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# On the $K$-Theoretic Classification of Topological Phases of Matter

Guo Chuan Thiang

**Abstract.** We present a rigorous and fully consistent $K$-theoretic framework for studying gapped phases of free fermions. It utilizes and profits from powerful techniques in operator $K$-theory, which from the point of view of symmetries such as time reversal, charge conjugation, and magnetic translations, is more general and natural than the topological version. In our model-independent approach, the dynamics are only constrained by the physical symmetries, which can be completely encoded using a suitable $C^*$-superalgebra. Contrary to existing literature, we do not use $K$-theory groups to classify phases in an absolute sense, but to classify topological obstructions between phases. The Periodic Table of Kitaev is exhibited as a special case within our framework, and we prove that the phenomena of periodicity and dimension shifts are robust against disorder and magnetic fields.

## 1. Introduction

In [30], it was proposed that gapped phases of non-interacting fermions can be classified using the techniques of topological $K$-theory. In this approach, there are $2 + 8 = 10$ classes of systems to consider in each spatial dimension $d$, based on the presence or absence of time reversal and/or $U(1)$ symmetry. The classification groups exhibit a certain periodicity with respect to $d$ and were attributed, somewhat mysteriously, to Bott periodicity. A Periodic Table was partially drawn up, with each symmetry class in each spatial dimension having one of the $K$-theory groups of a point as its classification group. A series of authors provided their own accounts [1,15,42,45] of this story, but with considerable variations in their treatments. Subsequent work on crystalline and weak topological insulators revealed the existence of phases which are not directly accounted for by the Periodic Table. Despite the lack of a proper proof of (or even the necessary definitions or assumptions in) the Periodic Table, a consensus that it unambiguously provides a complete $K$-theoretic classification of free-fermion phases appears to have been reached; for instance, see the review papers [18,41].

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With the exception of the excellent Freed–Moore paper [15] (which does not address the matter of dimension shifts in the Periodic Table), there has been very little attempt to put $K$-theoretic classification ideas on a firm mathematical footing. Furthermore, there seems to be a number of inconsistencies in the existing literature on topological phases and $K$-theory (see Sect. 2). Consequently, the full machinery of $K$-theory has not been substantially utilized yet. While $K$-theory might appear abstract, it is really a generalization of cohomology and had actually been lurking in the background for some time. Examples can be found in rigorous work on the integer quantum Hall effect (IQHE) [6], bulk-edge correspondence [27–29] and Fermi surfaces [21]. In addition, the Chern numbers commonly used in the physics literature as (cohomological) topological invariants can more fundamentally be understood in $K$-theoretic terms [40].

This paper seeks to address these issues by providing a complete and consistent framework for the use of $K$-theory in the study of gapped topological phases. Our treatment of quantum mechanical symmetries borrows heavily from the comprehensive analysis in [15]. Subsequently, this paper diverges from existing work in several important ways. First, we utilize operator $K$-theory rather its commutative (topological) version, which makes available powerful theorems such as the Connes–Thom isomorphisms, the Packer–Raeburn decomposition and stabilization theorems, and various exact sequences for the $K$-theory of crossed product algebras. The second difference is physical: our representation spaces for the symmetries and Hamiltonians are single-particle Hilbert spaces for charged free fermions. In some cases, they may also be regarded as Dirac–Nambu spaces—this point of view is taken by some other authors [1, 15, 20, 42].

Conceptually, the Clifford algebras enter our $K$-theoretic framework in a fundamental way—as twisted group algebras of time-reversal and/or charge-conjugation symmetries—which generalizes to more complicated symmetry data. Our important physical definitions are completely new and relates to $K$-theory in a mathematically precise way. For instance, Definition 7.1 gives a precise notion of homotopic phases, and Definition 7.3 illustrates how a $K$-theoretic group $\mathbf{K}_0(\cdot)$ classifies obstructions in passing between phases. No unnatural Grothendieck group completion needs to be carried out, and inverses arise simply by taking differences in the opposite order. Indeed, one need not expect that physical phases form a group,[1] and even the idea of classifying phases up to homotopy in an absolute sense can be problematic (see Example 2.2 and [11, 46]). The philosophy of using $K$-theory to classify differences between phases appears, in any case, to be the original intention of Kitaev in [30]. Furthermore the concept of a *relative index* had already been studied in the context of the quantum Hall effect in [5]. The relative picture is also suited for generalizing bulk-edge correspondences to interfaces. After we have set up the crucial definitions, the machinery of $K$-theory takes over and allows

[1] For example, the basic thermodynamic phases {solid, liquid, gas} merely form an ordered three-element set.

us to derive the Periodic Table of Kitaev (when appropriately interpreted) as a simple corollary. Our main result is Theorem 10.5, which demonstrates that the phenomenon of "dimension shifts" and periodicity in the $K$-theoretic classification remains even in the presence of disorder and magnetic fields.

Recall that Wigner's theorem says that a topological symmetry group $G$ (which is not assumed to be compact) for the dynamics of a quantum mechanical system may be represented projectively on a complex Hilbert space $\mathscr{H}$ as unitary or antiunitary operators. A continuous homomorphism $\phi\colon G \to \{\pm 1\}$ distinguishes the unitarily implemented subgroup $G^u := \ker(\phi)$ from the antiunitarily implemented subset $G^a = G - G^u$. For any two $x, y \in G$, their representatives $\theta_x, \theta_y$ satisfy $\theta_x\theta_y = \sigma(x,y)\theta_{xy}$, with $\sigma\colon G \times G \to U(1)$ a generalized 2-cocycle,

$$\sigma(x,y)\sigma(xy,z) = \sigma(y,z)^x\sigma(x,yz), \tag{1}$$

where for $\lambda \in U(1)$, $\lambda^x := \lambda$ if $\phi(x) = +1$ and $\lambda^x := \overline{\lambda}$ if $\phi(x) = -1$. Thus, interesting topology resides not only in the group of symmetries, but also in the cohomological data of $(\phi, \sigma)$. We go a step further and consider charge-conjugating symmetries on the same fundamental level as other symmetries, leading to $\mathbb{Z}_2$-graded representations. This step is already suggested by the central role of charge-conjugation in relativistic quantum theories, and is vindicated in our context by the unifying role of super-algebra in $K$-theory.

Our approach may be summarized as follows: the topology which appears in free-fermion topological phases originates from symmetry data, defined broadly. The data consists of a topological group $G$, a 2-cocycle $\sigma$, a grading homomorphism $c\colon G \to \{\pm 1\}$, and an action $\alpha$ of $G$ on a disorder algebra. Associated to the algebra $\mathcal{A}$ determined by this data is a derived "topological space" (e.g. a Brillouin torus), which is generally *noncommutative* and is of *secondary importance*. Instead of forcing a vector bundle picture, we extract topological invariants directly from $\mathcal{A}$.

**Outline.** Section 2 contains some general comments on the existing literature. We establish our conventions for symmetry-compatible free-fermion dynamics in Sect. 3. Sections 4 and 5 review mathematical material on dynamical systems and twisted crossed products, and can be skimmed over by experts. The relationship between Clifford algebras, group algebras of $CT$-subgroups, and the tenfold way is explained in Sect. 6. We move on to the $K$-theoretic classification of symmetry-compatible gapped Hamiltonians proper in Sect. 7, whose computation is illustrated by examples in Sect. 8. The special case of topological band insulators is treated in Sect. 9. A Periodic Table in the general sense of Kitaev is derived in Sect. 10, and we prove that periodicity and dimension shifts persist under very general conditions in Theorem 10.5.

## 2. Remarks on the Existing Literature

We discuss some subtleties in the mathematics and physical interpretation of $K$-theory groups which seem to have been missed in the existing literature. They are further elaborated upon and rectified in the main body of the paper.

1. A general definition of a symmetry-compatible topological phase begins with the space $Y$ of Hamiltonians (gapped or otherwise) which are compatible with a certain given representation of some symmetry data. This space is sometimes called the "classifying space", which should be distinguished from the mathematical notion with the same name. Two Hamiltonians which are path-connected within this space are identified, so the *set* of phases is the set of path-components $\pi_0(Y)$. On the other hand, a $K$-theory invariant has the crucial and useful additional structure of an abelian *group*. It is important and non-trivial to check whether the algebraic structure (e.g. composition, inverses etc.) of mathematical invariants have natural counterparts in the physical interpretation.

   *Example* 2.1. For Class A systems in 0D, a spectrally flattened gapped Hamiltonian (see Sect. 3.1) is just a grading operator on a Hilbert space $\mathbb{C}^N$. The space of such Hamiltonians is the union of the Grassmannians of $k$-planes in $\mathbb{C}^N$, with $k = 0, 1, \ldots, N$, each of which is connected. Therefore the phases, up to homotopy, form an $(N+1)$-element *set*. Passing to a "large-$N$ limit", we obtain a countably infinite set of phases, which is a different object from the abelian group $\mathbb{Z} \cong K^0(\star)$.

2. In $d > 0$ with $\mathbb{Z}^d$ as a discrete group of translational symmetries, Bloch theory leads to the study of vector bundles over the (compact) Brillouin torus $X = \mathbb{T}^d$. The group $K^0(X)$ of a compact Hausdorff space $X$ can be realized as the Grothendieck completion of the monoid (under direct sum) of isomorphism classes of complex vector bundles over $X$, thus the $K^0$ functor enters naturally. Its *reduced* version $\tilde{K}^0$ ignores the rank of the bundle, and is sometimes motivated by restricting attention to *stable* isomorphism classes of vector bundles. Physically, stabilization entails identifying systems which differ only by some "topologically trivial" subsystem. In the $d = 0$ case, the reduced $K$-theory of a point is trivial since all (finite dimensional) complex vector spaces in question are stably equivalent. To recover the $\mathbb{Z}$-classification for $d = 0$ which appears in many tables [30,42], we should use the unreduced $K$-theory group. We note that higher reduced $K$-theory groups of $X$ do not admit an interpretation in terms of stabilized bundles over $X$. For example, $K^{-1}(X) \cong \tilde{K}^{-1}(X)$, whereas $K^0(X) \cong \tilde{K}^0(X) \oplus \mathbb{Z}$. It is sometimes assumed that the Brillouin zone is a $d$-sphere $S^d$ rather than a torus, so the $K$-theory groups of spheres are studied instead. This should be distinguished from the appearance of $\tilde{K}^0(S^d)$ when systems with continuous $\mathbb{R}^d$ translational symmetry are considered. There, the momentum space (topologically also $\mathbb{R}^d$) is non-compact, and its $K^0$ group is defined to be the reduced $\tilde{K}^0$ group of the one-point compactification $S^d$.

   In this paper, we will study the unreduced $K$-theory of the appropriate space or algebra as determined by the symmetry constraints at hand. Furthermore, we deal primarily with the zeroth $K$-theory functor, or more precisely, $\mathbf{K}_0$ as defined in Definition 7.3, and provide it with a uniform physical interpretation applicable for all choices of symmetry data. Higher

$K$-theory groups only appear secondarily though mathematical identities such as Proposition 7.8. The meaning of the groups which appear in Table 2 is thereby clarified.

3. The precise equivalence relation defining a topological phase should be clearly defined. Topological insulators are often modeled by graded vector bundles, possibly with extra structure dictated by symmetries. Such bundles can be organized into isomorphism classes, graded or otherwise. Intuitively, a notion of "homotopy classes of bundles" is desired.[2] Isomorphism classes of ungraded vector bundles correspond to homotopy classes of *maps* from the base space to an appropriate classifying space, not those of the bundle itself. On the other hand, a gapped Hamiltonian determines a grading on a vector bundle, and it is homotopy within the space of allowed gradings which actually captures the physical intuition of "homotopic gapped Hamiltonians".

   Within a single fixed realization of relevant symmetries on a given representation space, it makes sense to consider homotopies between the compatible Hamiltonians. It is less straightforward to compare Hamiltonians defined on *different* spaces: a *choice* of identification between the two spaces is first required.
4. A detailed treatment of the IQHE, which does not make the assumption of rational flux and includes the effects of disorder, utilizes tools from noncommutative geometry and operator $K$-theory [6]. Despite this, the IQHE is usually included in expositions on the Periodic Table which assume the presence of a meaningful Brillouin zone. Furthermore, it has already been recognized that the presence of point symmetries leads to different classification groups from those in the original table.
5. The Altland–Zirnbauer (AZ) classification of disordered fermionic systems [2, 20] is based on the compact classical symmetric spaces which provide spaces of symmetry-compatible time evolutions. While large-$N$ versions of symmetric spaces also feature in the classifying spaces of $K$-theory, the AZ classification (and indeed its Wigner–Dyson predecessor) made no explicit reference to *gapped* Hamiltonians, whereas $K$-theory is supposed to classify gapped phases. The general approach in the literature (with [15] being an exception) is to keep only the data of the negative-energy states (or valence bands of topological insulators) for the purposes of classification. As long as charge-conjugating (i.e. Hamiltonian reversing) symmetries are not present, this makes good sense and can even be motivated physically. In such cases, the valence and conduction bands *separately* determine topological invariants of the insulating system, with the former usually more interesting. However, one should note that the availability of an interpretation of $K$-theory groups referring only to the valence band, distinguishes the A, AI and AII classes from the other seven classes in the tenfold way (see Sect.

[2] Note that the total space of a bundle can always be retracted onto its base space, which is usually fixed to be $\mathbb{T}^d$, so this is not the relevant notion of homotopy.

9.3). These are the three classes whose topological invariants (the Chern numbers and Kane–Mele $\mathbb{Z}_2$ invariants) have been studied most closely.
6. The use of $KR$-theory to account for time-reversal and charge-conjugation operators is redundant in our operator $K$-theory approach. Actually, band insulators with such discrete symmetries are not yet the Real bundles required for a $KR$-theory analysis. The proper construction of an associated Real bundle is somewhat involved (e.g. see Corollary 10.25 of [15]).

*Example* 2.2 (*Homotopy and isomorphism of band insulators*). Consider a band insulator in one spatial dimension which has one valence band and one conduction band, as well as sublattice/chiral symmetry. Physically, this is a Class AIII insulator, which we model by a rank-two $\mathbb{Z}_2$-graded complex hermitian vector bundle $E$ over $S^1$, equipped with an *odd* fiberwise unitary involution $\mathsf{S}$ implementing the sublattice symmetry.

Fix a global trivialization $E \mapsto (\theta, v) \in S^1 \times \mathbb{C}^2 \cong E$ such that $\mathsf{S}$ acts diagonally as $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on each fiber. According to the usual prescription in the literature [40, 42], a compatible gapped Hamiltonian is a grading $\Gamma$ of $E$, such that the grading operator $\Gamma_\theta$ on each fiber $E_\theta$ anticommutes with $\mathsf{S}_\theta$. Since the $\Gamma_\theta$ are traceless and hermitian, they must be of the off-diagonal form $\Gamma_\theta = \begin{pmatrix} 0 & q(\theta) \\ q(\theta)^\dagger & 0 \end{pmatrix}$, for some continuous function $\theta \mapsto q(\theta) \in U(1) = S^1$. A homotopy between two such functions is precisely a homotopy between the two bundle gradings (AIII band insulator structures) that they determine. Therefore, the set of phases, up to homotopy, is $\pi_1(S^1) \cong \mathbb{Z}$, labeled by the integer winding number of $q$.

With respect to the pre-defined trivialization, the phase labeled by $n \in \mathbb{Z}$ has a representative $\Gamma_n$ defined by $\Gamma_n(\theta) = \begin{pmatrix} 0 & e^{-in\theta} \\ e^{in\theta} & 0 \end{pmatrix}$. A straightforward calculation shows that for any $n, n' \in \mathbb{Z}$, the unitary bundle map $U_{n'-n} : (\theta, (v_1, v_2)) \mapsto (\theta, (e^{-i(n'-n)\theta} v_1, v_2))$ preserves the $\mathsf{S}$-action and intertwines $\Gamma_n$ with $\Gamma_{n'}$. Thus $(E, \Gamma_n)$ and $(E, \Gamma_{n'})$ differ by a change of coordinates (gauge transformation) which respects $\mathsf{S}$. The reference (zero) phase $\Gamma_0$, represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, is a relative notion which depends on the initial choice of trivialization. More generally, to compare two two-band insulators $(E', \Gamma')$ and $(E, \Gamma)$, we need to first choose an identification between $E$ and $E'$ (which respects the $\mathsf{S}$-action). Different choices of identifications may lead to different conclusions on whether $\Gamma$ and $\Gamma'$ are homotopic. This suggests that it is more meaningful to classify *relative* phases. For a more detailed discussion of isomorphism and homotopy of Type AIII systems, see [46] and the detailed study of "chiral" vector bundles in [11].

### 2.1. Some Differences with the Freed–Moore Approach

In the Freed and Moore [15] approach, higher twisted $K$-theory groups are constructed using bundles of graded Clifford modules, quotiented by a certain algebraic relation, in analogy to the Atiyah–Bott–Shapiro (ABS) construction of the $K$-theory ring of a point. However, we make the important observation that there are two inequivalent ways of taking parity reversals in the construction, with each choice leading to *opposite* orderings of the classification groups.[3] Also, not all the standard (untwisted) $K$-theory groups are recovered using their approach. For example, $K^{-1}(S^1) \cong \mathbb{Z}$, whereas all bundles of graded $\mathbb{C}l_1$-modules over $S^1$ are "trivial" in the sense of Definition 8.5 in [15].

Some usual assumptions on the group of symmetries are: (i) distinguished time-reversing and charge-conjugating elements which are involutary in the full symmetry group, and (ii) a direct product factorization into translational symmetries, point group symmetries, and time-reversal or charge-conjugation symmetries. As emphasized by Freed–Moore, these assumptions do not hold in many realistic systems. Because of this, they are led to twisted equivariant $K$-theory, although only some special twistings occur. In our approach, twistings appear in the form of twisted group algebras, and only the *untwisted* $K$-theory of these algebras enters. Furthermore, abstract results of Packer and Raeburn [36] allow these twistings to be untwisted without compromising the $K$-theory (see Sect. 8).

