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Smoothness of Compact Horizons

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Abstract. We prove that compact Cauchy horizons in a smooth spacetime satisfying the null energy condition are smooth. As an application, we consider the problem of determining when a cobordism admits Lorentzian metrics with certain properties. In particular, we prove a result originally due to Tipler without the smoothness hypothesis necessary in the original proof.

Introduction

An intriguing question in the theory of general relativity is that of *topology change*: is it possible for a spacelike slice of spacetime at one time to have a different topology than that of a spacelike slice at some other time? One way of making this question precise is through the concept of a *Lorentzian cobordism* (see Definition 2.3); that is, a spacetime whose boundary consists of disjoint spacelike submanifolds. The question whether topology change is possible can then be interpreted as the question of whether physically interesting nontrivial Lorentzian cobordisms exist. For this question to be interesting the cobordism needs to have some compactness property. We will consider both the case when the cobordism is compact and the case when the cobordism has the weaker property of causal compactness (see Definition 2.4).

The existence of Lorentzian cobordisms when no geometrical conditions are imposed is essentially a problem of differential topology. It is equivalent to the existence of a cobordism with a vector field with a prescribed direction at the boundary, and the problem of characterizing pairs of manifolds which are cobordant in this sense was considered by Reinhart [29].

When geometrical conditions are imposed, it is significantly more difficult to construct Lorentzian cobordisms which are not diffeomorphic to $S \times [0, 1]$ for some manifold S. There are two classical results about the nonexistence of nontrivial Lorentzian cobordisms under certain hypotheses: in 1967, it was shown by Geroch [15] that nontrivial Lorentzian cobordisms can exist only if they contain closed timelike curves, and in 1977, it was shown by Tipler

[30] that nontrivial Lorentzian cobordisms satisfying certain energy conditions cannot exist.

In proving Tipler's theorem, one works with a compact Cauchy horizon, and the question arises which regularity a Cauchy horizon has. It was shown in [9] that Cauchy horizons can be far from smooth. In [2, Section IV] it was shown that a compact "almost everywhere C^2" horizon satisfying the null energy condition is everywhere C^1. In [4, Section 4] it was asked whether *compact* Cauchy horizons are always smooth. This question was answered in the negative in [5], where it was also suggested that an energy condition might be sufficient to conclude that a compact Cauchy horizon is smooth.

The main result of this paper is Theorem 1.42, where we prove that compact horizons in a spacetime satisfying the null energy condition are smooth, thereby significantly generalizing the theorem in [2, Section IV], and providing an answer to the question raised in [5, Section 4] of whether compact horizons which satisfy energy conditions are smooth. We then apply this theorem to obtain a complete proof of Tipler's theorem. It has been known for some time (see for instance [9, Section 1]) that the proof of Tipler's result in [30] makes an implicit smoothness assumption. The proof uses arguments from the proof of the Hawking Singularity Theorem [16, p.295-298], and the same implicit assumption can be found there as well. A similar oversight was made in the original proof of the Hawking area theorem and has since been corrected by Chruściel et al. [7]. Significant work was necessary to fill in the gaps in the proof of the Hawking area theorem, and the proof in [7] is technical. Fortunately, their methods may be adapted to the setting of Tipler's theorem and we do so in Sects. 1 and 2 to present a more careful proof of the nonexistence of Lorentzian cobordisms which satisfy certain energy conditions.

Theorem 1.42 is also interesting in relation to the papers [17,25,26] by Isenberg and Moncrief. In the first two, it is shown that analytic compact null hypersurfaces with certain properties admit a null Killing vector field. Theorem 1.42 allows us to drop the hypothesis that the hypersurface is analytic, and replace it with the hypothesis that the null energy condition holds and that the hypersurface is a Cauchy horizon. In fact, as is discussed in Remark 1.44, it is not necessary that the hypersurface is a Cauchy horizon, provided its generators are complete to the past.

Appendix A contains a summary of results from geometric measure theory.

A Comment

After these results were published as part of the author's Master's thesis [20], similar work by Minguzzi [23,24] has appeared. Following concerns raised in [24] about the existence of a timelike vector field V satisfying $\nabla_V V = 0$ in a neighborhood of a Cauchy horizon, a construction of a vector field having this property on a sufficiently large set has been added to Lemma 1.6.

1. Smoothness of Compact Cauchy Horizons

The purpose of this section is to prove Theorem 1.42. We begin by stating and proving some properties of C^2 null hypersurfaces in Sect. 1.1. We then define the concept of a "horizon" and summarize some previously known results about horizons in Sect. 1.3. Finally, we prove the smoothness theorem (Theorem 1.42) in Sect. 1.4.

1.1. C^2 Null Hypersurfaces

1.1.1. The Null Weingarten Map. In the following section we summarize properties of C^2 null hypersurfaces which we will need later. For details, see [14,19], [13, Section II.1] or [7, Appendix A].

A null hypersurface $\mathcal{H}$ in a spacetime M is characterized by the fact that each tangent space $T_p\mathcal{H}$ contains a unique (up to scaling) null vector K_p. The tangent space $T_p\mathcal{H}$ then consists of those vectors of T_pM which are orthogonal to K_p. This means that any normal vector field K of $\mathcal{H}$ consists entirely of null vectors. We will call the integral curves of these vector fields *generators* of $\mathcal{H}$. By [14, Proposition 3.1] these generators (when given a suitable parametrization) are geodesics. By straightforward computations it holds that

$$\langle X, Y\rangle = \langle X', Y'\rangle \quad \text{and} \quad \langle \nabla_X K, Y\rangle = \langle \nabla_{X'} K, Y'\rangle$$

whenever $X, Y \in T_p\mathcal{H}$ and $X - X' = \lambda_1 K$ and $Y - Y' = \lambda_2 K$ for some real numbers λ_1, λ_2. Inspired by this, we work instead with the quotient space $T_p\mathcal{H}/\mathbb{R}K$. This quotient is independent of the particular choice of null vector field K, since all such vector fields differ only by scaling. We now define the *null Weingarten map* of $\mathcal{H}$ with respect to K by

$$b_K\colon\ T\mathcal{H}/\mathbb{R}K \to T\mathcal{H}/\mathbb{R}K,$$

$$b_K(\overline{X}) = \overline{\nabla_X K}.$$

This map is not independent of the particular choice of null vector field K. However, if f is a smooth function without zeros then $b_{fK} = f b_K$ since K is null. Note that if $\mathcal{H}$ is C^2, then K can be chosen C^1 so that b_K is continuous. Since all our spacetimes are time-oriented we may restrict attention to future-directed null vector fields K. This means that we can associate to each null hypersurface a family of null Weingarten maps which differ only by positive scaling. Since K is null the spacetime metric induces an inner product on $T\mathcal{H}/\mathbb{R}K$. Using this inner product, we may define the *null second fundamental form* of $\mathcal{H}$ with respect to K by

$$B_K(\overline{X}, \overline{Y}) = \langle b_K(\overline{X}), \overline{Y}\rangle.$$

A straightforward computation shows that B_K is symmetric. We will need the following theorem, a proof of which can be found in [19, Theorem 30].

Theorem 1.1. *Let $\mathcal{H}$ be a smooth null hypersurface in a spacetime M. Then the null second fundamental form of $\mathcal{H}$ is identically zero if and only if $\mathcal{H}$ is a totally geodesic submanifold of M.*

Remark 1.2. The theorem as stated in [19, Theorem 30] applies to null submanifolds in general, regardless of codimension, and so requires the submanifold to be "irrotational". This condition is automatically satisfied for null hypersurfaces.

Finally, we define the *null mean curvature* θ_K of a null hypersurface with respect to a null vector field K as the trace of the null Weingarten map:

$$\theta_K = \operatorname{tr} b_K.$$

Recall that if $K' = \lambda K$ is another null vector field then $b_{K'} = \lambda b_K$. Hence $\theta_{K'} = \lambda\theta_K$. This means that the sign of θ_K is independent of the particular future-directed null vector field K used to compute θ_K. We will sometimes omit the vector field K from the notation.

Recall that the integral curves of a null vector field K are reparametrizations of geodesics. If K is chosen to agree with $\dot\gamma$ of a geodesic segment γ with affine parameter s, and $b(s)$ is the family of null Weingarten maps with respect to K along γ, then

$$\dot b + b^2 + \widetilde{R} = 0. \tag{1}$$

Here, $\dot b$ denotes the derivative of b along γ, and $\widetilde{R}$ denotes the fiberwise endomorphism $\widetilde{R}\colon T\mathcal{H}/\mathbb{R}K \to T\mathcal{H}/\mathbb{R}K$ defined from the Riemann curvature tensor R by letting $\widetilde{R}(\overline{X}) = \overline{R(X,\dot\gamma)\dot\gamma}$. Note that it is not obvious that the derivative $\dot b$ exists, since b is a priori only continuous. A proof of the fact that the derivative does exist and satisfies Eq. (1) can be found in [7, Proposition A.1].

From Eq. (1) one can derive the *Raychaudhuri equation*. In particular, one may derive a certain differential inequality for the null mean curvature. Let b be the null Weingarten map of a C^2 null hypersurface with respect to a future-directed null vector field K (scaled to give an affine parametrization of an integral curve), and let θ denote the trace of b. Let $S = b - \frac{\theta}{n-2}\,\mathrm{id}$. Then the trace of b^2 is $\operatorname{tr} b^2 = \theta^2/(n-2) + \operatorname{tr}(S^2)$ so taking the trace of Eq. (1) yields

$$\dot\theta + \frac{\theta^2}{n-2} + \operatorname{tr}(S^2) + \operatorname{Ric}(K,K) = 0. \tag{2}$$

Since b and id are self-adjoint, so is S. Hence $\operatorname{tr}(S^2) \geq 0$ so

$$\dot\theta + \frac{\theta^2}{n-2} + \operatorname{Ric}(K,K) \leq 0. \tag{3}$$

This is the differential inequality we will use later.

1.1.2. Generator Flow on C^2 Null Hypersurfaces. A null vector field on a C^2 null hypersurface gives rise to a family of local diffeomorphisms with flow points along the vector field. The integral curves of such a vector field are called generators, and we will refer to such a flow as a *generator flow*. The generator flow for time t will be denoted β_t. Given a Riemannian metric σ on M with volume form ω on the null hypersurface, the *Jacobian determinant* $J(\beta_t)$ with respect to σ is the real-valued function defined by $(\beta_t)^*\omega = J(\beta_t)\omega$. In this section, we will show that the Jacobian determinant of a generator flow with respect to Riemannian metrics of a certain form is related to the null mean curvature of the hypersurface. We choose to work with a past-directed vector field T since this is the case in which we will apply the lemma.

Lemma 1.3. *Let $\mathcal{H}$ be a C^2 null hypersurface in a spacetime (M, g) of dimension $n+1$. Let V be an arbitrary unit timelike vector field on M, and define a Riemannian metric σ on M by*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V).$$

Let T denote the unique past-directed lightlike σ-unit vector field on $\mathcal{H}$ and let ω denote the σ-volume form induced on $\mathcal{H}$. Let θ denote the null mean curvature of $\mathcal{H}$ with respect to the future-directed null vector field $-T$. Then the Lie derivative of ω with respect to T is $\mathcal{L}_T\omega = -\theta\omega$.

Proof. Choose some point $p \in \mathcal{H}$ at which to evaluate $\mathcal{L}_T\omega$. Let $e_1, e_2, \ldots, e_n$ be a g-orthogonal basis for $T_p\mathcal{H}$ such that

- $e_1 = T_p$,
- $g(e_i, V) = 0$ for $i = 2, 3, \ldots, n$,
- $g(e_i, e_i) = 1$ for $i = 2, 3, \ldots, n$.

Recall that integral curves of null vector fields on null hypersurfaces are geodesic segments. Let γ be a segment of the integral curve of T through p with an affine parametrization. Extend the basis $e_1, \ldots, e_n$ along γ by letting

- $e_1 = T$,
- $\nabla_{e_1} e_i = 0$ for $i = 2, 3, \ldots, n$.

Here ∇ denotes covariant derivative with respect to g. Note that we do not yet know that $(e_i)_{i=1}^n$ is a frame for $\mathcal{H}$, since we first need to show that the e_i are tangent to $\mathcal{H}$. This will follow from the properties below. Direct computations show that $(e_i)_{i=1}^n$ have the following properties on the image of γ.

- $g(e_1, V) = 1/\sqrt{2}$,
- $g(e_1, e_i) = 0$,
- $g(e_i, e_j) = \delta_{ij}$ for $i, j = 2, 3, \ldots, n$,
- $\sigma(e_1, e_i) = \sqrt{2}g(e_i, V)$ for $i = 2, 3, \ldots, n$,
- $\sigma(e_i, e_j) = \delta_{ij} + 2g(e_i, V)g(e_j, V)$ for $i, j = 2, 3, \ldots, n$.

Step I: $\det(\sigma(e_i, e_j)) = 1$

Let $a_i = \sigma(e_i, V)$ for $i = 1, 2, \ldots, n$. By the previous claims the matrix A with entries $\sigma(e_i, e_j)$ can then be written as

$$A = \begin{pmatrix} 1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \cdots \\ \sqrt{2}a_2 & 1 + 2a_2^2 & 2a_2a_3 & \cdots \\ \sqrt{2}a_3 & 2a_2a_3 & 1 + 2a_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\sqrt{2}a_2 & 1 & 0 & 0 & \cdots \\ -\sqrt{2}a_3 & 0 & 1 & 0 & \cdots \\ -\sqrt{2}a_4 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\det B = 1$. Moreover

$$BA = \begin{pmatrix} 1 & \sqrt{2}a_2 & \sqrt{2}a_3 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence $\det(BA) = 1$. This means that

$$\det A = \frac{\det(BA)}{\det B} = 1,$$

proving the claim.

Step II: Computation of $\mathcal{L}_T\omega$
The volume form ω induced on $\mathcal{H}$ by the Riemannian metric σ can be expressed in the frame $e_1, e_2, \ldots, e_n$ as

$$\omega = \sqrt{\det(\sigma(e_i, e_j))}e^1 \wedge e^2 \wedge \cdots \wedge e^n$$

where the e^i are the covectors defined by $e^i(e_i) = 1$ and $e^i(e_j) = 0$ for $i \neq j$. By the previous claim the determinant is equal to 1, so

$$\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n$$

on all of γ. We will use Cartan's formula to compute the Lie derivative $\mathcal{L}_T\omega$, so we need to extend the frame e_i to a neighborhood of γ. Extend $e_1, \ldots, e_n$ to a frame such that $e_1 = T$. Extend the dual frame $e^1, \ldots, e^n$ in the natural way by letting $e^i(e_i) = 1$ and $e^i(e_j) = 0$ for $i \neq j$. Rescale e_n if necessary so that $\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n$ holds everywhere. We will now use this frame to compute $\mathcal{L}_T\omega$ at the point p. By Cartan's formula

$$\mathcal{L}_T\omega = i_T\mathrm{d}\omega + \mathrm{d}(i_T\omega).$$

Since ω is an n-form on an n-manifold we have $d\omega = 0$. Hence

$$\mathcal{L}_T\omega = \mathrm{d}(i_T\omega).$$

Since $\omega = e^1 \wedge e^2 \wedge \cdots \wedge e^n$

$$\mathrm{d}(i_T\omega) = \mathrm{d}(e^1(T)e^2 \wedge \cdots \wedge e^n) = \mathrm{d}(e^1(e_1)e^2 \wedge \cdots \wedge e^n) = \mathrm{d}(e^2 \wedge \cdots \wedge e^n).$$

Hence

$$\mathcal{L}_T\omega = \sum_{k=2}^{n}(-1)^k e^2 \wedge \cdots \wedge e^{k-1} \wedge \mathrm{d}e^k \wedge e^{k+1} \wedge \cdots \wedge e^n.$$

We now compute $\mathrm{d}e^k$, or rather the part of $\mathrm{d}e^k$ which does not contain any e^j for $j \notin \{1, k\}$; all such terms are annihilated when we insert this expression into the large wedge product above. Since $\mathrm{d}e^k$ is a two form this means that only one of its terms, $(\mathrm{d}e^k)(e_1, e_k)e^1 \wedge e^k$, is interesting. Now

$$(\mathrm{d}e^k)(e_1, e_k) = e_1(e^k(e_k)) - e_k(e^k(e_1)) - e^k([e_1, e_k]) = -e^k([e_1, e_k])$$

since $e^k(e_k) = 1$ and $e^k(e_1) = 0$ close to γ. We express the Lie bracket, evaluated at the point p, using the spacetime metric g as

$$(\mathrm{d}e^k)(e_1, e_k) = -e^k([e_1, e_k]) = -e^k(\nabla_{e_1} e_k - \nabla_{e_k} e_1) = e^k(\nabla_{e_k} e_1).$$

Recall that ∇ denotes covariant derivative with respect to g. We have used that $e_2, \ldots, e_n$ have been chosen such that $\nabla_{e_1} e_k = 0$ for all k. We now know that

$$\mathrm{d}e^k = (e^k(\nabla_{e_k} e_1))e^1 \wedge e^k + \ldots$$

where the dots signify terms containing some e^j for $j \notin \{1, k\}$. At the point p it then holds that

$$\begin{aligned}
&(-1)^k e^2 \wedge \cdots \wedge e^{k-1} \wedge \mathrm{d}e^k \wedge e^{k+1} \wedge \cdots \wedge e^n \\
&\quad = (-1)^k e^2 \wedge \cdots \wedge e^{k-1} \wedge (e^k(\nabla_{e_k} e_1))e^1 \wedge e^k \wedge e^{k+1} \wedge \cdots \wedge e^n \\
&\quad = (-1)^k (-1)^{k-2} (e^k(\nabla_{e_k} e_1))e^1 \wedge e^2 \wedge \cdots \wedge e^{k-1} \wedge e^k \wedge e^{k+1} \wedge \cdots \wedge e^n \\
&\quad = (e^k(\nabla_{e_k} e_1))e^1 \wedge e^2 \wedge \cdots \wedge e^n
\end{aligned}$$

where the additional factor of $(-1)^{k-2}$ is due to commuting e^1 with the $e^2, \ldots, e^{k-1}$. Hence

$$\mathcal{L}_T \omega = \sum_{k=2}^{n} (e^k(\nabla_{e_k} e_1))e^1 \wedge e^2 \wedge \cdots \wedge e^n = \sum_{k=2}^{n} (e^k(\nabla_{e_k} e_1))\omega.$$

Since $e_1 = T$ and $e_2, \ldots, e_n$ are g-orthonormal and g-orthogonal to e_1,

$$\sum_{k=2}^{n} e^k(\nabla_{e_k} e_1) = \sum_{k=2}^{n} g(e_k, \nabla_{e_k} e_1) = -\sum_{k=2}^{n} g(e_k, \nabla_{e_k}(-T)).$$

Recall from Sect. 1.1.1 that the quotient $T_p\mathcal{H}/\mathbb{R}T$ has an inner product induced by g such that the image of $(e_i)_{i=2}^n$ under the projection $T_p\mathcal{H} \to T_p\mathcal{H}/\mathbb{R}T$ forms an orthonormal basis. This means that $\sum_{k=2}^n g(e_k, \nabla_{e_k}(-T))$ is the trace of the null Weingarten map b_{-T} defined in Sect. 1.1.1. This trace is the null mean curvature θ of $\mathcal{H}$ with respect to $-T$. We can then conclude that

$$\mathcal{L}_T \omega = -\theta \omega$$

at p. Since p was arbitrary, this completes the proof. □

The proof of the following lemma essentially consists of integrating the Lie derivative $\mathcal{L}_T \omega$ to relate the null mean curvature θ to the Jacobian determinant of the generator flow.

