Ann. Henri Poincaré 17 (2016), 497–535

1424-0637/16/030497-39
published online April 28, 2015
DOI 10.1007/s00023-015-0408-y

Annales Henri Poincaré

Renormalization Group and Stochastic PDEs

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Abstract. We develop a renormalization group (RG) approach to the study of existence and uniqueness of solutions to stochastic partial differential equations driven by space-time white noise. As an example, we prove well-posedness and independence of regularization for the ϕ^4 model in three dimensions recently studied by Hairer and Catellier and Chouk. Our method is "Wilsonian": the RG allows to construct effective equations on successive space-time scales. Renormalization is needed to control the parameters in these equations. In particular, no theory of multiplication of distributions enters our approach.

1. Introduction

Nonlinear parabolic PDEs driven by a space-time decorrelated noise are ubiquitous in physics. Examples are thermal noise in fluid flow, random deposition in surface growth and stochastic dynamics for spin systems and field theories. These equations are of the form

$$\partial_t u = \Delta u + F(u) + \Xi \tag{1}$$

where $u(t,x)$ is defined on $\Lambda \subset \mathbb{R}^d$, $F(u)$ is a function of u and possibly its derivatives which can also be non-local and Ξ is white noise on $\mathbb{R} \times \Lambda$, formally

$$\mathbb{E}\ \Xi(t',x')\Xi(t,x) = \delta(t'-t)\delta(x'-x). \tag{2}$$

Usually in these problems, one is interested in the behavior of solutions in large time and/or long distances in space. In particular, one is interested in stationary states and their scaling properties. These can be studied with regularized versions of the equations where the noise is replaced by a mollified version that is smooth in small scales. Often one expects the large scale behavior is insensitive to such regularization.

Supported by Academy of Finland.

From the mathematical point of view and sometimes also from the physical one it is of interest to inquire the short-time short-distance properties, i.e., the well-posedness of the equations without regularizations. Then, one is encountering the problem that the solutions are expected to have very weak regularity, they are distributions, and it is not clear how to set up the solution theory for the nonlinear equations in distribution spaces.

Recently, this problem was addressed by Martin Hairer [1] who set up a solution theory for a class of such equations, including the KPZ equation in one spatial dimension and the nonlinear heat equation with cubic nonlinearity in three spatial dimensions. The latter case was also addressed by Catellier and Chouk [2] based on the theory of paracontrolled distributions developed in [3]. The class of equations discussed in these works is subcritical in the sense that the nonlinearity vanishes in small scales in the scaling that preserves the linear and noise terms in the equation. In physics terminology, these equations are *superrenormalizable*. This means the following. Let Ξ_ϵ be a mollified noise with short scale cutoff ϵ. One can write a formal series solution to the mollified version of Eq. (1) by starting with the solution $\eta_\epsilon(t) = \eta_\epsilon(t, \cdot)$ of the linear ($F = 0$) equation and iterating:

$$u_\epsilon(t) = \eta_\epsilon(t) + \int_0^t \mathrm{e}^{(t-s)\Delta} F(\eta_\epsilon(s))\mathrm{d}s + \cdots \tag{3}$$

Typically, the random fields (apart from η) occurring in this expansion have no limits as $\epsilon \to 0$: even when tested by smooth functions their variances blow up. These divergencies are familiar from quantum field theory (QFT). Indeed, the correlation functions of u_ϵ have expressions in terms of Feynman diagrams and as in QFT, the divergencies can be canceled in this formal expansion by adding to F extra ϵ-dependent terms, so-called *counter terms*. In QFT, there is a well-defined algorithm for doing this and in the superrenormalizable case rendering the first few terms in the expansion finite cures the divergences in the whole expansion. Hairer's work can be seen as reformulating this perturbative renormalization theory as a rigorous solution theory for the subcritical equations. It should be stressed that [1] goes further by treating also rough non-random forces.

In QFT, there is another approach to renormalization pioneered by K. Wilson in the 1960s [4]. In Wilson's approach adapted to the SPDE, one would not try to solve Eq. (1), call it $\mathcal{E}$, directly but rather go scale by scale starting from the scale ϵ and deriving *effective* equations $\mathcal{E}_n$ for larger scales $2^n\epsilon := \epsilon_n$, $n = 1, 2, \ldots$. Going from scale ϵ_n to ϵ_{n+1} is a problem with $\mathcal{O}(1)$ cutoff when transformed to dimensionless variables. This problem can be studied by a standard Banach fixed point method. The possible singularities of the original problem are present in the large n behavior of the corresponding effective equation. One views $n \to \mathcal{E}_n$ as a dynamical system and attempts to find an initial condition at $n = 0$, i.e., modify $\mathcal{E}$ so that if we fix the scale $\epsilon_n = \epsilon'$ and then let $\epsilon \to 0$ (and as a consequence $n \to \infty$) the effective equation at scale ϵ' has a limit. It turns out that controlling this limit for the effective equations allows one then to control the solution to the original equation (1).

In this paper, we carry out Wilson's renormalization group analysis to the cubic nonlinear heat equation in three dimensions. This equation is a good test case since its renormalization is non-trivial in the sense that a simple Wick ordering of the nonlinearity is not sufficient. Our analysis is robust in the sense that it works for other subcritical cases like the KPZ equation. We prove almost sure local well-posedness for the mild (integral equation) version of (1) thereby recovering the results in [1]. Our renormalization group method is a combination of the one developed in [5] for parabolic PDEs and the one in [6] used for KAM theory. A similar scale decomposition appears also in [7].

The content of the paper is as follows. In Sect. 2, we define the model and state the result. The RG formalism is set up in a heuristic fashion in Sects. 3 and 4. Sections 5 and 6 discuss the leading perturbative solution and set up the fixed point problem for the remainder. Section 7 states the estimates for the perturbative noise contributions and in Sect. 8 the functional spaces for RG are defined and the fixed point problem solved. The main result is proved in Sect. 9. Finally, in Sects. 10 and 11 estimates for the covariances of the various noise contributions are proved.

2. The φ_3^4 Model

Let $\Xi(t,x)$ be space-time white noise on $x \in \mathbb{T}^3$, i.e., $\Xi = \dot{\beta}$ with $\beta(t,x)$ Brownian in time and white noise in space. Given a realization of the noise Ξ, we want to make sense and solve the equation

$$\partial_t \varphi = \Delta\varphi - \varphi^3 - r\varphi + \Xi, \quad \varphi(0) = \varphi_0 \tag{4}$$

on some time interval $[0,\tau]$ and show $\tau > 0$ almost surely.

Due to the nonlinearity, the Eq. (4) is not well defined. We need to define it through regularization. To do this, we first formally write it in its integral equation form

$$\varphi = G(-\varphi^3 - r\varphi + \Xi) + \mathrm{e}^{t\Delta}\varphi_0 \tag{5}$$

where

$$(Gf)(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} f(s)\mathrm{d}s.$$

(for $f = \Xi$ this stands for $(G\xi)(t) = \int_0^t \mathrm{e}^{(t-s)\Delta}\mathrm{d}\beta(s)$). Next, introduce a regularization parameter $\epsilon > 0$ and define

$$(G_\epsilon f)(t) = \int_0^t (1 - \chi((t-s)/\epsilon^2))\mathrm{e}^{(t-s)\Delta} f(s)\mathrm{d}s. \tag{6}$$

where $\chi \geq 0$ is a smooth bump, $\chi(t) = 1$ for $t \in [0,1]$ and $\chi(t) = 0$ for $t \in [2,\infty)$. The regularization of (5) with $\varphi_0 = 0$ is then defined to be

$$\varphi = G_\epsilon(-\varphi^3 - r_\epsilon\varphi + \Xi). \tag{7}$$

We look for r_ϵ such that (7) has a unique solution $\varphi^{(\epsilon)}$ which converges as $\epsilon \to 0$ to a non-trivial limit. Note that, since only $t - s \geq \epsilon^2$ contribute in (6) $G_\epsilon\Xi$ is a.s. smooth.

Our main result is

Theorem 1. *There exits r_ϵ s.t. the following holds. For almost all realizations of the white noise Ξ there exists $t(\Xi) > 0$ such that the Eq. (7) has for all $\epsilon > 0$ a unique smooth solution $\varphi^{(\epsilon)}(t,x)$, $t \in [0,t(\Xi)]$ and there exists $\varphi \in \mathcal{D}'([0,t(\Xi)] \times \mathbb{T}^3)$ such that $\varphi^{(\epsilon)} \to \varphi$ in $\mathcal{D}'([0,t(\Xi)] \times \mathbb{T}^3)$. The limit φ is independent of the regularization χ.*

Remark 2. We will find that the renormalization parameter is given by

$$r_\epsilon = r + m_1\epsilon^{-1} + m_2 \log\epsilon + m_3 \tag{8}$$

where the constants m_1 and m_3 depend on χ whereas the m_2 is universal, i.e., independent on χ. They of course agree with the mass renormalization needed to make sense of the formal stationary measure of (4)

$$\mu(\mathrm{d}\phi) = \mathrm{e}^{-\frac{1}{4}\int_{\mathbb{T}^3}\phi(x)^4}\nu(\mathrm{d}\phi) \tag{9}$$

where ν is Gaussian measure with covariance $(-\Delta + r)^{-1}$ [8–12].

Remark 3. This result can be extended to a large class of initial conditions, deterministic or random. As an example, we consider the random case where $\phi_0 = \eta_0$ where η_0 is the gaussian random field on $\mathbb{T}^3$ with covariance $-\frac{1}{2}\Delta^{-1}$, independent of Ξ (this is the stationary state of the linear equation). Then, Theorem 1 holds a.s. in the initial condition and Ξ, see Remark 7.

Remark 4. One can as well introduce the short scale cutoff only to the spatial dependence of the noise by replacing Ξ with $\Xi_\epsilon := \rho_\epsilon \star \Xi$ where $\rho_\epsilon(x) = \epsilon^{-3}\rho(x/\epsilon)$ with ρ smooth non-negative compactly supported bump integrating to one. Then, the regularized equation is

$$\varphi = G(-\varphi^3 - r_\epsilon\varphi + \Xi_\epsilon). \tag{10}$$

See Remark 12 for more discussion. This cutoff has the advantage that (10) represents a regular (Stochastic) PDE.

3. Effective Equation

Consider the cutoff problem

$$\varphi = G_\epsilon(V(\varphi) + \Xi) \tag{11}$$

for $\varphi(t,x)$ on $(t,x) \in [0,\tau] \times \mathbb{T}^3$ with

$$V(\varphi)(t,x) = -\varphi^3(t,x) - r_\epsilon\varphi(t,x).$$

Let us attempt increasing the cutoff ϵ to $\epsilon' > \epsilon$ by solving the Eq. (11) for scales between ϵ and ϵ'. To do this split

$$G_\epsilon = G_{\epsilon'} + \Gamma_{\epsilon,\epsilon'}$$

with

$$(\Gamma_{\epsilon,\epsilon'}f)(t) = \int_0^t (\chi((t-s)/{\epsilon'}^2) - \chi((t-s)/\epsilon^2))\mathrm{e}^{(t-s)\Delta}f(s)\mathrm{d}s.$$

Thus, $\Gamma_{\epsilon,\epsilon'}$ involves temporal scales between ϵ^2 and ${\epsilon'}^2$ (and due to the heat kernel spatial scales between ϵ and ϵ'). Next, write

$$\varphi = \varphi' + Z \tag{12}$$

and determine $Z = Z(\varphi')$ as a function of φ' by solving the small scale equation

$$Z = \Gamma_{\epsilon,\epsilon'}(V(\varphi' + Z) + \Xi). \tag{13}$$

Equation (11) will then hold provided φ' is a solution to the "renormalized" equation

$$\varphi' = G_{\epsilon'}(V'(\varphi') + \Xi) \tag{14}$$

where

$$V'(\varphi') = V(\varphi' + Z(\varphi')).$$

Equation (14) is of the same form as (11) except that the cutoff has increased and V is replaced by V'. Combining (13) and (14) we see that the new V' can be obtained by solving a fixed point equation

$$V' = V(\cdot + \Gamma_{\epsilon,\epsilon'}(V' + \Xi)). \tag{15}$$

Finally, the solution of (11) is gotten from (12) as

$$\varphi = \varphi' + \Gamma_{\epsilon,\epsilon'}(V'(\varphi') + \Xi) \tag{16}$$

Our aim is to study the flow of the *effective equation V' at scale ϵ'* as ϵ' increases from ϵ to $\tau^{\frac{1}{2}}$ where $[0,\tau]$ is the time interval where we try to solve the original equation. It will be convenient to do this step by step. We fix a number $\lambda < 1$ (taken to be small in the proof to kill numerical constants) and take the cutoff scale

$$\epsilon = \lambda^N. \tag{17}$$

The corresponding V is denoted as

$$V^{(N)}(\varphi) = -\varphi^3 - r_{\lambda^N}\varphi. \tag{18}$$

Let $V_n^{(N)}$ be the solution of (15) with $\epsilon' = \lambda^n$. We will construct these functions iteratively in n, i.e., derive the effective equation on scale λ^{n-1} from that of λ^n:

$$V_{n-1}^{(N)} = V_n^{(N)}(\cdot + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)} + \Xi)). \tag{19}$$

The solution of (11) is then also constructed iteratively: let

$$F_N^{(N)}(\varphi) := \varphi$$

and define

$$F_{n-1}^{(N)} = F_n^{(N)}(\cdot + \Gamma_{\lambda^n,\lambda^{n-1}}(V_{n-1}^{(N)} + \Xi)). \tag{20}$$

Then, the solution of (11) is

$$\varphi = F_n^{(N)}(\varphi_n).$$

where φ_n solves

$$\varphi_n = G_{\lambda^n}(V_n^{(N)}(\varphi_n) + \Xi). \tag{21}$$

Finally, we want to control the limit $N \to \infty$ where the regularization is removed, i.e., construct the limits $V_n = \lim_{N\to\infty} V_n^{(N)}$ and $F_n = \lim_{N\to\infty} F_n^{(N)}$ which are then shown to describe the solution of (4) on time interval $[0, \lambda^{2n}]$. How long this time interval will be, i.e., how small n we can reach in the iteration depends on the realization of the noise. In informal terms, let A_m be the event that these limits exist for $n \geq m$. We will show

$$\mathbb{P}(A_m) \geq 1 - \mathcal{O}(\lambda^{Rm}). \tag{22}$$

with large R, i.e., the set of noise s.t. the Eq. (4) is well posed on time interval $[0, \tau]$ has probability $1 - \mathcal{O}(\tau^R)$.

4. Renormalization Group

The Eq. (19) deals with scales between λ^n and λ^{n-1}. Instead of letting the scale of the equations vary it will be more convenient to rescale everything to fixed scale (of order unity) after which we need to iterate a $\mathcal{O}(1)$-scale problem. This is the "Wilsonian" approach to Renormalization Group.

Let us define the space-time scaling s_μ by

$$(s_\mu f)(t, x) = \mu^{\frac{1}{2}} f(\mu^2 t, \mu x).$$

The Green function of the heat equation and the space-time white noise transform in a simple way under this scaling:

$$s_\mu G s_\mu^{-1} = \mu^2 G, \quad s_\mu G_\epsilon s_\mu^{-1} = \mu^2 G_{\epsilon/\mu}, \quad s_\mu \Xi \overset{d}{=} \mu^{-2} \Xi_\mu$$

as one can easily verify by a simple changes of variables. Here, Ξ_μ is space-time white noise on $\mathbb{R} \times \mu^{-1}\mathbb{T}^3$. Set

$$\phi_n = s_{\lambda^n} \varphi_n \tag{23}$$

where φ_n solves (21). Note that, $\phi_n(t, x)$ is defined on $x \in \mathbb{T}_n$ with

$$\mathbb{T}_n := \lambda^{-n}\mathbb{T}^3 \tag{24}$$

Since φ_n was interpreted as a field involving spatial scales on $[\lambda^n, 1]$ ϕ_n may be thought as a field involving scales on $[1, \lambda^{-n}]$.

In these new variables, Eq. (21) becomes

$$\phi_n = G_1(v_n^{(N)}(\phi_n) + \xi_n) \tag{25}$$

provided we define

$$v_n^{(N)} := \lambda^{2n} s_{\lambda^n} \circ V_n^{(N)} \circ s_{\lambda^n}^{-1}$$

and

$$\xi_n := \lambda^{2n} s_{\lambda^n} \Xi.$$

ξ_n is distributed as a space-time white noise on $\mathbb{R} \times \mathbb{T}_n$. Defining

$$f_n^{(N)} := s_{\lambda^n} \circ F_n^{(N)} \circ s_{\lambda^n}^{-1}$$

the solution of (11) is given by

$$\varphi = s_{\lambda^n}^{-1} f_n^{(N)}(\phi_n). \tag{26}$$

The iterative equations (19) and (20) have their unit scale counterparts: recalling (17), denoting $s := s_\lambda$ and using $s_{\lambda^{n+1}} = s \circ s_{\lambda^n}$, Eq. (19) becomes

$$v_{n-1}^{(N)}(\phi) = \lambda^{-2} s^{-1} v_n^{(N)}(s(\phi + \Gamma(v_{n-1}^{(N)}(\phi) + \xi_{n-1}))) \tag{27}$$

where

$$(\Gamma f)(t) = \int_0^t (\chi(t-s) - \chi((t-s)/\lambda^2)) \mathrm{e}^{(t-s)\Delta} f(s) \mathrm{d}s. \tag{28}$$

Note the integrand is supported on the interval $[\lambda^2, 2]$.