## 3. Symmetries, Spectral-Flattening, and Positive Energy Quantization

Following the general arguments of [15], elements of the symmetry group $G$ for the dynamics of a quantum mechanical system are presumed to be endowed with Hamiltonian and/or time preserving/reversing properties, which are encoded by a pair of continuous homomorphisms $c, \tau \colon G \to \{\pm 1\}$. An element $g \in G$ preserves (resp. reverses) the arrow of time if $\tau(g) = +1$ (resp. $\tau(g) = -1$); it commutes (resp. anticommutes) with the Hamiltonian if $c(g) = +1$ (resp. $c(g) = -1$). A third homomorphism $\phi : G \to \{\pm 1\}$ specifies whether $g$ is implemented unitarily ($\phi(g) = +1$) or antiunitarily ($\phi(g) = -1$). Writing $u_t$ for the unitary dynamics generated by the Hamiltonian $H$, and $\mathsf{g}$ for the unitary/antiunitary representative of $g$, the time-reversal equation

$$\mathsf{g} u_t \mathsf{g}^{-1} = u_{\tau(g)t}$$

leads to $\phi \cdot \tau \cdot c \equiv 1$, so any two of $\phi, \tau, c$ specifies the third. Often, $c \equiv 1$ is assumed (all symmetries commute with the Hamiltonian), then $\phi = \tau$ and antiunitarity becomes synonymous with time-reversal. However, in our description of free-fermion dynamics, we want to consider symmetries that effect charge-conjugation (see Sect. 3.1), so we allow $c(g) = -1$. Then any two of $\phi, \tau, c$ may be independently specified. We also allow the symmetries to be projectively realized, i.e. there may be a non-trivial cocycle $\sigma$.

The possibility of charge-reversing symmetries (present or otherwise) for free-fermion dynamics requires, logically, a notion of *charged* dynamics and *charged* representations of the canonical anticommutation relations (CARs), as

[3] A closely related construction of *super representation rings* can be found in [32].

opposed to their *neutral* counterparts. The latter describes neutral (Majorana) fermions. Non-degeneracy of the dynamics (i.e. a gapped Hamiltonian) allows us to distinguish between particle and antiparticle sectors, and we would like both species to have positive energy in second quantization. For instance, the Fermi level of a band insulator (which may be set to 0) lies in a gap of the Hamiltonian, providing the particle–hole distinction. We recall the algebraic formalism of positive energy charged field quantization, and refer to [12,13,16] for the neutral case and technical details. Then, we establish our conventions for dynamical symmetries, including time and charge reversal.

### 3.1. Non-Degenerate Unitary Dynamics and Second Quantization

Let $u_t$ be a strongly continuous 1-parameter unitary group on a complex Hilbert space $\mathcal{Y}$ with inner product $h$, which is non-degenerate in the sense that its self-adjoint generator (Hamiltonian) $H$ is gapped ($\ker(H) = \{0\}$). We may define

$$\Gamma = \mathrm{sgn}(H), \quad J = i\,\mathrm{sgn}(H) = i\Gamma, \quad |H| = \sqrt{H^2} > 0,$$

and rewrite $u_t$ as $e^{tJ|H|}$. Note that $J$ is unitary, skew-adjoint and commutes with $H, \Gamma$ and $|H|$. Furthermore, $\mathcal{Y}$ is graded by the charge operator $\Gamma$ (or the *spectrally flattened Hamiltonian*) into $\mathcal{Y}^+ \oplus \mathcal{Y}^-$, where $\mathcal{Y}^\pm$ is the $\pm 1$ eigenspace of $\Gamma$. Writing $\mathcal{Z}$ for the space $\mathcal{Y}$ equipped with the modified complex unit $J$ instead of $i$, we have $\mathcal{Z} = \mathcal{Y}^+ \oplus \overline{\mathcal{Y}^-}$, where $\overline{\mathcal{Y}^-}$ is given the inner product dual to $h|_{\mathcal{Y}^-}$. The subspaces $\mathcal{Y}^\pm$ are invariant for $\Gamma, H$ and $|H|$, so we may regard these operators as self-adjoint operators on $\mathcal{Z}$, in which case a subscript is appended, e.g., $H_{\mathcal{Z}}$.

On the Fock space $\bigwedge^* \mathcal{Z}$, the *charged fields* are

$$\mathrm{a}^*(y) = a^*(y^+) + a(\overline{y^-}), \quad \mathrm{a}(y) = a(y^+) + a^*(\overline{y^-}), \quad y = (y^+, y^-) \in \mathcal{Y}^+ \oplus \mathcal{Y}^-,$$

where $a^*$ and $a$ are the standard creation and annihilation operators on $\bigwedge^* \mathcal{Z}$. The maps $\mathrm{a}^*$ and $\mathrm{a}$ furnish a charged CAR representation over $(\mathcal{Y}, h)$ on Fock space, called the *positive energy Fock quantization* for the non-degenerate unitary dynamics $u_t$. There are second quantized versions of the Hamiltonian and charge operators,

$$\mathcal{H} = \mathrm{d}\Lambda(|H|_{\mathcal{Z}}) \geq 0, \quad \mathcal{Q} = \mathrm{d}\Lambda(\Gamma_{\mathcal{Z}}),$$

which implement the dynamics and charge symmetry on Fock space,

$$e^{it\mathcal{H}}\mathrm{a}(y)e^{-it\mathcal{H}} = \mathrm{a}(u_t y), \quad e^{i\theta\mathcal{Q}}\mathrm{a}(y)e^{-i\theta\mathcal{Q}} = \mathrm{a}(e^{i\theta}y).$$

#### 3.1.1. Charge and Time Reversal in Non-Degenerate Unitary Dynamics.

A symmetry operator $\mathsf{g}$ on $(\mathcal{Y}, h)$ is required to be unitary or antiunitary according to $\phi(g)$, and time preserving or reversing [Eq. (3)] according to $\tau(g) = c \cdot \phi(g)$. A short computation leads to the following commutation relations

$$\mathsf{g}H\mathsf{g}^{-1} = c(g)H, \quad \mathsf{g}|H|\mathsf{g}^{-1} = |H|, \quad \mathsf{g}\Gamma\mathsf{g}^{-1} = c(g)\Gamma, \quad \mathsf{g}J\mathsf{g}^{-1} = \tau(g)J,$$

from which we find that $\mathsf{g}_{\mathcal{Z}}$ (i.e. the map $\mathsf{g}$ considered as an operator on $\mathcal{Z}$) is unitary or antiunitary according to $\tau(g)$. We may then amplify $\mathsf{g}_{\mathcal{Z}}$ to a unitary or antiunitary operator $\hat{\mathsf{g}} = \Lambda(\mathsf{g}_{\mathcal{Z}})$ on Fock space.

*Remark* 3.1. The modified imaginary unit $J$ is determined by the dynamics $u_t = e^{itH}$ only through the spectrally flattened Hamiltonian $\mathrm{sgn}(H)$.

*Remark* 3.2. Whether symmetries commute with the Hamiltonian (e.g. [20]), or are allowed to anticommute with the Hamitonian (e.g. [15,42]), depends on whether one is in a first-quantized (with Hamiltonian $H$) or second-quantized (with Hamiltonian $\mathcal{H} = \mathrm{d}\Lambda(|H|_{\mathcal{Z}})$) setting.

We stress that the presence of a time/charge revers*ing* symmetry does not imply that of a distinguished charge/time revers*al* operator. Indeed, Freed and Moore [15] have pointed out that there are physically relevant examples that do not fit into the tenfold way [20,42], as the latter requires distinguished charge/time reversal operators $\mathsf{C}, \mathsf{T}$. We prefer to work more generally, and think of time/charge reversal as *properties* of a symmetry $g \in G$. Under certain splitting assumptions on $G$, we can recover the usual $\mathsf{T}$ and/or $\mathsf{C}$ operators, see Sect. 6.

### 3.2. Other Conventions for Free-Fermion Dynamics with Symmetries

In many treatments of the tenfold way [1,2,15,20,30,42,43], the single-particle "Hamiltonian" in certain symmetry classes is taken to act on a *Nambu space* $W = V \oplus \overline{V}$ rather than a single-particle Hilbert space $V$. A Dirac–Nambu space is a vector space of *second-quantized* creation operators and annihilation operators, and is a useful auxiliary space often used for studying Bogoliubov de Gennes Hamiltonians. It is necessarily even-dimensional, and comes with a canonical real structure $\Sigma\colon (v_1, \overline{v_2}) \mapsto (v_2, \overline{v_1})$ which exchanges the creation operators with the annihilation operators. The fixed points of $\Sigma$ form the real mode space $\mathcal{M}$ of Majorana operators, and $\mathcal{M}$ inherits a real inner product from $W$ by restriction. The operator $J = i \oplus -i$ on $W = V \oplus \overline{V} = \mathcal{M} \otimes \mathbb{C}$ restricts to an orthogonal complex structure on $\mathcal{M}$, and $(\mathcal{M}, J|_{\mathcal{M}}) \cong V$. One considers dynamics on Fock space $\bigwedge^* V$, generated by a second-quantized Hamiltonian $H_{\mathrm{F}}$ which is required to be quadratic in the creation and annihilation operators. Such dynamics can be reformulated on Nambu space $V \oplus \overline{V}$, with generating "Hamiltonian" $H_{\mathrm{N}}$ subject to certain symmetry constraints. Alternatively, the dynamics can be specified by a skew-symmetric operator $A$ on $\mathcal{M}$, whose complexification is $iH_{\mathrm{N}}$. The gapped condition is sometimes imposed on $H_N$. An example is the Bogoliubov–de Gennes (BdG) Hamiltonian for the quasi-particle dynamics of a superconducting system. It is important to note that the polarization $J$, and thus the Fock space in second quantization, is already implicit in the Nambu space formulation, whereas it is determined by $H$ in positive energy Fock quantization. Also, particle number is not necessarily conserved [because $a(v)a(v')$ and $a^*(v)a^*(v')$ terms are allowed in the second-quantized Hamiltonian $H_{\mathrm{F}}$], so $A$ may not have a $U(1)$-symmetry (i.e. it may not commute with $J$). The definition of symmetries of a Hamiltonian, especially those of charge-conjugation and time-reversal, also differ between authors.

In our approach, the Hamiltonian $H$ generating the non-degenerate unitary dynamics on $(\mathcal{Y}, h)$ determines the particle–antiparticle distinction in second quantization. In practice, we impose a stronger *gapped* condition on $H$, i.e. $0 \notin \mathrm{spec}(H)$. We allow antiunitary symmetries, as well as charge-conjugating symmetries which reverse $H$, or equivalently, $\Gamma = \mathrm{sgn}(H)$. Two symmetry-compatible gapped Hamiltonians $H_1, H_2$ are identified if they have the same spectral flattening, $\mathrm{sgn}(H_1) = \Gamma_1 = \Gamma_2 = \mathrm{sgn}(H_2)$. Then, the specification of charged free-fermion dynamics respecting the symmetry data $(G, c, \phi, \sigma)$, up to spectral-flattening,[4] is precisely that of a *graded* projective unitary–antiunitary representation (PUA-rep) for $(G, c, \phi, \sigma)$, as defined in Sect. 4.3.

We remark that a graded PUA-rep for $(G, c, \phi, \sigma)$ may also be interpreted as an ordinary quantum mechanical system. We usually combine such quantum mechanical systems using the tensor product. On the other hand, at the one-particle level, we combine free-fermion systems using the direct sum operation, which only gets translated into the tensor product at the Fock space level. We are only concerned with describing free-fermion dynamics and its symmetries at the one-particle level, so the direct sum applies. Thus it makes sense to construct abelian monoids of free-fermion systems and to use $K$-theoretic methods in their classification. Certain graded PUA-reps can also be interpreted as describing BdG Hamiltonians [42,43], but we note that the extra structure ($\Sigma$ and even-dimension condition) are not generally kept track of in $K$-theory.

## 4. The General Notion of Twisted Covariant Representations

We outline the basic definitions and constructions of twisted covariant representations of twisted $C^*$-dynamical systems [8,34,36]. We make a simple generalization to $\mathbb{Z}_2$-graded twisted covariant representations, and show that they arise naturally as graded PUA-reps in the context of quantum systems with time/charge-reversing symmetries. All gradings will be $\mathbb{Z}_2$-gradings.

### 4.1. Ungraded Covariant Representations

Let $\mathcal{A}$ be a separable, possibly non-unital, real or complex $C^*$-algebra.[5] We denote its multiplier algebra by $\mathcal{MA}$, and its group of unitary elements $\{u \in \mathcal{MA}\colon u^*u = uu^* = 1_{\mathcal{MA}}\}$ by $\mathcal{UMA}$. If $\mathbb{F}$ is the ground field of $\mathcal{A}$, we write $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ for the group of $\mathbb{F}$-linear $*$-automorphisms of $\mathcal{A}$. Let $G$ be a locally compact, second countable, amenable[6] group, with left Haar measure $\mu$ and

[4] If the gapped $H$ is bounded, $\mathrm{sgn}(\cdot)$ is continuous on $\mathrm{spec}(H)$ and homotopic to the identity function. If $H$ is unbounded, $\mathrm{sgn}(H)$ can still be defined, but care must be taken in order to interpret spectral-flattening as a homotopy in a precise sense, see Appendix D of [15].

[5] A reference for basic facts about real $C^*$-algebras is Chapter 1 of [44].

[6] Amenability holds in all the physical examples that we consider in this paper, and is made in order to avoid having to distinguish between reduced and full crossed products later on.

identity element $e$. As in Section 2 of [36], we give $\mathcal{UMA}$ the strict topology, and $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ the point-norm topology.

**Definition 4.1** (*Twisted $C^*$-dynamical system* [8,34]). A pair $(\alpha, \sigma)$ of Borel maps $\alpha\colon G \to \mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ and $\sigma\colon G \times G \to \mathcal{UMA}$ satisfying

$$\alpha(x)\alpha(y) = \mathrm{Ad}(\sigma(x,y)) \circ \alpha(xy), \tag{2a}$$

$$\sigma(x,y)\sigma(xy,z) = \alpha(x)(\sigma(y,z))\sigma(x,yz), \tag{2b}$$

$$\sigma(x,e) = 1 = \sigma(e,x), \tag{2c}$$

$$\alpha(e) = \mathrm{id}_{\mathcal{A}}, \quad x,y,z \in G, \tag{2d}$$

is called a *twisting pair* for $(G, \mathcal{A})$. The map $\sigma$ is called a *2-cocycle* with values in $\mathcal{UMA}$, or simply a *cocycle*, and the quadruple $(G, \mathcal{A}, \alpha, \sigma)$ is called a *twisted $C^*$-dynamical system*.

For notational ease, we write $\alpha_x \equiv \alpha(x)$ and $a^x \equiv \alpha_x(a) \equiv \alpha(x)(a)$.

**Definition 4.2** (*Twisted covariant representation*). A *twisted covariant representation* of a twisted $C^*$-dynamical system $(G, \mathcal{A}, \alpha, \sigma)$ is a non-degenerate $*$-representation of $\mathcal{A}$ as bounded operators on a separable Hilbert space $\mathscr{H}$ over $\mathbb{F}$, along with a compatible Borel map $\theta\colon x \mapsto \theta_x$ from $G$ to the unitary[7] operators on $\mathscr{H}$, in the sense that

$$\theta_x\theta_y = \sigma(x,y)\theta_{xy}, \tag{3a}$$

$$\theta_x(am) = a^x(\theta_x m), \quad x,y, \in G,\ a \in \mathcal{A},\ m \in \mathscr{H}. \tag{3b}$$

Note that (3b) can be restated as $a^x = \mathrm{Ad}(\theta_x)(a)$, and then we see that (3a) is consistent with (2a). In the untwisted case, i.e. $\sigma \equiv 1$, the Borel map $\alpha$ is a homomorphism, hence continuous (Theorem D.11 of [49]). Then $(G, \mathcal{A}, \alpha, 1)$ is a (untwisted) $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$ in the usual sense (e.g. 7.4.1 of [39], 2.1 of [49], or 10.1 of [7]). Similarly, $\theta$ becomes a strongly continuous homomorphism from $G$ to the unitary group of $\mathscr{H}$. Thus, $\theta$ is a (untwisted) covariant representation of $(G, \mathcal{A}, \alpha)$ in the usual sense (e.g. 7.4.8 of [39] or 10.1 of [7]), and no harm is done by dropping the adjective "twisted" when $\sigma \equiv 1$. We say that two twisted covariant representations $(\theta, \mathscr{H}), (\theta', \mathscr{H}')$ of $(G, \mathcal{A}, \alpha, \sigma)$ are *equivalent* if there is a unitary $\mathcal{A}$-linear intertwiner $U\colon \mathscr{H} \to \mathscr{H}'$ such that $U\theta_x U^{-1} = \theta'_x$ for all $x \in G$.

There is an action of the group of Borel functions $\lambda\colon G \to \mathcal{UMA}$ on twisting pairs (Section 3 of [36]), defined by

$$\alpha'(x) = \mathrm{Ad}(\lambda(x)) \circ \alpha(x), \tag{4a}$$

$$\sigma'(x,y) = \lambda(x)\alpha_x(\lambda(y))\sigma(x,y)\lambda(xy)^{-1}. \tag{4b}$$

Two twisting pairs $(\alpha, \sigma)$ and $(\alpha', \sigma')$ are *exterior equivalent* if they are related by such a transformation, and there is a 1–1 correspondence between the covariant representations of $(G, \mathcal{A}, \alpha, \sigma)$ and those of $(G, \mathcal{A}, \alpha', \sigma')$, via the adjustments $\theta_x \mapsto \lambda(x)\theta_x$. This generalizes the familiar notion of equivalence of cocycles for projective unitary group representations (where $\mathcal{A} = \mathbb{C}$). If the

[7] When $\mathbb{F} = \mathbb{R}$, we also use "orthogonal" for emphasis.

cocycle $\sigma$ is assumed to be central in $\mathcal{A}$, there is no effect of $\lambda$ on $\alpha$ in (4a). The conjugation in (2a) and the condition (2d) are then redundant, and we also have $\alpha(x^{-1}) = \alpha(x)^{-1}$. A central cocycle is *trivial* if there is a Borel function $\lambda\colon G \to \mathcal{Z}(\mathcal{UMA})$ such that $\sigma(x,y) = \lambda(x)\lambda(y)^x\lambda(xy)^{-1}$, i.e. $\sigma$ is a coboundary in the sense of cohomology. We say that two central cocycles $\sigma_1, \sigma_2$ are *equivalent*, or in the same cocycle class, if $\sigma_1\sigma_2^{-1}$ is trivial. Note that if $\sigma$ is not necessarily central, $\alpha$ and $\sigma$ must be modified *concurrently* when making an adjustment $\theta_x \mapsto \lambda(x)\theta_x$. In many of our physical examples, the representative $\sigma$ in a cocycle class can be chosen to make certain computations more convenient, e.g. Proposition 6.2 and Lemma 10.4.

