. Our results concern mainly the rate of this decay, for example we give conditions such that $|\psi_u(t)|^2$ is really dominated by $\langle t\rangle^{-1}$, not only in an L^2 sense.

Since ψ_u is (modulo a constant factor) the Fourier transform of E_u, there is a strong relation between the decay of ψ_u and the smoothness of E_u. If u is absolutely continuous with respect to H then $\psi_u \in C_0(\mathbb{R})$ (space of continuous functions which tend to zero at infinity). However, the decay may be quite slow if E_u is not regular enough.

Example 2.2. *Let Λ be a real compact set with empty interior and strictly positive Lebesgue measure and let H be the operator of multiplication by x in $\mathcal{H}=L^2(\Lambda, \mathrm{d}x)$. Then the spectrum of H is purely absolutely continuous but $\psi_u \notin L^1(\mathbb{R})$ for all $u \in \mathcal{H}\backslash\{0\}$.* Indeed, if $u \in \mathcal{H}$ and we extend it by zero outside Λ then $\psi_u(t) = \int \mathrm{e}^{\mathrm{i}tx}|u(x)|^2\mathrm{d}x$ hence if ψ_u is integrable then $|u|^2$ is the Fourier transform of an integrable function, so it is continuous, so the set where $|u(x)|^2 \neq 0$ is open and contained in Λ, hence it is empty.

On the other hand, if H has an absolutely continuous component then there are plenty of u such that $\psi_u \in L^2(\mathbb{R})$: indeed, observe that $\psi_u \in L^2(\mathbb{R})$ *if and only if E_u is an absolutely continuous measure with derivative $E'_u \in L^2(\mathbb{R})$ and then $\|\psi_u\|_{L^2} = \sqrt{2\pi}\|E'_u\|_{L^2}$.*

More generally, if we denote $E_{v,u}$ the complex measure $E_{v,u}(J) = \langle v|E(J)u\rangle$ then $\langle v|\mathrm{e}^{\mathrm{i}tH}u\rangle = \int \mathrm{e}^{\mathrm{i}t\lambda}E_{v,u}(\mathrm{d}\lambda)$ hence the left-hand side belongs to $L^2(\mathbb{R})$ if and only if the measure $E_{v,u}$ is absolutely continuous and has square integrable derivative $E'_{v,u}$ and then we have $\int_{\mathbb{R}} |\langle v|\mathrm{e}^{\mathrm{i}tH}u\rangle|^2\mathrm{d}t = 2\pi\int |E'_{v,u}(\lambda)|^2\mathrm{d}\lambda$. It is easy to prove the inequality $|E'_{v,u}(\lambda)|^2 \leq E'_v(\lambda)E'_u(\lambda)$, see ([2, Section 3.5] or [36, page 1002]) and this implies $E'^{1/2}_{u+v} \leq E'^{1/2}_u + E'^{1/2}_v$. Thus, if we set for any $u \in \mathcal{H}$

$$[u]_H = \left(\int_{\mathbb{R}} |\psi_u(t)|^2\mathrm{d}t\right)^{1/4} = \left(2\pi\int_{\mathbb{R}} E'_u(\lambda)^2\mathrm{d}\lambda\right)^{1/4}$$

then

$$\mathcal{E} \equiv \mathcal{E}(H) = \{u \in \mathcal{H} \mid [u]_H < \infty\} \tag{2.1}$$

is a dense linear subspace of the absolutely continuous subspace of H and $[\cdot]_H$ is a complete norm on it. We mention that the relation $|E'_{v,u}(\lambda)|^2 \leq E'_v(\lambda)E'_u(\lambda)$ also implies

$$\left(\int_{\mathbb{R}} |\langle v|\mathrm{e}^{\mathrm{i}tH}u\rangle|^2\mathrm{d}t\right)^{1/2} \leq [v]_H[u]_H. \tag{2.2}$$

Lemma 2.3. *If $J \in B(\mathcal{H})$ commutes with H then $J\mathcal{E} \subset \mathcal{E}$ and $[Ju]_H \leq \|J\|[u]_H$. If $J_n = \theta_n(H)$ with $\{\theta_n\}$ a uniformly bounded sequence of Borel functions such that $\lim_n \theta_n(\lambda) = 1$ for all $\lambda \in \mathbb{R}$, then for any $u \in \mathcal{E}$ we have $\lim_n [J_n u]_H = [u]_H$.*

Proof. For the first part we use $E'_{Ju}(\lambda) \leq \|J\|^2 E'_u(\lambda)$ (which is obvious) while for the second part $E'_{\theta_n(H)u}(\lambda) = \theta_n^2(\lambda)E'(\lambda)$ and the dominated convergence theorem. □

The quantity $\int_{\mathbb{R}} |\psi_u(t)|^2 \mathrm{d}t$ has a simple physical interpretation in the quantum setting: *if u, v are two state vectors then $\int_{\mathbb{R}} |\langle v|\mathrm{e}^{\mathrm{i}tH}u\rangle|^2 \mathrm{d}t$ is the total time spent by the system in the state v if the initial state is u*. Hence we may say that $\int_{\mathbb{R}} |\langle u|\mathrm{e}^{\mathrm{i}tH}u\rangle|^2 \mathrm{d}t$ is the *lifetime of the state u*. The elements of $\mathcal{E}(H)$ are those of finite lifetime, or states in which the system spends a finite total time. We might call them *self evanescent states*, and they are absolutely continuous with respect to H. Note that there is a Schrödinger Hamiltonian H and there is a state u in the singularly continuous subspace of H such that $\psi_u(t) = \mathcal{O}(|t|^{-1/2+\varepsilon})$ for any $\varepsilon > 0$ [48].

Another interesting class $\mathcal{E}_\infty \equiv \mathcal{E}_\infty(H)$ is that of *evanescent states* defined by the condition $\int_{\mathbb{R}} |\langle v|\mathrm{e}^{\mathrm{i}tH}u\rangle|^2 \mathrm{d}t < \infty$ for any v: *such a state u spends a finite time in any state v*. The evanescent states disappear (or go to infinity) in a natural quantum mechanical sense, which explains the fundamental role they play in the Rosenblum Lemma [36] and later on in the Birman–Kato trace class scattering theory. A simple argument shows that $\mathcal{E}_\infty$ is the linear subspace of $\mathcal{E}$ consisting of vectors u such that E'_u is a bounded function. In particular, $\mathcal{E}_\infty$ is dense in the absolutely continuity subspace of H.

Example 2.4. If $H = q =$ operator of multiplication by x in $L^2(\mathbb{R}, \mathrm{d}x)$ then $\mathcal{E}(q) = L^4(\mathbb{R})$ and $\mathcal{E}_\infty(q) = L^\infty(\mathbb{R})$. Indeed, $\langle u|\mathrm{e}^{\mathrm{i}tq}u\rangle = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}|u(x)|^2 \mathrm{d}x$ is an L^2 function of t if and only if $|u|^2 \in L^2$ and then $[u]_q = (2\pi)^{1/4}\|u\|_{L^4}$. On the other hand, $\langle v|\mathrm{e}^{\mathrm{i}tq}u\rangle = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx}\bar{v}(x)u(x)\mathrm{d}x$ is an L^2 function of t for any $v \in L^2$ if and only if $\bar{v}u \in L^2$ for any $v \in L^2$ hence if and only if $u \in L^\infty$.

3. Notes on Commutators

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. If S is a bounded operator on $\mathcal{H}$ then we denote $[A, S]_\circ$ the sesquilinear form on $D(A)$ defined by $[A, S]_\circ(u, v) = \langle Au|Sv\rangle - \langle u|SAv\rangle$. As usual, we set $[S, A]_\circ = -[A, S]_\circ$, $[S, \mathrm{i}A]_\circ = \mathrm{i}[S, A]_\circ$, etc. We say that *$S$ is of class $C^1(A)$*, and we write $S \in C^1(A)$, if $[A, S]_\circ$ is continuous for the topology induced by $\mathcal{H}$ on $D(A)$ and then we denote $[A, S]$ the unique bounded operator on $\mathcal{H}$ such that $\langle u|[A, S]v\rangle = \langle Au|Sv\rangle - \langle u|SAv\rangle$ for all $u, v \in D(A)$. It is easy to show that $S \in C^1(A)$ if and only if $SD(A) \subset D(A)$ and the operator $SA - AS$ with domain $D(A)$ extends to a bounded operator $[A, S] \in B(\mathcal{H})$. Moreover, S is of class $C^1(A)$ if and only if the following equivalent conditions are satisfied

1. the function $t \mapsto \mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA}$ is Lipschitz in the norm topology in $B(\mathcal{H})$
2. the function $t \mapsto \mathrm{e}^{-\mathrm{i}tA}S\mathrm{e}^{\mathrm{i}tA}$ is of class C^1 in the strong operator topology

and then we have $[S, \mathrm{i}A] = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{-\mathrm{i}tA} S \mathrm{e}^{\mathrm{i}tA}|_{t=0}$.

Clearly $C^1(A)$ is a $*$-subalgebra of $B(\mathcal{H})$ and the usual commutator rules hold true: for any $S, T \in C^1(A)$ we have $[A, S]^* = -[A, S^*]$ and $[A, ST] = [A, S]T + S[A, T]$, and if S is bijective then $S^{-1} \in C^1(A)$ and $[A, S^{-1}] = -S^{-1}[A, S]S^{-1}$.

We often abbreviate $S' = [S, \mathrm{i}A]$ if the operator A is obvious from the context. Then we may write $(S')^* = (S^*)'$, $(ST)' = S'T + ST'$, and $(S^{-1})' = -S^{-1}S'S^{-1}$.

We consider now the rather subtle case of unbounded operators. Note that we always equip the domain of an operator with its graph topology. If H is a self-adjoint operator on $\mathcal{H}$ then $[A, H]_\circ$ is the sesquilinear form on $D(A) \cap D(H)$ defined by $[A, H]_\circ(u, v) = \langle Au|Hv\rangle - \langle Hu|Av\rangle$. By analogy with the bounded operator case, one would expect that requiring denseness of $D(A) \cap D(H)$ in $D(H)$ and continuity of $[A, H]_\circ$ for the graph topology of $D(H)$ would give a good $C^1(A)$ notion. For example, this should imply the validity of the virial theorem, nice functions of H (at least the resolvent) should also be of class C^1, etc. However this is not true, as the following example from [16] shows.

Example 3.1. In $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$ let $q =$ operator of multiplication by x and $p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. Let $A = \mathrm{e}^{\omega p} - p$ and $H = \mathrm{e}^{\omega q}$ with $\omega = \sqrt{2\pi}$. This value of ω is chosen because $\mathrm{e}^{\omega p}\mathrm{e}^{\omega q} = \mathrm{e}^{\omega q}\mathrm{e}^{\omega p}$ on a very large set although the operators $\mathrm{e}^{\omega p}$ and $\mathrm{e}^{\omega q}$ do not commute. Then $D(A) \cap D(H)$ is dense in both $D(A)$ and $D(H)$ (moreover, $D(H) \cap D(HA)$ is dense in $D(H)$), one has $[H, \mathrm{i}A]_\circ = \omega H$ on $D(A) \cap D(H)$, but $(H + \mathrm{i})^{-1} \notin C^1(A)$.