Equation (20) in turn becomes

$$f_{n-1}^{(N)}(\phi) = s^{-1} f_n^{(N)}(s(\phi + \Gamma(v_{n-1}^{(N)}(\phi) + \xi_{n-1}))). \tag{29}$$

Our task then is to solve the Eq. (27) for $v_{n-1}^{(N)}$ to obtain the RG map

$$v_{n-1}^{(N)} = \mathcal{R}_n v_n^{(N)} \tag{30}$$

and then iterate this and (29) starting with

$$\begin{aligned} v_N^{(N)}(\phi) &= -\lambda^N \phi^3 - \lambda^{2N} r_{\lambda^N} \phi \\ &= -\lambda^N \phi^3 - (\lambda^N m_1 + \lambda^{2N}(m_2 \log \lambda^N + m_3))\phi \end{aligned} \tag{31}$$

$$f_N^{(N)}(\phi) = \phi \tag{32}$$

If the noise is in the set A_m (to be defined) we then show the functions $v_n^{(N)}$ and $f_n^{(N)}$ have limits as $N \to \infty$ and $n \geq m$ and allow to construct the solution to our original equation on time interval $[0, \lambda^{2m}]$.

Equations (27), (29) and (25) involve the operators Γ and G_1, respectively. These operators are infinitely smoothing and their kernels have fast decay in space-time. In particular, the noise $\zeta = \Gamma \xi_n$ entering Eqs. (27) and (29) has a smooth covariance which is short range in time

$$\mathbb{E}\zeta(t,x)\zeta(s,y) = 0 \quad \text{if} \quad |t-s| > 2\lambda^{-2}, \tag{33}$$

and it has gaussian decay in space. Hence, the fixed point problem (27) turns out to be quite easy.

As usual in RG studies one needs to keep track of the leading "relevant" terms of $v_n^{(N)}$ which are revealed by a first- and second-order perturbative study of (27) to which we turn now.

5. Linearized Renormalization Group

We will now study the fixed point Eq. (27) to first order in v. Define the map

$$(\mathcal{L}_n v)(\phi) := \lambda^{-2} s^{-1} v(s(\phi + \Gamma \xi_{n-1})) \tag{34}$$

Then, (27) can be written as

$$v_{n-1}^{(N)}(\phi) = (\mathcal{L}_n v_n^{(N)})(\phi + \Gamma v_{n-1}^{(N)}(\phi)) \tag{35}$$

and we see that $\mathcal{L}_n = D\mathcal{R}_n(0)$, the derivative of the RG map. The linear flow $u_{n-1} = \mathcal{L}_n u_n$ from scale N to scale n is easy to solve by just replacing λ by λ^{N-n}:

$$u_n = (\mathcal{L}_{n+1}\mathcal{L}_{n+2}\dots\mathcal{L}_N u_N)(\phi) = \lambda^{-2(N-n)} s^{n-N} u_N(s^{N-n}(\phi + \eta_n^{(N)}))$$

where

$$\eta_n^{(N)} = \Gamma_n^N \xi_n. \tag{36}$$

Here, ξ_n is space-time white noise on $\mathbb{T}_n = \lambda^{-n}\mathbb{T}^3$ and Γ_n^N is given by (28) with λ replaced by λ^{N-n}, i.e., its integral kernel is

$$\Gamma_n^N(t,s,x,y) = \chi_{N-n}(t-s) H_n(t-s, x-y) \tag{37}$$

where we denoted

$$H_n(t, x-y) = \mathrm{e}^{t\Delta}(x,y)$$

the heat kernel on $\mathbb{T}_n$ and

$$\chi_{N-n}(s) := \chi(s) - \chi(\lambda^{-2(N-n)}s) \tag{38}$$

is a smooth indicator of the interval $[\lambda^{2(N-n)}, 2]$.

The linearized flow is especially simple for a local u as the one we start with (31). For $u_N = \phi^k$ we get

$$u_n = \lambda^{n(k-5)/2}(\phi + \eta_n^{(N)})^k$$

and so we have "eigenfunctions"

$$\mathcal{L}_n(\phi + \eta_n^{(N)})^k = \lambda^{(k-5)/2}(\phi + \eta_{n-1}^{(N)})^k.$$

For $k < 5$, these are "relevant", for $k > 5$ they are "irrelevant" and for $k = 5$ "marginal".

The covariance of $\eta_n^{(N)}$ is readily obtained from (37) (let $t' \geq t$):

$$\begin{aligned} &\mathbb{E}\eta_n^{(N)}(t',x')\eta_n^{(N)}(t,x) \\ &\quad = \int_0^t H_n(t'-t+2s, x'-x)\chi_{N-n}(t'-t+s)\chi_{N-n}(s)\mathrm{d}s \\ &\quad := C_n^{(N)}(t',t,x',x) \end{aligned} \tag{39}$$

In particular, we have

$$\mathbb{E}\eta_n^{(N)}(t,x)^2 = \int_0^t H_n(2s,0)\chi_{N-n}(s)^2 \mathrm{d}s. \tag{40}$$

This integral diverges as $N - n \to \infty$ and is the source of the first renormalization constant in (31).

We need to study the N and χ dependence of the solution to (7). Since these dependencies are very similar we deal with them together. Thus, let $\Gamma_n^{\prime(N)}$ the operator (37) where the lower cutoff in (38) is modified to another bump χ':

$$\chi'_{N-n}(s) = \chi(s) - \chi'(\lambda^{-2(N-n)}s). \tag{41}$$

Varying χ' allows to study cutoff dependence of our scheme. Taking $\chi'(s) = \chi(\lambda^{-2}s)$ in turn implies $\Gamma_n'^{(N)} = \Gamma_n^{(N+1)}$ and this allows us to study the N dependence and the convergence as $N \to \infty$. The following lemma controls these dependencies for (40):

Lemma 5. *Let*

$$\rho := \int_0^\infty (8\pi s)^{-3/2}(1-\chi(s)^2)\mathrm{d}s. \tag{42}$$

Then,

$$\mathbb{E}\eta_n^{(N)}(t,x)^2 = \lambda^{-(N-n)}\rho + \delta_n^{(N)}(t)$$

with

$$|\delta_n^{(N)}(t)| \leq C(1+t^{-\frac{1}{2}}) \tag{43}$$

and

$$|\delta_n^{(N)}(t) - \delta_n'^{(N)}(t)| \leq C(t^{-\frac{1}{2}}1_{[0,2\lambda^{2(N-n)}]}(t) + \mathrm{e}^{-c\lambda^{-2N}})\|\chi - \chi'\|_\infty \tag{44}$$

The Lemma is proved in Sect. 10. We will fix in (31) the first renormalization constant

$$a = -3\rho \tag{45}$$

Defining

$$\rho_k = \lambda^{-k}\rho, \tag{46}$$

the first-order solution to our problem is

$$u_n^{(N)} := -\lambda^n((\phi + \eta_n^{(N)})^3 + 3\rho_{N-n}(\phi + \eta_n^{(N)})). \tag{47}$$

Remark 6. Note that, since we have the factor λ^n in $u_n^{(N)}$, the counting of what terms are relevant, marginal or irrelevant depends on the order in λ^n. Thus, $\lambda^n(\phi + \eta_n^{(N)})^k$ is relevant for $k < 3$, marginal for $k = 3$ and irrelevant for $k > 3$. Similarly, $\lambda^{2n}(\phi + \eta_n^{(N)})^k$ is marginal for $k = 1$ which is the source of the renormalization constant b in (8). The terms of order λ^{3n} are all irrelevant. For a precise statement, see Proposition 10.

Remark 7. Consider the random initial condition discussed in Remark 3. We realize it in terms of the white noise on $(-\infty, 0] \times \mathbb{T}_n$. In a regularized form, we replace $\mathrm{e}^{t\Delta}\varphi_0$ in(5) by

$$\int_{-\infty}^0 (1 - \chi((t-s)/\epsilon^2))\mathrm{e}^{(t-s)\Delta}\Xi(s)\mathrm{d}s. \tag{48}$$

This initial condition can be absorbed to $\eta_n^{(N)}$. Indeed, the covariance (39) is just replaced by the stationary one

$$\begin{aligned}\mathbb{E}\eta_n^{(N)}(t',x')\eta_n^{(N)}(t,x) \\ = \int_{-\infty}^t H_n(t'-t+2s, x'-x)\chi_{N-n}(t'-t+s)\chi_{N-n}(s)\mathrm{d}s\end{aligned} \tag{49}$$

and in particular $\delta_n^{(N)} = 0$ which makes the analysis in Sect. 11 actually less messy.

6. Second-Order Calculation

We will solve Eq. (27) by a fixed point argument in a suitable space of $v_n^{(N)}$. Before going to that we need to spell out explicitly the leading relevant (in the RG sense) terms. By Remark 6, this requires looking at the second-order terms in $v_n^{(N)}$. To avoid too heavy notation, we will drop the superscript $^{(N)}$ in $v_n^{(N)}$, $\eta_n^{(N)}$ and other expressions unless needed for clarity.

Let us first separate in (27) the linear part:

$$v_{n-1} = \mathcal{L}_n v_n + \mathcal{G}_n(v_n, v_{n-1}) \tag{50}$$

where we defined

$$\mathcal{G}_n(v, \bar{v})(\phi) = (\mathcal{L}_n v)(\phi + \Gamma \bar{v}(\phi)) - (\mathcal{L}_n v)(\phi) \tag{51}$$

Recalling the first-order expression u_n in (47), we write

$$v_n = u_n + w_n$$

so that w_n satisfies

$$w_{n-1} = \mathcal{L}_n w_n + \mathcal{G}_n(u_n + w_n, u_{n-1} + w_{n-1}), \tag{52}$$

with the initial condition

$$w_N(\phi) = -\lambda^{2N}(m_2 \log \lambda^N + m_3)\phi. \tag{53}$$

The reader should think about u_n as $\mathcal{O}(\lambda^n)$ and w_n as $\mathcal{O}(\lambda^{2n})$.

Next, we separate from the $\mathcal{G}_n$-term in (52) the $\mathcal{O}(\lambda^{2n})$ contribution. Since $\mathcal{L}_n u_n = u_{n-1}$, we have

$$\mathcal{G}_n(u_n, u_{n-1})(\phi) = u_{n-1}(\phi + \Gamma u_{n-1}(\phi)) - u_{n-1}(\phi) \tag{54}$$

which to $\mathcal{O}(\lambda^n)$ equals $Du_{n-1}\Gamma u_{n-1}$ where

$$Du_{n-1} = -3\lambda^{n-1}((\phi + \eta_{n-1})^2 - \rho_{N-n+1}).$$

Hence,

$$w_{n-1} = \mathcal{L}_n w_n + Du_{n-1}\Gamma u_{n-1} + \mathcal{F}_n(w_{n-1}) \tag{55}$$

where

$$\mathcal{F}_n(w_{n-1}) = \mathcal{G}_n(u_n + w_n, u_{n-1} + w_{n-1}) - Du_{n-1}\Gamma u_{n-1} \tag{56}$$

$\mathcal{F}_n$ is $\mathcal{O}(\lambda^{3n})$ and will turn out to be irrelevant under the RG (i.e., it will contract in a suitable norm under the linear RG map $\mathcal{L}_n$).

It is useful to solve (55) without the $\mathcal{F}_n$ term: let U_n satisfy

$$U_{n-1} = \mathcal{L}_n U_n + Du_{n-1}\Gamma u_{n-1}, \quad U_N = -\lambda^{2N}(m_2 \log \lambda^N + m_3)\phi \tag{57}$$

The solution is

$$U_n = Du_n \Gamma_n^N u_n - \lambda^{2n}(m_2 \log \lambda^N + m_3)(\phi + \eta_n) \tag{58}$$

where Γ_n^N is defined in (38). This can be seen by a direct calculation or, just noting that $u_{n-1}+U_{n-1}$ equals $\mathcal{R}_n \ldots \mathcal{R}_N(u_N+U_N)$ computed up to second order and since $\mathcal{R}_n$ is a semigroup we can do this is one step with λ replaced by λ^{N-n}. Now, write

$$w_n = U_n + \nu_n \tag{59}$$

so that ν_{n-1} satisfies the equation

$$\nu_{n-1} = \mathcal{L}_n \nu_n + \mathcal{F}_n(U_{n-1}+\nu_{n-1}), \quad \nu_N = 0 \tag{60}$$

Equation (60) is a fixed point equation that we will solve by contraction in a suitable space.

7. Noise Estimates

The time of existence for the solution depends on the size of the noise. In this Section, we state probabilistic estimates for this size.

The noise enters the fixed point Eq. (60) in the form $\Gamma\xi_{n-1}$ that enters the definition of $\mathcal{L}_n$ in (34) and in the polynomials u_n (47) and U_n (58) of the random field η_n (36). According to Remark 6 in the second-order term U_n only the constant in ϕ term should be relevant under the linear RG and the linear in ϕ term should be marginal (neutral), the rest being irrelevant (contracting). We will see this indeed is the case and accordingly write

$$U_n(\phi) = U_n(0) + DU_n(0)\phi + V_n(\phi) \tag{61}$$

where explicitly

$$\begin{aligned} U_n(0) &= 3\lambda^{2n}(\eta_n^2 - \rho_{N-n})\Gamma_n^N(\eta_n^3 - 3\rho_{N-n}\eta_n) - \lambda^{2n}(m_2\log\lambda^N + m_3)\eta_n \\ &:= \lambda^{2n}\omega_n \end{aligned} \tag{62}$$

and

$$(DU_n(0)\phi)(t,x) = \lambda^{2n}\mathfrak{z}_n(t,x)\phi(t,x) + \lambda^{2n}\int z_n(t,x,s,y)\phi(s,y)\mathrm{d}s\mathrm{d}y \tag{63}$$

where

$$\mathfrak{z}_n = 6\eta_n\Gamma_n^N(\eta_n^3 - 3\rho_{N-n}\eta_n) \tag{64}$$

and

$$z_n = 9(\eta_n^2-\rho_{N-n})\Gamma_n^N(\eta_n^2-\rho_{N-n}) - m_2\log\lambda^N - m_3 \tag{65}$$

(here, $\eta_n^2 - \rho_{N-n}$ is viewed as a multiplication operator).

The random fields whose size we need to constrain probabilistically are then

$$\eta_n, \quad \eta_n^2-\rho_{N-n}, \quad \eta_n^3 - 3\rho_{N-n}\eta_n, \quad \omega_n, \quad \mathfrak{z}_n, \quad z_n \tag{66}$$

They belong to the Wiener chaos of white noise of bounded order and their size and regularity are controlled by studying their covariances. For finite cutoff parameter N, these noise fields are a.s. smooth, but in the limit $N\to\infty$ they become distribution valued. These fields enter in the RG iteration (35) in the combination Γv_{n-1}, i.e., they are always acted upon by the operator Γ which is infinitely smoothing. Therefore, we estimate their size in suitable (negative index) Sobolev-type norms which we now define. In addition to the fields (66), we also need to constrain the Gaussian field $\Gamma\xi_n$.

Let K_1 be the operator $(-\partial_t^2+1)^{-1}$ on $L^2(\mathbb{R})$, i.e., it has the integral kernel

$$K_1(t,s)=\frac{1}{2}\mathrm{e}^{-|t-t'|} \tag{67}$$

Let $K_2=(-\Delta+1)^{-2}$ on $L^2(\mathbb{T}_n)$ which has a continuous kernel $K_2(x,y)=K_2(x-y)$ satisfying

$$K_2(x)\le C\mathrm{e}^{-|x|}. \tag{68}$$

Set

$$K:=K_1K_2. \tag{69}$$

Define $\mathcal{V}_n$ to be the completion of $C_0^\infty(\mathbb{R}_+\times\mathbb{T}_n)$ with the norm

$$\|v\|_{\mathcal{V}_n}=\sup_i\|Kv\|_{L^2(c_i)} \tag{70}$$

where c_i is the unit cube centered at $i\in\mathbb{Z}\times(\mathbb{Z}^3\cap\mathbb{T}_n)$. To deal with the bi-local field z_n in (65) we define for $z(t,x,s,y)$ in $C_0^\infty(\mathbb{R}_+\times\mathbb{T}_n\times\mathbb{R}_+\times\mathbb{T}_n)$

$$\|z\|_{\mathcal{V}_n}=\sup_i\sum_j\|K\otimes Kz\|_{L^2(c_i\times c_j)} \tag{71}$$

Now, we can specify the admissible set of noise. Let $\gamma>0$ and define events $\mathcal{A}_m$, $m>0$ in the probability space of the space-time white noise Ξ as follows. Let $\zeta_n^{(N)}$ denote any one of the fields (66). We want to constrain the size of $\zeta_n^{(N)}$ on the time interval $[0,\tau_{n-m}]$ where we will denote $\tau_{n-m}=\lambda^{-2(n-m)}$. To do this, choose a smooth bump h on $\mathbb{R}$ with $h(t)=1$ for $t\le-\lambda^2$ and $h(t)=0$ for $t\ge-\frac{1}{2}\lambda^2$ and set $h_k(t)=h(t-\tau_k)$ so that $h_k(t)=1$ for $t\le\tau_k-\lambda^2$ and $h_k(t)=0$ for $t\ge\tau_k-\frac{1}{2}\lambda^2$ (the reason for these strange choices will become clear in Sect. 8). The first condition on $\mathcal{A}_m$ is that for all $N\ge n\ge m$ the following hold:

$$\|h_{n-m}\zeta_n^{(N)}\|_{\mathcal{V}_n}\le\lambda^{-\gamma n} \tag{72}$$

We need also to control the N and χ dependence of the noise fields $\zeta_n^{(N)}$. Recall that we can study both by varying the lower cutoff in the operator $\Gamma_n^{(N)}$ in (37). We denote by $\zeta_n'^{(N)}$ any of the resulting noise fields. Our second condition on $\mathcal{A}_m$ is that for all $N\ge n\ge m$ and all cutoff functions χ,χ' with bounded C^1 norm

$$\|h_{n-m}(\zeta_n'^{(N)}-\zeta_n^{(N)})\|_{\mathcal{V}_n}\ \le\lambda^{\gamma(N-n)}\lambda^{-\gamma n}. \tag{73}$$

The final condition concerns the fields $\Gamma\xi_n$ entering the RG iteration (27). Note that, these fields are N independent and smooth and we impose on them a smoothness condition given in (74). We have:

Proposition 8. *There exist renormalization constants m_2 and m_3 such that for some $\gamma>0$ almost surely $\mathcal{A}_m$ holds for some $m<\infty$.*

On $\mathcal{A}_m$ we will control the RG iteration for scales $n \geq m$. This will enable us to solve the Eq. (4) on the time interval $[0, \lambda^{2m}]$.