### 4.2. Graded Covariant Representations

Let $\mathcal{A}$ be a *graded* real or complex $C^*$-algebra, i.e. $\mathcal{A}$ has a direct sum decomposition into two self-adjoint closed subspaces $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$, satisfying $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j\,(\mathrm{mod}\,2)}$. Let $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$ now denote its group of *even* $\mathbb{F}$-linear $*$-automorphisms, i.e. $*$-automorphisms that preserve the decomposition $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$. We assume that the cocycles $\sigma$ take values in the *even* elements $\mathcal{UMA}_0$ of $\mathcal{UMA}$. These grading restrictions are consistent with Eqs. (2a) and (2b) for a twisting pair $(\alpha, \sigma)$. Suppose that the group $G$ is also equipped with a continuous homomorphism $c\colon G \to \{\pm 1\}$. The quintuple $(G, c, \mathcal{A}, \alpha, \sigma)$ is called a *graded twisted $C^*$-dynamical system.*

**Definition 4.3** (*Graded twisted covariant representation*). A *graded* covariant representation of a graded twisted $C^*$-dynamical system $(G, c, \mathcal{A}, \alpha, \sigma)$ is a graded $*$-representation of $\mathcal{A}$ on a graded Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ over $\mathbb{F}$ (i.e. $\mathcal{A}_i\mathscr{H}_j \subset \mathscr{H}_{i+j\,(\mathrm{mod}\,2)}$), along with a compatible Borel map $\theta\colon G \to U(\mathscr{H})$, in the sense of (3a) and (3b), with the additional condition that $\theta_x$ is an even (resp. odd) operator if $c(x) = +1$ [resp. $c(x) = -1$]. Two such graded covariant representations $(\theta, \mathscr{H}), (\theta', \mathscr{H}')$ are *graded equivalent*, or simply *equivalent*, if there is an *even* unitary $\mathcal{A}$-linear map $U\colon \mathscr{H} \to \mathscr{H}'$ intertwining $\theta$ with $\theta'$.

In most of our applications, $\mathcal{A}$ is trivially graded, i.e. purely even, and the only complication comes from the data of $c\colon G \to \{\pm 1\}$.

### 4.3. Special Cases I: Graded Projective Unitary–Antiunitary Representations and Gapped Hamiltonians

A complex Hilbert space $(\mathscr{H}, h)$ is equivalently a real Hilbert space $(\mathscr{H}, b)$ with real inner product $b = \mathrm{Re}(h)$, along with a $b$-orthogonal complex structure $J$ (i.e. $J^2 = -1$) playing the role of multiplication by $i$. The complex inner product $h$ may be recovered from $b$ and $J$ by setting $h(u,v) = b(u,v) + ib(Ju,v)$. Note that $h$ induces the same norm on $\mathscr{H}$ as $b$ does. An orthogonal operator on $(\mathscr{H}, b)$ is (anti)-unitary as an operator on $(\mathscr{H}, h)$, iff it (anti)-commutes with $J$.

Let $\phi\colon G \to \{\pm 1\}$ be a continuous homomorphism, and $\sigma$ be a $U(1)$-valued 2-cocycle as in (1). A projective unitary–antiunitary representation[8] (PUA-rep) $\theta$ of $(G, \phi, \sigma)$ on a complex Hilbert space $(\mathscr{H}, h)$ is a Borel map

[8] See [38,49] for some topological matters.

$x \mapsto \theta_x$ such that $\theta_x$ is a unitary (resp. antiunitary) operator on $(\mathscr{H}, h)$ if $\phi(x) = +1$ [resp. $\phi(x) = -1$], and $\theta_x\theta_y = \sigma(x, y)\theta_{xy}$. By regarding $(\mathscr{H}, b)$ as a real Hilbert space, and $i$ as a complex structure $J$ as above, we can equivalently define a PUA-rep of $(G, \phi, \sigma)$ as a map $\theta$ from $G$ to the orthogonal operators on $(\mathscr{H}, b)$, subject to

$$\begin{aligned} \theta_x\theta_y &= \sigma(x, y)\theta_{xy}, \quad x, y \in G, \\ \theta_x J &= \phi(x)J\theta_x. \end{aligned}$$

Suppose $\phi$ is surjective, and let $\mathcal{A} = \mathbb{C}$ as a purely even real $C^*$-algebra. Thus $\mathcal{A} = \mathbb{R} \oplus i\mathbb{R}$ as a real vector space, with basis $\{1, i\}$, $i^2 = -1$, and the $*$-operation taking $i$ to $-i$. There are two elements of $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$, namely complex conjugation $K$ and the identity $\mathrm{id}_{\mathbb{C}}$. A $*$-representation of $\mathcal{A} = \mathbb{C}$ is a real Hilbert space $(\mathscr{H}, b)$ along with a linear operator $J$ representing $i$, such that $J^2 = -1$ and $J^* = -J$, i.e. $J$ is an orthogonal complex structure. Define the map $\alpha\colon G \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{C})$ by

$$\alpha_x \equiv \alpha(x) := \begin{cases} \mathrm{id}_{\mathbb{C}} & \text{if } \phi(x) = +1, \\ K & \text{if } \phi(x) = -1. \end{cases}$$

Equations (3a) and (3b) say that a covariant representation $\theta$ of $(G, \mathbb{C}, \alpha, \sigma)$ on a real Hilbert space $(\mathscr{H}, b)$, is precisely a PUA-rep of $(G, \phi, \sigma)$ on $\mathscr{H}$.

Suppose that a PUA-rep for $(G, \phi, \sigma)$ has, additionally, a gapped self-adjoint Hamiltonian $H$, and that $G$ has a second continuous homomorphism $c\colon G \to \{\pm 1\}$ such that $\theta_x H = c(x)H\theta_x$, or equivalently,

$$\theta_x\Gamma = c(x)\Gamma\theta_x, \quad \forall x \in G, \tag{5}$$

where $\Gamma = \mathrm{sgn}(H)$. Thus, $H$ is compatible with the symmetries specified by the data $(G, c, \phi, \sigma)$. A *graded* PUA-rep $\theta$ of $(G, c, \phi, \sigma)$ on a graded complex Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$ is a PUA-rep of $(G, \phi, \sigma)$, along with a self-adjoint grading operator $\Gamma$ satisfying (5). Such a $\Gamma$ represents the family of $(G, c, \phi, \sigma)$-compatible Hamiltonians on $\mathscr{H}$ whose spectral flattening is $\Gamma$. Note that a graded covariant representation of $(G, c, \mathbb{C}, \alpha, \sigma)$ is precisely a graded PUA-rep of $(G, c, \phi, \sigma)$.

#### 4.4. Special Case II: Disordered Systems and Covariant Representations

Disordered systems are often modeled on a disorder space $\Omega$, on which the group $G$ acts by homeomorphisms. More generally, the disorder space can be non-commutative, and so $G$ acts as automorphisms on an algebra $\mathcal{A}$. We can generalize PUA-reps to include disorder, by replacing $\mathbb{C}$ with the algebra $\mathcal{A}$ and working with twisted dynamical systems and their covariant representations. Such objects were considered in the analysis of the IQHE in [6], but without the additional data of $\phi$ or $c$.

# 5. Graded Twisted Crossed Products and Covariant Representations

In the previous section, we explained how the implementation of symmetry and compatible gapped Hamiltonians leads to graded twisted $C^*$-dynamical systems $(G, c, \mathcal{A}, \alpha, \sigma)$ and their covariant representations. In this section, we explain how all the symmetry data can be concisely and faithfully encoded in a graded twisted crossed product $C^*$-algebra $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$, which we may simply call the *symmetry algebra*. This device will be very convenient for the application of $K$-theory in the later sections.

## 5.1. Twisted Crossed Products and Covariant Representations

Let $L^1(G, \mathcal{A}, \alpha, \sigma)$ be the Banach $*$-algebra of integrable functions[9] $F\colon G \to \mathcal{A}$ with the $L^1$-norm $\|F\|_1 = \int_G \|F(x)\|\,\mathrm{d}x$, equipped with a $(\alpha, \sigma)$-twisted convolution product $\star$ and involution $*$,

$$(F_1 \star F_2)(y) := \int_G F_1(x)\left(F_2(x^{-1}y)\right)^x \sigma(x, x^{-1}y)\,\mathrm{d}x, \tag{6a}$$

$$F^*(x) := \sigma(x, x^{-1})^* \left(F(x^{-1})^*\right)^x \Delta(x^{-1}), \tag{6b}$$

where $\Delta$ is the modular function on $G$. There is a one-to-one correspondence between covariant representations $\theta$ of $(G, \mathcal{A}, \alpha, \sigma)$, and non-degenerate $*$-representations of $L^1(G, \mathcal{A}, \alpha, \sigma)$, given by taking the "integrated form" $\tilde{\theta}$ of $\theta$, see Theorem 3.3 of [8] and Remark 2.6 of [36]. A pre-$C^*$-norm is defined on $L^1(G, \mathcal{A}, \alpha, \sigma)$ by

$$\|F\|_{\max} = \sup\{\|\tilde{\theta}(F)\|\colon \theta \text{ is a covariant representation of } (G, \mathcal{A}, \alpha, \sigma)\}.$$

**Definition 5.1** (*Twisted crossed product $C^*$-algebra* [8]). Let $(G, \mathcal{A}, \alpha, \sigma)$ be a twisted dynamical system. The *twisted crossed product $C^*$-algebra* associated with $(G, \mathcal{A}, \alpha, \sigma)$, denoted by $\mathcal{A} \rtimes_{(\alpha,\sigma)} G$, is defined to be the completion of $L^1(G, \mathcal{A}, \alpha, \sigma)$ in the norm $\|\cdot\|_{\max}$.

The group $G$ and the algebra $\mathcal{A}$ are embedded into $\mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$ via the Borel map (not necessarily homomorphic) $j_G\colon G \to \mathcal{UM}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$ and the $*$-homomorphism $j_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{M}(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$, which are defined by

$$(j_G(g)(F))(x) := (F(g^{-1}x))^g \sigma(g, g^{-1}x), \tag{7a}$$

$$(j_{\mathcal{A}}(a)(F))(x) := a(F(x)), \tag{7b}$$

where $F \in L^1(G, \mathcal{A}, \alpha, \sigma),\ g, x \in G, a \in \mathcal{A}$.

When $\sigma \equiv 1$, we recover the untwisted crossed product $C^*$-algebra associated with the untwisted $C^*$-dynamical system $(G, \mathcal{A}, \alpha)$. If $\alpha \equiv 1$, we call $\mathbb{R} \rtimes_{(1,\sigma)} G$ (resp. $\mathbb{C} \rtimes_{(1,\sigma)} G$) the real (resp. complex) *twisted group $C^*$-algebra*

[9] The integral of a $\mathcal{A}$-valued function on $G$ is a Bochner integral, see Appendix B of [49] and the preliminary section of [36].

of $(G,\sigma)$. If $\sigma\equiv 1$ as well, we use a shortened notation[10] $\mathbb{R}\rtimes G$ (resp. $\mathbb{C}\rtimes G$) for the real (resp. complex) *group* $C^*$*-algebra* of $G$. More generally, when $\alpha\equiv 1$ and $\sigma\equiv 1$, we will write $\mathcal{A}\rtimes G:=\mathcal{A}\rtimes_{(1,1)}G$ to ease notation.

Although there are universal characterizations of twisted crossed products, (see 2.4 of [36], as well as [37]), we only need the following result.

**Proposition 5.2** [8,36,37]. *There is a one-to-one correspondence between the covariant representations of $(G,\mathcal{A},\alpha,\sigma)$, and the non-degenerate $*$-representations of $\mathcal{A}\rtimes_{(\alpha,\sigma)}G$.*

### 5.2. Graded Twisted Crossed Products

For a graded twisted $C^*$-dynamical system $(G,c,\mathcal{A},\alpha,\sigma)$, we assign a grading to $\mathcal{A}\rtimes_{(\alpha,\sigma)}G$ as follows. Let $G_0:=\ker(c)$ and $G_1:=G-G_0$. The even subalgebra of $\mathcal{A}\rtimes_{(\alpha,\sigma)}G$ is the completion in $\|\cdot\|_{\max}$ of $L^1(G_0,\mathcal{A}_0,\alpha,\sigma)\oplus L^1(G_1,\mathcal{A}_1,\alpha,\sigma)$, and the odd subspace is the completion of $L^1(G_0,\mathcal{A}_1,\alpha,\sigma)\oplus L^1(G_1,\mathcal{A}_0,\alpha,\sigma)$. Note that (6a) and (6b) respect this grading due to the restriction to even automorphisms $\alpha_x$ and even cocycles $\sigma$. Graded $*$-representations of the graded twisted crossed product $\mathcal{A}\rtimes_{(\alpha,\sigma)}G$ then correspond one-to-one to graded covariant representations of $(G,c,\mathcal{A},\alpha,\sigma)$.

## 6. $CT$-Symmetries, Clifford Algebras, and the Tenfold Way

Let $G=\{\pm 1\}^n, n\geq 0$, where $\{\pm 1\}^0$ means the trivial group. Suppose $m$ of the $\{\pm 1\}$ generators $U_i, 1\leq i\leq m$ are to be represented unitarily ($\phi(U_i)=+1$), while the other $n-m$ generators $A_k, 1\leq k\leq n-m$ are to be represented antiunitarily ($\phi(A_k)=-1$). We will write $\mathsf{U}_i:=\theta_{U_i}$ and $\mathsf{A}_k:=\theta_{A_k}$ for their representatives in a graded PUA-rep $\theta$ of $(\{\pm 1\}^n,c,\phi,\sigma)$. As always, $\mathsf{U}_i$ and $\mathsf{A}_k$ are odd/even according to $c$.

**Lemma 6.1.** *We may assume, without loss of generality, that there are at most two antiunitaries $\mathsf{A}_1,\mathsf{A}_2$.*

*Proof.* Let $A$ be the image of $(\phi,c)\colon\{\pm 1\}^n\to\{\pm 1\}\times\{\pm 1\}$, and $B$ be its kernel. Every non-identity element of $A, B$ and $\{\pm 1\}^n$ has order 2. Regarding the groups as finite-dimensional vector spaces over the two-element field $\mathbb{F}_2$, we have $\{\pm 1\}^n\cong B\times A$. Any $\mathbb{F}_2$-basis for $B$ provides a set of even unitarily implemented generators $U_i$. Since $\dim_{\mathbb{F}_2}(A)\leq 2$, there are at most two $A_k\in A$ with $\phi(A_k)=-1$ providing antiunitary operators $\mathsf{A}_k$. □

A basis for $A$ can be chosen to be one of the following: (i) empty, (ii) {odd $U_n$}, (iii) {even $A_1$}, (iv) {odd $A_1$}, or (v) {even $A_1$, odd $A_2$}.

The graded PUA-rep theory of $(\{\pm 1\}^n,c,\phi,\sigma)$ can be simplified by using the $U(1)$ phase freedom for the operators $\theta_x, x\in G$ to make certain standardizations. Since $\mathsf{U}_i^2=\sigma(U_i,U_i)\theta_e=\sigma(U_i,U_i)$, we can modify $\mathsf{U}_i\mapsto\pm\sigma(U_i,U_i)^{-1/2}\mathsf{U}_i$ to fix $\mathsf{U}_i^2=+1$. This does not work for $\mathsf{A}_k$, since $(\lambda\mathsf{A}_k)(\lambda\mathsf{A}_k)$

[10] It is also standard to write $\mathbb{R}\rtimes G$ for a semi-direct product of the *group* $\mathbb{R}$ with $G$. Nevertheless, the correct meaning should be clear from the context.

