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Open Quantum Random Walks: Reducibility, Period, Ergodic Properties

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Abstract. We study the analogues of irreducibility, period, and communicating classes for open quantum random walks, as defined in (J Stat Phys 147(4):832–852, 2012). We recover results similar to the standard ones for Markov chains, in terms of ergodic behaviour, decomposition into irreducible subsystems, and characterization of invariant states.

1. Introduction

Open quantum random walks were recently defined by Attal et al. [3]. These processes have a simple definition, implementing a Markovian dynamics influenced by internal degrees of freedom, and can be useful to model a variety of phenomena: quantum algorithms (see [23]), transfer in biological systems (see [17]) and possibly quantum exclusion processes. In addition, a continuous-time version can be defined (see [19]). Therefore, open quantum random walks seem to be good quantum analogues of Markov chains.

The usefulness of (classical) Markov chains, however, comes not only from the vast number of situations they can model, but also from the many properties implied by their simple definition. A textbook description of Markov chains, for instance, can start with the notion of irreducibility, which is easily characterized through the connectedness of the associated graph, and implies mean-ergodic convergence in law if an invariant probability exists (which is the case when the state space is finite). The next notion, the aperiodicity of an irreducible chain, is not as easy to characterize, but has simple sufficient conditions (e.g. the existence of loops) and implies convergence in law, at least when the state space is finite. Last, the notion of connected subsets of the initial graph allows one to decompose a Markov chain into irreducible ones and to characterize its invariant states as convex combinations of invariant states for restricted chains.

On the other hand, the only general properties of open quantum random walks proven so far are the central limit theorem for the position process (see [2]) and the general Kümmerer–Maassen theorem for quantum trajectories (see [16]). In the present paper, we discuss an analogue of the above textbook description of Markov chains, for open quantum random walks. The non-commutative nature of the objects under study, and specifically the fact that the transition probabilities are replaced by operators acting on a Hilbert space, are the cause of higher mathematical complexity. Some intuitive aspects of classical Markov chains, however, remain true, and we can recover a vision of irreducibility, period, and accessibility, in terms of paths. This is of interest for the study of more general quantum Markov processes, as it gives indications on the relevant extensions of classical concepts, and on techniques of proofs for associated results. We view this as an additional justification for the study of open quantum random walks.

Our theory will be constructed starting from pre-existing tools:

- a notion of irreducibility for general positive maps on non-commutative algebras, together with an associated Perron–Frobenius theorem, that was developed by various authors in the late seventies and early eighties [1,7,8, 14];
- a notion of period, together with associated results on the peripheral spectrum, which were defined in the same setting by Groh [14] and extended by Fagnola and Pellicer [9];
- some old and new inspiring ergodic results [12,16] and a decomposition of the support of invariant states proposed more recently by Baumgartner and Narnhofer [4] for quantum discrete time processes acting on finite-dimensional spaces.

We briefly describe the structure of the article and the main contents. Section 2 recalls the definitions, notations and basic results regarding open quantum random walks from [3]. We also introduce the classical processes associated with an OQRW. Sections 3 and 4 discuss, respectively, irreducibility and aperiodicity for OQRWs. Both follow the same structure: they start by recalling standard definitions and properties of irreducibility or aperiodicity for positive maps on operator algebras, then study the application to the special case of OQRWs. Some immediate consequences on the ergodic behaviour of the evolution are underlined. Section 5 applies the results of the previous two sections to obtain, for irreducible, or irreducible aperiodic, open quantum random walks, convergence properties of the processes described in Sect. 2. Section 6 expands on reducible open quantum random walks, characterizing in different ways their irreducible components. The resulting decomposition can be seen as related to a "quantum communication relation" among vectors of the underlying Hilbert space. Section 7 states the general form of invariant states for reducible open quantum random walks. Its central point is an extension of some results from [4]. Section 8 mentions a natural extension of open quantum random walks. For this extension, we discuss without proof a characterization of irreducibility, periodicity, communication classes, and their consequences: as we will see, all previous results will remain with paths on a graph replaced by paths on a multigraph. We conclude with Sect. 9, which is dedicated to examples and applications. We start a study of translation-invariant open quantum random walks on $\mathbb{Z}^d$ continued in [5], and extending that of [2]. We study examples which illustrate our most practical convergence results, namely Corollaries 5.2, 5.4, and 5.6, as well as our decomposition result, Theorem 7.13.

2. Open Quantum Random Walks

In this section, we recall basic results and notations about open quantum random walks. For a more detailed exposition of OQRWs and related notions, we refer the reader to [3].

We consider a Hilbert space $\mathcal{H}$ of the form $\mathcal{H} = \bigoplus_{i\in V} \mathfrak{h}_i$ where V is a countable set of vertices, and each $\mathfrak{h}_i$ is a separable Hilbert space (making $\mathcal{H}$ separable). This is a generalization with respect to standard OQRWs where the space $\mathcal{H}$ is $\mathfrak{h}\otimes\mathbb{C}^V$, or equivalently $\mathfrak{h}_i = \mathfrak{h}$ for all $i \in V$. This generalization will be useful when we consider decompositions of OQRWs, especially in Sect. 6. We view $\mathcal{H}$ as describing the degrees of freedom of a particle constrained to move on V: the "V-component" describes the spatial degrees of freedom (the position of the particle) while $\mathfrak{h}_i$ describes the internal degrees of freedom of the particle, when it is located at site $i \in V$.

For clarity, whenever a vector $x \in \mathcal{H}$ belongs to the subspace $\mathfrak{h}_i$, we will denote it by $x \otimes |i\rangle$, and drop the (implicit) assumption that $x \in \mathfrak{h}_i$. This will allow us to use the same notation as in the literature on open quantum random walks. Similarly, when an operator A on $\mathcal{H}$ satisfies $\mathfrak{h}_j^\perp \subset \operatorname{Ker} A$ and $\operatorname{Ran} A \subset \mathfrak{h}_i$, we denote it by $A = L_{i,j} \otimes |i\rangle\langle j|$ where $L_{i,j}$ is viewed as an operator from $\mathfrak{h}_j$ to $\mathfrak{h}_i$. Therefore, for i, j, k in V, we have the relation

$$\big(L_{i,j} \otimes |i\rangle\langle j|\big)\ \big(x \otimes |k\rangle\big) = \begin{cases} 0 & \text{if } j \neq k,\\ \big(L_{i,j}\, x\big) \otimes |i\rangle & \text{if } j = k.\end{cases}$$

All these notations are consistent with the special case of $\mathcal{H} = \mathfrak{h}\otimes\mathbb{C}^V$, and in particular with the notation used in [3].

We consider a map on the space $\mathcal{I}_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$,

$$\mathfrak{M} \,:\, \rho \mapsto \sum_{i,j\in V} A_{i,j}\, \rho\, A_{i,j}^* \tag{2.1}$$

where, for any i, j in V, the operator $A_{i,j}$ is of the form $L_{i,j} \otimes |i\rangle\langle j|$ and the operators $L_{i,j}$ satisfy

$$\forall j \in V \quad \sum_{i\in V} L_{i,j}^* L_{i,j} = \mathrm{Id}, \tag{2.2}$$

where the series is meant in the strong convergence sense. The $L_{i,j}$ are thought of as encoding both the probability of a transition from site j to site i, and the effect of that transition on the internal degrees of freedom. Equation (2.2), therefore, encodes the "stochasticity" of the transitions $L_{i,j}$.

Let us recall a few definitions: an operator X on $\mathcal{H}$ is called positive, denoted $X \geq 0$, if for $\phi \in \mathcal{H}$, one has $\langle \phi, X\,\phi\rangle \geq 0$. It is called strictly positive, denoted $X > 0$, if for $\phi \in \mathcal{H}\backslash\{0\}$, one has $\langle \phi, X\,\phi\rangle > 0$. A map

$\Phi : \mathcal{I}_1(\mathcal{H}) \to \mathcal{I}_1(\mathcal{H})$ is said to be positive if it maps positive operators to positive operators; it is n-positive if its extension $\Phi \otimes \mathrm{Id}$ to $\mathcal{I}_1(\mathcal{H}) \otimes \mathcal{B}(\mathbb{C}^n)$ is a positive map; it is completely positive if it is n-positive for all n in $\mathbb{N}$.

With these definitions (2.1) defines a trace-preserving (TP) and completely positive (CP) map $\mathcal{I}_1(\mathcal{H}) \to \mathcal{I}_1(\mathcal{H})$. In particular, such a map transforms states (defined here as positive elements of $\mathcal{I}_1(\mathcal{H})$ with trace one) into states. A completely positive, trace-preserving map will be called a CP-TP map. We shall call a map $\mathfrak{M}$ as defined by (2.1) an open quantum random walk, or OQRW. Note that (2.2) implies that $\|\mathfrak{M}\| = 1$ as an operator on $\mathcal{I}_1(\mathcal{H})$ (see Remark 2.2 below).

Remark 2.1. In our interpretation of $L_{i,j}$ above, it would be more precise to say that the transition from site j to site i is encoded by the completely positive map $\rho_j \mapsto L_{i,j}\, \rho_j\, L_{i,j}^*$. A natural extension would be to replace this with a more general completely positive map $\rho_j \mapsto \Phi_{i,j}(\rho_j)$. This will be discussed in Sect. 8.

Remark 2.2. Let us recall that the topological dual $\mathcal{I}_1(\mathcal{H})^*$ can be identified with $\mathcal{B}(\mathcal{H})$ through the duality

$$(\rho, X) \mapsto \mathrm{Tr}(\rho\, X).$$

Trace preservation of a map Φ is equivalent to $\Phi^*(\mathrm{Id}) = \mathrm{Id}$. The adjoint Φ^* is then a positive, unital (i.e. $\Phi^*(\mathrm{Id}) = \mathrm{Id}$) map on $\mathcal{B}(\mathcal{H})$, and by the Russo–Dye theorem [21] one has $\|\Phi^*\| = \|\Phi^*(\mathrm{Id})\|$ so that $\|\Phi\| = \|\Phi^*\| = 1$.

Definition 2.3. We say that an open quantum random walk $\mathfrak{M}$ is finite if V is finite and every $\mathfrak{h}_i$ is finite dimensional.

Remark 2.4. If an open quantum random walk is finite, then $\mathfrak{M}^*(\mathrm{Id}) = \mathrm{Id}$ implies that 1 is an eigenvalue of $\mathfrak{M}$. Since $\mathfrak{M}$ preserves the trace and the positivity, this implies that there exists an invariant state.

Remark 2.5. As noted in [3], classical Markov chains can be written as open quantum random walks. More precisely, if the transition matrix is $P = (P_{i,j})$ then, taking $L_{i,j} = \sqrt{P_{j,i}}\, U_{i,j}$ with any $U_{i,j}$ satisfying $U_{i,j}U_{i,j}^* = \mathrm{Id}_{\mathfrak{h}_i}$, will map any state $\sum_{i\in V} p_i \mathrm{Id}_{\mathfrak{h}_i} \otimes |i\rangle\langle i|$, to a state of the form $\sum_{i\in V} q_i \mathrm{Id}_{\mathfrak{h}_i} \otimes |i\rangle\langle i|$, and the induced dynamics $(p_i)_{i\in V} \mapsto (q_i)_{i\in V}$ will be described by the transition matrix P. However, if $\dim \mathfrak{h}_i > 1$, we will run into possible non-uniqueness problems, e.g. for the invariant states of $\mathfrak{M}$ (see Sect. 6). We feel this is an artificial degeneracy, not related to the properties of the Markov chain, but rather to the choice of the realization as an OQRW. We, therefore, define a *minimal OQRW realization* of a classical Markov chain to be the OQRW with $\mathfrak{h}_i = \mathbb{C}$ for all i in V, and $L_{i,j} = \sqrt{P_{j,i}}$.

A crucial remark is that, for any initial state ρ on $\mathcal{H}$, which can be expanded as

$$\rho = \sum_{i,j\in V} \rho(i,j) \otimes |i\rangle\langle j|$$

and, for any $n \geq 1$, the evolved state $\mathfrak{M}^n(\rho)$ is of the form

$$\mathfrak{M}^n(\rho) = \sum_{i\in V} \mathfrak{M}^n(\rho, i) \otimes |i\rangle\langle i|, \tag{2.3}$$

where, e.g. for $n = 1$,

$$\mathfrak{M}^1(\rho, i) = \sum_{j\in V} L_{i,j}\, \rho(j, j)\, L_{i,j}^*. \tag{2.4}$$

Each $\mathfrak{M}^n(\rho, i)$ is a positive, trace-class operator on $\mathfrak{h}_i$ and $\sum_{i\in V} \operatorname{Tr} \mathfrak{M}^n(\rho, i) = 1$. Therefore, the range of $\mathfrak{M}$ is included in the set $\mathcal{I}_D$ of block-diagonal trace-class operators,

$$\mathcal{I}_D = \left\{ \rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|, \sum_{i\in V} \operatorname{Tr}(|\rho(i)|) < +\infty \right\},$$

and $\mathcal{I}_D^*$ can be identified with

$$\mathcal{B}_D = \left\{ X = \sum_{i\in V} X(i) \otimes |i\rangle\langle i|,\ \sup \|X(i)\|_{\mathcal{B}(\mathfrak{h}_i)} < \infty \right\}.$$

This feature will have a great importance in the characterization of many properties of OQRWs, e.g.:

1. the invariant states of an OQRW $\mathfrak{M}$ belong to $\mathcal{I}_D$,
2. the reducibility of $\mathfrak{M}$ can be established considering only block-diagonal projections (see Sect. 3),
3. the cyclic projections defining the period have block-diagonal form (see Sect. 4),
4. the only meaningful enclosures—the vector spaces which will play the role of communicating classes—are generated by vectors of the form $x \otimes |i\rangle$ (see Sect. 6).

In addition, we remark from (2.4) that $\mathfrak{M}^n(\rho)$ depends only on the diagonal elements $\rho(i, i)$. Therefore, from now on, we will only consider states of the form $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$. Equation (2.4) remains valid, replacing $\rho(i, i)$ with $\rho(i)$.

We now describe the (classical) processes of interest, associated with $\mathfrak{M}$: here we remain at a heuristic level and speak loosely of the laws of random variables, without specifying the probability space (this will be done in Sect. 5). We start from a state ρ which we assume to be of the form $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$. We evolve ρ for a time n, obtaining the state $\mathfrak{M}^n(\rho)$ as in (2.3). We then make a measurement of the position observable, i.e. of the degree of freedom in $\mathbb{C}^V$. According to standard rules of quantum measurement, we obtain the result $i \in V$ with probability $\operatorname{Tr} \mathfrak{M}^n(\rho, i)$. Therefore, the result of this measurement is a random variable X_n, with law $\mathbb{P}(X_n = i) = \operatorname{Tr} \mathfrak{M}^n(\rho, i)$ for $i \in V$. In addition, if the position $X_n = i \in V$ is observed, then the state is transformed to $\frac{\mathfrak{M}^n(\rho,i)}{\operatorname{Tr} \mathfrak{M}^n(\rho,i)}$. We call this process $\left(X_n, \frac{\mathfrak{M}^n(\rho,X_n)}{\operatorname{Tr} \mathfrak{M}^n(\rho,X_n)}\right)_n$ the process "without measurement" to emphasize the fact that virtually only one measurement is done, at time n.

Now, assume that we make a measurement at every time $n \in \mathbb{N}$, applying the evolution by $\mathfrak{M}$ between two measurements. Again assume that we start from a state ρ of the form $\sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$. Suppose that at time n, the position was measured at $Q_n = j$ and the state (after the measurement) is $\rho_n \otimes |j\rangle\langle j|$. Then after the evolution, the state becomes

$$\mathfrak{M}(\rho_n \otimes |j\rangle\langle j|) = \sum_{i\in V} L_{i,j}\, \rho_n\, L_{i,j}^* \otimes |i\rangle\langle i|$$

so that a measurement at time $n+1$ gives a position $Q_{n+1} = i$ with probability $\operatorname{Tr} L_{i,j}\, \rho_n\, L_{i,j}^*$, and then the state becomes $\rho_{n+1} \otimes |i\rangle\langle i|$ with $\rho_{n+1} = \frac{L_{i,j}\, \rho_n\, L_{i,j}^*}{\operatorname{Tr} L_{i,j}\, \rho_n\, L_{i,j}^*}$. The sequence of random variables (Q_n, ρ_n) is therefore a Markov process with transitions defined, for $i, j \in V$, by

$$\mathbb{P}\left((Q_{n+1}, \rho_{n+1}) = \left(i, \frac{L_{i,j}\, \rho_n\, L_{i,j}^*}{\operatorname{Tr}(L_{i,j} \rho L_{i,j}^*)}\right) \Big| (Q_n, \rho_n) = (j, \rho_n)\right) = \operatorname{Tr}(L_{i,j}\, \rho_n\, L_{i,j}^*),$$

and initial law $\mathbb{P}\big((Q_0, \rho_0) = (i, \frac{\rho(i)}{\operatorname{Tr} \rho(i)})\big) = \operatorname{Tr} \rho(i)$. Note that the sequence of positions $Q_0 = i_0, \ldots, Q_n = i_n$ is observed with probability

$$\operatorname{Tr} L_{i_n,i_{n-1}} \ldots L_{i_1,i_0}\, \rho(i_0)\, L_{i_1,i_0}^* \ldots L_{i_n,i_{n-1}}^* \tag{2.5}$$

and completely determines the state ρ_n:

$$\rho_n = \frac{L_{i_n,i_{n-1}} \ldots L_{i_1,i_0}\, \rho(i_0)\, L_{i_1,i_0}^* \ldots L_{i_n,i_{n-1}}^*}{\operatorname{Tr} L_{i_n,i_{n-1}} \ldots L_{i_1,i_0}\, \rho(i_0)\, L_{i_1,i_0}^* \ldots L_{i_n,i_{n-1}}^*}. \tag{2.6}$$

Equation (2.5) implies for any n the relation

$$\mathbb{E}(\rho_n \otimes |X_n\rangle\langle X_n|) = \mathfrak{M}^n(\rho). \tag{2.7}$$

As emphasized in [3], it also implies that for every n the laws of X_n and Q_n are the same, i.e.

$$\mathbb{P}(X_n = i) = \mathbb{P}(Q_n = i) \quad \forall i \in V.$$

For this reason, from now on we will drop the notation Q_n and the only "position" process we consider will be $(X_n)_n$. On the other hand, $(\mathfrak{M}^n(\rho, X_n))_n$ and $(\rho_n)_n$ correspond to physically different quantities and we keep separate notations for them. The processes $(X_n)_n$, $(\mathfrak{M}^n(\rho, X_n))_n$ and $(\rho_n)_n$ are among our main interests when it comes to open quantum random walks: Sect. 5 is devoted to the study of their ergodic properties.