Lemma 1.4. *Let $\mathcal{H}$ be a C^2 null hypersurface in a spacetime (M, g). Let σ be a Riemannian metric on M of the form*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V)$$

for some g-unit timelike vector field V on M. Let T be the unique past-directed σ-unit null vector field on $\mathcal{H}$, and let θ be the null mean curvature of $\mathcal{H}$ with respect to $-T$. Fix $t > 0$ and let $\beta_t \colon \mathcal{H} \to \mathcal{H}$ denote the flow along T for time t (whenever defined). Suppose that p is such that $\beta_s(p)$ is defined for all $s \in [0, t]$. Let $J(\beta_t)$ denote the Jacobian determinant of β_t with respect to σ. Then

$$J(\beta_t)(p) = \exp\left(-\int_0^t \theta(\beta_s(p))\,\mathrm{d}s\right).$$

Proof. Let ω denote the volume form of σ. The Jacobian determinant $J(\beta_t)(p)$ is characterized by

$$\beta_t^*(\omega_{\beta_t(p)}) = J(\beta_t)(p)\omega_p.$$

For simpler notation, let $\alpha\colon [0, t] \to \mathbb{R}$ denote the function $\alpha(s) = J(\beta_s)(p)$. Note that $\alpha(0) = 1$ since β_0 is the identity. By [21, Proposition 12.36] it holds that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=s} \beta_\tau^*(\omega_{\beta_\tau(p)}) = \beta_s^*(\mathcal{L}_T\omega_{\beta_s(p)}).$$

Since $\beta_\tau^*(\omega_{\beta_\tau(p)}) = \alpha(\tau)\omega_p$ it holds that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\right|_{\tau=s} \beta_\tau^*(\omega_{\beta_\tau(p)}) = \alpha'(s)\omega_p.$$

Since $\mathcal{L}_T\omega_{\beta_s(p)} = -\theta(\beta_s(p))\omega_{\beta_s(p)}$ by Lemma 1.3, it holds that

$$\beta_s^*(\mathcal{L}_T\omega_{\beta_s(p)}) = -\theta(\beta_s(p))\beta_s^*(\omega_{\beta_s(p)}) = -\theta(\beta_s(p))\alpha(s)\omega_p.$$

Hence

$$\alpha'(s) = -\theta(\beta_s(p))\alpha(s).$$

Solving this differential equation subject to the initial condition $\alpha(0) = 1$ we see that

$$\alpha(t) = \exp\left(-\int_0^t \theta(\beta_s(p))\,\mathrm{d}s\right).$$

Since $\alpha(t) = J(\beta_t)(p)$ this completes the proof. □

1.1.3. Geodesically Spanned Null Hypersurfaces.

Proposition 1.5. *Let (M, g) be a spacetime of dimension $n+1$ and let $N \subset M$ be a spacelike C^2 submanifold of codimension 2. Let $\mathbf{n}$ denote a C^1 normal null vector field along N. Consider the normal exponential map* $\exp\colon \mathbb{R} \times N \to M$ *defined by*

$$\exp(t, p) = \exp_p(t\mathbf{n}_p)$$

where $\exp_p$ *is the exponential map at the point* p*. Suppose that* $\mathcal{O} \subset \mathbb{R} \times N$ *is an open subset such that* $\mathcal{H} := \exp(\mathcal{O})$ *is an embedded* C^1 *hypersurface in* M *and the tangent map* $\exp_*$ *is injective on* $\mathcal{O}$*. Then* $\mathcal{H}$ *is a null hypersurface.*

Proof. Choose a point $q = \exp(t, p) \in \mathcal{H}$ and let γ denote the null geodesic $s \mapsto \exp(s, p)$. Our goal is to show that every vector $W \in T_q\mathcal{H}$ is orthogonal to $\dot\gamma(t)$, thereby proving that $T_q\mathcal{H}$ is a null hyperplane.

Since $\exp_* : T(\mathbb{R} \times N) \to T\mathcal{H}$ is injective at (t, p), it is also surjective for dimensional reasons. This means that W has some preimage in $T_{(t,p)}(\mathbb{R} \times N)$. Denote this preimage by (ζ, Z), where we make use of the canonical isomorphism $T_{(t,p)}(\mathbb{R} \times N) \cong T_t\mathbb{R} \times T_pN$. The pushforward is linear so

$$\exp_*(\zeta, Z) = \exp_*(\zeta, 0) + \exp_*(0, Z).$$

Note that $\exp_*(\zeta, 0)$ is tangent to the null curve γ, so $g(\exp_*(\zeta, 0), \dot\gamma(t)) = 0$. Hence

$$g(W, \dot\gamma(t)) = g(\exp_*(\zeta, 0) + \exp_*(0, Z), \dot\gamma(t)) = g(\exp_*(0, Z), \dot\gamma(t)).$$

Let $\alpha : (-1, 1) \to N$ be a curve with $\alpha(0) = p$ and $\dot\alpha(0) = Z$. Consider the two-parameter map

$$\mathbf{x}(s, u) = \exp(st, \alpha(u))$$

defined for $s \in [0, 1]$ and $u \in (-1, 1)$. Let V be a vector field along γ defined by

$$V(s) = \mathbf{x}_u(s, 0).$$

Each curve $s \mapsto \mathbf{x}(s, u)$ is a geodesic, so the map $\mathbf{x}$ is a variation through geodesics. Hence V is a Jacobi vector field. The curve $u \mapsto \mathbf{x}(0, u)$ is contained in N so $V(0)$ is tangent to N. By assumption on $\mathbf{n}$, the vector $\dot\gamma(0)$ is orthogonal to N, so

$$g(V(0), \dot\gamma(0)) = 0.$$

Let T denote the vector field $\mathbf{x}_s$ along the map $\mathbf{x}$. Partial derivatives of two-parameter maps commute by [28, Proposition 44, Chapter 4] so

$$V'(0) = \mathbf{x}_{us}(0, 0) = \mathbf{x}_{su}(0, 0) = \nabla_Z T.$$

Hence

$$g(V'(0), T) = g(\nabla_Z T, T) = \frac{1}{2} Z g(T, T) = 0$$

since T is tangent to null curves. Since $\mathbf{x}_s(0, 0) = \dot\gamma(0)$ we have shown that

$$g(V'(0), \dot\gamma(0)) = 0.$$

By [28, Lemma 7, Chapter 8], the fact that $V(0)$ and $V'(0)$ are both orthogonal to the geodesic γ, together with the fact that V is a Jacobi field along γ, implies that $V(s)$ is orthogonal to γ for all s. In particular,

$$g(V(1), \dot\gamma(t)) = 0.$$

Computing $V(1)$ we see that

$$V(1) = \mathbf{x}_u(1, 0) = \exp_*(0, \dot\alpha(0)) = \exp_*(0, Z).$$

Hence

$$g(W, \dot{\gamma}(t)) = 0$$

for all $W \in T_q\mathcal{H}$. Since $q = \exp(t, p)$ was arbitrary, this shows that each tangent plane of $\mathcal{H}$ is a null hyperplane, so that $\mathcal{H}$ is a null hypersurface. $\square$

1.2. Complete Generators

The following is a straightforward generalization of Lemma 8.5.5 in [16], and the proof follows that of [16] but contains significantly more details.

Lemma 1.6. *Let S be an achronal hypersurface in a spacetime (M, g) of dimension $n+1$. Let γ be a null geodesic segment contained in $H^+(S)$. Suppose that γ has no past endpoint and is totally past imprisoned in some compact set K. Suppose moreover that each point $p \in \operatorname{im}\gamma$ has some spacetime neighborhood U_p such that $U_p \cap \operatorname{im}\gamma$ is contained in a $C^{1,1}$ hypersurface N_p. Then γ is complete in the past direction.*

Proof. Let γ have an affine parametrization. Suppose to get a contradiction that γ is incomplete to the past, i.e. that the domain of γ has some infimum v_0. We may without loss of generality (by translation of the parameter of γ and restriction of γ to a smaller domain to the future) assume that γ has domain $(v_0, 0]$ and that $\gamma(t) \in K$ for all $t \in (v_0, 0]$. Then the set $\overline{\operatorname{im}\gamma}$ is compact, so we may assume without loss of generality that $K = \overline{\operatorname{im}\gamma}$. Let $\mathcal{W}$ be a neighborhood of $K \cap H^+(S)$ with compact closure.

The idea is now to show that if γ is past incomplete, then a small perturbation of it yields a past inextendible timelike curve with contradictory properties. To help with this, we will introduce a timelike vector field V. For the construction of V, we will need an auxiliary distance function compatible with the manifold topology, for instance one given by a Riemannian metric η. Fix such a distance function and call it d_η. Since M is time-orientable, it admits a future-directed timelike vector field. Fix such a vector field and call it Z. For each $p \in \operatorname{im}\gamma$, we will define a vector field V^p in a neighborhood of p with the following properties.

- V^p is timelike and future-directed.
- $V^p = Z$ on $\operatorname{im}\gamma \cap \operatorname{dom}(V^p)$.
- $\nabla_{V^p} V^p = 0$.

To do this, consider a small neighborhood of p whose intersection with γ is contained in a $C^{1,1}$ hypersurface N. Consider the restriction of the exponential map to the restriction to N of the subbundle of TM spanned by Z. In other words, consider the map $\exp_Z \colon N \times \mathbb{R} \to M$ defined by

$$\exp_Z(q, t) = exp_q(tZ).$$

This map is submersive at $(p, 0)$, and hence for dimensional reasons also immersive at $(p, 0)$. By the inverse function theorem, it is then a $C^{1,1}$ diffeomorphism on some open neighborhood $(p, 0)$. Let $\mathcal{U}_p$ be the image of this neighborhood under $\exp_Z$. Let $\rho(p) > 0$ be a real number such that all points $r \in M$ with $d_\eta(r, p) < 4\rho(p)$ belong to $\mathcal{U}_p$. (Note that we will not need any continuity of

ρ.) Let $\mathcal{W}_p$ be the set of points $r \in M$ with $d_\eta(r,p) < \rho(p)$. Define V^p on $\mathcal{W}_p$ to be the tangent vectors of the curves $s \mapsto \exp_Z(q,s)$. Since $\exp_Z$ is a diffeomorphism onto $\mathcal{W}_p$, this is well defined.

We will now show that if $r \in \mathcal{W}_p \cap \mathcal{W}_q$ for some $p, q \in \operatorname{im}\gamma$ is such that the integral curves of both V^p and V^q through r both intersect γ, then $V^p_r = V^q_r$. Suppose that $r \in \mathcal{W}_p \cap \mathcal{W}_q$, that the integral curve of V^p through r intersects $\operatorname{im}\gamma$ in r_p, and that the integral curve of V^q through r intersects $\operatorname{im}\gamma$ in r_q. Suppose without loss of generality that $\rho(p) \geq \rho(q)$. Then $d_\eta(p, r_q) \leq d_\eta(p,r) + d_\eta(r,q) + d_\eta(q, r_q) < \rho(p) + 2\rho(q) \leq 3\rho(p)$. Hence $r_q \in \mathcal{U}_p$. Since $V^p_{r_q} = V^q_{r_q} = Z_{r_q}$ and $\exp_Z$ is a diffeomorphism onto $\mathcal{U}_p$, this means that $r_q = r_p$ and, by following the geodesic from $r_q = r_p$ with initial velocity $V^p_{r_q} = V^q_{r_q} = Z_{r_q}$, that $V^p_r = V^q_r$.

Choose a countable subset C of $\operatorname{im}\gamma$ such that the sets $\{U_p\}_{p\in F}$ cover $\operatorname{im}\gamma$. Combine the vector field V^p for $p \in C$ using a partition of unity corresponding to this cover to obtain a vector field V. This vector field is Lipschitz, timelike and future-directed since the V^p are. Moreover, $\nabla_V V = 0$ on each integral curve of V passing through $\operatorname{im}\gamma$, since this holds for the V^p and they agree on such curves. By a further partition of unity, we may extend V to a future-directed timelike Lipschitz vector field on all of M. Note, however, that V has larger regularity than Lipschitz on a 2-dimensional surface close to $\operatorname{im}\gamma$. More precisely, there is a subset $\Omega = \{(t,u) \in \mathbb{R}^2 \mid |u| < \psi(t)\}$ for some positive function ψ such that the map $\Omega \to TM$ defined by

$$(t,u) \mapsto V_{\exp_{\gamma(t)}(uZ_{\gamma(t)})}$$

is smooth. However, V is not necessarily smooth when viewed as a vector field on the spacetime.

Define a metric g' by

$$g'(X,Y) = g(X,Y) + 2g(X,V)g(Y,V).$$

This metric is positive definite. To see this, let X be nonzero and compute $g'(X,X)$ in a basis $(V, e_1, e_2, \ldots, e_n)$ orthonormal for g:

$$\begin{aligned} g'(X,X) &= g(X,X) + 2(g(X,V))^2 \\ &= \left(-(X^0)^2 + (X^1)^2 + (X^2)^2 + \cdots + (X^n)^2\right) + 2(X^0)^2 > 0. \end{aligned}$$

Let $\alpha_0(t) = \gamma(v(t))$ be a reparametrization of γ such that $g(\dot{\alpha_0}, V) = -1/\sqrt{2}$. Note that v is a smooth function, since V is smooth when viewed as a vector field along γ. This means that α_0 is a smooth curve. The definition of v implies that v is strictly increasing. For convenience, suppose also that $v(0) = 0$. Note that α_0 is parameterized by arc length in the Riemannian metric g':

$$\begin{aligned} \int_a^b \sqrt{g'(\dot\alpha_0(t), \dot\alpha_0(t))}\,\mathrm{d}t &= \int_a^b \sqrt{g(\dot\alpha_0(t), \dot\alpha_0(t)) + 2(g(\alpha_0(t), V))^2}\,\mathrm{d}t \\ &= \int_a^b \sqrt{0 + 2\frac{1}{2}}\,\mathrm{d}t = b - a. \end{aligned}$$

Since γ has no past endpoint, α_0 does not have one either.

We will later construct a variation α of α_0, and the computations will be done along the two-parameter map α.

Step I: The domain of α_0 is not bounded from below

Suppose for contradiction that the domain of α_0 is bounded below. Let $a > -\infty$ be the infimum of the domain of α_0. Recall that a Riemannian metric induces a distance function defined as the infimum of the lengths of curves from one point to another. Then for any sequence $a_n \to a$ it holds that $\alpha_0(a_n)$ is a Cauchy sequence with respect to the distance function induced by g' (for α_0 is a curve of length $|a_n - a_m| < |a_{\min(m,n)} - a| \to 0$ connecting $\alpha_0(a_n)$ to $\alpha_0(a_m)$). The sequence $\alpha_0(a_n)$ is also contained in the compact set K, and so has a convergent subsequence. These two statements together imply that $\alpha_0(a_n)$ is convergent for any sequence $a_n \to a$ so the limit $\lim_{t\to a^+} \alpha_0(t)$ exists contradicting the fact that α_0 has no past endpoint. Hence, the domain of α_0 is not bounded from below.

Step II: Relations between α_0 and γ

Since α_0 is a reparametrization of a geodesic, $\nabla_{\dot\alpha_0}\dot\alpha_0$ is parallel to $\dot\alpha_0$. In other words, there is a function $f\colon (-\infty, 0) \to \mathbb{R}$ such that

$$\nabla_{\dot\alpha_0(t)}\dot\alpha_0(t) = f(t)\dot\alpha_0(t), \quad \forall t \in (-\infty, 0).$$

Note that f is a smooth function. It also holds that

$$v'(t)\dot\gamma(v(t)) = \dot\alpha_0(t), \quad \forall t \in (-\infty, 0).$$

Now

$$f(t)\dot\alpha_0(t) = \nabla_{\dot\alpha_0}\dot\alpha_0 = \nabla_{\dot\alpha_0}(v'\dot\gamma) = \alpha_0(v')\dot\gamma + v'\nabla_{\dot\gamma}\dot\gamma = \frac{v''(t)}{v'(t)}\dot\alpha_0(t)$$

so

$$f = \frac{v''}{v'}.$$

Note also that f is bounded. This can be seen by the following computation.

$$\begin{aligned} f &= -\sqrt{2}g(f\dot\alpha_0, V) = -\sqrt{2}g(\nabla_{\dot\alpha_0}\dot\alpha_0, V) = -\sqrt{2}\left(\nabla_{\dot\alpha_0}g(\dot\alpha_0, V) - g(\dot\alpha_0, \nabla_{\dot\alpha_0}V)\right) \\ &= -\sqrt{2}\left(\dot\alpha_0(-1/\sqrt{2}) - g(\dot\alpha_0, \nabla_{\dot\alpha_0}V)\right) = \sqrt{2}g(\dot\alpha_0, \nabla_{\dot\alpha_0}V). \end{aligned}$$

This shows that f can be defined in terms of g, $\dot\alpha$ and V. The coordinate representations of these objects in coordinate patches are all bounded since $\dot\alpha$ is a unit vector field in g'. Since $H^+(S) \cap K$ is compact it can be covered by finitely many coordinate patches, and hence f is bounded.

Step III: v' is bounded

Since γ is incomplete to the past, v is bounded below. In other words, the integral

$$v(t) = \int_0^t v'(\tau)\mathrm{d}\tau$$

is bounded. This implies that $\liminf_{t\to-\infty} v'(t) = 0$, since v is strictly increasing. We will now show that boundedness of v on $(-\infty, 0]$ together with boundedness of $f = \frac{v''}{v'}$ implies that v' is bounded. Suppose not. Since v' is continuous, it can only be unbounded on $(-\infty, 0]$ if $\limsup_{t\to-\infty} v'(t) = \infty$. Since we also know that $\liminf_{t\to-\infty} v'(t) = 0$ and that v' is continuous there are, for arbitrarily large $C > 0$, sequences $t_n, s_n \to -\infty$ such that

$$\begin{aligned} t_{n+1} < s_n < t_n \quad &\text{for all } n, \\ v'(t_n) &= 2C, \\ v'(s_n) &= C \end{aligned}$$

and

$$C \leq v'(t) \leq 2C \quad \text{if } t \in (s_n, t_n).$$

By the mean value theorem of calculus, there is for each n some $\tau_n \in [s_n, t_n]$ such that

$$v''(\tau_n) = \frac{v'(t_n) - v'(s_n)}{t_n - s_n} = \frac{C}{t_n - s_n}.$$

However

$$\sum_{n=0}^{\infty} C(t_n - s_n) \leq \left| \int_0^{-\infty} v'(\tau)\mathrm{d}\tau \right| < \infty$$

so $(t_n - s_n) \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} f(\tau_n)v'(\tau_n) = \lim_{n\to\infty} v''(\tau_n) = \infty.$$

Since $v'(\tau_n) \in [C, 2C]$ for all n, this implies that $f(\tau_n) \to \infty$, contradicting the fact that f is bounded. Hence v' must be bounded.

Step IV: Construction of a variation α of α_0

We will now construct a variation α of α_0. The idea is to push α_0 to the past and make it timelike, and then derive a contradiction from the resulting curve. Let $x\colon (-\infty, 0) \to \mathbb{R}$ denote a smooth positive function which will be fixed later. Let

$$\begin{aligned} \alpha\colon (-\delta, \delta) \times (-\infty, 0) &\to H^+(S) \\ (u, t) &\mapsto \alpha(u, t) \end{aligned}$$

be a smooth map such that

$$\alpha(0, \cdot) = \alpha_0 \quad \text{and} \quad \frac{\partial \alpha}{\partial u}(u, t) = -x(t)V_{\alpha(u,t)}. \tag{4}$$

Recall that $V_{\alpha(u,t)}$ is smooth as a function of u and t, even though V is not a smooth vector field on the spacetime. To see that such a variation exists, note that the conditions can be viewed as a family of ordinary differential equations in u, parameterized by t. As a consequence of the existence theorem and theorem about smooth dependence on initial values for ordinary differential equations there is, for each t, a smooth solution with existence time $\delta_t > 0$. To claim that the necessary variation exists, we need to show that the existence times δ_t can be uniformly bounded from below by some $\delta > 0$ independent of

t. However, we know that a solution to the differential equation exists as long as it stays in the compact set $\overline{\mathcal{W}}$. Since $H^+(S) \cap K$ is compact and $\mathcal{W}$ open, the g' distance between $H^+(S) \cap K$ and $M \backslash \mathcal{W}$ is positive. Since V is bounded, and x will be bounded when we choose it, there is a positive uniform lower bound for the time after which a solution may leave $\mathcal{W}$. This means that there is a uniform lower bound for the existence times of the solutions of the family of ordinary differential equations defining the variation. Hence we may choose a suitable $\delta > 0$ uniformly, and a variation with the desired properties exists.

Let α_u denote the curve $\alpha(u, \cdot)$. Note that each curve α_u is smooth. We now wish to choose the positive function x in such a way that some curve α_ϵ is timelike. In other words, we want there to be some $\epsilon > 0$ such that the function

$$y(u,t) = g(\dot{\alpha}_u(t), \dot{\alpha}_u(t))$$

is negative for $u = \epsilon$ and all $t \in (-\infty, 0)$. To show that this is the case, we will compute $\frac{\partial y}{\partial u}\Big|_{u=0}$ and a bound for $\frac{\partial^2 y}{\partial u^2}$, and from this obtain an upper bound for y. Choosing a suitable function x will make this upper bound negative for small values of u.

