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Semi-Classical Analysis of Non-Self-Adjoint Transfer Matrices in Statistical Mechanics I

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Abstract. We propose a way to study one-dimensional statistical mechanics models with complex-valued action using transfer operators. The argument consists of two steps. First, the contour of integration is deformed so that the associated transfer operator is a perturbation of a normal one. Then the transfer operator is studied using methods of semi-classical analysis. In this paper, we concentrate on the second step, the main technical result being a semi-classical estimate for powers of an integral operator which is approximately normal.

1. Introduction

Operator-theoretic methods are known to be of great help in one-dimensional statistical mechanics.

Consider the following prototypical example. Let $V : \mathbb{R} \to \mathbb{R}$ be a potential growing sufficiently fast at infinity. It is known that, for any value of $W > 0$, there exists a unique probability measure (Gibbs measure) $\mu_{V,W}$ on the space of configurations in one dimension $\mathbb{R}^{\mathbb{Z}}$ such that, for every $M, N \in \mathbb{N}$, the conditional probability density at $\phi \in \mathbb{R}^{\mathbb{Z}}$ given $\phi|_{\{-M,\cdots,N\}^c}$ is proportional to

$$\exp\left\{-\sum_{j=-M}^{N} V(\phi_j) - \sum_{j=-M-1}^{N} W^2(\phi_j - \phi_{j+1})^2\right\}. \tag{1.1}$$

The existence of $\mu_{V,W}$ is a consequence of general theory, independent of the dimension of the lattice (see [9, Chapter 7]). The uniqueness can be proved using the transfer matrix formalism described below; it also follows for example from the van Hove theorem as stated in the book of Ruelle [9, Section 5.6.6], combined with the Dobrushin–Shlosman theorem [4].

One says that the measure $\mu_{V,W}$ corresponds to the (real-valued) action

$$S(\phi) = \sum_j V(\phi_j) + \sum_j W^2(\phi_j - \phi_{j+1})^2. \tag{1.2}$$

The properties of $\mu_{V,W}$, such as exponential decay of correlations (between ϕ_j and ϕ_k as $|j-k| \to \infty$), are encoded in the spectral structure of the self-adjoint operator K (called the transfer operator, or transfer matrix), acting on $L_2(\mathbb{R})$ as an integral operator with kernel given by

$$K(x,y) = \exp\left\{ -W^2(x-y)^2 - \frac{V(x)+V(y)}{2} \right\}. \tag{1.3}$$

When $W \gg 1$ is large, semi-classical analysis allows to relate the spectral properties of K to those of a simpler operator $\widetilde{K}$, which depends only on the behaviour of V at its minima. If the minima of V are non-degenerate, the potential corresponding to $\widetilde{K}$ is quadratic, and thus $\widetilde{K}$ is referred to as the harmonic approximation to K. In this case, computations can be performed explicitly; the small quantity W^{-1} plays the rôle of the semi-classical parameter $\hbar$.

In the case when V has a unique non-degenerate minimum $V(0) = 0$, $\widetilde{K}$ is given by the harmonic oscillator

$$\widetilde{K}(x,y) = \exp\left\{ -W^2(x-y)^2 - \frac{V''(0)}{4}(x^2+y^2) \right\}. \tag{1.4}$$

When V has several minima, K is approximated by a direct sum of several harmonic oscillators.

The semi-classical approach to various problems of one-dimensional statistical mechanics is presented in detail in the monograph of Helffer [5].

On the other hand, in many questions in statistical mechanics the potential V is complex-valued. This problem, often referred to as the "sign problem" or "complex action problem", is inherent, for example, to lattice quantum chromodynamics (see for example Muroya et al. [8] and Splittorff and Verbaarschot [12]), and also arises in supersymmetric models appearing in the study of random operators (see the reviews of Spencer [10,11]).

A naive attempt to apply the methods tailored for real-valued action to this situation encounters immediate obstacles. In the context of transfer operators, neither K nor $\widetilde{K}$ is self-adjoint; thus perturbation theory is not easily set on a rigorous basis, and on the other hand the spectrum of K is not directly connected to the semigroup $(K^n)_{n\geq 0}$. We refer to the articles of Davies [1,2] and further to the monograph of Helffer [6, Chapters 13–15] and to the PhD thesis of Henry [7], where difficulties in the semi-classical analysis of non-self-adjoint operators are discussed, along with some positive results.

Our goal in this paper is to suggest a strategy which allows to apply semi-classical analysis to models with complex-valued action, in spite of these difficulties. Here we apply it to a toy model (with one saddle); in a subsequent work, we hope to apply it to a statistical mechanics model arising from the supersymmetric analysis of a class of random band matrices; see [3] for an analysis of a related three-dimensional model, and the review of Spencer [10] for a discussion of supersymmetric models arising from random band matrices, and the possible transfer matrix approach.

The strategy we suggest is as follows. Before setting up the transfer operator, we deform the contour of integration so that the harmonic approximation $\widetilde{K}$ is almost normal (in appropriate sense). Then we set up the transfer operator and analyse it using (semi-) classical tools. In this paper, we restrict ourself to the simplest deformation

$$\phi \leftarrow \zeta\phi, \quad |\zeta| = 1; \tag{1.5}$$

in general, a more complicated deformation (similar to (1.5) near each saddle point but different away from the saddles) may be required.

To motivate the idea, let us consider the differential operator

$$L = -\frac{1}{W^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (a+ib)x^2, \quad a > 0.$$

One can always find ζ, $|\zeta| = 1$, so that after the change of variables $x \leftarrow \zeta x$ the operator L becomes normal, i.e. a scalar multiple of

$$\hat{L} = -\frac{1}{W^2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + |a+ib|x^2.$$

In a similar way, for an integral operator K, one can rotate the contour so that the harmonic approximation to K near the saddle point (i.e. an operator with quadratic V) becomes normal.

The main result of this paper justifies the approximation $K \approx \widetilde{K}$ for a class of operators K of a form similar to (1.3) by the corresponding harmonic approximation $\tilde{K}$, in the case when $\tilde{K}$ is almost normal. The precise statement and conditions are given in Sect. 2 below. The proof of this result occupies the central part of this paper, and appears in Sect. 4. It is preceded by Sect. 3, in which several properties of the non-self-adjoint harmonic oscillator are collected.

In Sect. 5, we show an application to a statistical mechanics model with complex-valued potential; namely, we find the sharp exponential decay of correlations for this model.

2. Main Result

2.1. Statement of the Main Technical Result

Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be an operator defined by its kernel

$$K(x,y) = \exp\left\{-W^2\zeta^2(x-y)^2 - \frac{1}{2}U(x) - \frac{1}{2}U(y)\right\}, \tag{2.1}$$

where $\zeta \in \mathbb{C}$ is a complex number with $|\zeta| = 1$ and $\Re\zeta^2 > 0$; $W > 0$ is a large parameter, and the potential $U : \mathbb{R} \to \mathbb{C}$ satisfies the following assumptions:

U1) U is smooth and $U(0) = U'(0) = 0$;

U2) $\zeta^2 U''(0) > 0$; [this condition ensures that the operator is approximately normal near the saddle]

U3) $\Re U(x) \geq \frac{1}{C}\min(1, |x|^2)$ for all $x \in \mathbb{R}$; [in particular, 0 is the unique minimum of $\Re U$]

U4) U has an analytic extension to a strip $|\Im z| \leq c$ about the real axis which satisfies $|U'(z)| \leq C(\max(1, \Re[U(z)]))^\gamma$ for some $\gamma > 1$ and all z in this strip.

We use the analyticity assumption to justify saddle point approximations; the second red part of the condition is a mild regularity assumption which rules out wildly oscillating potentials such as $x^2(1 + \exp(\sin e^x))$.