A convenient definition of the $C^1(A)$ class for any self-adjoint operator is as follows. Let $R(z) = (H - z)^{-1}$ for z in the resolvent set $\rho(H)$ of H. We say that H *is of class* $C^1(A)$ if $R(z) \in C^1(A)$ for some (hence for all) $z \in \rho(H)$. In this case the space $R(z)D(A)$ is independent of $z \in \rho(H)$, it is a core of H, and is a dense subspace of $D(A) \cap D(H)$ for the intersection topology, i.e. for the norm $\|u\| + \|Au\| + \|Hu\|$. Moreover:

Proposition 3.2. *Let A, H be self-adjoint operators on a Hilbert space $\mathcal{H}$.*

1. *H is of class $C^1(A)$ if and only if the next two conditions are satisfied:*
   (a) *$[A, H]_\circ$ is continuous for the topology induced by $D(H)$ on $D(A) \cap D(H)$,*
   (b) *there is $z \in \rho(H)$ such that $\{u \in D(A) \mid R(z)u \in D(A)\}$ is a core for A.*
2. *If $H \in C^1(A)$ then $D(A) \cap D(H)$ is dense in $D(H)$ hence $[A, H]_\circ$ has a unique extension to a continuous sesquilinear form $[A, H]$ on $D(H)$. We have:*

$$[A, R(z)] = -R(z)[A, H]R(z) \quad \forall\, z \in \rho(H). \tag{3.3}$$

This is Theorem 6.2.10 in [1]. The condition (a) above is quite easy to check in general but not condition (b) because it involves a certain knowledge of the resolvent of H, which is a complicated object. We now describe criteria which allow one to avoid this problem.

We denote $\mathcal{H}^1 = D(H)$ (equipped with the graph topology) and $\mathcal{H}^{-1} = D(H)^*$ its adjoint space. The identification of the adjoint space $\mathcal{H}^*$ of $\mathcal{H}$ with itself via the Riesz Lemma gives us a scale $\mathcal{H}^1 \subset \mathcal{H} \subset \mathcal{H}^{-1}$ with continuous and dense embeddings. If we define $\mathcal{H}^s := [\mathcal{H}^1, \mathcal{H}^{-1}]_{(1-s)/2}$ for $-1 \leq s \leq 1$ by complex interpolation then $(\mathcal{H}^s)^* = \mathcal{H}^{-s}$ for any s and $\mathcal{H}^{1/2}$ is just $D(|H|^{1/2})$. Finally, we have continuous and dense embeddings

$$\mathcal{H}^1 \subset \mathcal{H}^{1/2} \subset \mathcal{H} \subset \mathcal{H}^{-1/2} \subset \mathcal{H}^{-1}.$$

If $H \in C^1(A)$ the continuous sesquilinear form $[A, H]$ on $D(H)$ is then identified with a linear continuous operator $\mathcal{H}^1 \to \mathcal{H}^{-1}$ and this is useful for example because it gives a simple interpretation to supplementary conditions like $[A, H]\mathcal{H}^1 \subset \mathcal{H}$. Observe that

$$H' := [H, \mathrm{i}A] : \mathcal{H}^1 \to \mathcal{H}^{-1}$$

is a continuous symmetric operator. Now the following assertions are consequences of [1, Theorem 6.3.4, Lemma 7.5.3] and [16, Lemma 2].

1. If $\mathrm{e}^{\mathrm{i}tA}\mathcal{H}^1 \subset \mathcal{H}^1$ $(\forall t)$ then $H \in C^1(A)$ if and only if condition (a) in part (1) of Proposition 3.2 is satisfied.
2. If $H \in C^1(A)$ and $H'\mathcal{H}^1 \subset \mathcal{H}$ then $\mathrm{e}^{\mathrm{i}tA}\mathcal{H}^1 \subset \mathcal{H}^1$ $(\forall t)$ and the restrictions $\mathrm{e}^{\mathrm{i}tA}|\mathcal{H}^1$ give a strongly continuous group of operators on the Hilbert space $\mathcal{H}^1$.
3. If $\mathrm{e}^{\mathrm{i}tA}\mathcal{H}^1 \subset \mathcal{H}^1$ $(\forall t)$ then $D(A, \mathcal{H}^1) = \{u \in \mathcal{H}^1 \cap D(A) \mid Au \in \mathcal{H}^1\}$ is a dense subspace of $\mathcal{H}^1$ and H is of class $C^1(A)$ with $H'\mathcal{H}^1 \subset \mathcal{H}$ if and only if $|\langle Au|Hv\rangle - \langle Hu|Av\rangle| \leq C\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}^1}$ for all $u, v \in D(A, \mathcal{H}^1)$.
4. Assume $\mathrm{e}^{\mathrm{i}tA}\mathcal{H}^{1/2} \subset \mathcal{H}^{1/2}$ $(\forall t)$. Then $D(A, \mathcal{H}^{1/2}) := \{u \in \mathcal{H}^{1/2} \cap D(A) \mid Au \in \mathcal{H}^{1/2}\}$ is dense in $\mathcal{H}^{1/2}$ and if the quadratic form $\langle Hu|Au\rangle - \langle Au|Hu\rangle$ on $D(A, \mathcal{H}^{1/2})$ is continuous for the topology induced by $\mathcal{H}^{1/2}$ then $H \in C^1(A)$.

We mention that Hypotheses 1, 2′ and 3 on page 62 of [9] imply that H is of class $C^1(A)$, cf. relation (4.10) there.

We now give some "pathological" examples which clarify the notion of C^1 regularity.

Example 3.3. Let $\mathcal{H} = L^2(\mathbb{R})$ and $A = p$. It is clear that the operator of multiplication by a rational real function is of class $C^1(p)$, in fact of class $C^\infty(p)$ in a natural sense. For example, if $H = q^{-m}$ then $(H + \mathrm{i})^{-1} = q^m(1 + \mathrm{i}q^m)^{-1}$ is clearly a bounded operator of class $C^1(p)$ if $m \in \mathbb{N}$ and $[q^{-m}, \mathrm{i}p] = mq^{-m-1}$ as continuous forms on $D(q^{-m})$. The worst case is attained when $m = 1$: then $H' = H^2$ hence $H' + \mathrm{i} : \mathcal{H}^1 \to \mathcal{H}^{-1}$ is an isomorphism, in particular $H'\mathcal{H}^1$ is not included in any of the smaller spaces $\mathcal{H}^s$ with $s > -1$. If $m \geq 1$ is an odd integer then H is of class $C^1(A)$ and $H' = mH^{1+1/m}$ where $x^{1/m} := -|x|^{1/m}$ if $x < 0$; now we have $H'\mathcal{H}^1 \subset \mathcal{H}^{-1/m}$ and this is optimal.

Remark 3.4. Example 3.3 shows that *if* $H \in C^1(A)$ *then neither* $\mathrm{e}^{\mathrm{i}tA}$ *nor* $(A + \mathrm{i}\lambda)^{-1}$ *leave invariant* $D(H)$ *in general.*

If $H \in C^1(A)$ then $D(A) \cap D(H)$ is dense in $D(H)$ but *is not dense in* $D(A)$ in general.

Example 3.5. Let $H = q^{-m}$ with $m \geq 1$ and $A = p$ as in Example 3.3. Then $D(A)$ is the Sobolev space consisting of functions $u \in L^2(\mathbb{R})$ with derivative $u' \in L^2(\mathbb{R})$, so we have $D(A) \subset C_0(\mathbb{R})$ continuously. Thus, if $u \in D(A) \cap D(H)$ then u is a continuous function such that $\int |u(x)|^2 x^{-2m} \mathrm{d}x < \infty$ which implies $u(0) = 0$. But $\{u \in D(A) \mid u(0) = 0\}$ is a closed hyperplane of codimension one in the Hilbert space $D(A)$.

Given $\varepsilon > 0$, taking m large in the preceding example we see that for any $(\varepsilon > 0)$ there is a self-adjoint operator H of class $C^1(A)$ with $H'\mathcal{H}^1 \subset \mathcal{H}^{-\varepsilon}$ such that $D(A) \cap D(H)$ is not dense in $D(A)$. Thus, *the next result is optimal.*

Proposition 3.6. *If $H \in C^1(A)$ and $H'\mathcal{H}^1 \subset \mathcal{H}$ then $D(A) \cap D(H)$ is dense in $D(A)$. More precisely, if we set $R_\varepsilon = (1 + \mathrm{i}\varepsilon H)^{-1}$ for $\varepsilon > 0$ then $R_\varepsilon D(A) \subset D(A) \cap D(H)$ and $\mathrm{s} - \lim R_\varepsilon = 1$ in the Hilbert space $D(A)$.*

Proof. We have $R_\varepsilon D(A) \subset D(A) \cap D(H)$ and $[A, R_\varepsilon] = \varepsilon R_\varepsilon H' R_\varepsilon$ by Proposition 3.2. Then $\varepsilon \|H' R_\varepsilon\| \leq \|H' R_1\| \|(\varepsilon + \mathrm{i}\varepsilon H) R_\varepsilon\| \leq C$ and $\varepsilon H' R_\varepsilon u = \varepsilon H' R_1 R_\varepsilon (1 + \mathrm{i}H) u \to 0$ if $u \in D(H)$. Thus $\mathrm{s} - \lim_{\varepsilon \to 0} [A, R_\varepsilon] = 0$ hence $A R_\varepsilon u \to Au$ for any $u \in D(A)$. □

This $C^1(A)$ property transfers from H to some functions of H: for example, it is easy to prove that $\varphi(H) \in C^1(A)$ if $\varphi \in C^2(\mathbb{R})$ and $|\varphi(\lambda)| + |\varphi'(\lambda)| + |\varphi''(\lambda| \leq C \langle \lambda \rangle^{-2}$. But obviously $\mathrm{e}^{\mathrm{i}H} \notin C^1(A)$ in general.

Theorem 3.7. *Let H be a self-adjoint operator of class $C^1(A)$ and $t \in \mathbb{R}$. Then the restriction of $[A, \mathrm{e}^{\mathrm{i}tH}]_\circ$ to $D(A) \cap D(H)$ extends to a continuous form $[A, \mathrm{e}^{\mathrm{i}tH}]$ on $D(H)$ and, in the strong topology of the space of sesquilinear forms on $D(H)$, we have:*

$$[\mathrm{e}^{\mathrm{i}tH}, A] = \int_0^t \mathrm{e}^{\mathrm{i}(t-s)H} H' \mathrm{e}^{\mathrm{i}sH} \mathrm{d}s. \tag{3.4}$$

Proof. Clearly it suffices to assume $t = 1$. For $n \geq 1$ integer let $R_n = (1 - \mathrm{i}H/n)^{-1}$. Then R_n has norm ≤ 1 and $\mathrm{e}^{\mathrm{i}H} = \mathrm{s} - \lim_{n \to \infty} R_n^n$ in both spaces $\mathcal{H}$ and $D(H)$. Since H is of class $C^1(A)$ we have $R_n \in C^1(A)$ and $[A, R_n] = \frac{\mathrm{i}}{n} R_n [A, H] R_n$, so $R_n^n \in C^1(A)$ and

$$[A, R_n^n] = \sum_{k=0}^{n-1} R_n^k [A, R_n] R_n^{n-1-k} = \frac{\mathrm{i}}{n} \sum_{k=1}^{n} R_n^k [A, H] R_n^{n+1-k}.$$

It is clear that $\langle u | [A, R_n^n] v \rangle \to \langle u | [A, \mathrm{e}^{\mathrm{i}H}]_0 v \rangle$ as $n \to \infty$ for all $u, v \in D(A)$. Thus it remains to be shown that for all $u, v \in D(H)$:

$$\frac{1}{n} \sum_{k=1}^{n} \langle R_n^{*k} u | [A, H] R_n^{n+1-k} v \rangle \to \int_0^1 \langle \mathrm{e}^{-\mathrm{i}sH} u | [A, H] \mathrm{e}^{\mathrm{i}(1-s)H} v \rangle \mathrm{d}s. \tag{3.5}$$

We have

$$\Big\| \sum_{k=1}^{n} R_n^k [A, H] R_n^{n+1-k} \Big\|_{\mathcal{H}^1 \to \mathcal{H}^{-1}} \leq n \|[A, H]\|_{\mathcal{H}^1 \to \mathcal{H}^{-1}}$$

hence it suffices to prove that (3.5) holds for u, v in a dense subspace of $D(H)$. So we may assume that u, v have compact support with respect to H.