Step V: Computation of $\frac{\partial y}{\partial u}$

Let U denote the pushforward through α of the coordinate vector field $\frac{\partial}{\partial u}$ on $(-\delta, \delta) \times (-\infty, 0)$. We will not always write out the dependence on t and u. The first partial derivative of y can be computed as

$$\begin{aligned}\frac{\partial y}{\partial u}(u,t) &= \frac{\partial}{\partial u} g\left(\dot{\alpha}_u(t), \dot{\alpha}_u(t)\right) = U g(\dot{\alpha}_u, \dot{\alpha}_u) = 2g(\nabla_U \dot{\alpha}_u, \dot{\alpha}_u) \\ &= 2g(\nabla_{\dot{\alpha}_u} U, \dot{\alpha}_u) = 2\left(\nabla_{\dot{\alpha}_u} g(U, \dot{\alpha}_u) - g(U, \nabla_{\dot{\alpha}_u} \dot{\alpha}_u)\right)\end{aligned}$$

where $\nabla_U \dot{\alpha}_u = \nabla_{\dot{\alpha}_u} U$ since U and $\dot{\alpha}_u$ are pushforwards of coordinate vector fields. Evaluating this at $u = 0$ we see that

$$\begin{aligned}\frac{\partial y}{\partial u}(0,t) &= 2\left(\nabla_{\dot{\alpha}_0} g(-xV, \dot{\alpha}_0) - g(-xV, \nabla_{\dot{\alpha}_0} \dot{\alpha}_0)\right) \\ &= 2\left(-\nabla_{\dot{\alpha}_0}(x g(V, \dot{\alpha}_0)) + x g(V, \nabla_{v'\dot{\gamma}}(v'\dot{\gamma}))\right) \\ &= 2\left(\frac{1}{\sqrt{2}} \dot{\alpha}_0(x) + x v' g(V, \nabla_{\dot{\gamma}}(v'\dot{\gamma}))\right) \\ &= 2\left(\frac{1}{\sqrt{2}} \dot{\alpha}_0(x) + x (v')^2 g(V, \nabla_{\dot{\gamma}}(\dot{\gamma})) + \frac{x}{v'} \dot{\alpha}_0(v') g(V, \dot{\alpha}_0)\right) \\ &= \sqrt{2} x'(t) - \sqrt{2} \frac{x(t) v''(t)}{v'(t)} \\ &= \sqrt{2} v'(t) \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{x(t)}{v'(t)}\right),\end{aligned}$$

where we have used that $\dot{\alpha}_0 = v'\dot{\gamma}$, $g(V, \dot{\alpha}_0) = -1/\sqrt{2}$ and $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

Step VI: An upper bound for $\frac{\partial^2 y}{\partial u^2}$

We now compute an upper bound for the second partial derivative of y with respect to u. For convenient notation, we use the vector fields

$$T = \alpha^* \left(\frac{\partial}{\partial t} \right),$$
$$U = \alpha^* \left(\frac{\partial}{\partial u} \right).$$

Note that

$$T_{\alpha(u,t)} = \dot{\alpha}_u(t)$$

and

$$U_{\alpha(u,t)} = -x(t) V_{\alpha(u,t)}.$$

Now

$$\begin{aligned}\frac{1}{2}\frac{\partial^2}{\partial u^2} y(u,t) &= \frac{1}{2}\frac{\partial^2}{\partial u^2} g\left(\dot{\alpha}_u(t), \dot{\alpha}_u(t)\right) = \frac{1}{2}\frac{\partial^2}{\partial u^2} g\left(T,T\right) = \frac{\partial}{\partial u} g(\nabla_U T, T)\\ &= g(\nabla_U T, \nabla_U T) + g(\nabla_U \nabla_U T, T) = g(\nabla_T U, \nabla_T U) + g(\nabla_U \nabla_T U, T)\\ &= g(\nabla_T U, \nabla_T U) + g(\nabla_T \nabla_U U, T) + g(R(U,T)U, T)\end{aligned}$$

where we have used that $\nabla_U T = \nabla_T U$ since U and T are coordinate vector fields and

$$\nabla_U \nabla_T = \nabla_T \nabla_U + R(U,T).$$

We now compute each term separately.

Evaluating the first term at $\alpha(0,t)$ and using that $T(x) = \alpha_0(x) = x'$ we get

$$\begin{aligned}g(\nabla_T U, \nabla_T U) &= g(\nabla_T(xV), \nabla_T(xV)) = g(T(x)V + x\nabla_T V, T(x)V + x\nabla_T V)\\ &= \left(x'(t)\right)^2 g(V,V) + 2x(t)x'(t) g(V, \nabla_T V) + x^2(t) g(\nabla_T V, \nabla_T V)\\ &= -\left(x'(t)\right)^2 + (x(t))^2 g(\nabla_T V, \nabla_T V).\end{aligned}$$

We have used that $g(V, \nabla_T V) = 0$. That this is true is seen by noting that

$$g(V, \nabla_T V) = T g(V,V) - g(\nabla_T V, V) = T(-2^{-1/2}) - g(V, \nabla_T V) = -g(V, \nabla_T V)$$

so that $g(V, \nabla_T V) = -g(V, \nabla_T V)$. For the second term, note that

$$\nabla_U U = \nabla_U(-xV) = xU(x)V + x^2 \nabla_V V = x(t)\frac{\partial x}{\partial u} V + 0 = 0$$

(since $\nabla_V V = 0$ on the image of α by choice of V, and x is independent of u) so that

$$g(\nabla_T \nabla_U U, T) = g(\nabla_T 0, T) = 0.$$

The third term is simply

$$g(R(U,T)U,T) = g(R(-xV,T)(-xV),T) = x^2(t) g(R(V,T)V,T).$$

Hence

$$\frac{1}{2}\frac{\partial^2}{\partial u^2} g\left(\dot\alpha_u(t), \dot\alpha_u(t)\right) \begin{aligned}[t] &= -\left(x'(t)\right)^2 + (x(t))^2 \left(g(\nabla_T V, \nabla_T V) + g(R(V,T)V,T)\right) \\ &\le x^2 \left(g(\nabla_T V, \nabla_T V) + g(R(V,T)V,T)\right). \end{aligned}$$

We wish to bound this by $C^2x^2g'(T,T)$ for some constant C on the neighborhood $\mathcal{W}$ of $H^+(S)$, which we chose to have compact closure. (Recall that g' is the Riemannian metric constructed from the vector field V in the beginning of the proof.) To see that this is possible, view $g(\nabla_T V, \nabla_T V) + g(R(V,T)V,T)$ as a quadratic form in T. Its components in coordinates depend on g, V and R, all of which are bounded in coordinate neighborhoods, and $H^+(S) \cap K$ can be covered by finitely many such neighborhoods. Since the quadratic form g' is positive definite, there is some C such that $g(\nabla_T V, \nabla_T V) + g(R(V,T)V,T) \le Cg'(T,T)$. Hence

$$\frac{\partial^2}{\partial u^2} g\left(\dot\alpha_u(t), \dot\alpha_u(t)\right) \le C^2x^2g'(T,T)$$

for some constant C. We want a bound in terms of $g\left(\dot\alpha_u(t), \dot\alpha_u(t)\right)$ instead, so we compute

$$g'(T,T) = g(T,T) + 2\left(g(V,T)\right)^2.$$

Since

$$\begin{aligned} \frac{\partial}{\partial u} g(V,T) &= Ug(V,T) = g(-x\nabla_V V, T) + g(V, \nabla_U T) = 0 + g(V, \nabla_T U) \\ &= -g(V, T(x)V - x\nabla_T V) = T(x) + xg(V, \nabla_T V) = x'(t) \end{aligned}$$

(where as earlier $g(V, \nabla_T V) = 0$) we know that

$$g(V,T) = ux'(t) + g(V,T)|_{u=0} = ux'(t) - \frac{1}{\sqrt{2}}.$$

When we choose x, we will make sure that $\frac{\mathrm{d}x}{\mathrm{d}t}$ is bounded, and then $2\left(g(V,T)\right)^2$ is bounded by some constant d for all small u. Hence, we can convert our bound in terms of $g'(T,T)$ to a bound in terms of $g(T,T)$:

$$\frac{\partial^2}{\partial u^2} g\left(\dot\alpha_u(t), \dot\alpha_u(t)\right) \le C^2x^2g'(T,T) \le C^2x^2(g(T,T) + d).$$

In the notation of the function y, we now know that

$$\frac{\partial^2 y}{\partial u^2}(u,t) \le (y(u,t) + d)C^2(x(t))^2$$

for all sufficiently small $u > 0$.

Step VII: For all sufficiently small $\epsilon > 0$, the curve α_ϵ is timelike

From our previous computations we know that

$$\begin{aligned} \frac{\partial y}{\partial u}(0,t) &= \frac{v'(t)}{\sqrt{2}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x(t)}{v'(t)}\right), \\ \frac{\partial^2 y}{\partial u^2}(u,t) &\le (y(u,t) + d)C^2(x(t))^2. \end{aligned}$$

Moreover, $y(0,t) = 0$ since α_0 is a lightlike curve. For each fixed t, this is a differential inequality in the variable u. Let z be the solution of the differential equation resulting from replacing the inequality with equality:

$$\begin{aligned}\frac{\partial^2 z}{\partial u^2}(u,t) &= C^2 x^2(0,t)(z(u,t)+d),\\ \frac{\partial z}{\partial u}(0,t) &= \frac{\partial y}{\partial u}(0,t),\\ z(0,t) &= y(0,t) = 0.\end{aligned}$$

Integrating the inequality $\frac{\partial^2 y}{\partial u^2}(u,t) \le \frac{\partial^2 z}{\partial u^2}(u,t)$ we see that

$$\frac{\partial y}{\partial u}(u,t) - \frac{\partial y}{\partial u}(0,t) \le \frac{\partial z}{\partial u}(u,t) - \frac{\partial z}{\partial u}(0,t)$$

so that

$$\frac{\partial y}{\partial u}(u,t) \le \frac{\partial z}{\partial u}(u,t).$$

Integrating once again and using the fact that $z(0,t) = y(0,t) = 0$ we have

$$y(u,t) \le z(u,t).$$

Solving the differential equation for z we see that

$$z(u,t) = \mathrm{d}\cos h(Cx(t)u) + a(t)\sin h(Cx(t)u) - d$$

where

$$a(t) = \frac{v'(t)}{\sqrt{2}Cx(t)}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x(t)}{v'(t)}\right).$$

Since d is nonnegative, an upper bound for z is

$$\begin{aligned}z(u,t) &= \mathrm{d}\cos h(Cx(t)u) + a(t)\sin h(Cx(t)u) - d\\ &= (\mathrm{d}\tan h(Cx(t)u) + a(t))\sin h(Cx(t)u) - d\\ &\le (\mathrm{d}\tan h(Cx(t)u) + a(t))\sin h(Cx(t)u).\end{aligned}$$

Hence

$$y(u,t) \le (\mathrm{d}\tan h(Cx(t)u) + a(t))\sin h(Cx(t)u).$$

Recall that the idea was to choose the function x in such a way that there exists some $\epsilon > 0$ such that $y(\epsilon,t) < 0$ for all t. We claim that an example of such a function x is

$$x(t) = \frac{v'(t)}{v(t) - 2v_0}.$$

Recall that

$$v_0 = \lim_{t\to-\infty} v(t)$$

and that v is increasing so that

$$v_0 \le v(t) \le 0 \quad \forall t \in (-\infty, 0].$$

We begin by making good on the promises we made about the function x: It should be positive, bounded, and have bounded derivative. The denominator in the definition of x is bounded from below by $-v_0$ and from above by $-2v_0$, and $-v_0$ is positive, so boundedness and positivity of x follow from boundedness and positivity of v'. Computing the derivative of x we see that

$$x'(t) = \frac{v''(t)}{v(t) - 2v_0} - \frac{(v'(t))^2}{(v(t) - 2v_0)^2} = \frac{v'(t)f(t)}{v(t) - 2v_0} - x^2(t).$$

Since x, v' and $f = v''/v'$ are bounded, so is x'. Having chosen x, we can now fix the number $\delta > 0$ defining the domain of α such that the image of α is contained in $\mathcal{W}$.

Recall that

$$y(u,t) \leq (\mathrm{d} \tan h(Cx(t)u) + a(t)) \sin h(Cx(t)u)$$

where

$$a(t) = \frac{v'(t)}{\sqrt{2}Cx(t)} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{x(t)}{v'(t)} \right).$$

With our present choice of x,

$$a(t) = -\frac{x(t)}{\sqrt{2}Cv'(t)}.$$

The objective is to ensure that $y(u,t) < 0$ for some positive u and for all t. Since $\sin h(Cxu) \geq 0$ for positive u, a sufficient condition is that

$$\mathrm{d} \tan h(Cx(t)u) - \frac{x(t)}{\sqrt{2}Cv'(t)} < 0$$

for some $u > 0$ and all t. A series expansion tells us that

$$\tan h(Cx(t)u) = Cx(t)u + \mathcal{O}((ux(t))^3)$$

for small $ux(t)$, so that

$$\mathrm{d} \tan h(Cx(t)u) + a(t) = \left(\mathrm{d}Cu - \frac{1}{\sqrt{2}Cv'(t)} \right) x(t) + \mathcal{O}(u^3 x^3(t)).$$

Since v' is bounded, there is some positive lower bound for $1/v'$. Hence it holds for all sufficiently small u such that $\mathrm{d}Cu - 1/(\sqrt{2}Cv'(t))$ is negative for all t. Since x is bounded, it further holds for all sufficiently small u that the $\mathcal{O}(u^3x^3(t))$ term does not affect the sign: with such a choice of u, it holds that $\mathrm{d} \tan h(Cxu) + a$ is negative for all t, and hence

$$y(\epsilon,t) \leq (\mathrm{d} \tan h(Cx(t)\epsilon) + a(t)) \sin h(Cx(t)\epsilon) < 0$$

for all values of t and all sufficiently small $\epsilon > 0$. Since

$$y(\epsilon,t) = g(\dot{\alpha}_\epsilon(t), \dot{\alpha}_\epsilon(t))$$

this shows that the curve α_ϵ is timelike.

Step VIII: For all sufficiently small $\epsilon > 0$, the curve α_ϵ has infinite g'-length
For each (negative) integer k, let $L_k(u)$ be the g'-length of the restriction of α_u to $[k, k+1]$. By the formula for the first variation of arc length ([28, Proposition 2, Chapter 10])

$$\begin{aligned} L_k'(0) &= -\int_k^{k+1} g'(\nabla_{\dot\alpha_0}\dot\alpha_0, V)\,\mathrm{d}t + g'(\dot\alpha_0, V)\big|_k^{k+1} \\ &= -\int_k^{k+1} f(t) g'(\dot\alpha_0, V)\,\mathrm{d}t + g'(\dot\alpha_0, V)\big|_k^{k+1} \\ &= -\int_k^{k+1} \frac{f(t)}{\sqrt{2}}\,\mathrm{d}t. \end{aligned}$$

We here used that $g'(\alpha_0, V) = 1/\sqrt{2}$ by definition of g' and α_0. Since f is bounded, we know that $L_k'(0)$ is bounded uniformly in k. This means that for all sufficiently small $\epsilon > 0$ it holds that $L_k(\epsilon) > 1/2$ for all k. (Recall that $L_k(0) = 1$ since α_0 is parameterized by arc length.) This means that the length of α_ϵ is

$$\sum_{k<0} L_k(\epsilon) \geq \sum_{k<0} 1/2 = \infty.$$

Step IX: For all sufficiently small $\epsilon > 0$, the curve α_ϵ belongs to the interior of $D^+(S_0)$
Since the curve α_ϵ for $\epsilon > 0$ is a variation to the past of α_0, it belongs to the open set $I^-(H^+(S_0))$. We will first show that $I^-(H^+(S_0)) \cap I^+(S_0) \subseteq D^+(S_0)$, and then show that α_ϵ belongs to $I^-(H^+(S_0)) \cap I^+(S_0)$ for all sufficiently small $\epsilon > 0$. We will then have shown that α_ϵ belongs to an open set contained in $D^+(S_0)$, and hence it must belong to the interior of $D^+(S_0)$.

Let $p \in I^-(H^+(S_0)) \cap I^+(S_0)$. We will first show that $p \in \overline{D^+(S_0)}$, and then that $p \in D^+(S_0)$. That $p \in I^-(H^+(S_0))$ means that there is some future-directed timelike curve λ from p to $H^+(S_0)$. This curve cannot pass S_0, since p lies to the future of S_0 and S_0 is achronal. Suppose now that κ is a future-directed past inextendible timelike curve with future endpoint p. By concatenating κ and λ and smoothing (in a neighborhood of p which is disjoint from S_0, which exists since $p \in I^+(S_0)$) we obtain a past-inextendible timelike curve with future endpoint in $H^+(S_0)$. Since $H^+(S_0) \subseteq \overline{D^+(S_0)}$, this combined curve must intersect S_0. Since the curve λ does not intersect S_0, the curve κ must do so. This proves that every past-inextendible timelike curve κ through p must intersect S_0, so that $p \in \overline{D^+(S_0)}$. By the same argument, all points in the interior of λ belong to $\overline{D^+(S_0)}$. Let q be some point in the interior of λ. Since λ is timelike, $q \in I^+(p)$. Since $I^+(p)$ is open, it is a neighborhood of q. Since $q \in \overline{D^+(S_0)}$ it is a limit point of $D^+(S_0)$. This means that the neighborhood $I^+(p)$ of q must contain some point $r \in D^+(S_0) \cap I^+(p)$. Let $\widehat{\lambda}$ be a future-directed timelike curve from p to r. Now let κ be a future-directed past inextendible causal curve with future endpoint p. Concatenating κ with $\widehat{\lambda}$ and smoothing (again in a neighborhood of p which is disjoint from

S_0) we obtain a past-inextendible causal curve with future endpoint r. Since $r \in D^+(S_0)$, this curve must intersect S_0. Since $r \in I^+(p) \subseteq I^+(S_0)$ and S_0 is achronal, the curve $\widehat{\lambda}$ cannot intersect S_0. This means that κ must intersect S_0. This proves that every past-inextendible causal curve κ through p must intersect S_0, so that $p \in D^+(S_0)$.

Since α_0 is a curve in $H^+(S_0)$ and α_ϵ is a variation to the past for $\epsilon > 0$ we know that α_ϵ belongs to $I^-(H^+(S_0))$. Since $g'(V, V) = 1$ and x is bounded by $|2v_0|$, Eq. (4) implies that the distance from a point on α_ϵ to α_0 cannot exceed $|2v_0\epsilon|$. Since $H^+(S_0) \cap K$ is compact and disjoint from the closed set S_0, the g'-distance from $H^+(S_0)$ to S_0 is positive. Choosing $\epsilon > 0$ so small that $|2v_0\epsilon|$ is smaller than this distance, we know that α_ϵ does not intersect S_0. To see that $\alpha_\epsilon(t) \in I^+(S_0)$ for some t, note that no curve α_u with $0 \le u \le \epsilon$ can intersect S_0 so that the timelike curve $\lambda\colon [-\epsilon, 0] \to M$ defined by $\lambda(u) = \alpha_{-u}(t)$ does not intersect S_0. Extend λ to some past inextendible timelike curve. Then λ is a past inextendible timelike curve with future endpoint $\lambda(0) = \alpha_0(t) \in H^+(S_0)$, so λ must intersect S_0. Since λ passes through $\alpha_\epsilon(t)$, we know that $\alpha_\epsilon(t) \in I^+(S_0)$. Since t was arbitrary, we have now shown that the image of α_ϵ belongs to $I^-(H^+(S_0)) \cap I^+(S_0)$ for all sufficiently small $\epsilon > 0$. As noted previously, this together with the fact that $I^-(H^+(S_0)) \cap I^+(S_0) \subseteq D^+(S_0)$ shows that the image of α_ϵ belongs to the interior of $D^+(S_0)$.

Step X: Contradiction ensues
We have now shown that if we choose $\epsilon > 0$ small enough, then α_ϵ is a timelike curve of infinite g'-length, contained in the interior of $D^+(S_0)$. Since it has infinite g'-length and belongs to the compact set $\overline{\mathcal{W}}$, it must have a limit point. Since it is timelike, the existence of this limit point implies that the strong causality condition cannot hold in any open neighborhood of α_ϵ. However, the interior of $D^+(S_0)$ is an open neighborhood of α_ϵ satisfying the strong causality condition (by [6, Proposition 2.9.9]), so we have arrived at a contradiction. Hence γ cannot be incomplete in the past direction. □

1.3. Structure of Horizons

We begin by defining the abstract concept of a "horizon" (following [7]) and state some previously known results about the regularity of horizons. We then prove that the Cauchy horizons we will work with are horizons in this sense.

1.3.1. Abstract Horizons.

Definition 1.7. We say that an embedded topological hypersurface in a spacetime is *past null geodesically ruled* if every point on the hypersurface belongs to a past inextendible null geodesic contained in the hypersurface. These geodesics are called *generators*.

Remark 1.8. Note that if a past null geodesically ruled hypersurface is a C^2 null hypersurface, then these generators are the same as those defined in Sect. 1.1.1.

Definition 1.9. A *horizon* in a spacetime is an embedded, achronal, past null geodesically ruled, closed (as a set) topological hypersurface.

Remark 1.10. One may just as well define a horizon to be future null geodesically ruled. Indeed, in [7] the distinction is made between a "past horizon" and a "future horizon". However, since we will work only with future Cauchy horizons it is convenient to restrict our attention to past null geodesically ruled horizons.

Remark 1.11. If an open subset of a horizon is past null geodesically ruled, then its generators are the restrictions of the generators of the horizon.

Note that we have assumed no smoothness in the definition. Note also that the generators through a point of a horizon need not be unique. In fact, we have the following theorem (see Theorem 3.5 in [2] and Proposition 3.4 in [9]).