$= \lambda\bar{\lambda}\mathsf{A}_k^2 = \mathsf{A}_k^2$ for any $\lambda \in U(1)$. Setting $x = y = z = A_k$ in (1) leads to $\sigma(A_k, A_k) = \overline{\sigma(A_k, A_k)}$, so $\mathsf{A}_k^2 = \sigma(A_k, A_k) = \pm 1$ are invariants of the cocycle class of $\sigma$. For two unitaries $\mathsf{U}_i, \mathsf{U}_j$, $i < j$, we write $\lambda_{ij} := \sigma(U_i, U_j)/\sigma(U_j, U_i)$ so that $\mathsf{U}_i\mathsf{U}_j = \lambda_{ij}\mathsf{U}_j\mathsf{U}_i$. Then $\mathsf{U}_j = \mathsf{U}_i^2\mathsf{U}_j = \lambda_{ij}^2\mathsf{U}_j\mathsf{U}_i^2 = \lambda_{ij}^2\mathsf{U}_j$, so $\lambda_{ij} = \pm 1$. For any $a_1, a_2 \in \mathbb{C}$, we have $(a_1\mathsf{A}_1)(a_2\mathsf{A}_2) = a_1\bar{a}_2\sigma(A_1, A_2)\theta_{A_1A_2}$, and $(a_2\mathsf{A}_2)(a_1\mathsf{A}_1) = a_2\bar{a}_1\sigma(A_2, A_1)\theta_{A_1A_2}$. Let $a_1 = \pm\sigma(A_2, A_1)^{1/2}, a_2 = \pm\sigma(A_1, A_2)^{1/2}$, and redefine $\mathsf{A}_1' = a_1\mathsf{A}_1, \mathsf{A}_2' = a_2\mathsf{A}_2$ and $\theta'_{A_1A_2} = a_1a_2\theta_{A_1A_2}$, then $\mathsf{A}_1'\mathsf{A}_2' = \theta'_{A_1A_2} = \mathsf{A}_2'\mathsf{A}_1'$. Finally, $\mathsf{U}_i\mathsf{A}_k = \nu_{ik}\mathsf{A}_k\mathsf{U}_i$, where $\nu_{ik} := \sigma(U_i, A_k)/\sigma(A_k, U_i)$. The equation $\mathsf{A}_k = \mathsf{U}_i^2\mathsf{A}_k = \nu_{ik}^2\mathsf{A}_k\mathsf{U}_i^2 = \nu_{ik}^2\mathsf{A}_k$ means that $\nu_{ik} = \pm 1$, which cannot be fixed by utilizing the residual $\pm 1$ phase freedom in $\mathsf{U}_i, \mathsf{A}_k$. In summary:

**Proposition 6.2.** *Let $\theta$ be a graded PUA-rep of $(\{\pm 1\}^n, c, \phi, \sigma)$, with $0 \le m \le n$ (resp. $0 \le n - m \le 2$) unitarily (resp. antiunitarily) implemented group generators $U_i$ (resp. $A_k$), as in Lemma 6.1. We can adjust $\mathsf{U}_i$ and $\mathsf{A}_k$, while staying in the same cocycle class, so that $\mathsf{U}_i^2 = +1$, $\mathsf{A}_k^2 = \pm 1$, $\mathsf{A}_k\mathsf{A}_l = \mathsf{A}_l\mathsf{A}_k$, $\mathsf{U}_i\mathsf{U}_j = \pm\mathsf{U}_j\mathsf{U}_i$, and $\mathsf{U}_i\mathsf{A}_k = \pm\mathsf{A}_k\mathsf{U}_i$, for all $1 \le i, j \le m$ and $1 \le k, l \le n - m$.*

Let $\mathbf{i} = \{i_1, \ldots, i_p\}$ and $\mathbf{k} = \{k_1, \ldots, k_q\}$ be (possibly empty) increasing subsets of $\{1, \ldots, n\}$ and $\{1, \ldots, n - m \le 2\}$ respectively. Let $U_{\mathbf{i}}$ and $A_{\mathbf{k}}$ denote the group elements $U_{i_1}, \ldots, U_{i_p}$ and $A_{k_1}, \ldots, A_{k_q}$ respectively, with $U_\emptyset = 1 = A_\emptyset$. In particular, $U_i = U_{\{i\}}$ and $A_k = A_{\{k\}}$. Every element of $\{\pm 1\}^n$ can be uniquely written as $U_{\mathbf{i}}A_{\mathbf{k}}$ for some $\mathbf{i}, \mathbf{k}$. The phase freedom for the representatives of $U_{\mathbf{i}}A_{\mathbf{k}}$ with $|\mathbf{i}| + |\mathbf{k}| \ge 2$ can be used to fix the condition $\theta_{U_{\mathbf{i}}A_{\mathbf{k}}} = \mathsf{U}_{i_1}, \ldots, \mathsf{U}_{i_p}\mathsf{A}_{k_1}, \ldots, \mathsf{A}_{k_q}$. The cocycle for this standardized $\theta$ is then completely determined by the set of $\pm 1$ in Proposition 6.2.

### 6.1. Clifford Algebras Associated with $CT$-Subgroups: The Tenfold Way

We define the $CT$-group to be $\{\pm 1\}^2 = \{1, T, C, S\}$, which has $(\phi, c)(T) = (-1, +1), (\phi, c)(C) = (-1, -1), (\phi, c)(S) = (+1, -1)$. The elements $T, C$ and $S = CT = TC$ refer to time-reversal, charge-conjugation, and sublattice (or chiral) symmetries respectively. We are interested in the graded PUA-reps of $(A, \sigma)$, where $A \subset \{1, T, C, S\}$ is a $CT$-subgroup and the homomorphisms $\phi, c$ on $A$ are implicit. The representatives of $T$, $C$ and $S$ (where present) are denoted by $\mathsf{T}$, $\mathsf{C}$ and $\mathsf{S}$, respectively.

First, consider the full $CT$-group $A = \{1, T, C, S\}$. In the standard form of Proposition 6.2, there are four choices $\mathsf{T}^2 = \pm 1, \mathsf{C}^2 = \pm 1$, and we may assume that $\mathsf{TC} = \mathsf{S} = \mathsf{CT}$. The mutually anticommuting set of odd operators $\{\mathsf{C}, i\mathsf{C}, i\mathsf{CT}\}$ generates a *graded* real Clifford algebra $\mathrm{Cl}_{r,s}$ with $r$ (resp. $s$) determined by the number of negative skew-adjoint (resp. positive self-adjoint) Clifford generators in this set. Thus, a graded PUA-rep for the full $CT$-group is precisely a graded $*$-representation of an appropriate $\mathrm{Cl}_{r,s}$. For the subgroup $\{1, C\}$, there are two choices for $\mathsf{C}^2 = \pm 1$, and we take $\{\mathsf{C}, i\mathsf{C}\}$ as the set of odd Clifford generators for $\mathrm{Cl}_{0,2}$ or $\mathrm{Cl}_{2,0}$. For the subgroup $A = \{1, S\}$, there is only one standard choice $\mathsf{S}^2 = +1$, and $\{\mathsf{S}\}$ generates the *complex* graded Clifford algebra $\mathbb{C}\mathrm{l}_1$. For $A = \{1, T\}$, there are two choices $\mathsf{T}^2 = \pm 1$. The anticommuting set $\{i, \mathsf{T}, i\mathsf{T}\Gamma\}$ generates the *ungraded* Clifford algebra $\mathrm{Cl}_{1,2}$ when $\mathsf{T}^2 = +1$ and $\mathrm{Cl}_{3,0}$ when $\mathsf{T}^2 = -1$.

There are Morita equivalences between Clifford algebras,

$$\begin{aligned}\mathrm{Cl}_{r,s} \otimes M_2(\mathbb{R}) &\cong \mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{1,1} \cong \mathrm{Cl}_{r+1,s+1}\\ \mathbb{C}\mathrm{l}_n \otimes M_2(\mathbb{C}) &\cong \mathbb{C}\mathrm{l}_{n+2}\\ \mathrm{Cl}_{n,0} \otimes M_{16}(\mathbb{R}) &\cong \mathrm{Cl}_{n+8,0}\\ \mathrm{Cl}_{0,n} \otimes M_{16}(\mathbb{R}) &\cong \mathrm{Cl}_{0,n+8},\end{aligned}$$

as well as a 1–1 correspondence between graded representations of $\mathrm{Cl}_{r,s}$ and ungraded representations of $\mathrm{Cl}_{r,s+1}$ [4,33]. Using these, we may summarize the above discussion in Table 1.

*Remark* 6.3. The Clifford algebras constructed in this section can also be interpreted as twisted group algebras for $\mathbb{Z}_2^n$ (additive notation), where the group generators are taken from a subset of $\{C, T, i, \Gamma\}$ in the real case, and a subset of $\{S, \Gamma\}$ in the complex case. By redefining the group generators as above, the Clifford algebras can be written as $\mathrm{Cl}_{r,s} \cong \mathbb{R} \rtimes_{(1,\sigma_{r,s})} \mathbb{Z}_2^{r+s}$ and $\mathbb{C}\mathrm{l}_n \cong \mathbb{C} \rtimes_{(1,\sigma_n)} \mathbb{Z}_2^n$, where for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_2^n$ or $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_2^{r+s}$ as appropriate,

$$\sigma_{r,s}(\mathbf{x}, \mathbf{y}) = \big(-1\big)^{\sum\limits_{j<i} x_i y_j + \sum\limits_{i \leq r} x_i y_i}, \quad \sigma_n(\mathbf{x}, \mathbf{y}) = \big(-1\big)^{\sum\limits_{j<i} x_i y_j}.$$

## 7. The $K$-Theoretic Difference-Group of Symmetry-Compatible Gapped Hamiltonians

### Prelude: Representation Group of a Locally Compact Group

Let $G$ be a compact group. Its unitary representation theory can be summarized by the Peter–Weyl theorem. In $C^*$-algebraic language, this says that the group $C^*$-algebra $\mathbb{C} \rtimes G$ decomposes as a (possibly countably infinite) direct sum of matrix algebras over the unitary dual $\hat{G}$,

$$\mathbb{C} \rtimes G \cong \bigoplus_{[V] \in \hat{G}} M_{\dim(V)}(\mathbb{C}).$$

For a proof, see Proposition 3.4 of [49]. The complex representation group (we do not need the ring structure) $\mathcal{R}_{\mathbb{C}}(G)$ of $G$ is the Grothendieck group of the monoid of isomorphism classes of finite-dimensional unitary representations of $G$ under the direct sum. By complete reducibility, $\mathcal{R}_{\mathbb{C}}(G)$ is freely generated by the elements of the unitary dual $\hat{G}$. In terms of $K$-theory, there is an isomorphism $\mathcal{R}_{\mathbb{C}}(G) \cong K_0(\mathbb{C} \rtimes G) \cong K_0^G(\mathbb{C})$ (see Section 11.1 of [7]), where $K_0^G$ denotes the equivariant $K$-theory group. For projective unitary representations of $(G, \sigma)$, we can define the (twisted) representation group $\mathcal{R}_{\mathbb{C}}(G, \sigma) := K_0(\mathbb{C} \rtimes_{(1,\sigma)} G)$. If $G$ is locally compact but not compact, we choose the $K$-theoretic option and *define* $\mathcal{R}_{\mathbb{C}}(G, \sigma)$ to be $K_0(\mathbb{C} \rtimes_{(1,\sigma)} G)$, generalizing the compact case.[11] Finally, for a general twisted dynamical system

[11] This is a departure from the unitary representation theory of $G$, but is well-motivated by Bloch theory in condensed matter physics: $\mathbb{C} \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d)$ and its $K_0$-group is generated

$(G, \mathcal{A}, \alpha, \sigma)$, we *define* the (twisted) representation group of $(G, \mathcal{A}, \alpha, \sigma)$ to be $K_0(\mathcal{A} \rtimes_{(\alpha,\sigma)} G)$, which subsumes all the earlier definitions.

**Preliminaries on Graded Modules**

A graded module for a graded algebra $\mathcal{A}$ is an (ungraded) $\mathcal{A}$-module $W$ which admits a direct sum decomposition into $W = W_0 \oplus W_1$, such that $\mathcal{A}_i W_j \subset W_{i+j\,(\mathrm{mod}\,2)}$. The *right* parity-reversed module $W^\Pi$ has the same underlying vectors as $W$ but with the reversed grading, $W_0^\Pi = W_1, W_1^\Pi = W_0$, and retains the same graded action of $\mathcal{A}$. The *left* parity reversal ${}^\Pi W$ also has ${}^\Pi W_0 = W_1, {}^\Pi W_1 = W_0$ but the $\mathcal{A}$-action is, on homogeneous elements,

$$a \cdot (\pi_{\mathrm{L}}(w)) = \pi_{\mathrm{L}}((-1)^{|a|} a \cdot w), \quad a \in \mathcal{A}_0 \cup \mathcal{A}_1, w \in W,$$

where $|a| \in \mathbb{Z}_2$ is the parity of $a$. Both $\pi_{\mathrm{R}} \colon W \to W^\Pi$ and $\pi_{\mathrm{L}} \colon W \to {}^\Pi W$, which fix the underlying vectors, are odd maps and involutary operations on graded $\mathcal{A}$-modules: $\pi_{\mathrm{R}}^2 = \mathrm{id} = \pi_{\mathrm{L}}^2$. We write $w^\pi := \pi_{\mathrm{R}}(w)$ and ${}^\pi w := \pi_{\mathrm{L}}(w)$, which distinguishes them from $w \in W$. As graded $\mathcal{A}$-modules, $W^\Pi$ and ${}^\Pi W$ are equivalent under the even map $W^\Pi \ni w^\pi \equiv w_0^\pi + w_1^\pi \mapsto {}^\pi w_0 - {}^\pi w_1 \in {}^\Pi W$. Despite this, $\pi_{\mathrm{R}}$ commutes with $\mathcal{A}$, whereas $\pi_{\mathrm{L}}$ *graded* commutes with $\mathcal{A}$.

For a graded unital algebra $\mathcal{A}$, a graded finitely generated free $\mathcal{A}$-module is one of the form $\mathcal{A}^m \oplus (\mathcal{A}^\Pi)^n =: \mathcal{A}^{m|n}$, where $\mathcal{A}$ is regarded as a graded $\mathcal{A}$-module by left multiplication on itself, and $\mathcal{A}^\Pi$ is its right parity reverse.[12] A graded finitely generated projective (f.g.p.) $\mathcal{A}$-module is defined to be a graded $\mathcal{A}$-module which is a direct summand of $\mathcal{A}^{m|n}$ for some $(m, n)$. It can be shown [19] that a graded f.g.p. $\mathcal{A}$-module is the same thing as a graded $\mathcal{A}$-module which is f.g.p. in the ungraded sense. *In what follows, all modules are assumed to be f.g.p. unless otherwise stated.*

### 7.1. *K*-Theory as Topological Obstructions Between Gapped Hamiltonians

Standard presentations of $K$-theory in terms of Grothendieck completions and suspension constructions (e.g., in [7,33,48]) do not directly relate to the study of gapped phases, because the latter generally entails studying *graded* symmetry algebras. We have already seen this in Sect. 6, when studying graded representations of $CT$-subgroups. The ABS isomorphisms [4],

$$K_n(\mathbb{R}) \cong KO^{-n}(\star) \cong \mathcal{GR}(\mathrm{Cl}_{r,s})/i^*\mathcal{GR}(\mathrm{Cl}_{r+1,s}), \quad n = r - s\,(\mathrm{mod}\,8), \tag{8a}$$

$$K_n(\mathbb{C}) \cong K^{-n}(\star) \cong \mathcal{GR}(\mathbb{C}\mathrm{l}_n)/i^*\mathcal{GR}(\mathbb{C}\mathrm{l}_{n+1}), \tag{8b}$$

provide an example of the central unifying role of super-algebra in $K$-theory. Here $\mathcal{GR}(\mathrm{Cl}_{r,s})$ is the free abelian group generated by the irreducible graded $\mathrm{Cl}_{r,s}$-modules and $i^*$ is restriction-of-scalars induced by $i \colon \mathrm{Cl}_{r,s} \hookrightarrow \mathrm{Cl}_{r,s+1}$ (similarly for the complex case). We will instead utilize a version of $K$-theory due to Karoubi [23], which remains well defined for graded $C^*$-algebras, and is consistent with more commonly used definitions for ungraded $C^*$-algebras.

Footnote 11 continued
by finitely generated projective $C(\mathbb{T}^d)$-modules, which the Serre–Swan theorem identifies as the sections of finite-rank vector bundles over $\mathbb{T}^d$ (Bloch bundles).

[12] Note that the left parity reverse ${}^\Pi\mathcal{A}$ can also be used in this definition.

The central object is the $K$-theoretic *difference-group*. Although the definitions in this section are guided by the mathematics, we will motivate and interpret them physically, with the connection to Karoubi's $K$-theory only made towards the end.

Recall that the grading operator $\Gamma$ on a graded f.g.p. module $W$ for the graded algebra $\mathcal{A} = \mathcal{B} \rtimes_{(\alpha,\sigma)} G$ can be interpreted as a spectrally flattened gapped Hamiltonian which is compatible with the symmetry data $(G, c, \mathcal{B}, \alpha, \sigma)$. The difference-group $\mathbf{K}_0(\mathcal{A})$ will be defined in terms of graded f.g.p. modules for $\mathcal{A}$, which may be regarded as graded Hilbert $\mathcal{A}$-modules[13] on which there is a notion of self-adjointness for the adjointable (and bounded) operators $\mathscr{B}^*(W)$. In fact, $\mathscr{B}^*(W)$, along with the grading operator $\Gamma$, form a (evenly) graded $C^*$-algebra under the operator norm. The self-adjoint unitary grading operator for $\mathscr{B}^*(W)$ is $\Gamma$, and it also makes sense to talk about continuous functions of $\Gamma$ and homotopies between them.

*Example* 7.1 (*Noncommutative Bloch theory* [17]). When $\mathcal{A} = \mathbb{C} \rtimes \mathbb{Z}^d \cong C(\mathbb{T}^d)$, a graded f.g.p. $C(\mathbb{T}^d)$-module is, as a Hilbert $C(\mathbb{T}^d)$-module, the set of continuous sections of some graded Hermitian vector bundle over $\mathbb{T}^d$, on which there is a continuous [i.e. $C(\mathbb{T}^d)$-valued] fiberwise inner product. The restriction of the grading operator $\Gamma$ to a fiber can be viewed as the (flattened and gapped) Bloch Hamiltonian for that fiber. Here, we are assuming that the Bloch Hamiltonians have a band structure, giving rise to a continuous family of fiberwise grading operators. The positively graded sub-bundle is the conduction band, while the negatively graded sub-bundle is the valence band. The usual Bloch–Floquet picture of a direct integral decomposition of $L^2(\mathbb{R}^d)$ over the character space $\mathbb{T}^d$, can be recovered by passing to the GNS representation induced by a faithful trace on $C(\mathbb{T}^d)$. In non-commutative Bloch theory, $C(\mathbb{T}^d)$ is replaced by a possibly non-commutative $\mathcal{A}$. The grading operator $\Gamma$ for a f.g.p. graded $\mathcal{A}$-module $W$ is interpreted as a (flattened and gapped) Hamiltonian on the non-commutative graded "vector bundle" $W$.

Given an ungraded $\mathcal{A}$-module $W$, we can consider the set $\text{Grad}_{\mathcal{A}}(W)$ of possible grading operators on $W$ turning it into a graded $\mathcal{A}$-module. There is a standard Banach space structure on $W$ (either from its Hilbert $\mathcal{A}$-module structure or induced from the free module $\mathcal{A}^n$ which it is a direct summand of), which determines a norm topology on the bounded linear maps $W \to W$. Thus, there is a natural induced topology on $\text{Grad}_{\mathcal{A}}(W) \subset \text{End}_{\mathcal{A}}(W) \subset \text{End}(W)$ (e.g. see I.6.22 of [23], 11.2 of [7], or Chapter 15 of [48]).[14] We can then talk about the homotopy classes of symmetry-compatible gapped Hamiltonians on $W$:

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[13] A *Hilbert $C^*$-module over $\mathcal{A}$* [7,31,48], is an $\mathcal{A}$-module with an $\mathcal{A}$-valued "inner product" $\langle \cdot, \cdot \rangle$, whose associated norm $|||x||| = \|\langle x, x\rangle\|^{1/2}$ is complete. A f.g.p. $\mathcal{A}$-module can be endowed with the structure of a Hilbert $\mathcal{A}$-module, see Theorem 15.4.2 in [48].