3. Irreducibility for OQRWs

In this and in the following sections, Φ is assumed to be a positive map on the ideal $\mathcal{I}_1(\mathcal{H})$ of trace operators on some given Hilbert space $\mathcal{H}$. We recall that such a map is automatically bounded as a linear map on $\mathcal{I}_1(\mathcal{H})$ (see, e.g. Lemma 2.2 in [22]), so that it is also weakly continuous. In most practical cases, we will additionally assume that $\|\Phi\| = 1$; as we noted in Remark 2.2, this will be the case, in particular, if Φ is trace preserving.

We describe here the definition of irreducibility introduced by Davies (see [6]). We give an equivalent property, considered by some authors (see [22]) under the name *ergodicity*, in Proposition 3.4.

Definition 3.1. We say that the positive map Φ is irreducible if the only orthogonal projections P reducing Φ, i.e. such that $\Phi\big(P\mathcal{I}_1(\mathcal{H})P\big) \subset P\mathcal{I}_1(\mathcal{H})P$, are $P = 0$ and Id.

Remark 3.2. The condition $\Phi\big(P\mathcal{I}_1(\mathcal{H})P\big) \subset P\mathcal{I}_1(\mathcal{H})P$ is equivalent to the condition $\Phi(P) \leq \alpha P$ for some $\alpha > 0$ whenever $P \in \mathcal{I}_1(\mathcal{H})$, i.e. whenever P is finite dimensional. In the infinite-dimensional case one can prove that P reduces Φ if and only if for any finite-dimensional projection Q with $Q \leq P$, one has $\Phi(Q) \leq \alpha P$ for some $\alpha > 0$.

There is a possible confusion here due to the fact that some authors [9, 14] work in the Heisenberg representation, i.e. in our notation consider Φ^*, while others [8, 22], like us, work in the Schrödinger representation. For completeness we give the next proposition, which connects the two representations:

Proposition 3.3. *Let Φ be a positive, trace-preserving map on $\mathcal{I}_1(\mathcal{H})$. Then an orthogonal projection P reduces Φ if and only if $P \leq \Phi^*(P)$, i.e.* $(\mathrm{Id} - P)$ *reduces Φ^*.*

Proof. Assume first that P reduces Φ, i.e. $\Phi(P\mathcal{I}_1(\mathcal{H})P) \subset P\mathcal{I}_1(\mathcal{H})P$. Then, for any trace-class operator σ, using the trace-preserving property and the reduction assumption for Φ, we have

$$\mathrm{Tr}(\sigma P) = \mathrm{Tr}(P\sigma P) = \mathrm{Tr}\big(\Phi(P\sigma P)\big) = \mathrm{Tr}\big(P\,\Phi(P\sigma P)\big) = \mathrm{Tr}(\sigma P\Phi^*(P)P)$$

so that $P = P\Phi^*(P)P$ and $P\Phi^*(P^\perp)P = P(\mathrm{Id} - \Phi^*(P))P = 0$, where $P^\perp = \mathrm{Id} - P$. Since $\Phi^*(P^\perp)$ is positive, we also deduce $P\Phi^*(P^\perp)P^\perp = P^\perp\Phi^*(P^\perp)P = 0$ and $\Phi^*(P^\perp) = P^\perp\Phi^*(P^\perp)P^\perp$. Then

$$\Phi^*(P) = \mathrm{Id} - \Phi^*(P^\perp) = P + P^\perp\Phi^*(P)P^\perp \geq P.$$

Conversely, if $P \leq \Phi^*(P)$, then, for any trace class $\rho \geq 0$,

$$\mathrm{Tr}(P\rho P) \leq \mathrm{Tr}(P\rho P\,\Phi^*(P)) = \mathrm{Tr}(P\,\Phi(P\rho P)\,P) \leq \mathrm{Tr}(\Phi(P\rho P)) = \mathrm{Tr}(P\rho P).$$

We, therefore, have the equality $\mathrm{Tr}(P\,\Phi(P\rho P)\,P) = \mathrm{Tr}(\Phi(P\rho P))$ which implies the inclusion $\Phi(P\rho P) \in P\mathcal{I}_1(\mathcal{H})P$ for $\rho \geq 0$, hence for any $\rho \in \mathcal{I}_1(\mathcal{H})$. □

Proposition 3.4. *A positive map Φ on $\mathcal{I}_1(\mathcal{H})$ is irreducible if and only if*

$$\textit{for any } \rho \geq 0 \textit{ in } \mathcal{I}_1(\mathcal{H})\backslash\{0\} \textit{ there exists } t > 0 \textit{ such that } \mathrm{e}^{t\Phi}(\rho) > 0. \quad (3.1)$$

Proof. If Φ is not irreducible, then by definition there exists a non-trivial projection P and a non-negative trace-class operator ρ such that $\Phi(P\rho P)$ is of the form $P\sigma P$ for some σ in $\mathcal{I}_1(\mathcal{H})$. Then, with $\alpha = \|\sigma\|$, we have $\Phi(P\rho P) \leq \alpha P$ and consequently, for any t, $\mathrm{e}^{t\Phi}(P\rho P) \leq \mathrm{e}^{t\alpha}P$, so that $\mathrm{e}^{t\Phi}(P\rho P)$ is not strictly positive for all t. Therefore, condition (3.1) implies irreducibility.

For the converse implication, assume Φ, hence Φ^*, is irreducible, consider $X \geq 0$, $X \neq 0$ in $\mathcal{B}(\mathcal{H})$; for a fixed $t > 0$ let

$$e_p(X) = \sum_{k=0}^{p} \frac{t^k}{k!} \Phi^{*k}(X).$$

Define P to be the support projection of $\mathrm{e}^{t\Phi^*}(X)$ and $P_p = \mathbb{1}_{[1/p,+\infty[}(e_p(X))$. Obviously, $P_p \leq P$ and $P_p \leq p\, e_p(X)$ for all p, and $P_p \to P$ in the sense of strong convergence as $p \to \infty$, thanks to the properties of bounded measurable functional calculus (see, e.g. Theorem VII.2 in [20]). We have:

$$\begin{aligned} \frac{1}{p}\, \Phi^*(P_p) \leq \Phi^*(e_p(X)) &= \sum_{k=1}^{p+1} \frac{k}{t} \frac{t^k}{k!}\, \Phi^{*k}(X) \\ &\leq \frac{p+1}{t}\, e_{p+1}(X) \leq \frac{p+1}{t}\, \mathrm{e}^{t\Phi^*}(X) \end{aligned}$$

so that $\operatorname{supp} \Phi^*(P_p) \subset \operatorname{supp} P$ and, by the weak-$*$ continuity of Φ^*, one has $\operatorname{supp} \Phi^*(P) \subset \operatorname{supp} P$, i.e. P reduces Φ^*. Since $\mathrm{e}^{t\Phi^*}(X) \geq X$, the projector P cannot be zero, so by irreducibility P is Id and $\mathrm{e}^{t\Phi^*}(X) > 0$.

Take now X of the form $|\phi\rangle\langle\phi|$ and consider $\rho \geq 0$, $\rho \neq 0$ in $\mathcal{I}_1(\mathcal{H})$; then $0 < \operatorname{Tr}(\mathrm{e}^{t\Phi^*}(X)\,\rho) = \operatorname{Tr}(X\mathrm{e}^{t\Phi}(\rho)) = \langle \phi, \mathrm{e}^{t\Phi}(\rho)\phi\rangle$. So $\mathrm{e}^{t\Phi}(\rho) > 0$ for all $t > 0$. □

Remark 3.5. Condition (3.1) is called "ergodicity" by Schrader [22]. In finite dimension, it is equivalent to the condition that $(\mathrm{Id} + \Phi)^{\dim \mathcal{H} - 1}$ maps any positive, non-zero $\rho \in \mathcal{I}_1(\mathcal{H})$ to a strictly positive operator (see the remark following Lemma 3.1 in [22]). This latter condition is the definition of ergodicity in [8], where the equivalence of irreducibility and ergodicity is proven in the finite-dimensional case. Note also that, in (3.1), "for all $t > 0$" can be equivalently replaced with "for some $t > 0$". This follows from the observation that the support projection of $\mathrm{e}^{t\Phi}(\rho)$ does not depend on $t > 0$.

When speaking about reducibility/irreducibility of quantum maps, one enters a jungle of different approaches and terminologies, which, in many cases, are essentially equivalent. Concerning this, we recall that a reducing projection P is called by some authors a *subharmonic projection* for Φ^*, following the line common to the classical literature on Markov chains.

Also, more recently (in [4], as far as we know), the notion of enclosure has been introduced in the context of CP-TP maps. To define it, we recall the notion of support of a state ρ: it is the range of the projection $\mathrm{Id} - P_0(\rho)$, where

$$P_0(\rho) = \sup\{P \text{ orthogonal projection s.t. } \rho(P) = 0\}.$$

Definition 3.6. A closed vector space $\mathcal{V}$ is called an *enclosure* (for $\mathfrak{M}$) if for any state ρ $\operatorname{supp} \rho \subset \mathcal{V}$ implies $\operatorname{supp} \mathfrak{M}(\rho) \subset \mathcal{V}$.

It is immediate that a space $\mathcal{V}$ is an enclosure if and only if the projection P on $\mathcal{V}$ reduces $\mathfrak{M}$. So, an equivalent way to define irreducibility is asking that there exist no non-trivial enclosures. The notion of enclosure will be crucial in the discussion of decompositions of reducible open quantum random walks (see Sect. 6).

Next, we characterize irreducibility in terms of unravellings. We consider a completely positive trace-preserving map Φ and fix an unravelling $(A_\kappa)_{\kappa\in K}$ of Φ, provided by Kraus' representation theorem (see [15] or [18], where this is called the operator-sum representation):

$$\Phi(\rho) = \sum_{\kappa\in K} A_\kappa \rho A_\kappa^*. \tag{3.2}$$

We will characterize irreducibility in terms of an unravelling $(A_\kappa)_{\kappa\in K}$. We denote by $\mathbb{C}[A]$ the set of polynomials in A_κ, i.e. the algebra (*not* the *-algebra) generated by the operators A_κ, $\kappa \in K$. The following result is a straightforward consequence of a result of Schrader ([22, Lemma 3.4]—note that that lemma considers the equivalent property that for every $\phi \in \mathcal{H}\backslash\{0\}$, the set $\mathbb{C}[A^*]\,\phi$, associated with the operators A_k^*,is dense in $\mathcal{H}$):

Lemma 3.7. *A completely positive map Φ of the form (3.2) is irreducible if and only if one of the following equivalent conditions holds for any $\phi \in \mathcal{H}\backslash\{0\}$:*

- *the set $\mathbb{C}[A]\,\phi$ is dense in $\mathcal{H}$,*
- *for any ψ in $\mathcal{H}\backslash\{0\}$, there exist $\kappa_1 \ldots \kappa_\ell$ in K such that $\langle\psi, A_{\kappa_1}\cdots A_{\kappa_\ell}\,\phi\rangle \neq 0$,*

We will now characterize irreducibility for open quantum random walks, i.e. for CP-TP maps such that each $A_{i,j}$ is of the form $L_{i,j}\otimes|i\rangle\langle j|$. Let us introduce some notation: for i, j in V we call a *path from i to j* any finite sequence $i_0, \ldots, i_\ell$ in V with $\ell \geq 1$, such that $i_0 = i$ and $i_\ell = j$. Such a path is said to be *of length ℓ*. We denote by $\mathcal{P}(i,j)$ (resp. $\mathcal{P}_\ell(i,j)$) the set of paths from i to j of arbitrary length (resp. of length ℓ). A path from i to i will be called a loop; by convention we consider the sequence $\{i\}$ as a loop (with length one), i.e. an element of $\mathcal{P}(i,i)$. For $\pi = (i_0, \ldots, i_\ell)$ in $\mathcal{P}(i,j)$ we denote by L_π the operator from $\mathfrak{h}_i$ to $\mathfrak{h}_j$:

$$L_\pi = L_{i_\ell,i_{\ell-1}}\ldots L_{i_1,i_0} = L_{j,i_{\ell-1}}\ldots L_{i_1,i}.$$

We can now prove:

Proposition 3.8. *The CP-TP map $\mathfrak{M}$ is irreducible if and only if, for every i and j in V, one of the following equivalent conditions holds:*

- *for any x in $\mathfrak{h}_i\backslash\{0\}$, the set $\{L_\pi x \,|\, \pi\in\mathcal{P}(i,j)\}$ is total in $\mathfrak{h}_j$,*
- *for any x in $\mathfrak{h}_i\backslash\{0\}$ and y in $\mathfrak{h}_j\backslash\{0\}$ there exists a path π in $\mathcal{P}(i,j)$ such that $\langle y, L_\pi x\rangle \neq 0$.*

Proof. This is an immediate application of Lemma 3.7, and the observation that, if $A_{j,i} = L_{j,i}\otimes|j\rangle\langle i|$, then

$$A_{j_\ell,i_\ell}\ldots A_{j_1,i_1} = \begin{cases} L_{j_\ell,i_\ell}\ldots L_{i_2,i_1}\otimes|j_\ell\rangle\langle i_1| & \text{if } i_\ell = j_{\ell-1},\ldots,i_2 = j_1,\\ 0 & \text{otherwise.}\end{cases}$$

□

Remark 3.9. Considering the irreducibility property (3.1), it would be natural, in analogy with the theory of Markov chains, to define a CP-TP map to be N-regular when there exists $N \in \mathbb{N}$ such that $\Phi^N(\rho) > 0$ for any $\rho \geq 0$ in $\mathcal{I}_1(\mathcal{H})\backslash\{0\}$. We can then prove that $\mathfrak{M}$ is N-regular if and only if for every i and j in V, for any x in $\mathfrak{h}_i\backslash\{0\}$, the set $\{L_\pi x \,|\, \pi \in \mathcal{P}_N(i,j)\}$ is total in $\mathfrak{h}_j$. Therefore, a necessary condition for N-regularity is $\operatorname{card} \mathcal{P}_N(i,j) \geq \dim \mathfrak{h}_j$ and in particular $\mathfrak{M}$ can be 1-regular (in which case it is also called positivity improving, see, e.g. [22]) only if $\dim \mathfrak{h}_j = 1$ for all j in V.

We can therefore give the following definition for an irreducible OQRW, which emphasizes our interpretation in terms of paths.

Definition 3.10. Let $\mathfrak{M}$ be an open quantum random walk. We say that two sites i, j in V are connected by $\mathfrak{M}$, which we denote by $i \stackrel{\mathfrak{M}}{\rightarrow} j$, if one of the equivalent conditions of Proposition 3.8 holds. As we have shown, $\mathfrak{M}$ is irreducible if and only if, for any two i and j in V, one has $i \stackrel{\mathfrak{M}}{\rightarrow} j$ and $j \stackrel{\mathfrak{M}}{\rightarrow} i$.

Remark 3.11. A minimal OQRW realization of a classical Markov chain is irreducible if and only if the Markov chain is irreducible in the classical sense.

Until now, we have basically found necessary and sufficient conditions for irreducibility of an open quantum random walk. In Sect. 6, we will discuss decompositions of reducible open quantum random walks into irreducible ones.

The following proposition essentially comes from [22]:

Proposition 3.12. *Assume a 2-positive map Φ on $\mathcal{I}_1(\mathcal{H})$ has an eigenvalue λ of modulus $\|\Phi\|$, with eigenvector ρ. Then:*

- *$\|\Phi\|$ is also an eigenvalue, with eigenvector $|\rho|$,*
- *if Φ is irreducible, then λ is a simple eigenvalue.*

In particular, if Φ is irreducible and has an eigenvalue of modulus $\|\Phi\|$, then $\|\Phi\|$ is a simple eigenvalue, with an eigenvector that is strictly positive.

Remark 3.13. The proof of the above result relies on the 2-positivity and irreducibility of Φ only. Therefore by Proposition 3.3, the same statement holds when the map Φ on $\mathcal{I}_1(\mathcal{H})$ is replaced with the map Φ^* on $\mathcal{B}(\mathcal{H})$ (alternatively, see [14, Theorem 3.1]).