Theorem 1.12. *A horizon is differentiable precisely at those points which belong to a single generator.*

We also note that horizons are null hypersurfaces whenever they are differentiable, so that the generators of a C^2 horizon are precisely the integral curves of the null vector fields:

Proposition 1.13. *If a horizon* $\mathcal{H}$ *is differentiable at a point* p*, then* $T_p\mathcal{H}$ *is a null hyperplane.*

Proof. Since p belongs to a lightlike geodesic segment contained in $\mathcal{H}$, we know that $T_p\mathcal{H}$ contains null vectors. If $T_p\mathcal{H}$ were to contain a timelike vector, then there would be a timelike curve in $\mathcal{H}$ with this tangent vector. This would contradict achronality of $\mathcal{H}$, and hence $T_p\mathcal{H}$ must be a null hyperplane. □

Finally, we note that generators can only intersect in common endpoints.

Proposition 1.14. *Let* $\mathcal{H}$ *be a horizon, and suppose that* p *is an interior point of a generator* Γ*. Then there is no other generator containing* p*.*

Proof. Suppose that some other generator Γ' contained p. Let q be a point to the past of p along Γ', and let r be a point to the future of p along Γ. By following Γ' from q to p and then Γ from p to r we have connected q and r by a causal curve which not a null geodesic. By [6, Proposition 2.6.9] this curve cannot be achronal. Since the image of the curve belongs to $\mathcal{H}$, this contradicts achronality of $\mathcal{H}$. □

1.3.2. Cauchy Horizons. We now connect the statements in Sect. 1.3.1 about abstract horizons to the particular case of a Cauchy horizon in a spacetime. We begin by quoting [14, Proposition 2.7]. A similar statement can be found in [6, Proposition 2.10.6].

Proposition 1.15. *Let* S *be an achronal subset of a spacetime* M*. Then the set* $H^+(S)\setminus\mathrm{edge}(S)$*, if nonempty, is an achronal* C^0 *hypersurface of* M *ruled by null geodesics, each of which either is past inextendible in* M *or has a past endpoint on* edge(S)*.*

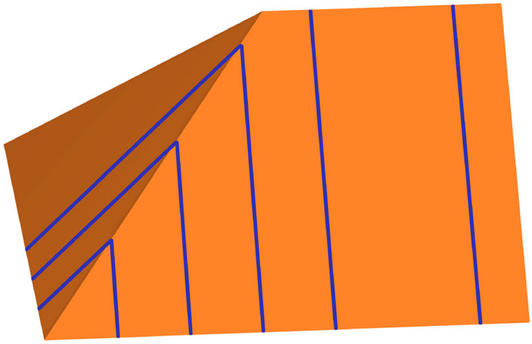

Figure 1. Part of the future Cauchy horizon of a spacelike rectangle in (2+1)-dimensional Minkowski space, with some of the generators shown. A more complicated example in which no open subset of the horizon is differentiable is given in [9]

Corollary 1.16. *Let M be a spacetime. Suppose that $S \subseteq M$ is an achronal set with* $\operatorname{edge}(S) = \emptyset$. *Then $H^+(S)$ is a horizon in the sense of Definition 1.9.*

Proof. The proposition tells us that $H^+(S)$ is a topological hypersurface which is achronal and past null geodesically ruled. To see that $H^+(S)$ is closed, note that it by definition is the difference of a closed set and an open set. This completes the proof. □

We conclude with a lemma allowing us to apply Corollary 1.16 to closed spacelike hypersurfaces. The lemma follows from [18, Lemma 8.3.3].

Lemma 1.17. *Let M be a spacetime and let S be a spacelike hypersurface which is closed as a set. Then* $\operatorname{edge}(S) = \emptyset$.

1.3.3. Properties of Nonsmooth Horizons. In general, Cauchy horizons are not C^2 hypersurfaces: Fig. 1 shows an example of a non-C^2 Cauchy horizon. This particular example is "almost C^2" in the sense that it has a dense open subset which is C^2, so we would expect that many results about C^2 hypersurfaces are applicable to this example. However, it was shown in [9] that Cauchy horizons are not necessarily almost C^2. This means that the proofs of theorems like Tipler's theorem need to deal with horizons of lower regularity. For this reason, and in particular to prove Theorem 1.42 about smoothness of compact Cauchy horizons, we need some results about general horizons. The definitions and results in this section can be found in [7].

Definition 1.18. Let M be a spacetime, and let $(a, b) \times \Sigma \cong \mathcal{O} \subseteq M$ be an open subset such that each slice $\{t\} \times \Sigma$ is spacelike and each curve $(a, b) \times \{p\}$ is timelike. Let $N \subseteq M$ be a hypersurface. A function $f : \Sigma \to (a, b)$ is said to be a *graphing function* of N if $N \cap \mathcal{O} = \{(f(x), x) \mid x \in \Sigma\}$.

Theorem 2.2 of [7] says that any locally achronal hypersurface, in particular any horizon, is semi-convex (see Definition A.2). This implies (see [7, Proposition 2.1]) that every point on the horizon has a globally hyperbolic spacetime neighborhood $(-a, a) \times \Sigma$ in which the horizon has a graphing function f for which there is a subset $\Sigma_{\mathcal{A}l} \subseteq \Sigma$ such that

- $\Sigma \backslash \Sigma_{\mathcal{A}l}$ has measure zero,
- f is differentiable at all points of $\Sigma_{\mathcal{A}l}$,
- f is *twice-Alexandrov-differentiable* at all points $x \in \Sigma_{\mathcal{A}l}$. In other words, there is a quadratic form $D^2 f(x)$ such that for all $y \in \Sigma$

$$f(y) - f(x) - \mathrm{d}f(x)(y - x) = \frac{1}{2} D^2 f(x)(x - y, x - y) + o(|x - y|^2).$$

Moreover, it is shown in [7] that this notion is coordinate invariant: if $p = (f(x), x)$ with $x \in \Sigma_{\mathcal{A}l}$ for one globally hyperbolic neighborhood of p, then $p = (\tilde{f}(\tilde{x}), \tilde{x})$ with $\tilde{x} \in \tilde{\Sigma}_{\mathcal{A}l}$ for any other neighborhood $(-\tilde{a}, \tilde{a}) \times \tilde{\Sigma}$ of p with corresponding graphing function $\tilde{f}$ satisfying the above conditions. Hence the following definition makes sense.

Definition 1.19. Let $\mathcal{H}$ be a horizon in a spacetime. Denote by $\mathcal{H}_{\mathcal{A}l}$ the set of all points $p \in \mathcal{H}$ which are images under a graphing function of one of the corresponding sets $\Sigma_{\mathcal{A}l}$. We will call $\mathcal{H}_{\mathcal{A}l}$ the set of *Alexandrov points* of the horizon.

Remark 1.20. By the definition of semi-convexity a semi-convex function is the sum of a C^2 function and a convex function, and hence locally Lipschitz. This means that horizons are Lipschitz hypersurfaces.

Following [7] we will now define the null mean curvature $\theta_{\mathcal{A}l}$ and the null second fundamental form $B_{\mathcal{A}l}$ of $\mathcal{H}_{\mathcal{A}l}$. More precisely, we will define $\theta_{\mathcal{A}l}$ and $B_{\mathcal{A}l}$ on the intersection of $\mathcal{H}_{\mathcal{A}l}$ with a globally hyperbolic coordinate neighborhood $\mathcal{O}$. This definition is *not* coordinate invariant. However, $\theta_{\mathcal{A}l}$ and $B_{\mathcal{A}l}$ are defined up to pointwise scaling by a positive function, so the sign of $\theta_{\mathcal{A}l}$ is globally well defined.

Definition 1.21. Let $\mathcal{H}$ be a horizon in a spacetime (M, g). Choose a globally hyperbolic coordinate neighborhood $\mathcal{O} = (-a, a) \times \Sigma$ of some point in $\mathcal{H}$, and let $f\colon \Sigma \to (-a, a)$ be the graphing function of $\mathcal{H}$ in this neighborhood. Let the function $t\colon (-a, a) \times \Sigma \to (-a, a)$ be the projection. For each point $p \in \mathcal{O} \cap \mathcal{H}_{\mathcal{A}l}$, with x such that $p = (f(x), x)$, define $k(p) = -\mathrm{d}t + \mathrm{d}f(x)$. This makes sense since f is differentiable at all such points. Let K be the vector field dual to k with respect to g. Let $e_0 \in T_p\mathcal{O}$ be the vector which is g-dual to $\mathrm{d}t$. Choose a basis $e_1, \ldots, e_n$ for $T_p\mathcal{H}$ such that

- $e_n = K_p$,
- $g(e_i, e_i) = 1$ if $1 \le i \le n - 1$,
- $g(e_i, e_j) = 0$ if $i \ne j$,
- $g(e_i, e_0) = 0$ if $1 \le i \le n - 1$.

We now define $\theta_{\mathcal{A}l}$ and $B_{\mathcal{A}l}$ using the coordinate formulae

$$\theta_{\mathcal{A}l} = \sum_{l=1}^{n-1} e_l^i e_l^j \left(D^2_{ij} f - \Gamma^\mu_{ij} k_\mu \right),$$
$$B_{\mathcal{A}l}(X^a e_a, Y^b e_b) = X^a Y^b e_a^i e_b^j \left(D^2_{ij} f - \Gamma^\mu_{ij} k_\mu \right).$$

In the definitions above, we have constructed an "artificial" covariant derivative $D^2_{ij} f - \Gamma^\mu_{ij} k_\mu$ using the Alexandrov second derivative $D^2 f$ of f. We emphasize again that this definition of $\theta_{\mathcal{A}l}$ is not independent of the coordinate system. However, the definitions using different coordinate systems differ only by a positive multiplicative constant. In particular, the sign of $\theta_{\mathcal{A}l}$ is invariantly defined (see [7, Proposition 2.5]).

Remark 1.22. If $\mathcal{H}$ is C^2, then $\theta_{\mathcal{A}l}$ and $B_{\mathcal{A}l}$ agree with the null mean curvature θ_K and null second fundamental form B_K defined in Sect. 1.1.1.

Remark 1.23. By [7, Theorem 5.1] the $(1,1)$-tensor $b_{\mathcal{A}l}$ associated to $B_{\mathcal{A}l}$ satisfies Eq. (1) from Sect. 1.1.1.

We will later derive a formula involving $\theta_{\mathcal{A}l}$ for the area of a horizon, and knowing the sign of $\theta_{\mathcal{A}l}$ will yield inequalities between different areas. In our case, the generators of the horizon will be past complete, so we will have use of the following result. It is a generalization of [7, Proposition 4.17], and the proof of that proposition is sufficient for proving the generalization as well, since the proof considers one point at a time.

Proposition 1.24. *Let M be a spacetime, and let $\mathcal{H}$ be a horizon in M. Let $A \subset \mathcal{H}_{\mathcal{A}l}$ be a set of Alexandrov points of $\mathcal{H}$ such that each point in A belongs to a generator which is complete to the past. Suppose that the null energy condition holds. Then*

$$\theta_{\mathcal{A}l} \leq 0 \quad \text{on } A.$$

Remark 1.25. The result in [7] is expressed with the opposite time orientation compared to our setting. Consequently, we obtain the inequality $\theta_{\mathcal{A}l} \leq 0$ instead of $\theta_{\mathcal{A}l} \geq 0$.

1.4. A Smoothness Theorem

In [4, Section 4] the question was posed whether a compact Cauchy horizon is necessarily smooth. A negative answer was given by the same authors in [5, Section 4], where it was also mentioned that compactness together with some energy condition might be sufficient to guarantee smoothness. In this section, we show using methods from [7] that this is indeed the case. When our proofs parallel those in [7], we will adhere to the notation in [7].

1.4.1. Outline of the Proof. The theorem which will be proved in this section is Theorem 1.42, stating that compact Cauchy horizons in a spacetime which satisfies the null energy condition are smooth. We first give an outline of the proof. Horizons are Lipschitz null hypersurfaces, and so differentiable almost everywhere. At the points of differentiability there is a unique (up to scaling) null tangent vector, giving rise to an almost everywhere defined vector field on the horizon. By restricting to a suitably chosen subset of the horizon, we can define a flow along this vector field. One may then construct a $C^{1,1}$ manifold containing, locally, this chosen subset, and extend the flow to a Lipschitz flow on the $C^{1,1}$ manifold. This is sufficient regularity to express how the area of a set changes under the flow, and this change of area is the central idea of the proof. To measure area, we introduce a Riemannian metric σ on the spacetime. With a suitably chosen such metric, the change in area is related to the Alexandrov null mean curvature $\theta_{\mathcal{A}l}$ of the Cauchy horizon. The argument for this relation between area change and $\theta_{\mathcal{A}l}$ proceeds via a C^2 approximation of a part of the local $C^{1,1}$ approximation of the original horizon. Once the relation between $\theta_{\mathcal{A}l}$ and area change has been established, knowledge of the sign of $\theta_{\mathcal{A}l}$ gives an inequality for area change under the flow. A sufficient condition under which the sign of $\theta_{\mathcal{A}l}$ may be determined is that all null geodesics in the horizon are complete in the past direction, together with an energy condition. Lemma 1.6 tells us that the generators are complete. By these arguments, we determine that the flow increases area. However, the flow maps a subset of the horizon into itself, thereby decreasing area. Hence the only possibility is that the flow conserves area. We show in Proposition 1.40 that this implies that the horizon is smooth.

1.4.2. Flow Sets and Generator Flow. We wish to generalize the notion of the generator flow on C^2 null hypersurfaces discussed in Sect. 1.1.2 to possibly nonsmooth horizons. In other words, we want a flow along generators of a horizon $\mathcal{H}$. However, since some points belong to several generators it is in general not possible to do this on all of $\mathcal{H}$. Instead, we construct a smaller subset on which to define the flow.

Definition 1.26. Let $\mathcal{H}$ be a horizon in a spacetime M. Define the *total flow set* of $\mathcal{H}$ to be the set $A_0(\mathcal{H})$ of points $p \in \mathcal{H}$ such that the following conditions are satisfied:

- There is a unique generator Γ of $\mathcal{H}$ passing through p.
- The point p belongs to the interior of Γ.
- Each interior point of Γ is an Alexandrov point.

Let σ be a Riemannian metric on M. For $\delta > 0$ define the *δ-flow set* of $\mathcal{H}$ with respect to σ to be the set

$$A_\delta(\mathcal{H}, \sigma) = \{p \in A_0(\mathcal{H}) \mid \text{The generator through } p \text{ exists for a } \sigma\text{-distance greater than } \delta \text{ to the past and to the future}\}.$$

Remark 1.27. Note that the total flow set is the union of all δ-flow sets:

$$A_0(\mathcal{H}) = \bigcup_{\delta > 0} A_\delta(\mathcal{H}, \sigma).$$

Remark 1.28. When the context allows it, we will sometimes drop $\mathcal{H}$ and σ from the notation and write A_0 or A_δ.

For the next proposition, we will need the following result, a proof of which can be found in [7, Theorem 5.6].

Lemma 1.29. *Let $\mathcal{H}$ be a horizon in a spacetime M of dimension $n+1$. Suppose that $\mathcal{S}$ is a C^2 hypersurface intersecting $\mathcal{H}$ properly transversally (in the sense that if $q \in \mathcal{S} \cap \mathcal{H}$ and the tangent space $T_q\mathcal{H}$ exists then $T_q\mathcal{S}$ is transverse to $T_q\mathcal{H}$). Define*

$$S_0 = \{q \in \mathcal{S} \cap \mathcal{H} \mid q \text{ is an interior point of a generator of } \mathcal{H}\},$$

$$S_1 = \{q \in S_0 \mid \text{all interior points of the generator through } q \text{ are Alexandrov points of } \mathcal{H}\}.$$

Then S_1 has full $(n-1)$-dimensional Hausdorff measure in S_0.

Proposition 1.30. *Let $\mathcal{H}$ be a horizon in a spacetime (M, g) of dimension $n+1$, and let σ be a Riemannian metric on M. Let $\mathfrak{h}^n$ be the n-dimensional Hausdorff measure induced by the distance function induced by σ. Then the total flow set A_0 of $\mathcal{H}$ has full $\mathfrak{h}^n$-measure in the sense that*

$$\mathfrak{h}^n(\mathcal{H}\backslash A_0) = 0.$$

Proof. To show that $\mathcal{H}\backslash A_0$ has measure zero, it is sufficient to show that each point $p \in \mathcal{H}$ has an open neighborhood $U \subseteq M$ such that $\mathfrak{h}^n\left(U \cap (\mathcal{H}\backslash A_0)\right) = 0$, for $\mathcal{H}$ can be covered by countably many such neighborhoods since it is second-countable. The idea of the proof is to construct a spacetime of one dimension greater than M and apply Lemma 1.29 in this higher-dimensional spacetime.

To this end, choose a globally hyperbolic neighborhood $U \subseteq M$ of $p \in \mathcal{H}$ diffeomorphic to $(-a, a) \times \Sigma$, where $\Sigma \subseteq \mathbb{R}^n$ and each slice $\{t\} \times \Sigma$ is spacelike. We may choose the zero slice to be such that $p \in \{0\} \times \Sigma$. Let $\widehat{M} = M \times I$ denote the product manifold which is equipped with the metric $\widehat{g} = g + \mathrm{d}s^2$, where s refers to the coordinate in the open interval I. Let $\widehat{\mathcal{H}} = \mathcal{H} \times I$. Let $\pi\colon \widehat{M} \to M$ denote the projection. We now verify that $\widehat{\mathcal{H}}$ is a horizon in $\widehat{M}$. Since $\mathcal{H}$ is an embedded topological hypersurface, so is $\widehat{\mathcal{H}}$. By definition of the product topology, $\widehat{\mathcal{H}} = \pi^{-1}(\mathcal{H})$ is closed. If there were some timelike curve γ between two points of $\widehat{\mathcal{H}}$, then $\pi \circ \gamma$ would be a timelike curve between two points of $\mathcal{H}$ contradicting achronality of $\mathcal{H}$, so $\widehat{\mathcal{H}}$ must also be achronal. To see that $\pi \circ \gamma$ is indeed timelike, note that by definition of $\widehat{g}$ it holds that $g(\pi_* V) \leq \widehat{g}(V)$ for all vectors V. Finally $\widehat{\mathcal{H}}$ is past null geodesically ruled since if Γ is a past inextendible null M-geodesic contained in $\mathcal{H}$, then $\Gamma \times \{s\}$ is a past inextendible null $\widehat{M}$-geodesic contained in $\widehat{\mathcal{H}}$ for each $s \in I$.

We now wish to construct, after possibly decreasing a or shrinking I, a diffeomorphism $\rho\colon I \to (-a, a)$ such that the hypersurface

$$\mathcal{S} := \{(t, q, s) \in (-a, a) \times \Sigma \times I \mid t = \rho(s)\}$$

is spacelike. A possible choice of basis for the tangent space $T_{(t,q,s)}\mathcal{S} \subseteq T_q\mathbb{R} \times \Sigma \times \mathbb{R}$ of $\mathcal{S}$ at some point (t, q, s) consists of a basis for the tangent space of Σ together with the vector $(\rho'(s), 0, 1)$. The basis of $T_q\Sigma$ consists of spacelike vectors since Σ is spacelike, and if $\rho'(s)$ is sufficiently close to zero then $(\rho'(s), 0, 1)$ is also spacelike (since $(0, 0, 1)$ is spacelike by definition of $\widehat{g}$, and the set of spacelike vectors at a point is open). This means that for each $(t, q) \in (-a, a) \times \Sigma$, there is some $c(t, q) > 0$ such that if $\zeta < c(t, q)$ then for any $s \in I$ the vector $(0, \zeta, 1) \in T_{(t,q,s)}$ is spacelike. Since g is smooth, c can be chosen smooth. Hence c takes some minimum on every compact subset of $(-a, a) \times \Sigma$. This minimum is positive since c is positive on $(-a, a) \times \Sigma$, after possibly shrinking Σ and a. We may then find some real number ζ such that $0 < \zeta < c(t, q)$ for all $(t, q) \in (-a, a) \times \Sigma$. Letting $\rho(s) = \zeta(s - s_0)$ where s_0 is the midpoint of I, and subsequently shrinking I or a to make ρ bijective, we have found a diffeomorphism ρ making $\mathcal{S}$ spacelike. Since ρ is a diffeomorphism, the restriction of the projection $\pi \colon \widehat{M} \to M$ to $\mathcal{S}$ is also a diffeomorphism.

Now $\mathcal{S}$ is a smooth hypersurface in $\widehat{M}$, which intersects $\widehat{\mathcal{H}}$ properly transversally in the sense that if $q \in \mathcal{S} \cap \widehat{\mathcal{H}}$ and the tangent space $T_q\widehat{\mathcal{H}}$ exists then $T_q\mathcal{S}$ is transverse to $T_q\widehat{\mathcal{H}}$. Let

$$\widehat{S_0} = \{q \in \mathcal{S} \cap \widehat{\mathcal{H}} \mid q \text{ is an interior point of a generator of } \widehat{\mathcal{H}}\},$$
$$S_0 = \{q \in U \cap \mathcal{H} \mid q \text{ is an interior point of a generator of } \mathcal{H}\}.$$

Note that $\pi(\widehat{S_0}) = S_0$ since if p is an interior point of a generator Γ then $\pi(p)$ is an interior point of the generator $\pi(\Gamma)$ and vice versa. Note further that it holds that $\pi(\mathcal{S} \cap \widehat{\mathcal{H}}) = U \cap \mathcal{H}$. Moreover, the projection π restricted to $\mathcal{S}$ is bijective and hence $\pi((\mathcal{S} \cap \widehat{\mathcal{H}}) \setminus \widehat{S_0}) = (U \cap \mathcal{H}) \setminus S_0$.