Main Proposition. *Let K be an operator given by (2.1), where U satisfies the assumptions U1)–U4). Denote*

$$\alpha = W\sqrt{\zeta^2 U''(0)/2}, \quad \mu = \sqrt{\frac{\pi}{W^2\zeta^2 + \alpha}}, \tag{2.2}$$

and

$$g_\alpha(x) = \left(\frac{2\alpha}{\pi}\right)^{1/4} \exp(-\alpha x^2). \tag{2.3}$$

Let

$$K = \sqrt{\frac{\pi}{W^2\zeta^2 + \alpha}} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \mu \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.4}$$

be the block representation of K corresponding to the decomposition $L_2(\mathbb{R}) = \mathbb{C}g_\alpha \oplus (\mathbb{C}g_\alpha)^\perp$; more formally, if $\hat{K} = \mu^{-1}K$, and P is the orthogonal projection to $g_\alpha^\perp$,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (1-P)\hat{K}(1-P) & (1-P)\hat{K}P \\ P\hat{K}(1-P) & P\hat{K}P \end{pmatrix}.$$

Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 + O(W^{-1-\delta}) & O(W^{-1-\delta}) \\ O(W^{-1-\delta}) & \text{of norm} \leq 1 - \sqrt{\frac{|U''(0)|}{2}}\frac{\Re\zeta^2}{W} + O(W^{-1-\delta}) \end{pmatrix},$$

meaning that

$$A = 1 + O(W^{-1-\delta}), \tag{2.5}$$

$$\|B\|, \|C\| = O(W^{-1-\delta}), \tag{2.6}$$

$$\|D\| \leq 1 - \sqrt{\frac{|U''(0)|}{2}}\frac{\Re\zeta^2}{W} + O(W^{-1-\delta}); \tag{2.7}$$

the exponent $\delta > 0$ depends only on γ, and the implicit constants in the O-notation depend on γ and the implicit constants in the assumptions.

Note that by assumption U2) above $\alpha > 0$, hence g_α is a real-valued function. Here and forth, we slightly abuse notation and identify scalar multiples of $1 - P$ with complex numbers.

2.2. The Main Corollary

Let λ_0 be the largest eigenvalue (in absolute value) of K; the existence of λ_0 is part of the statement of Corollary 2.1 below. Let u_0 be the corresponding eigenfunction. Since u_0 is complex valued, we can fix the following normalisation conditions:

$$\|u_0\|^2 = \langle u_0, u_0\rangle = 1, \qquad \text{and} \qquad \langle u_0, g_\alpha\rangle \geq 0, \tag{2.8}$$

where g_α was defined in (2.3) above. The Main Proposition yields the following corollary:

Corollary 2.1. *In the setting of the Main Proposition, K has a largest eigenvalue (in absolute value), which satisfies*

$$\lambda_0 = \sqrt{\frac{\pi}{W^2\zeta^2 + \alpha}}(1 + O(W^{-1-\delta})) = \mu(1 + O(W^{-1-\delta})), \tag{2.9}$$

and the corresponding eigenfunction u_0 (with the normalisation conditions (2.8)) satisfies $\|u_0 - g_\alpha\| \leq CW^{-\delta}$. For any natural n and any u in the invariant subspace $\bar{u}_0^\perp$ of K,

$$\|K^n u\| \leq |\lambda_0|^n \left(1 - \sqrt{\frac{|U''(0)|}{2}}\frac{\Re\zeta^2}{W} + O(W^{-1-\delta})\right)^n \|u\|. \tag{2.10}$$

We remark that (2.9) can be restated as

$$\lambda_0 = \widetilde{\lambda}_0(1 + O(W^{-1-\delta})),$$

where $\widetilde{\lambda}_0$ is the largest eigenvalue of the harmonic approximation

$$\widetilde{K}(x, y) = \exp\left\{-W^2\zeta^2(x-y)^2 - \frac{U''(0)}{4}(x^2 + y^2)\right\}. \tag{2.11}$$

This remark is justified by the formulæ of Sect. 3 below.

Proof of Corollary 2.1. According to the Main Proposition, $\hat{K} = \mu^{-1}K$ (with the normalising factor μ given by (2.2)) has the block structure

$$\hat{K} = \begin{pmatrix} 1 + O(W^{-1-\delta}) & O(W^{-1-\delta}) \\ O(W^{-1-\delta}) & D \end{pmatrix}$$

with respect to the decomposition $L_2(\mathbb{R}) = \mathbb{C}g_\alpha \oplus (\mathbb{C}g_\alpha)^\perp$. Set

$$\hat{K}_t = \begin{pmatrix} A_t & B_t \\ C_t & D \end{pmatrix} = t\hat{K} + (1-t)\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}, \quad 0 \leq t \leq 1.$$

From (2.7), the largest eigenvalue of $\hat{K}_0$ is equal to 1, whereas the rest of the spectrum lies in a disc of radius $1 - \mathrm{const}/W$. Let us show that, as t varies from 0 to 1, the resolvent $R_z[\hat{K}_t] = (\hat{K}_t - z)^{-1}$ of $\hat{K}_t$ remains bounded on a circle $\mathcal{C}$ of radius $O(W^{-1-\delta})$ about 1. This will imply that the spectral projection

$$P_t = \frac{1}{2\pi i}\oint_{\mathcal{C}} R_z[\hat{K}_t]\mathrm{d}z$$

is a continuous function of $t \in [0, 1]$, whence the rank of P_t is identically equal to 1, its value at $t = 0$.

Let $R > 0$ be a large positive number, to be chosen shortly. We shall verify that the operator $D - z$ and the Schur complement

$$S_t(z) = (A_t - z) - B_t(D - z)^{-1}C_t$$

are invertible on the circle $|z - 1| = RW^{-1-\delta}$ for all $t \in [0, 1]$ (actually, the norm of the inverse is bounded uniformly in t). Therefore, we can apply the Schur–Banachiewicz formula for the inverse of a block operator, according to which the resolvent $R_z[\hat{K}_t]$ is given by

$$R_z[\hat{K}_t] = \begin{pmatrix} S_t^{-1} & -S_t^{-1}B_t(D-z)^{-1} \\ -(D-z)^{-1}C_tS_t^{-1} & (D-z)^{-1} + (D-z)^{-1}C_tS_t^{-1}B_t(D-z)^{-1} \end{pmatrix}.$$

One can choose $R > 0$ such that for all z on the circle $|z - 1| = RW^{-1-\delta}$ and all $t \in [0, 1]$ one has $|A_t - z| \geq \frac{R}{2}W^{-1-\delta}$. By (2.5), this choice is independent of W. Hence the Schur complement admits the bound

$$|S_t(z)| \geq \frac{R}{2}W^{-1-\delta} - O(W^{-2-2\delta} \times W) \geq \frac{R}{4}W^{-1-\delta}.$$

Then we have:

$$|S_t(z)^{-1}| \leq \frac{4}{R}W^{1+\delta}.$$

Similarly, the norms of the other three blocks of $R_z[\hat{K}_t]$ are bounded, thus the application of the Schur–Banachiewicz formula is justified, and the norm of the resolvent is bounded for these z, uniformly in $0 \leq t \leq 1$. Hence $\hat{K}_0$ and $\hat{K}_1$ have the same number of eigenvalues in the interior of the circle $|z - 1| \leq RW^{-1-\delta}$, i.e. K has a unique eigenvalue λ_0 satisfying (2.9). The eigenvalue λ_0 is the largest one in absolute value due to the estimate (2.10) which we shall prove shortly.

Next, let u_0 be an eigenfunction corresponding to λ_0, with normalisation conditions (2.8). Then we can decompose u_0 as

$$u_0 = pg_\alpha + u_0^\perp, \quad \text{where} \quad p = \langle u_0, g_\alpha \rangle, \quad \langle u_0^\perp, g_\alpha \rangle = 0;$$

the Main Proposition implies

$$u_1 = \hat{K}u_0 = \left[(1 + O(W^{-1-\delta}))p + O(W^{-1-\delta})\sqrt{1 - |p|^2}\right] g_\alpha + u_1^\perp,$$

where

$$\|u_1^\perp\| \leq O(W^{-1-\delta})|p| + \left(1 - \frac{1}{CW}\right)\sqrt{1 - |p|^2}.$$

On the other hand,

$$u_1 = \hat{\lambda}_0 u_0 = p\hat{\lambda}_0 g_\alpha + \hat{\lambda}_0 u_0^\perp,$$

where $\hat{\lambda}_0 = \mu^{-1}\lambda_0$; comparing the norms of $u_1^\perp$ and $\hat{\lambda}_0 u_0^\perp$, using (2.9) and $p = \langle u_0, g_\alpha \rangle \geq 0$, we obtain:

$$\sqrt{1 - |p|^2} \leq CW^{-\delta}, \quad \text{whence} \quad \|u_0 - g_\alpha\| \leq \|u_0^\perp\| + \|g_\alpha(p - 1)\| \leq C'W^{-\delta}.$$