8. Fixed Point Problem

We will now fix the noise $\Xi \in \mathcal{A}_m$ for some $m > 0$ and set up a suitable space of functions $\nu_n(\phi)$ for the fixed point problems (60) for $n \geq m$.

Since the noise contributions take values in $\mathcal{V}_n$ we let ν_n take values there as well. The noise enters in (27) in the argument of v_n in the combination $\Gamma(v_{n-1} + \xi_{n-1})$. Since Γ is infinitely smoothing this means that we may take the domain of $\nu_n(\phi)$ to consist of suitably smooth functions ϕ. Since the noise contributions become distributions in the limit $N \to \infty$ and enter multiplicatively with ϕ, e.g., in (47) we need to match the smoothness condition for ϕ with that of the noise. Finally, since (25) implies $\phi_n \equiv 0$ on $[0, 1]$ we let the ϕ be defined on $[1, \tau_{n-m}]$.

With these motivations, we take the domain Φ_n of v_n to consist of $\phi : [1, \tau_{n-m}] \times \mathbb{T}_n \to \mathbb{C}$ which are C^2 in t and C^4 in x with $\partial_t^i \phi(1, x) = 0$ for $0 \leq i \leq 2$ and all $x \in \mathbb{T}_n$. We equip Φ_n with the sup norm

$$\|\phi\|_{\Phi_n} := \sum_{i \leq 2, |\alpha| \leq 4} \|\partial_t^i \partial_x^\alpha \phi\|_\infty.$$

We will now set up the RG map (35) in a suitable space of v_n, v_{n-1} defined on Φ_n and Φ_{n-1}, respectively.

First, note that for $\phi \in \Phi_{n-1}$, $s\phi(t, x) = \lambda^{\frac{1}{2}} \phi(\lambda^2 t, \lambda x)$ is defined on $[\lambda^{-2}, \tau_{n-m}] \times \mathbb{T}_n$. We extend it to $[1, \tau_{n-m}] \times \mathbb{T}_n$ by setting $s\phi(t, x) = 0$ for $1 \leq t \leq \lambda^{-2}$. Next, $\Gamma\xi_{n-1}$ vanishes (a.s.) on $[0, \lambda^2]$ and thus $s\Gamma\xi_{n-1}$ vanishes on $[0, 1]$. We can now state the final condition for the set $\mathcal{A}_m$: for all $n > m$ we demand

$$\|s\Gamma\xi_{n-1}\|_{\Phi_n} \leq \lambda^{-\gamma n}. \tag{74}$$

Let $B_n \subset \Phi_n$ be the open ball centered at origin of radius $r_n = \lambda^{-2\gamma n}$ and $\mathcal{W}_n(B_n)$ be the space of analytic functions from B_n to $\mathcal{V}_n$ equipped with the supremum norm which we denote by $\| \cdot \|_{B_n}$ (see [6] for a summary of basic facts on analytic functions on Banach spaces). We will solve the fixed point problem (35) in this space. We collect some elementary properties of these norms in the following lemma, proven in Sect. 10:

Lemma 9. (a) *$s\Gamma : \mathcal{V}_{n-1} \to \Phi_n$ and $h_{n-1-m}\Gamma : \mathcal{V}_{n-1} \to \mathcal{V}_{n-1}$ are bounded operators with norms bounded by $C(\lambda)$. Moreover, $s\Gamma h_{n-1-m} v = s\Gamma v$ as elements of Φ_n.*

(b) *$s : \Phi_{n-1} \to \Phi_n$ and $s^{-1} : \mathcal{V}_n \to \mathcal{V}_{n-1}$ are bounded with*

$$\|s\| \leq \lambda^{\frac{1}{2}}, \quad \|s^{-1}\| \leq C\lambda^{-\frac{1}{2}}.$$

(c) *Let $\phi \in C^{2,4}(\mathbb{R} \times \mathbb{T}_n)$ and $v \in \mathcal{V}_n$. Then, $\phi v \in \mathcal{V}_n$ and $\|\phi v\|_{\mathcal{V}_n} \leq C\|\phi\|_{C^{2,4}} \|v\|_{\mathcal{V}_n}$.*

The linear RG (34) is controlled by

Proposition 10. *Given $\lambda < 1$, $\gamma > 0$ there exists $n(\gamma,\lambda)$ s.t. for $n \geq n(\gamma,\lambda)$ $\mathcal{L}_n$ maps $\mathcal{W}_n(B_n)$ into $\mathcal{W}_{n-1}(\lambda^{-\frac{1}{2}}B_{n-1})$ with norm $\|\mathcal{L}_n\| \leq C\lambda^{-\frac{5}{2}}$.*

Proof. Let $v \in \mathcal{W}_n(B_n)$ and $\phi \in \lambda^{-\frac{1}{2}}B_{n-1}$. By (74) and Lemma 9(a,b)

$$\|s(\phi + \Gamma\xi_{n-1})\|_{\Phi_n} \leq \lambda^{-2\gamma(n-1)} + \lambda^{-\gamma n} \leq \lambda^{-2\gamma n}$$

for $n \geq n(\gamma,\lambda)$. Hence, $v(s(\phi+\Gamma\xi_{n-1}))$ is defined and analytic in $\phi \in \lambda^{-\frac{1}{2}}B_{n-1}$, i.e., $\mathcal{L}_n$ maps $\mathcal{W}_n(B_n)$ into $\mathcal{W}_{n-1}(\lambda^{-\frac{1}{2}}B_{n-1})$ and by Lemma 9(b)

$$\|\mathcal{L}_n v\|_{\lambda^{-\frac{1}{2}}B_{n-1}} \leq C\lambda^{-\frac{5}{2}}\|v\|_{B_n}$$

□

As a corollary of Lemma 9(c) and (72) we obtain for $n \geq m$ and $N \geq n$ (recall (47) and (62)):

$$\|h_{n-m}u_n^{(N)}\|_{RB_n} \leq CR^3\lambda^{(1-6\gamma)n} \tag{75}$$

and

$$\|h_{n-m}(U_n^{(N)}(0) + DU_n^{(N)}(0)\phi)\|_{RB_n} \leq CR\lambda^{(2-3\gamma)n} \tag{76}$$

for all $R \geq 1$ (since they are polynomials in ϕ with coefficients the noise fields ζ_n).

Next, we will rewrite the fixed point Eq. (35) in a localized form. Define $\tilde{v}_n^{(N)} = h_{n-m}v_n^{(N)}$ so that

$$\tilde{v}_{n-1}^{(N)}(\phi) = h_{n-1-m}(\mathcal{L}_n v_n^{(N)})(\phi + \Gamma\tilde{v}_{n-1}^{(N)}(\phi)) \tag{77}$$

where we used Lemma 9(a) in the argument. By (34)

$$h_{n-m-1}\mathcal{L}_n = \mathcal{L}_n h_{n-m-1}(\lambda^2\cdot) \tag{78}$$

and $h_{n-1-m}(\lambda^2\cdot)$ is supported on $[0,\tau_{n-m} - \frac{1}{2}]$. Since $h_{n-m} = 1$ on $[0,\tau_{n-m} - \lambda^2]$ we get

$$h_{n-m-1}(\lambda^2 t) = h_{n-m-1}(\lambda^2 t)h_{n-m}(t) \tag{79}$$

so that (77) can be written as

$$\tilde{v}_{n-1}^{(N)}(\phi) = h_{n-1-m}(\mathcal{L}_n\tilde{v}_n^{(N)})(\phi + \Gamma\tilde{v}_{n-1}^{(N)}(\phi)). \tag{80}$$

Hence, the ν fixed point problem (60) becomes

$$\tilde{\nu}_{n-1} = h_{n-1-m}(\mathcal{L}_n\tilde{\nu}_n + \tilde{\mathcal{F}}_n(\tilde{U}_{n-1} + \tilde{\nu}_{n-1})), \quad \tilde{\nu}_N = 0 \tag{81}$$

where $\tilde{\mathcal{F}}_n$ is as in (56), i.e.,

$$\tilde{\mathcal{F}}_n(w) = \mathcal{G}_n(\tilde{u}_n + \tilde{w}_n, \tilde{u}_{n-1} + w) - D\tilde{u}_{n-1}\Gamma\tilde{u}_{n-1} \tag{82}$$

Thus, with only a slight abuse of notation we will drop the tildes and h factors in the norms in the following:

Proposition 11. *There exist $\lambda_0 > 0$, $\gamma_0 > 0$ so that for $\lambda < \lambda_0$, $\gamma < \gamma_0$ and $m > m(\gamma, \lambda)$ if $\Xi \in A_m$ then then for all $N \geq n-1 \geq m$ the Eq. (81) has a unique solution $\nu_{n-1}^{(N)} \in \mathcal{W}_n(B_{n-1})$. These solutions satisfy*

$$\|\nu_n^{(N)}\|_{B_n} \leq \lambda^{(3-\frac{1}{4})n} \tag{83}$$

and $\nu_n^{(N)}$ converge in $\mathcal{W}_n(B_n)$ to a limit $\nu_n \in \mathcal{W}_n(B_n)$ as $N \to \infty$. ν_n is independent on the small scale cutoff: $\nu_n = \nu_n'$.

Proof. We solve (81) by Banach fixed point theorem in the ball $\|\nu_{n-1}\|_{B'} \leq \lambda^{(3-\frac{1}{4})(n-1)}$ where $B' = \lambda^{-\frac{1}{2}} B_{n-1}$ (we only need to prove analyticity in B_{n-1}, but for bounding U_n the larger region is needed). By Proposition 10 we have

$$\|\mathcal{L}_n \nu_n\|_{B'} \leq C\lambda^{-\frac{5}{2}} \lambda^{(3-\frac{1}{4})n} = C\lambda^{\frac{1}{4}} \lambda^{(3-\frac{1}{4})(n-1)} \tag{84}$$

Next, we estimate the $\mathcal{F}_n$ term in (81). $\mathcal{F}_n$ is given in (82), so we need to start with $\mathcal{G}_n$ given in (54). Let $v \in \mathcal{W}_n(B_n)$ and $\bar{v} \in \mathcal{W}_{n-1}(B')$ and define

$$f(v, \bar{v})(\phi) := \lambda^{-5/2} s^{-1} v(s(\phi + \Gamma\xi_{n-1} + \Gamma\bar{v}(\phi)). \tag{85}$$

For $\phi \in B'$ we have by Lemma 9(a) and (74)

$$\|s(\phi + \Gamma\xi_{n-1} + \Gamma\bar{v}(\phi))\|_{\Phi_n} \leq \lambda^{-2\gamma(n-1)} + \lambda^{-\gamma n} + C(\lambda)\|\bar{v}\|_{B'}$$

Hence, $f(v, \bar{v}) \in \mathcal{W}_{n-1}(B')$ provided

$$\|\bar{v}\|_{B'} \leq c(\lambda)\lambda^{-2\gamma n} \tag{86}$$

We use this first to estimate

$$g_n := \mathcal{G}(u_n, u_{n-1}) - Du_{n-1}\Gamma u_{n-1}.$$

We have $g_n = f(1) - f(0) - f'(0)$ where $f(z) = f(u_n, zu_{n-1})$. By (86) f is analytic in

$$|z| < c(\lambda)\lambda^{-2\gamma n}\|u_{n-1}\|_{B'}^{-1}$$

and so by a Cauchy estimate and (75)

$$\|g_n\|_{B'} \leq C(\lambda)\lambda^{4\gamma n}\|u_n\|_{B_n}\|u_{n-1}\|_{B'}^2 \leq C(\lambda)\lambda^{(3-14\gamma)n} \tag{87}$$

Next, we write

$$\mathcal{F}_n(w) = g_n + \mathcal{G}_n(w_n, u_{n-1} + w) + h_n(w) \tag{88}$$

with

$$h_n(w) = \mathcal{G}_n(u_n, u_{n-1} + w) - \mathcal{G}(u_n, u_{n-1})$$

We have $h_n(w) = \tilde{f}(1) - \tilde{f}(0)$ with $\tilde{f}(z) = f(u_n, u_{n-1} + zw)$ which is analytic in

$$|z| < c(\lambda)\lambda^{-2\gamma n}\|w\|_{B'}^{-1}.$$

Hence, by a Cauchy estimate

$$\|h_n(w)\|_{B'} \leq C(\lambda)\lambda^{2\gamma n}\|u_n\|_{B_n}\|w\|_{B'}. \tag{89}$$

Finally, in the same way,

$$\|\mathcal{G}_n(w_n, u_{n-1} + w)\|_{B'} \leq C(\lambda)\|w_n\|_{B_n}(\|u_{n-1}\|_{B'} + \|w\|_{B'}). \tag{90}$$

Recalling (59), we have

$$\|w_n\|_{B_n} \le \|U_n\|_{B_n} + \|\nu_n\|_{B_n} \tag{91}$$

so to proceed we need a bound for U_n. It is defined iteratively in (57) which we write as

$$U_{n-1} = \mathcal{L}U_n + \tilde{U}_{n-1}$$

where $\tilde{U}_{n-1} = Du_{n-1}\Gamma u_{n-1} = f'(0)$ where $f(z) = f(u_n, zu_{n-1})$ as above. Again by Cauchy, we get

$$\|\tilde{U}_{n-1}\|_{B'} \le C(\lambda)\lambda^{2(1-6\gamma)n+2\gamma n}. \tag{92}$$

Recall the definition of V_n in (61). It satisfies

$$V_{n-1}(\phi) = (\mathcal{L}_n V_n)(\phi) - (\mathcal{L}_n V_n)(0) - D(\mathcal{L}_n V_n)(0)\phi + \tilde{V}_{n-1}(\phi).$$

where $\tilde{V}_{n-1} = \tilde{U}_{n-1} - \tilde{U}_{n-1}(0) - D\tilde{U}_{n-1}(0)\phi$. From (92) we get

$$\|\tilde{V}_{n-1}\|_{B'} \le C(\lambda)\lambda^{(2-10\gamma)n}. \tag{93}$$

Assume inductively

$$\|V_n\|_{B_n} \le \lambda^{(2-11\gamma)n}. \tag{94}$$

Proposition 10 combined with a Cauchy estimate (here we use $B' = \lambda^{-\frac{1}{2}} B_{n-1}$) and (93) gives

$$\|V_{n-1}\|_{B_{n-1}} \le C\lambda^{-3/2}\|V_n\|_{B_n} + C(\lambda)\lambda^{(2-10\gamma)n} \tag{95}$$

which proves the induction step taking γ small enough and $n \ge n(\lambda)$. Since U_N is linear by (57) the induction starts with $V_N = 0$. Combining (94) with (76) and the initial condition in (57), we then arrive at

$$\|U_n\|_{B_n} \le 2\lambda^{(2-11\gamma)n}.$$

Combining this bound with (87), (89) and (90) gives, for γ small enough

$$\begin{aligned}&\|\mathcal{F}_n(U_{n-1} + \nu_{n-1})\|_{B_{n-1}} \\ &\quad \le \lambda^{(3-\frac{1}{4})n} + \lambda^{\frac{1}{2}n}(\|\nu_{n-1}\|_{B_{n-1}} + \|\nu_n\|_{B_n}) + \|\nu_{n-1}\|_{B_{n-1}}\|\nu_n\|_{B_n}\end{aligned}$$

Recalling (84) and Lemma 9(c) to bound the h_{n-1-m} factor in (81) we conclude that the ball $\|\nu_{n-1}\|_{B_{n-1}} \le \lambda^{(3-\frac{1}{4})(n-1)}$ is mapped by the RHS of (81) to itself. The map is also a contraction if $n \ge n(\lambda)$ since by (84) this holds for $\mathcal{L}$ and

$$\|\mathcal{F}_n(U_{n-1} + \nu_1) - \mathcal{F}_n(U_{n-1} + \nu_2)\|_{B_{n-1}} \le C(\lambda)\lambda^{(1-6\gamma)n}\|\nu_1 - \nu_2\|_{B_{n-1}}.$$

Let us address the convergence as $N \to \infty$ and cutoff dependence of $\nu_n = \nu_n^{(N)}$. Recall we can deal with both questions together with ν_n'. Since u_n and $U_n - V_n$ are polynomials in ϕ with coefficients ζ_n satisfying the estimate (73) we get

$$\|u_n - u_n'\|_{B_n} \le C\lambda^{\gamma(N-n)}\lambda^{(1-6\gamma)n} \tag{96}$$

$$\|(U_n - V_n) - (U_n' - V_n')\|_{B_n} \le C\lambda^{\gamma(N-n)}\lambda^{(2-3\gamma)n}. \tag{97}$$

To study $\mathcal{V}_n := V_n - V_n'$ we need to estimate (recall (93)) $\tilde{V}_n - \tilde{V}_n'$ which in turn is determined by $\tilde{U}_n - \tilde{U}_n'$. This is again estimated by Cauchy and we get

$$\|\tilde{U}_n - \tilde{U}_n'\|_{B'} \leq C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n}.$$

Proceeding as in the derivation of (95) we get

$$\|\mathcal{V}_{n-1}\|_{B_{n-1}} \leq C\lambda^{-3/2}\|\mathcal{V}_n\|_{B_n} + C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n}$$

leading to

$$\|\mathcal{V}_n\|_{B_n} \leq C(\lambda)\lambda^{\gamma(N-n)}\lambda^{(2-10\gamma)n}.$$

Combining this with (97) we arrive at

$$\|U_n - U_n'\|_{B_n} \leq \lambda^{\gamma(N-n)}\lambda^{(2-11\gamma)n}. \tag{98}$$

Finally, using (96) and (98) it is now straightforward to prove, for γ suitably small,

$$\|\mathcal{F}_n - \mathcal{F}_n'\|_{B_{n-1}} \leq \lambda^{\gamma(N-n)}\lambda^{(3-\frac{1}{4})n} + \lambda^{\frac{1}{2}n}\|\nu_{n-1} - \nu_{n-1}'\|_{B_{n-1}}.$$

As in (84) we get

$$\|\mathcal{L}_n(\nu_n - \nu_n')\|_{B_{n-1}} \leq C\lambda^{-\frac{5}{2}}\|\nu_n - \nu_n'\|_{B_n} \tag{99}$$

Hence, for small γ we obtain inductively for $m \leq n \leq N$

$$\|\nu_n - \nu_n'\|_{B_n} \leq C\lambda^{\gamma(N-n)}\lambda^{(3-\frac{1}{4})n}.$$

This establishes the convergence of $\nu_n^{(N)}$ to a limit that is independent on the short-time cutoff. □

Remark 12. Let us briefly indicate how the cutoff (10) can be accommodated to our scheme. We only need to modify the first RG step. For $n = N$ in (27) the noise is replaced by the spatially smooth noise $\tilde{\xi}_{N-1} = \rho * \xi_{n-1}$ and Γ by $\tilde{\Gamma}$ where we use the cutoff $\chi(t-s)$ in (28). $\tilde{\Gamma}$ is not infinitely smoothing, but $s\tilde{\Gamma}v_{N-1}^N \in B_N$ nevertheless since at this scale v_{N-1}^N is as smooth as ϕ is.