Let a be a number such that $|\log(1+z)-z| \le a|z|^2$ if $z \in \mathbb{C}$ and $|z| < 1/2$. If $\phi_n(x) = (1 - \mathrm{i}x/n)^{-1}$ then for x in a real compact set, $1 \le k \le n$, and n large, we have

$$|\phi_n(x)^k - \mathrm{e}^{\mathrm{i}\frac{kx}{n}}| = |\mathrm{e}^{k\log(1-\mathrm{i}\frac{x}{n})+\mathrm{i}\frac{kx}{n}} - 1| \le Ck|\log(1-\mathrm{i}x/n)+\mathrm{i}x/n| \le Cka|x/n|^2$$

where C is a number depending only on the set where x varies. Thus, the last term above is an $\mathcal{O}(x^2/n)$ and so we get $\|R_n^{*k}u - \mathrm{e}^{-\mathrm{i}\frac{k}{n}H}u\|_{D(H)} = \mathcal{O}(n^{-1})$. A similar argument gives $\|R_n^{n+1-k}v - \mathrm{e}^{\mathrm{i}\frac{n+1-k}{n}H}V\|_{D(H)} = \mathcal{O}(n^{-1})$. Hence:

$$\frac{1}{n}\sum_{k=1}^{n}\langle R_n^{*k}u|[A,H]R_n^{n+1-k}v\rangle = \frac{1}{n}\sum_{k=1}^{n}\langle \mathrm{e}^{-\mathrm{i}\frac{k}{n}H}u|[A,H]\mathrm{e}^{-\mathrm{i}\frac{k}{n}H}\mathrm{e}^{\mathrm{i}\frac{n+1}{n}H}v\rangle + \mathcal{O}(n^{-1}).$$

Finally, we have $\mathrm{e}^{\mathrm{i}\frac{n+1}{n}H}v \to \mathrm{e}^{\mathrm{i}H}v$ in $D(H)$ and the $D(H)$-valued functions $s \mapsto \mathrm{e}^{-\mathrm{i}sH}u$ and $s \mapsto \mathrm{e}^{-\mathrm{i}sH}v$ are continuous. This proves (3.4). □

The relation (3.4) also holds in $B(D(H), D(H)^*)$ in the strong topology and then one may easily prove relations like the next one hold in $B(\mathcal{H})$:

$$[A, \mathrm{e}^{\mathrm{i}tH}R(z)^2] = R(z)[A, \mathrm{e}^{\mathrm{i}tH}]R(z) + [A, R(z)]\mathrm{e}^{\mathrm{i}tH}R(z) + \mathrm{e}^{\mathrm{i}tH}R(z)[A, R(z)]. \tag{3.6}$$

If $H'D(H) \subset \mathcal{H}$ then the right-hand side of (3.4) will clearly belong to $B(D(H), \mathcal{H})$ hence we shall also have $[A, \mathrm{e}^{\mathrm{i}tH}] \in B(D(H), \mathcal{H})$ and (3.4) will hold strongly in $B(D(H), \mathcal{H})$.

We say that H'*, or* $[A, H]$*, commutes with* H if for any $t \in \mathbb{R}$ the relation $H'\mathrm{e}^{\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}tH}H'$ holds in $B(D(H), D(H)^*)$. This is clearly equivalent to $H'\varphi(H) = \varphi(H)H'$ for any bounded Borel function $\varphi : \mathbb{R} \to \mathbb{C}$. Note also that H' commutes with H if and only if there is $z \in \rho(H)$ such that $[A, R(z)]$ commutes with $R(z)$ (this condition is independent of z). If we set $R = R(z)$, we then have $R' = -RH'R = -H'R^2$.

If H' commutes with H then Theorem 3.7 can be significantly improved. If $k \in \mathbb{N}$ let $C^k_{\mathrm{b}}(\mathbb{R})$ be the space of functions in $C^k(\mathbb{R})$ whose derivatives of orders $\le k$ are bounded.

Proposition 3.8. *Let H be self-adjoint of class $C^1(A)$ such that H' commutes with H and let $\varphi \in C^1_{\mathrm{b}}(\mathbb{R})$. Then the restriction of $[A, \varphi(H)]_\circ$ to $D(A) \cap D(H)$ extends to a continuous form $[A, \varphi(H)]$ on $D(H)$ and $[A, \varphi(H)] = [A, H]\varphi'(H) = \varphi'(H)[A, H]$. In other terms:*

$$\varphi(H)' = \varphi'(H)H' = H'\varphi'(H), \quad \textit{in particular } (\mathrm{e}^{\mathrm{i}tH})' = \mathrm{i}tH'\mathrm{e}^{\mathrm{i}tH}. \tag{3.7}$$

Proof. Due to Theorem 3.7 we have $[A, \mathrm{e}^{\mathrm{i}tH}] = \mathrm{i}t[A, H]\mathrm{e}^{\mathrm{i}tH}$ for any real t, hence the proposition is true if $\varphi(\lambda) = \mathrm{e}^{\mathrm{i}t\lambda}$. This clearly implies the proposition if φ is the Fourier transform of a bounded measure $\widehat{\varphi}$ such that $\int |x\widehat{\varphi}(x)|\mathrm{d}x < \infty$. The general case follows by a standard limiting procedure. □

Example 3.9. Consider once again the situation from Example 3.3. Then $[p, \mathrm{e}^{\mathrm{i}H}]_\circ$ is a restriction of $-mq^{-m-1}\mathrm{e}^{\mathrm{i}H}$ hence is not a bounded operator but it extends to a continuous form on $D(H)$. In the worst case $m = 1$ we get $[\mathrm{e}^{\mathrm{i}H}, A] = H^2\mathrm{e}^{\mathrm{i}H}$, hence *the result of Theorem 3.7 is optimal.* If φ is a C^1 function then $\varphi(H)' = H^2\varphi'(H)$ hence $\varphi(H)'$ *cannot be bounded unless* $|\varphi'(\lambda)| \le C\langle\lambda\rangle^{-2}$.

4. Commutators and Decay

From Proposition 3.8 we get the following decay result.

Proposition 4.1. *Let $H \in C^1(A)$ such that $H' := [H, \mathrm{i}A]$ commutes with H and let $u \in D(H) \cap D(A)$. Then $|\langle u|H'\mathrm{e}^{\mathrm{i}tH}u\rangle| \le 2|t|^{-1}\|Au\|\|u\|$. In particular, if $H' = B^*B$ for some continuous $B : D(H) \to \mathcal{H}$ commuting with H, then $|\psi_{Bu}(t)| \le C_u\langle t\rangle^{-1}$ for $u \in D(H) \cap D(A)$. If B is bounded on $\mathcal{H}$ then this holds for all $u \in D(A)$.*

Proof. The first part is obvious. The fact that B commutes with H means $\mathrm{e}^{\mathrm{i}tH}B = B\mathrm{e}^{\mathrm{i}tH}$ for any t and this clearly implies that $[A, H]$ commutes with H. Then $\langle Bu|\mathrm{e}^{\mathrm{i}tH}Bu\rangle = \langle u|[H, \mathrm{i}A]\mathrm{e}^{\mathrm{i}tH}u\rangle$ hence the second and the third part are consequences of the first one. □

Remark 4.2. Some of the next results are abstract versions of the following estimate: if $H = h(q)$ and $A = -p$ in $L^2(\mathbb{R})$ then $H' = h'(q)$ hence if $|h'| \ge c > 0$ and h''/h'^2 is bounded then an integration by parts gives $|\int \mathrm{e}^{\mathrm{i}th(x)}|u(x)|^2\mathrm{d}x| \le C_u\langle t\rangle^{-1}$ if $u \in D(p)$.

We shall say that a densely defined operator S on $\mathcal{H}$ is *boundedly invertible* if S is injective, its range is dense, and its inverse extends to a continuous operator on $\mathcal{H}$. If S is symmetric this means that S is essentially self-adjoint and 0 is in the resolvent set of its closure.

Proposition 4.3. *Let $H \in C^1(A)$ such that H' commutes with H and $H'D(H) \subset \mathcal{H}$. Assume that H', when considered as operator on $\mathcal{H}$, is boundedly invertible and H'^{-1} extends to a bounded operator of class $C^1(A)$. Then $|\psi_u(t)| \le C_u\langle t\rangle^{-1}$ if $u \in D(A)$.*

Proof. From Proposition 3.8 we get $[\mathrm{e}^{\mathrm{i}tH}, A] = tH'\mathrm{e}^{\mathrm{i}tH}$ as operators $D(H) \to D(H)^*$ hence $[\mathrm{e}^{\mathrm{i}tH}, A]$ is a bounded operator $D(H) \to \mathcal{H}$ and we have $[\mathrm{e}^{\mathrm{i}tH}, A]H'^{-1} = t\mathrm{e}^{\mathrm{i}tH}$ on the range of H'. We denote K the continuous extension to $\mathcal{H}$ of H'^{-1} and note that K commutes with H because $H'\mathrm{e}^{\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}tH}H'$ hence $H'^{-1}\mathrm{e}^{\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}tH}H'^{-1}$ for all t. If $u \in D(A)$ and $Ku \in D(H)$ then $Ku \in D(A)$ because $K \in C^1(A)$ hence

$$t\psi_u(t) = \langle u|[\mathrm{e}^{\mathrm{i}tH}, A]Ku\rangle = \langle \mathrm{e}^{-\mathrm{i}tH}u|AKu\rangle - \langle Au|\mathrm{e}^{\mathrm{i}tH}Ku\rangle.$$

This implies

$$|t\psi_u(t)| \le \|u\|\|AKu\| + \|Au\|\|Ku\| \le \|[A, K]\|\|u\|^2 + 2\|K\|\|u\|\|Au\|.$$

Now let u be an arbitrary element of $D(A)$. Let $R_\varepsilon = (1 + \mathrm{i}\varepsilon H)^{-1}$ and $u_\varepsilon = R_\varepsilon u$. Then $u_\varepsilon \in D(A)$ because $R_\varepsilon D(A) \subset D(A)$ and $Ku_\varepsilon = R_\varepsilon Ku \in D(H)$ hence we have

$$|t\psi_{u_\varepsilon}(t)| \le \|[A, K]\|\|u_\varepsilon\|^2 + 2\|K\|\|u_\varepsilon\|\|Au_\varepsilon\|.$$

Since $[A, R_\varepsilon] = \varepsilon H' R_\varepsilon^2$ we get $|t\psi_u(t)| \leq \|[A, K]\|\|u\|^2 + 2\|K\|\|u\|\|Au\|$ by making $\varepsilon \to 0$ in the preceding inequality. $\square$

Remark 4.4. We may restate the assumptions of Proposition 4.3 as follows: H is of class $C^2(A)$, $H'D(H) \subset \mathcal{H}$, and H' when seen as operator on $\mathcal{H}$ is essentially self-adjoint and 0 is not in its spectrum.

Remark 4.5. The good decay $\psi_u(t) = \mathcal{O}(t^{-1})$ obtained in Proposition 4.3 depends on a quite strong condition on H' which in particular forces H' to be an essentially self-adjoint operator on $\mathcal{H}$ whose spectrum does not contain zero. In the "classical" case mentioned in the Remark 4.2 this means $|h'(x)| \geq c > 0$ which is rather natural when one has to estimate an integral like $\psi(t) = \int e^{ith(x)} f(x) dx$ for large positive t: *points of stationary phase should be avoided, otherwise we cannot expect more than* $\psi(t) = \mathcal{O}(t^{-1/2})$.

We now consider operators satisfying some special commutation relations but allow H' to have zeros, e.g. we treat the simplest case $H' = cH$. Note that Example 3.1 shows that requiring only an algebraic relation like $[H, iA] = cH$ is highly ambiguous; the property $H \in C^1(A)$ is then necessary and is not automatically satisfied.

In many of the applications of the conjugate operator method, see for example Sect. 7, the operator A is unbounded in energy space. However, it is possible to introduce an energy cutoff for A that does not alter the $C^1(A)$ condition for H and preserves the behaviour of the commutation relation at thresholds. For instance, consider $H \in C^1(A)$ bounded from below and such that $H' = cH$. Define the operators $g(H) = (H + c)^{-1/4}$ and $\tilde{A} = g(H)Ag(H)$. Then, it is easy to see that $H \in C^1(\tilde{A})$ and $[H, i\tilde{A}] = H'g(H)^2$.