Finally, let u be a unit vector lying in the invariant subspace $\bar{u}_0^\perp$, of $\hat{K}$. Then

$$|\langle u, g_\alpha\rangle| \le |\langle u, \bar{u}_0\rangle| + |\langle u, \bar{u}_0 - g_\alpha\rangle| \le CW^{-\delta}.$$

Therefore,

$$u = p' g_\alpha + u^\perp, \quad u^\perp \perp g_\alpha, \quad |p'| \le CW^{-\delta}.$$

Denote $q' = \|u^\perp\| = \sqrt{1 - |p'|^2}$. Then

$$\begin{aligned}\|\hat{K}u\|^2 \le &\left\{(1 + O(W^{-1-\delta}))|p'| + O(W^{-1-\delta})|q'|\right\}^2 \\ &+ \left\{O(W^{-1-\delta})|p'| + \left(1 - \frac{c_0}{W} + O(W^{-1-\delta})\right)|q'|\right\}^2,\end{aligned}$$

where $c_0 = \sqrt{\frac{|U''(0)|}{2}\Re\zeta^2}$. Therefore,

$$\begin{aligned}\|\hat{K}u\|^2 &\le \left[1 + O(W^{-1-\delta})\right]|p'|^2 + \left[(1 - \frac{c_0}{W})^2 + O(W^{-1-\delta})\right]|q'|^2 + O(W^{-1-\delta}) \\ &\le (1 - \frac{c_0}{W})^2(|p'|^2 + |q'|^2) + O(W^{-1-\delta}) \\ &\le \left(1 - \frac{c_0}{W} + O(W^{-1-\delta})\right)^2.\end{aligned}$$

Recalling that

$$\langle \hat{K}u, \bar{u}_0\rangle = \langle u, \hat{K}^*\bar{u}_0\rangle = \langle u, \bar{\lambda}_0\bar{u}_0\rangle = 0,$$

we can iterate this estimate, thus obtaining

$$\|\hat{K}^n u\| \le \left(1 - \frac{c_0}{W} + O(W^{-1-\delta})\right)^n$$

for any n, as claimed. □

3. Preliminaries: Harmonic Oscillator

In this section, we collect the properties of the harmonic oscillator, defined by

$$K_{\mathrm{hr}}(x,y) = \exp\left\{-W^2\zeta^2(x-y)^2 - \frac{a+ib}{2}(x^2+y^2)\right\} \tag{3.1}$$

for

$$a > 0, \quad b \in \mathbb{R}, \quad |\zeta| = 1, \quad \Re\zeta^2 > 0. \tag{3.2}$$

The operator K_{hr} defined by (3.1) is compact under the conditions (3.2), hence it has pure point spectrum.

We are especially interested in the case when

$$\zeta^2(a+ib) \in \mathbb{R}_+, \tag{3.3}$$

since this is consistent with assumption U2) for the operator K defined in (2.1); the properties stated here hold however in the generality of (3.2).

The eigenvalues of K_{hr} are given by the formula:

$$\lambda_j^{\mathrm{hr}} = \sqrt{\frac{\pi}{W^2\zeta^2 + \alpha_{\mathrm{hr}} + \frac{a+ib}{2}}}\left(\frac{W^2\zeta^2}{W^2\zeta^2 + \alpha_{\mathrm{hr}} + \frac{a+ib}{2}}\right)^j, \tag{3.4}$$

where $\alpha_{\rm hr}$ is the solution of

$$\alpha_{\rm hr}^2 = W^2\zeta^2(a+ib) + \frac{(a+ib)^2}{4} = \alpha^2 + \frac{(a+ib)^2}{4} = \alpha^2\left[1 + O(W^{-2})\right], \quad (3.5)$$

with $\Re\alpha_{\rm hr} > 0$, and $\alpha^2 = W^2\zeta^2(a+ib)$. This definition is consistent with (2.2). Note that a solution with $\Re\alpha_{\rm hr} > 0$ exists as long as the right-hand side of (3.5) is not real negative (this is ensured by (3.2)). In the special case when both (3.2) and (3.3) hold we have

$$\zeta^2(a+ib) = \Re[\zeta^2(a+ib)] = a/\Re\zeta^2 > 0,$$

hence for large W the right-hand side of (3.5) has positive real part.

The eigenfunction corresponding to $\lambda_0^{\rm hr}$ is exactly the function $g_{\alpha_{\rm hr}}$ given by (2.3) (with α replaced by $\alpha_{\rm hr}$); if $\alpha_{\rm hr}$ is real, the L_2 norm of $g_{\alpha_{\rm hr}}$ is equal to one.

Let us comment on the validity of (3.4) for complex parameters. For any $z \in \mathbb{C}$, the Fredholm determinant of the operator $zK_{\rm hr}$ is equal to

$$\det(1 - zK_{\rm hr}) = \prod_{j=0}^{\infty}(1 - z\lambda_j^{\rm hr}). \quad (3.6)$$

Also, $K_{\rm hr}$ is an analytic function of the parameters ζ and $a+ib$, therefore, by Vitali's theorem, this remains true also for the left-hand side of (3.6), since it is the limit of a sequence of analytic functions converging locally uniformly with respect to ζ and $a+ib$. On the other hand, for real ζ and $a+ib$, the identity (3.4) is well known (see [5, Section 5.2]); thus the right-hand side of (3.6) is equal to

$$\prod_{j=0}^{\infty}\left(1 - z\sqrt{\frac{\pi}{W^2\zeta^2 + \alpha_{\rm hr} + \frac{a+ib}{2}}}\left(\frac{W^2\zeta^2}{W^2\zeta^2 + \alpha_{\rm hr} + \frac{a+ib}{2}}\right)^j\right),$$

which is also an analytic function of ζ and $a+ib$. By analytic continuation, we have:

$$\prod_{j=0}^{\infty}(1 - z\lambda_j^{\rm hr}) = \prod_{j=0}^{\infty}\left(1 - z\sqrt{\frac{\pi}{W^2\zeta^2 + \alpha_{\rm hr} + \frac{a+ib}{2}}}\left(\frac{W^2\zeta^2}{W^2\zeta^2 + \alpha_{\rm hr} + \frac{a+ib}{2}}\right)^j\right)$$

in the full range of parameters (3.2), and this implies (3.4).
Now we turn to $K_{\rm hr}^* K_{\rm hr}$. Set $A = 2W^2\Re\zeta^2 + a$. One may check that

$$\begin{aligned}(K_{\rm hr}^* K_{\rm hr})(x,y) &\\ = \sqrt{\tfrac{\pi}{A}}\exp\Bigg\{&-\frac{W^4 + W^2(a\bar{\zeta}^2 + (a-ib)\Re\zeta^2) + \frac{a}{2}(a-ib)}{A}x^2 \\ &-\frac{W^4 + W^2(a\zeta^2 + (a+ib)\Re\zeta^2) + \frac{a}{2}(a+ib)}{A}y^2 + \frac{2W^4}{A}xy\Bigg\}\end{aligned}$$

$$= \sqrt{\tfrac{\pi}{A}} \exp\left\{ - \frac{\bar{\alpha}_{\rm hr}^2 + \Re[W^2\zeta^2(a-ib)] + \frac{1}{4}(a^2+b^2)}{A} x^2 \right.$$
$$\left. - \frac{\alpha_{\rm hr}^2 + \Re[W^2\zeta^2(a-ib)] + \frac{1}{4}(a^2+b^2)}{A} y^2 - \frac{W^4}{A}(x-y)^2 \right\}. \quad (3.7)$$

In particular, $K_{\rm hr}$ is normal ($K_{\rm hr}^* K_{\rm hr} = K_{\rm hr} K_{\rm hr}^*$) if and only if $\alpha_{\rm hr}^2$ is real (which happens if and only if $\alpha_{\rm hr} > 0$). More generally, two operators of the form (3.1) commute if and only if they share the same $\alpha_{\rm hr}$.