[14] The use of Hilbert $C^*$-modules is motivated physically by Example 7.1, and the topology for $\text{Grad}_{\mathcal{A}}(W)$ is convenient for the application of Karoubi's $K$-theory later. It should be contrasted with the approach in [15], where a continuous family of gapped quantum mechanical systems was defined in terms of families of non-spectrally flattened and possibly unbounded Hamiltonians.

**Definition 7.1** (*Symmetry compatible gapped Hamiltonians*). Consider the graded twisted $C^*$-dynamical system $(G, c, \mathcal{B}, \alpha, \sigma)$, and suppose the algebra $\mathcal{A} = \mathcal{B} \rtimes_{(\alpha,\sigma)} G$ is unital. Let $W$ be an ungraded f.g.p. $\mathcal{A}$-module. We call $\mathrm{Grad}_{\mathcal{A}}(W)$ the set of $(G, c, \mathcal{B}, \alpha, \sigma)$-*compatible*, or $\mathcal{A}$-*compatible*, or simply *symmetry-compatible* (*flattened*) *gapped Hamiltonians on* $W$. Two grading operators $\Gamma_1, \Gamma_2 \in \mathrm{Grad}_{\mathcal{A}}(W)$ are *homotopic* if there is a norm-continuous path between $\Gamma_1$ and $\Gamma_2$ within $\mathrm{Grad}_{\mathcal{A}}(W)$; in this case, we write $\Gamma_1 \sim_h \Gamma_2$.

Intuitively, $\Gamma_1 \sim_h \Gamma_2$ means that the two Hamiltonians can be continuously deformed into one another, while respecting the symmetries encoded by the algebra $\mathcal{A}$, and maintaining the gapped condition. Regarding $W$ as a Hilbert $C^*$-module, we may further assume that the Hamiltonians $\Gamma_i$ are self-adjoint (and unitary), and that a homotopy between $\Gamma_1$ and $\Gamma_2$ takes place within such self-adjoint grading operators (see 4.6 of [7]).

The set $\pi_0(\mathrm{Grad}_{\mathcal{A}}(W))$ of homotopy classes is interesting in itself [30, 42, 45], but it does not come with any additional structure, much less that of an abelian group. To make the connection to $K$-theory groups, we do not follow the usual arguments in these references—they typically involve taking some "large-$N$ limit" of symmetric spaces as the classifying spaces for $K$-theory—because the connection to the Hamiltonians is difficult to make precise. Rather, we use a modified version of Karoubi's $K$-theory, in which the elements of the $K$-theory of $\mathcal{A}$ represent *differences* or *topological obstructions* between pairs of $\mathcal{A}$-compatible Hamiltonians.

**Definition 7.2** (*Trivial differences between Hamiltonians*). Let $W$ be a graded f.g.p. module for a graded unital $C^*$-algebra $\mathcal{A}$, and let $\Gamma_1, \Gamma_2 \in \mathrm{Grad}_{\mathcal{A}}(W)$ be a pair of compatible gapped Hamiltonians. We call $(W, \Gamma_1, \Gamma_2)$ a *trivial triple* if $\Gamma_1 \sim_h \Gamma_2$ in $\mathrm{Grad}_{\mathcal{A}}(W)$.

A triple $(W, \Gamma_1, \Gamma_2)$ represents the (ordered) difference between two $\mathcal{A}$-compatible gapped Hamiltonians on $W$, and we do not distinguish between two Hamiltonians which can be continuously deformed into one another. We want to be able to consider all graded modules concurrently, and to combine two or more systems with the same symmetry constraints. The direct sum gives a natural abelian monoid structure to the collection $\mathrm{Grad}_{\mathcal{A}}$ of all triples, where some obvious identifications have been made to ensure commutativity and associativity. The collection of trivial triples forms a submonoid $\mathrm{Grad}^t_{\mathcal{A}}$.

**Definition 7.3** (*Difference-group of Hamiltonians*). Let $\mathbf{K}_0(\mathcal{A})$ be the quotient monoid of $\mathrm{Grad}_{\mathcal{A}}$ by the congruence generated by $\mathrm{Grad}^t_{\mathcal{A}}$, i.e. $[W, \Gamma_1, \Gamma_2] = [W', \Gamma'_1, \Gamma'_2]$ in $\mathbf{K}_0(\mathcal{A})$ if there are trivial triples $(F, \zeta_1, \zeta_2)$ and $(F', \zeta'_1, \zeta'_2)$ such that $(W \oplus F, \Gamma_1 \oplus \zeta_1, \Gamma_2 \oplus \zeta_2) = (W' \oplus F', \Gamma'_1 \oplus \zeta'_1, \Gamma'_2 \oplus \zeta'_2)$ in $\mathrm{Grad}_{\mathcal{A}}$. We call $\mathbf{K}_0(\mathcal{A})$ the *difference-group of* $\mathcal{A}$-*compatible gapped Hamiltonians*.

**Proposition 7.4.** $\mathbf{K}_0(\mathcal{A})$ *is an abelian group, with* $[W, \Gamma_1, \Gamma_2] = -[W, \Gamma_2, \Gamma_1]$. *Furthermore, two isomorphic triples (in the natural sense) define the same class in* $\mathbf{K}_0(\mathcal{A})$.

*Proof.* $\Gamma_1 \oplus \Gamma_2 \sim_h \Gamma_2 \oplus \Gamma_1$ in $\mathrm{Grad}_{\mathcal{A}}(W \oplus W)$ via the homotopy

$$\Gamma(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \left[0, \frac{\pi}{2}\right], \tag{9}$$

so $(W \oplus W, \Gamma_1 \oplus \Gamma_2, \Gamma_2 \oplus \Gamma_1)$ is trivial, i.e. $[W, \Gamma_2, \Gamma_1] = -[W, \Gamma_1, \Gamma_2]$ in $\mathbf{K}_0(\mathcal{A})$. For isomorphic triples $(W, \Gamma_1, \Gamma_2)$ and $(W', \Gamma'_1, \Gamma'_2)$, let $\alpha\colon W \to W'$ be the isomorphism of ungraded $\mathcal{A}$-modules, such that $\Gamma'_i = \alpha\Gamma_i\alpha^{-1}$, $i = 1, 2$. Then $\Gamma_2 \oplus \Gamma'_1 \sim_h \Gamma_1 \oplus \Gamma'_2$ in $\mathrm{Grad}_{\mathcal{A}}(W \oplus W')$ via the homotopy

$$\Gamma(\theta) = \begin{pmatrix} \cos\theta & -\alpha^{-1}\sin\theta \\ \alpha\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma'_2 \end{pmatrix} \begin{pmatrix} \cos\theta & \alpha^{-1}\sin\theta \\ -\alpha\sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \left[0, \frac{\pi}{2}\right].$$

so $0 = [W \oplus W', \Gamma_1 \oplus \Gamma'_2, \Gamma_2 \oplus \Gamma'_1] = [W, \Gamma_1, \Gamma_2] - [W', \Gamma'_1, \Gamma'_2]$. Since $\mathcal{A}$ acts diagonally, it follows that it graded commutes with $\Gamma(\theta)$ in both cases.

**Proposition 7.5.** (Path and homotopy independence of differences) *The equation* $[W, \Gamma_1, \Gamma_2] + [W, \Gamma_2, \Gamma_3] = [W, \Gamma_1, \Gamma_3]$ *holds in* $\mathbf{K}_0(\mathcal{A})$*. Furthermore, the class* $[W, \Gamma_1, \Gamma_2]$ *depends only on the homotopy class of* $\Gamma_i$ *in* $\mathrm{Grad}_{\mathcal{A}}(W)$*.*

*Proof.* We need to show that $[W \oplus W \oplus W, \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3, \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_1]$ is trivial. Since $\Gamma_2 \oplus \Gamma_3 \oplus \Gamma_1$ can be obtained from $\Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3$ by conjugation with the permutation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{SO}(3)$, and $\mathrm{SO}(3)$ is path-connected, it follows that $\Gamma_2 \oplus \Gamma_3 \oplus \Gamma_1 \sim_h \Gamma_1 \oplus \Gamma_2 \oplus \Gamma_3$ in $\mathrm{Grad}_{\mathcal{A}}(W \oplus W \oplus W)$. If $\Gamma'_i \sim_h \Gamma_i$, $i = 1, 2$, then $(W, \Gamma'_1, \Gamma_1)$ and $(W, \Gamma_2, \Gamma'_2)$ are trivial, so $[W, \Gamma'_1, \Gamma'_2] = [W, \Gamma'_1, \Gamma_1] + [W, \Gamma_1, \Gamma_2] + [W, \Gamma_2, \Gamma'_2] = [W, \Gamma_1, \Gamma_2]$. ☐

For non-unital $\mathcal{A}$, its unitization $\mathcal{A}^+$ is assigned the grading $(\mathcal{A}^+)_0 = \{(a, \lambda)\colon a \in \mathcal{A}_0, \lambda \in \mathbb{F}\}$, $(\mathcal{A}^+)_1 = \{(a, 0)\colon a \in \mathcal{A}_1\}$, then $\mathcal{A}$ is a graded two-sided ideal in $\mathcal{A}^+$. We define $\mathbf{K}_0(\mathcal{A})$ to be the kernel of the homomorphism $p_*\colon \mathbf{K}_0(\mathcal{A}^+) \to \mathbf{K}_0(\mathbb{F}) \cong \mathbb{Z}$ induced by the even projection $p\colon \mathcal{A}^+ \to \mathbb{F}$. Note that a triple $(W, \Gamma_1, \Gamma_2)$ gets mapped to $(p_*(W), p_*(\Gamma_1), p_*(\Gamma_2))$, where $p_*(W) = \mathbb{F}\hat{\otimes}_{\mathcal{A}^+}W$, $p_*(\Gamma_i) = 1\hat{\otimes}\Gamma_i$, so $\Gamma_1 \sim_h \Gamma_2$ implies $p_*(\Gamma_1) \sim_h p_*(\Gamma_2)$ as is required for consistency.

*Remark* 7.6. The classes in the difference-group which can be written as $[W, \Gamma, -\Gamma]$ form a subgroup of $\mathbf{K}_0(\mathcal{A})$. Such classes can be "represented" by the *single* graded $\mathcal{A}$-module $(W, \Gamma)$. The right parity reversal $W^\Pi = (W, -\Gamma)$ "represents" the class $[W, -\Gamma, \Gamma]$, and "cancels" $(W, \Gamma)$ in the sense that $[W, \Gamma, -\Gamma] = -[W, -\Gamma, \Gamma]$. The left parity reversal $({}^\Pi W, -\Gamma)$ "represents" $[{}^\Pi W, -\Gamma, \Gamma]$, which also "cancels" $(W, \Gamma)$ using the homotopy (9).

If $W$ is homotopic to $W^\Pi$, i.e. $\Gamma \sim_h -\Gamma$, then $[W, \Gamma, -\Gamma] = 0$. For example, suppose $(W, \Gamma)$ admits an extra "supersymmetry", in the sense that there is an odd involution $\mathcal{I}$ on $W$ which graded commutes with $\mathcal{A}$,

$$a \cdot (\mathcal{I}w) = (-1)^{|a|}\mathcal{I}(a \cdot w), \quad a \in \mathcal{A}_0 \cup \mathcal{A}_1,\ w \in W.$$

Then $\Gamma(\theta) = (\cos\theta)\Gamma + (\sin\theta)\mathcal{I}$, $\theta \in [0, \pi]$ provides a homotopy $\Gamma \sim_h -\Gamma$.

*Remark* 7.7. When an arbitrary reference Hamiltonian $\Gamma_0$ on $W$ has been chosen, all other $\Gamma \in \mathrm{Grad}_{\mathcal{A}}(W)$ may be measured in relation to $\Gamma_0$ through the difference class $[W, \Gamma_0, \Gamma]$. Then two homotopic $\Gamma, \Gamma' \in \mathrm{Grad}_{\mathcal{A}}(W)$ differ from $\Gamma_0$ by the same amount: $0 = [W, \Gamma, \Gamma'] = [W, \Gamma_0, \Gamma'] - [W, \Gamma_0, \Gamma]$. They may be said to be in the same phase *relative* to $\Gamma_0$. A canonical $\Gamma_0$ can sometimes (but not always) be chosen (see Example 2.2).

**Relation to Ordinary *K*-Theory Groups.** For purely even $\mathcal{A}^{\mathrm{ev}}$, our definition of $\mathbf{K}_0(\mathcal{A}^{\mathrm{ev}})$ is equivalent to either of Karoubi's two definitions of $K'^{0,0}(\mathcal{A}^{\mathrm{ev}})$ in III.4.15 and III.4.19 of [23], and $K'^{0,0}(\mathcal{A}^{\mathrm{ev}})$ is itself isomorphic to his $K^{0,0}(\mathcal{A}^{\mathrm{ev}})$ as defined in III.4.11 and II.2.13 of the same reference. Both $K'^{0,0}(\mathcal{A}^{\mathrm{ev}})$ and $K^{0,0}(\mathcal{A}^{\mathrm{ev}})$ were shown to be isomorphic to the ordinary $K$-theory group $K_0$ $(\mathcal{A}^{\mathrm{ev}})$ (Theorem III.4.12 of [23]). In particular, a virtual module $[W_0 \ominus W_1] \in K_0(\mathcal{A}^{\mathrm{ev}})$ corresponds to $[W_0 \oplus W_1, 1 \oplus -1, -1 \oplus 1]$ in $\mathbf{K}_0(\mathcal{A}^{\mathrm{ev}})$. Since Karoubi's $K'^{0,0}(\cdot)$ and our $\mathbf{K}_0(\cdot)$ continue to make sense and coincide for graded algebras $\mathcal{A}$, we shall take the difference-group $\mathbf{K}_0(\mathcal{A})$ to be a *definition* of the $K_0$-group of a graded algebra $\mathcal{A}$ [25]. We denote this group using bold-faced notation $\mathbf{K}_0(\mathcal{A})$, to avoid confusion with the ordinary $K_0(\mathcal{A})$ in which $\mathcal{A}$ is regarded as an ungraded algebra.

If we define $\mathbf{K}_{s,r}(\mathcal{A}) := \mathbf{K}_0(\mathcal{A} \hat{\otimes} \mathrm{Cl}_{r,s})$ for real graded algebras $\mathcal{A}$, we obtain an alternative definition of Karoubi's $K'^{r,s}(\mathcal{A})$ as introduced (initially for ungraded $\mathcal{A}$) in III.4.11 of [23]. Due to the periodicity properties of the Clifford algebras, the $\mathbf{K}_{s,r}(\mathcal{A})$ and $K'^{r,s}(\mathcal{A})$ groups depend only on $(r-s) \pmod 8$. Thus, the singly indexed groups $\mathbf{K}_n(\mathcal{A}) := \mathbf{K}_0(\mathcal{A} \hat{\otimes} \mathrm{Cl}_{0,n})$ are 8-periodic. In the complex case, we can similarly define $\mathbf{K}_n(\mathcal{A}) := \mathbf{K}_0(\mathcal{A} \hat{\otimes} \mathbb{C}\mathrm{l}_n)$, which are 2-periodic. Karoubi also showed the following difficult result:

**Proposition 7.8** (Clifford suspension [22, 23, 25]). *There are isomorphisms* $K'^{0,1}(\mathcal{A}) \equiv K'^{0,0}(\mathcal{A} \hat{\otimes} \mathrm{Cl}_{0,1}) \cong K'^{0,0}(C_0(\mathbb{R}, \mathcal{A}))$.

Thus, tensoring $\mathcal{A}^{\mathrm{ev}}$ by $\mathrm{Cl}_{0,n}$ shifts the index of $\mathbf{K}_\star$ in the same way that the ordinary suspension of $\mathcal{A}^{\mathrm{ev}}$ shifts the index of $K_\star$,

$$\mathbf{K}_n(\mathcal{A}^{\mathrm{ev}}) \equiv \mathbf{K}_0(\mathcal{A}^{\mathrm{ev}} \hat{\otimes} \mathrm{Cl}_{0,n}) \cong K_0(C_0(\mathbb{R}^n, \mathcal{A}^{\mathrm{ev}})) \equiv K_n(\mathcal{A}^{\mathrm{ev}});$$

similarly for the complex case.

## 8. Computing $\mathbf{K}_0(\mathcal{A})$ by Decomposing $\mathcal{A}$

We state a very useful decomposition theorem for twisted crossed products, which facilitates the computation of some $K$-theoretic difference-groups arising from physical examples. In certain cases, a description of these groups in terms of topological $K$-theory is possible, but this is not generic.

**Theorem 8.1** (Packer–Raeburn decomposition theorem [36]). *Consider a graded twisted $C^*$-dynamical system $(G, c, \mathcal{A}, \alpha, \sigma)$, and let $N \subset \ker(c)$ be a closed normal subgroup of $G$. There is an isomorphism of graded $C^*$-algebras*

$$\mathcal{A} \rtimes_{(\alpha,\sigma)} G \cong (\mathcal{A} \rtimes_{(\alpha,\sigma)} N) \rtimes_{(\beta,\nu)} G/N, \tag{10}$$

*where the twisting pair* $(\beta, \nu)$ *is determined by a choice of Borel section* $s\colon G/N \ni p \mapsto s_p \in G$ *such that* $s_{eN} = 1$*. For each* $x \in G$*, there is a* $\gamma_x \in \mathrm{Aut}_{\mathbb{F}}(\mathcal{A} \rtimes_{(\alpha,\sigma)} N)$ *such that*

$$\gamma_x(a) = \alpha_x(a) \equiv a^x, \qquad a \in \mathcal{A}, \tag{11a}$$

$$\gamma_x(n) = \sigma(x, n)\sigma(xnx^{-1}, x)^{-1}xnx^{-1}, \qquad n \in N, \tag{11b}$$

*where the canonical embeddings* $j_{\mathcal{A}}, j_N$ [*Eqs.* (7a), (7b)] *are implied. The formulae for* $(\beta, \nu)$ *are, for* $p, q \in G/N$*,*

$$\beta_p := \beta(p) = \gamma_{s_p}, \tag{12a}$$

$$\nu(p, q) = \sigma(s_p, s_q)\sigma(s_p s_q s_{pq}^{-1}, s_{pq})^{-1} s_p s_q s_{pq}^{-1}. \tag{12b}$$

Part of the theorem says that up to isomorphism, the iterated crossed product does not depend on the choice of section $s$. Theorem 8.1 was proved in [36] for ungraded complex twisted crossed products, but the generalization to the real and/or graded cases still holds. We have required $N \subset \ker(c)$ to ensure that $c$ descends to the quotient group $G/N$, and that the automorphisms $\beta_p$ and cocycle $\nu(\cdot, \cdot)$ are even, independently of $s$. Then one checks that the standard grading on either side of (10) agrees with the other.