Remark 4.6. The subsequent results will hold for self evanescent states $u \in \mathcal{E}$, but they also rely on the condition $Au \in \mathcal{E}$. The latter assumption is not satisfied in general, in fact, it is implied by a stronger localization condition for u. To elude this, it can be assumed that there is a projection P which commutes with H, and such that $u \in \mathrm{Ran}P$. Then the condition $Au \in \mathcal{E}$ can be replaced by $PAu \in \mathcal{E}$, which is easier to satisfy. This idea will be explored further in forthcoming work. For instance, one can choose P as the projection on the continuous spectrum of H and the proofs presented below can be slightly modified to obtain the same decay estimates.

Proposition 4.7. *Let $H \in C^1(A)$ such that $H' = cH$ with $c \in \mathbb{R}\backslash\{0\}$ and let $u \in D(A)$ such that $u, Au \in \mathcal{E}$. Then $|\psi_u(t)| \leq C_u \langle t \rangle^{-1/2}$.*

Proof. We have $\psi_u \in L^2(\mathbb{R})$ because $u \in \mathcal{E}$ hence, according to Corollary 8.2, it suffices to show that the function $(\delta\psi)(t) = t\psi_u'(t)$ also belongs to $L^2(\mathbb{R})$. If $u \in D(|H|^{1/2})$ then $t\psi_u'(t) = \langle u | itHe^{itH} u\rangle$ so that using Proposition 3.8 we get:

$$ict\psi_u'(t) = \langle u|tcHe^{itH}u\rangle = \langle u|[A, itH]e^{itH}u\rangle = \langle u|[A, e^{itH}]u\rangle.$$

Then, if $u \in D(A)$ we get $ict\psi_u'(t) = \langle Au|e^{itH}u\rangle - \langle u|e^{itH}Au\rangle$ hence (2.2) implies:

$$c^2\|\delta\psi_u\|^2 \leq 2\|\psi_u\|_{L^2}\|\psi_{Au}\|_{L^2} = 2[u]_H^2[Au]_H^2. \tag{4.8}$$

So the proposition is proved under the supplementary condition $u \in D(H)$ and the estimate (4.8) depends only on c.

Now consider an arbitrary $u \in D(A)$ such that $u, Au \in \mathcal{E}$ and for $\varepsilon > 0$ let $R_\varepsilon = (1 + \mathrm{i}\varepsilon H)^{-1}$. Then from Proposition 3.2 we get $R_\varepsilon u \in D(A)$ and $[A, R_\varepsilon] = R_\varepsilon[\mathrm{i}\varepsilon H, A]R_\varepsilon = c\varepsilon H R_\varepsilon^2$. If we set $u_\varepsilon = R_\varepsilon u$ then the estimate (4.8) gives $c^2\|\delta\psi_{u_\varepsilon}\|^2 \le 2[u_\varepsilon]_H^2[Au_\varepsilon]_H^2$. Finally, let $\varepsilon \to 0$ and use Fatou's lemma in the left-hand side and Lemma 2.3 on the right-hand side to get (4.8) without the condition $u \in D(H)$. □

Theorem 4.8. *Let $H \in C^1(A)$ such that $H' = \theta(H)$ with θ real of class C^1 with bounded derivative and such that:* (1) *if $|\lambda| \ge \varepsilon > 0$ then $|\theta(\lambda)| \ge c_\varepsilon > 0$,* (2) *$\lambda/\theta(\lambda)$ extends to a C^1 function on $\mathbb{R}$. If $u \in D(A)$ and $u, Au \in \mathcal{E}$ then $|\psi_u(t)| \le C_u\langle t\rangle^{-1/2}$.*

Proof. Let $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R})$ real and equal to one on a neighbourhood of zero and let us set $\phi = \varphi(H), \phi^\perp = 1 - \phi^2$, so that $\psi_u(t) = \psi_{\phi u}(t) + \langle u|\phi^\perp \mathrm{e}^{\mathrm{i}tH}u\rangle$. We first show that the second term is $\mathcal{O}(t^{-1})$. We have $\phi^\perp H'^{-1} = \xi(H)$ with $\xi(\lambda) = (1 - \varphi^2(\lambda))/\theta(\lambda)$ hence

$$\begin{aligned} t\langle u|\phi^\perp \mathrm{e}^{\mathrm{i}tH}u\rangle &= \langle \phi^\perp H'^{-1}u|tH'\mathrm{e}^{\mathrm{i}tH}u\rangle = \langle \xi(H)u|tH'\mathrm{e}^{\mathrm{i}tH}u\rangle \\ &= \langle \xi(H)u|[\mathrm{e}^{\mathrm{i}tH}, A]u\rangle. \end{aligned}$$

Until here u was an arbitrary element of $\mathcal{H}$. If $u \in D(H) \cap D(A)$ then we can expand the commutator and get

$$\begin{aligned} |t\langle u|\phi^\perp \mathrm{e}^{\mathrm{i}tH}u\rangle| &= |\langle \mathrm{e}^{-\mathrm{i}tH}\xi(H)u|Au\rangle - \langle A\xi(H)u|\mathrm{e}^{\mathrm{i}tH}u\rangle| \\ &\le \|Au\|\|\xi(H)u\| + \|A\xi(H)u\|\|u\|. \end{aligned}$$

Since ξ is a bounded function of class C^1 with bounded derivative we can use Proposition 3.6 and get $\xi(H)' = \xi'(H)H' = (\xi'\theta)(H)$. We have $\xi'\theta = -\theta'/\theta$ outside a compact neighbourhood of zero, hence $\xi(H)'$ is a bounded operator, so $\xi(H)$ is of class $C^1(A)$, hence $\xi(H)D(A) \subset D(A)$. Then, since $H'D(H) \subset \mathcal{H}$, this estimate remains true for any $u \in D(A)$ by Proposition 3.6. Thus $|\langle u|\phi^\perp \mathrm{e}^{\mathrm{i}tH}u\rangle| \le C_u\langle t\rangle^{-1}$ for any $u \in D(A)$.

From now on we change notations: ϕu will be denoted u. So we may assume $\operatorname{supp} E_u \subset [-1, 1]$ and $u, Au \in \mathcal{E}$, cf. Lemma 2.3, and we want to prove that the function $t\psi_u'(t)$ belongs to $L^2(\mathbb{R})$. Let η be the C^1 function on $\mathbb{R}$ which extends $\lambda/\theta(\lambda)$, let $\zeta \in C_{\mathrm{c}}^\infty(\mathbb{R})$ such that $\zeta(H)u = u$, and let us set $\tilde{\eta} = \eta\zeta$. Then $\tilde{\eta}(H)u \in D(A) \cap \mathcal{E}$ and $A\tilde{\eta}(H)u = [A, \tilde{\eta}(H)]u + \tilde{\eta}(H)Au \in \mathcal{E}$. Finally

$$\begin{aligned} -\mathrm{i}t\psi_u'(t) &= \langle u|tH\mathrm{e}^{\mathrm{i}tH}u\rangle = \langle u|\eta(H)tH'\mathrm{e}^{\mathrm{i}tH}u\rangle = \langle \tilde{\eta}(H)u|[\mathrm{e}^{\mathrm{i}tH}, A]\zeta(H)u\rangle \\ &= \langle \mathrm{e}^{-\mathrm{i}tH}\tilde{\eta}(H)u|A\zeta(H)u\rangle - \langle A\tilde{\eta}(H)u|\mathrm{e}^{\mathrm{i}tH}\zeta(H)u\rangle \end{aligned}$$

and from (2.2) we get the square integrability of $t\psi_u'(t)$. □

Example 4.9. Let $H \in C^1(A)$ such that $H \ge 0$ and $H' = H(1 + H)^{-1}$. If $u \in D(A)$ and $u, Au \in \mathcal{E}$ then $|\psi_u(t)| \le C_u\langle t\rangle^{-1/2}$.

5. Higher-Order Commutators

The decay estimates obtained so far on $\psi_u(t)$ are at most of order $\mathcal{O}(t^{-1})$ and it is clear that to obtain $\mathcal{O}(t^{-k})$ for some integer $k > 1$ we need conditions of the form $u \in D(A^k)$ and assumptions on the higher-order commutators of A with H. We recall here the necessary formalism.

Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $k \in \mathbb{N}$. We say that S *is of class* $C^k(A)$, and we write $S \in C^k(A)$, if the map $\mathbb{R} \ni t \mapsto \mathrm{e}^{-\mathrm{i}tA} S \mathrm{e}^{\mathrm{i}tA} S \in B(\mathcal{H})$ is of class C^k in the strong operator topology. It is clear that $S \in C^{k+1}(A)$ if and only if $S \in C^1(A)$ and $S' \in C^k(A)$. If $S \in C^2(A)$ we set $(S')' = S'' = S^{(2)}$, etc. Clearly $C^k(A)$ is a $*$-subalgebra of $B(\mathcal{H})$ and if $S \in B(\mathcal{H})$ is bijective and $S \in C^k(A)$ then $S^{-1} \in C^k(A)$.

For any $S \in B(\mathcal{H})$ let $\mathcal{A}(S) = [S, \mathrm{i}A]$ considered as a sesquilinear form on $D(A)$. We may iterate this and define a sesquilinear form on $D(A^k)$ by:

$$S^{(k)} \equiv \mathcal{A}^k(S) = \mathrm{i}^k \sum_{i+j=k} \frac{k!}{i!j!}(-A)^i S A^j.$$

Then $S \in C^k(A)$ if and only if this form is continuous for the topology induced by $\mathcal{H}$ on $D(A^k)$. We keep the notation $\mathcal{A}^k(S)$ or $S^{(k)}$ for the bounded operator associated to its continuous extension to $\mathcal{H}$.

Strictly speaking, the operator $\mathcal{A}$ acting in $B(\mathcal{H})$ must be defined as the infinitesimal generator of the group of automorphisms $\mathcal{U} = \{\mathcal{U}_t\}_{t\in\mathbb{R}}$ of $B(\mathcal{H})$ given by $\mathcal{U}_t(S) \equiv \mathrm{e}^{t\mathcal{A}}(S) = \mathrm{e}^{-\mathrm{i}tA} S \mathrm{e}^{\mathrm{i}tA}$. This group is not of class C_0 and so $\mathcal{A}$ is not densely defined. Then $C^k(A)$ is just the domain of $\mathcal{A}^k$. One may also define $C^\alpha(A)$ if α is not an integer as the Besov space of order (α, ∞) associated to $\mathcal{U}$.

We denote $B_1(\mathcal{H})$ the Banach algebra of trace class operators on $\mathcal{H}$. Its dual is identified with the space $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ with the help of the bilinear form $\mathrm{Tr}(S\rho)$. It is clear that the restrictions of the $\mathcal{U}_t$ to $B_1(\mathcal{H}) \subset B(\mathcal{H})$ give a group of automorphisms of $B_1(\mathcal{H})$ and that this group is of class C_0. We do not distinguish in notation between $\mathcal{U}$ and $\mathcal{A}$ and their restrictions to $B_1(\mathcal{H})$ but note that for example the domain of $\mathcal{A}$ in $B_1(\mathcal{H})$ is the set of $S \in C^1(A) \cap B_1(\mathcal{H})$ such that $\mathcal{A}(S) \in B_1(\mathcal{H})$. Moreover, if $S = |u\rangle\langle v|$ and $u, v \in D(A^k)$ then S belongs to the domain of $\mathcal{A}^k$ in $B_1(\mathcal{H})$.