Proof. Theorems 4.1 and 4.2 from [22] give us the first two statements. The third one follows from the fact that $\exp\|\Phi\| \times |\rho| = (\exp\Phi)(|\rho|) > 0$ by irreducibility. □

The following theorem is a direct application of Proposition 3.12:

Theorem 3.14. *An irreducible open quantum random walk $\mathfrak{M}$ has an invariant state if and only if 1 is an eigenvalue of $\mathfrak{M}$. If this is the case, then $\mathfrak{M}$ has only one invariant state, which in addition is faithful.*

By a simple application, we can obtain the following ergodic convergence result, which can be seen as a discrete time version of the Frigerio–Verri ergodic theorem ([12], Theorem 1.1):

Theorem 3.15. *Assume that an open quantum random walk* $\mathfrak{M}$ *is irreducible and has an invariant state* $\rho^{\rm inv}$*. For any state* ρ*, one has* $\frac{1}{n}\sum_{k=0}^{n-1}\mathfrak{M}^k(\rho) \to \rho^{\rm inv}$ *weakly.*

Proof. Let ρ be trace class, and define $\mathfrak{E}_n = \frac{1}{n}\sum_{k=0}^{n-1}\mathfrak{M}^k$. One has $\mathrm{Tr}[\mathfrak{E}_n(\rho)\,X] = \mathrm{Tr}[\rho\,\mathfrak{E}_n^*(X)]$. By the Banach–Alaoglu theorem, $\mathfrak{E}_n^*(X)$ has weak-$*$ convergent subsequences. Denote by Y the weak-$*$ limit of a subsequence $\mathfrak{E}_{n_k}^*(X)$; one has $\mathfrak{M}^*\circ\mathfrak{E}_{n_k}^*(X) \to \mathfrak{M}^*(Y)$, so that

$$\begin{aligned}\mathrm{Tr}[\rho\,(\mathrm{Id}-\mathfrak{M}^*)(Y)] &= \lim_k \mathrm{Tr}[\rho\,(\mathrm{Id}-\mathfrak{M}^*)\,\mathfrak{E}_{n_k}^*(X)]\\ &= \lim_k \mathrm{Tr}\left[\rho\,\frac{1}{n_k}\sum_{j=0}^{n_k-1}(\mathfrak{M}^{*j}-\mathfrak{M}^{*(j+1)})(X)\right]\\ &= \lim_k \mathrm{Tr}\left[\rho\,\frac{1}{n_k}(\mathrm{Id}-\mathfrak{M}^{*n_k})(X)\right] = 0\end{aligned}$$

so that $\mathfrak{M}^*(Y) = Y$. Since $\mathfrak{M}^*$ is a 2-positive irreducible unital operator, $Y = \lambda_X \mathrm{Id}$ (recall Remark 3.13) and we have $\lim_k \mathrm{Tr}\,[\mathfrak{E}_{n_k}(\rho)\,X] \to \lambda_X$ for any trace class ρ. Writing this for ρ equal to the eigenvector $\rho^{\rm inv}$ leads to $\lambda_X = \mathrm{Tr}(\rho^{\rm inv}X)$, showing that λ is independent of the subsequence $(n_k)_k$. When ρ is a state we obtain the desired convergence. This concludes the proof. □

4. Period and Aperiodicity for OQRWs

As in the previous section, we start with a review of the notion of period for a positive trace-preserving map Φ. Here, we follow Fagnola and Pellicer [9] and Groh [14]. We define $\overset{\mathrm{d}}{-}$ to be subtraction *modulo d*.

Definition 4.1. Let Φ be a positive, trace-preserving, irreducible map and let $(P_0,\ldots,P_{d-1})$ be a resolution of identity, i.e. a family of orthogonal projections such that $\sum_{k=0}^{d-1}P_k = \mathrm{Id}$. One says that $(P_0,\ldots,P_{d-1})$ is Φ-cyclic if $\Phi^*(P_k) = P_{k\overset{\mathrm{d}}{-}1}$ for $k = 0,\ldots,d-1$. The supremum of all d for which there exists a Φ-cyclic resolution of identity $(P_0,\ldots,P_{d-1})$ is called the period of Φ. If Φ has period 1 then we call it aperiodic.

Remark 4.2. We recall that a characterization of a cyclic resolution of the identity was already given, even if in an embryonic stage (and in the Schrödinger picture), in [8, Theorem 3.4].

The following result is a combination of Theorems 3.7 and 4.3 of Fagnola and Pellicer in [9] (the latter was also partially proven by Groh in [14]). Note that these results are proven in finite dimension, but they immediately extend to infinite dimension. Before we state it, let us recall that the point spectrum of an operator is its set of eigenvalues. The point spectrum of Φ^* will be denoted $\mathrm{Sp}_{\rm pp}\,\Phi^*$.

Proposition 4.3. *If* Φ *is an irreducible, 2-positive map on* $\mathcal{I}_1(\mathcal{H})$ *then the set* $\mathrm{Sp}_{pp}\Phi^*\cap\mathbb{T}$*, is a subgroup of the circle group* $\mathbb{T}$*. If in addition* Φ *has finite* *period d, then a primitive root of unity $e^{i2\pi/d}$ belongs to $\mathrm{Sp}_{pp}\Phi^*$ if and only if Φ is d-periodic.*

The following result is an immediate consequence of Proposition 4.3.

Proposition 4.4. *If a 2-positive TP map Φ on $\mathcal{I}_1(\mathcal{H})$ is irreducible and aperiodic with invariant state ρ^{inv}, and $\mathcal{H}$ is finite dimensional then*

- $\mathrm{Sp}_{\mathrm{pp}}\,\Phi \cap \mathbb{T} = \{1\}$
- *for any $\rho \in \mathcal{I}_1(\mathcal{H})$ one has $\Phi^n(\rho) \to \rho^{\mathrm{inv}}$ as $n \to \infty$.*

Remark 4.5. For a Φ^*-invariant weight (not necessarily a state) ρ and a cyclic resolution of identity $(P_0, \ldots, P_{d-1})$, the above definition implies that $\rho(P_k) = \rho(P_0)$ for all k.

Now, we consider once again the special case of an OQRW $\mathfrak{M}$.

Proposition 4.6. *A resolution of the identity $(P_0, \ldots, P_{d-1})$ is cyclic for an irreducible open quantum random walk $\mathfrak{M}$ if and only if $P_k = \sum_{j\in V} P_{k,j} \otimes |j\rangle\langle j|$ for every k, with projectors $P_{k,j}$ satisfying the relation*

$$P_{k,i} L_{i,j} = L_{i,j} P_{k \overset{d}{-} 1, j}. \tag{4.1}$$

Proof. Assume that there exists an $\mathfrak{M}$-cyclic resolution of identity $(P_0, \ldots, P_{d-1})$. Since $\mathfrak{M}^*(P_k) = P_{k\overset{d}{-}1}$, every P_k is block diagonal, i.e. $P_k = \sum_j P_{k,j} \otimes |j\rangle\langle j|$. By Theorem 5.4 from [9] (which is stated in the finite-dimensional case but can be immediately extended to infinite dimension), a resolution of the identity is $\mathfrak{M}$-cyclic if and only, for any i, j in V: $P_k\, L_{i,j} \otimes |i\rangle\langle j| = L_{i,j} \otimes |i\rangle\langle j|\, P_{k\overset{d}{-}1}$. Relation (4.1) follows by inspection. The converse is obvious. □

Remark 4.7. For classical, irreducible, d-periodic Markov chains with stochastic matrix K, the cyclic components are uniquely determined and coincide with the irreducible communication classes $C_0, \ldots, C_{d-1}$ of the (aperiodic) Markov chain with transition matrix K^d. In the quantum context, the role of the partition $C_0, \ldots, C_{d-1}$, or, better yet, of the corresponding indicator functions $\mathbb{1}_{C_0}, \ldots, \mathbb{1}_{C_{d-1}}$, is played by the cyclic projections $P_0, \ldots, P_{d-1}$. Indeed, notice that in the classical case $K\mathbb{1}_{C_k} = \mathbb{1}_{C_{k-1}}$ and, for the minimal OQRW realization of this Markov chain, the cyclic projections $P_0, \ldots, P_{d-1}$ are uniquely determined as $P_k = \sum_{j\in C_k} |j\rangle\langle j|$. However, an important difference should be underlined, with respect to the classical case: in general, the resolution of the identity which verifies the definition of the period is not uniquely determined, since the decomposition of Φ^d into minimal irreducible components is not unique in general, as we will see in Sect. 7. An example of this fact can be easily constructed, as we now describe.

Example 4.8. Take an OQRW $\mathfrak{M}$ with two sites and $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathbb{C}^2$, and introduce the matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then we consider

$$L_{11} = L_{21} = \frac{\mathrm{i}}{\sqrt{2}}\, R, \qquad L_{12} = L_{22} = -\frac{\mathrm{i}}{\sqrt{2}}\, R.$$

This $\mathfrak{M}$ is an irreducible OQRW (by a direct application of Proposition 3.8) with period 2, and the cyclic projections P_0, P_1 can be chosen in different ways:

$$P_0^{(x)} = |x\rangle\langle x| \otimes |1\rangle\langle 1| + |x\rangle\langle x| \otimes |2\rangle\langle 2|,$$
$$P_1^{(x)} = |Rx\rangle\langle Rx| \otimes |1\rangle\langle 1| + |Rx\rangle\langle Rx| \otimes |2\rangle\langle 2|$$

is a cyclic decomposition of the OQRW for any norm-one vector x in $\mathbb{C}^2$. As mentioned above, this is due to the fact that the map $\mathfrak{M}^2$ does not have a unique decomposition in irreducible components: $\mathfrak{M}^2$ is the OQRW with all transition operators L equal to $\mathrm{Id}/\sqrt{2}$. The decomposition of $\mathfrak{M}^2$ into irreducible components, however, is unique up to unitary equivalence. See [4] for related results (in the finite case).

We now discuss some results which will give us simple sufficient criteria for aperiodicity of an open quantum random walk.

Lemma 4.9. *Let $\mathfrak{M}$ be a d-periodic open quantum random walk. Let $i, j \in V$ and $x \in \operatorname{Ran} P_{k,i}$, $y \in \operatorname{Ran} P_{k',j}$ for some $k, k' \in \{0, \ldots, d-1\}$. For any path $\pi \in \mathcal{P}(i,j)$ of length ℓ one has $\langle y, L_\pi x\rangle = 0$ unless $k' - k = \ell \bmod d$.*

Proof. Relation (4.1) implies that $L_\pi x$ belongs to the range of $P_{k\overset{d}{+}\ell,j}$. □

Theorem 4.10. *Consider an irreducible open quantum random walk. For i in V, x in $\mathfrak{h}_i$, define*

$$D(i,x) = \mathrm{GCD}\{\ell \geq 1, \exists \pi \in \mathcal{P}_\ell(i,i) \text{ s.t. } \langle x, L_\pi x\rangle \neq 0\} \tag{4.2}$$

(GCD *denotes the greatest common divisor*). *Then, for every x in the range of $P_{k,i}$, the period d is a divisor of $D(i,x)$. In particular, if there exists i in V such that, for all $x \in \mathfrak{h}_i$, $D(i,x) = 1$, then the open quantum random walk is aperiodic.*

Proof. Irreducibility implies that the defining set of ℓ's is nonempty, so that $D(i,x)$ is well defined. The result follows from Lemma 4.9. □

Corollary 4.11. *Consider an irreducible open quantum random walk $\mathfrak{M}$. If there exists i in V such that*

$$\forall x \in \mathfrak{h}_i,\ \langle x, L_{i,i}\, x\rangle \neq 0 \tag{4.3}$$

then the open quantum random walk is aperiodic.

Remark 4.12. The definition of the quantity $D(i,x)$ in Theorem 4.10 has an interpretation in terms of paths, and is reminiscent of the definition of the period for a state i of a classical Markov chain with transition matrix K, i.e. $D(i) = \mathrm{GCD}\,\{\ell \geq 1 \,|\, K_{ii}^\ell > 0\}$. In addition, $D(i)$ coincides with (4.2) when applied to an OQRW which is a minimal OQRW realization of the Markov chain. In the quantum context, however, $D(i,x)$ does not always coincide with the period, and, in particular, is not invariant with the argument (i,x) even if the OQRW is irreducible (see Example 9.7). Even worse, the relation $d \,|\, D(i,x)$ may not hold if x does not belong to the range of some $P_{i,k}$. Since the P_k are a priori unknown, the practical study of the period of an OQRW is difficult when simple sufficient conditions (such as the condition for aperiodicity given in Theorem 4.10) do not hold.

In such cases, the following result can be helpful:

Proposition 4.13. *Consider an irreducible, finite, d-periodic open quantum random walk $\mathfrak{M}$. If for some i in V, and some ℓ prime with d, there exists a loop $\pi \in \mathcal{P}_\ell(i,i)$ of length ℓ, such that L_π is invertible, then d is a divisor of* $\dim \mathfrak{h}_i$.

Proof. By Bézout's lemma, for any k in $0,\ldots,d-1$ there exists an integer a such that $a\ell = k \bmod d$. Then $L_\pi^a P_{0,i} L_\pi^{-a} = P_{k,i}$, so that $\dim P_{k,i}$ does not depend on k. Therefore, $\dim \mathfrak{h}_i = d \dim P_{0,i}$ and the conclusion follows. □

Remark 4.14. As a consequence of Corollary 4.11, starting from a finite irreducible periodic open quantum random walk $\mathfrak{M}$ we can perturb it into an aperiodic one, $\mathfrak{M}_{(\varepsilon)}$, in different ways. If there exists i_0 in V such that $L_{i_0,i_0} = 0$ then one possible way is to define, for some $\varepsilon \in]0,1[$,

$$L^{(\varepsilon)}_{i,j} = L_{i,j} \quad \text{if } j \neq i_0 \quad \text{and} \quad L^{(\varepsilon)}_{i,i_0} = \begin{cases} \sqrt{\varepsilon}\,\mathrm{Id} & \text{if } i = i_0, \\ \sqrt{1-\varepsilon}\, L_{i,i_0} & \text{if } i \neq i_0. \end{cases}$$

This is the analogue of "adding a loop" for classical Markov chains. Another way is to "add a loop" at every site, a method we will use in Example 9.5.

For clarity, we restate Proposition 4.4 specifically for OQRWs:

Theorem 4.15. *Consider an irreducible, aperiodic and finite open quantum random walk $\mathfrak{M}$. For any state ρ, the sequence $(\mathfrak{M}^n(\rho))_n$ converges to the invariant state ρ^{inv} (which is unique, and faithful).*

5. Ergodic Properties of Irreducible OQRWs

We will now discuss the consequences of the previous theoretical results in terms of ergodic properties of irreducible open quantum random walks. We recall briefly the definitions of the (classical) random processes introduced in Sect. 2. Fixing an open quantum random walk $\mathfrak{M}$ on V defined by operators $(L_{i,j})_{i,j\in V}$ we define the set $\Omega = V^{\mathbb{N}}$, equipped with the σ-field generated by cylinder sets. An element of Ω is denoted by $\omega = (\omega_n)_{n\in\mathbb{N}}$. We define, for any state ρ on $\bigoplus_{i\in V} \mathfrak{h}_i$ of the form $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$, a probability $\mathbb{P}^{(n)}_\rho$ on V^{n+1} by

$$\mathbb{P}^{(n)}_\rho(\omega_0 = i_0,\ldots,\omega_n = i_n) = \mathrm{Tr}(L_{i_n,i_{n-1}}\ldots L_{i_1,i_0}\rho(i_0)L^*_{i_1,i_0}\ldots L^*_{i_n,i_{n-1}}).$$

One easily shows, using the stochasticity property (2.2), that the family $(\mathbb{P}^{(n)}_\rho)_n$ is consistent, and can therefore be extended uniquely to a probability $\mathbb{P}_\rho$ on Ω. We denote by $(X_n)_{n\in\mathbb{N}}$ the coordinate maps. Then $\mathfrak{M}^n(\rho, X_n)$ is defined by (2.3), and we let

$$\rho_n = \frac{L_{X_n,X_{n-1}}\ldots L_{X_1,X_0}\rho(X_0)L^*_{X_1,X_0}\ldots L^*_{X_n,X_{n-1}}}{\mathrm{Tr}\left(L_{X_n,X_{n-1}}\ldots L_{X_1,X_0}\rho(X_0)L^*_{X_1,X_0}\ldots L^*_{X_n,X_{n-1}}\right)}.$$

These processes mimic the behaviour of the measurement outcomes and of the associated resulting states, described in Sect. 2. In particular, all the statements in Sect. 2 describing laws remain true with $\mathbb{P}_\rho$ replacing $\mathbb{P}$. We will, from now on, usually drop the ρ in $\mathbb{P}_\rho$ in the proofs.

A first result regarding the ergodic behaviour of these quantities is the following, which is a consequence of the ergodic theorem due to Kümmerer and Maassen [16]. For completeness, we give a self-contained proof in the present framework.