From (3.7), $K_{\rm hr}^* K_{\rm hr}$ is similar (conjugate) to the operator

$$T_{\rm hr}(x,y) = e^{-i\frac{\Im \alpha_{\rm hr}^2}{A}x^2}(K_{\rm hr}^* K_{\rm hr})(x,y) e^{+i\frac{\Im\alpha_{\rm hr}^2}{A}y^2}$$
$$= \sqrt{\tfrac{\pi}{A}} \exp\left\{ -\frac{W^4}{A}(x-y)^2 - 2a\left(1 - \frac{a}{2A}\right)\frac{x^2+y^2}{2}\right\}$$

of the form (3.1). This allows to compute the singular values

$$s_0^{\rm hr} \geq s_1^{\rm hr} \geq \cdots$$

of K_{hr}:

$$\left(s_j^{\rm hr}\right)^2 = \sqrt{\frac{\pi^2}{W^4 + 2aA + \sqrt{[2W^4 + aA]aA}}}\left(\frac{W^4}{W^4 + 2aA + \sqrt{[2W^4 + aA]aA}}\right)^j.$$

If $\alpha_{\rm hr} > 0$, we have $s_j^{\rm hr} = |\lambda_j^{\rm hr}|$ for any j. If instead we require $\alpha > 0$, then $\alpha_{\rm hr}$ is real up to an error term of order $O(W^{-2})$. The corresponding operator is almost normal. More precisely, we have the following result.

Lemma 3.1. *If K_{hr} is an operator of the form (3.1) with real (positive) $\zeta^2(a+ib)$, then for any fixed j*

$$\frac{s_j^{\rm hr}}{|\lambda_j^{\rm hr}|} = 1 + O(W^{-2}),$$

where the implicit constant may depend on j. Moreover, for any $0 < \epsilon < 1$,

$$\|K_{\rm hr} g_\alpha - \mu g_\alpha\| = |\mu| O(W^{-2+2\epsilon}) \qquad and \qquad \|\tilde{g} - g_\alpha\| \leq O(W^{-2}),$$

where $\tilde{g}$ is the top normalised eigenfunction for $K_{\rm hr}^* K_{\rm hr}$, and g_α and μ are given by (2.3) and (2.2), respectively.

Proof. By the formulæ for $\lambda_j^{\rm hr}$ and $s_j^{\rm hr}$ given above and using $\zeta^2(a+ib) > 0$

$$\frac{|\lambda_0^{\rm hr}|^2}{\left(s_0^{\rm hr}\right)^2} = \sqrt{\frac{W^4 + 2W^3\sqrt{a\Re\zeta^2} + O(W^2)}{W^4 + 2W^3\Re\zeta^2\sqrt{\zeta^2(a+ib)} + O(W^2)}}.$$

Using the constraints $\Re\zeta^2 > 0$, $|\zeta^2| = 1$ and $\zeta^2(a+ib) > 0$ we see that $\zeta^2(a+ib)\Re\zeta^2 = a$, whence $|\lambda_0^{\rm hr}|^2/\left(s_0^{\rm hr}\right)^2 = (1 + O(W^{-2}))$. The same proof applies to the case $j > 0$. To prove the second part, we see that

$$(K_{\rm hr} g_\alpha)(x) = \mu g_\alpha(x)\, c(\alpha) e^{-x^2 d(\alpha)},$$

where

$$c(\alpha) = \frac{1}{\sqrt{1 + \frac{a+ib}{2(\alpha+W^2\zeta^2)}}} = 1 + O(W^{-2}), \quad d(\alpha) = \frac{\frac{a+ib}{2}}{1 + \frac{2[\alpha+W^2\zeta^2]}{a+ib}} = O(W^{-2}).$$

Then, using the exponential decay of g_α,

$$\|K_{\mathrm{hr}} g_\alpha - \mu g_\alpha\| \le |\mu||c(\alpha) - 1| + |\mu c(\alpha)| \left\| g_\alpha [e^{-x^2 d(\alpha)} - 1] \mathbf{1}_{|x| \le W^\epsilon} \right\| + |\mu c(\alpha)| \left\| g_\alpha [e^{-x^2 d(\alpha)} - 1] \mathbf{1}_{|x| > W^\epsilon} \right\| = |\mu|\, O(W^{-2+2\epsilon}).$$

Finally, to prove the last inequality, we remark that

$$\tilde{g}(x) = \left(\tfrac{2\alpha_T}{\pi}\right)^{1/4} e^{i \frac{\Im \alpha_{\mathrm{hr}}^2}{A} x^2} e^{-\alpha_T x^2},$$

where $\exp[-\alpha_T x^2]$ is a top eigenfunction for T_h, and α_T is the real-positive solution of

$$\alpha_T^2 = \frac{W^4}{A} 2a \left(1 - \frac{a}{2A}\right) + a^2 \left(1 - \frac{a}{2A}\right)^2 = \alpha^2 [1 + O(W^{-2})].$$

By assumption, $\Im \alpha_{\mathrm{hr}}^2 = O(1)$ and $\alpha_T = \alpha(1 + O(1/W^2)) = \alpha + O(1/W)$, therefore,

$$\tilde{g}(x) = (1 + O(W^{-2})) e^{O(W^{-1})x^2} g_\alpha(x).$$

Hence

$$\|\tilde{g} - g_\alpha\|^2 \le \int g_\alpha(x)^2 \left|1 - e^{O(W^{-1})x^2}\right|^2 \mathrm{d}x + O(W^{-4}) \le O(W^{-2}) \int g_\alpha(x)^2 x^4 e^{O(W^{-1})x^2} \mathrm{d}x + O(W^{-4}) = O(W^{-4}),$$

where in the last line we applied $|1 - e^x| \le |x| e^x$. □

4. Proof of the Main Proposition

Similarly to the semi-classical arguments in the self-adjoint case (see [5, (5.6.1)]), we separate the contribution of the vicinity of the saddle point and the rest of the real line as follows. Let $T(x, y)$ be a kernel, and suppose $\chi_1^2 + \chi_2^2 = 1$ is a partition of unity. Then

$$T(x, y) = \sum_{j=1}^{2} \chi_j(x) T(x, y) \chi_j(y) + \sum_{j=1}^{2} R_j(x, y), \tag{4.1}$$

where

$$R_j(x, y) = \frac{1}{2} (\chi_j(x) - \chi_j(y))^2 T(x, y).$$

In operator notation,

$$T = \sum_{j=1}^{2} \chi_j T \chi_j + \sum_{j=1}^{2} R_j.$$

Another ingredient is Schur's bound (see [5, Lemma 4.4.1] for a proof)

$$\|T\| \leq \sqrt{\sup_x \int \mathrm{d}y |T(x,y)|} \sqrt{\sup_y \int \mathrm{d}x |T(x,y)|}, \tag{4.2}$$

which, in the case when $|T(x,y)| = |T(y,x)|$, assumes the form

$$\|T\| \leq \sup_x \int \mathrm{d}y\, |T(x,y)|.$$

The difference from the usual setting stems from the fact that K is not self-adjoint. This is why we work with the self-adjoint operator K^*K, and our main effort will be invested in decent bounds on the kernel.

The Main Proposition will follow from the next three lemmata, which are applied to estimate the four blocks A, B, C, D of (2.4). We shall compare our operator K with its harmonic approximation $\widetilde{K}$ introduced in (2.11), which is approximately normal due to assumption U2) of Sect. 2.1 and Lemma 3.1.

Lemma 4.1. *Let K be an operator given by (2.1), so that U satisfies the assumptions U1) and U3). If $\alpha > 0$ is such that*

$$\left| \alpha^2 - W^2 \zeta^2 \frac{U''(0)}{2} \right| \leq C W^{3/2}, \tag{4.3}$$

then the asymptotics of the integral

$$I(\alpha) = \iint \mathrm{d}x \mathrm{d}y \exp\left\{ -W^2\zeta^2(x-y)^2 - \frac{1}{2}U(x) - \alpha x^2 - \frac{1}{2}U(y) - \alpha y^2 \right\}$$

is given by

$$I(\alpha) = (1 + O(W^{-3/2+\epsilon})) \sqrt{\frac{\pi}{W^2\zeta^2 + \alpha}} \sqrt{\frac{\pi}{2\alpha}},$$

for any $\epsilon > 0$.

Remark 4.2. Although the bound is valid for any $\alpha > 0$ satisfying (4.3), we shall only apply it to $\alpha = W\sqrt{\zeta^2 U''(0)/2}$ of (2.2).

Lemma 4.3. *In the setting of the Main Proposition,*

$$\|(K - \widetilde{K}) g_\alpha\| = O(W^{-3/2+\epsilon} |\mu|)$$

for any $\epsilon > 0$, where μ, g_α and $\widetilde{K}$ were introduced in (2.2) (2.3) and (2.11).