Now let H be a self-adjoint operator on $\mathcal{H}$ and $R(z) = (H - z)^{-1}$ for z in the resolvent set $\rho(H)$ of H. We say that H *is of class* $C^k(A)$ if $R(z_0) \in C^k(A)$ for some $z_0 \in \rho(H)$; then we shall have $R(z) \in C^k(A)$ for all $z \in \rho(H)$ and more generally $\varphi(H) \in C^k(A)$ for a large class of functions φ (e.g. rational and bounded on the spectrum of H).

For each real m let $S^m(\mathbb{R})$ be the set of symbols of class m on $\mathbb{R}$, i.e. the set of functions $\varphi : \mathbb{R} \to \mathbb{C}$ of class C^∞ such that $|\varphi^{(k)}(\lambda)| \le C_k \langle\lambda\rangle^{m-k}$ for all $k \in \mathbb{N}$. Note that $S^m \cdot S^n \subset S^{m+n}$ and $\varphi^{(j)} \in S^{m-j}$ if $\varphi \in S^m$ and $j \in \mathbb{N}$.

Proposition 5.1. *Let H be a self-adjoint operator of class $C^1(A)$ with $H' = \theta(H)$ for some $\theta \in S^2(\mathbb{R})$. Then H is of class $C^\infty(A)$. Let δ_θ be the first-order differential operator given by $\delta_\theta = \theta(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}$. If $\theta \in S^1(\mathbb{R})$ and $\varphi \in S^0(\mathbb{R})$ then $\varphi(H)$ is of class $C^\infty(A)$ and*

$$\mathcal{A}^k(\varphi(H)) = (\delta_\theta^k \varphi)(H) \quad \forall k \in \mathbb{N}. \tag{5.9}$$

Proof. We begin with a general remark. Using Proposition 3.8 we see that if H is of class $C^1(A)$ and $H' = \theta(H)$ for some real Borel function θ, and if $\varphi \in C^1_{\mathrm{b}}(\mathbb{R})$, then $\varphi(H)' = \mathcal{A}(\varphi(H)) = \theta(H)\varphi'(H) = (\delta_\theta\varphi)(H)$. In particular, if $\delta_\theta\varphi = \theta\varphi'$ is a bounded function then $\varphi(H)$ is of class $C^1(A)$.

If we take $\theta \in S^2$ and $\varphi(\lambda) = (\lambda + \mathrm{i})^{-1}$ then $\varphi \in S^{-1}$ hence $\theta\varphi' \in S^0$. Thus, the operator $R = (H + \mathrm{i})^{-1} = \varphi(H)$ satisfies $R' = \psi(H)$ with $\psi \in S^0$. Now we may apply the preceding argument with φ replaced by ψ and get $\psi \in C^1(A)$, so $R' \in C^1(A)$, etc. This proves that H is of class $C^\infty(A)$.

In the preceding argument we clearly may take any $\varphi \in S^{-1}$. If $\theta \in S^1$ then the same argument works for any $\varphi \in S^0$ and gives the last assertion of the proposition. □

Remark 5.2. If $\theta \in S^m$ and $\varphi \in S^{-(m-1)}$ with $1 \le m \le 2$ the last assertion of the proposition remains true (with the same proof).

We finish this section with some comments in connection with relation (5.9). At a formal level (5.9) means

$$\mathrm{e}^{-\mathrm{i}tA}\varphi(H)\mathrm{e}^{\mathrm{i}tA} \equiv \mathrm{e}^{t\mathcal{A}}(\varphi(H)) = (\mathrm{e}^{t\delta_\theta}\varphi)(H). \tag{5.10}$$

We shall explain without going into details how one may rigorously interpret this relation and how one may use it to get decay estimates.

Let ξ_t be the flow of diffeomorphisms of the real line defined by the vector field $\delta_\theta = \theta(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}$. This means that $\frac{\mathrm{d}}{\mathrm{d}t}\xi_t(\lambda) = \theta(\xi_t(\lambda))$ and $\xi_0(\lambda) = \lambda$ for all $\lambda \in \mathbb{R}$ (we assume that such a global flow exists). Then if $\varphi : \mathbb{R} \to \mathbb{C}$ is a smooth function we have $\frac{\mathrm{d}}{\mathrm{d}t}\varphi \circ \xi_t = (\delta_\theta\varphi) \circ \xi_t$ or $\varphi \circ \xi_t = \mathrm{e}^{t\delta_\theta}\varphi$. Hence (5.10) may be written $\mathrm{e}^{-\mathrm{i}tA}\varphi(H)\mathrm{e}^{\mathrm{i}tA} = (\varphi \circ \xi_t)(H)$. This can be easily checked independently of what we have done before.

Let $M(\mathbb{R})$ be the space of all bounded Borel measures on $\mathbb{R}$. We associate to H a continuous linear map $\Phi : B_1(\mathcal{H}) \to M(\mathbb{R})$ defined as follows: if $\rho \in B_1(\mathcal{H})$ then $\int \varphi\Phi(\rho) = \mathrm{Tr}(\varphi(H)\rho)$ for any bounded Borel function φ. Then

$$\mathrm{Tr}(\varphi(H)\mathcal{U}_{-t}(\rho)) = \mathrm{Tr}(\mathrm{e}^{-\mathrm{i}tA}\varphi(H)\mathrm{e}^{\mathrm{i}tA}\rho) = \mathrm{Tr}((\varphi \circ \xi_t)(H)\rho)$$

which means that the measure $\Phi(\mathcal{U}_{-t}(\rho))$ is equal to the image of the measure $\Phi(\rho)$ through the map ξ_t. Or, if we denote V_t the map $M(\mathbb{R}) \to M(\mathbb{R})$ which sends a measure μ into its image $\xi_t^*(\mu)$ through ξ_t, we have $\Phi \circ \mathcal{U}_{-t} = V_t \circ \Phi$.

Thus, if ρ belongs to the Besov space $B_1(\mathcal{H})_{s,p}$ associated to the group of automorphisms $\mathcal{U}_t$ of $B_1(\mathcal{H})$ then $\Phi(\rho)$ belongs to the Besov space $M(\mathbb{R})_{s,p}$ associated to the group of automorphisms V_t of $M(\mathbb{R})$ (notations as in [1]). This gives smoothness properties of the measure $\Phi(\rho)$ with respect to the differential operator δ_θ in terms of smoothness properties of ρ with respect to the operator A. In particular, since $\mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}\rho) = \int \mathrm{e}^{\mathrm{i}t\lambda}\Phi_\rho(\mathrm{d}\lambda)$ is just the Fourier transform of the measure $\Phi_\rho \equiv \Phi(\rho)$, this allows us to control the decay as $t \to \infty$ of $t \mapsto \mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}\rho)$ in terms of the local behaviour of the measure Φ_ρ.

The operators V_t can be explicitly computed in many situations and the preceding strategy gives optimal results. For example, in the simplest case $[H, \mathrm{i}A] = 1$ we get for any $s > 0$

$$\|\langle A\rangle^{-s} \mathrm{e}^{\mathrm{i}tH} \langle A\rangle^{-s}\| \le C_s \langle t\rangle^{-s} \tag{5.11}$$

If $[H, \mathrm{i}A] = H$ then such a good decay is impossible because zero is a threshold (see Remark 6.2). On the other hand, if η is a smooth function equal to zero near zero and to one near infinity then (see [8]):

$$\|\langle A\rangle^{-s} \mathrm{e}^{\mathrm{i}tH} \eta(H) \langle A\rangle^{-s}\| \le C_s \langle t\rangle^{-s}. \tag{5.12}$$

Estimates of this nature hold in fact for a large class of commutation relations $[H, \mathrm{i}A] = \theta(H)$. Moreover, if the function η is of compact support and such that a strict Mourre estimate holds on a neighbourhood of its support then $\langle A\rangle^{-s} \mathrm{e}^{\mathrm{i}tH} \eta(H) \langle A\rangle^{-s}$ may be controlled in terms of the regularity of the boundary values of the resolvent $R(\lambda \pm \mathrm{i}0)$ via a Fourier transformation argument. The higher-order continuity properties of the operators $R(\lambda \pm \mathrm{i}0)$ as functions of λ in a region where one has a strict Mourre estimate have been studied by commutator methods first in [24] and then in [7] where the optimal regularity result has been obtained. This gives the following decay (see [8]): if the self-adjoint operator H has a spectral gap and is of class $C^{s+1/2}(A)$ for some real $s > 1/2$, and if η is a C^∞ function with compact support in an open set where A is strictly conjugate to H, then there is a number C such that

$$\|\langle A\rangle^{-s} \mathrm{e}^{\mathrm{i}tH} \eta(H) \langle A\rangle^{-s}\| \le C_s \langle t\rangle^{-(s-1/2)}. \tag{5.13}$$

This decay is the best possible for H of class $C^{s+1/2}(A)$. This may be compared with the corresponding result in [24, p. 222] but one should take into account the remark in [7, p. 13]. Note that (5.13) is an endpoint estimate and it can be improved by interpolation at intermediary points. For example, if $H \in C^\infty(A)$ and $s > \varepsilon > 0$ then we have

$$\|\langle A\rangle^{-s} \mathrm{e}^{\mathrm{i}tH} \eta(H) \langle A\rangle^{-s}\| \le C_{s,\varepsilon} \langle t\rangle^{-(s-\varepsilon)}.$$

The problem with these estimates is that the cutoff function η eliminates the thresholds of H, i.e. exactly the energies in which we are interested. We have explained before that the behaviour of $\|\langle A\rangle^{-s} \mathrm{e}^{\mathrm{i}tH} \langle A\rangle^{-s}\|$ may be very bad because of the thresholds.

We emphasize that in this article we are mainly interested in global estimates which take into account the existence of thresholds. To get some decay we consider self evanescent states $u \in \mathcal{E}(H)$ and show how one can get a better decay of the physically meaningful quantity $\psi_u(t)$. Our results are obtained by a direct study of the evolution operator $\mathrm{e}^{\mathrm{i}tH}$ and do not involve regularity properties of the resolvent of H.

6. Higher-Order Decay

The expressions $\psi_u(t) = \langle u|\mathrm{e}^{\mathrm{i}tH} u\rangle$ that we considered until now are quadratic in u and this complicates the computations of higher order. To elude this we note that $\psi_u(t) = \mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}\rho)$ with $\rho = |u\rangle\langle u|$, expression which makes sense for any $\rho \in B_1(\mathcal{H})$ and is linear in ρ.

We begin with an extension to higher orders of Proposition 4.3.