9. Proof of Theorem 1

We are now ready to construct the solution $\phi^{(\epsilon)}$ of the ϵ cutoff Eq. (7). Recall that formally $\phi^{(\epsilon)}$ is given on time interval $[0, \lambda^{2m}]$ by Eq. (26) (with $n = m$) with ϕ_m given as the solution of Eq. (25) on time interval $[0, 1]$. Hence, we first need to study the f iteration Eq. (29). This is very similar to the v iteration (27) except there is no fixed point problem to be solved and there is no multiplicative λ^{-2} factor. As in (80) for $v_n^{(N)}$, we study instead of (29) the localized iteration

$$\tilde{f}_{n-1}^{(N)}(\phi) = h_{n-1-m}s^{-1}\tilde{f}_n^{(N)}(s(\phi + \Gamma(\tilde{v}_{n-1}^{(N)}(\phi) + \xi_{n-1}))) \tag{100}$$

for $\tilde{f}_n^{(N)} = h_{n-m}f_n^{(N)}$. The following Proposition is immediate :

Proposition 13. *Let* $\tilde{\nu}_n^{(N)} \in \mathcal{W}_n(B_n)$*,* $m \leq n \leq N$ *be as in Proposition* 11*. Then, for* $m \leq n \leq N$ $\tilde{f}_n^{(N)} \in \mathcal{W}_n(B_n)$ *and*

$$f_n^{(N)}(\phi) = \phi + \eta_n^{(N)} + g_n^{(N)}(\phi). \tag{101}$$

with

$$\|\tilde{g}_n^{(N)}\|_{B_n} \leq \lambda^{\frac{3}{4}n}. \tag{102}$$

and $g_n^{(N)}$ *converge in* $\mathcal{W}_n(B_n)$ *as* $N \to \infty$ *to a limit* $\tilde{g}_n \in \mathcal{W}_n(B_n)$ *which is independent on the short-time cutoff.*

Proof. We have

$$\tilde{g}_{n-1}^{(N)}(\phi) = h_{n-1-m}(\Gamma \tilde{v}_{n-1}^{(N)}(\phi) + s^{-1}\tilde{g}_n^{(N)}(s(\phi + \Gamma(\tilde{v}_{n-1}^{(N)}(\phi) + \xi_{n-1}))))$$

Since $\|\tilde{v}_{n-1}^{(N)}\|_{B_{n-1}} \leq C\lambda^{(1-3\gamma)(n-1)}$ Lemma 9(b) implies

$$\|\tilde{g}_{n-1}^{(N)}\|_{B_{n-1}} \leq C(\lambda)\lambda^{(1-3\gamma)(n-1)} + C\lambda^{-\frac{1}{2}}\lambda^{\frac{3}{4}n} \leq \lambda^{\frac{3}{4}(n-1)}$$

The convergence and cutoff independence follows from that of $v_n^{(N)}$. □

We need the following lemma:

Lemma 14. G_1 *is a bounded operator from* $\mathcal{V}_n$ *to* Φ_n *and* $G_1(h_{n-1-m}(\lambda^2\cdot)v) = G_1 v$*.*

Proof of Theorem 1*.* We claim that if $\Xi \in \mathcal{A}_m$ the solution $\varphi^{(N)}$ of equation (7) with $\epsilon = \lambda^N$ is given by (recall (26))

$$\varphi^{(N)} = s^{-m}\tilde{f}_m^{(N)}(0) \tag{103}$$

on the time interval $[0, \frac{1}{2}\lambda^{-2m}]$. Let $\phi_n \in \Phi_n$ be defined inductively by $\phi_m = 0$ and for $n > m$

$$\phi_n = s(\phi_{n-1} + \Gamma(\tilde{v}_{n-1}^{(N)}(\phi_{n-1}) + \xi_{n-1})). \tag{104}$$

We claim that for all $m \leq n \leq N$ $\phi_n \in B_n$ and

$$\phi_n = G_1(\tilde{v}_n^{(N)}(\phi_n) + \xi_n). \tag{105}$$

Indeed, this holds trivially for $n = m$ since the RHS vanishes identically on $[0, 1]$. Suppose $\phi_{n-1} \in B_{n-1}$ satisfies

$$\phi_{n-1} = G_1(\tilde{v}_{n-1}^{(N)}(\phi_{n-1}) + \xi_{n-1}). \tag{106}$$

Then, first by Lemma 9(b)

$$\|\phi_n\|_{\Phi_n} \leq \lambda^{\frac{1}{2}}\|\phi_{n-1}\|_{\Phi_{n-1}} + C(\lambda)\lambda^{-\gamma n} \leq \lambda^{-2\gamma n}$$

so that $\phi_n \in B_n$. Second, we have by (106) and (104)

$$\begin{aligned}\phi_n &= s((G_1 + \Gamma)(\tilde{v}_{n-1}^{(N)}(\phi_{n-1}) + \xi_{n-1})) = G_1\lambda^2 s(\tilde{v}_{n-1}^{(N)}(\phi_{n-1}) + \xi_{n-1}) \\ &= G_1(h_{n-1-m}(\lambda^2\cdot)\tilde{v}_n^{(N)}(\phi_n) + \xi_n) = G_1(\tilde{v}_n^{(N)}(\phi_n) + \xi_n) \end{aligned} \tag{107}$$

where in the third equality we used the RG iteration (77) and in the last equality Lemma 14.

From (100) we have since $\phi_m = 0$

$$\tilde{f}_m^{(N)}(0) = h_0 s^{-1} \tilde{f}_{m+1}^{(N)}(\phi_{m+1}) = h_0 h_1(\cdot/\lambda^2) s^{-2} \tilde{f}_{m+2}^{(N)}(\phi_{m+2})$$
$$= h_0 s^{-2} \tilde{f}_{m+2}^{(N)}(\phi_{m+2})$$

where we used (79). Iterating we get

$$\tilde{f}_m^{(N)}(0) = h_0 s^{-(N-m)} \tilde{f}_N^{(N)} = h_0 h_{N-m}(\cdot/\lambda^{2(N-m)}) s^{-(N-m)} \phi_N$$
$$= h_0 s^{-(N-m)} \phi_N \tag{108}$$

again by (79). Now $\phi_N \in B_N$ solves (105) with $\tilde{v}_N^{(N)}(\phi) = h_{N-m} v_N^{(N)}(\phi)$ with $v_N^{(N)}$ given by (31). Since $h_{N-m} = 1$ on $[0, \tau_{N-m} - \lambda^2]$ we obtain

$$\phi_N = G_1(\tilde{v}_n^{(N)}(\phi_n) + \xi_n) = G_1(v_n^{(N)}(\phi_n) + \xi_n).$$

and thus $\varphi^{(N)} = s^{-N}\phi_N$ solves (103) on the time interval $[0, \lambda^{-2m}]$. (108) then gives

$$h_0(\lambda^{-2m}\cdot)\varphi^{(N)} = s^{-m} \tilde{f}_m^{(N)}(0)$$

so that (103) holds on the time interval $[0, \frac{1}{2}\lambda^{-2m}]$.

By Proposition 13 $f_m^{(N)}(0)$ converges in $\mathcal{V}_m$ to a limit ψ_m which is independent on the short-distance cutoff. Convergence in $\mathcal{V}_m$ implies convergence in $\mathcal{D}'([0,1]\times\mathbb{T}_m)$. The claim follows from continuity of $s^{-m} : \mathcal{D}'([0,1]\times\mathbb{T}_m) \to \mathcal{D}'([0,\lambda^{2m}]\times\mathbb{T}_1)$. □

10. Kernel Estimates

In this Section, we prove Lemmas 9, 14 and 5 and give bounds for the various kernels entering the proof of Proposition 8.

10.1. Proof of Lemmas 9 and 14

Lemma 9 *(a) and Lemma* 14 Let $v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_{n-1})$. Then,

$$s\Gamma v(t) = \int_0^\infty k(\lambda^2 t - s) \mathrm{e}^{(\lambda^2 t - s)\Delta} v(s) \mathrm{d}s$$

where $k(\tau) = \lambda^{\frac{1}{2}}(\chi(\tau) - \chi(\tau/\lambda^2))$ vanishes for $\tau \le \lambda^2$. Hence, $s\Gamma v \in C_0^\infty([1,\infty) \times \mathbb{T}_n)$. Next, write $v = (-\partial_t^2 + 1)(-\Delta + 1)^2 K v$ so that setting $w = Kv$ we have

$$s\Gamma v(t) = \int_{\mathbb{R}} k(\lambda^2 t - s)(-\Delta + 1)^2 \mathrm{e}^{(\lambda^2 t - s)\Delta}(-\partial_s^2 + 1) w(s) \mathrm{d}s \tag{109}$$

Integrating by parts we get

$$s\Gamma v(t) = \int_{\mathbb{R}} ((-\partial_s^2 + 1) k(\lambda^2 t - s)(-\Delta + 1)^2 \mathrm{e}^{(\lambda^2 t - s)\Delta}) w(s) \mathrm{d}s.$$

The kernels $\partial_t^a(k(\lambda^2 t - s)\partial_x^\alpha \mathrm{e}^{(\lambda^2 t - s)\Delta}(x-y))$ are smooth, exponentially decreasing in $|x-y|$ and supported on $\lambda^2 t - s \in [\lambda^2, 2]$ for all a and α. Hence, the corresponding operators $O_{a\alpha}$ satisfy

$$|(O_{a\alpha} 1_{c_i} w)(t,x)| \le C(\lambda) \mathrm{e}^{-cd(i,(t,x))} \|w\|_{L^2(c_i)}$$

which in turn yields our claim $\|s\Gamma v\|_{\Phi_n} \leq C(\lambda)\|v\|_{\mathcal{V}_{n-1}}$. Hence, $s\Gamma$ extends to a bounded operator from $\mathcal{V}_{n-1}$ to Φ_n.

$G_1 v$ is given by (109) with k is replaced by $1-\chi(t-s)$. The time integral is confined to $[0, \tau_{n-m}]$ so that $\|G_1 v\|_{\Phi_n} \leq C(\lambda, n)\|v\|_{\mathcal{V}_n}$.

Lemma 9 (b). Recall $(s\phi)(t,x) = \lambda^{\frac{1}{2}}\phi(\lambda^2 t, \lambda x)$ on $[\lambda^{-2}, \tau_{n-1-m}] \times \mathbb{T}_{n-1}$. Since $\lambda < 1$ we then get $\|s\phi\|_{\Phi_n} \leq \lambda^{\frac{1}{2}}\|\phi\|_{\Phi_{n-1}}$. Next, let $v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_n)$ and $w = Kv$. First, write

$$Ks^{-1}v = Ks^{-1}(-\partial_t^2 + 1)(-\Delta + 1)^2 w = K_1(-\lambda^4\partial_t^2 + 1)K_2(-\lambda^2\Delta + 1)^2 s^{-1}w.$$

Next, $K_1(-\lambda^4\partial_t^2 + 1) = \lambda^4 + (1-\lambda^4)K_1$ and

$$K_2(-\lambda^2\Delta + 1)^2 = \lambda^4 + 2\lambda^2(1-\lambda^2)(-\Delta + 1)^{-1} + (1-\lambda^2)^2 K_2.$$

Thus, we need to show the operators K_i, $(-\Delta + 1)^{-1}$ and $\lambda^{\frac{1}{2}}s^{-1}$ are bounded in the norm $\sup_i \|\cdot\|_{L^2(c_i)}$ uniformly in λ. For K_i this follows from the bounds (67) and (68) and for the third one from $(-\Delta+1)^{-1}(x,y) \leq C\mathrm{e}^{-c|x-y|}|x-y|^{-1}$. Finally, let $c_i/\lambda := \{(t/\lambda^2, x/\lambda)|(t,x) \in c_i\}$

$$\|\lambda^{\frac{1}{2}}s^{-1}w\|_{L^2(c_i)}^2 = \lambda^5 \int_{c_i/\lambda} |w|^2 \leq \lambda^5 \sum_{c\cap(c_i/\lambda)\neq\emptyset} \|w\|_{L^2(c)}^2 \leq C$$

Lemma 9 (c) Let $\phi \in C^{2,4}(\mathbb{R} \times \mathbb{T}_n)$ and $v \in C_0^\infty(\mathbb{R}_+ \times \mathbb{T}_n)$ and set again $w = Kv$ so that

$$\phi v = \phi(-\partial_t^2 + 1)(-\Delta + 1)^2 w$$

Using $\phi\partial_t^2 f = \partial_t^2(\phi f) - 2\partial_t(\partial_t\phi f) + \partial_t^2\phi f$ and similar commuting for Δ we get

$$K(\phi v) = \phi w + \sum_a \mathcal{O}_a(\phi_a w)$$

where the operators $\mathcal{O}_a$ belong to the set $\{\partial_t^n K_1, \partial_x^\alpha K_2, \partial_t^n K_1 \partial_x^\alpha K_2\}$ with $n \leq 1$ and $|\alpha| \leq 3$. The functions ϕ_a are multiples of $\partial_t^m \partial_x^\beta \phi$ with $m \leq 2$ and $|\beta| \leq 4$ and hence bounded in sup norm by $C\|\phi\|_{C^{2,4}}$. We get

$$\|\phi v\|_{\mathcal{V}_n} \leq C\|\phi\|_{C^{2,4}} \max_a \sup_i \sum_j \|\mathcal{O}_a\|_{L^2(c_j)\to L^2(c_i)}$$

The operators $\partial_t K_1$ and K_1 have bounded exponentially decaying kernels. The operators $\partial_x^\alpha K_2$ are bounded in L^2 hence from $L^2(A) \to L^2(B)$ with $A, B \subset \mathbb{T}_n$. Moreover, their kernels $\partial_x^\alpha K_2(x-y)$ are smooth for $x - y \neq 0$ and exponentially decaying. We conclude

$$\|\mathcal{O}_a\|_{L^2(c_j)\to L^2(c_i)} \leq C\mathrm{e}^{-c|i-j|}$$

and then

$$\|\phi v\|_{\mathcal{V}_n} \leq C\|\phi\|_{C^{2,4}}\|v\|_{\mathcal{V}_n}.$$

□

10.2. Proof of Lemma 5

Let $H(t,x) = \mathrm{e}^{t\Delta}(0,x) = (4\pi s)^{-3/2}\mathrm{e}^{-x^2/4t}$ be the heat kernel on $\mathbb{R}^3$. Then,

$$H_n(t,x) = \sum_{i\in\mathbb{Z}^3} H(t, x+\lambda^{-n}i). \tag{110}$$

Denoting $\epsilon = \lambda^{2(N-n)}$ and separating the $i=0$ term, we have

$$\mathbb{E}\eta_n^{(N)}(t,x)^2 = \int_0^t (8\pi s)^{-3/2}(\chi(s)^2 - \chi(s/\epsilon^2)^2)\mathrm{d}s + \alpha(t) \tag{111}$$

where

$$|\alpha(t)| \leq \sum_{i\neq 0}\int_0^2 (8\pi s)^{-3/2}\mathrm{e}^{-i^2/(4s\lambda^{2n})} \leq C\mathrm{e}^{-c\lambda^{-2n}}.$$