Since π restricted to $\mathcal{S}$ is a diffeomorphism, both $\pi|_{\mathcal{S}}$ and its inverse $(\pi|_{\mathcal{S}})^{-1}$ are locally Lipschitz so that $\mathfrak{h}^n((\mathcal{S} \cap \widehat{\mathcal{H}}) \setminus \widehat{S_0}) = 0$ if and only if $\mathfrak{h}^n((U \cap \mathcal{H}) \setminus S_0) = 0$. The latter set $(U \cap \mathcal{H}) \setminus S_0$ is the set of endpoints of generators of $\mathcal{H}$ contained in U. It is shown in [2, Theorem 3.5] and [8, Theorem 1] that this set has zero $\mathfrak{h}^n$-measure. This means that we can conclude that $\mathfrak{h}^n\left((\mathcal{S} \cap \widehat{\mathcal{H}}) \setminus \widehat{S_0}\right) = 0$. In other words, $\widehat{S_0}$ has full measure in $\mathcal{S} \cap \widehat{\mathcal{H}}$.

Let

$$\widehat{S_1} = \{q \in \widehat{S_0} \mid \text{all interior points of the generator}$$
$$\text{through } q \text{ are Alexandrov points of } \widehat{\mathcal{H}}\}.$$

By Lemma 1.29 the set $\widehat{S_1}$ has full $\mathfrak{h}^n$ measure in $\widehat{S_0}$. Hence, it also has full $\mathfrak{h}^n$-measure in $\mathcal{S} \cap \widehat{\mathcal{H}}$. Since π is bi-Lipschitz, $\pi(\widehat{S_1})$ has full $\mathfrak{h}^n$-measure in $U \cap \mathcal{H}$.

The projection $\pi \colon \widehat{M} \to M$ maps generators to generators, and Alexandrov points of $\widehat{\mathcal{H}}$ to Alexandrov points of $\mathcal{H}$, so each point of $\widehat{S_1}$ belongs to A_0. We have then shown that $A_0 \cap U$ contains a subset $\pi(\widehat{S_1})$ which has full measure in $\mathcal{H} \cap U$. Hence A_0 itself has full measure in $\mathcal{H} \cap U$. As noted in the beginning of the proof, $\mathcal{H}$ may be covered by countably many such sets U, so we have shown that A_0 has full $\mathfrak{h}^n$-measure in $\mathcal{H}$. This completes the proof. $\square$

Definition 1.31. Let $\mathcal{H}$ be a horizon in a spacetime (M, g), and let A_0 be its total flow set. Let σ be a Riemannian metric on M. Since $\mathcal{H}$ is differentiable at all points in A_0, there is a unique σ-unit past-directed null vector tangent to $\mathcal{H}$ at each point in A_0. This defines a vector field T on A_0, which is tangent to the generators of $\mathcal{H}$. Recall that A_0 contains full generators, and hence full integral curves of T. We will call the flow of T the *generator flow* of $\mathcal{H}$ with respect to σ, and denote it by $(t, p) \mapsto \beta_t(p)$.

Note that β_t is not in general defined on all of A_0 for any $t > 0$. However, it will be defined on all of A_0 for all $t > 0$ in the case considered in our main theorem, so we will mainly be concerned with this case. Note also that the choice of A_0 was made so that A_0 flows into itself, in the sense that if $p \in A_0$ and $t \geq 0$ are such that $\beta_t(p)$ is defined, then $\beta_t(p) \in A_0$.

1.4.3. Generator Flow is Area-Preserving. The purpose of this section is to prove that the generator flow on a horizon with respect to a certain family of Riemannian metrics preserves the associated Hausdorff measure if the null mean curvature is nonpositive. Our first goal is to construct a $C^{1,1}$ approximation of the horizon to be able to express the volume change. We do this in Lemmas 1.33 and 1.34. We then construct a C^2 approximation of the horizon to compute the volume change in Propositions 1.36 and 1.39. The complicated constructions necessary are contained in Lemma 1.38.

We begin by stating an extension result, which is proved in [7, Proposition 6.6].

Lemma 1.32. *Let $B \subseteq \mathbb{R}^n$ be an arbitrary subset and $f: B \to \mathbb{R}$ be an arbitrary function. Suppose that there is some constant $C > 0$, and some function $B \to \mathbb{R}^n$, $p \mapsto a_p$, (not necessarily continuous) such that the following two conditions hold:*

1. *f has global upper and lower support paraboloids of opening C. Explicitly, for all $x, p \in B$,*

$$|f(x) - f(p) - \langle x - p, a_p \rangle| \leq C||x - p||^2.$$

2. *The upper and lower support paraboloids of f are disjoint. Explicitly, for all $p, q \in B$ and all $x \in \mathbb{R}^n$,*

$$f(p) + \langle x - p, a_p \rangle - C||x - p||^2 \leq f(q) + \langle x - q, a_q \rangle + C||x - q||^2.$$

Then there is a function $F: \mathbb{R}^n \to \mathbb{R}$ of class $C^{1,1}_{loc}$ such that f is the restriction of F to B.

Using this lemma, we may prove the following.

Lemma 1.33. *Let $\mathcal{H}$ be a horizon in an $(n+1)$-dimensional spacetime M. Let σ be any Riemannian metric on M, let $\delta > 0$ and let A_δ be the δ-flow set of $\mathcal{H}$ with respect to σ. Let $p \in A_\delta$. Then there is some open globally hyperbolic neighborhood $V \subseteq M$ of p and a $C^{1,1}$ hypersurface $N \subseteq V$ in M such that $A_\delta \cap V \subseteq N$.*

Proof. For each point $q \in A_\delta$, let q^+ denote the point a σ-distance δ to the future along the unique generator through q. Similarly, let q^- denote the point along the generator a distance δ to the past. By one of the defining properties of A_δ, we have $q^+, q^- \in \mathcal{H}$.

By the same reasoning as used in the proof of Lemma 6.9 in [7], one may obtain a globally hyperbolic neighborhood $V \subseteq W$ of p and a constant $C > 0$ with the following properties:

- V is diffeomorphic to $(-a, a) \times B^n(r)$ with the slices $\{t\} \times B^n(r)$ spacelike and the curves $(-a, a) \times \{x\}$ timelike and future-directed for all $t \in (-a, a)$ and all $x \in B^n(r)$.
- Let f denote the graphing function of the horizon over $B^n(r)$, i.e. the function such that $V \cap \mathcal{H} = \{(f(x), x) \mid q \in B^n(r)\}$. For each $q = (f(x_q), x_q) \in V \cap A_\delta$, the graph of the function

$$f_q^-(x) = f(x_q) + \mathrm{d}f(x_q)(x - x_q) - C||x - x_q||^2,$$

  with the exception of the point $q = (f(x_q), x_q)$ itself lies in the timelike past $I^-(q^+, V)$ of q^+.
- For each $q = (f(x_q), x_q) \in V \cap A_\delta$, the graph of the function

$$f_q^+(x) = f(x_q) + \mathrm{d}f(x_q)(x - x_q) + C||x - x_q||^2,$$

  with the exception of the point $q = (f(x_q), x_q)$ itself lies in the timelike future $I^+(q^-, V)$ of q^-.

Note that if this holds for some value of C, it holds for all larger values of C as well.

We will now show that these conditions imply the first hypothesis of Lemma 1.32. Suppose that the condition is violated. Then either $f(x) > f_q^+(x)$ or $f(x) < f_q^-(x)$ for some $q = (f(x_q), x_q) \in A_\delta$ and $x \in B^n(r)$. The argument is the same for both cases, so suppose without loss of generality that the first is the case. Since $f(x_q) = f_q^+(x_q)$ we must have $x \neq x_q$. Then $(f_q^+(x), x)$ belongs to the timelike future of q^-, by the choice of C. However, since $f(x) > f_q^+(x)$, the point $(f(x), x)$ lies to the timelike future of $(f_q^+(x), x)$. This means that we can connect q^- to $(f_q^+(x), x)$ to $(f(x), x)$ by a timelike curve. Hence $(f(x), x)$ belongs to the timelike future of q^-. Since both points belong to the horizon, this violates achronality of the horizon. This proves the first hypothesis of Lemma 1.32.

For the second hypothesis, note that the first continues to hold if we increase C. By making sure that C is sufficiently large compared to the Lipschitz constant of f and the values of f, one may conclude as in the proof of Lemma 6.9 in [7] that the second hypothesis is satisfied as well.

Let B denote the projection of $A_\delta \cap V$ on $B^n(r)$. We can then apply the extension theorem described in Lemma 1.32 to obtain a $C^{1,1}$ extension $\mathbb{R}^n \to \mathbb{R}$ of $f|_B : B \to (-a, a)$. Let $F \colon \mathbb{R}^n \to \mathbb{R}$ denote the restriction to $B^n(r)$ of this extension. By definition F agrees with f on B. In particular, the graph of F contains $p = (f(x_p), x_p)$, so $F(x_p) \in (-a, a)$. Since F is continuous, there is some neighborhood $B^n(\epsilon) \subseteq B^n(r)$ of x_p such that $F(B^n(\epsilon)) \subseteq (-a, a)$. Hence by shrinking the neighborhood V to $(-a,a)\times B^n(\epsilon)$ and letting N be the graph of F there, we have obtained a $C^{1,1}$ hypersurface containing $A_\delta \cap V$. □

Lemma 1.34. *Let $\mathcal{H}$ be a horizon in an $(n+1)$-dimensional spacetime (M,g) equipped with a Riemannian metric σ, let $\delta > 0$, let A_δ be the δ-flow set of $\mathcal{H}$ with respect to σ, let $\tilde{A}_\delta$ be the full-density subset (in the sense of Definition A.9) of A_δ, let V be a globally hyperbolic open neighborhood of p and let $N \subseteq V$ be a $C^{1,1}$ hypersurface containing $A_\delta \cap V$, which can be represented by a graphing function in V.*

Fix $t \geq 0$. Let $\beta_t\colon A_\delta \cap V \to A_0$ be the restriction of the generator flow (with respect to σ) to $A_\delta \cap V$, and suppose that this flow is defined on all of $A_\delta \cap V$. Then there is a neighborhood $U \subseteq V$ of p such that the restriction of β_t to $\tilde{A}_\delta \cap U$ is the restriction of a locally Lipschitz function $\widehat{\beta}_t\colon N \cap U \to M$.

Proof. In the trivial case $t = 0$ we can let $\widehat{\beta}_t$ be the identity on N. Hence, we can assume for the remainder of the proof that $t > 0$.

Let $(a,b)\times\Sigma$ be a decomposition in space and time of the globally hyperbolic neighborhood V of p, and let f denote the graphing function of N with respect to this decomposition.

We wish to construct a Lipschitz vector field normal to N in a neighborhood of $p = (f(x), x)$. Choose a frame $(e_i)_{i=1}^n$ close to p consisting of the pushforward of a frame of Σ close to x under the map $y \mapsto (f(y), y)$. This frame is Lipschitz since f is $C^{1,1}$. By shrinking V we may assume that the frame covers all of N. The condition that a vector field $\mathbf{n}$ along N is normal to N with respect to the spacetime metric g and consists of unit vectors with respect to the Riemannian metric σ can be expressed by saying that $\mathbf{n}$ satisfies the $n+1$ equations

$$\begin{aligned}\sigma(\mathbf{n},\mathbf{n}) - 1 &= 0,\\ g(e_1,\mathbf{n}) &= 0,\\ g(e_2,\mathbf{n}) &= 0,\\ &\vdots\\ g(e_n,\mathbf{n}) &= 0.\end{aligned}$$

Choose a trivialization $N \times \mathbb{R}^{n+1}$ of $TM|_N$. Define $F\colon N \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by the above equations. Explicitly

$$F(\mathbf{n}) = (\sigma(\mathbf{n},\mathbf{n}) - 1, g(e_1,\mathbf{n}), \ldots, g(e_n,\mathbf{n})).$$

Let $\mathbf{n}$ be a zero of F. The tangent map of F at $\mathbf{n}$ with respect to the $\mathbb{R}^{n+1}$ component is

$$\mathbf{k} \mapsto (2\sigma(\mathbf{n},\mathbf{k}), g(e_1,\mathbf{k}), \ldots, g(e_n,\mathbf{k})).$$

We wish to show that this tangent map has full rank. For dimensional reasons, this is equivalent to its kernel being trivial. If $\mathbf{k}$ belongs to the kernel, then $g(e_i,\mathbf{k}) = 0$ for all $1 \leq i \leq n$. Hence $\mathbf{k}$ is a normal vector to N, and hence parallel to $\mathbf{n}$. If $\mathbf{k}$ belongs to the kernel of the tangent map then it also holds that $\sigma(\mathbf{n},\mathbf{k}) = 0$. When $\mathbf{k}$ is parallel to $\mathbf{n}$, this can only happen when $\mathbf{k} = 0$.

This shows that the tangent map of the $\mathbb{R}^{n+1}$ component of F has full rank at zeros of F. Clearly there is a vector at p which is a zero of F; simply take a normal vector and rescale it. This means that we (after choosing a local trivialization of $TM|_N$ around p) can apply Clarke's Implicit Function Theorem (Corollary, p. 256 in [10]) to conclude that there is a Lipschitz function $\mathbf{n}$ satisfying $F(q, \mathbf{n}(q)) = 0$ in a neighborhood of p. By shrinking V if necessary, we have then found a Lipschitz normal (with respect to g) vector field to N which is of unit length (with respect to σ). By shrinking V further and replacing $\mathbf{n}$ with $-\mathbf{n}$ if necessary, we may assume that $\mathbf{n}$ is everywhere past-directed.

By considering graphing functions of N and $\mathcal{H}$ and applying the result about tangent spaces at full-density points described in Proposition A.12 we see that the tangent spaces T_qN and $T_q\mathcal{H}$ agree at all $q \in \tilde{A}_\delta$. Consider now a point $q \in \tilde{A}_\delta$. By Proposition 1.13 the tangent space $T_q\mathcal{H}$ is a null hyperplane. Since $q \in \tilde{A}_\delta$ we have $T_qN = T_q\mathcal{H}$, so T_qN is also a null hyperplane. The normal vector $\mathbf{n}_q$ to the null hyperplane T_qN is then null, and any two null vectors in a null hyperplane are parallel, so the vector $\mathbf{n}_q$ for a point $q \in \tilde{A}_\delta$ is parallel to any tangent vector of the null geodesic generator through q.

Once again, consider some point $q \in \tilde{A}_\delta \subseteq N$. Since $\beta_t(q)$ is a point along the geodesic (with respect to the spacetime metric g) with initial velocity parallel to $\mathbf{n}_q$ (which is nonzero and past-directed) it holds that there is some function $r\colon \tilde{A}_\delta \to (0, \infty)$ such that

$$\beta_t(q) = \exp^g(r(q)\mathbf{n}_q).$$

For each $q \in N$ there is a unique positive real number $\hat{r}(q)$ such that the σ-distance between q and $\exp^g(\hat{r}(q)\mathbf{n}_q)$ along the curve $\tau \to \exp^g(\tau\mathbf{n}_q)$ is precisely t. By definition r and $\hat{r}$ coincide on $\tilde{A}_\delta$.

We now want to use Clarke's implicit function theorem (Corollary, p. 256 in [10]) again, this time to conclude that $\hat{r}$ is locally Lipschitz. By definition, the choice $\xi = \hat{r}$ solves the equation

$$\forall q \in N \quad \int_0^1 \left|\left| \frac{\partial}{\partial \tau} \exp^g(\tau\xi(q)\mathbf{n}_q) \right|\right|_\sigma \, \mathrm{d}\tau = 1,$$

and this solution is of course unique if we require that $\xi(q) > 0$ everywhere. In other words $\xi = \hat{r}$ is the unique positive function satisfying $F(q, \xi(q)) = 0$ where $F\colon N \times \mathbb{R} \to \mathbb{R}$ is defined by

$$F(q, t) = \left(\int_0^1 \left|\left| \frac{\partial}{\partial \tau} \exp^g(\tau t\mathbf{n}_q) \right|\right|_\sigma \, \mathrm{d}\tau \right) - 1.$$

Since $\mathbf{n}$ is locally Lipschitz, so is F. Note that F has a partial derivative with respect to t and that $\frac{\partial F}{\partial t} \neq 0$ everywhere since $\mathbf{n}$ is nowhere zero. Clarke's implicit function theorem now tells us that there is a solution ξ of $F(q, \xi(q)) = 0$ with $\xi(p) = r(p)$ which is Lipschitz in a neighborhood of p. Since $r(p)$ is positive, so is ξ in a neighborhood of p. Since we already know that the only positive solution of this equation is $\hat{r}$, this shows that $\hat{r}$ is Lipschitz in some neighborhood U of p.

Since $\hat{r}$ is Lipschitz on U, the function $\widehat{\beta}_t\colon N \cap U \to M$ defined by

$$\widehat{\beta}_t(q) = \exp^g(\hat{r}(q)\mathbf{n}_q)$$

is also Lipschitz. The restriction of this function to $\tilde{A}_\delta \cap U$ agrees with β_t, completing the proof. $\square$

For future reference, we note the following corollary.

Corollary 1.35. *Let $\tilde{A}_\delta$ be the full-density subset of a δ-flow set A_δ, and let U be any set such that the generator flow with respect to some Riemannian metric σ is defined on all of $U \cap A_\delta$. Let $\mathfrak{h}^n$ be the n-dimensional Hausdorff measure associated to σ. Then $\beta_t(\tilde{A}_\delta \cap U)$ has full $\mathfrak{h}^n$-measure in $\beta_t(A_\delta \cap U)$.*

Proof. When U is contained in a sufficiently small open set, Lemma 1.34 tells us that β_t is the restriction of a Lipschitz function. Hence $\beta_t(A_\delta \cap U) \backslash \beta_t(\tilde{A}_\delta \cap U)$ has $\mathfrak{h}^n$-measure zero, since $A_\delta \backslash \tilde{A}_\delta$ has $\mathfrak{h}^n$-measure zero. If U is not sufficiently small, it may be covered by countably many such small open sets since it is second-countable, giving the same conclusion. $\square$

Proposition 1.36. *Let $\mathcal{H}$ be a horizon in an $(n+1)$-dimensional spacetime (M, g) equipped with a Riemannian metric σ of the form*

$$\sigma(X, Y) = g(X, Y) + 2g(X, V)g(Y, V)$$

for some timelike g-unit vector field V. Let $\mathfrak{h}^n$ be the n-dimensional Hausdorff measure associated to σ, let $\delta > 0$, let A_δ be the δ-flow set, let $\tilde{A}_\delta$ be the full-density subset of A_δ, let $t > 0$, let β_t be the restriction to A_δ of the generator flow with respect to σ. Suppose that β_t is defined on all of A_δ and that $\theta_{\mathcal{A}l} \leq 0$ on all of $\mathcal{H}_{\mathcal{A}l}$.

Then every $p \in A_\delta$ has a neighborhood Z which is open in A_δ such that there is a measurable function Ψ with $\Psi \geq 1$ almost everywhere such that

$$\int_{\tilde{A}_\delta} \varphi \Psi \, \mathrm{d}\mathfrak{h}^n = \int_{\beta_t(\tilde{A}_\delta)} \varphi(\beta_t^{-1}(y)) \, \mathrm{d}\mathfrak{h}^n(y)$$

for every $\varphi\colon A_\delta \to \mathbb{R}$ which is $\mathfrak{h}^n$-integrable and supported in $Z \cap \tilde{A}_\delta$. Moreover, if $\Psi = 1$ almost everywhere on Z, then $\theta_{\mathcal{A}l} = 0$ almost everywhere on Z.

Remark 1.37. A sufficient condition for having $\theta_{\mathcal{A}l} \leq 0$ is that the generators of $\mathcal{H}$ are complete in the past direction, together with the null energy condition (see Proposition 1.24).

Proof. Choose some point $p \in A_\delta$. By Lemma 1.33, there is a globally hyperbolic open spacetime neighborhood U of p and a $C^{1,1}$ hypersurface $N \subseteq U$ such that $A_\delta \cap U \subseteq N$. Lemma 1.34 tells us that after possibly shrinking U, the generator flow $\beta_t\colon A_\delta \to A_0$ is the restriction of a Lipschitz function $\widehat{\beta}_t\colon N \cap U \to M$.