Theorem 6.1. *Let $k \in \mathbb{N}$ and $s \in [0, k]$ real. Assume that H is of class $C^{k+1}(A)$ and H' commutes with H, satisfies $H'D(H) \subset \mathcal{H}$, and is boundedly invertible. Then for each vector $u \in D(|A|^s)$ we have $\psi_u(t) = \mathcal{O}(t^{-s})$.*

Proof. By an interpolation argument it suffices to prove $|\psi_u(t)| \le C_k(\|u\| + \|A^k u\|)^2 \langle t\rangle^{-k}$ for u in a dense subspace of $D(A^k)$. Formally this is quite straightforward starting with the formula $(\mathrm{i}tH')^{-1}\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}) = \mathrm{e}^{\mathrm{i}tH}$ and then iterating it k times; we next sketch the rigorous proof. We change slightly the notations from the proof of Proposition 4.3 and denote K the continuous extension to $\mathcal{H}$ of $-\mathrm{i}H'^{-1}$. Then K commutes with H, is of class $C^k(A)$, and we have $K\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}) = \mathcal{A}(\mathrm{e}^{\mathrm{i}tH})K = t\mathrm{e}^{\mathrm{i}tH}$. Let $u \in D(A^k)$ and $\rho = |u\rangle\langle u|$ or a more general trace class operator. Let L_K and R_K be the operators of right and left multiplication by K, which act both in $B(\mathcal{H})$ and in $B_1(\mathcal{H})$. Then $R_K\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}) = t\mathrm{e}^{\mathrm{i}tH}$ hence

$$\begin{aligned} t\psi_u(t) &= \mathrm{Tr}((R_K\mathcal{A})(\mathrm{e}^{\mathrm{i}tH})\rho) = \mathrm{Tr}(\mathcal{A}(\mathrm{e}^{\mathrm{i}tH})(K\rho)) \\ &= -\,\mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}\mathcal{A}(K\rho)) = -\,\mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}(\mathcal{A}L_K)\rho) \end{aligned}$$

This is easy to justify since $Ku \in D(A^k)$ because K is of class $C^k(A)$. In exactly the same way, starting with $(R_K\mathcal{A})^k(\mathrm{e}^{\mathrm{i}tH}) = t^k\mathrm{e}^{\mathrm{i}tH}$ we get

$$t^k\psi_u(t) = \mathrm{Tr}((R_K\mathcal{A})^k(\mathrm{e}^{\mathrm{i}tH})\rho) = (-1)^k\,\mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}(\mathcal{A}L_K)^k\rho).$$

Finally, it remains to note that $\|(\mathcal{A}L_K)^k\rho\|_{B_1(\mathcal{H})} \le C_k(\|u\| + \|A^k u\|)^2$. □

Remark 6.2. The following example shows that such a good decay as in Theorem 6.1 cannot be expected if H' is not boundedly invertible. In the Hilbert space $\mathcal{H} = L^2(0,\infty)$ let H be the operator of multiplication by the independent variable x and let A be the self-adjoint realization of $\frac{\mathrm{i}}{2}(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}x)$. Then H is of class $C^\infty(A)$ and $H' = [H, \mathrm{i}A] = H$. Let u be a C^∞ function on $(0,\infty)$ which is zero for $x > 2$ and equal to $x^{-\theta}$ for $x < 1$ with $0 < \theta < 1/2$. Then $u \in D(|A|^s)$ for all $s > 0$ but $\psi_u(t) \sim \int_0^1 \mathrm{e}^{\mathrm{i}tx}x^{-2\theta}\mathrm{d}x \sim t^{2\theta-1}$ for $t \to \infty$, hence the decay can be made as bad as possible. On the other hand, Example 2.4 explains why the space $\mathcal{E}$ helps to improve the behaviour.

We now give a higher-order version of Theorem 4.8. Recall that $\theta \in S^m(\mathbb{R})$ is an *elliptic symbol* if there is $c > 0$ such that $|\theta(\lambda)| \ge c|\lambda|^m$ near infinity. Then $\eta/\theta \in S^{-m}(\mathbb{R})$ for any C^∞ function η with support in the region where $\theta \ne 0$ and equal to one near infinity.

Theorem 6.3. *Let $H \in C^1(A)$ such that $H' = \theta(H)$ for some elliptic symbol $\theta \in S^m(\mathbb{R})$ with $0 \le m \le 1$. Assume: (1) $\theta(\lambda) \ne 0$ if $\lambda \ne 0$ and (2) $\lambda/\theta(\lambda)$ extends to a C^∞ function on $\mathbb{R}$. Let k be an odd integer and let $u \in \mathcal{H}$ be of the form $|H|^{(k-1)/4}v$ for some $v \in D(A^k)$ such that $A^j v \in \mathcal{E}$ for $0 \le j \le k$. Then $|\psi_u(t)| \le C_u\langle t\rangle^{-k/2}$.*

Proof. Denote $S^0_{(0)}(\mathbb{R})$ the set of $a \in S^0(\mathbb{R})$ such that $a(\lambda) = 0$ near zero. We first prove the following: if $n \in \mathbb{N}$ and $a \in S^0_{(0)}(\mathbb{R})$ then there are $a_0, a_1, \ldots, a_n \in S^0(\mathbb{R})$ such that:

$$t^n a(H)\mathrm{e}^{\mathrm{i}tH} = \sum_{j=0}^{n} \mathcal{A}^j(a_j(H)\mathrm{e}^{\mathrm{i}tH}). \tag{6.14}$$

Of course, the a_j also depend on n. If $n = 1$ we write (see also the proof of Theorem 4.8):

$$ta(H)\mathrm{e}^{\mathrm{i}tH} = -\mathrm{i}\frac{a(H)}{\theta(H)}\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}) = \mathcal{A}(a_1(H)\mathrm{e}^{\mathrm{i}tH}) + a_0(H)\mathrm{e}^{\mathrm{i}tH} \tag{6.15}$$

where $a_1 = \frac{a}{\mathrm{i}\theta}$ and $a_0 = -\theta a_1'$ (use Proposition 5.1). We mention that we use without comment the relation $\mathcal{A}(ST) = \mathcal{A}(S)T + S\mathcal{A}(T)$ with the further simplification that in our context S and T are functions of H hence commute. Now assume (6.14) is true and let us prove it with n replaced by $n+1$. Let $b \in C^\infty$ equal to zero near zero and to 1 near infinity and such that $a_j = a_j b$ for all j. Then

$$t^{n+1}a(H)\mathrm{e}^{\mathrm{i}tH} = \sum_{j=0}^{n} \mathcal{A}^j(a_j(H)tb(H)\mathrm{e}^{\mathrm{i}tH})$$

Now we use (6.15) and replace $tb(H)\mathrm{e}^{\mathrm{i}tH} = \mathcal{A}(b_1(H)\mathrm{e}^{\mathrm{i}tH}) + b_0(H)\mathrm{e}^{\mathrm{i}tH}$. Thus

$$\begin{aligned} a_j(H)tb(H)\mathrm{e}^{\mathrm{i}tH} &= a_j(H)\mathcal{A}(b_1(H)\mathrm{e}^{\mathrm{i}tH}) + a_j(H)b_0(H)\mathrm{e}^{\mathrm{i}tH} \\ &= \mathcal{A}(a_j(H)b_1(H)\mathrm{e}^{\mathrm{i}tH}) + (a_j(H)b_0(H) - \mathcal{A}(a_j(H))b_1(H))\mathrm{e}^{\mathrm{i}tH} \end{aligned}$$

which clearly gives the required result.

Now we begin the proof of the theorem. As in the proof of Theorem 4.8 we consider separately the case when u is zero near energy zero and that when $u = \varphi(H)u$ for some $\varphi \in C^\infty_{\mathrm{c}}$. The first case is an immediate consequence of (6.14) because there is $a \in S^0_{(0)}(\mathbb{R})$ such that $a(H)u = u$ hence (recall the notation $\rho = |u\rangle\langle u|$)

$$t^k\langle u|\mathrm{e}^{\mathrm{i}tH}u\rangle = \mathrm{Tr}(t^k a(H)\mathrm{e}^{\mathrm{i}tH}\rho) = \sum_{j=0}^{k}(-1)^j\,\mathrm{Tr}(a_j(H)\mathrm{e}^{\mathrm{i}tH}\mathcal{A}^j(\rho)) \tag{6.16}$$

which implies $\psi_u(t) = \mathcal{O}(t^{-k})$ because obviously $\rho \in D(\mathcal{A}^k)$ if $u \in D(A^k)$.

Note that the facts established above hold for an arbitrary $u \in \mathcal{H}$. The condition involving v is needed to have some control on the behaviour of u at zero energy, which cannot be arbitrary as explained in Remark 6.2. When we localize near zero energy we replace u by $\varphi(H)u$ with $\varphi \in C^\infty_{\mathrm{c}}$ equal to one on a neighbourhood of zero. If $u = |H|^m v$ with $m = (k-1)/4$ and $v \in D(A^k)$ such that $A^j v \in \mathcal{E}$ for $0 \le j \le k$ then $\varphi(H)u = |H|^m\varphi(H)v$. By Proposition 5.1 H is of class $C^\infty(A)$ so $\varphi(H)D(A^j) \subset D(A^j)$ for any j and $A^j\varphi(H)v \in \mathcal{E}$ by Lemma 2.3.

Thus for the rest of the proof we may assume that the support of u in a spectral representation of H is included in $[-1,1]$ and $u = |H|^m v$ with

$v \in D(A^k)$ such that $A^j v \in \mathcal{E}$ for $0 \le j \le k$. It is clear that v has the same H-support as u. Our purpose is to check the assumptions of the Corollary 8.3 for $\psi = \psi_u$. There are two conditions to be verified: the functions $t^{\frac{k-1}{2}}\psi_u(t)$ and $t^{\frac{k+1}{2}}\psi_u'(t)$ should be in $L^2(\mathbb{R})$. We treat only the second one, the first is treated similarly. If $\ell = 2m+1 = (k+1)/2$ then

$$\begin{aligned} t^\ell \psi_u'(t) = \langle u | \mathrm{i} t^\ell H \mathrm{e}^{\mathrm{i}tH} u\rangle &= \langle |H|^m v | \mathrm{i}t^\ell H \mathrm{e}^{\mathrm{i}tH} |H|^m v\rangle \\ &= \langle v | \mathrm{i}t^\ell H^\ell \mathrm{sgn}^{\ell+1}(H) \mathrm{e}^{\mathrm{i}tH} v\rangle. \end{aligned}$$

Let η be a C^∞ function with compact support such that $\eta(\lambda) = \lambda/\theta(\lambda)$ on $[-1,1]$. Then $\lambda = \eta(\lambda)\theta(\lambda)$ on $[-1,1]\setminus\{0\}$ hence on $[-1,1]$ so we have

$$\mathrm{i}^{\ell-1} t^\ell \psi_u'(t) = \langle \eta(H)^\ell v | \mathrm{i}^\ell t^\ell \theta(H)^\ell \mathrm{e}^{\mathrm{i}tH} v\rangle = \langle \eta(H)^\ell v | (\mathrm{i}tH')^\ell \mathrm{e}^{\mathrm{i}tH} v\rangle.$$

Recall that we have $\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}) = \mathrm{i}tH'\mathrm{e}^{\mathrm{i}tH}$ in a sense described in Proposition 3.8. But under the present conditions we have much more because $H'D(H) \subset \mathcal{H}$ hence $\mathrm{e}^{\mathrm{i}\tau A}$ leaves invariant the domain of H and induces there a C_0-group [see the assertion (2) page 7]. In particular, the set of $u \in D(H) \cap D(A^j)$ such that $A^j u \in D(H)$ for any $j \in \mathbb{N}$ is dense in $D(H)$ (and is a core for A). Moreover, the $\mathcal{A}^j(H)$ are bounded operators if $j \ge 2$. This allows us to compute $\mathcal{A}^\ell(\mathrm{e}^{\mathrm{i}tH})$ inductively as usual. Our next computations look slightly formal but it is straightforward, although a little tedious, to rigorously justify each step.