Theorem 5.1 (Kümmerer–Maassen) *If the open quantum random walk $\mathfrak{M}$ is finite then, for any initial state ρ, there exists a random variable $\rho^{\text{inv}} = \sum_{i\in V} \rho^{\text{inv}}(i) \otimes |i\rangle\langle i|$ with values in the set of invariant states on $\mathcal{H} = \bigoplus_{i\in V} \mathfrak{h}_i$ such that $\mathbb{P}_\rho$-almost-surely,*

$$\frac{1}{n}\sum_{k=0}^{n} \rho_k \otimes |X_k\rangle\langle X_k| \underset{n\to\infty}{\longrightarrow} \sum_{i\in V} \rho^{\text{inv}}(i) \otimes |i\rangle\langle i|.$$

Proof. Let η_n be the state $\rho_n \otimes |X_n\rangle\langle X_n|$. Denote by $\mathcal{F}_n$ the σ-algebra generated by η_k for $k \le n$, and let

$$m_n = \sum_{k=0}^{n} \eta_k - \sum_{k=0}^{n-1} \mathfrak{M}(\eta_k).$$

We have, from (2.7), that $\mathbb{E}(m_{n+1} - m_n|\mathcal{F}_n) = 0$ so that $(m_n)_n$ is a martingale, and since $\|m_{n+1} - m_n\| = \|\eta_{n+1} - \mathfrak{M}(\eta_n)\|$ is uniformly bounded, we can apply the law of large numbers for martingales with uniformly bounded increments. Therefore, $\frac{1}{n}\sum_{k=0}^{n} \eta_k - \frac{1}{n}\sum_{i=0}^{n-1} \mathfrak{M}(\eta_k) \to 0$ where convergence is meant in the almost-sure sense. In turn, this implies for any $N \in \mathbb{N}^*$,

$$\frac{1}{n}\sum_{k=0}^{n} \eta_k - \frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{M}^N(\eta_k) \to 0$$

so that

$$\frac{1}{n}\sum_{k=0}^{n} \eta_k - \frac{1}{n}\sum_{k=0}^{n-1} \frac{\text{Id} + \mathfrak{M} + \cdots + \mathfrak{M}^{N-1}}{N}(\eta_k) \to 0.$$

For any state η, $\frac{\text{Id}+\mathfrak{M}+\cdots+\mathfrak{M}^{N-1}}{N}(\eta)$ converges when N goes to infinity to an invariant state. This can be seen viewing $\mathfrak{M}$ as a contraction on the Hilbert–Schmidt space $\mathcal{I}_2(\mathcal{H})$, i.e. $\mathcal{B}(\mathcal{H})$ equipped with the scalar product $\text{Tr}(A^*B)$. This invariant state must be of the form $P\eta$, where P is a linear operator on $\mathcal{I}_1(\mathcal{H})$. The operator P can be approximated uniformly by $\frac{I+\mathfrak{M}+\cdots+\mathfrak{M}^{N-1}}{N}$, therefore, $\frac{1}{n}\sum_{k=0}^{n} \eta_k - \frac{1}{n}\sum_{k=0}^{n-1} P\eta_k \to 0$. On the other hand $P\mathfrak{M} = P$ implies that $\mathbb{E}(P\eta_{n+1}|\mathcal{F}_n) = P\eta_n$, i.e. $(P\eta_n)_n$ is a bounded martingale, so $\frac{1}{n}\sum_{k=0}^{n} P\eta_k$ converges almost-surely to some invariant state. This concludes the proof. □

A direct consequence of Theorem 5.1 (of which we shall preserve the notations) and of our previous observations on the form of ρ^{inv} is the following:

Corollary 5.2. *If the open quantum random walk $\mathfrak{M}$ is finite and irreducible with invariant (and faithful) state $\rho^{\mathrm{inv}} = \sum_{i\in V} \rho^{\mathrm{inv}}(i) \otimes |i\rangle\langle i|$, then for a given initial state ρ, all i in V, define $N_n(i) = \mathrm{card}\,\{k \le n\,|\,X_k = i\}$. We have for any initial state ρ*

$$\frac{N_n(i)}{n} \underset{n\to\infty}{\longrightarrow} \mathrm{Tr}\,\rho^{\mathrm{inv}}(i) \quad \mathbb{P}_\rho\text{-almost-surely,}$$

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}(X_k = i) \underset{n\to\infty}{\longrightarrow} \mathrm{Tr}\,\rho^{\mathrm{inv}}(i),$$

$$\frac{1}{N_n(i)}\sum_{k=0}^{n-1} \rho_k\, \mathbb{1}_{X_k=i} \underset{n\to\infty}{\longrightarrow} \frac{\rho^{\mathrm{inv}}(i)}{\mathrm{Tr}\,\rho^{\mathrm{inv}}(i)} \quad \mathbb{P}_\rho\text{-almost-surely.}$$

Proof. This is simply obtained by examination from Theorem 5.1. □

Remark 5.3. Theorem 3.14 tells us that the state ρ^{inv} is unique and faithful, and in particular $\mathrm{Tr}\,\rho^{\mathrm{inv}}(i) > 0$ for any i in V. This implies that, for any irreducible open quantum random walk with an invariant state ρ^{inv}, one has

$$\text{for all } i \in V, \quad \mathbb{P}(X_n = i \text{ infinitely often}) = 1,$$
$$\text{for all } i \in V,\ x \in \mathfrak{h}_i, \quad \mathbb{P}(\langle x, \rho_n(i)\, x\rangle\, \mathbb{1}_{X_n=i} > 0 \text{ infinitely often}) = 1.$$

The first statement has an immediate interpretation in terms of "spatial recurrence" (every site i in V is visited infinitely often), the second one is stronger and can be seen as "spatial and internal recurrence".

The second ergodic result of this section is a consequence of Theorem 3.15.

Corollary 5.4. *If the open quantum random walk $\mathfrak{M}$ is irreducible with invariant (and faithful) state $\rho^{\mathrm{inv}} = \sum_{i\in V} \rho^{\mathrm{inv}}(i) \otimes |i\rangle\langle i|$, then for any initial state ρ, for all i in V,*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathbb{P}(X_k = i) \underset{n\to\infty}{\longrightarrow} \mathrm{Tr}\rho^{\mathrm{inv}}(i),$$

$$\frac{1}{n}\sum_{k=0}^{n-1} \mathfrak{M}^k(\rho, i) \underset{n\to\infty}{\longrightarrow} \rho^{\mathrm{inv}}(i) \text{ in the weak-}*\text{ sense.}$$

Remark 5.5. The assumption that there exists an invariant state is necessary in Corollary 5.4 (contrary to Corollaries 5.2 and 5.6, where it is always true and only stated to establish notations), because we do not assume finiteness of $\mathfrak{M}$. The first statement of Corollary 5.4 is a refinement of the second statement of Corollary 5.2.

Our third corollary is a consequence of Theorem 4.15. It is an improvement of the previous result in the case where we have aperiodicity.

Corollary 5.6. *If the open quantum random walk $\mathfrak{M}$ is finite, irreducible and aperiodic with invariant (and faithful) state $\rho^{\mathrm{inv}} = \sum_{i\in V} \rho^{\mathrm{inv}}(i) \otimes |i\rangle\langle i|$, then* *for any initial state* ρ*, for all* i *in* V*,*

$$\mathbb{P}(X_n = i) \underset{n\to\infty}{\longrightarrow} \mathrm{Tr}\rho^{\mathrm{inv}}(i),$$
$$\mathfrak{M}^n(\rho, i) \underset{n\to\infty}{\longrightarrow} \rho^{\mathrm{inv}}(i).$$

Remark 5.7.
- Corollary 5.4 seems rather useless, from an operational point of view: there is no joint realization of the different $\mathfrak{M}^k(\rho, i)$ for different k; or, in other terms, measuring X_k disrupts the existence of $\mathfrak{M}^{k'}(\rho, i)$ for $k' > k$. Corollary 5.6, on the other hand, is operational, and tells us that, if the system is left to evolve for a large time, then a single measurement will give the position i with approximate probability $\mathrm{Tr}\rho^{\mathrm{inv}}(i)$, and the state after that unique measurement will be approximately $\frac{\rho^{\mathrm{inv}}(i)}{\mathrm{Tr}\rho^{\mathrm{inv}}(i)}$. These limiting quantities are the same as those that appear for limits with measurements. These results display evident similarities with the behaviour of classical Markov chains.
- Example 9.5 suggests that aperiodicity is a necessary assumption for Corollary 5.6, as one could expect from analogous results for classical Markov chains.

6. Reducible OQRWs and Communication Classes

In this section, we study the failure of irreducibility for an open quantum random walk. Considering reducible OQRWs, the first natural problem one has to face is how to characterize reducibility and how to determine reducing, possibly minimal, components.

A reasonable way to proceed, mimicking what happens for classical Markov chains, is to define a communication relation between vectors of the Hilbert space $\mathcal{H}$: this relation should be an equivalence relation constructed in such a way that the induced equivalence classes are the irreducible components of the map $\mathfrak{M}$. We will see that it is possible to do this in a way which is consistent with the classical case.

However, it is important to immediately underline that the quantum case displays peculiar features: the decomposition of the Hilbert space $\mathcal{H}$ as the direct sum of irreducible components is not unique in general. This is not at all surprising if one thinks about the structure of invariant states for a CP-TP map (see [4], from which we take much of our inspiration): essentially, the quantum peculiarity is related to the fact that there can exist invariant states which are not simple convex combinations of the invariant states on each irreducible component.

We recall (see Definition 3.6) that, following Baumgartner and Narnhofer [4], we call a closed vector space $\mathcal{V}$ an *enclosure* for $\mathfrak{M}$ if $\mathrm{supp}\,\rho \subset \mathcal{V}$, with ρ a positive trace-class operator, implies $\mathrm{supp}\,\mathfrak{M}(\rho) \subset \mathcal{V}$. The next proposition will be extremely useful.

From now on, we fix an OQRW $\mathfrak{M}$ with the same notation as in Sect. 2.

Proposition 6.1. 1. *A closed subspace $\mathcal{V}$ of $\mathcal{H}$ is an enclosure if and only if $L_\pi \otimes |j\rangle\langle i|\,\mathcal{V} \subset \mathcal{V}$ for any i, j in V and $\pi \in \mathcal{P}(i,j)$. In particular, if a vector $x = \sum_{i\in V} x_i \otimes |i\rangle$ is in an enclosure $\mathcal{V}$ then $L_\pi x_i \otimes |j\rangle \in \mathcal{V}$.*

2. *A projection P reduces $\mathfrak{M}$ if and only if*
$$P(L_\pi \otimes |j\rangle\langle i|)P = (L_\pi \otimes |j\rangle\langle i|)P \quad \text{for all } i, j \text{ in } V \text{ and } \pi \in \mathcal{P}(i,j).$$
3. *If $\mathcal{V}$ is an enclosure, then $(L_\pi \otimes |j\rangle\langle i|)(\mathcal{V}) \subset \mathfrak{h}_j \cap \mathcal{V}$ for any i, j in V and $\pi \in \mathcal{P}(i,j)$, and $\bigoplus_{j\in V} \mathrm{Vect}\{(L_\pi \otimes |j\rangle\langle i|)(\mathcal{V}), i \in V, \pi \in \mathcal{P}(i,j)\}$ is also an enclosure.*
4. *The support of an invariant state is an enclosure.*

Proof. 1. Suppose that $\mathcal{V}$ is an enclosure. Remark that, for any positive integer ℓ and any $x = \sum_{i\in V} x_i \otimes |i\rangle$ in $\mathcal{H}$, one has
$$\mathfrak{M}^\ell(|x\rangle\langle x|) = \sum_{i,j\in V} \sum_{\pi\in\mathcal{P}_\ell(i,j)} |L_\pi x_i\rangle\langle L_\pi x_i| \otimes |j\rangle\langle j|; \tag{6.1}$$
so, if x is in $\mathcal{V}$, then every $L_\pi x_i \otimes |j\rangle$ is in $\mathcal{V}$. Conversely, let us now suppose that $\mathcal{V}$ is stable under the action of the operators $L_\pi \otimes |j\rangle\langle i|$. Starting from ρ with $\operatorname{supp}\rho$ in $\mathcal{V}$, then considering its spectral decomposition and (6.1) above shows that $\operatorname{supp}\mathfrak{M}^\ell(\rho) \subset \mathcal{V}$.

2. Just recall that a subspace of $\mathcal{H}$ is the support of a reducing projection if and only if it is an enclosure. This point is then an immediate consequence of the previous one.
3. This point also follows immediately from 1.
4. Consider an invariant state ρ_0 and a state ρ with support contained in $\operatorname{supp}\rho_0$. Then there exists a weak approximation of ρ by an increasing sequence of finite-dimensional operators ρ_n with $\operatorname{supp}\rho_n \subset \operatorname{supp}\rho_0$. Furthermore, for every n, there exists a λ_n such that $\rho_n \leq \lambda_n \rho_0$, so that $\mathfrak{M}(\rho_n) \leq \lambda_n\rho_0$ and $\operatorname{supp}\mathfrak{M}(\rho_n) \subset \operatorname{supp}\rho_0$. The sequence $\mathfrak{M}(\rho_n)$ is increasing and weakly convergent to $\mathfrak{M}(\rho)$ so that $\operatorname{supp}\mathfrak{M}(\rho) \subset \operatorname{supp}\rho_0$, which proves that $\operatorname{supp}\rho_0$ is an enclosure. □

In general, it is not true that all the reducing projections are diagonal, i.e. of the form $\sum_{i\in V} P_i \otimes |i\rangle\langle i|$, but by the previous Proposition, point 3, if $\mathfrak{M}$ is reducible, then it admits at least one block-diagonal reducing projection. So the reducibility of an OQRW can be established considering only block-diagonal projections. Moreover, notice that the support projection of an invariant state is block diagonal, i.e. of the form $P = \sum_{i\in V} P_i \otimes |i\rangle\langle i|$.

We can characterize the block-diagonal projections reducing $\mathfrak{M}$ using the unravelling of $\mathfrak{M}$.

Proposition 6.2. *An orthogonal block-diagonal projection $P = \sum_j P_j \otimes |j\rangle\langle j|$ reduces $\mathfrak{M}$ if and only if $\operatorname{Ran} L_{i,j}P_j \subset \operatorname{Ran} P_i$, (i.e. $L_{i,j}P_j = P_i L_{i,j} P_j$) for all i and j in V. Equivalently, a closed subspace of the form $\mathcal{V} = \oplus_{i\in V}\mathcal{V}_i$, with $\mathcal{V}_i \subset \mathfrak{h}_i$, is an enclosure if and only if $L_{i,j}\mathcal{V}_j \subset \mathcal{V}_i$ for all i and j.*

Proof. It is clear that it is sufficient to prove only the first statement. So, take a reducing projection $P = \sum_{j\in V} P_j \otimes |j\rangle\langle j|$. By point 2 in the previous proposition, it is necessary that the range of P is invariant under the action of all operators of the form $L_{i,j} \otimes |i\rangle\langle j|$ and so the relation $L_{i,j}P_j = P_i L_{i,j} P_j$ for all i and j in V immediately follows. The reverse implication is also easy to obtain using again the characterization in point 2 of previous proposition and the fact that any operator $L_\pi \otimes |i\rangle\langle j|$, for a path $\pi = (i_0, i_1, ...i_\ell) \in \mathcal{P}(i,j)$ (with $i = i_0$, $j = i_\ell$) of length ℓ, is the composition of the operators $L_{i_{k+1},i_k} \otimes |i_{k+1}\rangle\langle i_k|$, with $k = 0, \ldots, \ell - 1$. □

Corollary 6.3. *When $P = \sum_{j\in V} P_j \otimes |j\rangle\langle j|$ is a reducing projection, each P_j is a projection on a subspace preserved by L_{jj}.*

Remark 6.4. Suppose that, for all sites i and j, there exists "a path of invertible operators which connects them", i.e. a path $\pi \in \mathcal{P}(i,j)$ such that L_π is invertible. In this case, the previous proposition proves that, if $P = \sum_{j\in V} P_j \otimes |j\rangle\langle j|$ is a reducing projection, then $\operatorname{rank} P_i = \operatorname{rank} P_j$ for any i, j in V. In particular, if a state $\rho = \sum_{i\in V} \rho_i \otimes |i\rangle\langle i|$ is invariant, then $\rho_i \neq 0$ for all $i \in V$ (i.e. an invariant state is supported by all sites), and if ρ_i is faithful on $\mathfrak{h}_i$ for some index i, then ρ_j is faithful on $\mathfrak{h}_j$ for any $j \in V$.

The next notion, of enclosure generated by a single vector in $\mathcal{H}$, will be crucial in our analysis of decompositions of reducible OQRWs:

Definition 6.5. *For ϕ in $\mathcal{H}$, we denote by* $\mathrm{Enc}(\phi)$ *the closed vector space*

$$\mathrm{Enc}(\phi) = \overline{\mathrm{Vect} \bigcup_{i,j\in V} \{(L_\pi \otimes |j\rangle\langle i|)\, \phi \,|\, \pi \in \mathcal{P}(i,j)\}}.$$

Consistently with Proposition 6.1, we will consider specifically enclosures of vectors $x \otimes |i\rangle$, which take the form

$$\mathrm{Enc}(x \otimes |i\rangle) = \overline{\mathrm{Vect} \bigcup_{j\in V} \{L_\pi\, x \otimes |j\rangle \,|\, \pi \in \mathcal{P}(i,j)\}}.$$

We will be mostly interested in enclosures that are minimal but non-trivial (i.e. not equal to $\{0\}$). From now on, the term *minimal enclosure* will refer to minimal, non-trivial enclosures. The following lemma contains relevant properties of enclosures.

Lemma 6.6.
- *The space* $\mathrm{Enc}(x \otimes |i\rangle)$ *is the smallest enclosure containing $x \otimes |i\rangle$.*
- *Any minimal enclosure is of the form* $\mathrm{Enc}(x \otimes |i\rangle)$*.*
- *If two minimal enclosures* $\mathrm{Enc}(x \otimes |i\rangle)$ *and* $\mathrm{Enc}(y \otimes |j\rangle)$ *are distinct then they are in direct sum.*

Proof. All statements follow from Proposition 6.1. □

Remark 6.7. In the same way that the specific form of $\mathfrak{M}(\rho)$ led us to consider only states ρ of the form $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$, Proposition 6.1 shows that vectors of the form $x \otimes |i\rangle$ are of particular interest. In particular, any minimal enclosure will be generated by a vector $x \otimes |i\rangle$. It is not true, however, that any $\mathrm{Enc}(x \otimes |i\rangle)$ is a minimal enclosure, as the following example shows.

Example 6.8. Take $V = \{1,2,3\}$ with

$$L_{1,2} = L_{2,3} = L_{3,1} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \qquad L_{2,1} = L_{3,2} = L_{1,3} = \frac{1}{\sqrt{5}}\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}.$$

One can see that for $k = 1,2$, $\mathrm{Enc}(e_k \otimes |1\rangle)$, are minimal enclosures, equal to $\mathbb{C}\, e_k \otimes \mathbb{C}^V$, but the space $\mathrm{Enc}((e_1+e_2)\otimes|1\rangle)$ is equal to $\mathbb{C}^2 \otimes \mathbb{C}^V$.