Let $Z = A_\delta \cap U$. Let $\varphi\colon A_\delta \to \mathbb{R}$ be $\mathfrak{h}^n$-integrable and supported in $Z \cap \tilde{A}_\delta$. Since $\widehat{\beta}_t$ is Lipschitz on N, Theorem 3.1 of [11] tells us that

$$\int_N \varphi J(\widehat{\beta}_t)\mathrm{d}\mathfrak{h}^n = \int_N \left(\sum_{x \in \widehat{\beta}_t^{-1}(y)} \varphi(x) \right) \mathrm{d}\mathfrak{h}^n(y).$$

Here, $J(\widehat{\beta}_t)$ is the Jacobian determinant of $\widehat{\beta}_t$ with respect to σ at points where $\widehat{\beta}_t$ is differentiable. Since $\widehat{\beta}_t$ is Lipschitz, $J(\widehat{\beta}_t)$ is thus almost everywhere defined on N, which is sufficient for the integral to make sense.

Note that φ is zero outside of $Z \cap \tilde{A}_\delta$, so

$$\sum_{x \in \widehat{\beta}_t^{-1}(y)} \varphi(x) = \sum_{x \in \widehat{\beta}_t^{-1}(y) \cap Z \cap \tilde{A}_\delta} \varphi(x).$$

Note that $\widehat{\beta}_t^{-1}(y) \cap Z \cap \tilde{A}_\delta$ is the inverse image of y under the restriction of $\widehat{\beta}_t$ to $Z \cap \tilde{A}_\delta$. This restriction agrees with the restriction of β_t to the same set. Since β_t is injective on A_δ, its restriction to $Z \cap \tilde{A}_\delta$ is injective as well. This means that

$$\widehat{\beta}_t^{-1}(y) \cap Z \cap \tilde{A}_\delta = \begin{cases} \{\beta_t^{-1}(y)\} & \text{if } y \in \beta_t(Z \cap \tilde{A}_\delta), \\ \emptyset & \text{otherwise.} \end{cases}$$

Using this together with the fact that φ is zero outside of Z we see that

$$\sum_{x \in \widehat{\beta}_t^{-1}(y)} \varphi(x) = \begin{cases} \varphi(\beta_t^{-1}(y)) & \text{if } y \in \beta_t(Z \cap \tilde{A}_\delta), \\ 0 & \text{otherwise,} \end{cases}$$
$$= \begin{cases} \varphi(\beta_t^{-1}(y)) & \text{if } y \in \beta_t(\tilde{A}_\delta), \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\int_{\tilde{A}_\delta} \varphi J(\widehat{\beta}_t)\mathrm{d}\mathfrak{h}^n = \int_{\beta_t(\tilde{A}_\delta)} \varphi(\beta_t^{-1}(y))\mathrm{d}\mathfrak{h}^n(y).$$

To complete the proof of the theorem, we need to show that $J(\widehat{\beta}_t) \geq 1$ almost everywhere on $\tilde{A}_\delta \cap U$, and that $J(\widehat{\beta}_t) = 1$ almost everywhere on $\tilde{A}_\delta \cap U$ only if $\theta_{\mathcal{A}l} = 0$ almost everywhere on $A_\delta \cap U$, possibly after shrinking U. We can then choose Ψ to be $J(\widehat{\beta}_t)$. Since the argument for proving these statements is quite long, we prove them separately as Lemma 1.38. □

Lemma 1.38. *Fix t. Let p, U, A_δ, $\tilde{A}_\delta$, N and $\widehat{\beta}_t$ be as in Proposition 1.36. After possibly shrinking U to a smaller neighborhood of p, it holds that $J(\widehat{\beta}_t) \geq 1$ almost everywhere (with respect to $\mathfrak{h}^n$) on $A_\delta \cap U$. Moreover, if $C \subset U$ is a closed set in the subspace topology on U and $J(\widehat{\beta}_t) = 1$ almost everywhere on $A_\delta \cap C$ then $\theta_{\mathcal{A}l} = 0$ almost everywhere on $A_\delta \cap C$.*

Proof. **Step I Construction of the set $\widehat{B}$**
After possibly shrinking U to a smaller neighborhood of p and decomposing it as $U = (a,b) \times \Sigma$ (with the curves $(a,b) \times \{q\}$ timelike and the slices $\{\tau\} \times \Sigma$ spacelike) where Σ is an open subset of $\mathbb{R}^n$, we may view $\mathcal{H}$ as the graph of a semi-convex function f, as noted in Sect. 1.3.1. Similarly, N is the graph of a $C^{1,1}$ function g. Let $\mathfrak{L}^n$ denote Lebesgue measure on Σ. By [12, Theorem 3.1.15], for each positive integer k there is a C^2 function $g_k\colon \Sigma \to \mathbb{R}$ such that

$$\mathfrak{L}^n\left(\{x \in \Sigma \mid g_k(x) \neq g(x)\}\right) < 1/k.$$

Let $\mathrm{pr}\,\tilde{A}_\delta$ be the projection of $\tilde{A}_\delta$ on Σ. Let B be the full-density subset of $\mathrm{pr}\,\tilde{A}_\delta$. By Proposition A.7 the set B has full $\mathfrak{L}^n$-measure in $\mathrm{pr}\,\tilde{A}_\delta$. Letting

$$B_k = B \cap \{x \in \Sigma \mid g_k(x) = g(x)\}$$

we then have

$$\mathfrak{L}^n(B \backslash B_k) < 1/k.$$

Once again, we discard low-density points: let $\tilde{B}_k$ be the full-density subset of B_k. Then $\tilde{B}_k$ has full measure in B_k by Proposition A.7 so

$$\mathfrak{L}^n(B \backslash \tilde{B}_k) < 1/k.$$

Let Σ_{Rad} denote the points of Σ where g is twice differentiable in the sense that it has second-order expansions of the form

$$\begin{aligned} g(x) &= g(x_0) + \mathrm{d}g(x_0)(x - x_0) + \frac{1}{2}D^2 g(x_0)(x - x_0, x - x_0) + o(|x - x_0|^2),\\ \mathrm{d}g(x) &= \mathrm{d}g(x_0) + D^2 g(x_0)(x - x_0, \cdot) + o(|x - x_0|). \end{aligned} \tag{5}$$

Rademacher's theorem tells us that since g is $C^{1,1}$, the set Σ_{Rad} has full measure in Σ. Defining

$$\widehat{B}_k := \tilde{B}_k \cap \Sigma_{\mathrm{Rad}},$$
$$\widehat{B} := \bigcup_{k \in \mathbb{N}} \widehat{B}_k = \Sigma_{\mathrm{Rad}} \cap \bigcup_{k \in \mathbb{N}} \tilde{B}_k$$

we then know that $\widehat{B}$ has full $\mathfrak{L}^n$-measure in $\mathrm{pr}\,\tilde{A}_\delta$. Since g is Lipschitz, the graph of g over $\widehat{B}$ has full $\mathfrak{h}^n$-measure in $\tilde{A}_\delta \cap U$ by Proposition A.3. It is now sufficient to show that $J(\widehat{\beta}_t)(p_0) \geq 1$ whenever $p_0 = (g(x_0), x_0)$ is such that $x_0 \in \widehat{B}$.

Choose some $x_0 \in \widehat{B}$. Since $\widehat{B} \subseteq \mathrm{pr}\,\tilde{A}_\delta$, the functions f and g agree at x_0, and x_0 is an Alexandrov point of f. Hence we have the expansion

$$f(x) = f(x_0) + \mathrm{d}f(x_0)(x - x_0) + \frac{1}{2}D^2 f(x_0)(x - x_0, x - x_0) + o(|x - x_0|^2).$$

Since $x_0 \in \widehat{B}$, we know that g is twice differentiable at x_0 so that we have the expansions of Eq. (5). Moreover, $x_0 \in B_i$ for some $i \in \mathbb{N}$ so $g_i(x_0) = g(x_0)$. Fix this value of i for the remainder of the proof. Moreover, by definition of $\widehat{B}$, the point x_0 is a full-density point of $\mathrm{pr}\,A_\delta$, and g and f agree on $\mathrm{pr}\,A_\delta$. Similarly,

x_0 is a full-density point of B_i, and g and g_i agree on B_i. This means that we can use Proposition A.12 to conclude that

$$\mathrm{d}f(x_0) = \mathrm{d}g(x_0) = \mathrm{d}g_i(x_0) \tag{6}$$

and

$$D^2 g_i(x_0) = D^2 g(x_0) = D^2 f(x_0).$$

□

Step II: Construction of the C^2 null hypersurface $\mathcal{H}_i$
Let N_i denote the graph of g_i over Σ. Let $\widehat{N}_i$ denote a C^2 spacelike hypersurface in N_i containing p_0. Thus $\widehat{N}_i$ has codimension 2 in the spacetime M. Let $\mathbf{n}_i$ denote the past-directed σ-unit normal null vector field of $\widehat{N}_i$ such that $\mathbf{n}_i(p_0)$ is the σ-unit tangent of the (unique since $p_0 \in A_\delta$) generator of $\mathcal{H}$ passing through p_0. Note that $\mathbf{n}_i$ is C^1. Let $\mathcal{G}_i$ be the union of the geodesics starting from $\widehat{N}_i$ with initial velocities given by $\mathbf{n}_i$. Let $\gamma\colon [0,t] \to \mathcal{G}_i$ denote the curve $s \mapsto \beta_s(p_0)$. We wish to choose a subset of $\mathcal{G}_i$ which is a C^2 hypersurface containing γ. Define $\exp\colon \Omega \to M$ by $\exp(\tau, q) = \exp_q(\tau \mathbf{n}_i(q))$, where $\Omega \subseteq \mathbb{R} \times \widehat{N}_i$ is the largest subset on which exp may be defined. Proposition A.3 of [7] says that if $\exp_*$ is injective at (τ, q) then there is an open neighborhood $\mathcal{O}$ of (τ, q) such that $\exp(\mathcal{O})$ is a C^2 submanifold of M. This, together with the fact that exp is injective when restricted to $[0,t]\times\{p_0\}$, shows that some neighborhood of γ in $\mathcal{G}_i$ is a C^2 hypersurface in M. Hence we need to show that $\exp_*$ is injective at (s, p_0) for each $s \in [0,t]$. Note that $\widehat{N}_i$ is a C^2 spacelike submanifold of M of codimension 2, and that $\widehat{N}_i$ is *second-order tangent* to $\mathcal{H}$ at p_0 in the sense of [7, Section 4.2] since $D^2 g_i(x_0) = D^2 f(x_0)$. By [7, Lemma 4.15] there can then be no focal point of $\widehat{N}_i$ along γ. By [28, Proposition 30, Chapter 10] this means that $\exp_*$ is injective at (s, p_0) for all $s \in [0,t]$. As pointed out previously, [7, Proposition A.3] then tells us that some open neighborhood of γ in $\mathcal{G}_i$ is a C^2 submanifold. Since $\exp_*$ is injective at each point on γ, it is injective in a neighborhood of each such point. Since $\gamma([0,t])$ is compact, finitely many such neighborhoods suffice to cover γ. Hence there is a neighborhood of γ where $\mathcal{G}_i$ is C^2 and $\exp_*$ is injective. Denote this neighborhood by $\mathcal{H}_i$.

By an application of Proposition 1.5 we see that $\mathcal{H}_i$ is a null hypersurface.

Step III: Definition of a map $\widehat{\beta}_t^{\,i} : \mathcal{H}_i \to M$
By definition of $\mathcal{H}_i$, the vector field $\mathbf{n}_i$ (where defined) is tangent to $\mathcal{H}_i$. Since $\mathcal{H}_i$ is a null hypersurface, it has a unique σ-unit normal null vector field, which must then be an extension of $\mathbf{n}_i$. Call this extension $\mathbf{n}_i$ as well.

Note that $\mathcal{H}_i$ contains both $\widehat{N}_i$ and the generator passing through p_0, these being submanifolds of N_i transverse to each other. Hence the first and second derivatives of the graphing functions of $\mathcal{H}_i$ and N_i must agree at p_0.

Define the map $\widehat{\beta}_t^{\,i} : \mathcal{H}_i \to M$ by

$$\widehat{\beta}_t^{\,i}(q) = \exp^g(r(q)\mathbf{n}_i)$$

where $r(q)$ is the unique nonnegative real number such that the σ-distance from q to $\exp^g(r(q)\mathbf{n}_i)$ along the g-geodesic $\exp^g(\tau\mathbf{n}_i)$ is precisely t. Then by definition

$$\widehat{\beta}_t^{\,i}(p_0) = \widehat{\beta}_t(p_0).$$

Note that $\widehat{\beta}_t$ and $\widehat{\beta}_t^{\,i}$ are defined by the formula $\exp^g(r(q)\mathbf{k})$ where $\mathbf{k}$ is the normal vector field of N and $\mathcal{H}_i$, respectively. The derivative of $\exp^g(r(q)\mathbf{k})$ is determined by the first derivatives of r and $\mathbf{k}$. These in turn are determined by the second derivatives of the graphing functions of N and $\mathcal{H}_i$. Since $\mathrm{d}g(p_0) = \mathrm{d}g_i(p_0)$ and $D^2g(p_0) = D^2g_i(p_0)$ this means that the tangent maps of $\widehat{\beta}_t^{\,i}$ and $\widehat{\beta}_t$ agree at p_0. Hence

$$J(\widehat{\beta}_t^{\,i})(p_0) = J(\widehat{\beta}_t)(p_0).$$

We have now reduced the problem to showing that $J(\widehat{\beta}_t^{\,i})(p_0) \geq 1$.

Step IV: Computation of $J(\widehat{\beta}_t^{\,i})(p_0)$

Let $b_{\mathcal{H}_i}$ be the one-parameter family of Weingarten maps (defined in Sect. 1.1.1) along the generator of $\mathcal{H}_i$ through p_0 with its affine parametrization, and let $\dot{b}_{\mathcal{H}_i}$ denote the covariant derivative of b along this generator. Equation (1) in Sect. 1.1.1 tells us that $b_{\mathcal{H}_i}$ satisfies the equation

$$\dot{b} + b^2 + \widetilde{R} = 0. \tag{7}$$

Recall from Remark 1.23 that $\mathcal{H}$ has a null Weingarten map $b_{\mathcal{A}l}$, defined in terms of Alexandrov derivatives, on all points to the past of p_0 on the generator of $\mathcal{H}$ through p_0, and that this map also satisfies Eq. (7). Since the null Weingarten map of a null hypersurface can be expressed in the first two derivatives of a graphing function, and $\mathcal{H}$ shares these derivatives with N_i which in turn shares them with $\mathcal{H}_i$, it holds that $b_{\mathcal{H}_i}(p_0) = b_{\mathcal{A}l}(p_0)$. By the uniqueness of solutions to the ordinary differential equation (7), these two maps must agree on all of the past of p_0 along the generator through p_0. Let $\theta_{\mathcal{H}_i}$ denote the null mean curvature of $\mathcal{H}_i$, as defined in Sect. 1.1.1. Then we have

$$\theta_{\mathcal{H}_i} = \operatorname{tr} b_{\mathcal{H}_i} = \operatorname{tr} b_{\mathcal{A}l} = \theta_{\mathcal{A}l}.$$

Lemma 1.4 implies that

$$J(\widehat{\beta}_t^{\,i})(p_0) = \exp\left(-\int_0^t \theta_{\mathcal{H}_i}(\widehat{\beta}_s^{\,i}(p_0))\,\mathrm{d}s\right).$$

Since $J(\widehat{\beta}_t^{\,i})(p_0) = J(\widehat{\beta}_t)(p_0)$ and $\theta_{\mathcal{A}l} = \theta_{\mathcal{H}_i}$ along the curve $s \mapsto \widehat{\beta}_s^{\,i}(p_0) = \widehat{\beta}_s(p_0)$, we then know that

$$J(\widehat{\beta}_t)(p_0) = \exp\left(-\int_0^t \theta_{\mathcal{A}l}(\widehat{\beta}_s(p_0))\,\mathrm{d}s\right).$$

Recall that $g(\widehat{B})$ has full $\mathfrak{h}^n$-measure in $N \cap U$ and that p_0 was an arbitrary point of $g(\widehat{B})$. Since we have assumed that $\theta_{\mathcal{A}l} \leq 0$, we can conclude that

$J(\widehat{\beta_t}) \geq 1$ almost everywhere on the neighborhood U of p. This completes the proof of the first part of the lemma.

To prove the last part of the lemma, some measure theoretical technicalities remain.

Step V: $J(\widehat{\beta_t}) = 1$ almost everywhere on $\tilde{A}_\delta \cap C$ only if $\theta_{\mathcal{A}l} = 0$ almost everywhere on $\tilde{A}_\delta \cap C$

Recall that C is an arbitrary closed subset of U. Suppose that $J(\widehat{\beta_t}) = 1$ almost everywhere on $\tilde{A}_\delta \cap C$ with respect to $\mathfrak{h}^n$. Then in particular it holds that $J(\widehat{\beta_t}) = 1$ almost everywhere on $\tilde{A}_\delta \cap C \cap g(\widehat{B})$. (Recall that $g(\widehat{B})$ has full measure in $N \cap U$.) We have seen that on this set

$$J(\widehat{\beta_t})(q) = \exp\left(-\int_0^t \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))\,\mathrm{d}\tau\right).$$

Recall that we have assumed that $\theta_{\mathcal{A}l} \leq 0$. This means that if $J(\widehat{\beta_t}) = 1$ almost everywhere, then it holds for $\mathfrak{h}^n$-almost every $q \in \tilde{A}_\delta \cap C \cap g(\widehat{B})$ that

$$\int_0^t \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))\,\mathrm{d}\tau = 0.$$

In other words

$$\int_{C\cap N}\int_0^t \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{h}^n(q) = 0.$$

Note that the Hausdorff measure $\mathfrak{h}^n$ differs only by a C^1 function from the Lebesgue measure in coordinates on $C \cap N$. Hence Fubini's theorem [12, Theorem 2.6.2] applied in coordinates implies that

$$\int_0^t\int_{C\cap N} \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))\,\mathrm{d}\mathfrak{h}^n(q)\,\mathrm{d}\tau = 0.$$

In other words, it holds for almost every $\tau \in [0, t]$ that

$$\int_{C\cap N} \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))\,\mathrm{d}\mathfrak{h}^n(q) = 0.$$

This means that

$$\int_{C\cap N} \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))J(\widehat{\beta_\tau})(q)\,\mathrm{d}\mathfrak{h}^n(q) = 0$$

for almost every $\tau \in [0, 1]$, where $J(\widehat{\beta_\tau})$ denotes the determinant of the Jacobian of $\widehat{\beta_\tau}$. Using Theorem 3.1 of [11] we see that

$$\int_{C\cap N} \theta_{\mathcal{A}l}(\widehat{\beta_\tau}(q))J(\widehat{\beta_\tau})(q)\,\mathrm{d}\mathfrak{h}^n(q) = \int_{\widehat{\beta_\tau}(C\cap N)} \theta_{\mathcal{A}l}(q)\,\mathrm{d}\mathfrak{h}^n(q)$$

so that

$$\int_{C\cap N\cap\widehat{\beta_\tau}(C\cap N)} \theta_{\mathcal{A}l}(q)\,\mathrm{d}\mathfrak{h}^n(q) = 0$$

for almost every $\tau \in [0,t]$. In particular, there is a decreasing sequence $\tau_k \to 0$ such that

$$\int_{C\cap N\cap\widehat{\beta_{\tau_k}}(C\cap N)} \theta_{\mathcal{A}l}(q)\,\mathrm{d}\mathfrak{h}^n(q) = 0\,.$$

Hence $\theta_{\mathcal{A}l} = 0$ almost everywhere on each of the sets $C\cap N\cap\widehat{\beta_{\tau_k}}(C\cap N)$. Since these form a countable increasing sequence with union $C\cap N$, it holds that $\theta_{\mathcal{A}l} = 0$ almost everywhere on $C\cap N$ with respect to $\mathfrak{h}^n$, which completes the proof. □

Proposition 1.39. *Let $\mathcal{H}$ be a horizon in an $(n+1)$-dimensional spacetime M equipped with a Riemannian metric σ of the form*

$$\sigma(X,Y) = g(X,Y) + 2g(X,V)g(Y,V)$$

for some timelike g-unit vector field V. Let $\delta > 0$ and let A_δ be the δ-flow set of $\mathcal{H}$. Let $\mathfrak{h}^n$ denote the n-dimensional Hausdorff measure induced by σ. Let $\widehat{\mathcal{H}}$ be a past null geodesically ruled open subset of $\mathcal{H}$. Suppose that $\theta_{\mathcal{A}l} \le 0$ on all of $\mathcal{H}_{\mathcal{A}l}\cap\widehat{\mathcal{H}}$. Let $t>0$ be such that the generator flow β_t is defined on all of $\widehat{\mathcal{H}}\cap A_\delta$.

Then

$$\mathfrak{h}^n(\widehat{\mathcal{H}}\cap A_\delta) = \mathfrak{h}^n(\beta_t(\widehat{\mathcal{H}}\cap A_\delta)).$$

Moreover, $\theta_{\mathcal{A}l} = 0$ almost everywhere on $\widehat{\mathcal{H}}\cap A_\delta$.