Let $\alpha'(t)$ have the lower cutoff replaced by χ'. Then,

$$\begin{aligned}|\alpha(t) - \alpha'(t)\| &\leq C\int s^{-3/2}|\chi(s/\epsilon'^2)^2 - \chi(s/\epsilon^2)^2|\mathrm{e}^{-\lambda^{-2n}/4s}\mathrm{d}s\\ &\leq C\mathrm{e}^{-c\lambda^{-2N}}\|\chi-\chi'\|_\infty.\end{aligned}$$

Denote the first term in (111) by $\beta(t,\epsilon)$ and $\beta'(t,\epsilon)$ where the lower cutoff is χ'. Then,

$$\beta'(\infty,\epsilon) = \epsilon^{-1}\int_0^\infty (8\pi s)^{-3/2}(\chi(\epsilon^2 s)^2 - \chi'(s)^2)\mathrm{d}s = \epsilon^{-1}\rho' - \rho$$

So

$$\delta_n^{(N)}(t) = \alpha(t) + \rho - \gamma(t,\epsilon)$$

with

$$\gamma(t,\epsilon) = \beta(\infty,\epsilon) - \beta(t,\epsilon) = \int_t^\infty (8\pi s)^{-3/2}(\chi(s)^2 - \chi(s/\epsilon^2)^2)\mathrm{d}s \leq Ct^{-\frac{1}{2}}.$$

Moreover,

$$\begin{aligned}|\gamma(t,\epsilon) - \gamma'(t,\epsilon)| &= \int_t^\infty (8\pi s)^{-3/2}|\chi(s/\epsilon^2)^2 - \chi'(s/\epsilon^2)^2|\mathrm{d}s\\ &\leq Ct^{-\frac{1}{2}}\|\chi-\chi'\|_\infty 1_{[0,\epsilon]}(t).\end{aligned}$$

□

10.3. Covariance and Response Function Bounds

We prove now bounds for the covariance and response kernels (37) and (39) that are needed for the probabilistic estimates in Sect. 11. These kernels are translation invariant in the spatial variable and we will denote $C_n^{(N)}(t',t,x',x)$ simply by $C_n^{(N)}(t',t,x'-x)$ and similarly for the other kernels. As before primed kernels and fields have the lower cutoff χ'. We need to introduce the mixed covariance

$$C_n'^{(N)}(t',t,x'-x) := \mathbb{E}\eta_n'^{(N)}(t',x')\eta_n^{(N)}(t,x) \tag{112}$$

Let us define

$$\begin{aligned}\mathcal{C}_n(\tau,x) &= \sup_{|t'-t|=\tau}\sup_{N\geq n} C_n'^{(N)}(t',t,x)\\ \mathcal{C}_n^N(\tau,x) &= \sup_{|t'-t|=\tau}|C_n'^{(N)}(t',t,x)-C_n^{(N)}(t',t,x)|\\ \mathcal{G}_n(t,x) &= \sup_{N\geq n}\Gamma_n^N(t,x)\\ \mathcal{G}_n^N(t,x) &= |\Gamma_n'^N(t,x)-\Gamma_n^N(t,x)|\end{aligned}$$

The regularity of these kernels is summarized in

Lemma 15. *$\mathcal{C}_n\in L^p(\mathbb{R}\times\mathbb{T}_n)$ uniformly in n for $p<5$ and*

$$\|\mathcal{C}_n^N\|_p\leq C\lambda^{\gamma_p(N-n)}\|\chi-\chi'\|_\infty. \tag{113}$$

for $\gamma_p>0$ for $p<5$. $\mathcal{G}_n\in L^p(\mathbb{R}\times\mathbb{T}_n)$ uniformly in n for $p<5/3$ and

$$\|\mathcal{G}_n^N\|_p^p\leq C\lambda^{\gamma_p'(N-n)}\|\chi-\chi'\|_\infty. \tag{114}$$

for $\gamma_p'>0$ for $p<5/3$.

Proof. As in (39) we have, for $t'\geq t$:

$$C_n'^{(N)}(t',t,x'-x)=\int_0^t H_n(|t'-t|+2s,x'-x)\chi^1_{N-n}(t'-t+s)\chi^2_{N-n}(s)\mathrm{d}s \tag{115}$$

where $\chi^1=\chi'$, $\chi^2=\chi$ or vice versa depending on $t'>t$ or $t'<t$. The heat kernel H_n is pointwise positive. Using $\chi^1_{N-n}(t'-t+s)\chi^2_{N-n}(s)\leq 1_{[0,2]}(s)1_{[0,2]}(t'-t)$, we may bound

$$C_n'^{(N)}(t',t,x)\leq C_n(t'-t,x)1_{[0,2]}(t'-t) \tag{116}$$

where

$$C_n(\tau,x)=\int_0^2 H_n(\tau+2s,x)\mathrm{d}s$$

From (110) we get

$$H_n(\tau,x)=\sum_{m\in\mathbb{Z}^3}(4\pi t)^{-3/2}\mathrm{e}^{-\frac{(x+\lambda^{-n}m)^2}{4t}}. \tag{117}$$

Therefore,

$$C_n(\tau,x)=\sum_{m\in\mathbb{Z}^3}c(\tau,x+\lambda^{-n}m) \tag{118}$$

where

$$\begin{aligned}c(\tau,x) &= \int_0^2(4\pi(\tau+2s))^{-3/2}\mathrm{e}^{-\frac{x^2}{4(\tau+2s)}}\mathrm{d}s\\ &\leq C(\mathrm{e}^{-cx^2}(x^2+\tau)^{-\frac12}1_{\tau\in[0,2]}+\tau^{-3/2}\mathrm{e}^{-cx^2/\tau}1_{\tau>2})).\end{aligned} \tag{119}$$

Combining with (118) and (116) (with $\chi'=\chi$)

$$\mathcal{C}_n(\tau,x)\leq C\mathrm{e}^{-cx^2}(x^2+\tau)^{-\frac12}1_{[0,2]}(\tau) \tag{120}$$

and so $\mathcal{C}_n \in L^p$ if $p < 5$. To get (113) use

$|\chi_k^1(t'-t+s)\chi_k^2(s) - \chi_k^1(t'-t+s)\chi_k^2(s)| \leq 1_{[0,2\lambda^{2k}]}(s)1_{[0,2]}(t'-t)\|\chi-\chi'\|_\infty$.

Hence,

$$\mathcal{C}_n^N(\tau,x) \leq \sum_{m\in\mathbb{Z}^3} c_{N-n}(\tau, x+\lambda^{-n}m)1_{[0,2]}(\tau)\|\chi-\chi'\|_\infty$$

where

$$c_M(\tau,x) := \int_0^{2\lambda^{2M}} (4\pi(\tau+2s))^{-3/2}\mathrm{e}^{-\frac{x^2}{4(\tau+2s)}}\,\mathrm{d}s = \lambda^{-M}c_0(\lambda^{-2M}\tau, \lambda^{-M}x). \tag{121}$$

Hence, using (119)

$$\begin{aligned}\|c_M 1_{[0,2]}\|_p^p &= \lambda^{(5-p)M}\|c_0 1_{[0,2\lambda^{-2M}]}\|_p^p \\ &\leq \lambda^{(5-p)M}(1+\int \tau^{\frac{3}{2}(1-p)}1_{[2,2\lambda^{-2M}]}\mathrm{d}t) \leq C\lambda^{\gamma M}\end{aligned}$$

with $\gamma > 0$ for $p < 5$. This yields (113).

In the same way,

$$\mathcal{G}_n(t,x) \leq Ct^{-3/2}\mathrm{e}^{-cx^2/t}1_{[0,2]}(t) \tag{122}$$

which is in L^p for $p < 5/3$. (113) follows then as above. □

We will later also need the properties of the kernel

$$S_n^{(N)}(t',t,x) := C_n^{(N)}(t',t,x)\Gamma_n^N(t',t,x). \tag{123}$$

Set

$$\mathcal{S}_n(\tau,x) := \sup_{|t'-t|=\tau}\sup_{N\geq n}\mathcal{S}_n^{(N)}(t',t,x) \tag{124}$$

We get from (120) and (122)

$$\mathcal{S}_n(t,x) \leq Ct^{-2}\mathrm{e}^{-cx^2/t}1_{[0,2]}(t) \in L^p, \quad p < 5/4.$$

Finally, let $S_n'^N$ have χ' in all the lower cutoffs in C_n^N and Γ_n^N and set

$$\mathcal{S}_n'^{(N)}(\tau,x) = \sup_{|t'-t|=\tau}|S_n'^{(N)}(t',t,x) - S_n^{(N)}(t',t,x)|.$$

Then,

$$\|\mathcal{S}_n^N\|_p^p \leq C\lambda^{\gamma_p''(N-n)}\|\chi-\chi'\|_\infty. \tag{125}$$

for some $\gamma_p'' > 0$ for $p < 5/4$.

Remark 16. Recall from Remark 7 that for the initial condition (48) the covariance of $\eta_n^{(N)}$ is the stationary one (49). Hence, Lemma 15 holds for it as well.

11. Proof of Proposition 8

In this section, we prove Proposition 8. The strategy is straightforward. We need to compute the covariances of the various fields in (66) and establish enough regularity for them. Covariance estimate is all we need since the probabilistic bounds are readily derived from it. The covariances have of course expressions in terms of Feynman diagrams only one of which is diverging as $N \to \infty$. The renormalization constant b is needed to cancel that divergence. We do not introduce the terminology of diagrams since the ones that enter are simple enough to be expressed without that notational device.

11.1. Covariance Bound

We will deduce Proposition 8 from a covariance bound for the fields in (66). Let $\zeta_n^{(N)}(t,x)$ or $\zeta_n^{(N)}(t,x,s,y)$ be any of the fields in (66). Let

$$\tilde{K}(t',t,x) = \mathrm{e}^{\frac{1}{2}\mathrm{dist}(t',I)} K(t'-t,x) h_{n-m}(t) \tag{126}$$

where $I = [0, \lambda^{-2(n-m)}]$ and define

$$\rho_n^{(N)} = \tilde{K}\zeta_n^{(N)} \quad \textit{or} \quad \rho_n^{(N)} = \tilde{K} \otimes \tilde{K}\zeta_n^{(N)}.$$

Then,

$$\|K\tilde{\zeta}_n^{(N)}\|_{L^2(c_i)} \le C\mathrm{e}^{-\frac{1}{2}\mathrm{dist}(i_0,I)} \|\rho_n^{(N)}\|_{L^2(c_i)}. \tag{127}$$

where i_0 is the time component of i and similarly for the bi-local case. We bound the covariance of $\rho_n^{(N)}$:

Proposition 17. *There exist renormalization constants m_1, m_2, m_3 and $\gamma > 0$ s.t. for all $0 \le n \le N < \infty$*

$$\mathbb{E}\rho_n^{(N)}(t,x)^2 \le C \tag{128}$$

$$\mathbb{E}({\rho'}_n^{(N)}(t,x) - \rho_n^{(N)}(t,x))^2 \le C\lambda^{\gamma(N-n)} \|\chi - \chi'\|_\infty \tag{129}$$

$$\mathbb{E}\rho_n^{(N)}(t,x,s,y)^2 \le C\mathrm{e}^{-c(|t-s|+|x-y|)} \tag{130}$$

$$\mathbb{E}({\rho'}_n^{(N)}(t,x,s,y) - \rho_n^{(N)}(t,x,s,y))^2 \le C\lambda^{\gamma(N-n)}\mathrm{e}^{-c(|t-s|+|x-y|)} \|\chi - \chi'\|_\infty \tag{131}$$

Proof of Proposition 8. $\rho_n^{(N)}(t,x)^2$ belongs to the inhomogeneous Wiener Chaos of bounded order m (in fact $m \le 10$). Thus, we get for all $p > 1$

$$\mathbb{E}\rho_n^{(N)}(t,x)^{2p} \le (2p-1)^{pm} (\mathbb{E}\rho_n^{(N)}(t,x)^2)^p \tag{132}$$

(see [14], page 62). Using Hölder, (132) and (128) in turn we deduce

$$\begin{aligned}\mathbb{E}(\|\rho_n^{(N)}\|_{L^2(c_i)}^{2p}) &\le \mathbb{E}(\|\rho_n^{(N)}\|_{L^{2p}(c_i)}^{2p}) = \int_{c_i} \mathbb{E}(\rho_n^{(N)}(t,x))^{2p} \\ &\le C_p \int_{c_i} ((\mathbb{E}\rho_n^{(N)}(t,x))^2)^p \le C_p\end{aligned} \tag{133}$$

and thus by (127)

$$\mathbb{E}(\|K\tilde{\zeta}_n^{(N)}\|_{L^2(c_i)}^{2p}) \le C_p \mathrm{e}^{-p\mathrm{dist}(i_0,I_n)}$$

so that

$$\mathbb{P}(\|K\tilde{\zeta}_n^{(N)}\|_{L^2(c_i)} \geq R) \leq C_p(R^{-1}\mathrm{e}^{-\frac{1}{2}\mathrm{dist}(i_0,I_n)})^{2p}$$

and finally

$$\mathbb{P}(\|\tilde{\zeta}_n^{(N)}\|_{\mathcal{V}_n} \geq \lambda^{-\gamma n}) \leq \sum_i C_p(\lambda^{\gamma n}\mathrm{e}^{-\frac{1}{2}\mathrm{dist}(i_0,I_n)})^{2p} \leq C_p\lambda^{2\gamma pn}\lambda^{2m}\lambda^{-5n}. \tag{134}$$

For the bi-local fields, we proceed in the same way. First, as in (133) we deduce

$$\mathbb{E}(\|\rho_n^{(N)}\|^{2p}_{L^2(c_i\times c_j)}) \leq C_p\int_{c_i\times c_j}((\mathbb{E}\rho_n^{(N)}(t,x,s,y)^2)^p \leq C_p(C\mathrm{e}^{-c\mathrm{dist}(c_i,c_j)})^{2p}$$

and then use exponential decay to control

$$\begin{aligned}\mathbb{P}(\|\tilde{\zeta}_n^{(N)}\|_{\mathcal{V}_n} &\geq \lambda^{-\gamma n})\\ &\leq \sum_{i,j} C_p(\lambda^{\gamma n}\mathrm{e}^{-c(\mathrm{dist}(i_0,I_n)+\mathrm{dist}(c_i,c_j))})^{2p} \leq C_p\lambda^{2p\gamma n}\lambda^{2m}\lambda^{-5n}.\end{aligned}$$

Next we turn to (73) which we recall we want to hold for all cutoff functions χ, χ' with bounded C^1 norm. We proceed by a standard Kolmogorov continuity argument.

Let $f(\chi) := K\tilde{\zeta}_n^{(N)}$. Without loss we may consider χ in the ball B_r of radius r at origin in $C^1[0,1]$. As above we conclude from (129)

$$\mathbb{E}(\|f(\chi)-f(\chi')\|^{2p}_{L^2(c_i)}) \leq C_p(\epsilon\|\chi-\chi'\|_\infty\mathrm{e}^{-\mathrm{dist}(i_0,I_n)})^p \tag{135}$$

with $\epsilon = \lambda^{\gamma(N-n)}$. Let χ_n, $n = 1, 2, \dots$ be the Fourier coefficients of χ in the basis 1, $\sin 2\pi x$, $\cos 2\pi x$, $\sin 4\pi x$, $\cos 4\pi x, \dots$. Hence, $|\chi_n| \leq Crn^{-1}$. Let $\chi_n^m = \chi_n$ for $n \leq m$ and $\chi_n^m = \chi'_n$ for $n > m$. Let Q_N be the dyadic rationals in $[0,1]$. Then, for $\beta \in (0,1)$

$$\begin{aligned}\|f(\chi)-f(\chi')\|_{L^2(c_i)} &\leq \sum_{m=1}^\infty \|f(\chi^m)-f(\chi^{m+1})\|_{L^2(c_i)}\\ &\leq \sum_{m=1}^\infty |\chi_m-\chi'_m|^\beta\sum_{N=0}^\infty 2^{\beta N}\Delta_{N,m}\end{aligned}$$

where

$$\Delta_{N,m} = \sup_{t,t'\in Q_N,|t-t'|=2^{-N}}\|g_m(t)-g_m(t')\|_{L^2(c_i)}$$

and $g_m(t) = f(\chi_1,\dots,\chi_{m-1},t,\chi'_{m+1},\dots)$. Using (135) for g_m we get

$$\mathbb{P}(\Delta_{N,m} > R) \leq C2^N(R^{-2}\epsilon 2^{-N}\mathrm{e}^{-\mathrm{dist}(i_0,I_n)})^p$$

and then, taking $\beta < \frac{1}{2}$ and $2\beta p > 1$

$$\begin{aligned}\mathbb{P}\left(\sup_{\chi,\chi'\in B_r}\|f(\chi)-f(\chi')\|_{L^2(c_i)} > R\right) &\leq \sum_{m,N}\mathbb{P}(\Delta_{N,m} > R|\chi_m-\chi'_m|^{-\beta}2^{-\beta N})\\ \leq \sum_{m,N} C2^N(R^{-2}\epsilon 2^{-N(1-2\beta)}r^\beta\mathrm{e}^{-\mathrm{dist}(i_0,I_n)}m^{-2\beta})^p &\leq C(R^{-2}\epsilon r^\beta\mathrm{e}^{-\mathrm{dist}(i_0,I_n)})^p\end{aligned}$$

since $|\chi_m - \chi'_m| \le Crm^{-1}$. We conclude then

$$\mathbb{P}\left(\sup_{\chi,\chi'\in B_r} \|\tilde{\zeta}'^{(N)}_n - \tilde{\zeta}^{(N)}_n\|_{\mathcal{V}_n} \ge \lambda^{\frac{1}{2}\gamma(N-n)}\lambda^{-\gamma n}\right) \le C_p r^{\beta p}\lambda^{p\gamma(N-n)}\lambda^{(2\gamma p-5)n}\lambda^{2m}. \tag{136}$$

We still need to deal with the last condition on $\mathcal{A}_m$ in (74). By (28) $\zeta := \Gamma\xi_{n-1}$ is a Gaussian field with covariance

$$\mathbb{E}\zeta(t',x')\zeta(t,x) = \int_0^t H_n(t'-t+2s, x'-x)\chi(t'-t+s)\chi(s)\mathrm{d}s$$

where χ is smooth with support in $[\lambda^2, 2]$. $\mathbb{E}\zeta(t',x')\zeta(t,y)$ is smooth, compactly supported in $t'-t$ and exponentially decaying in $x-y$. We get then by standard Gaussian estimates [13] for $a \le 2, |\alpha| \le 4$

$$\mathbb{P}(\|\partial_t^a\partial_x^\alpha\Gamma\xi_{n+1}\|_{L^\infty(c_i)} > r) \le C(\lambda)\mathrm{e}^{-c(\lambda)r^2}$$

and thus

$$\mathbb{P}(\|s\Gamma\xi_{n-1}\|_{\Phi_n} > \lambda^{-2\gamma n}) \le C(\lambda)\lambda^{2m}\lambda^{-5n}\mathrm{e}^{-c(\lambda)\lambda^{-4n}} \tag{137}$$

Combining (134), (136) and (137) with a Borel–Cantelli argument gives the claims (72) and (73). □

11.2. Normal Ordering

Since the fields (66) are polynomials in gaussian fields, the computation of their covariances is straightforward albeit tedious. To organize the computation it is useful to express them in terms of "normal-ordered" expressions in the field η_n. This provides also a transparent way to see why the renormalization constant b is needed. We suppress again the superscript $^{(N)}$ unless needed for clarity.