Remark 6.9. Let us return to the notion of irreducibility, as introduced in Definition 3.10: an open quantum random walk $\mathfrak{M}$ is irreducible if for any i,j in V, one has $i \stackrel{\mathfrak{M}}{\to} j$, which by Proposition 3.8 is defined by the equivalent conditions

$$\forall\, x \in \mathfrak{h}_i,\ y \in \mathfrak{h}_j,\ \exists \pi \in \mathcal{P}(i,j) \text{ such that } \langle y, L_\pi x\rangle \neq 0, \tag{6.2}$$

$$\forall\, x \in \mathfrak{h}_i,\ y \in \mathfrak{h}_j,\ y \in \overline{\mathrm{Vect}\,\{L_\pi\, x \,|\, \pi \in \mathcal{P}(i,j)\}}. \tag{6.3}$$

From the above discussion, it is clear that both conditions can be characterized using enclosures:

$$\exists \pi \in \mathcal{P}(i,j) \text{ such that } \langle y, L_\pi x\rangle \neq 0 \Leftrightarrow y \notin \mathrm{Enc}(x\otimes|i\rangle)^\perp$$
$$y \in \overline{\mathrm{Vect}\,\{L_\pi\, x \,|\, \pi \in \mathcal{P}(i,j)\}} \Leftrightarrow y \in \mathrm{Enc}(x\otimes|i\rangle).$$

As we will see below, in Proposition 7.3, the orthogonal of an enclosure can be related to another enclosure. This will allow us to strengthen the connection between the two notions above.

The above discussion gives immediately:

Lemma 6.10. *An open quantum random walk $\mathfrak{M}$ is irreducible if and only if $\mathcal{H}$ is a minimal enclosure, or equivalently, if $\mathcal{H} = \mathrm{Enc}(x\otimes|i\rangle)$ for any $x\otimes|i\rangle$ in $\mathcal{H}$.*

To emphasize the picturesque aspect of our definition of irreducibility, we define the following notion of accessibility among vectors, denoted by $\stackrel{\mathfrak{M}}{\to}$. We remark that the notation, $\stackrel{\mathfrak{M}}{\to}$, is the same we used in Definition 3.10, but this should not generate confusion, the difference being clear in the arguments we use: in the previous case, the connection $\stackrel{\mathfrak{M}}{\to}$ is between sites i, j in V, whereas here it is between vectors ϕ, ψ of the Hilbert space $\mathcal{H}$.

Definition 6.11. *For ϕ, ψ in $\mathcal{H}$, we denote $\phi \stackrel{\mathfrak{M}}{\to} \psi$ if $\psi \in \mathrm{Enc}(\phi)$, and $\phi \stackrel{\mathfrak{M}}{\leftrightarrow} \psi$ if $\phi \stackrel{\mathfrak{M}}{\to} \psi$ and $\psi \stackrel{\mathfrak{M}}{\to} \phi$.*

Again, we will be specifically interested in the relation $\stackrel{\mathfrak{M}}{\to}$ between vectors of the form $x\otimes|i\rangle$, $y\otimes|j\rangle$ and we have immediately

$$x\otimes|i\rangle \stackrel{\mathfrak{M}}{\to} y\otimes|j\rangle \Leftrightarrow y \in \overline{\{L_\pi\, x \,|\, \pi \in \mathcal{P}(i,j)\}}.$$

The following Proposition can easily be proven:

Proposition 6.12. *The relation $\stackrel{\mathfrak{M}}{\to}$ on $\mathcal{H}$ is transitive, and $\stackrel{\mathfrak{M}}{\leftrightarrow}$ is an equivalence relation. Any minimal enclosure deprived of 0 is an equivalence class of $\stackrel{\mathfrak{M}}{\leftrightarrow}$.*

Remark 6.13. Every equivalence class of a vector $x \otimes |i\rangle$ by $\overset{\mathfrak{M}}{\leftrightarrow}$ is a subset of $\mathcal{H}$ contained in $\mathrm{Enc}(x \otimes |i\rangle)$, but it may fail to be an enclosure and even a subspace. A minimal OQRW realization of a classical Markov chain with a proper transient class easily gives an example of an equivalence class that is not an enclosure. For an example where an equivalence class is not a subspace, consider Example 6.14 below.

Example 6.14. Consider $V = \{1,2\}$, $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathbb{C}^2$ with canonical basis denoted by (e_1, e_2), and introduce the OQRW $\mathfrak{M}$ with transitions

$$L_{1,1} = L_{2,2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad L_{1,1} = L_{2,2} = \frac{1}{\sqrt{2}}\,\mathrm{Id}.$$

Then, the only minimal enclosures are

$$\begin{aligned} E_+ &= \mathrm{Enc}\big((e_1 + e_2) \otimes |1\rangle\big) = \mathbb{C}\,(e_1 + e_2) \otimes \mathbb{C}^V \\ E_- &= \mathrm{Enc}\big((e_1 - e_2) \otimes |1\rangle\big) = \mathbb{C}\,(e_1 - e_2) \otimes \mathbb{C}^V \end{aligned}$$

and for any $x \notin \mathbb{C}(e_1 + e_2) \cup \mathbb{C}(e_1 - e_2)$ one has

$$\mathrm{Enc}(x \otimes |1\rangle) = \mathrm{Enc}(x \otimes |2\rangle) = \mathcal{H}.$$

Therefore, for such an x, the equivalence class of $x \otimes |1\rangle$ is $\mathcal{H} \setminus (E_+ \cup E_-)$.

7. Decompositions of OQRWs and Invariant States

In this section, we wish to focus on the behaviour of an OQRW $\mathfrak{M}$ on the so-called fast recurrent subspace, i.e. the support of the $\mathfrak{M}$-invariant states. We decompose the corresponding restriction of $\mathfrak{M}$ into a "direct sum" of irreducible OQRWs $\mathfrak{M}_k$, and study how the different irreducible components interact. We follow the lines traced in [4] for quantum evolutions on finite-dimensional Hilbert spaces; we will state and prove generalizations to infinite dimension. As we will see in Proposition 7.3, the form of invariant states is dictated by the uniqueness or non-uniqueness of the decompositions into minimal enclosures, and Lemma 7.7 shows that non-uniqueness is related to the existence of mutually non-orthogonal minimal enclosures. We will need to consider the closed vector space generated by non-orthogonal subspaces E and F with $E \cap F = \{0\}$; we denote by $E + F$ this closed vector space, while we use the symbol "$\oplus$" when the subspaces E and F are necessarily orthogonal.

To further study the invariant states of an OQRW, we recall some notation. Inspired by [12], we denote:

$$\mathcal{R} = \sup\{\mathrm{supp}\,\rho \,|\, \rho \text{ an invariant state}\}.$$

This space is often called the fast recurrent space, see [13,24] (it is the counterpart of the fast recurrent, or positive recurrent, sets for Markov chains). We let $\mathcal{D} = \mathcal{R}^{\perp}$, which is characterized as

$$\mathcal{D} = \{\phi \in \mathcal{H} \,|\, \langle \phi, \rho\, \phi\rangle = 0 \text{ for any invariant state } \rho\}. \tag{7.1}$$

Remark 7.1. The above definition of $\mathcal{R}$ is unfortunately not explicit, and makes a (small) part of Theorem 7.13 describing invariant states tautological. In the finite-dimensional case, $\mathcal{R}$ can be described without reference to the set of invariant states, as $\mathcal{R} = \mathcal{D}^\perp$, where $\mathcal{D}$ is defined by

$$\mathcal{D} = \{\phi \in \mathcal{H} \,|\, \langle \phi, \mathfrak{M}^n(\rho)\, \phi\rangle \underset{n\to\infty}{\longrightarrow} 0 \text{ for any state } \rho\}. \tag{7.2}$$

In the infinite-dimensional case, however, it is not clear that (7.1) and (7.2) coincide. This is similar to the classical difficulty arising with infinite recurrent points in classical Markov chains with infinite state space. We will return to this problem in a future paper.

The following Lemma is an immediate consequence of Proposition 6.1.

Lemma 7.2. *The subspace $\mathcal{R}$ is an enclosure.*

From the block-diagonal structure of $\mathfrak{M}^n(\rho)$ for ρ any state, we clearly have

$$\mathcal{R} = \bigoplus_{i\in V} \mathcal{R}_i \text{ with } \mathcal{R}_i \subset \mathfrak{h}_i, \qquad \mathcal{D} = \bigoplus_{i\in V} \mathcal{D}_i \text{ with } \mathcal{D}_i \subset \mathfrak{h}_i.$$

Since our main interest is to investigate the invariant states of an open quantum random walk $\mathfrak{M}$ on $\mathcal{H}$, we will be interested in decomposing $\mathcal{R}$, not $\mathcal{D}$, into irreducible subsystems.

We will use the following results, which were stated in [4] in the finite-dimensional case. We extend them here to infinite dimension. We say that a vector space $\mathcal{V}$ has a unique decomposition into a direct sum of minimal enclosures if $\mathcal{V} = \sum_{i\in I} E_i = \sum_{j\in J} E'_j$ with all E_i and E'_j enclosures, implies that card I = card J and there exists a permutation σ from I onto J such that $E'_i = E_{\sigma(i)}$ for all i.

Proposition 7.3. *Assume that $\mathcal{V}$ and $\mathcal{W}$ are two subspaces of $\mathcal{H}$ such that $\mathcal{V}\cap\mathcal{W} = \{0\}$ and denote by $P_\mathcal{V}$ and $P_\mathcal{W}$ the respective orthogonal projections. For a state ρ with support in $\mathcal{V}+\mathcal{W}$, we introduce the decomposition $\rho = \rho_\mathcal{V} + \rho_\mathcal{W} + \rho_\mathcal{C} + \rho'_\mathcal{C}$ with*

$$\rho_\mathcal{V} = P_\mathcal{V}\,\rho\,P_\mathcal{V}, \quad \rho_\mathcal{W} = P_\mathcal{W}\,\rho\,P_\mathcal{W} \quad \rho_\mathcal{C} = P_\mathcal{V}\,\rho\,P_\mathcal{W}, \quad \rho'_\mathcal{C} = P_\mathcal{W}\,\rho\,P_\mathcal{V}.$$

Decompose $\mathfrak{M}(\rho)$ in a similar way. Then the following facts hold.

1. *If $\mathcal{V}$ is an enclosure, then $P_\mathcal{W}\,\mathfrak{M}(\rho_C + \rho'_C)\,P_\mathcal{W} = 0$.*
2. *If $\mathcal{V}$ is an enclosure, then so is $\mathcal{V}^\perp \cap \mathcal{R}$.*
3. *If $\mathcal{V}$ and $\mathcal{W}$ are enclosures, then*

$$\mathfrak{M}(\rho)_\mathcal{V} = \mathfrak{M}(\rho_\mathcal{V}) \quad \mathfrak{M}(\rho)_\mathcal{W} = \mathfrak{M}(\rho_\mathcal{W}) \quad \mathfrak{M}(\rho)_\mathcal{C} = \mathfrak{M}(\rho_\mathcal{C}) \quad \mathfrak{M}(\rho)'_\mathcal{C} = \mathfrak{M}(\rho'_\mathcal{C}).$$

4. *A subspace of $\mathcal{R}$ is a minimal enclosure if and only if it is the support of an extremal invariant state. In particular, if $\mathcal{V} \subset \mathcal{R}$ is an enclosure, then it contains a (non-trivial) minimal enclosure.*
5. *If ρ is $\mathfrak{M}$-invariant and $\mathcal{V}$ and $\mathcal{W}$ are two minimal enclosures contained in $\mathcal{R}$, such that the decomposition of $\mathcal{V}+\mathcal{W}$ into a sum of minimal enclosures is unique, then $\rho_\mathcal{C} = 0$ and $\rho'_\mathcal{C} = 0$.*

Proof. We essentially borrow the main ideas of the proofs from [4], adding some variations when required by the infinite-dimensional setting.

1. To prove the first point, we define $\kappa_{\pm\varepsilon} = \frac{1}{\varepsilon}\,\rho_{\mathcal{V}} \pm (\rho_{\mathcal{C}} + \rho'_{\mathcal{C}}) + \varepsilon\,\rho_{\mathcal{W}}$, for $\varepsilon > 0$. We have $\kappa_{\pm\varepsilon} \geq 0$ (as can be checked from $\langle u, \kappa_{\pm\varepsilon}\,u\rangle = \langle u_{\pm\varepsilon}, \rho\,u_{\pm\varepsilon}\rangle$ where $u_{\pm\varepsilon} = \frac{1}{\sqrt{\varepsilon}}\,P_{\mathcal{V}}u + \sqrt{\varepsilon}\,P_{\mathcal{W}}u$), so that $\mathfrak{M}(\kappa_{\pm\varepsilon}) \geq 0$, and, because $\mathcal{V}$ is an enclosure, the support of $\mathfrak{M}(\rho_{\mathcal{V}})$ is contained in $\mathcal{V}$, so that
$$P_{\mathcal{W}}\,\mathfrak{M}(\kappa_{\pm\varepsilon})\,P_{\mathcal{W}} = \pm P_{\mathcal{W}}\,\mathfrak{M}(\rho_{\mathcal{C}} + \rho'_{\mathcal{C}})\,P_{\mathcal{W}} + \varepsilon\,P_{\mathcal{W}}\,\mathfrak{M}(\rho_{\mathcal{W}})\,P_{\mathcal{W}}.$$
This is non-negative for any ε, and by necessity $P_{\mathcal{W}}\,\mathfrak{M}(\rho_{\mathcal{C}} + \rho'_{\mathcal{C}})\,P_{\mathcal{W}} = 0$.
2. Consider $\mathcal{W} = \mathcal{V}^{\perp}$ and η any invariant state; then
$$\eta_{\mathcal{V}} + \eta_{\mathcal{W}} + \eta_{\mathcal{C}} + \eta'_{\mathcal{C}} = \mathfrak{M}(\eta_{\mathcal{V}}) + \mathfrak{M}(\eta_{\mathcal{W}}) + \mathfrak{M}(\eta_{\mathcal{C}}) + \mathfrak{M}(\eta'_{\mathcal{C}}).$$
Projecting by $P_{\mathcal{W}}$ this yields $\eta_{\mathcal{W}} = P_{\mathcal{W}}\mathfrak{M}(\eta_{\mathcal{W}})P_{\mathcal{W}}$. Since
$$\operatorname{Tr}\eta_{\mathcal{W}} = \operatorname{Tr}\mathfrak{M}(\eta_{\mathcal{W}}) = \operatorname{Tr}P_{\mathcal{W}}\mathfrak{M}(\eta_{\mathcal{W}})P_{\mathcal{W}} + \operatorname{Tr}P_{\mathcal{V}}\mathfrak{M}(\eta_{\mathcal{W}})P_{\mathcal{V}},$$
this implies that $P_{\mathcal{V}}\,\mathfrak{M}(\eta_{\mathcal{W}})\,P_{\mathcal{V}}$ is positive with zero trace. Therefore $P_{\mathcal{V}}\,\mathfrak{M}(\eta_{\mathcal{W}})\,P_{\mathcal{V}} = 0$ which implies $P_{\mathcal{V}}\,\mathfrak{M}(\eta_{\mathcal{W}}) = \mathfrak{M}(\eta_{\mathcal{W}})\,P_{\mathcal{V}} = 0$ and so $\eta_{\mathcal{W}} = \mathfrak{M}(\eta_{\mathcal{W}})$. As the support of a invariant state, $\operatorname{supp}\eta_{\mathcal{W}} = \operatorname{supp}\eta \cap \mathcal{V}^{\perp}$ is an enclosure. Taking the supremum over all possible invariant states η, this tells us that $\mathcal{R} \cap \mathcal{V}^{\perp}$ is also an enclosure.
3. If both $\mathcal{V}$ and $\mathcal{W}$ are enclosures, then by point 1, and the fact that $\operatorname{supp}\mathfrak{M}(\rho_{\mathcal{V}}) \subset \mathcal{V}$ and $\operatorname{supp}\mathfrak{M}(\rho_{\mathcal{W}}) \subset \mathcal{W}$, we have
$$\mathfrak{M}(\rho_{\mathcal{C}}) + \mathfrak{M}(\rho'_{\mathcal{C}}) = \mathfrak{M}(\rho)_{\mathcal{C}} + \mathfrak{M}(\rho)'_{\mathcal{C}}. \tag{7.3}$$
Now, remark that if, e.g. $\phi \in \mathcal{V}$ and $\psi \in \mathcal{W}$, then for any i and j in V we have
$$\big(L_{i,j} \otimes |i\rangle\langle j|\big)\phi \in \mathcal{V} \quad \text{and} \quad \big(L_{i,j} \otimes |i\rangle\langle j|\big)\psi \in \mathcal{W}.$$
Therefore, (7.3) actually implies $\mathfrak{M}(\rho_{\mathcal{C}}) = \mathfrak{M}(\rho)_{\mathcal{C}}$ and $\mathfrak{M}(\rho'_{\mathcal{C}}) = \mathfrak{M}(\rho)'_{\mathcal{C}}$.
4. If $\mathcal{V}$ is a minimal enclosure contained in $\mathcal{R}$, then there exists an $\mathfrak{M}$-invariant state ρ such that $\rho_{\mathcal{V}} = P_{\mathcal{V}}\rho P_{\mathcal{V}} \neq 0$. By point 3, we have $\rho_{\mathcal{V}} = \mathfrak{M}(\rho)_{\mathcal{V}} = \mathfrak{M}(\rho_{\mathcal{V}})$, and so $\rho_{\mathcal{V}}$ is (up to normalization) an invariant state of $\mathfrak{M}_{|\mathcal{I}_1(\mathcal{V})}$. Since $\mathcal{V}$ is irreducible, by Theorem 3.14, $\mathfrak{M}_{|\mathcal{I}_1(\mathcal{V})}$ has a unique invariant state, which has support equal to $\mathcal{V}$. Therefore, $\rho_{\mathcal{V}}$ is a state with support $\mathcal{V}$. This $\rho_{\mathcal{V}}$ must be extremal since $\rho_{\mathcal{V}} = t\,\rho_1 + (1-t)\,\rho_2$ with ρ_1, ρ_2 invariant states and $t \in]0,1[$ would imply that ρ_1, ρ_2 are invariant states with support in $\mathcal{V}$ but then by uniqueness, $\rho_{\mathcal{V}} = \rho_1 = \rho_2$.