Lemma 4.4. *In the setting of the Main Proposition, let $u \in L_2$ be a function of unit norm. Then there exists $\delta > 0$ so that*

$$\|Ku\| \leq |\mu| \left(1 + O(W^{-1-\delta})\right).$$

Moreover, if $u \perp g_\alpha$, then

$$\|Ku\| \leq |\mu| \left(1 - \sqrt{\frac{|U''(0)|}{2}} \frac{\Re \zeta^2}{W} + O(W^{-1-\delta}) \right).$$

The same estimates hold for $\|K^*u\|$.

Proof of Main Proposition. The estimate on A follows from Lemma 4.1:

$$\mu A = \langle K g_\alpha, g_\alpha \rangle = \sqrt{\frac{2\alpha}{\pi}} I(\alpha) = (1 + O(W^{-3/2+\epsilon}))\mu. \tag{4.4}$$

The estimate on B and C follows from Lemmas 3.1 and 4.3

$$\begin{aligned}\|\mu C\| = \|PKg_\alpha\| &= \inf_{w \in \mathbb{C}} \|(K - w)g_\alpha\| \\ &\leq \|(K - \widetilde{K})g_\alpha\| + \|\widetilde{K}g_\alpha - \mu g_\alpha\| = O(W^{-3/2+\epsilon}|\mu|).\end{aligned}$$

In a similar way $\|\mu B\| = \|PK^* g_\alpha\| \leq \|(K^* - \bar{\mu})g_\alpha\| = \|(K - \mu)g_\alpha\|$ since g_α is real. Therefore, the arguments for C apply. Finally, the bound on $\|D\|$ follows from the second statement of Lemma 4.4, since

$$\|PKP\| \leq \sup_{u \perp g_\alpha, \|u\|=1} \|Ku\|.$$

□

Now we turn to the proofs of the lemmata.

Proof of Lemma 4.1. Changing variables

$$y \leftarrow \frac{y + x}{\sqrt{2}}, \quad x \leftarrow \frac{y - x}{\sqrt{2}},$$

we obtain:

$$I(\alpha) = \iint \mathrm{d}x\mathrm{d}y \exp\left\{-(2W^2\zeta^2 + \alpha)x^2 - \alpha y^2 - \frac{1}{2}U(\frac{y+x}{\sqrt{2}}) - \frac{1}{2}U(\frac{y-x}{\sqrt{2}})\right\}.$$

The integration over the complement of the rectangle defined by the inequalities $|x| \leq W^{-1+\epsilon/3}$, $|y| \leq W^{-1/2+\epsilon/3}$ is exponentially suppressed according to U3), where we took into account that α is of order W. Inside the rectangle, we expand about $y/\sqrt{2}$ (since x is typically smaller in absolute value):

$$U\left(\frac{y \pm x}{\sqrt{2}}\right) = U\left(\frac{y}{\sqrt{2}}\right) \pm U'\left(\frac{y}{\sqrt{2}}\right)\frac{x}{\sqrt{2}} + O(x^2),$$

whence by U1)

$$\frac{1}{2}\left(U\left(\frac{y+x}{\sqrt{2}}\right) + U\left(\frac{y-x}{\sqrt{2}}\right)\right) = U\left(\frac{y}{\sqrt{2}}\right) + O(x^2) = \frac{U''(0)}{4}y^2 + O(x^2 + |y|^3).$$

Therefore,

$$I(\alpha) = (1 + O(W^{-3/2+\epsilon}))\sqrt{\frac{\pi}{2W^2\zeta^2 + \alpha}}\sqrt{\frac{\pi}{\alpha + \frac{U''(0)}{4}}}.$$

We have:

$$(2W^2\zeta^2 + \alpha)\left(\alpha + \frac{U''(0)}{4}\right) = 2W^2\zeta^2\alpha + \alpha^2 + W^2\zeta^2\frac{U''(0)}{2} + O(W),$$

whereas

$$2\alpha(W^2\zeta^2 + \alpha) = 2W^2\zeta^2\alpha + 2\alpha^2.$$

Under the assumption (4.3) on α, the two expressions differ by $O(W^{3/2})$. □

Proof of Lemma 4.3. We start with the identity

$$\|(K - \widetilde{K})g_\alpha\|^2 = \int_{-\infty}^{\infty} \mathrm{d}r \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \, (K^* - \widetilde{K}^*)(x, r)(K - \widetilde{K})(r, y) g_\alpha(x) g_\alpha(y).$$

This time the integral over the complement of the polytope

$$|x - r|, |r - y| \le W^{-1+\epsilon/3}, \ |x|, |r| \le W^{-1/2+\epsilon/3}$$

is exponentially suppressed, whereas inside the polytope

$$|(K^* - \widetilde{K}^*)(x, r)| \le CW^{-3/2+\epsilon} |\widetilde{K}^*(x, r)|$$

and

$$|(K - \widetilde{K})(r, y)| \le CW^{-3/2+\epsilon} |\widetilde{K}(r, y)|;$$

the statement follows from these inequalities. □

To prove Lemma 4.4, we need several estimates on the kernel of $K^* K$, which are collected in the next lemma. We shall apply the first estimate when $|x|, |y| \le W^{-1/2+\delta}$, the second one when either $\Re U \ge W^\eta$ or $|x - y| \ge W^{-1+\eta}$ (for a small $\eta > 0$ to be chosen later), and the third one in the remaining range of parameters.

Lemma 4.5. *The kernel $(K^* K)(x, y)$ satisfies the following estimates.*

1. *For $|x|, |y| \le c_0$ (where $c_0 > 0$ may depend on ζ and on the width of the strip in which U is analytic),*
$$(K^* K)(x, y) = \left[1 + O(|x|^3 + |y|^3 + W^{-3+\epsilon})\right](\widetilde{K}^* \widetilde{K})(x, y),$$
where $\epsilon > 0$ is an arbitrary positive number, and $\widetilde{K}$ was defined in (2.11).
2. *For any x, y,*
$$|(K^* K)(x, y)| \le \sqrt{\frac{\pi}{2W^2 \Re \zeta^2}} \exp\left\{ -\frac{\Re U(x) + \Re U(y)}{2} - \frac{W^2}{2} \Re \zeta^2 (x - y)^2 \right\}.$$
3. *Let γ be the parameter appearing in U4). If $|x - y| \le W^{-1+\eta}$, $\Re U(x) \le W^\eta$ and $\Re U(y) \le W^\eta$, where $\eta > 0$ is sufficiently small, then we have*
$$(K^* K)(x, y) = (1 + O(W^{-1+5\eta\gamma})) \sqrt{\frac{\pi}{2W^2 \Re \zeta^2}} \times \exp\left\{ -\frac{\bar{U}(x) + U(y)}{2} - \Re U\left(\frac{x + y}{2}\right) - \frac{W^2}{2\Re \zeta^2}(x - y)^2 \right\}.$$

We postpone the proof of Lemma 4.5 and start with

Proof of Lemma 4.4. Let $\delta > 0$ be a small number. Construct a partition of unity $\chi_1^2 + \chi_2^2 = 1$. We choose

$$\chi_1, \chi_2 : \mathbb{R} \to \mathbb{R}_+$$

such that

1. χ_1 is supported on $[-W^{-1/2+\delta}, W^{-1/2+\delta}]$ and is identically equal to one in $[-\frac{1}{2} W^{-1/2+\delta}, \frac{1}{2} W^{-1/2+\delta}]$;

2. χ_2 is supported outside $[-\frac{1}{2}W^{-1/2+\delta}, \frac{1}{2}W^{-1/2+\delta}]$ and is identically equal to one outside $[-W^{-1/2+\delta}, W^{-1/2+\delta}]$;
3. the two functions are differentiable, and $|\chi_1'|, |\chi_2'| \leq CW^{1/2-\delta}$.