Define the "normal-ordered" random fields

$$:\eta_n := \eta_n, \quad :\eta_n^2 := \eta_n^2 - \mathbb{E}\eta_n^2, \quad :\eta_n^3 := \eta_n^3 - 3\eta_n\mathbb{E}\eta_n^2 \tag{138}$$

and (recall Lemma 5)

$$\delta_n(t) := \mathbb{E}(\eta_n^{(N)}(t,x))^2 - \rho_{N-n}. \tag{139}$$

Then,

$$\eta_n^2 - \rho_{N-n} =: \eta_n^2 : +\delta_n, \quad \eta_n^3 - 3\rho_{N-n}\eta_n =: \eta_n^3 : +3\delta_n\eta_n$$

In virtue of Lemma 5 δ_n terms will turn out to give negligible contribution.

These normal-ordered fields have zero mean and the following covariances:

$$\mathbb{E} : \eta_n^2 : (t,x) : \eta_n^2 : (s,y) = 2C_n(t,s,x,y)^2 \tag{140}$$

and

$$\mathbb{E} : \eta_n^3 : (t,x) : \eta_n^3 : (s,y) = 6C_n(t,s,x,y)^3. \tag{141}$$

where C_n is defined in (39).

Next, we process ω_n, ζ_n and z_n:

$$\omega_n = 3 : \eta_n^2 : \Gamma_n^N : \eta_n^3 : +(m_2 \log \lambda^N + m_3)\eta_n + 3\delta_n \Gamma_n^N : \eta_n^3 : +9 : \eta_n^2 : \Gamma_n^N \delta_n \eta_n + 9\delta_n : \Gamma_n^N \delta_n \eta_n$$

$$\mathfrak{z}_n = 6\eta_n \Gamma_n^N : \eta_n^3 : +18\eta_n \Gamma_n^N \delta_n \eta_n$$

$$z_n = 9 : \eta_n^2 : \Gamma_n^N : \eta_n^2 : +(m_2 \log \lambda^N + m_3) + 9\delta_n \Gamma_n^N : \eta_n^2 : +9 : \eta_n^2 : \Gamma_n^N \delta_n$$

(Recall that $\omega_n(t,x)$ and $\mathfrak{z}_n(t,x)$ are functions, whereas $z_n(t,x,s,y)$ is a kernel). Consider first the terms not involving δ_n, call them $\tilde{\omega}_n$, $\tilde{\mathfrak{z}}_n$ and $\tilde{z}_n$. We normal order $\tilde{\omega}_n$ and $\tilde{z}_n$:

$$\tilde{\omega}_n = (18C_n^2\Gamma_n^N - m_2 \log \lambda^N - m_3)\eta_n + 18 : \eta_n C_n \Gamma_n^N \eta_n^2 : +3 : \eta_n^2 \Gamma_n^N \eta_n^3 : \tag{142}$$

$$\tilde{z}_n = (18C_n^2\Gamma_n^N - m_2 \log \lambda^N - m_3) + 36 : \eta_n C_n \Gamma_n^N \eta_n : +9 : \eta_n^2 \Gamma_n^N \eta_n^2 : \tag{143}$$

where the product means pointwise multiplication of kernels, e.g., the operator $C_n^2\Gamma_n^N$ has the kernel

$$B_n(t,x,s,y) := C_n(t,x,s,y)^2 \Gamma_n^N(t,x,s,y). \tag{144}$$

For Proposition 17, it suffices to bound separately the covariances of all the fields in (142) and (143) as well as the δ_n-dependent ones.

11.3. Regular Fields

We will now prove the claims of Proposition 17 for the fields ζ_n not requiring renormalization.

It will be convenient to denote the space-time points (t,x) by the symbol z. We recall from (67) and (68) the bounds for the kernel $K(z'-z)$ of the operator K which imply a similar bound for $\tilde{K}$ of (126):

$$0 \le \tilde{K}(z',z) \le C\mathrm{e}^{-\frac{1}{2}|z'-z|} := \mathcal{K}(z'-z). \tag{145}$$

We start with the local fields. Let $\zeta_n(z)$ be one of them and

$$\mathbb{E}\zeta_n(z_1)\zeta_n(z_2) := H_n(z_1,z_2)$$

Then, (H_n is non-negative, see below)

$$\mathbb{E}\rho_n(z)^2 = \int \tilde{K}(z,z_1)\tilde{K}(z,z_2)H_n(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \le C\int \mathrm{e}^{-\frac{1}{2}(|z-z_1|+|z-z_2|)}H_n(z_1,z_2)\mathrm{d}z_1\mathrm{d}z_2 \le C\|\mathcal{H}_n\|_1 \tag{146}$$

where

$$\mathcal{H}_n(z) := \sup_{z_1-z_2=z} H_n(z_1,z_2)$$

Since H_n is given in terms of products of the covariances C_n and Γ_n^N we get an upper bound for $\mathcal{H}$ by replacing C_n and Γ_n^N by the translation invariant upper bounds $\mathcal{C}_n$ and $\mathcal{G}_n^N$ defined and bounded in Lemma 15. Let us proceed case by case.

(a) $\zeta_n =: \eta_n^i :$. Using (140) and (141) we have

$$\mathcal{H}_n(z) \le \mathcal{C}_n(z)^i$$

so by Lemma 15 $\|\mathcal{H}_n\|_1 \le \|\mathcal{C}_n\|_i^i < \infty$. In the same way, we get

$$\mathbb{E}(\rho_n'(z) - \rho_n(z))^2 \le C\|\mathcal{C}_n\|_i^{i-1}\|\mathcal{C}_n^{(N)}\|_i \le C\lambda^{\frac{5-i}{i}(N-n)} \le C\lambda^{\frac{2}{3}(N-n)}$$

(b) $\zeta_n =: \eta_n C_n \Gamma_n^{(N)} \eta_n^2 :$. We have

$$\begin{aligned}\mathbb{E} : \eta_n(z_1)\eta_n(z_2)^2 :: \eta_n(z_3)\eta_n(z_4)^2 \\ := 2C_n(z_1,z_3)C_n(z_2,z_4)^2 + 4C_n(z_1,z_4)C_n(z_2,z_3)C_n(z_2,z_4)\end{aligned}$$

and so

$$\begin{aligned}\mathbb{E}\zeta_n(z_1)\zeta_n(z_3) = \int &(2C_n(z_1,z_3)C_n(z_2,z_4)^2 + 4C_n(z_1,z_4)C_n(z_2,z_3)C_n(z_2,z_4)) \\ &\cdot\ S(z_1,z_2)S(z_3,z_4)\mathrm{d}z_2\mathrm{d}z_4 \end{aligned} \tag{147}$$

where

$$S(z_1,z_2) := C_n(z_1,z_2)\Gamma_n(z_1,z_2) \le \mathcal{S}_n(z_{12})$$

where we use the notation $z_{12} = z_1 - z_2$ and $\mathcal{S}_n$ is defined in (124).

Thus,

$$\begin{aligned}0 \le \mathbb{E}\zeta_n(z_1)\zeta_n(z_3) \le 2\mathcal{C}_n(z_{13}) &\int \mathcal{S}_n(z_{12})\mathcal{S}_n(z_{34})\mathcal{C}_n(z_{24})^2\mathrm{d}z_2\mathrm{d}z_4 \\ &+ 4\int \mathcal{S}_n(z_{12})\mathcal{S}_n(z_{34})\mathcal{C}_n(z_{14})\mathcal{C}_n(z_{23})\mathcal{C}_n(z_{24})\mathrm{d}z_2\mathrm{d}z_4 \\ &:= \mathcal{A}(z_{13}) + \mathcal{B}(z_{13})\end{aligned} \tag{148}$$

so that

$$\mathbb{E}\rho_n(z)^2 \le C(\|\mathcal{A}\|_1 + \|\mathcal{B}\|_1). \tag{149}$$

Since $\mathcal{A} = \mathcal{C}_n(\mathcal{S}_n * \mathcal{S}_n * \mathcal{C}_n^2)$ we get by Hölder and Young's inequalities

$$\|\mathcal{A}\|_1 \le \|\mathcal{C}_n\|_p\|\mathcal{S}_n * \mathcal{S}_n * \mathcal{C}_n^2\|_q \le \|\mathcal{C}_n\|_p\|\mathcal{S}_n\|_r^2\|\mathcal{C}_n^2\|_s$$

where $2 + \frac{1}{q} = \frac{2}{r} + \frac{1}{s}$. We can take, e.g., $p = 3$, $q = 3/2$, $r = 1$ and $s = 3/2$ and by Lemma 15 this is finite. As for $\mathcal{B}$, we write

$$\begin{aligned}\|\mathcal{B}\|_1 &= 4\int \mathcal{S}_n(z-z_2)\mathcal{S}_n(z_4)\mathcal{C}_n(z-z_4)\mathcal{C}_n(z_2)\mathcal{C}_n(z_{24})\mathrm{d}z\mathrm{d}z_2\mathrm{d}z_4 \\ &= (\mathcal{S}_n * (\mathcal{C}_n(\mathcal{S}_n * \mathcal{C}_n)) * \mathcal{C}_n)(0) \le \|\mathcal{S}_n\|_r\|\mathcal{C}_n(\mathcal{S}_n * \mathcal{C}_n)\|_p\|\mathcal{C}_n\|_q\end{aligned}$$

by Young's inequality with $2 = \frac{1}{r} + \frac{1}{p} + \frac{1}{q}$. Since $\mathcal{C}_n \in L^a$ for $a < 5$ and $\mathcal{S}_n \in L^b$ for $b < 5/4$ we have $\mathcal{S}_n * \mathcal{C}_n \in L^c$ for $c < \infty$ and so $\mathcal{C}_n(\mathcal{S}_n * \mathcal{C}_n) \in L^p$ for $p < 5$. So, we may take, e.g., $p = q = 2$ and $r = 1$.

In the same way, using (113) and (125) we obtain

$$\mathbb{E}(\rho_n'(z) - \rho_n(z))^2 \le C\lambda^{\gamma(N-n)}\|\chi - \chi'\|_\infty \tag{150}$$

for some $\gamma > 0$.

(c) $\zeta_n =: \eta_n^2 \Gamma_n^N \eta_n^3 :$. Now

$$\mathbb{E} : \eta_n(z_1)^2 \eta_n(z_2)^3 :: \eta_n(z_3)^2 \eta_n(z_4)^3 := 12 C_n(z_1,z_3)^2 C_n(z_2,z_4)^3 + 36 C_n(z_1,z_4)^2 C_n(z_2,z_3)^2 C_n(z_2,z_4) + 36 C_n(z_1,z_3) C_n(z_1,z_4) C_n(z_2,z_3) C_n(z_2,z_4)^2. \tag{151}$$

The first two terms have the same topology (as Feynman diagrams!) as the $\mathcal{A}$ and $\mathcal{B}$ above and we call their contributions with those names again. Thus,

$$\|\mathcal{A}\|_1 \leq \|\mathcal{C}_n^2\|_p \|\Gamma_n^N * \Gamma_n^N * \mathcal{C}_n^3\|_q \leq \|\mathcal{C}_n^2\|_p \|\Gamma_n^N\|_r^2 \|\mathcal{C}_n^3\|_s$$

where $2+\frac{1}{q} = \frac{2}{r}+\frac{1}{s}$. Now Γ_n^N is in L^r for $r < 5/3$. So, we may take for instance $p = 2$, $q = 2$, $s = 4/3$, $r = 8/7$ so that $\|\mathcal{A}\|_1 \leq C\|\mathcal{C}_n\|_4^5 \|\Gamma_n^N\|_{8/7}^2$. For $\mathcal{B}$ we get

$$\|\mathcal{B}\|_1 = 36(\Gamma_n^N * (\mathcal{C}_n(\Gamma_n^N * \mathcal{C}_n^2)) * \mathcal{C}_n^2)(0) \leq 36\|\Gamma_n^N\|_r \|\mathcal{C}_n(\Gamma_n^N * \mathcal{C}_n^2)\|_p \|\mathcal{C}_n^2\|_q$$

with $2 = \frac{1}{r}+\frac{1}{p}+\frac{1}{q}$. Since $\mathcal{C}_n \in L^a$ for $a < 5$ and $\Gamma_n^N \in L^b$ for $b < 5/4$ we have by Young $\Gamma_n^N * \mathcal{C}_n^2 \in L^c$ for $c < 5$ and so $\mathcal{C}_n(\Gamma_n^N * \mathcal{C}_n) \in L^p$ for $p < 5/2$. So we may take, e.g., $p = q = 2$ and $r = 1$.

Finally, the last term in (151), call it $\mathcal{D}$, is bounded by

$$\|\mathcal{D}\|_1 \leq 36 \int \mathcal{C}_n(z)\Gamma_n^N(z-z_2)\Gamma_n^N(z_4)\mathcal{C}_n(z-z_4)\mathcal{C}_n(z_2)\mathcal{C}_n(z_{24})^2 \mathrm{d}z\mathrm{d}z_2\mathrm{d}z_4 := \int f(z_2,z_4)g(z_2,z_4)\mathrm{d}z_2\mathrm{d}z_4$$

where $f(z_2,z_4) = \int \mathcal{C}_n(z)\Gamma_n^N(z-z_2)\mathcal{C}_n(z-z_4)\mathrm{d}z$. We have

$$\int f(z_2,z_4)\mathrm{d}z_2 = (\mathcal{C}_n * \mathcal{C}_n)(z_4) \tag{152}$$

which is in L^∞ since $\mathcal{C}_n \in L^p$, $p < 5$. Hence,

$$\|\mathcal{D}\|_1 \leq C \int g(z_2,z_4)\mathrm{d}z_2\mathrm{d}z_4 = C(\mathcal{C}_n * \mathcal{C}_n^2 * \Gamma_n^N)(0) < \infty$$

since $\Gamma_n^N \in L^1$ and $\mathcal{C}_n * \mathcal{C}_n^2 \in L^\infty$.

Now, we turn to the bi-local fields $\zeta(z_1,z_2)$ and set

$$\mathbb{E}\zeta(z_1,z_2)\zeta(z_3,z_4) := H_n(\mathbf{z})$$

We proceed as in (146)

$$\mathbb{E}\rho_n(z',z)^2 \leq C \int_{(\mathbb{R}\times\mathbb{T}_n)^4} \mathrm{e}^{-\frac{1}{2}(|z-z_1|+|z-z_3|+|z'-z_2|+|z'-z_4|)} H_n(\mathbf{z})\mathrm{d}\mathbf{z}. \tag{153}$$

Let

$$Y := \sup_i \sum_j \sup_{z'\in c_i, z\in c_j} \mathbb{E}\rho_n(z',z)^2 \mathrm{e}^{c|z-z'|}.$$

(130) follows from $Y < \infty$. Let

$$\mathcal{H}_n(\mathbf{z}) := \sup_u H_n(z_1+u, z_2+u, z_3+u, z_4+u)\mathrm{e}^{\frac{1}{2}(|z_1-z_2|+|z_3-z_3|)}$$

We have

$$Y \le C \sup_z \times \int_{(\mathbb{R}\times\mathbb{T}_n)^4} \mathrm{e}^{-\frac{1}{2}(|z-z_1|+|z-z_3|+|z'-z_2|+|z'-z_4|+|z_1-z_2|+|z_3-z_3|)} \mathrm{e}^{c|z-z'|} \mathcal{H}_n(\mathbf{z}) \mathrm{d}\mathbf{z}.$$

The integrand is actually independent on z as $\mathcal{H}_n$ is translation invariant:

$$\mathcal{H}_n(\mathbf{z}) = \tilde{\mathcal{H}}_n(z_{14}, z_{24}, z_{34})$$

with $\tilde{\mathcal{H}}_n(z_1, z_2, z_3) = \mathcal{H}(z_1, z_2, z_3, 0)$. We can then conclude

$$Y \le C\|\tilde{\mathcal{H}}_n\|_1$$

i.e., we need to show for the various bi-local fields that $\|\tilde{\mathcal{H}}_n\|_1 < \infty$. Let us again proceed by cases.