   Conversely, if $\mathcal{V} = \operatorname{supp}\rho$ with ρ an extremal invariant state, then $\mathcal{V}$ must be a minimal enclosure. Indeed, the restriction of ρ to $\mathcal{V}$ is by definition a faithful state, and by Lemma 1 in [11] it is the only invariant of $\mathfrak{M}_{|\mathcal{I}_1(\mathcal{V})}$. If we suppose $\mathcal{V}$ is not minimal, then there exist an enclosure $\mathcal{W} \neq \mathcal{V}$ with $\mathcal{W} \subset \mathcal{V}$ and therefore, as in the preceding point, an invariant state $\rho_{\mathcal{W}}$ of $\mathfrak{M}$ with support in $\mathcal{W}$. Therefore, $\rho_{\mathcal{W}} \neq \rho$ is an invariant state also for $\mathfrak{M}_{|\mathcal{I}_1(\mathcal{V})}$, leading to a contradiction.

   To prove the last statement in point 4, observe that by definition there exists an invariant ρ such that $\mathcal{V} \cap \operatorname{supp}\rho \neq \{0\}$. By point 3, $\mathcal{V}$ contains the support of the invariant state $\rho_{\mathcal{V}}$. By the Krein–Milman theorem, $\rho_{\mathcal{V}}$

is a convex combination of extremal invariant states, so there exists an invariant state η such that $\operatorname{supp}\eta \subset \operatorname{supp}\rho_{\mathcal{V}}$, and the minimal enclosure $\operatorname{supp}\eta$ is contained in $\mathcal{V}$.

5. If $\mathcal{V}$ and $\mathcal{W}$ are minimal enclosures contained in $\mathcal{R}$, then, as in the proof of point 4, they are the supports of extremal invariant states $\rho_{\mathcal{V}}$ and $\rho_{\mathcal{W}}$. Because the decomposition of $\mathcal{V}+\mathcal{W}$ into minimal enclosures is unique, $\rho_{\mathcal{V}}$ and $\rho_{\mathcal{W}}$ are the unique extremal invariant states of $\mathfrak{M}_{|\mathcal{I}_1(\mathcal{V}+\mathcal{W})}$. Since the set of invariant states is convex, then by the Krein–Milman theorem, ρ is a convex combination of $\rho_{\mathcal{V}}$ and $\rho_{\mathcal{W}}$, so $\rho_{\mathcal{C}}$ and $\rho'_{\mathcal{C}}$ must be zero. □

We can now return to the study of enclosures generated by vectors of the form $x \otimes |i\rangle$. Remark that "non-connectedness of i and j through $\overset{\mathfrak{M}}{\to}$" (Definition 3.10), when stated in terms of enclosures, is related to the existence of $x \in \mathfrak{h}_i$, $y \in \mathfrak{h}_j$, such that one of the following holds:

(a1) $y \otimes |j\rangle \notin \operatorname{Enc}(x \otimes |i\rangle)^\perp$ and $x \otimes |i\rangle \in \operatorname{Enc}(y \otimes |j\rangle)^\perp$,
(a2) $y \otimes |j\rangle \in \operatorname{Enc}(x \otimes |i\rangle)^\perp$ and $x \otimes |i\rangle \in \operatorname{Enc}(y \otimes |j\rangle)^\perp$.

Our first task will be to show that, when restricting to the subspace $\mathcal{R}$, the situation (a1) cannot appear:

Lemma 7.4. *If $x \otimes |i\rangle$ and $y \otimes |j\rangle$ are in $\mathcal{R}$, then one of the following situations holds:*

1. $x \otimes |i\rangle \notin \operatorname{Enc}(y \otimes |j\rangle)^\perp$ *and* $y \otimes |j\rangle \notin \operatorname{Enc}(x \otimes |i\rangle)^\perp$
2. $\operatorname{Enc}(x \otimes |i\rangle) \perp \operatorname{Enc}(y \otimes |j\rangle)$.

Proof. It is sufficient to notice that, if $y \otimes |j\rangle \in \operatorname{Enc}(x \otimes |i\rangle)^\perp \cap \mathcal{R}$, then the minimal enclosures containing $x \otimes |i\rangle$ and $y \otimes |j\rangle$ are orthogonal. Indeed, by point 2 in Proposition 7.3, the subspace $\operatorname{Enc}(x \otimes |i\rangle)^\perp \cap \mathcal{R}$ is an enclosure, and it contains $y \otimes |j\rangle$ by assumption. □

Remark 7.5. Beware that, in situation 1 of Lemma 7.4, one may still have $\operatorname{Enc}(x \otimes |i\rangle)$ and $\operatorname{Enc}(y \otimes |j\rangle)$ non-orthogonal but in direct sum, as the following example shows.

Example 7.6. We consider an OQRW $\mathfrak{M}$ with two sites, i.e. $V = \{1, 2\}$, and $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathbb{C}^2$, and, for a fixed $p \in]0, 1[$,

$$L_{11} = L_{22} = \sqrt{p}\,\mathrm{Id}, \quad L_{12} = L_{21} = \sqrt{1-p}\,B \qquad \text{with} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We denote the canonical basis of $\mathfrak{h}$ by e_1, e_2. Then

$$\operatorname{Enc}(e_1 \otimes |1\rangle) = \operatorname{Vect}\{e_1 \otimes |1\rangle, e_2 \otimes |2\rangle\}$$
$$\operatorname{Enc}((e_1 + e_2) \otimes |1\rangle) = \operatorname{Vect}\{(e_1 + e_2) \otimes |1\rangle, (e_1 + e_2) \otimes |2\rangle\}$$

are non-orthogonal but have trivial intersection.

As we noticed at the beginning of this section, the form of invariant states for an OQRW $\mathfrak{M}$ will depend on the uniqueness, or non-uniqueness, of its decompositions into minimal enclosures. This will be proven in the next results.

Lemma 7.7. *Let $\mathcal{V} = E_1 + E_2$, where E_1 and E_2 are minimal enclosures contained in $\mathcal{R}$. The decomposition of $\mathcal{V}$ in a direct sum of minimal enclosures is unique if and only if any minimal enclosure $\mathcal{W}$ such that $\mathcal{W} \not\perp E_1$ and $\mathcal{W} \not\perp E_2$ satisfies $\mathcal{W} \cap \mathcal{V} = \{0\}$. If the latter statement holds, then the two enclosures are orthogonal.*

Proof. Assume the decomposition of $\mathcal{V}$ as a direct sum of minimal enclosures is unique. Then $E_1 \perp E_2$, otherwise by Proposition 7.3, $\mathcal{V} \cap E_1^{\perp}$ would be an enclosure that does not contain E_2, leading to a different decomposition of $\mathcal{V}$. Now consider a minimal enclosure $\mathcal{W}$ with $\mathcal{W} \not\perp E_1$ and $\mathcal{W} \not\perp E_2$. This implies $\mathcal{W} \neq E_1$ so by Lemma 6.6, $\mathcal{W} \cap E_1 = \{0\}$. If $\mathcal{W} \cap \mathcal{V} \neq \{0\}$ then it is an enclosure in $\mathcal{W}$ so by minimality, $\mathcal{W} \subset \mathcal{V}$. Then $\mathcal{W} + E_1$ is a direct sum of minimal enclosures contained in $\mathcal{V}$, so, by point 2 in Proposition 7.3, one can complete this as a decomposition of $\mathcal{V}$ into a direct sum of minimal enclosures. This is a contradiction, leading to $\mathcal{W} \cap \mathcal{V} = \{0\}$.

Now assume that any enclosure $\mathcal{W}$ such that $\mathcal{W} \not\perp E_1$ and $\mathcal{W} \not\perp E_2$ satisfies $\mathcal{W} \cap \mathcal{V} = \{0\}$. Taking first $\mathcal{W} = E_2$, which obviously has a non-trivial intersection with $\mathcal{V}$, we obtain that $E_1 \perp E_2$. Now, consider some minimal enclosure E_3 contained in $\mathcal{V}$. Then by assumption one has, e.g. $E_3 \perp E_1$ and $E_3 \not\perp E_2$. But then one has $E_3 \subset E_1^{\perp} \cap \mathcal{V}$, which, as proved above, is E_2. This proves the uniqueness of the decomposition. □

The following remark shows that Lemma 7.7 is consistent with the uniqueness of the irreducible decomposition for classical Markov chains:

Remark 7.8. Consider a minimal OQRW realization $\mathfrak{M}$ of a classical Markov chain. By Proposition 6.1 and Lemma 6.6, any minimal enclosure is of the form $\mathbb{C} \otimes \mathbb{C}^{V_i}$ for $V_i \subset V$. Therefore, for such an OQRW $\mathfrak{M}$, any distinct minimal enclosures $\mathcal{V}$ and $\mathcal{W}$ are always orthogonal.

Once again, the following result is proven in [4] in finite dimension. We extend the proof to infinite dimension.

Corollary 7.9. *Let E_1 and E_2 be two minimal enclosures contained in $\mathcal{R}$. Assume that the decomposition of $\mathcal{V} = E_1 + E_2$ in a direct sum of minimal enclosures is not unique. Then* $\dim E_1 = \dim E_2$.

If, in addition, $E_1 \perp E_2$, then there exists a partial isometry Q from E_1 to E_2 satisfying

$$Q^* Q = \mathrm{Id}_{|E_1} \qquad Q\, Q^* = \mathrm{Id}_{|E_2} \tag{7.4}$$

and for any ρ in $\mathcal{I}_1(\mathcal{H})$, for $R = Q + Q^*$, and $P_i = P_{E_i}$, $i = 1, 2$:

$$R\, \mathfrak{M}(\rho)\, P_i + P_i\, \mathfrak{M}(\rho)\, R = \mathfrak{M}\big(R\, \rho\, P_i + P_i\, \rho\, R\big). \tag{7.5}$$

Proof. Assume that there exists a minimal enclosure $\mathcal{W}$ that is distinct from E_1 and non-orthogonal to it. Then by point 2 of Proposition 7.3, $E_1 \cap \mathcal{W}^{\perp}$ is an enclosure contained in E_1. By minimality of E_1 and non-orthogonality between those two enclosures, $E_1 \cap \mathcal{W}^{\perp} = \{0\}$. Therefore, $\dim E_1 \leq \dim \mathcal{W}$, and by symmetry one has the equality $\dim E_1 = \dim \mathcal{W}$.

If $E_1 \not\perp E_2$, this yields $\dim E_1 = \dim E_2$. If $E_1 \perp E_2$, the non-uniqueness of the decomposition implies the existence of minimal enclosures $\widetilde{E}_1$ and $\widetilde{E}_2$ such that

$$E_1 + E_2 = \widetilde{E}_1 + \widetilde{E}_2$$

and one can assume that, e.g. $\widetilde{E}_1$ is distinct from both E_1 and E_2. Necessarily, $\widetilde{E}_1$ is also non-orthogonal to both E_1 and E_2, for, e.g. $\widetilde{E}_1 \perp E_1$ would imply $\widetilde{E}_1 \subset E_2$ and contradict the minimality of E_2. Therefore, taking $\mathcal{W} = \widetilde{E}_1$ we recover the equality $\dim E_1 = \dim E_2$.

Assume now that $E_1 \perp E_2$. By the above discussion there exists a minimal enclosure $\mathcal{W}$ distinct from E_1 and non-orthogonal to it. Denote by P_1, P_2, $P_{\mathcal{W}}$ the orthogonal projections on E_1, E_2, $\mathcal{W}$, respectively. Define the map $\mathfrak{N}$ on $\mathcal{B}(\mathcal{H})$ by

$$\mathfrak{N} : X \mapsto P_{\mathcal{R}}\, \mathfrak{M}^*(X)\, P_{\mathcal{R}}.$$

One sees immediately that if $E = E_1$, E_2 or $\mathcal{W}$, then P_E is (up to multiplication) the unique invariant of $\mathfrak{N}_{|\mathcal{B}(E)}$. Consider the decomposition of $P_{\mathcal{W}} = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$ in the splitting $\mathcal{V} = E_1 + E_2$, where necessarily $B \neq 0$. A simple consequence of Proposition 7.3 is that in the same decomposition, $\mathfrak{N}(P_{\mathcal{W}}) = \begin{pmatrix} \mathfrak{N}(A) & \mathfrak{N}(B)^* \\ \mathfrak{N}(B) & \mathfrak{N}(C) \end{pmatrix}$. Therefore, A is proportional to P_1 and C to P_2. Writing relations $P = P^* = P^2$ satisfied by $P_{\mathcal{W}}$, one sees that B must be proportional to an operator Q satisfying the relations (7.4). In addition, fixing that same operator Q, for $\theta \in [0, \pi]$, the operator that has the form

$$P_\theta = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta\, Q^* \\ \sin\theta\cos\theta\, Q & \sin^2\theta \end{pmatrix}$$

is an orthogonal projection preserved by the map $\mathfrak{N}$. So its range is an enclosure and, by point 3 of Proposition 7.3, P_θ will satisfy the relation

$$\mathfrak{M}(P_\theta\, \rho\, P_\theta) = P_\theta\, \mathfrak{M}(\rho)\, P_\theta.$$

Differentiating this relation with respect to the θ variable, we have

$$\mathfrak{M}\left(\frac{\mathrm{d}P_\theta}{\mathrm{d}\theta}\,\rho\, P_\theta + P_\theta\, \rho\, \frac{\mathrm{d}P_\theta}{\mathrm{d}\theta}\right) = \frac{\mathrm{d}P_\theta}{\mathrm{d}\theta}\,\mathfrak{M}(\rho)\, P_\theta + P_\theta\, \mathfrak{M}(\rho)\, \frac{\mathrm{d}P_\theta}{\mathrm{d}\theta}$$

Computing the derivatives at $\theta = 0$ and $\theta = \pi/2$, we obtain relation (7.5). □

Corollary 7.10. *Assume that $\mathcal{V} = E_1 + E_2$ where E_1 and E_2 are mutually orthogonal minimal enclosures, contained in $\mathcal{R}$, but that the decomposition into a direct sum of minimal enclosures is non-unique. Denote by ρ_i^{inv} the unique invariant state with support in E_i, $i = 1, 2$, and by Q the partial isometry defined in Corollary 7.9. Then $\rho_2^{\mathrm{inv}} = Q\, \rho_1^{\mathrm{inv}}\, Q^*$.*

If ρ is an invariant state with support in $\mathcal{V}$, write $\rho = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ \rho_{2,1} & \rho_{2,2} \end{pmatrix}$. Then:

- *$\rho_{1,1}$ is proportional to ρ_1^{inv},*

- $\rho_{2,2}$ *is proportional to* $\rho_2^{\rm inv}$,
- $\rho_{1,2}$ *is proportional to* $\rho_1^{\rm inv} Q^* = Q^* \rho_2^{\rm inv}$,
- $\rho_{2,1}$ *is proportional to* $\rho_2^{\rm inv} Q = Q\rho_1^{\rm inv}$.

Proof. The first identity is obtained by applying relation (7.5) to $\rho = \rho_1^{\rm inv}$ with P_1, then applying it again to the resulting relation, this time with P_2.

That each $\rho_{i,j}$ is an invariant is an immediate consequence of Proposition 7.3. The relation satisfied by $\rho_{1,2}$ and $\rho_{2,1}$ is then obtained by applying relation (7.5) to, e.g. $\rho_{1,2}$, with P_1 or P_2. □

We are now in a position to state the relevant decomposition associated with an open quantum random walk $\mathfrak{M}$.

Proposition 7.11. *Let* $\mathfrak{M}$ *be an OQRW on* $\mathcal{H} = \bigoplus_{i\in V} \mathfrak{h}_i$. *There exists an orthogonal decomposition of* $\mathcal{H}$ *in the form*

$$\mathcal{H} = \mathcal{D} \oplus \bigoplus_{\alpha\in A} \mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle) \oplus \bigoplus_{\beta\in B} \bigoplus_{\gamma\in C_\beta} \mathrm{Enc}(x_{\beta,\gamma} \otimes |i_{\beta,\gamma}\rangle) \qquad (7.6)$$

such that the sets A, B, C_β *are at most countable,* A *and* B *can be empty (but not simultaneously, unless* $R = \{0\}$*), any* C_β *has cardinality at least two, and:*

- *every* $\mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$ *or* $\mathrm{Enc}(x_{\beta,\gamma} \otimes |i_{\beta,\gamma}\rangle)$ *in this decomposition is a minimal enclosure, and therefore an equivalence class for* $\overset{\mathfrak{M}}{\leftrightarrow}$,
- *for* α *in* A, *the only minimal enclosure not orthogonal to* $\mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$ *is* $\mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$ *itself,*
- *for* β *in* B *and* $\gamma \in C_\beta$, *any minimal enclosure that is not orthogonal to* $\mathrm{Enc}(x_{\beta,\gamma} \otimes |i_{\beta,\gamma}\rangle)$ *is contained in* $\bigoplus_{\gamma\in C_\beta} \mathrm{Enc}(x_{\beta,\gamma} \otimes |i_{\beta,\gamma}\rangle)$.