Also denote $\mathbb{1}_1 = \mathbb{1}_{\chi_1>0}$, $\mathbb{1}_2 = \mathbb{1}_{\chi_2>0}$; then $\chi_1\mathbb{1}_1 = \chi_1$ and $\chi_2\mathbb{1}_2 = \chi_2$. According to the decomposition (4.1),

$$\|Ku\|^2 = \langle K^*Ku, u\rangle = \sum_{j=1}^{2}\langle \chi_j K^*K\chi_j u, u\rangle + \sum_{j=1}^{2}\langle R_j u, u\rangle. \tag{4.5}$$

Let $\widetilde{s}_0 \geq \widetilde{s}_1 \geq \cdots$ be the singular values of $\widetilde{K}$, and let $\widetilde{g}$ be the top eigenfunction of $\widetilde{K}^*\widetilde{K}$. From the properties of the harmonic oscillator collected in Lemma 3.1,

$$\widetilde{s}_0 = |\mu|(1 + O(W^{-2})), \quad \frac{\widetilde{s}_1}{\widetilde{s}_0} = 1 - \sqrt{\frac{|U''(0)|}{2}}\frac{\Re\zeta^2}{W} + O(W^{-2}). \tag{4.6}$$

We shall prove the following estimates:

$$\langle \chi_1 K^*K\chi_1 u, u\rangle \leq \widetilde{s}_1^2\|\chi_1 u\|^2 + C\widetilde{s}_0^2 W^{-3/2+3\delta}\|u\|^2 \quad (u \perp g_\alpha), \tag{4.7}$$

$$\langle \chi_1 K^*K\chi_1 u, u\rangle \leq \widetilde{s}_0^2(1 + CW^{-3/2+3\delta})\|\chi_1 u\|^2, \tag{4.8}$$

$$|\langle \chi_2 K^*K\chi_2 u, u\rangle| \leq \widetilde{s}_0^2(1 - \frac{1}{C}W^{-1+2\delta})\|\chi_2 u\|^2, \tag{4.9}$$

$$|\langle R_j u, u\rangle| \leq CW^{-1-2\delta}\widetilde{s}_0^2\|u\|^2. \tag{4.10}$$

Once these bounds are established, the proof of the lemma is concluded as follows. For $u \perp g_\alpha$, we use (4.7), (4.9), (4.10) to estimate the addends in (4.5); then from the inequality

$$\widetilde{s}_0^2\left(1 - \frac{1}{CW^{1-2\delta}}\right) \leq \widetilde{s}_0^2(1 - 2\sqrt{\tfrac{|U''(0)|}{2}}\tfrac{\Re\zeta^2}{W} + O(W^{-2})) = \widetilde{s}_1^2$$

and the identity

$$\|\chi_1 u\|^2 + \|\chi_2 u\|^2 = \|u\|^2$$

we obtain:

$$|\langle K^*Ku, u\rangle| \leq (1 + O(W^{-1-2\delta}))\widetilde{s}_1^2\|u\|^2.$$

For arbitrary u we apply (4.8) in place of (4.7), and obtain:

$$|\langle K^*Ku, u\rangle| \leq (1 + O(W^{-1-2\delta}))\widetilde{s}_0^2\|u\|^2.$$

Inserting now (4.6), the proof is concluded.

Proof of (4.7) starts from

$$\begin{aligned}&\langle \chi_1 K^*K\chi_1 u, u\rangle \\ &\quad = \langle \widetilde{K}^*\widetilde{K}\chi_1 u, \chi_1 u\rangle + \langle(\mathbb{1}_1 K^*K\mathbb{1}_1 - \mathbb{1}_1\widetilde{K}^*\widetilde{K}\mathbb{1}_1)\chi_1 u, \chi_1 u\rangle.\end{aligned} \tag{4.11}$$

From the decomposition

$$\chi_1 u = \langle \chi_1 u, \widetilde{g}\rangle\widetilde{g} + \{\chi_1 u - \langle \chi_1 u, \widetilde{g}\rangle\widetilde{g}\}$$

we obtain

$$\langle \widetilde{K}^*\widetilde{K}\chi_1 u, \chi_1 u\rangle \leq \widetilde{s}_0^2|\langle \chi_1 u, \widetilde{g}\rangle|^2 + \widetilde{s}_1^2\|\chi_1 u\|^2.$$

If $u \perp g_\alpha$,

$$\begin{aligned}|\langle \chi_1 u, \widetilde{g}\rangle| &\le \|\chi_1 u\| \|\widetilde{g} - g_\alpha\| + |\langle u - \chi_1 u, g_\alpha\rangle| \\ &\le \|u\| \|\widetilde{g} - g_\alpha\| + \|u\| \|(1-\chi_1) g_\alpha\| \le C_1 W^{-2} \|u\|,\end{aligned}$$

where in the first term of the sum we applied Lemma 3.1. Finally

$$\|\mathbb{1}_1 K^* K \mathbb{1}_1 - \mathbb{1}_1 \widetilde{K}^* \widetilde{K} \mathbb{1}_1\| \le C_2 W^{-3/2+3\delta} \widetilde{s}_0^2 \tag{4.12}$$

according to item 1. of Lemma 4.5 and Schur's bound (4.2). Hence

$$\begin{aligned}\langle \chi_1 K^* K \chi_1 u, u\rangle &\le C_1^2 W^{-4} \|u\|^2 + \widetilde{s}_1^2 \|\chi_1 u\|^2 + C_2 \widetilde{s}_0^2 W^{-3/2+3\delta} \|u\|^2 & (4.13)\\ &\le \widetilde{s}_1^2 \|\chi_1 u\|^2 + C \widetilde{s}_0^2 W^{-3/2+3\delta} \|u\|^2. & (4.14)\end{aligned}$$

Proof of (4.8) also starts from (4.11). From

$$\langle \chi_1 \widetilde{K}^* \widetilde{K} \chi_1 u, u\rangle \le \widetilde{s}_0^2 \|\chi_1 u\|^2$$

and (4.12), we obtain the bound

$$\langle \chi_1 K^* K \chi_1 u, u\rangle \le \widetilde{s}_0^2 (1 + C W^{-3/2+3\delta}) \|\chi_1 u\|^2. \tag{4.15}$$

Proof of (4.9) We plug into Schur's bound (4.2) the estimates on the kernel of $(K^*K)(x,y)$ obtained in Lemma 4.5, as follows. Set $\eta = \delta/(5\gamma)$. We use the estimate given in item 2 when either $|x-y| > W^{-1+\eta}$, or $\Re U(x) > W^\eta$, or $\Re U(y) > W^\eta$. In the complementary region

$$|x-y| \le W^{-1+\eta} \quad \wedge \quad \Re U(x) \le W^\eta \quad \wedge \quad \Re U(y) \le W^\eta$$

we use the estimate given in item 3. Then

$$\begin{aligned}\|\mathbb{1}_2 K^* K \mathbb{1}_2\| &\le \tfrac{\pi}{W^2} e^{-\frac{W^{-1+2\delta}}{C}} (1 + O(W^{-1+5\eta\gamma})) + e^{-C_1 W^\eta} \\ &\le \widetilde{s}_0^2 \left(1 - \frac{1}{C} W^{-1+2\delta}\right),\end{aligned}$$

where in the first term we used $\Re U(x) \ge \frac{|x|^2}{C} \ge \frac{W^{-1+2\delta}}{C}$ (from U3) and the definition of χ_2) and the relations between $|\mu| = \frac{\sqrt{\pi}}{W}(1+O(W^{-1}))$ and $\tilde{s}_0$ given in (4.6). The second term comes from the estimate in item 2 of Lemma 4.5. Hence

$$|\langle \chi_2 K^* K \chi_2 u, u\rangle| \le \widetilde{s}_0^2 \left(1 - \frac{1}{C} W^{-1+2\delta}\right) \|u\|^2. \tag{4.16}$$

Proof of (4.10) is similar: Schur's bound, item 2 of Lemma 4.5 and the bounds

$$|\chi_j(x) - \chi_j(y)| \le C W^{1/2-\delta} |x-y|$$

are used to show that

$$\|R_j\| \le C W^{-1-2\delta} \widetilde{s}_0^2.$$

□

Proof of Lemma 4.5. First,

$$(K^*K)(x,y) = E(x,y)I(x,y),$$

where

$$E(x,y) = \exp\left\{-\frac{\bar{U}(x)+U(y)}{2} - W^2\bar{\zeta}^2x^2 - W^2\zeta^2y^2\right\},$$

and

$$I(x,y) = \int \mathrm{d}r \exp\left\{-2W^2\left[\Re\zeta^2r^2 - (\bar{\zeta}^2x+\zeta^2y)r\right] - \Re U(r)\right\}.$$

On the real line, $\Re U(z)$ coincides with the analytic function

$$U_{\mathrm{Schw}}(z) = (U(z) + \overline{U(\bar{z})})/2, \tag{4.17}$$

therefore, we replace $\Re U$ with U_{Schw}.