Above we fixed ℓ to the value $(k+1)/2$ but now we allow it to take any value smaller than this one. For the case $\ell = 1$ see the proof of Theorem 4.8. For $\ell = 2$ we write

$$\begin{aligned} \mathcal{A}^2(\mathrm{e}^{\mathrm{i}tH}) &= \mathcal{A}(\mathrm{i}tH'\mathrm{e}^{\mathrm{i}tH}) = \mathrm{i}tH''\mathrm{e}^{\mathrm{i}tH} + (\mathrm{i}tH')^2\mathrm{e}^{\mathrm{i}tH} \\ &= \frac{H''}{H'}\mathrm{i}tH'\mathrm{e}^{\mathrm{i}tH} + (\mathrm{i}tH')^2\mathrm{e}^{\mathrm{i}tH} = \frac{H''}{H'}\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}) + (\mathrm{i}tH')^2\mathrm{e}^{\mathrm{i}tH} \end{aligned}$$

By "localising" Proposition 5.1 we get $H'' = \mathcal{A}(H') = \mathcal{A}(\theta(H)) = \theta(H)\theta'(H) = H'\theta'(H)$ hence $\frac{H''}{H'} = \theta'(H)$. Thus

$$(\mathrm{i}tH')^2\mathrm{e}^{\mathrm{i}tH} = \mathcal{A}^2(\mathrm{e}^{\mathrm{i}tH}) - \theta'(H)\mathcal{A}(\mathrm{e}^{\mathrm{i}tH}).$$

Then if we set $\rho = |\eta^2(H)v\rangle\langle v|$ we get

$$\begin{aligned} \mathrm{i}t^2\psi_u'(t) &= \mathrm{Tr}((\mathrm{i}tH')^2\mathrm{e}^{\mathrm{i}tH}\rho) = \mathrm{Tr}(\mathcal{A}^2(\mathrm{e}^{\mathrm{i}tH})\rho) - \mathrm{Tr}(\theta'(H)\mathcal{A}(\mathrm{e}^{\mathrm{i}tH})\rho) \\ &= \mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}\mathcal{A}^2(\rho)) - \mathrm{Tr}(\mathrm{e}^{\mathrm{i}tH}\mathcal{A}(\rho\theta'(H))). \end{aligned}$$

The right-hand side belongs to $L^2(\mathbb{R})$ by the argument from Theorem 4.8, which finishes the proof in the case $\ell = 2$. The general case does not involve any new idea: by writing conveniently $\frac{H^{(\ell)}}{H'}$ one may express $(\mathrm{i}tH')^\ell\mathrm{e}^{\mathrm{i}tH}$ as a linear combination of functions of H times commutators $\mathcal{A}^j(\mathrm{e}^{\mathrm{i}tH})$ and one may proceed as above. □

7. Applications

We will use the previous results to obtain decay estimates for $\psi_u(t) = \langle u|\mathrm{e}^{\mathrm{i}tH}u\rangle$ in several situations. Note that Example 3.1 and Proposition 3.2 show that the commutation relation is not enough to prove the $C^1(A)$ condition for H. For instance, in addition to the continuity of $[A,H]_0$ on $D(A)\cap D(H)$, it suffices to verify the invariance of domain $R(z)D(A)\subset D(A)$. In other cases, it is convenient to verify the simplified assumptions of Mourre [30], which are stronger than the $C^1(A)$ property [1]:

(a) $\mathrm{e}^{\mathrm{i}\theta A}D(H)\subset D(H)$

(b) There is a subspace $\mathscr{S}\subset D(A)\cap D(H)$ which is a core for H such that $\mathrm{e}^{\mathrm{i}\theta A}\mathscr{S}\subset\mathscr{S}$ and the form $[H,\mathrm{i}A]$ on $\mathscr{S}$ extends to a continuous operator $D(H)\to\mathcal{H}$.

Recall that $\mathcal{E}=\{u\in\mathcal{H}\mid [u]_H<\infty\}$, where $[u]_H=\left(\int_{\mathbb{R}}|\psi_u(t)|^2\mathrm{d}t\right)^{1/4}$.

7.1. Example 1: Laplacian in $\mathbb{R}^n$

Let $H=-\Delta$ in $L^2(\mathbb{R}^n)$ with domain the Sobolev space $\mathcal{H}^2(\mathbb{R}^n)$ and $A=-\frac{\mathrm{i}}{2}(x\cdot\nabla+\nabla\cdot x)$ the generator of dilations which is essentially self-adjoint on the Schwartz space $\mathscr{S}=\mathcal{S}(\mathbb{R}^n)$. Condition (a) is a consequence of the formula $\mathrm{e}^{\mathrm{i}\theta A}(H+\mathrm{i})^{-1}=(\mathrm{e}^{-2\theta}H+\mathrm{i})^{-1}\mathrm{e}^{-\mathrm{i}\theta A}$, and (b) is satisfied since $\mathscr{S}$ is a core for H which is trivially invariant under the dilation group. Integration by parts on $\mathscr{S}$ shows that $[H,\mathrm{i}A]=2H$. We conclude from Proposition 4.7 that for $u\in D(A)$ such that $u, Au\in\mathcal{E}$, ψ_u satisfies the decay estimate $|\psi_u(t)|\leq C_u\langle t\rangle^{-1/2}$. Higher-order decay estimates follow from Theorem 6.3.

7.2. Example 2: $H=-\partial_{xx}+\partial_{yy}$ in $\mathbb{R}^2$

Let $H=-\partial_{xx}+\partial_{yy}$ and $A=-\frac{\mathrm{i}}{2}(x\cdot\nabla+\nabla\cdot x)$ in $L^2(\mathbb{R}^2)$. With the help of a Fourier transformation we see that H is essentially self-adjoint on $\mathcal{S}(\mathbb{R}^2)$. Clearly $[H,\mathrm{i}A]=2H$, hence the estimate of Example 1 holds. One may treat similarly the case when the operator H in $L^2(\mathbb{R}^n)$ is an arbitrary homogeneous polynomial of order m in the derivatives $\mathrm{i}\partial_1,\dots,\mathrm{i}\partial_n$ with constant coefficients: then $[H,\mathrm{i}A]=mH$.

7.3. Example 3: Electric Field in $\mathbb{R}^n$

Here we study the case $H=-\Delta+\vec{h}\cdot x$ and $A=\mathrm{i}\vec{h}\cdot\nabla$ in $\mathbb{R}^n$, where $\vec{h}$ is a fixed unitary vector. We take again $\mathscr{S}=\mathcal{S}(\mathbb{R}^n)$ as a core for H and then it is easy to check the commutation relation $[H,\mathrm{i}A]=1$. Therefore, Proposition 4.1 provides the estimate $|\psi_u(t)|\leq C_u\langle t\rangle^{-1}$ for $u\in D(A)$, where $C_u=2\|u\|\|Au\|$. Further estimates follow from Theorem 6.1.

7.4. Example 4: $H=-x^{2-\theta}\Delta-\Delta x^{2-\theta}$ in $\mathbb{R}_+$

For $0<\theta<2$ consider $H=-x^{2-\theta}\Delta-\Delta x^{2-\theta}$ and $A=-\frac{\mathrm{i}}{2}(x\cdot\nabla+\nabla\cdot x)$ in $\mathbb{R}_+$. Then $\mathscr{S}=C_{\mathrm{c}}^\infty(\mathbb{R}_+)$ is a core for H and the domain conditions follow from the formula $\mathrm{e}^{-\mathrm{i}\alpha A}H\mathrm{e}^{\mathrm{i}\alpha A}=\mathrm{e}^{\theta\alpha}H$. The commutation relation is $[H,\mathrm{i}A]=\theta H$, which yields the estimate of Example 1.

7.5. Example 5: Fractional Laplacian in $\mathbb{R}^n$

For $0 < s < 2$, let $H = (-\Delta)^{s/2}$ with domain the Sobolev space $\mathcal{H}^s(\mathbb{R}^n)$ and consider $A = -\frac{\mathrm{i}}{2}(x \cdot \nabla + \nabla \cdot x)$. Then $\mathscr{S} = C_\mathrm{c}^\infty(\mathbb{R}_+)$ is a core for H and homogeneity of H with respect to A implies that $[H, \mathrm{i}A] = sH$. The estimate of Example 1 follows.

7.6. Example 6: Multiplication by λ in $L^2(\mathbb{R}_+, \mathrm{d}\mu)$

Let $H = \lambda$ and $A = -\frac{\mathrm{i}}{2}(\lambda\partial_\lambda + g(\lambda))$ on $L^2(\mathbb{R}_+, \mathrm{d}\mu)$, where g is to be determined. Assume that $\mathrm{d}\mu = h(\lambda)\mathrm{d}\lambda$, for some non-vanishing function h of class $C^1(\mathbb{R}_+)$. It can be shown that if g satisfies the relation $g(\lambda) = \lambda\frac{h'}{h} + 1$, then A is self-adjoint in $L^2(\mathbb{R}_+, \mathrm{d}\mu)$. For instance, if $h(\lambda) = \lambda^N$ then choose $g(\lambda) = N+1$. If g is a bounded function, $\mathscr{S} = C_\mathrm{c}^\infty(\mathbb{R}_+)$ is a core for A and the commutation relation is $[H, \mathrm{i}A] = 2H$. For $z \in \rho(H)$ the function $(\lambda - z)^{-1}$ is smooth and has bounded derivative on $\mathbb{R}_+$, hence the domain invariance $R(z)D(A) \subset D(A)$ can be easily checked. Therefore, H is of class $C^1(A)$, which gives the estimate of Example 1.

7.7. Example 7: Dirac Operator in $L^2(\mathbb{R}^3; \mathbb{C}^4)$

We consider the Dirac operator for a spin-1/2 particle of mass $m > 0$ given by $H = \alpha \cdot P + \beta m$ on $\mathcal{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β denote the 4×4 Dirac matrices. The domain of H is the Sobolev space $\mathcal{H}^1(\mathbb{R}^3; \mathbb{C}^4)$ and it is known that $\sigma(H) = \sigma_{\mathrm{ac}}(H) = (-\infty, -m] \cup [m, \infty)$. See the book of Thaller [53].

The Foldy–Wouthuysen transformation U_{FW} maps the free Dirac operator into a 2×2 block form. Consider the Newton–Wigner position operator Q_{NW} defined as the inverse FW-transformation of multiplication by x, that is, $Q_{\mathrm{NW}} = U_{\mathrm{FW}}^{-1} Q U_{\mathrm{FW}}$. Using $A = Q_{\mathrm{NW}}$, then H is of class $C^1(A)$ and direct calculation shows that $[H, \mathrm{i}A] = \sqrt{H^2 - m^2}H^{-1}$ [44]. The following decay estimate follows from this commutation relation.

Proposition 7.1. *Let H and A as above. Then for $u \in D(A) \cap \mathcal{E}$ such that $Au \in \mathcal{E}$, one has the estimate $|\psi_u(t)| \le C_u\langle t\rangle^{-1/2}$.*

Proof. Let $\varphi \in C_c^\infty([m, \infty))$ real and equal to one on a small interval $[m, m+\epsilon]$ and set $\phi = \varphi(H)$, $\phi^\perp = 1 - \phi^2$. For simplicity we assume u in the subspace of positive energies, then $\psi_u = \psi_{\phi u} + \langle \phi^\perp u | \mathrm{e}^{\mathrm{i}tH} u\rangle$. For the high-energy region

$$\begin{aligned} t\langle u|\phi^\perp \mathrm{e}^{\mathrm{i}tH} u\rangle &= \langle \phi^\perp u| t\mathrm{e}^{\mathrm{i}tH} u\rangle \\ &= \langle \phi^\perp u| H(H^2 - m^2)^{-1/2}[\mathrm{e}^{\mathrm{i}tH}, A]u\rangle \\ &= \langle H(H^2 - m^2)^{-1/2}\phi^\perp \mathrm{e}^{-\mathrm{i}tH} u|Au\rangle \\ &\quad - \langle AH(H^2 - m^2)^{-1/2}\phi^\perp u|\mathrm{e}^{\mathrm{i}tH} u\rangle, \end{aligned}$$

and it follows that $|\langle u|\phi^\perp \mathrm{e}^{\mathrm{i}tH} u\rangle| \le C\langle t\rangle^{-1}$.

For energy close to m, assume that the support of u in a spectral representation of H is contained in a compact interval.

Note that $[\mathrm{e}^{\mathrm{i}t(H-m)}, A] = t\sqrt{H^2 - m^2}H^{-1}\mathrm{e}^{\mathrm{i}t(H-m)}$ as continuous forms on $D(H)$.