(d) $\zeta_n =: \eta_n C_n \Gamma_n^N \eta_n :$. We have

$$\mathrm{e}^{\frac{1}{2}(|z_{12}|+|z_{34}|)} \mathcal{H}_n(\mathbf{z})) \le \tilde{\mathcal{S}}_n(z_{12}) \tilde{\mathcal{S}}_n(z_{34}) (\mathcal{C}_n(z_{13})\mathcal{C}_n(z_{24}) + \mathcal{C}_n(z_{14})\mathcal{C}_n(z_{23}))$$

where $\tilde{\mathcal{S}}_n(z) = \mathrm{e}^{c|z|}\mathcal{S}_n(z)$ so that

$$\|\tilde{\mathcal{H}}_n\|_1 \le 2\|\tilde{\mathcal{S}}_n * \mathcal{C}_n\|_2^2 < \infty$$

since by Lemma 15, $\tilde{\mathcal{S}}_n$ is in L^p, $p < 5/4$ and $\mathcal{C}_n$ is in L^p, $p < 5$ so that $\tilde{\mathcal{S}}_n * \mathcal{C}_n \in L^p$, $p < \infty$. (131) goes in the same way where at least one of the $\mathcal{C}_n$ or $\mathcal{S}_n$ is replaced by $\mathcal{C}_n^{(N)}$ or $\mathcal{S}_n^{(N)}$.

(e) $\zeta_n =: \eta_n^2 \Gamma_n^N \eta_n^2 :$. We get

$$\begin{aligned}\mathrm{e}^{\frac{1}{2}(|z_{12}|+|z_{34}|)} \mathcal{H}_n(\mathbf{z})) \le{}& 4\tilde{\Gamma}_n^N(z_{12})\tilde{\Gamma}_n^N(z_{23})(\mathcal{C}_n(z_{13})^2\mathcal{C}_n(z_{24})^2 \\ &+ \mathcal{C}_n(z_{14})^2\mathcal{C}_n(z_{23})^2 + \mathcal{C}_n(z_{13})\mathcal{C}_n(z_{24})\mathcal{C}_n(z_{14})\mathcal{C}_n(z_{23}))\end{aligned}$$

The first two terms on the RHS have the same topology as in (d): their contribution to $\|\tilde{\mathcal{H}}_n\|_1$ is bounded by $C\|\tilde{\Gamma}_n^N * \mathcal{C}_n^2\|_2^2$ which is finite since $\tilde{\Gamma}_n^N$ is in L^1 and $\mathcal{C}_n^2$ in L^2.

The third term is treated as the analogous one $\mathcal{D}$ in (c), let us call it $\mathcal{D}$ again. Again its L^1 norm is given by $\|\mathcal{D}\|_1 = \int f(z_2, z_3) g(z_2, z_3) \mathrm{d}z_2 \mathrm{d}z_3$ where f is as above and is in L^∞. $g(z_2, z_3) = \mathcal{C}_n(z_2 - z_3)\mathcal{C}_n(z_2)\Gamma_n^N(z_3)$and thus

$$\|\mathcal{D}\|_1 \le C \int g(z_2, z_4) \mathrm{d}z_2 \mathrm{d}z_4 = C(\mathcal{C}_n * \mathcal{C}_n * \Gamma_n^N)(0) < \infty.$$

Let us finally turn to the δ_n terms in ω_n, ζ_n and z_n. Starting with ω_n, and the term $\zeta_n = \delta_n \Gamma_n^N : \eta_n^3 :$ we have

$$H_n(z_1, z_2) \le \delta_n(t_1)\delta_n(t_2)(\Gamma_n^N * \mathcal{C}_n^3 * \Gamma_n^N)(z_{12}).$$

The function $g_n = \Gamma_n^N * \mathcal{C}_n^3 * \Gamma_n^N$ is in L^p, $p < 5/3$. Letting $f_n(z_i) = \mathrm{e}^{-c|z-z_i|}\delta_n(t_i)$ we see from Lemma 5 that $f_n \in L^p$, $p < 2$ and so

$$\mathbb{E}\rho_n(z)^2 \le C(f_n * g_n * f_n)(0)$$

is finite by Young. To get the bound (131) we use (43) to get for $p < 2$

$$\|f_n' - f_n\|_p \le C\lambda^{\frac{2-p}{2p}(N-n)}\|\chi - \chi'\|_\infty.$$

Consider next the case where δ_n is on the "other side": $\zeta_n =: \eta_n^2 : \Gamma_n^N \delta_n \eta_n$. Replacing C_n by the translation invariant upper bound $\mathcal{C}_n$ we have

$$H_n(z_1, z_2) \leq \int G_n(z_1, z_2, z_3, z_4)\delta_n(t_3)\delta_n(t_4)\mathrm{d}z_3\mathrm{d}z_4$$

with

$$G_n(\mathbf{z}) = \Gamma_n^N(z_{13})\Gamma_n^N(z_{24})(2\mathcal{C}_n(z_{12})^2\mathcal{C}_n(z_{34}) + 4\mathcal{C}_n(z_{12})\mathcal{C}_n(z_{13})\mathcal{C}_n(z_{24}) \\ + 4\mathcal{C}_n(z_{12})\mathcal{C}_n(z_{14})\mathcal{C}_n(z_{23}))$$

At the cost of replacing $\mathcal{C}_n$ and Γ_n^N by $\tilde{\mathcal{C}}_n$ and $\tilde{\Gamma}_n^N$ we may replace δ_n by the f in the upper bound for $\mathbb{E}\rho_n(z)^2$:

$$\mathbb{E}\rho_n(z)^2 \leq \int \tilde{G}_n(z_1, z_2, z_3, z_4) f(z_3) f(z_4)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\mathrm{d}z_4 \\ = \int g_n(z_3 - z_4) f(z_3) f(z_4)\mathrm{d}z_3\mathrm{d}z_4$$

This is bounded if g_n is in L^p, $p > 1$ which is now straightforward.

As the last case consider an example of a bi-local field the second last term in z_n: $\zeta_n = \delta_n \Gamma_n^N : \eta_n^2 :$. By the now familiar steps

$$\mathbb{E}\rho_n(z)^2 \leq \int g(z_1 - z_2) f(z_1) f(z_2)\mathrm{d}z_1\mathrm{d}z_2$$

with $g = \tilde{\Gamma}_n^N * \tilde{\mathcal{C}}^2 * \tilde{\Gamma}_n^N$ in L^p, $p < 5/2$. □

11.4. Renormalization

We are left with the first terms in (142) and (143) that require fixing the renormalization constants m_2 and m_3. Define (product is of kernels as usual)

$$B_n^{(N)} = (C_n^{(N)})^2 \Gamma_n^N \tag{154}$$

Let $\mathcal{B}_\infty$ denote the set of bounded operators $L^\infty(\mathbb{R} \times \mathbb{T}_n) \to L^\infty(\mathbb{R} \times \mathbb{R}^3)$ and $\mathcal{B}_1$ the ones $L^{\infty,1}(\mathbb{R} \times \mathbb{T}_n) \to L^\infty(\mathbb{R} \times \mathbb{R}^3)$ (here $L^{\infty,1}$ has L^∞-norm in t and L^1-norm in x). We prove:

Proposition 18. *There exist constants β_i s.t. the operator*

$$R_n^{(N)} := \tilde{K}(B_n^{(N)} - (\beta_2 \log \lambda^{N-n} + \beta_3)\ id)$$

satisfies

$$\|R_n^{(N)}\|_{\mathcal{B}_i} \leq C, \quad \|R'^{(N)}_n - R_n^{(N)}\|_{\mathcal{B}_i} \leq C\lambda^{(N-n)}\|\chi - \chi'\|_\infty, \quad i = 1, \infty \tag{155}$$

uniformly in $0 \leq n \leq N < \infty$. β_2 universal (i.e., independent on χ).

We fix the renormalization constants $m_i = 18\beta_i$. This means that, e.g., the first term in (142) becomes

$$18R_n^{(N)}\eta_n + (m_2 \log \lambda^n + m_3)\eta_n.$$

The second term once multiplied by λ^{2n} fits into the bound (76).

The $i = \infty$ case of Proposition 18 takes care of the deterministic term in (143) since

$$\|h_{n-m}(B_n^{(N)} - \beta_1 \log \lambda^{N-n} + \beta_2)\phi\|_{\mathcal{V}_n} \le \|R_n^{(N)}\|_{\mathcal{B}_\infty} \|\phi\|_\infty.$$

and $\|\phi\|_\infty \le \|\phi\|_{\Phi_n}$. The $i = 1$ case is needed for the first term in (142). Indeed, we have

$$\mathbb{E}\rho_n^{(N)}(z)^2 = \int (R_n^{(N)}(t,t_1) * R_n^{(N)}(t,t_2) * C_n^{(N)}(t_1,t_2))(0)\mathrm{d}t_1\mathrm{d}t_2$$

where $*$ is spatial convolution. We have

$$\sup_{t_1,t_2} \|C_n^{(N)}(t_1,t_2)\|_1 < \infty$$

so that indeed

$$\mathbb{E}\rho_n^{(N)}(z)^2 \le C\|R_n^{(N)}\|_{\mathcal{B}_\infty} \| \|R_n^{(N)}\|_{\mathcal{B}_1}.$$

Recall also that by (146) a sufficient condition for $\|\tilde{K}B\|_{\mathcal{B}_i}$ to be bounded is

$$\sup_t |B(t+\cdot,t,\cdot)| \in L^1(\mathbb{R} \times \mathbb{T}_n). \tag{156}$$

Proof. Let us first remark that it suffices to work in $\mathbb{R}^3$ instead of $\mathbb{T}_n$. Indeed, recall $B_n^{(N)} = 18(C_n^{(N)})^2 B_n^{(N)}$ where the product is defined as pointwise multiplication of kernels. Let $\tilde{C}_n^{(N)}$ and $\tilde{\Gamma}_n^{(N)}$ be given by (39) and (37) where the $\mathbb{T}_n$ heat kernels H_n are replaced by the $\mathbb{R}^3$ heat kernel H and similarly let

$$\delta B_n^{(N)}(t',t,x) := B_n^{(N)}(t',t,x) - \tilde{B}_n^{(N)}(t',t,x)$$

From (118) and a similar representation for $\Gamma_n^{(N)}$, we infer that $C_n^{(N)} - \tilde{C}_n^{(N)}$ and $\Gamma_n^{(N)} - \tilde{\Gamma}_n^{(N)}$ are in $L^p(\mathbb{R} \times \mathbb{T}_n)$ for all $1 \le p \le \infty$. Proceeding as in Lemma 15 we then conclude $\sup_t |\delta B_n^{(N)}(t+\tau,t,x)|$ is in L^p for $p < 5/4$ and the analog of (125) holds. Hence, (156) holds.

For the rest of this Section, we work in $\mathbb{R} \times \mathbb{R}^3$ and fix the UV cutoff $\lambda^{N-n} := \epsilon$ and denote the operators simply by C and Γ. Also, we set $\chi_\epsilon(s) := \chi(s) - \chi(s/\epsilon^2)$. With these preliminaries, we will start to work towards extracting from the operator $B_n^{(N)}$ (144) the divergent part responsible for the renormalization. First, we will derive a version of the fluctuation–dissipation relation relating C and Γ:

$$\partial_{t'} C(t',t) = -\frac{1}{2}\Gamma(t',t) + A(t',t). \tag{157}$$

where $A(t',t)$ will give a non-singular contribution to B. To derive (157) write (39) in operator form and differentiate in t':

$$\partial_{t'} C(t',t) = \mathrm{e}^{(t'-t)\Delta} \int_0^t (\Delta + \partial_{t'})\mathrm{e}^{2s\Delta} \chi_\epsilon(t'-t+s)\chi_\epsilon(s)\mathrm{d}s$$

Next, write $\Delta \mathrm{e}^{2s\Delta} = \frac{1}{2}\partial_s \mathrm{e}^{2s\Delta}$ and integrate by parts to get

$$\begin{aligned}
&\partial_t C(t',t) \\
&\quad = -\frac{1}{2}\mathrm{e}^{(t'+t)\Delta}\chi_\epsilon(t')\chi_\epsilon(t) + \mathrm{e}^{(t'-t)\Delta}\int_0^t \mathrm{e}^{2s\Delta}(\partial_t - \frac{1}{2}\partial_s)(\chi_\epsilon(t'-t+s)\chi_\epsilon(s))\mathrm{d}s \\
&\quad := a_0(t',t) + \tilde{a}(t',t)
\end{aligned}$$

Now, recall that $\chi_\epsilon(s) = \chi(s) - \chi(\epsilon^{-2}s)$ and write denoting $\tau = t' - t$:

$$\left(\partial_{t'} - \frac{1}{2}\partial_s\right)(\chi_\epsilon(t'-t+s)\chi_\epsilon(s)) = \rho_1(\tau,s) + \rho_2(\tau,s) + \rho_3(\tau,s) \tag{158}$$

with

$$\begin{aligned}
\rho_1(\tau,s) &= \frac{1}{2}(-\chi_\epsilon(\tau+s)\chi'(s) + \chi'(\tau+s)\chi_\epsilon(s)) \\
\rho_2(\tau,s) &= -\frac{1}{2}\chi_\epsilon(s)\partial_s\chi(\epsilon^{-2}(\tau+s)) \\
\rho_3(\tau,s) &= \frac{1}{2}\chi_\epsilon(\tau+s)\partial_s\chi(\epsilon^{-2}s)
\end{aligned}$$

and correspondingly

$$\tilde{a}(t',t) = a_1(t',t) + a_2(t',t) + a_3(t',t).$$

ρ_1 localizes s-integral to $s > \mathcal{O}(1)$ and gives a smooth contribution as $\epsilon \to 0$. $\mathrm{d}\mu(s) := -\partial_s\chi(\epsilon^{-2}s)\mathrm{d}s$ is a probability measure supported on $[\epsilon^2, 2\epsilon^2]$ so ρ_2 localizes s and τ to $\mathcal{O}(\epsilon^2)$. The main term comes from ρ_3

$$\begin{aligned}
a_3(t',t) &= -\frac{1}{2}\mathrm{e}^{\tau\Delta}\int_0^t \mathrm{e}^{2s\Delta}\chi_\epsilon(\tau+s)\mathrm{d}\mu(s) \\
&= -\frac{1}{2}\Gamma(t',t) + a_4(t',t)
\end{aligned} \tag{159}$$

where

$$a_4(t',t) = \frac{1}{2}\mathrm{e}^{\tau\Delta}\int_0^t (\chi_\epsilon(\tau) - \mathrm{e}^{2s\Delta}\chi_\epsilon(\tau+s))\mathrm{d}\mu(s) \tag{160}$$

(157) follows then with

$$A(t',t) = a_0(t',t) + a_1(t',t) + a_2(t',t) + a_4(t',t). \tag{161}$$

Inserting (157) into (144) and using

$$\begin{aligned}
&\int \mathrm{d}y \int_0^t \mathrm{d}s\partial_t C(t,s,x-y)^3\phi(s,y) \\
&\quad = \partial_t \int \mathrm{d}y \int_0^t \mathrm{d}s C(t,s,x-y)^3\phi(s,y) - \int C(t,t,x-y)^3\phi(t,y)\mathrm{d}y
\end{aligned}$$

we obtain

$$(B\phi)(t) = D(t) * \phi(t) + \partial_t(E\phi)(t) + (F\phi)(t) \tag{162}$$

with

$$D(t,x) = 12C(t,t,x)^3$$
$$(E\phi)(t,x) = -12\partial_t \int \mathrm{d}y \int_0^t \mathrm{d}s C(t,s,x-y)^3 \phi(s,y)$$
$$(F\phi)(t,x) = -36 \int \mathrm{d}y \int_0^t \mathrm{d}s A(t,s,x-y) C(t,s,x-y)^2 \phi(s,y).$$

We estimate these three operators in turn.
(a) D. The Fourier transform of D in x is given by

$$\hat{D}(t,p) = \int \hat{C}(t,t,p+k)\hat{C}(t,t,k+q)\hat{C}(t,t,q)dkdq \tag{163}$$

where

$$\hat{C}(t,t,q) = \int_0^t \mathrm{e}^{-sq^2} \chi_\epsilon(s)^2 \mathrm{d}s \tag{164}$$