To prove the first item of the lemma, let $|x|, |y| < c_0$ for a small constant c_0, and let $\widetilde{E}$ and $\widetilde{I}$ be expressions analogous to E and I which correspond to $\widetilde{K}$;

$$\widetilde{I}(x,y) = \sqrt{\frac{\pi}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}}}\exp\left\{\frac{W^4(\bar{\zeta}^2x+\zeta^2y)^2}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}}\right\}.$$

Let $\xi \geq 2|x| + 2|y|$ be a small number to be fixed later on in (4.18), and set

$$r_0 = \frac{W^2(\bar{\zeta}^2x+\zeta^2y)}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}}.$$

Then $|r_0| \leq \xi/4$. Deform the contour of integration to

$$(-\infty, \Re r_0 - \xi) \cup (\Re r_0 - \xi, r_0 - \xi) \cup (r_0 - \xi, r_0 + \xi) \\ \cup (r_0 + \xi, \Re r_0 + \xi) \cup (\Re r_0 + \xi, \infty).$$

Let $I = I_1 + I_2$, where I_1 is the integral over $(r_0 - \xi, r_0 + \xi)$, and I_2 is the integral over the remaining part of the contour. Let $\widetilde{I} = \widetilde{I}_1 + \widetilde{I}_2$ be the analogous decomposition of $\widetilde{I}$. (Observe that, for sufficiently small c_0, the deformed contour is within the domain of analyticity of U.) Then

$$|I_2|, |\widetilde{I}_2| \leq \exp\left\{-C^{-1}W^2\xi^2\right\}.$$

To estimate the difference between the dominant parts $I_1, \widetilde{I}_1$, we write

$$I_1 - \widetilde{I}_1 \\ = \exp\left\{\frac{W^4(\bar{\zeta}^2x+\zeta^2y)^2}{2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}}\right\}\int_{r_0-\xi}^{r_0+\xi} dr\, e^{-\left[2W^2\Re\zeta^2 + \frac{\Re U''(0)}{2}\right](r-r_0)^2}\left[e^{R(r)} - 1\right]$$

where

$$R(r) = \frac{U''_{\mathrm{Schw}}(0)}{2}r^2 - U_{\mathrm{Schw}}(r).$$

We obtain:

$$|I_1 - \widetilde{I}_1| = O((|r_0| + \xi)^3)|\widetilde{I}|,$$

To conclude the proof of the first item, set

$$\xi = 2(|x| + |y|) + W^{-1+\epsilon/3}, \tag{4.18}$$

and observe that

$$E(x, y) = (1 + O(|x|^3 + |y|^3))\widetilde{E}(x, y).$$

To prove the second item, we insert absolute values:

$$|E(x, y)| \le \exp\left\{-\frac{\Re U(x) + \Re U(y)}{2} - W^2 \Re\zeta^2(x^2 + y^2)\right\},$$

and

$$\begin{aligned} |I(x, y)| &\le \exp\left\{\frac{W^2}{2}\Re\zeta^2(x + y)^2\right\} \\ &\quad \times \int dr \exp\left\{-2W^2\Re\zeta^2\left(r - \frac{x + y}{2}\right)^2 - \Re U(r)\right\} \\ &\le \sqrt{\frac{\pi}{2W^2\Re\zeta^2}} \exp\left\{\frac{W^2}{2}\Re\zeta^2(x + y)^2\right\}. \end{aligned}$$

To prove the third item, let us rewrite $I(x, y)$ as

$$\exp\left\{\frac{W^2}{2}\frac{(\bar\zeta^2 x + \zeta^2 y)^2}{\Re\zeta^2}\right\} \int dr \exp\left\{-2W^2\Re\zeta^2(r - r_0)^2 - \Re U(r)\right\},$$

where

$$r_0 = \frac{y + x}{2} + \frac{\Im\zeta^2}{\Re\zeta^2}\frac{y - x}{2}\, i.$$

Then performing a contour deformation similar to the one in the proof of the first item we have

$$\int dr \exp\left\{-2W^2\Re\zeta^2(r - r_0)^2 - \Re U(r)\right\} = I_1 + I_2,$$

where

$$I_1 = e^{-U_{\text{Schw}}(r_0)} \int_{-\xi}^{\xi} dr e^{-2W^2\Re\zeta^2 r^2} e^{U_{\text{Schw}}(r_0) - U_{\text{Schw}}(r + r_0)} \tag{4.19}$$

$$|I_2| \le e^{-W^2(\xi^2 - |\Im r_0|^2)/C} \le e^{-W^2\xi^2/C'}, \tag{4.20}$$

if we choose $\xi > |\Im r_0|/4$. The imaginary part $|\Im r_0|$ may be as large as $W^{-1+\eta}$, therefore, we have to take $\xi = W^{-1+\eta+\epsilon}$ for some $\epsilon > 0$. We later set $\epsilon = 2\eta\gamma$.

Let us show that for any $\eta_1 \in (\eta, 2\eta)$ the following estimate holds for r in a complex neighbourhood of x:

$$|r - x| \le 2\xi = 2W^{-1+\eta+\epsilon} \implies \quad |U'(r)| < 2CW^{\eta_1\gamma}. \tag{4.21}$$

Indeed, by the inequalities $\Re[U(x)] \le W^\eta$ and U4), $U'(x)$ satisfies $|U'(r)| < 2CW^{\eta\gamma}$, and the smoothness of U guarantees there exists some constant $c_x > 0$ such that (4.21) holds inside the ball of radius c_x centred at x. Let $r_1 \in \mathbb{C}$ be a point such that $|r_1 - x| > c_x$, (4.21) holds for all $|r - x| < |r_1 - x|$ and fails at r_1. Then by U4)

$$\Re[U(r_1)] > W^{\eta_1}. \tag{4.22}$$

Performing a Taylor expansion with first-order integral remainder we have

$$\frac{U(r_1) - U(x)}{r_1 - x} = \int_0^1 U'(x + t(r_1 - x))\mathrm{d}t.$$

Inserting absolute values,

$$\frac{|U(r_1) - U(x)|}{|r_1 - x|} \leq \int_0^1 |U'(x + t(r_1 - x)|\mathrm{d}t \leq 2CW^{\eta_1\gamma}$$

since $|x+t(r_1-x)| < |r_1-x|$ for all $0 \leq t < 1$. From (4.22) and the assumptions $\Re[U(x)] \leq W^\eta \ll W^{\eta_1}$ and $|r_1 - x| \leq 2W^{-1+\eta+\epsilon}$ we get

$$\frac{1}{4}W^{\eta_1+1-\eta-\epsilon} \leq \frac{|\Re[U(r_1)] - \Re[U(x)]|}{|r_1 - x|} \leq \frac{|U(r_1) - U(x)|}{|r_1 - x|} \leq 2CW^{\eta_1\gamma}$$

hence

$$W^{1-\eta-\epsilon} \leq 8CW^{\eta_1(\gamma-1)} \tag{4.23}$$

as long as $\eta_1(\gamma - 1) < 1 - \eta - \epsilon$. When $\eta, \epsilon > 0$ are sufficiently small we have $\eta_1(\gamma - 1) < 1 - \eta - \epsilon$ for all $\eta_1 \in (\eta, 2\eta)$, in contradiction with (4.23). Thus (4.21) is established.

Applying the definition (4.17) of U_{Schw} we have $U'_{\mathrm{Schw}}(r) = (U'(r) + \overline{U'(\bar{r})})/2$, and from (4.21)

$$|r - x| \leq 2\xi = 2W^{-1+\eta+\epsilon} \Rightarrow \quad |U'_{\mathrm{Schw}}(r)| < 2CW^{\eta_1\gamma},$$

where $|\bar{r} - x| = |r - x|$ since $x \in \mathbb{R}$. Now set $\epsilon = 2\eta\gamma$. Then for any $\eta < \eta_1 < 2\eta$ we have $\epsilon > \eta_1\gamma$ and

$$\begin{aligned}|U_{\mathrm{Schw}}(r_0) - U_{\mathrm{Schw}}(r)| &\leq |r - r_0| \int_0^1 |U'_{\mathrm{Schw}}(r + t(r_0 - r))|\mathrm{d}t \\ &\leq O(W^{-1+\eta+\epsilon+\gamma\eta_1}) \ = O(W^{-1+5\gamma\eta}),\end{aligned}$$

hence I_1 of (4.19) satisfies

$$I_1 = e^{-\Re U(\frac{x+y}{2})}\sqrt{\frac{\pi}{2W^2\Re\zeta^2}}\left[1 + O(W^{-1+5\gamma\eta})\right].$$

This concludes the proof of the third item of Lemma 4.5. □

5. Application to a Complex Statistical Mechanics Model

In this section, we apply the results of the paper to a toy model. The model is tailored so that the conditions U1)–U4) of the Main Proposition will be satisfied after a rotation of the integration contour. The choice of the potential is partly inspired by supersymmetric models appearing in the study of random operators, but in our case the potential has only one minimum, instead of several minima as in the original models.