Proof. We start with the decomposition $\mathcal{H} = \mathcal{D} \oplus \mathcal{R}$, and proceed to decompose $\mathcal{R}$. Consider the set of all minimal enclosures $\mathrm{Enc}(x \otimes |i\rangle)$ with the property that the only minimal enclosure non-orthogonal to $\mathrm{Enc}(x \otimes |i\rangle)$ is $\mathrm{Enc}(x \otimes |i\rangle)$ itself. By separability this set is at most countable. We can label these enclosures $\mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$, $\alpha \in A$. Let $\mathcal{O} = \bigoplus_{\alpha\in A} \mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$. Then $\mathcal{O}$ is an enclosure, and if $\mathcal{R} \cap \mathcal{O}^\perp \neq \{0\}$ then, by point 2 of Proposition 7.3, it is also an enclosure and we proceed to decompose it. Consider families of minimal enclosures labelled by a set C, $\{\mathrm{Enc}(x_\gamma \otimes |i_\gamma\rangle), \gamma \in C\}$ with the property that any minimal enclosure that is not orthogonal to the space $\bigoplus_{\gamma\in C} \mathrm{Enc}(x_\gamma \otimes |i_\gamma\rangle)$ is contained in $\bigoplus_{\gamma\in C} \mathrm{Enc}(x_\gamma \otimes |i_\gamma\rangle)$; by the assumption that $\mathcal{R} \cap \mathcal{O}^\perp \neq \{0\}$ this set is not empty. Pick a maximal such family, and index it as $\{\mathrm{Enc}(x_{1,\gamma} \otimes |i_{1,\gamma}\rangle), \gamma \in C_1\}$. By point 2 of Proposition 7.3 and Lemma 6.6, one can assume that the different enclosures in this family are mutually orthogonal. If

$$\mathcal{R} \cap \mathcal{O}^\perp \cap \big(\bigoplus_{\gamma\in C_1} \mathrm{Enc}(x_{1,\gamma} \otimes |i_{1,\gamma}\rangle)\big)^\perp \neq \{0\}$$

we can iterate this process. □

Remark 7.12. By Remark 7.8 and Lemma 7.7, any minimal OQRW realization $\mathfrak{M}$ of a classical Markov chain is simply of the form $\mathcal{H} = \mathcal{D} \oplus \bigoplus_{\alpha\in A} \mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$.

We will use this decomposition to characterize the form of invariant states. Before we state our next result, let us give some notation. We fix a decomposition (7.6) as considered in Proposition 7.11. We define for every $\alpha \in A$ and $(\beta,\gamma) \in B \times C_\beta$ the following orthogonal projections (for $\mathcal{V}$ a subspace of $\mathcal{H}$, the orthogonal projection on $\mathcal{V}$ is denoted $P_{\mathcal{V}}$):

$$P_0 = P_{\mathcal{D}} \qquad P_\alpha = P_{\mathrm{Enc}(x_\alpha\otimes|i_\alpha\rangle)} \qquad P_{\beta,\gamma} = P_{\mathrm{Enc}(x_{\beta,\gamma}\otimes|i_{\beta,\gamma}\rangle)}$$

and for a state ρ, and indices i, j taking the values 0, $\alpha \in A$ or $(\beta,\gamma) \in B\times C_\beta$

$$\rho_i = P_i\,\rho\,P_i \qquad \rho_{i,j} = P_i\,\rho\,P_j. \tag{7.7}$$

When $\mathcal{V}$ is a subspace of $\mathcal{H}$ such that $\mathcal{I}_1(\mathcal{V})$ is stable by $\mathfrak{M}$, we will talk about the restriction $\mathfrak{M}_{|\mathcal{V}}$ of $\mathfrak{M}$ to $\mathcal{V}$ (instead of the restriction $\mathfrak{M}_{|\mathcal{I}_1(\mathcal{V})}$ of $\mathfrak{M}$ to $\mathcal{I}_1(\mathcal{V})$). In addition, for i taking the values $\alpha \in A$ or $(\beta,\gamma) \in B \times C_\beta$, we denote by ρ_i^{inv} the unique invariant state of $\mathfrak{M}_{|\mathrm{Enc}(x_\alpha\otimes|i_\alpha\rangle)}$ or $\mathfrak{M}_{|\mathrm{Enc}(x_{\beta,\gamma}\otimes|i_{\beta,\gamma}\rangle)}$.

Theorem 7.13. *Let ρ be a $\mathfrak{M}$-invariant state. With the notation (7.7) we have*

1. $\rho_0 = 0$,
2. *every ρ_α is proportional to $\rho_\alpha^{\mathrm{inv}}$, which has support* $\mathrm{Enc}(x_\alpha \otimes |i_\alpha\rangle)$,
3. *every $\rho_{(\beta,\gamma)}$ is proportional to $\rho_{(\beta,\gamma)}^{\mathrm{inv}}$, which has support* $\mathrm{Enc}(x_{\beta,\gamma}\otimes|i_{\beta,\gamma}\rangle)$,
4. *for $\gamma \neq \gamma'$ in C_β, the off-diagonal term $\rho_{((\beta,\gamma),(\beta,\gamma'))}$, which we simply denote by $\rho_{(\beta,\gamma,\gamma')}$, may be non-zero, and is $\mathfrak{M}$-invariant. In addition, there exists a partial isometry $Q_{(\beta,\gamma,\gamma')}$ from* $\mathrm{Enc}(x_{\beta,\gamma} \otimes |i_{\beta,\gamma}\rangle)$ *to* $\mathrm{Enc}(x_{\beta,\gamma'} \otimes |i_{\beta,\gamma'}\rangle)$ *such that:*
   - $\rho_{(\beta,\gamma')}^{\mathrm{inv}} = Q_{(\beta,\gamma,\gamma')}\,\rho_{(\beta,\gamma)}^{\mathrm{inv}}\,Q_{(\beta,\gamma,\gamma')}^*$,
   - *$\rho_{(\beta,\gamma,\gamma')}$ is proportional to $Q_{(\beta,\gamma,\gamma')}^*\,\rho_{(\beta,\gamma')}^{\mathrm{inv}} = \rho_{(\beta,\gamma)}^{\mathrm{inv}}\,Q_{(\beta,\gamma,\gamma')}^*$*,
5. *all other $\rho_{i,j}$ (i,j taking the values 0, $\alpha \in A$ or $(\beta,\gamma) \in B\times C_\beta$) are zero.*

Proof. This follows from a repeated application of Propositions 7.3 and 7.11, and Corollary 7.10. □

Remark 7.14. Our main comment here is that there may exist "coherences" between minimal blocks, i.e. non-zero off-diagonal blocks $\rho_{i,j}$, for i,j corresponding to distinct minimal irreducible blocks. Invariant states are not, contrarily to the classical case, just convex combinations of states invariant for the reduced (irreducible) dynamics. We will observe this in Example 9.8. Note however, that, according to Remark 7.12, this cannot happen for minimal OQRW realizations of classical Markov chains.

Remark 7.15. One might have hoped that a relevant decomposition of $\mathfrak{M}$ would separate sites, *i.e* that one could decompose $\mathcal{R}$ into a sum of minimal enclosures $\bigoplus \mathrm{Enc}(x_k \otimes |i_k\rangle)$ with $\mathrm{Enc}(x_k \otimes |i_k\rangle) \subset \bigoplus_{i\in I_k} \mathfrak{h}_i$ for disjoint I_k. This is not true, as Example 7.16 shows.

Example 7.16. Consider again Example 6.8. We have a unique decomposition of $\mathcal{H} = \mathfrak{h} \otimes \mathbb{C}^V$ as a sum of minimal enclosures,

$$\mathfrak{h} \otimes \mathbb{C}^V = \mathrm{Enc}(e_1 \otimes |1\rangle) \oplus \mathrm{Enc}(e_2 \otimes |1\rangle)$$

even though the two minimal enclosures

$$\mathrm{Enc}(e_k \otimes |1\rangle) = \mathbb{C}\, e_k \otimes \mathbb{C}^V, \quad k = 1, 2,$$

connect all three sites. Note also that, in accordance with Lemma 7.7, the two unique enclosures are mutually orthogonal.

Remark 7.17. Applying Theorem 7.13 and the Frigerio–Verri ergodic theorem (see [12]) one can obtain results about the ergodic behaviour of $(\mathfrak{M}^n(\rho))_n$, that extend Proposition 3.15 to the reducible case. This will be done in a forthcoming article. However, in certain cases, the results given in the present article can be enough to describe convergence in reducible OQRWs: see Example 9.4.

8. Extensions of Open Quantum Random Walks

In this section, we define an extension of open quantum random walks, already mentioned in Remark 2.1. We consider again a countable set of vertices V and a separable Hilbert space $\mathcal{H} = \bigoplus_{i\in V} \mathfrak{h}_i$. An extended open quantum random walk will be a map $\widetilde{\mathfrak{M}} : \mathcal{I}_1(\mathcal{H}) \to \mathcal{I}_1(\mathcal{H})$ such that if $\rho = \sum_{i,j\in V} \rho(i,j) \otimes |i\rangle\langle j|$ then

$$\widetilde{\mathfrak{M}}(\rho) = \sum_{i\in V} \left(\sum_{j\in V} \Phi_{i,j}\big(\rho(j,j)\big) \right) \otimes |i\rangle\langle i| \tag{8.1}$$

where each $\Phi_{i,j}$ is a completely positive map from $\mathcal{I}_1(\mathfrak{h}_j)$ to $\mathcal{I}_1(\mathfrak{h}_i)$ such that, for any j in V,

$$\sum_{i\in V} \Phi^*_{i,j}(\mathrm{Id}_{\mathfrak{h}_i}) = \mathrm{Id}_{\mathfrak{h}_j}. \tag{8.2}$$

Again this $\widetilde{\mathfrak{M}}$ maps $\mathcal{I}_1(\mathcal{H})$ to the set $\mathcal{I}_{\mathcal{D}}$ of block-diagonal trace-class operators (see Sect. 2). In addition, the Kraus representation associates to each $\Phi_{i,j}$ a countable set $E(j,i)$ and, for every $e \in E(j,i)$, a map L_e from $\mathfrak{h}_j$ to $\mathfrak{h}_i$ such that $\Phi_{i,j}$ can be written as

$$\Phi_{i,j}(\rho) = \sum_{e\in E(j,i)} L_e\, \rho\, L_e^* \quad \text{for any } \rho \in \mathcal{I}_1(\mathfrak{h}_j).$$

We view the operators L_e as associated with the edges of a directed multigraph (V,E) where $E = \cup_{i,j\in V} E(j,i)$. Then if we denote by $E(j) = \cup_{i\in V} E(j,i)$ the set of outgoing edges at j, the stochasticity condition (8.2) becomes similar to (2.2):

$$\forall j \in V \qquad \sum_{e\in E(j)} L_e^* L_e = \mathrm{Id}.$$

This reminds us that the present framework encompasses open quantum random walks as defined in the rest of this article. What's more, it should be noted that the power $\mathfrak{M}^n$ of an OQRW $\mathfrak{M}$ is not in general an OQRW, but is always an extended OQRW. All the results of the previous sections can be extended to this more general class of evolutions.

As in Sect. 2, starting from a state $\rho = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$ we can define processes "without measurement" $\Big(\widetilde{X}_n, \frac{\widetilde{\mathfrak{M}}^n(\rho,\widetilde{X}_n)}{\operatorname{Tr}\widetilde{\mathfrak{M}}^n(\rho,\widetilde{X}_n)}\Big)_{n\in\mathbb{N}}$: denote

$$\widetilde{\mathfrak{M}}^n(\rho) = \sum_{i\in V} \widetilde{\mathfrak{M}}^n(\rho, i) \otimes |i\rangle\langle i|.$$

Then the process "without measurement" is determined by the variable $\widetilde{X}_n$, with law

$$\mathbb{P}(\widetilde{X}_n = i) = \operatorname{Tr}\widetilde{\mathfrak{M}}^n(\rho, i)$$

and the process "with measurement" $(\widetilde{X}_n, \widetilde{\rho}_n)_{n\in\mathbb{N}}$ by

$$(\widetilde{X}_0, \widetilde{\rho}_0) = \big(j, \rho(j)\big) \text{ with probability } \operatorname{Tr}\rho(j)$$

$$\mathbb{P}\left((\widetilde{X}_{n+1}, \widetilde{\rho}_{n+1}) = \left(i, \frac{\Phi_{i,j}(\widetilde{\rho}_n)}{\operatorname{Tr}\Phi_{i,j}(\widetilde{\rho}_n)}\right) \Big| (\widetilde{X}_n, \widetilde{\rho}_n) = (j, \widetilde{\rho}_n) \right) = \operatorname{Tr}\Phi_{i,j}(\widetilde{\rho}_n) \quad \forall i \in V.$$

We claim that our vision of open quantum random walks in terms of paths π in $\mathcal{P}(i,j)$ on a directed graph extends to this framework, with paths $\tilde{\pi}$ in $\widetilde{\mathcal{P}}(i,j)$ on a directed multigraph.

In particular, we recover all results from Sects. 3 through 7, replacing $\mathcal{P}$ with $\widetilde{\mathcal{P}}$ in our assumptions, and $X_n, \mathfrak{M}^n(\rho,i), \rho_n$ with $\widetilde{X}_n, \widetilde{\mathfrak{M}}^n(\rho,i), \widetilde{\rho}_n$. More precisely, Proposition 3.8 and Definition 3.10 on irreducibility, as well as Lemma 4.9 and Theorem 4.10 on the period, extend to $\widetilde{\mathfrak{M}}$ by simply replacing every $\mathcal{P}$ with $\widetilde{\mathcal{P}}$. Proposition 4.6 holds if (4.6) is replaced with

$$P_{k,i} L_e = L_e P_{k \stackrel{\mathrm{d}}{-} 1, j} \quad \forall\, e \in E(j,i).$$

And similarly Corollary 4.11 holds if relation (4.3) becomes

$$\forall x \in \mathfrak{h}_i,\ \exists\, e \in E(i,i) \text{ such that } \langle x, L_e x\rangle \neq 0.$$

Then, the whole of Sect. 5 holds if the processes $X_n, \mathfrak{M}^n(\rho,i), \rho_n$ are replaced with $\widetilde{X}_n, \widetilde{\mathfrak{M}}^n(\rho,i), \widetilde{\rho}_n$. Similarly, Sects. 6 and 7 remain the same, replacing $\mathcal{P}$ with $\widetilde{\mathcal{P}}$ in the definition of enclosures.

9. Examples and Applications

Example 9.1. We start with an application to space-homogeneous open quantum random walks on a graph associated with a set of generators of a group. This applies in particular when this graph is the lattice $\mathbb{Z}^d$, a case which we study in [5].

To be more precise, we assume that V is a set of vertices in an additive (abelian) group G, that $\mathfrak{h}_i = \mathfrak{h}$ does not depend on i, and that there is a finite set $S \subset G$ such that $L_{i,j} = L_{j-i}$ depends only on $j-i$, and is zero unless $j - i \in S$.

We associate with this OQRW the map

$$\begin{aligned} \mathfrak{L} : \mathcal{I}(\mathfrak{h}) &\to \mathcal{I}(\mathfrak{h}) \\ \eta &\mapsto \textstyle\sum_{s\in S} L_s\, \eta\, L_s^*. \end{aligned} \tag{9.1}$$

If $\mathfrak{M}$ is irreducible, then clearly $\mathfrak{L}$ is also irreducible, and by Proposition 3.12, it has at most one invariant state which we then denote by η^{inv}. Note that, if $\mathfrak{h}$ is finite dimensional, then this η^{inv} exists.

Remark 9.2. From Lemma 3.7, one easily sees that $\mathfrak{L}$ is irreducible if and only if the operators L_s, $s \in S$, have no non-trivial common invariant subspace. This criterion is stated, in particular, in [10].

Proposition 9.3. *Assume $\mathfrak{M}$ as above is irreducible.*
- *If V is infinite, then $\mathfrak{M}$ has no invariant state.*
- *If V is finite, then $\mathfrak{L}$ has an invariant state η^{inv} and the unique invariant state of $\mathfrak{M}$ is*

$$\sum_{i\in V} \frac{\eta^{\mathrm{inv}}}{\mathrm{card}\, V} \otimes |i\rangle\langle i|.$$

Proof. Assume there exists an invariant state ρ^{inv} for $\mathfrak{M}$. Since $\mathfrak{M}$ is invariant by translation, any translation of that state is also an invariant state, so by Theorem 3.14, the state ρ^{inv} is translation invariant. It must therefore be of the form $\sum_{v\in V} \rho \otimes |v\rangle\langle v|$. If V is infinite, this has trace either infinite or null and in either case this is a contradiction. If V is finite then it is easy to see that ρ must be an invariant of $\mathfrak{L}$. ☐

The Perron–Frobenius theorem for CP maps, Proposition 3.12, allows us to obtain a large deviation principle and a central limit theorem for the position process $(X_n)_{n\in\mathbb{N}}$ associated with an open quantum random walk $\mathfrak{M}$ and an initial state ρ (see Sect. 2), therefore, extending the results of [2]. In addition, we can also make more precise the convergence of the sequence of states $(\rho_n)_{n\in\mathbb{N}}$ (still using the notations of Sect. 2). This will be done in a separate paper [5] studying in detail OQRWs on $\mathbb{Z}^d$.