The integral in (163) diverges at $p = 0$ logarithmically as $\epsilon \to 0$. Doing the gaussian integrals over k, q, we have

$$\hat{D}(t,p) = (4\pi)^{-3} \int_{[0,t]^3} \mathrm{e}^{-\alpha(\boldsymbol{s})p^2} d(\boldsymbol{s})^{-3/2} \prod_{i=1}^3 \chi_\epsilon(s_i)^2 \mathrm{d}s_i$$

where $\alpha(\boldsymbol{s}) := -\frac{s_1 s_2 s_3}{d(\boldsymbol{s})}$ and $d(\boldsymbol{s}) := s_1 s_2 + s_1 s_3 + s_2 s_3$. Let us study the cutoff dependence of $\hat{D}$. First, by differentiating and changing variables

$$\epsilon\partial_\epsilon \hat{D}(t,p) = -\frac{3}{32\pi^3} \int_{[0,t/\epsilon^2]^3} \mathrm{e}^{-\alpha(\boldsymbol{s})(\epsilon p)^2} d(\boldsymbol{s})^{-3/2} s_1 \partial_{s_1} \chi(s_1)^2 \mathrm{d}s_1$$
$$\times \prod_{i=2}^3 (\chi(\epsilon^2 s_i)^2 - \chi(s_i)^2) \mathrm{d}s_i. \tag{165}$$

Let

$$\alpha_\epsilon(t,p)$$
$$:= -\frac{3}{32\pi^3} \int_{[0,t/\epsilon^2]^3} \mathrm{e}^{-\alpha(\boldsymbol{s})(\epsilon p)^2} d(\boldsymbol{s})^{-3/2} s_1 \partial_{s_1} \chi(s_1)^2 \mathrm{d}s_1 \prod_{i=2}^3 (1 - \chi(s_i)^2) \mathrm{d}s_i. \tag{166}$$

and set $\tilde{\alpha}_\epsilon(t,p) = \epsilon\partial_\epsilon \hat{D}(t,p) - \alpha_\epsilon(t,p)$. Since the s_1 integral is supported on $[1,2]$ and the others on $s_i \geq 1$ we have $d(\boldsymbol{s}) \asymp s_2 s_3, \quad \alpha(\boldsymbol{s}) \asymp 1$ which leads to

$$|\tilde{\alpha}_\epsilon(t,p)| \leq C \int_{\mathbb{R}_+^3} (s_2 s_3)^{-3/2} 1_{[1,2]}(s_1) 1_{[\epsilon^{-2},\infty)}(s_2) 1_{[1,\infty)}(s_3) \leq C\epsilon. \tag{167}$$

Furthermore, let $\tilde{\alpha}'_\epsilon(t,p)$ be gotten by replacing the lower cutoffs $\chi(s_i)$ by another one $\chi'(s_i)$. Then,

$$|\tilde{\alpha}'_\epsilon(t,p) - \tilde{\alpha}_\epsilon(t,p)| \leq C\epsilon \|\chi - \chi'\|_\infty. \tag{168}$$

Since $\tilde{\alpha}_0(t,p) = 0$ we get that $\int_0^\epsilon \tilde{\alpha}_{\epsilon'}(t,p)\frac{\mathrm{d}\epsilon'}{\epsilon'}$ satisfies (167) and (168) as well. Thus, all the divergences come from $\alpha_\epsilon(t,p)$. Note that, $\alpha_0(t,p) = \alpha_0$ is independent on t and p. Set $a_\epsilon(t,p) = \alpha_\epsilon(t,p) - \alpha_0$. We get

$$|a_\epsilon(t,p)| \leq C((\epsilon^2 p^2 + \epsilon/\sqrt{t}) \wedge 1) \tag{169}$$

We fix the renormalization constant $\beta_2 = \alpha_0$ and define

$$d(t,p) := \hat{D}(t,p) - \alpha_0 \log \epsilon. \tag{170}$$

Combining above, we get

$$|d(t,p)| = \left| \int_\epsilon^1 (\tilde{\alpha}_{\epsilon'}(t,p) + a_{\epsilon'}(t,p))\frac{\mathrm{d}\epsilon'}{\epsilon'} \right| \leq C(1 + \log(1 + p^2 + 1/\sqrt{t})). \tag{171}$$

Next, we write

$$\begin{aligned}\alpha_\epsilon(t,p) &= -\frac{1}{32\pi^3}\int_{[0,t/\epsilon^2]^3} \mathrm{e}^{-\alpha(\boldsymbol{s})(\epsilon p)^2} d(\boldsymbol{s})^{-3/2} \sum_i s_i \partial_{s_i} \prod_{i=1}^3 (1-\chi(s_i)^2)\mathrm{d}s_i \\ &= -\frac{1}{32\pi^3}\int_0^\infty \mathrm{d}\lambda \int_{[0,t/\lambda\epsilon^2]^3} \delta\left(\sum_{i=1}^3 s_i - 1\right) \mathrm{e}^{-\lambda\alpha(\boldsymbol{s})(\epsilon p)^2} \mathrm{d}(\boldsymbol{s})^{-3/2}\partial_\lambda \\ &\quad \times \prod_{i=1}^3 (1-\chi(\lambda s_i)^2)\mathrm{d}s_i \end{aligned} \tag{172}$$

At $\epsilon = 0$ this implies after an integration by parts

$$\alpha_0 = -\frac{1}{32\pi^3}\int_{\mathbb{R}_+^3} \delta\left(\sum_{i=1}^3 s_i - 1\right) \mathrm{d}(\boldsymbol{s})^{-3/2} \prod_{i=1}^3 \mathrm{d}s_i.$$

That is α_0 is *universal* (i.e., independent of χ). Finally, let us vary the cutoff. Replace χ by $\chi_\sigma = \sigma\chi + (1-\sigma)\chi'$. Since $\partial_\sigma a_\epsilon = \partial_\sigma(a_\epsilon - a_0)$ we get from (172)

$$\begin{aligned}\partial_\sigma \alpha_\epsilon(t,p) &= \int_0^\infty \mathrm{d}\lambda \int_{[0,t/\lambda\epsilon^2]^3} \alpha(\boldsymbol{s})(\epsilon p)^2 \mathrm{e}^{-\lambda\alpha(\boldsymbol{s})(\epsilon p)^2} \mathrm{d}(\boldsymbol{s})^{-3/2} \mathrm{d}\mu(\boldsymbol{s}) \\ &\quad + \sum_{i=1}^3 \int_0^\infty \mathrm{d}\lambda (t/\lambda^2\epsilon^2) \int_{[0,t/\lambda\epsilon^2]^2} \mathrm{e}^{-\lambda\alpha(\boldsymbol{s})(\epsilon p)^2} \mathrm{d}(\boldsymbol{s})^{-3/2} \mathrm{d}\mu(\boldsymbol{s})|_{s_i = t/\lambda\epsilon^2}\end{aligned} \tag{173}$$

where

$$\mathrm{d}\mu(\boldsymbol{s}) = \frac{3}{16\pi^3}\delta\left(\sum_{i=1}^3 s_i - 1\right)(\chi_\sigma(\lambda s_1) - \chi'_\sigma(\lambda s_1))\chi_\sigma(\lambda s_1)\prod_{i=2}^3 (1-\chi_\sigma(\lambda s_i)^2).$$

Start with the first term in (173). Since $\lambda s_i \geq 1$ the λ-integral is supported in $\lambda \geq 3$. On the support of $\chi_\sigma(\lambda s_1) - \chi'_\sigma(\lambda s_1)$ $s_1 \in [\lambda^{-1}, 2\lambda^{-1}]$. By symmetry we may assume $s_2 \leq s_3$ and then $s_3 \geq 1/6$. Hence, in the support of the χ's $\alpha \asymp s_1$ and $\mathrm{d} \asymp s_2$. We get the bound

$$C(\epsilon p)^2 \mathrm{e}^{-(\epsilon p)^2} \int_{1/3}^{\infty} \mathrm{d}\lambda \int_{1/\lambda}^{2/\lambda} \mathrm{d}s_1 \int_{1/\lambda}^{1} \mathrm{d}s_2 s_2^{-3/2} \epsilon \|\chi - \chi'\|_\infty$$
$$\leq C(\epsilon p)^2 \mathrm{e}^{-(\epsilon p)^2} \|\chi - \chi'\|_\infty.$$

For the second term, if $i = 1$ then $s_1 = t/\lambda\epsilon^2 \in [1/\lambda, 2/\lambda]$ implies $t \in [\epsilon^2, 2\epsilon^2]$. Again $\alpha \asymp s_1$, $\mathrm{d} \asymp s_2 \leq s_3$ and we end up with the bound

$$C1(t \in [\epsilon^2, 2\epsilon^2])\|\chi - \chi'\|_\infty.$$

If $i \neq 1$ the same bound results. We may summarize this discussion in

$$|\mathrm{d}(t,p) - \mathrm{d}'(t,p)| \leq C(\epsilon + 1(t \in [\epsilon^2, 2\epsilon^2])\|\chi - \chi'\|_\infty. \tag{174}$$

The operator $K(t', t)\mathrm{d}(t)$ acts as a Fourier multiplier with $\frac{1}{2}\mathrm{e}^{-|t'-t|}(p^2 + 1)^{-2}\mathrm{d}(t,p)$. Since $\mathrm{d}(t,p)$ is analytic in a strip $|\mathrm{Im}\, p| \leq c$ we get in x-space from the above bounds

$$|(K(t',t)d(t))(x)| \leq C\mathrm{e}^{-|t'-t|-c|x|}(1 + \log(1 + t^{-\frac{1}{2}}))$$
$$|(K(t',t)(d'(t) - d(t)))(x)| \leq C\mathrm{e}^{-|t'-t|-c|x|}(\epsilon + 1(t \in [\epsilon^2, 2\epsilon^2]))\|\chi - \chi'\|_\infty.$$

Hence,

$$|(\tilde{K}\mathrm{d}\phi)(t', x)| \leq \int \mathrm{e}^{-\frac{1}{2}|t'-t|-c|x-y|}(1 + \log(1 + t^{-\frac{1}{2}}))\phi(t,y)\mathrm{d}t\mathrm{d}y$$

which is in L^∞ for $\phi \in L^\infty \times L^\infty$ and for $\phi \in L^\infty \times L^1$ as well. This gives the first bound in (155). The second is similar.

(b) E. We have $E\phi = \partial_t C^3\phi$ where $C^3(t,s,x) := C(t,s,x)^3$. Thus integrating by parts

$$Kh_{n-m}E\phi = K\partial_t h_{n-m}C^3\phi + \partial_t K h_{n-m}C^3\phi. \tag{175}$$

By (67) $|\partial_t K(z)| = K(z)$ so we may use (145) for the second term as well to get

$$\|\tilde{K}h_{n-m}E\phi\|_\infty \leq C\|\mathcal{K} * \mathcal{C}^3 * \phi\|_\infty.$$

By Lemma 15 $\mathcal{C}^3$ is in $L^1(\mathbb{R} \times \mathbb{T}_n)$ and hence $\mathcal{K} * \mathcal{C}^3$ is in $L^1(\mathbb{R} \times \mathbb{T}_n)$ and in $L^1(\mathbb{R}) \times L^\infty(\mathbb{T}_n)$ so that the first estimate of (155) follows. The second is similar.

(c) F. By (161), F has four contributions, call them F_0, F_1, F_2, F_4. Start with F_1. Since ρ_1 is supported in $s \geq 1$ the kernel is bounded (in fact smooth)

$$|a_1(t', t, x)| \leq C\mathrm{e}^{-c|x|}.$$

and so by (120) we get

$$|F_1(t+\tau, t, x)| \leq C\mathrm{e}^{-c|x|}(x^2 + \tau)^{-1}.$$

Hence, (156) holds for F_1 uniformly in ϵ and the first bound in (155) follows. For the second one we proceed as in Lemma 15 to get for $f_1(\tau, x) := \sup_t |F_1(t+\tau, t, x) - F_1'(t+\tau, t, x)|$ that $\|f_1\|_1 \leq C\epsilon^\gamma$ for some $\gamma > 0$.

Consider next $F_2 = -36a_2C^2$. We show:

$$F_2(t', t, x) = \epsilon^{-5}p((t'-t)/\epsilon^2, x/\epsilon) + r(t', t, x) \tag{176}$$

where

$$|p(\tau, x)| \le C\mathrm{e}^{-cx^2} 1_{\tau \le 2} \tag{177}$$

and r satisfies (155). To derive (176) we note that by a change of variables

$$a_2(t+\tau, t, x) = -\frac{1}{2}\int_0^{t/\epsilon^2} H(\tau + 2\epsilon^2 s, x)(1-\chi(s))\partial_s\chi(\tau/\epsilon^2 + s)\mathrm{d}s$$

where we also noted since $\partial_s\chi$ is supported on $[1, 2]$ $\chi(\epsilon^2 s) = 1$. Using scaling property of the heat kernel $H(\tau + \epsilon^2 s, x) = \epsilon^{-3}H(\tau/\epsilon^2 + 2s, x/\epsilon)$ we get then

$$a_2(t+\tau, t, x) = \epsilon^{-3}\alpha\left(\frac{\tau}{\epsilon^2}, \frac{t}{\epsilon^2}, \frac{x}{\epsilon}\right)$$

with

$$\alpha(\tau, t, x) = -\frac{1}{2}\int_0^{t} H(\tau + s, x)(1-\chi(s))\partial_s\chi(\tau + s)\mathrm{d}s. \tag{178}$$

Note that, α depends on t only on $t \le 2$ and is bounded by

$$|\alpha(\tau, t, x)| \le C\mathrm{e}^{-cx^2} 1_{\tau\le 2}, \quad |\alpha(\tau, t, x) - \alpha(\tau, \infty, x| \le C\mathrm{e}^{-cx^2} 1_{\tau \le 2}1_{t\le 2}.$$

Comparing two lower cutoffs χ and χ' we get

$$|\alpha(\tau, t, x) - \alpha'(\tau, t, x| \le C\mathrm{e}^{-cx^2} 1_{\tau\le 2}\|\chi - \chi'\|_\infty.$$

By similar manipulations, we obtain

$$C(t+\tau, t, x)^2 = \epsilon^{-2}c\left(\frac{\tau}{\epsilon^2}, \frac{t}{\epsilon^2}, \frac{x}{\epsilon}; \epsilon\right)^2$$

where

$$c(\tau, t, x; \epsilon) = \int_0^t H(\tau + 2s, x)(\chi(\epsilon^2(\tau + s)) - \chi(\tau + s))(\chi(\epsilon^2 s) - \chi(s))\mathrm{d}s.$$

Since $H(\tau + 2s, x) \le C(1 + s)^{-3/2}$ on support of the integrand we get

$$|c(\tau, t, x; \epsilon)| \le C(1 + |x|)^{-1}$$

and

$$|c(\tau, t, x; \epsilon) - c(\tau, \infty, x; 0)| \le C(\epsilon(1 + \epsilon|x|)^{-1} + (1 + |x| + t)^{-\frac{1}{2}})$$

with an extra $\|\chi - \chi'\|_\infty$ factor if we compare two lower cutoffs.

(176) follows with

$$p(\tau, x) = -36c(\tau, x, \infty, 0)^2\alpha(\tau, \infty, x).$$

The error term satisfies

$$|r(t+\tau, t, x)| \le C\epsilon^{-5}(\epsilon + (1 + t/\epsilon^2)^{-\frac{1}{2}} + 1_{t\le 2\epsilon^2})\mathrm{e}^{-cx^2/\epsilon^2} 1_{\tau\le 2\epsilon^2} \tag{179}$$

with an extra $\|\chi - \chi'\|_\infty$ factor if we compare two lower cutoffs. Hence,

$$|(\tilde{K}r\phi)(t', x')| \le \int \mathrm{e}^{-c|t'-\tau-t|+|x'-x|}|r(t+\tau, t, x-y)||\phi(t, y)|\mathrm{d}\tau dx \le C\epsilon\|\phi\|$$

both in the norm $L^\infty \times L^\infty$ and in $L^\infty \times L^1$. This and similar statement with $\|\chi - \chi'\|_\infty$ gives (155).

Let $\tilde{p}$ be the operator with the kernel $\epsilon^{-5}p((t'-t)/\epsilon^2,(x'-x)/\epsilon)-p_0\delta(z'-z)$ where $p_0=\int p(z)\mathrm{d}z$. Then,

$$(Kh_{n-m}\tilde{p})(z+v,z)=\int(K(v-u_\epsilon)h(z+u_\epsilon)-K(v)h_{n-m}(z))p(u)du \quad (180)$$

where $u_\epsilon=(\epsilon^2u_0,\epsilon\boldsymbol{u})$. The bound (177) then implies $\|\tilde{K}\tilde{p}\phi\|_\infty\leq C\epsilon\|\phi\|$ in both norms. A similar statement holds with $\|\chi-\chi'\|_\infty$. p_0 contributes the renormalization constant m_2.

The analysis of $F_4=-36a_4C^2$ parallels that of F_2 so we are brief:

$$F_2(t',t,x)=\epsilon^{-5}q((t'-t)/\epsilon^2,x/\epsilon)+s(t',t,x) \quad (181)$$

where s satisfies (155) and

$$q(\tau,x)=-36c(\tau,x,\infty,0)^2a(\tau,x)$$

with

$$a(\tau,x)=\int(H(\tau,x)(1-\chi(\tau))-H(\tau+2s,x)(1-\chi(\tau+s))\partial_s\chi(s)\mathrm{d}s.$$

Since $\partial_s\chi$ is supported on $[1,2]$ and $1-\chi$ on $\tau\geq 1$ we get $|a(\tau,x)|\leq(1+\tau)^{-5/2}\mathrm{e}^{-cx^2/\tau}$. Since $c(\tau,x,\infty,0)\leq C(1+|x|)^{-1}$ we end up with

$$|q(\tau,x|\leq C(1+\tau)^{-5/2}(1+|x|)^{-2}\mathrm{e}^{-cx^2/\tau}.$$

We may now proceed as in (180).

The analysis of the term F_0 proceeds along similar lines and is omitted. □

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Communicated by Abdelmalek Abdesselam.

Received: October 20, 2014.

Accepted: March 11, 2015.