Let $V(x) = a\log(1+bx^2)$, where $a > 0$ and $\Re b > 0$. We are interested in the statistical mechanics model corresponding to the action

$$W^2 \sum_j (\phi_j - \phi_{j+1})^2 + \sum_j V(\phi_j);$$

for simplicity of notation, we set the inverse temperature to one. Without going into the details of the construction of infinite-volume measures (which is impeded by several obstacles, see e.g. Remark 5.1 below) , let us define the "mean" of a local observable $F : \mathbb{R}^{n+1} \to \mathbb{C}$ as follows:

$$\langle F(\phi_0, \cdots, \phi_n)\rangle = \lim_{M,N\to\infty} \frac{\int \prod_{j=-M}^N \mathrm{d}\phi_j e^{-\sum_{j=-M}^N V(\phi_j) - W^2 \sum_{j=-M}^{N-1}(\phi_j-\phi_{j+1})^2} F(\phi_0,\cdots,\phi_n)}{\int \prod_{j=-M}^N \mathrm{d}\phi_j e^{-\sum_{j=-M}^N V(\phi_j) - W^2\sum_{j=-M}^{N-1}(\phi_j-\phi_{j+1})^2}}. \tag{5.1}$$

We are interested in the long-distance correlations, e.g.

$$\langle (F(\phi_0) - \langle F(\phi_0)\rangle)(G(\phi_n) - \langle G(\phi_n)\rangle)\rangle,$$

with $F, G : \mathbb{R} \to \mathbb{C}$.

Define $\zeta \in \mathbb{C}$ by

$$|\zeta| = 1, \quad |\arg\zeta| < \pi/4, \quad \zeta^4 V''(0) \left(= 2\zeta^4\, ab \right) > 0; \tag{5.2}$$

let $\Sigma = \mathrm{conv}\,(\mathbb{R} \cup \mathbb{R}\zeta)$, and set

$$U(x) = V(\zeta x). \tag{5.3}$$

Our transfer operator method can be applied to study observables $F : \mathbb{R}^{n+1} \to \mathbb{C}$ which have an analytic extension to Σ^{n+1} and do not grow too fast in this sector. For simplicity, let us focus on $n = 0$, i.e. on observables depending only on one variable: assume that

F1) $F : \Sigma \to \mathbb{C}$ is analytic;
F2) $|F(z)| \le C(1+|z|)^{2a-1-\epsilon}$ for some $C > 0$ and $\epsilon > 0$, and all $z \in \Sigma$.

Theorem 1. *Suppose that $V(x) = a\log(1+bx^2)$ for some $a > 0$ and $\Re b > 0$, and that F, G are observables which satisfy F1)–F2). Then*

$$|\langle (F(\phi_0) - \langle F(\phi_0)\rangle)(G(\phi_n) - \langle G(\phi_n)\rangle)\rangle| \le C_F C_G \left(1 - \sqrt{\frac{|V''(0)|}{2}\frac{\Re\zeta^2}{W}} + O(W^{-1-\delta})\right)^n,$$

where ζ is defined in (5.2).

Proof. Let F, G be observables satisfying F1)–F2). Note that the functions inside both integrals of (5.1) satisfy $f(\phi_{-M}, \ldots, \phi_N) \sim e^{-W^2\phi_j^2}$ as $|\phi_j| \to \infty$ while keeping the other variables fixed. Then using Cauchy's theorem we can rotate the contour of integration $\phi_j \to \phi_j \zeta$, as long as $\Re\zeta^2 > 0$. Repeating the argument for each variable ϕ_j we obtain

$$\frac{\int \prod_{j=-M}^{N} \mathrm{d}\phi_j e^{-\sum_{j=-M}^{N} V(\phi_j) - W^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2} F(\phi_0) G(\phi_n)}{\int \prod_{j=-M}^{N} \mathrm{d}\phi_j e^{-\sum_{j=-M}^{N} V(\phi_j) - W^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2}}$$
$$= \frac{\int \prod_{j=-M}^{N} \mathrm{d}\phi_j e^{-\sum_{j=-M}^{N} U(\phi_j) - W^2 \zeta^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2} F(\zeta\phi_0) G(\zeta\phi_n)}{\int \prod_{j=-M}^{N} \mathrm{d}\phi_j e^{-\sum_{j=-M}^{N} U(\phi_j) - W^2 \zeta^2 \sum_{j=-M}^{N-1} (\phi_j - \phi_{j+1})^2}}. \tag{5.4}$$

Remark 5.1. The argument presented above for free boundary conditions applies equally well to periodic boundary conditions. For more general boundary conditions the potential at the boundary is modified, therefore, additional requirements on U and the observables probably have to be imposed.

Now we set up an integral operator K, the kernel of which is given by (2.1); the Main Proposition and Corollary 2.1 are applicable, so the largest eigenvalue of K (in absolute value) λ_0 is given by (2.9). Let u_0 be a corresponding eigenfunction satisfying the normalisation conditions (2.8). Set $B(x) = \exp(-U(x)/2)$. Then the denominator of (5.4) is equal to

$$\int K^{M+N}(x_1, x_2) B(x_1) B(x_2) \mathrm{d}x_1 \mathrm{d}x_2.$$

From Corollary 2.1, we have

$$\langle u_0, \bar{u}_0 \rangle = \langle u_0, g_\alpha \rangle + \langle u_0, \bar{u}_0 - g_\alpha \rangle = 1 + O(W^{-\delta}) \neq 0.$$

Decomposing $B = \frac{\langle B, \bar{u}_0 \rangle}{\langle u_0, \bar{u}_0 \rangle} u_0 + B_1$ (this is well defined since $\langle u_0, \bar{u}_0 \rangle = 1 + O(W^{-\delta}) \neq 0$) and setting $\hat{K} = \lambda_0^{-1} K$, we obtain from Corollary 2.1:

$$\lim_{M,N\to\infty} \int \hat{K}^{M+N}(x_1, x_2) B(x_1) B(x_2) \mathrm{d}x_1 \mathrm{d}x_2 = \frac{\langle B, \bar{u}_0 \rangle^2}{\langle u_0, \bar{u}_0 \rangle^2}.$$

Similarly, for $F : \mathbb{R} \to \mathbb{C}$ satisfying F1)–F2),

$$\int \prod_{j=1}^{3} \mathrm{d}x_j B(x_1) \hat{K}^M(x_1, x_2) F(\zeta x_2) \hat{K}^N(x_2, x_3) B(x_3) \to \frac{\langle B, \bar{u}_0 \rangle^2}{\langle u_0, \bar{u}_0 \rangle^2} \langle F_\zeta u_0, \bar{u}_0 \rangle,$$

where $F_\zeta(x) = F(\zeta x)$. Hence we obtain:

$$\langle F(\phi_0) \rangle = \langle F_\zeta u_0, \bar{u}_0 \rangle. \tag{5.5}$$

Similarly,

$$\langle F(\phi_0) G(\phi_n) \rangle = \langle \hat{K}^n F_\zeta u_0, \bar{G}_\zeta \bar{u}_0 \rangle. \tag{5.6}$$

If $\langle F(\phi_0) \rangle = 0$ (i.e. $F_\zeta u_0 \in \bar{u}_0^\perp$), Corollary 2.1 yields:

$$\|\hat{K}^n F_\zeta u_0\| \leq \left(1 - \sqrt{\frac{|V''(0)|}{2}} \frac{\Re\zeta^2}{W} + O(W^{-1-\delta})\right)^n \|F_\zeta u_0\|,$$

thus for any F, G satisfying F1)–F2)

$$\begin{aligned}&|\langle (F(\phi_0) - \langle F(\phi_0)\rangle)(G(\phi_n) - \langle G(\phi_n)\rangle)\rangle| \\ &\quad \le C_F C_G \left(1 - \sqrt{\frac{|V''(0)|}{2}} \frac{\Re\zeta^2}{W} + O(W^{-1-\delta})\right)^n .\end{aligned}$$

□

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