### 8.1. Finitely Generated Projective Modules in Equivariant $K$-Theory

We make a short digression to define the notion of a f.g.p. $(G, \mathcal{A}, \alpha)$-module $W$, following Chapter 11.2 of [7]. Here, $\mathcal{A}$ is a (ungraded) unital $C^*$-algebra, and $\alpha$ is a Borel homomorphism (hence continuous) from a compact group $G$ to $\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$. Such modules are needed to define the $G$-equivariant $K$-theory of $\mathcal{A}$, and when $\mathcal{A}$ is commutative, they provide the link to the corresponding topological equivariant $K$-theory.

We write $\mathscr{L}(W)$ for the set of bounded linear operators on $W$, $\mathscr{GL}(W)$ for the subgroup of invertible operators, and $\mathscr{B}(W)$ for the subalgebra of module maps. A f.g.p. $(G, \mathcal{A}, \alpha)$-module is a f.g.p $\mathcal{A}$-module $W$, together with a strongly continuous homomorphism $\theta\colon G \to \mathscr{GL}(W)$, such that

$$\theta_x(aw) = a^x(\theta_x w), \quad x \in G,\, a \in \mathcal{A},\, w \in W.$$

The equivariant $K$-theory group $K_0^G(\mathcal{A})$ is defined to be the Grothendieck group of the monoid (under the direct sum) of equivalence classes of f.g.p. $(G, \mathcal{A}, \alpha)$-modules.[15] The Green–Julg theorem says that an equivariant $K_0$-group is isomorphic to the ordinary $K_0$-group of the crossed product,

$$K_0^G(\mathcal{A}) \cong K_0(\mathcal{A} \rtimes_{(\alpha,1)} G).$$

Graded f.g.p. $(G, c, \mathcal{A}, \alpha)$-modules $W$ are similarly defined, with the operators $\theta_x$ required to be odd or even according to $c(x)$.

*Example* 8.1 (*Decomposing group $C^*$-algebras over an abelian normal subgroup*). Consider the case where $\alpha = c = \sigma \equiv 1$, $N$ is a discrete abelian group, $P := G/N$ is compact, and $G$ is a topological semidirect product $G = N \rtimes P$.

[15] If $\mathcal{A}$ is non-unital, $K_0^G(\mathcal{A})$ is defined to be the kernel of the the induced map $p_*\colon K_0^G(\mathcal{A}^+) \to K_0^G(\mathbb{F})$, where $\mathcal{A}^+$ has the induced action from $\alpha$, $\mathbb{F}$ has the trivial $G$-action, and $p$ is the equivariant projection $\mathcal{A}^+ \to \mathbb{F}$.

The Fourier transform gives an isomorphism $\mathbb{C} \rtimes N \cong C(\hat{N})$, where $\hat{N}$ is the Pontryagin dual of $N$. In physical applications, $N$ is the subgroup of translational symmetries of a crystal lattice and $P$ is a compact point group. Thus $\hat{N}$ is the Brillouin torus, over which the Bloch bands of solid-state physics reside. Since $c \equiv 1$, $\mathcal{A} = \mathbb{C} \rtimes G$ is purely even and $K_0(\mathcal{A}) \cong \mathbf{K}_0(\mathcal{A})$.

The standard homomorphic section $s\colon p \mapsto (e,p) \in N \rtimes P$ satisfies $(e,p)(n,e)(e,p^{-1}) = (p \cdot n, e)$, where $n \mapsto p \cdot n$ is the defining automorphic action of $p \in P$ on $N$. From (11b) and (12a), the automorphisms $\beta_p$ act on $\delta_n \equiv j_N(n)$ by $\beta_p(\delta_n) = \delta_{p\cdot n}$; in terms of functions $\mathbb{C} \rtimes N \ni f\colon N \to \mathbb{C}$, this is $\beta_p(f)(n) = f(p^{-1} \cdot n)$. Under the Fourier transform $f \mapsto \hat{f} \in C(\hat{N})$, the automorphisms $\beta_p$ become $\hat{\beta}_p$, defined by $\hat{\beta}_p(\hat{f})(\chi) := \hat{f}(p^{-1} \cdot \chi)$, where $(p \cdot \chi)(n) := \chi(p^{-1} \cdot n)$ is the dual $P$-action on $\hat{N}$. Also, (12b) gives $\nu \equiv 1$, so we may decompose $\mathcal{A} = \mathbb{C} \rtimes G$ as $C(\hat{N}) \rtimes_{(\hat{\beta},1)} P$, then $K_0(\mathcal{A}) \cong K_0^P(C(\hat{N})) \cong K_P^0(\hat{N})$. Thus a f.g.p. $\mathcal{A}$-module corresponds to the sections of some finite-rank $P$-equivariant vector bundle over $\hat{N}$.

If $G$ is not assumed to be a semidirect product of $N$ and $P$ (this is the case when $G$ is a non-symmorphic space group), there is a non-trivial central cocycle $\nu(p,q) = s_p s_q s_{pq}^{-1} \in N$ according to (12b). The automorphism $\hat{\beta}_p$ is now the dual of conjugation by $s_p$, i.e. $\hat{\beta}_p(\hat{f})(\chi) = \hat{f}(p^{-1} \cdot \chi)$ where $(p \cdot \chi)(n) := \chi(s_p^{-1} n s_p)$. The isomorphism $\mathbb{C} \rtimes G \cong C(\hat{N}) \rtimes_{(\hat{\beta},\nu)} P$ suggests the interpretation of a "f.g.p. $(P, C(\hat{N}), \hat{\beta}, \nu)$-module" as the sections of a vector bundle $E \to \hat{N}$, equipped with a "$\nu$-twisted equivariant $P$-action". There is a family of *projective* representations of $P$, with $p \in P$ mapping the fiber over $\chi \in \hat{N}$ linearly to the fiber over $p \cdot \chi$.

Note that when $N$ is not discrete, $\hat{N}$ is non-compact. Topological equivariant $K$-theory groups for $\hat{N}$ must then be interpreted using vector bundles trivialized outside a compact subspace of $\hat{N}$, i.e. $K$-theory with "compact supports". Such a situation arises, for instance, when $N = \mathbb{R}^d$, which has a very different topological nature to $\mathbb{Z}^d$. If $N$ is projectively realized (so $\sigma \not\equiv 1$), then $\mathbb{C} \rtimes_{(1,\sigma)} N$ is *noncommutative* in general. This occurs when there is *magnetic* translational symmetry, e.g. in the IQHE [6]. Our non-commutative approach bears fruit here, since it still makes sense to study the $K$-theory of $\mathbb{C} \rtimes_{(1,\sigma)} G \cong (\mathbb{C} \rtimes_{(1,\sigma)} N) \rtimes_{(\beta,\nu)} P$.

*Example* 8.2 (*A Clifford algebra factorises in the symmetry algebra*). Consider the symmetry data $(G, c, \phi, \sigma)$, and assume that $G = (N \rtimes Q) \times A$ where $A \cong \mathrm{Im}(\phi, c) \subset \{\pm 1\}^2$, $\ker(\phi, c) = N \rtimes Q$, $N$ is abelian, and $Q$ is compact. $A$ should be thought of as one of the subgroups of the $CT$-group, by identifying $\{\pm 1\}^2$ with $\{1, T, C, S\}$ as in Sect. 6.1. We also assume that $\sigma(nq, \cdot) = 1 = \sigma(\cdot, nq)$ for all $nq$ in $N \rtimes Q$, then $\sigma$ is simply specified by its restriction to $A$. This is the setting (with $Q$ usually assumed trivial) that is often considered in the literature when studying band structures with time-reversal and/or charge-conjugation symmetry.

Let $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$, with $\alpha$ determined by $\phi$ as usual. There is a non-trivial grading on $\mathcal{A}$ if $c \not\equiv 1$. We denote the images, under $j_{\mathcal{A}}$, of $C, T, S$ in $\mathcal{MA}$

by $\mathsf{C},\mathsf{T},\mathsf{S}$. As in Sect. 6.1, we can choose $\sigma$ such that $\mathsf{CT}=\mathsf{TC}=\mathsf{S}$, then $\sigma$ is simply specified by $\mathsf{T}^2=\pm 1, \mathsf{C}^2=\pm 1$, while $\mathsf{S}^2=+1$ can be assumed. There are ten possibilities for $(A,\sigma)$, each with a corresponding Clifford algebra, as listed in Table 1. Decomposing $\mathcal{A}$ with respect to the subgroup $N\rtimes Q$, and noting that $\nu$ reduces to $\sigma$ in (12b), we obtain

$$\begin{aligned}\mathcal{A}&\cong(\mathbb{C}\rtimes(N\rtimes Q))\rtimes_{(\gamma,\sigma)}A\cong(\mathbb{C}\rtimes(N\rtimes Q))\rtimes_{(\gamma,\sigma)}A\\&\cong\left(C_0(\hat{N})\rtimes_{(\hat{\beta},1)}Q\right)\rtimes_{(\gamma,\sigma)}A,\end{aligned}$$

where $\hat{\beta}$ is determined as in Example 8.1, and $\gamma_r, r\in A$ are some automorphisms of $\mathbb{C}\rtimes(N\rtimes Q)$. Since $A$ appears as a direct product factor in $G$, it acts only on $\mathbb{C}$ in $\mathbb{C}\rtimes(N\rtimes Q)$, so $\gamma_r$ effects complex conjugation if $\phi(r)=-1$ and does nothing otherwise.

If $A\subset\{1,S\}$ so that $\phi\equiv 1$ (complex case), we have

$$\mathcal{A}\cong\left(C_0(\hat{N})\rtimes_{(\hat{\beta},1)}Q\right)\hat{\otimes}_{\mathbb{C}}\left(\mathbb{C}\rtimes_{(1,\sigma)}A\right)\cong\left(C_0(\hat{N})\rtimes_{(\hat{\beta},1)}Q\right)\hat{\otimes}\,\mathbb{C}\mathrm{l}_n,\quad(13)$$

where the complex Clifford algebra is $\mathbb{C}\mathrm{l}_1$ if $A=\{1,S\}$ and $\mathbb{C}\mathrm{l}_0$ if $A=\{1\}$. For discrete $N$, a graded f.g.p. $(Q,C(\hat{N}),\hat{\beta})$-module corresponds to a graded $Q$-equivariant complex vector bundle over $\hat{N}$, which is just the direct sum of two ungraded such bundles. When $n=1$, there is an additional graded action of $\mathbb{C}\mathrm{l}_1$ on the fibers which commutes with the $Q$-action.

If $\phi\not\equiv 1$ (real case) so that either of $C$ and $T$ is present, we first write $\mathbb{C}\rtimes(N\rtimes Q)=(\mathbb{R}\rtimes(N\rtimes Q))\otimes_{\mathbb{R}}\mathbb{C}$. Then we obtain

$$\begin{aligned}\mathcal{A}&\cong((\mathbb{R}\rtimes(N\rtimes Q))\otimes_{\mathbb{R}}\mathbb{C})\rtimes_{(\gamma,\sigma)}A=(\mathbb{R}\rtimes(N\rtimes Q))\otimes_{\mathbb{R}}\left(\mathbb{C}\rtimes_{(\alpha,\sigma)}A\right)\\&\cong(\mathbb{R}\rtimes(N\rtimes Q))\,\hat{\otimes}\,\mathrm{Cl}_{r,s},\end{aligned}\quad(14)$$

where the Clifford algebra $\mathrm{Cl}_{r,s}$ is determined by $(A,\sigma)$ as usual.

It is possible to formulate things in terms of Real bundles in the sense of Atiyah [3], with Clifford modules as fibers [15,47]. However, doing this directly by Fourier transforming $\mathbb{C}\rtimes N$ to $C(\hat{N})$ requires a fairly complicated and opaque auxiliary construction in the real case. The is because the real $C^*$-algebra $\mathbb{R}\rtimes N$ does not simply translate into $C_0(\hat{N},\mathbb{R})$ under the Fourier transform. Instead, we have to consider $\hat{N}$ as a Real space with involution given by the map taking a character $\chi$ to its complex conjugate $\bar{\chi}$. Upon taking the Fourier transform $\mathbb{C}\rtimes N\cong C_0(\hat{N})$, complex conjugation on $\mathbb{C}\rtimes N$ (which fixes $\mathbb{R}\rtimes N$) turns into the antilinear involution $\bar{\hat{f}}(\chi):=\overline{\hat{f}(\bar{\chi})}$. Thus,

$$\mathbb{R}\rtimes N\cong C_0(i\hat{N}):=\left\{\hat{f}\in C_0(\hat{N})\colon\overline{\hat{f}(\chi)}=\hat{f}(\bar{\chi})\right\}.\quad(15)$$

If we had performed a Fourier transform in (14), a Clifford algebra cannot be nicely factorized, and the analysis becomes unnecessarily obscured.

Equations (13) and (14) express $\mathcal{A}$ as the tensor product of a purely even algebra with a graded Clifford algebra. According to Sect. 7, this effects a degree shift in $K$-theory, so the difference-group $\mathbf{K}_0(\mathcal{A})$ is easy to compute:

$$\mathbf{K}_0(\mathcal{A}) \cong \begin{cases} K_n(C_0(\hat{N}) \rtimes_{(\hat{\beta},1)} Q) \cong K_Q^{-n}(\hat{N}) & \text{complex case,} \\ K_{s-r}(C_0(i\hat{N}) \rtimes_{(\hat{\beta},1)} Q) & \text{real case.} \end{cases} \tag{16}$$

There is a correspondence between a $C_0(i\hat{N})$-module and the sections of a Real bundle over $\hat{N}$ [with antilinear involution $\overline{(\cdot)}$ lifting $\chi \mapsto \bar{\chi}$] which are fixed under the induced involution $\overline{s(\chi)} = s(\bar{\chi})$; then we may rewrite

$$\mathbf{K}_0(\mathcal{A}) \cong KR_Q^{r-s\,(\mathrm{mod}\,8)}(\hat{N}), \quad \text{real case.} \tag{17}$$

## 9. Applications to Topological Band Insulators

### 9.1. Band Insulators and $K$-Theory

The computations in Example 8.1 include the special case of topological band insulators, in which $N = \mathbb{Z}^d$. When we assumed $G = \mathbb{Z}^d \rtimes P$ as well as $\sigma \equiv 1$, we obtained $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,1)} G \cong C(\mathbb{T}^d) \rtimes_{(\hat{\beta},1)} P$. Underlying a graded f.g.p. $\mathcal{A}$-module is a graded f.g.p. $C(\mathbb{T}^d)$-module, and thus a graded complex vector bundle $E \to \mathbb{T}^d$ which we call a graded *Bloch bundle*.

Allowing for non-trivial homomorphisms $(\phi, c)$ but with $N \subset \ker(\phi, c)$, the maps $(\phi, c)$ descend to $P$ and tell us whether $p \in P$ acts complex-linearly/antilinearly and preserves/reverses the particle–hole distinction. The initial and final fibers depend on the action of $p$ on $\mathbb{Z}^d$ and whether $\phi(p) = +1$ or $-1$. The positively graded subbundle can be interpreted as the conduction band lying above the Fermi level $\mathcal{E}_F$ (taken to be 0), while the negatively graded subbundle is the valence band. This description does not require any distinguished involutory time-reversal, charge-conjugation, or chiral symmetry element in $P$.

In Sect. 8.2, we allowed $\sigma \not\equiv 1$, and found that a significant simplification occurs when $G = (\mathbb{Z}^d \rtimes Q) \times A$ with $A \cong \mathrm{Im}(\phi, c) \subset \{1, T, C, S\}$ and $\sigma$ is only non-trivial between elements of $A$. The data of $(A, \sigma)$ are associated with one of ten possible Clifford algebras, which factorizes in the symmetry algebra $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$. In the two complex cases, the graded Bloch bundle underlying a graded f.g.p. $\mathcal{A}$-module is a graded $Q$-equivariant complex bundle $E \to \mathbb{T}^d$. If $A = \{1, S\}$, there is an additional *commuting* graded $\mathbb{C}l_1$-action on each fiber, generated by the complex-linear, odd, and involutory map $\mathsf{I}_\mathsf{S}$ representing $\mathsf{S}$. For the remaining eight real cases, the elements $T, C$ of $A$ (whenever present) act through the antilinear maps $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$ representing $\mathsf{T}, \mathsf{C}$, which are even and odd respectively. Unlike $\mathsf{I}_\mathsf{S}$, the maps $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$ do *not* commute with the bundle projection, but take the fiber over $\chi \in \mathbb{T}^d$ to the fiber over $\bar{\chi}$. The map $\mathsf{I}_\mathsf{T}$ is the standard time-reversal operator on the Bloch bundle $E$, and both cases $\mathsf{I}_\mathsf{T}^2 = \pm 1$ can occur. Similarly, $\mathsf{I}_\mathsf{C}$ is the standard particle–hole conjugation operator. Note that $\mathsf{I}_\mathsf{T}, \mathsf{I}_\mathsf{C}$ do *not* commute with $C(\mathbb{T}^d)$, so the graded Bloch bundle is not a bundle of Clifford modules.

The difference-groups of $(G, c, \phi, \sigma)$-compatible band insulators can be read off from (16) and (17),

$$\mathbf{K}_0(\mathcal{A}) = \begin{cases} K_Q^{-n}(\mathbb{T}^d) & \text{complex case,} \\ KR_Q^{r-s\,(\mathrm{mod}\,8)}(\mathbb{T}^d) & \text{real case,} \end{cases}$$

with $n, r, s$ determined by Table 1.