Proof. Note that the generators of $\widehat{\mathcal{H}}$ are the intersections of the generators of $\mathcal{H}$ with $\widehat{\mathcal{H}}$ since $\widehat{\mathcal{H}}$ is an open subset of the horizon $\mathcal{H}$. For each p, let Z_p and Ψ_p be the neighborhoods and functions given by Proposition 1.36. Since A_δ is second-countable, a countable number of neighborhoods $Z_1, Z_2, \ldots$ suffice to cover A_δ. For each $i \ge 1$, let

$$Y_i = Z_i \setminus \bigcup_{1\le j<i} Z_j.$$

Then

- each Y_i is measurable,
- $Y_i \subseteq Z_i$ for each i,
- the Y_i are pairwise disjoint,
- $\bigcup_{i\ge1} Y_i = \bigcup_{i\ge1} Z_i \supseteq A_\delta$.

For each $i \ge 1$, let φ_i be the indicator function of $Y_i\cap\widehat{\mathcal{H}}\cap\tilde{A}_\delta$. Then Proposition 1.36 says that

$$\int_{\tilde{A}_\delta} \varphi_i\Psi_i \mathrm{d}\mathfrak{h}^n = \int_{\beta_t(\tilde{A}_\delta)} \varphi_i(\beta_t^{-1}(y))\mathrm{d}\mathfrak{h}^n(y)$$

for each $i \ge 1$. Since each φ_i is zero outside of $\widehat{\mathcal{H}}$, this means that

$$\int_{\widehat{\mathcal{H}}\cap\tilde{A}_\delta} \varphi_i\Psi_i \mathrm{d}\mathfrak{h}^n = \int_{\beta_t(\widehat{\mathcal{H}}\cap\tilde{A}_\delta)} \varphi_i(\beta_t^{-1}(y))\mathrm{d}\mathfrak{h}^n(y)$$

Taking a sum over i, we see that

$$\int_{\widehat{\mathcal{H}}\cap\tilde{A}_\delta}\sum_{i\geq 1}\varphi_i\Psi_i \mathrm{d}\mathfrak{h}^n = \int_{\beta_t(\widehat{\mathcal{H}}\cap\tilde{A}_\delta)}\sum_{i\geq 0}\varphi_i(\beta_t^{-1}(y))\mathrm{d}\mathfrak{h}^n(y).$$

Since precisely one of the functions φ_i is nonzero at any point $p\in\widehat{\mathcal{H}}\cap\tilde{A}_\delta$, and takes the value 1 there, we have $\sum_{i\geq 1}\varphi_i\Psi_i\geq 1$ almost everywhere on $\widehat{\mathcal{H}}\cap\tilde{A}_\delta$. Moreover, we have $\sum_{i\geq 0}\varphi_i(\beta_t^{-1}(y)) = 1$ almost everywhere on $\beta_t(\widehat{\mathcal{H}}\cap\tilde{A}_\delta)$. Hence

$$\mathfrak{h}^n(\widehat{\mathcal{H}}\cap\tilde{A}_\delta)\leq\int_{\widehat{\mathcal{H}}\cap\tilde{A}_\delta}\sum_{i\geq 1}\varphi_i\Psi_i\mathrm{d}\mathfrak{h}^n = \mathfrak{h}^n(\beta_t(\widehat{\mathcal{H}}\cap\tilde{A}_\delta)).$$

Since $\tilde{A}_\delta$ has full measure in A_δ and $\beta_t(\tilde{A}_\delta)$ has full measure in $\beta_t(A_\delta)$ by Corollary 1.35, this means that

$$\mathfrak{h}^n(\widehat{\mathcal{H}}\cap A_\delta)\leq\mathfrak{h}^n(\beta_t(\widehat{\mathcal{H}}\cap A_\delta)).$$

However, $\beta_t(\widehat{\mathcal{H}}\cap A_\delta)\subseteq\widehat{\mathcal{H}}\cap A_\delta$ since the generators of $\widehat{\mathcal{H}}$ agree with the generators of $\mathcal{H}$, so we also know that

$$\mathfrak{h}^n(\widehat{\mathcal{H}}\cap A_\delta)\geq\mathfrak{h}^n(\beta_t(\widehat{\mathcal{H}}\cap A_\delta))$$

by additivity of the measure. Hence equality must hold, and the proof of the first statement is complete.

Equality can hold only if $\sum_{i\geq 1}\varphi_i\Psi_i = 1$ almost everywhere on $\widehat{\mathcal{H}}\cap A_\delta$. This means that each function Ψ_i must be equal to 1 almost everywhere on $Y_i\cap\widehat{\mathcal{H}}\cap A_\delta$. By Proposition 1.36 this implies that $\theta_{\mathcal{A}l}=0$ almost everywhere on $Y_i\cap\widehat{\mathcal{H}}\cap A_\delta$. Since these sets cover $\widehat{\mathcal{H}}\cap A_\delta$, we have shown that $\theta_{\mathcal{A}l}=0$ almost everywhere on $\widehat{\mathcal{H}}\cap A_\delta$ with respect to the measure $\mathfrak{h}^n$. This completes the proof. □

1.4.4. Smoothness from Area-Preserving Generator Flow.

Proposition 1.40. *Let $\mathcal{H}$ be a horizon in a spacetime of dimension $n+1$ equipped with a Riemannian metric σ and the corresponding Hausdorff measure $\mathfrak{h}^n$. Let Ω be a past null geodesically ruled open subset of $\mathcal{H}$. Let A_δ denote the δ-flow set of $\mathcal{H}$ with respect to σ. Suppose that*

$$\mathfrak{h}^n(\Omega\cap A_\delta)=\mathfrak{h}^n(\beta_t(\Omega\cap A_\delta))$$

for all $t>0$ and all $\delta>0$, and that $\mathfrak{h}^n(\Omega)<\infty$.

Then the following two statements hold.

1. *The union of the images of generators which are inextendible and completely contained in Ω is a dense subset of Ω.*
2. *No generator of $\mathcal{H}$ has any endpoint on Ω.*

Proof. Let A_δ denote the δ-flow set of $\mathcal{H}$ with respect to σ. Recall that the total flow set of $\mathcal{H}$ is the set

$$A_0=\bigcup_{\delta>0}A_\delta.$$

Note that if $\delta < \delta'$ then $A_\delta \supseteq A_{\delta'}$. Hence

$$A_0 = \bigcup_{\delta > 0} A_\delta = \bigcup_{k \in \mathbb{Z}^+} A_{1/k}.$$

Since the family $\Omega \cap A_{1/k}$ is increasing,

$$\mathfrak{h}^n(\beta_t(\Omega \cap A_0)) = \lim_{k\to\infty} \mathfrak{h}^n(\beta_t(\Omega \cap A_{1/k})) = \lim_{k\to\infty} \mathfrak{h}^n(\Omega \cap A_{1/k}) = \mathfrak{h}^n(\Omega \cap A_0).$$

The limits are finite, since by hypothesis $\mathfrak{h}^n(\Omega \cap A_{1/k})$ is uniformly bounded with respect to k by $\mathfrak{h}^n(\Omega)$.

Introduce the sets $\mathfrak{C}$ and $\mathfrak{D}$ defined by

$$\begin{aligned}
\mathfrak{C} &= \bigcap_{t \in \mathbb{Z}^+} \beta_t(\Omega \cap A_0), \\
\mathfrak{D} &= \{p \in \Omega \mid \text{ there is a unique generator through } p, \\
&\qquad\qquad \text{and this generator has no future endpoint}\}.
\end{aligned}$$

We first show that if $p \in \mathfrak{C}$ then the generator Γ_p through p is an inextendible geodesic contained in Ω. By choice of Ω, the part of the generator to the past of p belongs to Ω and is inextendible in the past direction. Parameterize Γ_p by an affine parameter such that $\Gamma_p(0) = p$. Suppose that the maximal future extension of the generator were to leave Ω at some point $q = \Gamma_p(s)$. Since Γ_p is smooth and the interval $[0, s]$ is compact, the curve segment $\Gamma_p([0, s])$ has finite length in the Riemannian metric σ. This means that $p \notin \beta_t(\Omega \cap A_0)$ whenever t is greater than this length, contradicting the assumption that $p \in \mathfrak{C}$. This shows that the set $\mathfrak{C}$ satisfies the conditions for the first statement in the conclusion, so that it is sufficient to show that $\mathfrak{C}$ is dense to complete the proof of that statement.

Note that the fact that generators through points of $\mathfrak{C}$ are inextendible means that they can have no endpoints. Hence $\mathfrak{C} \subseteq \mathfrak{D}$. Since $(\beta_t(\Omega \cap A_0))_{t=1}^\infty$ is a countable decreasing family of sets of equal measure it holds that $\mathfrak{h}^n(\mathfrak{C}) = \mathfrak{h}^n(\Omega \cap A_0)$. Since $A_0 \subseteq \mathcal{H}$, and $\mathfrak{h}^n(\mathcal{H} \backslash A_0) = 0$ by Proposition 1.30, this means that $\mathfrak{h}^n(\Omega) = \mathfrak{h}^n(\Omega \cap A_0)$ so that

$$\mathfrak{h}^n(\mathfrak{C}) = \mathfrak{h}^n(\Omega).$$

In particular, $\mathfrak{C}$ is dense in Ω. Since $\mathfrak{C} \subseteq \mathfrak{D}$, it follows that $\mathfrak{D}$ is also dense in Ω.

We will now show that $\mathfrak{D}$ is closed in Ω. Suppose that a sequence $(p_k)_{k\in\mathbb{N}}$ in $\mathfrak{D}$ converges to $p \in \Omega$. Let X_k denote the (unique, by definition of $\mathfrak{D}$) future-directed σ-unit tangent of a generator at p_k. The σ-unit tangent bundle over the compact countable set $\{p\} \cup \{p_1, p_2, \ldots\}$ is compact, so by passing to a subsequence we may assume that the X_k converges to some unit vector X at p. By [7, Lemma 6.4] the space of past-directed σ-unit generator tangents is closed in the unit tangent bundle, so we know that X is tangent to a generator. Let γ denote the inextendible geodesic with initial velocity X, and let γ_k denote the inextendible geodesic with initial velocity X_k. By definition of $\mathfrak{D}$, each geodesic γ_k avoids the open set $I^+(\mathcal{H})$. By continuous dependence on initial conditions for ordinary differential equations, γ must also avoid the open set

$I^+(\mathcal{H})$. Suppose to get a contradiction that γ leaves $\mathcal{H}$ at some point p. Choose coordinates around p of the form $(-a,a)\times\Sigma$, such that each curve $(-a,a)\times\{q\}$ is timelike and each slice $\{t\}\times\Sigma$ is spacelike. Let Σ' denote the projection of $\operatorname{im}\gamma\cap((-a,a)\times\Sigma)$ on Σ. Recall that $\mathcal{H}$ is an achronal hypersurface, so by possibly shrinking Σ we may represent $\mathcal{H}\cap((-a,a)\times\Sigma')$ as the graph of a function $f_{\mathcal{H}}\colon\Sigma'\to(-a,a)$. Let $f_\gamma\colon\Sigma'\to(-a,a)$ be the function the graph of which is the image of γ (on both sides of the point p). Recall that γ is a null curve, so if $f_{\mathcal{H}}(x)>f_\gamma(x)$ at some point $x\in\Sigma'$ then there is a timelike curve from $(f_{\mathcal{H}}(x),x)$ to p, contradicting achronality of $\mathcal{H}$. If $f_{\mathcal{H}}(x)<f_\gamma(x)$ at some point $x\in\Sigma'$ then γ intersects $I^+(\mathcal{H})$ which we saw earlier is impossible. Hence γ cannot leave $\mathcal{H}$ to the future, and so there is a generator through p without future endpoint. Moreover, p is then an interior point of a generator so this generator is unique. Hence $p\in\mathfrak{D}$ and we have shown that $\mathfrak{D}$ is closed in Ω.

We have now shown that $\mathfrak{D}$ is a closed dense subset of Ω. Hence $\mathfrak{D}=\Omega$. Since no point in $\mathfrak{D}$ lies on a generator with a future endpoint, no point in $\mathfrak{D}$ can be a future endpoint of a generator. This shows that no generator of $\mathcal{H}$ can have a future endpoint on Ω. Recall that no generator has any past endpoint either, since $\mathcal{H}$ is a horizon. This completes the proof. □

The condition of $\mathcal{H}$ containing no endpoints is very strong, as the next theorem (see [7, Theorem 6.18]) illustrates.

Theorem 1.41. *Suppose that Ω is an open subset of a horizon $\mathcal{H}$ in a spacetime M, such that Ω contains no endpoints of generators of $\mathcal{H}$. Suppose moreover that $\theta_{\mathcal{A}l}=0$ almost everywhere with respect to the n-dimensional Hausdorff measure $\mathfrak{h}^n$ induced by a Riemannian metric σ on M. Then Ω is a smooth submanifold of M. Moreover, if the metric on M is analytic then Ω is an analytic submanifold of M.*

We are now in a position to prove our main theorem. It was shown in [5, Section 4] that not all compact horizons are smooth. Our theorem shows that the additional hypothesis of the null energy condition is sufficient to guarantee smoothness. Note that an analogous result holds for past Cauchy horizons, as can be seen by reversing the time orientation.

Theorem 1.42. *Let M be a spacetime of dimension $n+1$ satisfying the null energy condition. Let $S\subset M$ be an achronal set with $\operatorname{edge}(S)=\emptyset$. Let $\widehat{\mathcal{H}}$ be an open subset of $H^+(S)$ with compact closure. Suppose that $\widehat{\mathcal{H}}$ is past null geodesically ruled. Then $\widehat{\mathcal{H}}$ is a smooth totally geodesic null hypersurface. If moreover the metric is analytic then $\widehat{\mathcal{H}}$ is an analytic hypersurface.*

Proof. First note that Corollary 1.16 tells us that $H^+(S)$ is a horizon in the sense of Definition 1.9. Further note that $\widehat{\mathcal{H}}$ is a Lipschitz hypersurface, since it is an open subset of the Lipschitz hypersurface $H^+(S)$. It is also assumed to be past null geodesically ruled, so the generators of $\widehat{\mathcal{H}}$ are the intersection of the generators of $H^+(S)$ with $\widehat{\mathcal{H}}$. Note that Alexandrov points of $H^+(S)$ are Alexandrov points of $\widehat{\mathcal{H}}$. Each generator of $\widehat{\mathcal{H}}$ is a part of a null geodesic contained in $H^+(S)$. By definition, each generator of $\widehat{\mathcal{H}}$ is completely contained in, and hence totally past imprisoned in, the compact set $\overline{\widehat{\mathcal{H}}}$. This means that the generator flow β_t is defined for all t on all of the part of the total flow set of $H^+(S)$ which lies in $\widehat{\mathcal{H}}$. Moreover, we can show using Lemma 1.6 that almost every point of $\widehat{\mathcal{H}}$ belongs to a generator which is complete in the past direction. To apply that lemma to a generator, we must show that the intersection of the generator with a sufficiently small spacetime neighborhood of any of its points is contained in a $C^{1,1}$ hypersurface. However, the flow set A_0 of the horizon contains full generators, and each point such point belongs to some A_δ for some sufficiently small $\delta > 0$. (To define A_δ we use a Riemannian metric, but it does not matter precisely which metric is chosen.) By Lemma 1.33, the intersection of A_δ with some spacetime neighborhood of any point of A_δ. belongs to a $C^{1,1}$ hypersurface. This means that Lemma 1.6 is applicable to all generators contained in A_0, so that all points on A_0 belong to a generator which is complete in the past direction. Since the null energy condition holds, this means that Proposition 1.24 is applicable, telling us that $\theta_{\mathcal{A}l} \leq 0$ on A_0, which has full measure in $\mathcal{H}$. Hence $\theta_{\mathcal{A}l} \leq 0$ almost everywhere on $\mathcal{H}$. Let V be an arbitrary unit timelike vector field on M, introduce the Riemannian metric σ on M defined by

$$\sigma(X,Y) = g(X,Y) + 2g(X,V)g(Y,V), \tag{8}$$

and let $\mathfrak{h}^n$ be the corresponding n-dimensional Hausdorff measure. By Proposition A.4, the set $\widehat{\mathcal{H}}$ and all its measurable subsets have finite $\mathfrak{h}^n$-measure since $\overline{\widehat{\mathcal{H}}}$ is compact. For each $\delta > 0$, let A_δ denote the δ-flow set of $H^+(S)$. By Proposition 1.39 we know that

$$\mathfrak{h}^n(\widehat{\mathcal{H}} \cap A_\delta) = \mathfrak{h}^n(\beta_t(\widehat{\mathcal{H}} \cap A_\delta))$$

and

$$\theta_{\mathcal{A}l} = 0 \text{ almost everywhere on } \widehat{\mathcal{H}} \cap A_\delta$$

for all $t > 0$ and all $\delta > 0$. Proposition 1.40 then tells us that no generator of $H^+(S)$ has any endpoint on $\widehat{\mathcal{H}}$. Moreover, since the total flow set $A_0 = \bigcup_{\delta>0} A_\delta$ has full $\mathfrak{h}^n$-measure by Proposition 1.30 we see that $\theta_{\mathcal{A}l} = 0$ almost everywhere on $\widehat{\mathcal{H}}$. Theorem 1.41 then says that $\widehat{\mathcal{H}}$ is a smooth submanifold of M, and that it is analytic if the metric is analytic. Since $\widehat{\mathcal{H}}$ is an open subset of $H^+(S)$ and the tangent space of $H^+(S)$ is a null hyperplane whenever it exists, $\widehat{\mathcal{H}}$ is a null hypersurface.

Let K be a tangent vector field of the generators of $\widehat{\mathcal{H}}$. Since $\widehat{\mathcal{H}}$ is smooth, its null mean curvature θ with respect to K is a smooth function and its sign agrees with that of the Alexandrov null mean curvature $\theta_{\mathcal{A}l}$. We saw previously that $\theta_{\mathcal{A}l} = 0$ almost everywhere, so by continuity $\theta = 0$ everywhere. Let b denote the null Weingarten map with respect to K, and let $S = b - \frac{\theta}{n-2}$. Since S is self-adjoint, $\mathrm{tr}(S^2) \geq 0$. Since $\theta = 0$ and $\mathrm{Ric}(K,K) \geq 0$ everywhere, Eq. (2) tells us that $\mathrm{tr}(S^2) = 0$. Since S is self-adjoint this implies that $S = 0$. Hence $b = 0$ everywhere, so that $\widehat{\mathcal{H}}$ has everywhere zero null second fundamental form. Theorem 1.1 then implies that $\widehat{\mathcal{H}}$ is totally geodesic, completing the proof. □

The following corollary is immediate.

Corollary 1.43. *Let M be a spacetime satisfying the null energy condition. Let $S \subset M$ be an achronal set with* $\operatorname{edge}(S) = \emptyset$. *Suppose that $H^+(S)$ is compact. Then $H^+(S)$ is a smooth totally geodesic null hypersurface. If moreover the metric is analytic then $H^+(S)$ is an analytic hypersurface.*

Proof. Let $\widehat{\mathcal{H}} = H^+(S)$ and apply Theorem 1.42. □

Remark 1.44. The methods used here are sufficient to prove a slightly stronger statement. The hypothesis that the horizon is a Cauchy horizon, rather than an arbitrary horizon in the sense of Definition 1.9, is used only to prove that its generators are complete in the past direction. If one were to know for some other reason that the generators are complete in the past direction, then the result can be applied to general horizons.

Moreover, the hypothesis that the horizon is compact is used only to prove that the generators are complete in the past direction and to ensure that the horizon has finite measure in the n-dimensional Hausdorff measure associated to the Riemanniann metric defined in (8) using an arbitrary unit timelike vector field V. This means that if it is known that the generators are complete in the past direction and that there is a vector field V which gives the horizon finite n-dimensional Hausdorff measure, then both the compactness hypothesis and the hypothesis that the horizon is a Cauchy horizon may be dropped.

2. Lorentzian Cobordisms Satisfying Energy Conditions are Trivial

2.1. Lorentzian Pseudocobordisms

In this section, we will define the concept of a Lorentzian pseudocobordism, and the various related notions we will use.

Definition 2.1. Let S and S' be smooth compact manifolds of dimension n. A *cobordism* between S and S' is a compact $(n+1)$-dimensional manifold-with-boundary M, the boundary of which is the disjoint union $S \sqcup S'$. If there is a cobordism between S and S', we say that they are *cobordant*.

We note in passing that two compact manifolds without boundary are cobordant if and only if their Stiefel–Whitney numbers agree (see [22, Corollary 4.11]). In particular (see [22, p. 203]) any pair of compact three-dimensional manifolds without boundary is cobordant.