Example 9.4. We consider the example given in section 12.1 of [3]. In our notation this example is given by $V = \{1, 2\}$, $\mathfrak{h} = \mathbb{C}^2$ (with canonical basis (e_1, e_2)) and transitions given by

$$L_{1,1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad L_{1,2} = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \quad L_{2,2} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{q} \end{pmatrix} \quad L_{2,1} = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

where we assume $q = 1 - p \in (0, 1)$, $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$, $0 < |a|^2, |c|^2 < 1$. Note that we do not need the additional assumptions $a \neq b$, $c \neq d$, $ab \neq \sqrt{q}$, $a^2 \neq q$, $b^2 \neq q$ made in [3]. First, observe that the only minimal enclosure is

$$\mathrm{Enc}(e_1 \otimes |2\rangle) = \mathrm{Vect}(e_1 \otimes |2\rangle).$$

Indeed,
- $\mathrm{Enc}(e_1 \otimes |1\rangle)$ obviously contains $\mathrm{Enc}(L_{2,1} e_1 \otimes |2\rangle) = \mathrm{Enc}(e_1 \otimes |2\rangle)$;

- $\mathrm{Enc}(x \otimes |2\rangle)$ contains $\mathrm{Enc}(L_{1,2}x \otimes |1\rangle)$ and if $x = x_1 e_1 + x_2 e_2$ with $x_2 \neq 0$, this contains $\mathrm{Enc}(e_1 \otimes |1\rangle)$.
- $\mathrm{Enc}(x \otimes |1\rangle)$ contains $\mathrm{Enc}(L_{2,1}x \otimes |2\rangle) = \mathrm{Enc}\big((cx_1 e_1 + dx_2 e_2) \otimes |2\rangle\big)$, and, if x_2 is non-null, then we fall in the previous case and conclude.

Therefore, the decomposition (7.6) is given by

$$\mathfrak{h} \otimes \mathbb{C}^V = \mathcal{D} \oplus \left\{ \begin{pmatrix} a \\ 0 \end{pmatrix} \otimes |2\rangle,\ a \in \mathbb{C} \right\}.$$

By the equivalent definition of $\mathcal{D}$ given in Remark 7.1, any eigenvector of $\mathfrak{M}$ associated with an eigenvalue of modulus one must be zero on $\mathcal{D}$. So the OQRW $\mathfrak{M}$ has a unique eigenvalue of maximum modulus, which is the simple eigenvalue 1 associated with the eigenvector

$$\rho^{\mathrm{inv}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes |2\rangle\langle 2|$$

and this implies that, for any initial state ρ, one has $\mathfrak{M}^n(\rho) \to \rho^{\mathrm{inv}}$ as $n \to \infty$.

Example 9.5. We consider a family of examples which extends the main example given in [3]. This family is indexed by $N \in \mathbb{N}^* \cup \{\infty\}$; every $\mathfrak{h}$ is $\mathbb{C}^2$ and V is either $V_N = \{0, \ldots, N-1\}$ or $V_\infty = \mathbb{Z}$, and the operators $L_{i,j}$ are defined by

$$L_{i \overset{N}{+} 1, i} = L_+ = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad L_{i \overset{N}{-} 1, i} = L_- = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

where here $\overset{N}{+}$, $\overset{N}{-}$ denote addition or substraction modulo N in the case where $N < \infty$, and standard addition or substraction if $N = \infty$. We denote by $\mathfrak{M}_{(N)}$ the above open quantum random walk.

To illustrate the method of "adding loops" described in Remark 4.14, for $\varepsilon \in]0,1[$ we define the open quantum random walk $\mathfrak{M}_{(N,\varepsilon)}$ with sites V_N and transition operators

$$L^{(\varepsilon)}_{i \overset{N}{+} 1, i} = L^{(\varepsilon)}_{+} = \sqrt{1-\varepsilon}\, L_+ \qquad L^{(\varepsilon)}_{i \overset{N}{-} 1, i} = L^{(\varepsilon)}_{-} = \sqrt{1-\varepsilon}\, L_- \quad L^{(\varepsilon)}_{i,i} = \sqrt{\varepsilon}\, \mathrm{Id}.$$

Note that we consider a perturbation that "adds a loop" at every site, because it simplifies both the computation of the invariant state and the simulation.

Proposition 9.6. *Consider the open quantum random walks $\mathfrak{M}_{(N)}$ and $\mathfrak{M}_{(N,\varepsilon)}$ as above. We have:*

- *for every N in $\mathbb{N}^* \cup \{\infty\}$, the OQRWs $\mathfrak{M}_{(N)}$ and $\mathfrak{M}_{(N,\varepsilon)}$ are irreducible,*
- *for N in $2\,\mathbb{N}^* \cup \{\infty\}$ the OQRW $\mathfrak{M}_{(N)}$ has period 2,*
- *for N in $2\,\mathbb{N}+1$ the OQRW $\mathfrak{M}_{(N)}$ is aperiodic,*
- *for N in $\mathbb{N}^* \cup \{\infty\}$, the OQRW $\mathfrak{M}_{(N,\varepsilon)}$ is aperiodic,*
- *for N in $\mathbb{N}^*$, the OQRWs $\mathfrak{M}_{(N)}$ and $\mathfrak{M}_{(N,\varepsilon)}$ have as unique invariant state*

$$\rho^{\mathrm{inv}} = \sum_{i \in V_N} \frac{1}{2N}\, \mathrm{Id} \otimes |i\rangle\langle i|.$$

Proof. We first show that, for any N, the chain $\mathfrak{M}_{(N)}$ is irreducible. This implies that $\mathfrak{M}_{(N,\varepsilon)}$ is irreducible as well. For this, fix i and j in $\mathbb{N}$, and let $\Delta = i-j$. For p large enough, consider $\pi \in \mathcal{P}(i,j)$ of the form $(i, i-1, \ldots, i-\Delta-p, i-\Delta-p+1, \ldots, j)$ (i.e. one first moves down $p+\Delta$ times, then up p times). Then by inspection one sees that if two vectors $x_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$ and $x_j = \begin{pmatrix} a_j \\ b_j \end{pmatrix}$ satisfy $\langle x_j, L_\pi\, x_i\rangle = 0$ for arbitrarily large p, necessarily one has $a_i = b_i = 0$ or $a_j = b_j = 0$. Therefore, for any $x_i \neq 0$ the set $L_\pi x_i$ is total in $\mathfrak{h}_j$, and $\mathfrak{M}_{(N)}$ is irreducible by Proposition 3.8.

For any N, the chain $\mathfrak{M}_{(N,\varepsilon)}$ is aperiodic by Remark 4.14. Let us now consider the period of $\mathfrak{M}_{(N)}$. For any non-null vector x in $\mathbb{C}^2$, we always have either $\langle x, L_+L_-x\rangle \neq 0$ or $\langle x, L_-L_+x\rangle \neq 0$. Relation (4.2) implies that $D(i,x) \in \{1,2\}$ for all $i \in V$ and all x, so, by Theorem 4.10, the period can be only 1 or 2.

If N is odd, then for $p \in \mathbb{N}^*$, consider the path $\pi \in \mathcal{P}(1,1)$ starting from 1 and going "up", doing p loops on V_N before stopping at 1, so that $L_\pi = L_+^{pN}$. For $x \neq 0$ we show by inspection that $\langle x, L_\pi\, x\rangle$ is zero for at most one p, so $D(1,x)$, defined in (4.2), divides pn for any large enough $p \in \mathbb{N}^*$. Consequently $D(1,x) = 1$ and, by Theorem 4.10, the period is 1.

On the other hand, if N is even or infinite, then the projections

$$P_{\text{even}} = \sum_{i\,\text{even}} \mathrm{Id} \otimes |i\rangle\langle i| \quad \text{and} \quad P_{\text{odd}} = \sum_{i\,\text{odd}} \mathrm{Id} \otimes |i\rangle\langle i|$$

are $\mathfrak{M}$-cyclic, so that the period is 2.

Last, with $\mathfrak{M}_{(N)}$ (respectively, $\mathfrak{M}_{(N,\varepsilon)}$) we associate a map $\mathfrak{L}_{(N)}$ (respectively, $\mathfrak{L}_{(N,\varepsilon)}$) on $\mathcal{I}_1(\mathbb{C}^2)$, as in (9.1). We can check that in all cases, the state $\frac{1}{2}\,\mathrm{Id}$ on $\mathbb{C}^2$ is the only invariant of that map. We conclude by Proposition 9.3. □

We now describe the results of numerical simulations. We always start from the initial state $\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes |1\rangle\langle 1|$, but the phenomena are insensitive to the particular choice of ρ. Figure 1 shows for the three chains $\mathfrak{M}_{(3)}$, $\mathfrak{M}_{(4)}$, $\mathfrak{M}_{(4,0.05)}$ (rows 1, 2, 3, respectively):

- the probability $\mathbb{P}(X_n = 1)$ for $n = 0, \ldots, 30$,
- the $(1,1)$ coefficient of $\mathfrak{M}^n(\rho, 1)$ for $n = 0, \ldots, 30$ (note that, by our choice of $\mathfrak{M}$ and ρ, this coefficient is real).

We observe that both quantities converge as n increases, for the aperiodic OQRW $\mathfrak{M}_{(3)}$ and $\mathfrak{M}_{(4,\varepsilon)}$, as expected by Corollary 5.6, but not for the periodic OQRW $\mathfrak{M}_{(4)}$. Clearly the averages of both quantities converge in all cases, in agreement with Corollary 5.4.

Example 9.7. We use $V_\infty = \mathbb{Z}$, $\mathfrak{h} = \mathbb{C}^2$ as in the previous example and change the transition matrices,

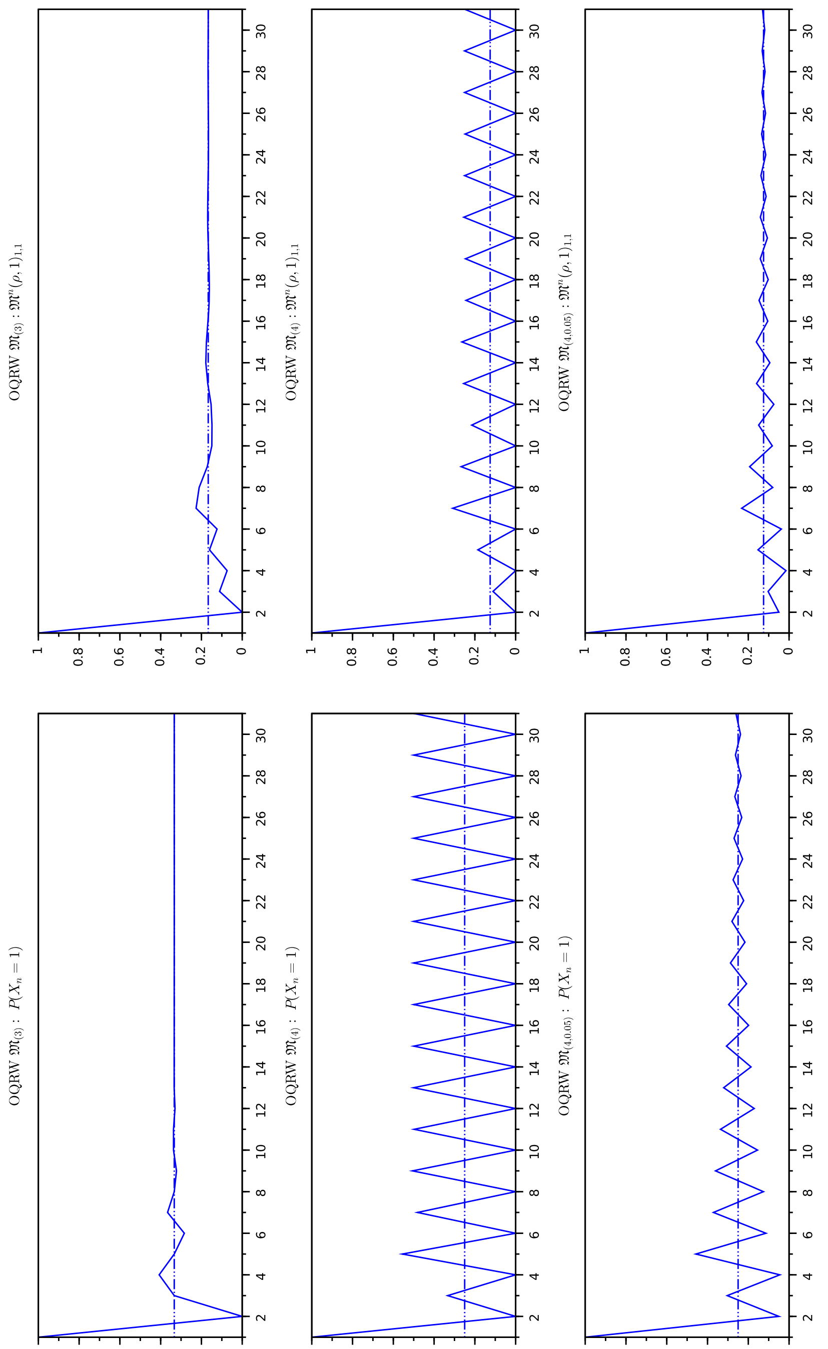

FIGURE 1. Data associated with Example 9.5

$$L_+ = p\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad L_- = q\begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

with $\alpha \in [0, 2\pi)$, $p, q \in \mathbb{C}\backslash\{0\}$, $|p|^2 + |q|^2 = 1$.

This OQRW is irreducible when $\alpha \neq 0, \pi$. To prove this consider the non-zero vector $v = \begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathfrak{h}_i$, for all $n > 0$ one sees by inspection that $\operatorname{span}\{L_+^n v, L_+^{n+1} L_- v, L_+^n L_- L_+ v\}$ coincides with $\mathfrak{h}_{i+n}$. We can proceed similarly for $n \leq 0$.

The period is 4: from the properties of this OQRW and Theorem 4.10, it cannot be greater than 4 and we can choose the resolution of the identity

$$P_k = \sum_{i\in\mathbb{Z}} |e_0\rangle\langle e_0| \otimes |4\,i+k\rangle\langle 4\,i+k| + \sum_{i\in\mathbb{Z}} |e_1\rangle\langle e_1| \otimes |4\,i+k+2\rangle\langle 4\,i+k+2|,$$

for $k = 0, \ldots, 3$. Finally, notice that the quantity $D(i, x)$ introduced in Theorem 4.10 is not the same for all vectors: $D(i, e_0) = D(i, e_1) = 4$ but, if we call $x = \begin{pmatrix} 1 \\ e^{i\alpha/2} \end{pmatrix}$, then x is an eigenvector for $L_- L_+$ and so the set of lengths ℓ introduced in the definition of $D(i, x)$ contains 2. Since it is clear that all those lengths are even, then $D(i, x) = 2$.

Example 9.8. We consider an OQRW $\mathfrak{M}$ as introduced in Example 7.6. Then $\mathfrak{M}$ does not have a unique decomposition in irreducible components. Indeed, it is easy to see that the $\mathfrak{M}$-invariant states are all the states of the form

$$\rho = \rho_1 \otimes |1\rangle\langle 1| + B\rho_1 B \otimes |2\rangle\langle 2|$$

for any 2×2 matrix ρ_1 such that $2\rho_1$ is a state in $M_2(\mathbb{C})$. So $\mathcal{R} = \mathcal{H}$ for this $\mathfrak{M}$, and the minimal enclosures are exactly all the enclosures generated by vectors of the form $x \otimes |1\rangle$, for $x = \begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{C}^2$,

$$\operatorname{Enc}(x \otimes |1\rangle) = \operatorname{Vect}\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \otimes |1\rangle, \begin{pmatrix} b \\ a \end{pmatrix} \otimes |2\rangle \right\}.$$

Therefore, the decomposition of $\mathcal{R} = \mathcal{H}$ into a sum of minimal enclosures is non-unique. To illustrate Theorem 7.13, consider an invariant state ρ; from the above discussion, it is of the form

$$\rho = \frac{1}{2}\begin{pmatrix} t & s \\ \overline{s} & 1-t \end{pmatrix} \otimes |1\rangle\langle 1| + \frac{1}{2}\begin{pmatrix} 1-t & \overline{s} \\ s & t \end{pmatrix} \otimes |2\rangle\langle 2|$$

with $t \in [0, 1]$, $|s|^2 \leq t(1-t)$. Writing this ρ in the decomposition

$$\mathcal{H} = \operatorname{Enc}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |1\rangle\right) \oplus \operatorname{Enc}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes |1\rangle\right),$$

which is a possible choice of decomposition (7.6), we obtain

$$\rho = \frac{1}{2}\begin{pmatrix} t & 0 & s & 0 \\ 0 & t & 0 & s \\ \overline{s} & 0 & 1-t & 0 \\ 0 & \overline{s} & 0 & 1-t \end{pmatrix}.$$

In agreement with Theorem 7.13, this ρ is of the form $t\,\rho_1^{\mathrm{inv}} + (1-t)\,\rho_2^{\mathrm{inv}} + s\;\eta_{1,2} + \overline{s}\;\eta_{2,1}$, where ρ_1^{inv} and ρ_2^{inv} are invariant states with support equal to $\mathrm{Enc}(\begin{pmatrix}1\\0\end{pmatrix}\otimes|1\rangle)$, $\mathrm{Enc}(\begin{pmatrix}0\\1\end{pmatrix}\otimes|1\rangle)$ respectively. In addition, the off-diagonal blocks $\eta_{1,2}$ and $\eta_{2,1}$ are also $\mathfrak{M}$-invariant, and with Q the partial isometry of the form

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}$$

we see that $\rho_2^{\mathrm{inv}} = Q\rho_1^{\mathrm{inv}}Q^*$ and $\eta_{1,2}$ is proportional to $Q^*\rho_2^{\mathrm{inv}} = \rho_1^{\mathrm{inv}}Q^*$.

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