### 9.2. The Three Special Purely Even Cases

If $A \subset \{1, T\}$, there are three possibilities for $\mathbb{C} \rtimes_{(\alpha,\sigma)} A$, all of which are purely even algebras: it is either the complex algebra $\mathbb{C}$, the real algebra $M_2(\mathbb{R})$, or the real algebra $\mathbb{H}$, with the latter two real algebras generated by $i$ and $\mathsf{T}$. This means that (13) and (14) should give[16]

$$\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G = \begin{cases} \mathbb{C} \rtimes (N \rtimes Q) & A = \{1\}, \\ \mathbb{R} \rtimes (N \rtimes Q) \otimes_{\mathbb{R}} M_2(\mathbb{R}) & A = \{1, T\}, \mathsf{T}^2 = +1, \\ (\mathbb{R} \rtimes (N \rtimes Q)) \otimes_{\mathbb{R}} \mathbb{H} & A = \{1, T\}, \mathsf{T}^2 = -1, \end{cases}$$

with difference-groups or *ordinary* $K$-theory groups

$$K_0(\mathcal{A}) \cong \mathbf{K}_0(\mathcal{A}) = \begin{cases} K_Q^0(\hat{N}) & A = \{1\}, \\ KR_Q^0(\hat{N}) & A = \{1, T\}, \mathsf{T}^2 = +1, \\ KR_Q^{-4}(\hat{N}) & A = \{1, T\}, \mathsf{T}^2 = -1. \end{cases}$$

For band insulators ($N = \mathbb{Z}^d$), these three cases are usually labeled by A, AI and AII, respectively.

The third case ($\mathsf{T}^2 = -1$), which resulted in a $KR^{-4}$ group, can be modified to resemble the first two cases more closely. In that case, we identify a f.g.p. $\mathcal{A}$-module with the sections of a $Q$-equivariant "Quaternionic" vector bundle over $\hat{N}$. "Quaternionic" bundles resemble the Real bundles of $KR$-theory in that they are complex vector bundles over a Real space $(X, \varsigma)$, but they are equipped with a lift of $\varsigma$ to an antilinear *anti-involution* $\Theta$ ($\Theta^2 = -1$) rather than an antilinear involution. Such bundles were considered in the context of Type AII topological insulators in [10], in which detailed definitions and references for "Quaternionic" bundles can be found. The corresponding topological $K$-theory is called $KQ$-theory, and there is a useful isomorphism $KR^{-4}(X, \varsigma) \cong KQ^0(X, \varsigma)$ (see [14] and the Appendix of [10]). This establishes a $KR^{-4}$ group as a Grothendieck group $KQ^0$ of "Quaternionic" bundles. Therefore, $\mathbf{K}_0(\mathcal{A})$ may be interpreted as a Grothendieck group of vector bundles in all the three cases where $\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} G$ is purely even *but not in the other seven.*

[16] Recall that in the third case, we replaced $\mathbb{H}$ by the graded Clifford algebra $\mathrm{Cl}_{4,0} \cong \mathrm{Cl}_{0,4}$ in (14), which we used to arrive at (17), c.f. Theorem III.4.12 of [23] which says that $K'^{0,0}(\mathcal{B} \otimes_{\mathbb{R}} \mathbb{H}) \cong K'^{0,4}(\mathcal{B})$ for any ungraded $\mathcal{B}$.

### 9.3. Finite Versus Infinite Rank Bundles

In many realistic Hamiltonians compatible with a subgroup $N = \mathbb{Z}^d \subset G$ of translational symmetries of a lattice, the Bloch Hamiltonians are unbounded operators on a bundle of infinite-dimensional Hilbert spaces over the Brillouin torus $\hat{N} = \mathbb{T}^d$. In our application of $K$-theory, we have confined ourselves to the finite-rank situation. We can motivate this physically by imposing an energy cut-off, or by assuming that there are finite-rank conduction and valence bands which we restrict attention to. As such, $\Gamma$ refers not to the flattened version of the full Hamiltonian, but to the flattened version of its restriction to the relevant low energy modes. Likewise, the symmetries act only on this restricted (invariant) subspace. Also, in certain tight-binding models, the Bloch Hamiltonians do have finite rank, so the $K$-theory classification makes sense for the full Hamiltonian. It is actually very important to distinguish the finite- and infinite-rank cases.

In the Hilbert $C^*$-module description of Bloch theory [17], the Bloch Hamiltonians act on a continuous field of infinite-dimensional separable Hilbert spaces over the Brillouin torus $\mathbb{T}^d$, whose sections form a countably generated Hilbert $C(\mathbb{T}^d)$-module. In [15], the authors considered graded bundles $E^+ \oplus E^-$ over $\mathbb{T}^d$, such that $E^+$ is infinite-dimensional, while $E^-$ is a finite-rank bundle. They called such bundles "Type $I$" insulators, while graded bundles with finite rank $E^+$ and $E^-$ were called "Type $F$" insulators. We can make this distinction in non-commutative language.

**Definition 9.1** (*Type I and Type F insulators*). Let $\mathcal{A}$ be a graded unital $C^*$-algebra. A *Type I insulator* is a graded $\mathcal{A}$-module $E = E_0 \oplus E_1 =: E^+ \oplus E^-$ such that $E^-$ is an ungraded f.g.p. $\mathcal{A}$-module and $E^+$ is a countably generated (but not finitely generated) Hilbert $\mathcal{A}$-module. A *Type F insulator* is a graded f.g.p. $\mathcal{A}$-module.

Suppose $\mathcal{A}$ arises from the symmetry data $(G, c, \mathcal{B}, \phi, \sigma)$. For Type $I$ insulators, there is no possibility of an invertible odd operator taking $E^+$ to the f.g.p. submodule $E^-$, so $c \equiv 1$ and $\mathcal{A}$ is purely even. We define $\mathcal{GV}^I(\mathcal{A})$ to be the abelian monoid, under the direct sum, of equivalence classes of Type $I$ insulators. Note that we have to formally introduce the zero module as a zero element in $\mathcal{GV}^I(\mathcal{A})$. This monoid is really the direct sum of the monoid $\mathcal{V}^+(\mathcal{A})$ of ungraded countably generated Hilbert $\mathcal{A}$-modules $E^+$, and the monoid $\mathcal{V}^-(\mathcal{A})$ of ungraded f.g.p. $\mathcal{A}$-modules. The Grothendieck completion $\mathcal{GR}^I(\mathcal{A})$ of $\mathcal{GV}^I(\mathcal{A}) = \mathcal{V}^+(\mathcal{A}) \oplus \mathcal{V}^-(\mathcal{A})$ simplifies upon performing an "Eilenberg swindle" on the $\mathcal{V}^+$ part:

**Proposition 9.2.** *The Grothendieck completion of $\mathcal{V}^+(\mathcal{A})$ is the trivial group.*

*Proof.* There is a "standard" countably generated Hilbert $\mathcal{A}$-module $\mathscr{H}_\mathcal{A}$ (see 15.1.7 of [48]), defined by

$$\mathscr{H}_\mathcal{A} := \left\{ (a_k) \in \prod_{k=1}^{\infty} A \colon \sum_{k=1}^{\infty} a_k^* a_k \text{ converges in norm in } \mathcal{A} \right\},$$

with component-wise operations and $\mathcal{A}$-valued inner product $\langle (a_k),(b_k)\rangle := \sum_k a_k^* b_k$. The Kasparov stabilization theorem (see 15.4.6 of [48] for a proof and original references), says that $\mathscr{H}_{\mathcal{A}}$ is absorbing in the sense that every countably generated Hilbert $\mathcal{A}$-module $E^+$ satisfies $E^+ \oplus \mathscr{H}_{\mathcal{A}} \cong \mathscr{H}_{\mathcal{A}}$. Then any virtual module $[E^+ \ominus E'^+]$ in the Grothendieck completion of $\mathcal{V}^+(\mathcal{A})$ satisfies $[E^+ \ominus E'^+] = [(E^+ \oplus \mathscr{H}_{\mathcal{A}}) \ominus (E'^+ \oplus \mathscr{H}_{\mathcal{A}})] = [\mathscr{H}_{\mathcal{A}} \ominus \mathscr{H}_{\mathcal{A}}] = 0$. □

**Corollary 9.3.** *$\mathcal{GR}^I(\mathcal{A})$ is isomorphic to the Grothendieck completion $K_0(\mathcal{A})$ of $\mathcal{V}^-(\mathcal{A})$.*

Corollary 9.3 provides yet another interpretation of $K_0(\mathcal{A})$ (with $\mathcal{A}$ necessarily purely even). Namely, a virtual class $[E^- \ominus 0] \in K_0(\mathcal{A}) \cong \mathcal{GR}^I(\mathcal{A})$ can be represented by a Type $I$ insulator whose "bundle of negative eigenstates" is $E^-$. Therefore, $\mathcal{GR}^I(\mathcal{A})$ retains only the information of $E^-$. More generally, $[E^- \ominus E'^-] \in K_0(\mathcal{A})$ is represented by the *formal difference* of (the negative eigenstates of) two Type $I$ insulators. This is more familiar than it looks. The Hall conductivity in the IQHE is related to the $K$-theory element $[E^- \ominus 0]$ associated to a Type $I$ Landau Hamiltonian [6]. Here, there is generally no Brillouin zone in the ordinary sense, so our definition of a "noncommutative" Type $I$ insulator has genuine physical relevance. Similarly, the Kane–Mele $\mathbb{Z}_2$ invariant for $T$-invariant insulators is determined by (formal differences of) the negative-energy bundles of Type $I$ insulators in Class AII, and was studied in [15]. On the other hand, the group $\mathcal{GR}^I(\mathcal{A})$ does not generally make sense for Type $F$ insulators, since the latter includes those with $\mathcal{A}$ having non-trivial grading.

In summary: *the interpretation of ordinary $K$-theory groups either as $\mathcal{GR}^I(\mathcal{A})$ in the Type $I$ case, or as virtual classes of ungraded $\mathcal{A}$-modules in the Type $F$ case, cannot be used in a unified picture which includes charge-conjugating symmetries ($c \not\equiv 1$). On the other hand, the difference-group has general applicability.*

## 10. Periodic Table of Difference-Groups and Dimension Shifts

### 10.1. Zero-Dimensional Gapped Phases

In Sect. 6.1, we showed that the graded PUA-reps of each of the ten choices of $(A,\sigma)$ are in 1–1 correspondence with the graded representations of an associated Clifford algebra. Based on the ABS isomorphisms (8a) and (8b), we can introduce a notion of trivial Clifford modules, namely those which admit the action of an extra negative Clifford generator—such modules may be interpreted as those which actually belong to a different symmetry class. Then we can say that 0-dimensional gapped topological phases compatible with $(A,\sigma)$, modulo the trivial ones, are classified by $K^{-n}(\star) \cong K_n(\mathbb{C})$ or $KO^{-n}(\star) \cong K_n(\mathbb{R})$, according to Table 1. The same $K$-theory groups also classify the differences of $(A,\sigma)$-compatible gapped Hamiltonians, due to the isomorphisms $\mathbf{K}_0(\mathrm{Cl}_{r,s}) \cong K_{s-r}(\mathbb{R})$ and $\mathbf{K}_0(\mathbb{C}\mathrm{l}_n) \cong K_n(\mathbb{C})$.

TABLE 1. Classification of 0-dimensional gapped topological phases and their difference-classes

| $n$ | Class | $(A,\sigma)$ | $\mathsf{C}^2$ | $\mathsf{T}^2$ | $Cl_{0,n}$ or $\mathbb{C}l_n$ | $K_n(\mathbb{R})$ or $K_n(\mathbb{C})$ |
|---|---|---|---|---|---|---|
| 0 | AI | $T$ | | $+1$ | $Cl_{0,0}$ | $\mathbb{Z}$ |
| 1 | BDI | $C,T$ | $+1$ | $+1$ | $Cl_{0,1}$ | $\mathbb{Z}_2$ |
| 2 | D | $C$ | $+1$ | | $Cl_{0,2}$ | $\mathbb{Z}_2$ |
| 3 | DIII | $C,T$ | $+1$ | $-1$ | $Cl_{0,3}$ | 0 |
| 4 | AII | $T$ | | $-1$ | $Cl_{0,4}$ | $\mathbb{Z}$ |
| 5 | CII | $C,T$ | $-1$ | $-1$ | $Cl_{0,5}$ | 0 |
| 6 | C | $C$ | $-1$ | | $Cl_{0,6}$ | 0 |
| 7 | CI | $C,T$ | $-1$ | $+1$ | $Cl_{0,7}$ | 0 |
| 0 | A | N/A | N/A | | $\mathbb{C}l_0$ | $\mathbb{Z}$ |
| 1 | AIII | $S$ | $\mathsf{S}^2=+1$ | | $\mathbb{C}l_1$ | 0 |

The next-to-last column lists the Clifford algebra $Cl_{0,n}$ or $\mathbb{C}l_n$ in the graded Morita class of the algebra $\mathcal{A}$ associated with $(A,\sigma)$. The $K$-theory group in the last column is isomorphic to $\mathbf{K}_0(\mathcal{A})$

### 10.2. Higher Dimensional Gapped Phases

In the literature, it has been suggested [30,42,45] that the $K$-theoretic classification of gapped topological phases in $d$ spatial dimensions is the same as that in 0 dimensions, except for a shift in the $K$-theory index by $d$. We have located the appropriate $K$-theoretic object (the difference-group) for which such a phenomenon might be plausible. We will prove a robust version of this dimension shift using some powerful results from the $K$-theory of crossed product $C^*$-algebras.

**Theorem 10.1** (Packer–Raeburn stabilization trick [36]). *Let $(G,\mathcal{A},\alpha,\sigma)$ be a twisted $C^*$-dynamical system, and let $\mathcal{K}$ denote the compact operators on the Hilbert space $L^2(G)$. There is an isomorphism*

$$(\mathcal{A}\rtimes_{(\alpha,\sigma)}G)\otimes\mathcal{K}\cong(\mathcal{A}\otimes\mathcal{K})\rtimes_{(\alpha',1)}G,$$

*for some untwisted action $\alpha'$ of $G$ on $\mathcal{A}\otimes\mathcal{K}$.*

**Theorem 10.2** (Connes–Thom isomorphism [9,26,44]). *Let $(\mathbb{R},\mathcal{A},\alpha)$ be a $C^*$-dynamical system, with $\mathcal{A}$ a real or complex (ungraded) $C^*$-algebra. Then $K_n(\mathcal{A}\rtimes_{(\alpha,1)}\mathbb{R})\cong K_{n-1}(\mathcal{A})$.*

Remarkably, Theorem 10.2 holds for any continuous $\alpha\colon\mathbb{R}\to\mathrm{Aut}_{\mathbb{F}}(\mathcal{A})$.

**Corollary 10.3** (Dimension shifts). *Let $(\mathbb{R}^d,\mathcal{A},\alpha,\sigma)$ be a twisted $C^*$-dynamical system. Then $K_n(\mathcal{A}\rtimes_{(\alpha,\sigma)}\mathbb{R}^d)\cong K_{n-d}(\mathcal{A})$.*

*Proof.* Iterating Theorem 8.1 yields

$$\begin{aligned}\mathcal{A}\rtimes_{(\alpha,\sigma)}\mathbb{R}^d&\cong(\mathcal{A}\rtimes_{(\alpha_1,\sigma_1)}\mathbb{R})\rtimes_{(\alpha_2,\sigma_2)}\mathbb{R}^{d-1}\\&\;\;\vdots\\&\cong\big(\ldots\big((\mathcal{A}\rtimes_{(\alpha_1,\sigma_1)}\mathbb{R})\rtimes_{(\alpha_2,\sigma_2)}\mathbb{R}\big)\rtimes_{(\alpha_3,\sigma_3)}\ldots\big)\rtimes_{(\alpha_d,\sigma_d)}\mathbb{R},\end{aligned}$$

for some sequence of twisting pairs $(\alpha_i,\sigma_i)$, $i=1,\ldots,d$. Then, Theorem 10.1 says that we can untwist each of the iterated crossed products, provided we stabilize the algebras. Using the fact that $K$-theory is invariant under stabilization, as well as the Connes–Thom isomorphisms, we have

$$\begin{aligned} K_n(\mathcal{A} \rtimes_{(\alpha,\sigma)} \mathbb{R}^d) &\cong K_n\left(\left(\ldots\left((\mathcal{A} \rtimes_{(\alpha'_1,1)} \mathbb{R}) \rtimes_{(\alpha'_2,1)} \mathbb{R}\right) \rtimes \ldots\right) \rtimes_{(\alpha'_d,1)} \mathbb{R}\right) \\ &\cong K_{n-d}(\mathcal{A}), \end{aligned}$$

so the effect on $K$-theory is to lower the index by $d$. □

**Lemma 10.4.** *Let $\sigma$ be a $U(1)$-valued 2-cocycle for $(G,\phi)$ and suppose there is an element $w \in G$ with $\phi(w) = -1$. Then the restriction of $\sigma$ to the centralizer $\mathcal{Z}_G(w)$ of $w$ in $G$ is equivalent to one which takes only $\{\pm 1\}$ values.*

*Proof.* Let $\varsigma(x,y) := \sigma(x,y)/\sigma(xyx^{-1},x)$, so that $\theta_x\theta_y\theta_x^{-1} = \varsigma(x,y)\theta_{xyx^{-1}}$ is an identity in any PUA-rep $\theta$ of $(G,\phi,\sigma)$, and let $y,z \in \mathcal{Z}_G(w)$. Then

$$\begin{aligned} \theta_w\theta_y\theta_w^{-1}\theta_w\theta_z\theta_w^{-1} &= \varsigma(w,y)\theta_{wyw^{-1}}\varsigma(w,z)\theta_{wzw^{-1}} \\ &= \varsigma(w,y)\varsigma(w,z)^y\theta_y\theta_z \\ &= \varsigma(w,y)\varsigma(w,z)^y\sigma(y,z)\theta_{yz}. \end{aligned} \tag{18}$$

The left-hand-side of (18) can also be written as

$$\theta_w\theta_y\theta_z\theta_w^{-1} = \sigma(y,z)^w\theta_w\theta_{yz}\theta_w^{-1} = \sigma(y,z)^{-1}\varsigma(w,yz)\theta_{yz},$$

so $\sigma(y,z)^{-2} = \varsigma(w,y)\varsigma(w,z)^y\varsigma(w,yz)^{-1}$ is the coboundary of the function $\lambda\colon y \mapsto \varsigma(w,y)$. The function $\lambda^{\frac{1}{2}}$ acts on $\sigma$ [Eq. (4b)] to give an equivalent 2-cocycle $\sigma_w$, i.e. $\sigma_w(y,z) := \{\lambda(y)\lambda(z)^y\lambda(yz)^{-1}\}^{\frac{1}{2}}\sigma(y,z)$, corresponding to the phase modification $\theta_y \mapsto \lambda^{\frac{1}{2}}\theta_y$ in terms of PUA-reps. Then

$$\sigma_w(y,z)^2 = \frac{\lambda(y)\lambda(z)^y}{\lambda(yz)}\sigma(y,z)^2 = 1,$$

and the new 2-cocycle $\sigma_w$ is $\{\pm 1\}$-valued when restricted to $\mathcal{Z}_G(w)$. □

We can now state the main result of this paper. Let $(\tilde{G}, c, \mathcal{B}^{\mathrm{ev}}, \alpha, \sigma)$ be a twisted $C^*$-dynamical system with $\mathcal{B}^{\mathrm{ev}}$ a purely even complex $C^*$-algebra. We assume that $\sigma$ is $U(1)$-valued so that $\alpha$ is a homomorphism. If there is any complex-antilinear automorphism in the range of $\alpha$, we further assume that $\mathcal{B}^{\mathrm{ev}} = \mathcal{B}^{\mathrm{ev}}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and that each $\alpha_x, x \in \tilde{G}$ is either the complex-linear or complex-antilinear extension of a real-linear automorphism (which we also denote by $\alpha_x$) of $\mathcal{B}^{\mathrm{ev}}_{\mathbb{R}}$. Thus, there is a homomorphism $\phi\colon \tilde{G} \to \{\pm 1\}$ keeping track of whether $\alpha_x$ is complex linear or antilinear.