Define the auxiliary function $\psi(t) = \langle u|\mathrm{e}^{\mathrm{i}t(H-m)}u\rangle$. Then

$$\begin{aligned}-\mathrm{i}t\psi'(t) &= \langle u|(H-m)t\mathrm{e}^{\mathrm{i}t(H-m)}u\rangle \\ &= \langle (H-m)^{1/2}u|H(H+m)^{-1/2}[\mathrm{e}^{\mathrm{i}t(H-m)}, A]u\rangle \\ &= \langle (H-m)^{1/2}H(H+m)^{-1/2}\mathrm{e}^{-\mathrm{i}t(H-m)}u|Au\rangle \\ &\quad - \langle A(H-m)^{1/2}H(H+m)^{-1/2}u|\mathrm{e}^{\mathrm{i}t(H-m)}u\rangle.\end{aligned}$$

The right-hand side is in L^2_t because $H \in C^1(A)$ and u is compactly supported so Lemma 2.3 applies. We conclude that $|\psi(t)| \leq C\langle t\rangle^{-1/2}$ for all t and since $|\psi_u| = |\psi|$ the result is proven. □

7.8. Example 8: Wave Equation in $\mathbb{R}^n$

For $H > 0$ consider the equation

$$(\mathrm{WE}) \begin{cases} \partial_{tt}u + H^2u = 0 \\ u(0) = f \\ \partial_t u(0) = g. \end{cases}$$

Assume $\mathcal{H} = L^2(\mathbb{R}^n)$. Define $u_1(t) := \cos(tH)$, $u_2(t) := \frac{\sin(tH)}{H}$. Then $u(t) := u_1(t)f + u_2(t)g$ is a solution to (WE).

For $f, g \in \mathcal{H}$ define the function $\psi_{f,g}(t) := \langle f|u_1(t)f\rangle + \langle f|u_2(t)g\rangle$ and the subspace $\mathcal{E} = \{u \in \mathcal{H} \mid [u]_H < \infty\}$, where $[h]_H = \|\langle h|u_1(t)h\rangle\|^{1/2}_{L^2_t} + \|\langle h|u_2(t)h\rangle\|^{1/2}_{L^2_t}$.

Proposition 7.2. *Let H and A be self-adjoint operators, assume $H \in C^1(A)$ and the commutation relation $[H, \mathrm{i}A] = cH$, with $c \neq 0$. Then for $f, g \in D(A) \cap \mathcal{E}$ such that $Af, Ag \in \mathcal{E}$, one has the estimate $|\psi_{f,g}(t)| \leq C_{f,g}\langle t\rangle^{-1/2}$.*

Proof. Similarly to Proposition 3.8, the following two sesquilinear forms restricted to $D(A) \cap D(H)$ extend to continuous forms on $D(H)$ satisfying the identities

$$[\cos(tH), \mathrm{i}A] = -ctH\sin(tH)$$
$$\left[\frac{\sin(tH)}{H}, \mathrm{i}A\right] = ct\cos(tH) - c\,\frac{\sin(tH)}{H}.$$

We will use Corollary 8.2 for $f, g \in D(|H|^{1/2})$. Clearly $\psi_{f,g} \in L^2(\mathbb{R})$. Now we calculate

$$\begin{aligned} ct\psi'_{f,g}(t) &= -\langle f|ctH\sin(tH)f\rangle + \langle f|ct\cos(tH)g\rangle \\ &= \langle f|[u_1, \mathrm{i}A]f\rangle + \langle f|[u_2, \mathrm{i}A]g\rangle + c\langle f, u_2g\rangle \\ &= \langle u_1f|\mathrm{i}Af\rangle + \langle \mathrm{i}Af|u_1f\rangle + \langle u_2f|\mathrm{i}Ag\rangle + \langle \mathrm{i}Af|u_2g\rangle + c\langle f|u_2g\rangle. \end{aligned}$$

Thus $c\|\delta\psi_{f,g}\|_{L^2} \leq C[f]_H([g]_H + [Ag]_H + [Af]_H)$. For f, g not necessarily in $D(H)$ we can proceed analogously to Proposition 4.7 using $u_\epsilon = R_\epsilon u$ and letting $\epsilon \to 0$. □

7.9. Example 9: Klein–Gordon Equation in $\mathbb{R}^n$

Now we draw our attention to (WE) in the case $H = \sqrt{-\Delta + m^2}$, for $m > 0$. The vector space is again defined as $\mathcal{E} = \{u \in \mathcal{H} \mid [u]_H < \infty\}$, where $[h]_H = \|\langle h|\mathrm{e}^{\mathrm{i}tH}h\rangle\|_{L^2_t}^{1/2}$. Let A be the generator of dilations, then H is of class $C^1(A)$ and it can be formally shown that $[H, \mathrm{i}A] = H - m^2 H^{-1}$.

Let u_1, u_2 be as in (WE), define $\psi^1_{f,g}(t) = \langle f|u_1(t)f\rangle$ and $\psi^2_{f,g}(t) = \langle f|u_2(t)g\rangle$. We are interested in the decay rate of $\psi_{f,g} := \psi^1_{f,g}(t) + \psi^2_{f,g}(t)$.

Proposition 7.3. *For H and A defined as above and $f, g \in D(A) \cap \mathcal{E}$ such that $Af, Ag \in \mathcal{E}$, then $|\psi_{f,g}(t)| \le C_{f,g}\langle t\rangle^{-1/2}$.*

Proof. Note that this result is a direct consequence of Proposition 4.8. Higher-order decay estimates follow from Proposition 6.3. Here we present a direct proof.

We define the auxiliary function $\psi(t) := \langle f|\mathrm{e}^{\mathrm{i}t(H-m)}g\rangle$ and we prove that the conditions of Corollary 8.2 are satisfied. It is clear that $\psi \in L^2(\mathbb{R})$ since $f, g \in \mathcal{E}$. Assume $g \in D(H)$ and we estimate

$$\begin{aligned}
-\mathrm{i}t\psi'(t) &= \langle f|t(H-m)\mathrm{e}^{\mathrm{i}t(H-m)}g\rangle \\
&= \langle f|[\mathrm{e}^{\mathrm{i}t(H-m)}, A]H(H+m)^{-1}g\rangle \\
&= \langle \mathrm{e}^{-\mathrm{i}t(H-m)}f|AH(H+m)^{-1}g\rangle + \langle Af|\mathrm{e}^{\mathrm{i}t(H-m)}H(H+m)^{-1}g\rangle.
\end{aligned}$$

By Lemma 2.3 we conclude that $\|\delta\psi\|_{L^2} \le C([f]_H[g]_H + [f]_H[Ag]_H + [Af]_H[g]_H)$. For general $g \in D(A)$, replace it by $g_\epsilon = R_\epsilon g$ and let $\epsilon \to 0$.

We conclude that $|\psi(t)| \le C_{f,g}\langle t\rangle^{-1/2}$. Notice that $|\langle f|\mathrm{e}^{\mathrm{i}tH}g\rangle| = |\psi(t)|$ satisfies the same bound.

Now we prove the desired estimate. Observe that $\psi^1_{f,g}(t) = \frac{1}{2}\big(\langle f|\mathrm{e}^{\mathrm{i}tH}f\rangle + \langle f|\mathrm{e}^{-\mathrm{i}tH}f\rangle\big)$, therefore $|\psi^1_{f,g}(t)| \le C_f\langle t\rangle^{-1/2}$.

For the second term, we write

$$\psi^2_{f,g}(t) = \frac{1}{2}\big(\langle f|H^{-1}\mathrm{e}^{\mathrm{i}tH}g\rangle + \langle f|H^{-1}\mathrm{e}^{-\mathrm{i}tH}g\rangle\big)$$

and redefine the auxiliary function $\psi(t) := \langle f|H^{-1}\mathrm{e}^{\mathrm{i}t(H-m)}g\rangle$, which is in $L^2(\mathbb{R})$ by the spectral theorem. Now

$$\begin{aligned}
-\mathrm{i}t\psi'(t) &= \langle f|tH^{-1}(H-m)\mathrm{e}^{\mathrm{i}t(H-m)}g\rangle \\
&= \langle f|[\mathrm{e}^{\mathrm{i}t(H-m)}, A](H+m)^{-1}g\rangle \\
&= \langle \mathrm{e}^{-\mathrm{i}t(H-m)}f|A(H+m)^{-1}g\rangle + \langle Af|\mathrm{e}^{\mathrm{i}t(H-m)}(H+m)^{-1}g\rangle,
\end{aligned}$$

which again yields the estimate $|\psi(t)| \le C_{f,g}\langle t\rangle^{-1/2}$, concluding the proof. □

Appendix

We prove here an auxiliary estimate. We consider functions g defined on $\mathbb{R}_+ = (0, \infty)$ and denote $\|g\|_p$ their L^p norms. Let δ the operator $(\delta g) = xg'(x)$ acting in the sense of distributions and set $\tilde{g}(t) = \int_0^\infty \mathrm{e}^{\mathrm{i}tx}g(x)\mathrm{d}x$ for $t > 0$ (improper integral).

Lemma 8.1. *$|\tilde{g}(t)| \leq |t|^{-1/2} 2^{3/2} (p-1)^{-1/2p} \|g\|_p^{1/2} \|\delta g\|_q^{1/2}$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. We may assume that $g \in L^p$ and $\delta g \in L^q$. For any $s > 0$ we have

$$\left| \int_0^s \mathrm{e}^{\mathrm{i}tx} g(x) \mathrm{d}x \right| \leq s^{1/q} \|g\|_p. \tag{8.17}$$

Since $g \in L^p$ with $p < \infty$, there is a sequence $a_n \to \infty$ such that $g(a_n) \to 0$ (otherwise $|g(x)| \geq c > 0$ on a neighbourhood of infinity, so $|g|^p$ cannot be integrable). Since $p > 1$, after integrating over (s, a_n) and then making $n \to \infty$, we also obtain

$$|g(s)| \leq \int_s^\infty |g'(x)| \mathrm{d}x \leq (p-1)^{-1/p} s^{1/p-1} \|\delta g\|_q \tag{8.18}$$

by Hölder inequality. Then

$$\begin{aligned} \int_s^\infty \mathrm{e}^{\mathrm{i}tx} g(x) \mathrm{d}x &= \lim_{a \to \infty} \int_s^a \left(\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{\mathrm{i}t} \mathrm{e}^{\mathrm{i}tx} \right) g(x) \mathrm{d}x \\ &= \lim_{a \to \infty} \left[\frac{\mathrm{e}^{\mathrm{i}ta} g(a) - \mathrm{e}^{\mathrm{i}ts} g(s)}{\mathrm{i}t} - \frac{1}{\mathrm{i}t} \int_s^a \mathrm{e}^{\mathrm{i}tx} g'(x) \mathrm{d}x \right]. \end{aligned}$$

We take here $a = a_n$ and make $n \to \infty$ to get

$$-\mathrm{i}t \int_s^\infty \mathrm{e}^{\mathrm{i}tx} g(x) \mathrm{d}x = \mathrm{e}^{\mathrm{i}ts} g(s) + \int_s^\infty \mathrm{e}^{\mathrm{i}tx} g'(x) \mathrm{d}x$$

and then using (8.18) two times we obtain

$$\left| \int_s^\infty \mathrm{e}^{\mathrm{i}tx} g(x) \mathrm{d}x \right| \leq 2(p-1)^{-1/p} s^{-1/q} t^{-1} \|\delta g\|_q.$$

Let $\varepsilon > 0$ and $s = \varepsilon^q / t$. Then (8.17) and the last inequality give

$$|\tilde{g}(t)| \leq \varepsilon t^{-1/q} \|g\|_p + 2(p-1)^{-1/p} \varepsilon^{-1} t^{-1/p} \|\delta g\|_q.$$

The infimum over $\varepsilon > 0$ of an expression $\varepsilon a + \varepsilon^{-1} b$ is $2\sqrt{ab}$. This finishes the proof. □

Corollary 8.2. *If $\psi \in L^2(\mathbb{R})$ and $t\psi'(t) \in L^2(\mathbb{R})$ then $|\psi(t)| \leq C_\psi |t|^{-1/2}$ for $t \in \mathbb{R} \backslash \{0\}$.*

Proof. We use Lemma 8.1 with $p = 2$ and g equal to the Fourier transform of ψ. □

Corollary 8.3. *If a function ψ is such that $t^{\frac{k-1}{2}} \psi(t)$ and $t^{\frac{k+1}{2}} \psi'(t)$ belong to $L^2(\mathbb{R})$ for some $k \geq 1$ then $|\psi(t)| \leq C_\psi |t|^{-k/2}$ for all $t \in \mathbb{R} \backslash \{0\}$.*

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