**Theorem 10.5** (General periodicity and dimension shift theorem). *Let $\tilde{\mathcal{A}}$ be the graded twisted crossed product for the twisted $C^*$-dynamical system $(\tilde{G}, c, \mathcal{B}^{\mathrm{ev}}, \alpha, \sigma)$ described in the above paragraph. Assume that $\tilde{G} = \tilde{G}_0 \times A$ where $\tilde{G}_0 = \ker(\phi, c)$, and that the $U(1)$-valued cocycle $\sigma$ is trivial between elements of $\tilde{G}_0$ and $A$. Suppose $\tilde{G}_0$ is an extension of a group $G_0$ by $\mathbb{R}^d$, i.e. $1 \to G_0 \to \tilde{G}_0 \to \mathbb{R}^d \to 1$ is exact. Let $\mathcal{A}^{\mathrm{ev}}_{(\mathbb{R})}$ be the even twisted crossed product associated with the subsystem $(G_0, \mathcal{B}^{\mathrm{ev}}_{(\mathbb{R})}, \alpha, \sigma)$. Then*

$$\mathbf{K}_0(\tilde{\mathcal{A}}) \cong \begin{cases} K_{s-r-d}(\mathcal{A}^{\mathrm{ev}}_{\mathbb{R}}), & \phi \not\equiv 1, \\ K_{n-d}(\mathcal{A}^{\mathrm{ev}}), & \phi \equiv 1, \end{cases} \tag{19}$$

*where $(r,s)$ or $n$ is determined by the Clifford algebra associated with the data $(A,c,\phi,\sigma)$.*

*Proof.* In the $\phi \not\equiv 1$ case, there is a $w \in A$ such that $\phi(w) = -1$. By Lemma 10.4, we may assume that $\sigma$ is $\{\pm 1\}$-valued on $\tilde{G}_0 \subset \mathcal{Z}_{\tilde{G}}(w)$, so the imaginary unit $i \in \mathcal{B}^{\mathrm{ev}}$ is completely decoupled from the sub-dynamical system $(\tilde{G}_0, \mathcal{B}^{\mathrm{ev}}_{\mathbb{R}}, \alpha, \sigma)$. Thus $\tilde{A}$ has the form

$$\tilde{\mathcal{A}} \cong \left(\mathcal{A}^{\mathrm{ev}}_{\mathbb{R}} \rtimes_{(\beta,\nu)} \mathbb{R}^d\right) \hat{\otimes} \mathrm{Cl}_{r,s}$$

for some $\mathrm{Cl}_{r,s}$ determined by $(A,c,\phi,\sigma)$ as in Sect. 6, and some twisting pair $(\beta,\nu)$ arising from the $\mathbb{R}^d$ extension of $G_0$ as in Theorem 8.1. Similarly, a complex Clifford algebra $\mathbb{C}\mathrm{l}_n$ can be factorized in the $\phi \equiv 1$ case, giving $\tilde{\mathcal{A}} \cong \left(\mathcal{A}^{\mathrm{ev}} \rtimes_{(\beta,\nu)} \mathbb{R}^d\right) \hat{\otimes} \mathbb{C}\mathrm{l}_n$. Using the dimension shift in $\mathbf{K}_\star$ effected by tensoring with a Clifford algebra, along with $\mathbf{K}_\star(\mathcal{A}^{\mathrm{ev}}_{(\mathbb{R})}) \cong K_\star(\mathcal{A}^{\mathrm{ev}}_{(\mathbb{R})})$ and Corollary 10.3, we arrive at (19). □

In Theorem 10.5, $\mathcal{B}^{\mathrm{ev}}$ might be an algebra used to model disorder [6], and $(\tilde{G}, c, \mathcal{B}^{\mathrm{ev}}, \alpha, \sigma)$ can be interpreted as the full set of symmetry data for the gapped dynamics in question. Even when the additional $\mathbb{R}^d$ symmetries are realized projectively (e.g. as *magnetic* translations in the presence of a constant magnetic field as in the IQHE), Eq. (19) continues to hold. Therefore, under fairly general assumptions about additional $\mathbb{R}^d$ symmetries and disorder, the difference-group for the symmetry algebra becomes that for $d = 0$, provided we *lower* the $K$-theory index by $d$.

Some other arguments for the dimension shift phenomenon are model-dependent and utilize suspensions [30,45], and only work if the extra $\mathbb{R}^d$ symmetries are assumed to enter in a trivial way. Furthermore, the ordinary suspension operation *raises* rather than lowers the index in $K$-theory. Although this is not a problem in 2-periodic complex $K$-theory, it matters greatly in real $K$-theory where there are two distinct notions of suspension (see Appendix A). A restricted notion of degree shifts can be explained by using the correct type of suspension,[17] and is a special case of our general result.

There is a weaker discretized version of the dimension shift phenomenon. To understand this, we first consider the simplest example of the graded twisted group $C^*$-algebra of $G = \mathbb{Z}^d \times A$, where $\mathbb{Z}^d$ acts trivially on $\mathbb{C}$, and $\sigma$ is only non-trivial between elements of $A$. Then

$$\mathcal{A} = \mathbb{C} \rtimes_{(\alpha,\sigma)} (\mathbb{Z}^d \times A) \cong \begin{cases} (\mathbb{R} \rtimes \mathbb{Z}^d)\hat{\otimes}\mathrm{Cl}_{r,s} \cong C(i\mathbb{T}^d)\hat{\otimes}\mathrm{Cl}_{r,s} & \text{real case,} \\ (\mathbb{C} \rtimes \mathbb{Z}^d)\hat{\otimes}\mathbb{C}\mathrm{l}_n \cong C(\mathbb{T}^d,\mathbb{C})\hat{\otimes}\mathbb{C}\mathrm{l}_n & \text{complex case,} \end{cases}$$

where $C(i\mathbb{T}^d)$ is defined in Appendix A. The $\mathbf{K}_0(\mathcal{A})$ groups reduce to ordinary $K$-theory groups, and when $d = 1$, we can use (21a) and (21b) to obtain

$$\mathbf{K}_0(\mathbb{C} \rtimes_{(\alpha,\sigma)} (\mathbb{Z} \times A)) \cong \begin{cases} K_{s-r}(\mathbb{R}) \oplus K_{s-r-1}(\mathbb{R}) & \text{real case,} \\ K_n(\mathbb{C}) \oplus K_{n-1}(\mathbb{C}) & \text{complex case.} \end{cases}$$

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[17] Topological $KR$-theory does admit two suspensions, which are related to $S$ and $\bar{S}$ in real operator $K$-theory (see Appendix A).

Thus, a trivial crossed product with $\mathbb{Z}$ results in $\mathbf{K}_0(\cdot)$ acquiring an extra $K$-theory group with index *lowered* by 1. As in Theorem 10.5, we can replace $\mathbb{C}$ by $\mathcal{B}^{\text{ev}}$, but it will then be necessary to assume that $\mathcal{B}^{\text{ev}}$ is trivially acted upon by $\mathbb{Z}$ (the Connes–Thom isomorphism does not apply). Assuming this and using (21a) and (21b) repeatedly, we obtain

$$\mathbf{K}_0\left(\mathcal{B}^{\text{ev}} \rtimes_{(\alpha,\sigma)} (\mathbb{Z}^d \times A)\right) \cong \begin{cases} \bigoplus_{k=0}^{d} \binom{d}{k} K_{s-r-k}(\mathcal{B}^{\text{ev}}_{\mathbb{R}}) & \text{real case,} \\ \bigoplus_{k=0}^{d} \binom{d}{k} K_{n-k}(\mathcal{B}^{\text{ev}}) & \text{complex case.} \end{cases} \tag{20}$$

Note that there is always a single extra $K_{s-r-d}(\mathcal{B}^{\text{ev}}_{\mathbb{R}})$ or $K_{n-d}(\mathcal{B}^{\text{ev}})$ factor, just as in the case of extra $\mathbb{R}^d$ symmetries. Equation (20) generalizes the topological $KR$-theory calculations of Kitaev in equations 25–27 of [30]. In particular, if we take $A = \{1, T\}$, $\mathsf{T}^2 = -1$, $\mathcal{B}^{\text{ev}} = \mathbb{C}$ and $d = 3$, we obtain the difference-group for 3D $T$-invariant insulators,

$$\begin{aligned} \mathbf{K}_0(\mathbb{C} \rtimes_{(\alpha,\sigma)} (\mathbb{Z}^d \times \{1, T\})) &\cong \bigoplus_{k=0}^{3} \binom{3}{k} K_{4-k}(\mathbb{R}) \\ &= K_4(\mathbb{R}) \oplus 3K_3(\mathbb{R}) \oplus 3K_2(\mathbb{R}) \oplus K_1(\mathbb{R}) \\ &= \mathbb{Z} \oplus 4\mathbb{Z}_2, \end{aligned}$$

a result obtained by $KR$-theory methods in Theorem 11.14 of [15]. We stress that the expression (20) assumes some specific structure of the symmetry data $(G, c, \mathcal{B}^{\text{ev}}, \alpha, \sigma)$, in particular, the way in which $\mathbb{Z}^d$ sits inside $G$. *These assumptions do not hold when there is spatial inversion symmetry, which acts on $\mathbb{Z}^d$ non-trivially.*

Table 2 summarizes the periodicities in the difference-groups, *in the special case where $G$ is a $CT$-subgroup.* The groups appearing there are the same as those in various Periodic Tables in the literature [30,42,43], *but their physical interpretation is very different.*

*Remark* 10.6. Despite the projectively realized translational symmetry in the physically important case of the IQHE, it fits nicely into our version of the Periodic Table. We also see, in a model-independent way, why time-reversal symmetry needs to be broken (by a magnetic field or otherwise) in order to exhibit a quantized Hall conductivity. For Type $I$ insulators in two dimensions with $\mathsf{T}^2 = +1$ (assuming spin-0), the relevant $K$-theory group is trivial; however, a $\mathbb{Z}_2$-invariant is possible if the spin-$\frac{1}{2}$ nature of electrons is taken into account, so that $\mathsf{T}^2 = -1$.

**Conclusion.** We have treated all physical symmetries on an equal footing in this paper: they include time/charge preserving/reversing symmetries, projectively realized symmetries, $\mathbb{Z}^d$-symmetries underlying band insulators, and $\mathbb{R}^d$ translations in extra spatial dimensions. We have also introduced the $K$-theoretic difference-group, which is well-defined and has a uniform physical interpretation for all symmetry classes. Furthermore, Theorem 10.5 shows that the phenomenon of "dimension shifts" is very robust, and does not depend on the details of how the extra spatial dimensions come into play.

TABLE 2. Periodic Table of difference-groups for gapped topological phases in $d$ dimensions *with only CT-type symmetries*, and their relation to the $K$-theory groups of a point ($d = 0$ column)

| $n$ | Class | $\mathsf{C}^2$ | $\mathsf{T}^2$ | $\mathbf{K}_0(\mathcal{A}) \cong K_{n-d}(\mathbb{R})$ or $K_{n-d}(\mathbb{C})$ | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | $d = 0$ | $d = 1$ | $d = 2$ | $d = 3$ | $d = 4$ |
| 0 | AI | | $+1$ | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| 1 | BDI | $+1$ | $+1$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| 2 | D | $+1$ | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| 3 | DIII | $+1$ | $-1$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| 4 | AII | | $-1$ | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| 5 | CII | $-1$ | $-1$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| 6 | C | $-1$ | | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| 7 | CI | $-1$ | $+1$ | 0 | 0 | 0 | $\mathbb{Z}$ | 0 |
| 0 | A | N/A | | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| 1 | AIII | $\mathsf{S}^2 = +1$ | | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |

The degree shifts also occur for more general symmetries, and are due to various isomorphisms in $K$-theory (Theorem 10.5). Vertically, it is due to the effect on $\mathbf{K}_0(\mathcal{A})$ of tensoring with a Clifford algebra; horizontally, it is due to the Connes–Thom isomorphism and the Packer–Raeburn theorems. The two- and eightfold periodicities are due to Bott periodicity in the Clifford algebras and $K$-theory

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## Appendix A. *K*-Theory Results

A reference for the topological $K$-theory of spaces, which discusses Clifford algebras and the ABS isomorphisms, is [33]. For the ordinary (ungraded) $K$-theory of $C^*$-algebras, we refer to [48] which also treats Hilbert $C^*$-modules and f.g.p. modules, and [7] which discusses graded $C^*$-algebras and $KK$-theory. The $K$-theory of real $C^*$-algebras is covered in detail in [44]; many complex $K$-theory results carry over to the real case, but require rather different proofs. A different approach emphasizing Clifford algebras, from which we have borrowed heavily in defining $\mathbf{K}_n(\cdot)$ in Sect. 7, can be found in Karoubi's [23] (especially Ch. III) and Chapter 2 of [35], as well as references therein. Some of the connections between the ABS constructions in [3], Karoubi's $K$-theory, as well as twisted $K$-theory, are described in [24,25]. The following results are taken from these references.

Let $C_0(\mathbb{R},\mathbb{C})$ (resp. $C_0(\mathbb{R},\mathbb{R})$) be the complex (resp. real) non-unital $C^*$-algebra of continuous functions $\mathbb{R}\to\mathbb{C}$ (resp. $\mathbb{R}\to\mathbb{R}$) vanishing at infinity. The *suspension* $S\mathcal{A}$ of a complex (resp. real) $C^*$-algebra $\mathcal{A}$ is defined to be $S\mathcal{A}:=\mathcal{A}\otimes C_0(\mathbb{R},\mathbb{C})$ (resp. $S\mathcal{A}:=\mathcal{A}\otimes C_0(\mathbb{R},\mathbb{R})$). The higher $K$-theory groups are defined to be $K_n(\mathcal{A})\equiv K_0(S^n\mathcal{A})$. In the complex case, we have $\mathcal{A}\rtimes\mathbb{R}\cong\mathcal{A}\otimes C_0(\hat{\mathbb{R}},\mathbb{C})\cong S\mathcal{A}$, where $\hat{\mathbb{R}}$ is the Fourier transform (dual) of $\mathbb{R}$. In the real case, we have $\mathcal{A}\rtimes\mathbb{R}\cong\mathcal{A}\otimes C_0(i\hat{\mathbb{R}})$ instead [see Eq. (15)], which suggests the definition $\bar{S}\mathcal{A}:=\mathcal{A}\otimes C_0(i\hat{\mathbb{R}})$. It turns out that $\bar{S}$ is the $K$-theoretic "inverse" operation to $S$,

$$K_n(S\bar{S}\mathcal{A})\cong K_n(\bar{S}S\mathcal{A})=K_n(\mathcal{A}\otimes C_0(i\hat{\mathbb{R}})\otimes C_0(\mathbb{R},\mathbb{R}))\cong K_n(\mathcal{A}),$$

and so $K_n(\bar{S}\mathcal{A})\cong K_{n-1}(\mathcal{A})$, which is a special case of Theorem 10.2. Bott periodicity in $K$-theory says that

$$\begin{aligned} K_n(\mathcal{A})&\cong K_{n+8}(\mathcal{A}) \quad \text{real case},\\ K_n(\mathcal{A})&\cong K_{n+2}(\mathcal{A}) \quad \text{complex case},\end{aligned}$$

and leads to cyclic long exact sequences in $K$-theory, with six terms in the complex case and 24 terms in the real case.

Let $(\mathbb{T}^d,\varsigma)$ be the Real space $\mathbb{T}^d$ with involution $\mathbf{z}\mapsto\varsigma(\mathbf{z}):=\bar{\mathbf{z}}$, and let $C(i\mathbb{T}^d)$ be the real $C^*$-algebra of continuous functions $f\colon\mathbb{T}^d\to\mathbb{C}$ such that $f(\varsigma(\mathbf{z}))=\overline{f(\mathbf{z})}$. The group $C^*$-algebras of $\mathbb{Z}^d$ are $\mathbb{R}\rtimes\mathbb{Z}^d\cong C(i\mathbb{T}^d)$ and $\mathbb{C}\rtimes\mathbb{Z}^d\cong C(\mathbb{T}^d,\mathbb{C})$. There are isomorphisms

$$K_n(\mathcal{A}\otimes C(i\mathbb{T}^1))\cong K_n(\mathcal{A})\oplus K_{n-1}(\mathcal{A}) \quad \text{real case}, \tag{21a}$$

$$K_n(\mathcal{A}\otimes C(\mathbb{T}^1,\mathbb{C}))\cong K_n(\mathcal{A})\oplus K_{n-1}(\mathcal{A}) \quad \text{complex case}, \tag{21b}$$

which follow from the Pimsner–Voiculescu exact sequence (Theorem 10.2.1 of [7], Theorem 1.5.5 of [44]) for the $K$-theory of crossed products by $\mathbb{Z}$.

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Guo Chuan Thiang
Mathematical Institute
University of Oxford
Oxford OX2 6GG, UK

and

*Present Address:*
School of Mathematical Sciences
University of Adelaide
Adelaide, SA, 5005
Australia
e-mail: guo.thiang@adelaide.edu.au

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