We will need several different notions of Lorentzian cobordisms. Since the word "cobordism" generally refers to a compact space, we will define the notion of a "Lorentzian pseudocobordism":

Definition 2.2. Let S_1 and S_2 be manifolds of dimension n without boundary. A *Lorentzian pseudocobordism* between S_1 and S_2 is a Lorentzian $(n+1)$-manifold M, the boundary of which is the disjoint union $S_1 \sqcup S_2$, such that S_1 and S_2 are spacelike, and M admits a (nowhere zero) timelike vector field which is inward-directed on S_1 and outward-directed on S_2.

The classical notion of a Lorentzian cobordism is the following.

Definition 2.3. A Lorentzian pseudocobordism M between S_1 and S_2 is a *compact Lorentzian cobordism* (or simply *Lorentzian cobordism*) if M is compact.

It turns out, as was noted by Borde (see [3]), that many of the theorems about Lorentzian cobordisms continue to hold when the property of compactness is replaced by the property of "causal compactness". We will call the resulting object a "causally compact Lorentzian cobordism".

Definition 2.4. A spacetime M is *causally compact* if $\overline{I(p)}$ is compact for each $p \in M$.

Causal compactness captures the concept of "compact in time", while allowing the spacetime to be noncompact in the spatial directions.

Definition 2.5. A Lorentzian pseudocobordism M between S_1 and S_2 is called a *causally compact Lorentzian pseudocobordism* if M is causally compact.

Of course, we immediately see that every (compact) Lorentzian cobordism is also a causally compact Lorentzian pseudocobordism.

Definition 2.6. A Lorentzian pseudocobordism M between S_1 and S_2 is *topologically trivial* if it is diffeomorphic to $S_1 \times [0, 1]$.

2.2. Tipler's Theorem

A theorem due to Geroch [15, Theorem 2] states that a topologically nontrivial Lorentzian cobordism cannot satisfy the chronology condition. A result from 1977 by Tipler [30, Theorems 3 and 4] further implies that a nontrivial Lorentzian cobordism cannot satisfy certain energy conditions. Unfortunately Tipler's original proof, the methods of which are also used in [16, p.295–298] for proving Hawking's singularity theorem, is flawed in that it is implicitly assumed that a certain Cauchy horizon is C^2. In this section, we will apply Theorem 1.42 to prove Tipler's theorem (Theorem 2.9) without needing this assumption.

Note that Geroch's theorem is not quite true as stated and that a detail is missing in the proof. See [20, Section 1.2] for a correct statement and a complete proof. For completeness, we quote a statement of Geroch's theorem from [20, Theorem 1.2.1].

Theorem 2.7. *Let $n \geq 1$, let S_1, S_2 be n-manifolds without boundary (not necessarily compact, nor necessarily a priori connected). Let (M, g) be a connected causally compact Lorentzian pseudocobordism between S_1 and S_2 which satisfies the chronology condition. Then there is a diffeomorphism $\varphi \colon S_1 \times [0, 1] \to M$ such that the submanifold $\varphi(\{x\} \times [0, 1])$ is timelike for every $x \in S_1$; in particular M is topologically trivial, and S_1 and S_2 are diffeomorphic.*

The following is the theorem about cobordisms which is proved (but not stated in this form) by Tipler [30, Theorems 3 and 4]. Tipler did not mention the need for the condition that $H^+(S_1)$ is C^2. However, his proof works when this hypothesis is added.

Theorem 2.8. *Let $n \geq 2$, let S_1, S_2 be compact n-dimensional manifolds and let (M, g) be a compact connected Lorentzian cobordism between S_1 and S_2 which satisfies the either the strict null energy condition (i.e. that* $\mathrm{Ric}(X, X) > 0$ *for all lightlike vectors X) or the null energy condition together with the lightlike generic condition.* **Suppose moreover that $H^+(S_1)$ is C^2.** *Then there exists a diffeomorphism $\varphi\colon S_1 \times [0, 1] \to M$ such that the submanifold $\varphi(\{x\} \times [0, 1])$ is g-timelike for every $x \in S_1$; in particular, S_1 is diffeomorphic to S_2.*

2.3. Tipler's Theorem Without Smoothness Hypothesis

We will prove the generalization of Tipler's theorem, suggested in [3], to causally compact Lorentzian pseudocobordisms.

Theorem 2.9. *Let $n \geq 2$. Let S_1, S_2 be n-dimensional manifolds and let (M, g) be a causally compact Lorentzian pseudocobordism between S_1 and S_2 which satisfies the null energy condition (i.e. that* $\mathrm{Ric}(X, X) \geq 0$ *for all lightlike vectors X) and the lightlike generic condition (i.e. that each lightlike geodesic γ contains at least one point at which $\dot\gamma^e \dot\gamma^f \dot\gamma_{[a} R_{b]ef[c} \dot\gamma_{d]} \neq 0$).*

Then M is globally hyperbolic. In particular, $M \cong S_1 \times [0, 1]$ so that S_1 and S_2 are diffeomorphic.

Proof. Extend M to a manifold without boundary $\hat{M}$ by glueing copies of $S_1 \times [0, \epsilon)$ and $S_2 \times [0, \epsilon)$ to the respective boundaries. Consider the future Cauchy horizon $H^+(S_1)$ of S_1 in $\hat{M}$. If it is empty, then S_1 would be a Cauchy surface for M which would mean that M is globally hyperbolic. Hence it is sufficient to show that $H^+(S_1)$ is empty. Suppose for contradiction that this is not the case.

Step I: $H^+(S_1)$ is a horizon

The set S_1 is a smooth hypersurface in $\hat{M}$, closed as a set. Note that it is also achronal since a timelike curve intersecting S_1 more than once would have to do so with the wrong time orientation. Hence we can use Lemma 1.17 and Corollary 1.16 to conclude that $H^+(S_1)$ is a horizon in the sense of Definition 1.9.

Choose a point $p \in I^+(H^+(S_1), M)$. To see that such a point exists, note that $I^+(q, M)$ is nonempty if $q \notin S_2$, and that $H^+(S_1) \backslash S_2$ is nonempty since $H^+(S_1)$ is past null geodesically ruled and hence cannot be completely contained in a spacelike hypersurface. With this choice, $H^+(S_1) \cap I^-(p, M)$ is nonempty.

Step II: Every generator of $H^+(S_1)$ which intersects $I^-(p, M)$ stays in $I^-(p, M)$ when followed to the past

Let γ be a generator of $H^+(S_1)$. We will show that if $\gamma(t) \in I^-(p, M)$ for some

t then $\gamma((-\infty, t]) \subseteq \overline{I^-(p, M)}$. This will mean that γ is totally past imprisoned in the set $\overline{I^-(p, M)}$ which is compact since M is causally compact. To this end, let $\gamma\colon (a, t] \to \hat{M}$ be the maximal past geodesic extension of a generator of $H^+(S_1)$, with $\gamma(t) \in I^-(p)$. By [28, Proposition 53, Chapter 14], $H^+(S_1)$ and S_1 are disjoint, so γ does not intersect S_1. Moreover, when followed to the past γ cannot intersect S_2 since it would need to do so with the wrong time orientation. Hence γ stays in M. Let $s < t$. Then there is a causal curve in M from $\gamma(s)$ to p formed by concatenating γ with a timelike curve from $\gamma(t)$ to p. Such a curve exists since we assumed that $\gamma(t) \in I^-(p, M)$. Since this curve is not everywhere lightlike, there is a timelike curve from $\gamma(s)$ to p. Hence $\gamma(s) \in I^-(p)$, proving the claim.

Step III: $\mathcal{H} := H^+(S_1) \cap I^-(p, M)$ is past null geodesically ruled
Let $\mathcal{H}$ denote the set $H^+(S_1) \cap I^-(p, M)$. To find a past complete null geodesic segment through a point $q \in \mathcal{H}$, consider the intersection of $\mathcal{H}$ and the generator of $H^+(S_1)$ through q. This curve is a geodesic segment, and is connected and past complete by the previous claim. Hence $\mathcal{H}$ is past null geodesically ruled.

Step IV: The existence of $\mathcal{H}$ is contradictory
Since $I^-(p, M)$ is open, the set $\mathcal{H}$ is an open subset of $H^+(S_1)$. Since M is causally compact, the set $\overline{I^-(p, M)}$ is compact. Hence $\mathcal{H}$ is contained in a compact set and so has compact closure. Theorem 1.42 tells us that $\mathcal{H}$ is a totally geodesic smooth null hypersurface. The inequality (3) from Sect. 1.1.1 then reads

$$\mathrm{Ric}(K, K) \leq 0$$

for all null tangent vectors K to $\mathcal{H}$. Combining this with the null energy condition, we can conclude that

$$\mathrm{Ric}(K, K) = 0.$$

We will now derive a contradiction from this.

If the spacetime were to satisfy the strict null energy condition, then the contradiction is immediate. When we assume that the lightlike generic condition holds, a further argument is needed. By Proposition 1.40 there is a dense, and in particular nonempty, subset of $\mathcal{H}$ consisting of points on maximal null geodesics which are contained in $\mathcal{H}$. Choose one such maximal geodesic γ. Let b denote the null Weingarten map with respect to a null vector field K which agrees with the tangent vector field of γ with an affine parametrization. Since $\mathcal{H}$ is totally geodesic, $b = 0$ along γ. Equation (1) then implies that $R(X, K)K$ is parallel to the null vector K for every vector X, where R denotes the curvature tensor of the spacetime. By [1, Proposition 2.2] this is equivalent to $K^e K^f K_{[a} R_{b]ef[c} K_{d]} = 0$. However, the lightlike generic condition says that this tensor is nonzero at some point along each maximal lightlike geodesic. We have now obtained a contradiction, so the assumption that the Cauchy horizon $H^+(S_1)$ is nonempty must be false. Hence M is globally hyperbolic. □

The above theorem is in one sense the strongest result one may hope for: the null energy condition is the weakest of the commonly used energy conditions, and in the setting of cobordisms it implies global hyperbolicity, which is the strongest of the commonly used causality conditions.

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Appendix A. Geometric Measure Theory

General references for geometric measure theory are [12,27].

A.1. Regularity of Functions

Definition A.1. A function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is $C^{1,1}$ if it is C^1 and its differential $\mathrm{d}f$ is Lipschitz. A submanifold of a smooth manifold is $C^{1,1}$ if it is locally the graph of a $C^{1,1}$ function in coordinates.

Definition A.2. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *semi-convex* if it is the sum of a convex function and a C^2 function. A submanifold of a smooth manifold is semi-convex if it is locally the graph of a semi-convex function in coordinates.

A.2. Measure Zero

Let Σ be some smooth manifold of dimension m, and let M be a smooth manifold of dimension at least m. Let $\psi\colon \Sigma \to M$ be a topological embedding. We will consider two notions of "measure zero":

- Since Σ is a smooth manifold (and in particular second-countable), any two Riemannian metrics on Σ give rise to the same family of sets having measure zero in the associated volume measure on Σ.
- Let σ be an arbitrary Riemannian metric on M. This metric induces a distance function, which in turn induces Hausdorff measures of any dimension. Let $\mathfrak{h}^m$ denote the m-dimensional Hausdorff measure induced by σ. Then we say that $A \subseteq \psi(\Sigma)$ has measure zero if $\mathfrak{h}^m(A) = 0$.

These two notions are related in the following way.

Proposition A.3. *Let Σ be some smooth manifold of dimension m, and let M be a smooth manifold of dimension n with $n \geq m$. Let $\psi\colon \Sigma \to M$ be a topological embedding. Suppose that ψ is locally Lipschitz. Then $\mathfrak{h}^m(\psi(A)) = 0$ if A has measure zero viewed as a subset of Σ.*

Proof. After representing ψ coordinates by $\psi\colon U \to V$ with open sets $U \subseteq \Sigma$ and $V \subseteq M$ identified with subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ it holds that

$$\mathfrak{h}^m(\psi(A \cap U)) \leq L\mu(A \cap U)$$

where L is the Lipschitz constant of ψ over U, and μ denotes the m-dimensional Hausdorff measure on U. (This can be proved by bounding the volume change of images of unit balls using the Lipschitz constant, or it can be seen as a special case of the much more powerful [12, Theorem 2.10.25].) Since U is a subset of $\mathbb{R}^m$, this Hausdorff measure agrees up to pointwise scaling by a smooth function with the Lebesgue measure in coordinates. In particular, if $A \cap U$ has measure zero in Σ, then $\mathfrak{h}^m(\psi(A \cap U)) = 0$. By second countability, countably many charts suffice to cover $\psi(\Sigma)$, and so $\mathfrak{h}^m(\psi(A \cap U)) = 0$ if A has measure zero. □

In general, we will mostly be interested in the notions of "measure zero" and "finite measure", so it will not matter precisely which Riemannian metric is used to induce a measure.

Proposition A.4. *Let M be a smooth manifold of dimension at least n and let N be a Lipschitz submanifold of M with dimension n. Let K be a compact subset of N. Let σ be a Riemannian metric on M, and let $\mathfrak{h}^n$ be the corresponding n-dimensional Hausdorff measure. Then all measurable subsets of K have finite $\mathfrak{h}^n$-measure.*

Proof. It is sufficient to show that K has finite measure, since subsets of K have smaller measure than K. Take a finite subcover of the cover $K \subseteq \bigcup_{p \in N} B_\sigma(p, 1)$. Each $N \cap B_\sigma(p, 1)$ has finite $\mathfrak{h}^n$-measure since N is a Lipschitz hypersurface. Hence we can conclude that K has finite measure. □

A.3. Density Functions

A reference for density functions is [27, Chapter 2]. We will use the same idea, but with somewhat different notation.

Definition A.5. Let $\mathfrak{L}^n$ denote Lebesgue measure on $\mathbb{R}^n$. For each measurable subset $A \subseteq \mathbb{R}^n$ define the *density function of A (with respect to $\mathfrak{L}^n$)* to be the function

$$\Theta(A, \cdot)\colon A \to [0, 1],$$
$$\Theta(A, q) = \lim_{r \to 0} \frac{\mathfrak{L}^n(A \cap B^n(q, r))}{\mathfrak{L}^n(B^n(q, r)}.$$

Here $B^n(q, r)$ denotes the ball of radius r centered at q.

Definition A.6. Let A be a measurable subset of $\mathbb{R}^n$. We will call the set

$$\tilde{A} = \{a \in A \mid \Theta(A, a) = 1\}$$

the *full-density subset* of A.

Proposition A.7. *Let $\tilde{A}$ be the full-density subset of some set $A \subseteq \mathbb{R}^n$. Then $\tilde{A}$ has full Lebesgue measure in A.*

Proof. By [27, Corollary 2.9] the density function $\Theta(A, \cdot)$ is equal to the characteristic function of A almost everywhere, yielding the conclusion. □

We now generalize the notion of full-density subsets to hypersurfaces in Riemannian manifolds.

Lemma A.8. *Let (M, σ) be a Riemannian manifold of dimension $n+1$ and let N be a Lipschitz hypersurface. Let U be an open subset of N and let $\varphi\colon U \to \mathbb{R}^n$ and $\psi\colon U \to \mathbb{R}^n$ be charts. Let A be a subset of U and let $\tilde{A}_\varphi$ and $\tilde{A}_\psi$ denote the full-density subsets of $\varphi(A)$ and $\psi(A)$, respectively. Then*

$$\tilde{A}_\psi = \psi(\varphi^{-1}(\tilde{A}_\varphi)).$$

Proof. Abbreviate $\psi \circ \varphi^{-1}$ by f. Since f is bi-Lipschitz, it holds for any measurable subset X of $\operatorname{im}\varphi$ that

$$\frac{1}{L^n_{f^{-1}}}\mathfrak{L}^n(X) \leq \mathfrak{L}^n(f(X)) \leq L^n_f \mathfrak{L}^n(X)$$

where $L_{f^{-1}}$ and L_f denote the Lipschitz constants of f^{-1} and f. In particular,

$$\frac{\mathfrak{L}^n(B^n(q,r)\backslash\varphi(A))}{\mathfrak{L}^n(B^n(q,r))} \leq \frac{L^n_{f^{-1}}}{L^n_f}\frac{\mathfrak{L}^n(f(B^n(q,r))\backslash\psi(A))}{\mathfrak{L}^n(f(B^n(q,r)))}.$$

By letting $R(r)$ be a positive real number such that

$$f(B^n(q,r)) \subseteq B^n(f(q), R(r))$$

and $\rho(r)$ a positive real number such that

$$f(B^n(q,r)) \supseteq B^n(f(q), \rho(r))$$

we see that

$$\frac{\mathfrak{L}^n(B^n(q,r)\backslash\varphi(A))}{\mathfrak{L}^n(B^n(q,r))} \leq \frac{L^n_{f^{-1}}}{L^n_f}\frac{\mathfrak{L}^n(B^n(f(q),R(r))\backslash\psi(A))}{\mathfrak{L}^n(B^n(f(q),\rho(r)))}.$$

Since f and f^{-1} are Lipschitz, we may choose R and ρ to be bounded from above and below by linear functions of positive derivative, so there are positive constants D and D' such that

$$D\mathfrak{L}^n(B^n(f(q),\rho(r))) \leq \mathfrak{L}^n(B^n(f(q),R(r))) \leq D'\mathfrak{L}^n(B^n(f(q),\rho(r))).$$

This together with the previous inequality means that there is a positive real number C independent of r such that

$$\frac{\mathfrak{L}^n(B^n(q,r)\backslash\varphi(A))}{\mathfrak{L}^n(B^n(q,r))} \leq C\frac{\mathfrak{L}^n(B^n(f(q),R(r))\backslash\psi(A))}{\mathfrak{L}^n(B^n(f(q),R(r)))}.$$

In particular, if $\Theta(\psi(A), f(q)) = 1$ so that

$$\lim_{r\to 0}\frac{\mathfrak{L}^n(B^n(f(q),R(r))\backslash\psi(A))}{\mathfrak{L}^n(B^n(f(q),R(r)))} = 0$$

then

$$\lim_{r\to 0}\frac{\mathfrak{L}^n(B^n(q,r)\backslash\varphi(A))}{\mathfrak{L}^n(B^n(q,r))} = 0$$

so that $\Theta(\phi(A), q) = 1$. Hence

$$\Theta(\psi(A), f(q)) = 1 \implies \Theta(\phi(A), q) = 1.$$

Repeating this argument for the inverse of f we see that

$$\Theta(\phi(A), q) = 1 \implies \Theta(\psi(A), f(q)) = 1.$$

This proves that $\tilde{A}_\psi = f(\tilde{A}_\varphi)$, completing the proof. □

Definition A.9. Consider a Riemannian manifold (M, σ) of dimension $n+1$ and let N be a Lipschitz hypersurface. In light of the previous proposition, we may define the *full-density subset* of a set $A \subseteq N$ to be the set $\tilde{A}$ such that if $\varphi \colon U \to \mathbb{R}^n$ is a chart on N then $\varphi(\tilde{A} \cap U)$ is the full-density subset of $\varphi(A \cap U)$.

Definition A.10. If q belongs to the full-density subset of a set A, we say that q is a *full-density point* of A.

Proposition A.11. *Let (M, σ) be a Riemannian manifold of dimension $n+1$ and let N be a Lipschitz hypersurface. Let $\mathfrak{h}^n$ be the n-dimensional Hausdorff measure constructed from σ. Let $A \subseteq N$ be any subset and let $\tilde{A}$ be its full-density subset. Then $\mathfrak{h}^n(A \backslash \tilde{A}) = 0$.*

Proof. Since N is second-countable, it is sufficient to prove this locally. This can be done by the use of charts and Proposition A.7. □

Proposition A.12. *Let Ω be an open subset of $\mathbb{R}^n$ and let $f, g \colon \Omega \to \mathbb{R}$ be Lipschitz functions. Let $A \subset \Omega$ be a measurable subset of Ω and suppose that f and g agree on A. Let q be a full-density point of A and suppose that f and g are both differentiable at q. Then $\mathrm{d}f(q) = \mathrm{d}g(q)$.*

If, moreover, q is a point where f and g have second-order expansions of the form

$$f(q+\xi) = f(q) + \mathrm{d}f(q)(\xi) + \frac{1}{2} D^2 f(q)(\xi, \xi) + o(|\xi|^2),$$
$$g(q+\xi) = g(q) + \mathrm{d}g(q)(\xi) + \frac{1}{2} D^2 g(q)(\xi, \xi) + o(|\xi|^2),$$

then $D^2 f(q) = D^2 g(q)$.

Proof. Let $h = f - g$ and note that h is differentiable at q and zero on A. Suppose that $\mathrm{d}h(q)(V) \neq 0$ for some vector V at q. By continuity $\mathrm{d}h(q)(W)$ for all W in some open neighborhood U of V in $T_q\mathbb{R}^n$. Then for all sufficiently small $\epsilon > 0$ and all $W \in U$ it holds that $h(q + \epsilon U) \neq 0$. This means that h is nonzero on some small open cone in the direction of V, which in turn means that $\Theta(A, q)$ cannot be equal to 1, contradicting the fact that q is a full-density point of A. This shows that $\mathrm{d}h(q) = 0$, proving the first part of the proposition.

Suppose now that q is a point where f and g have second-order expansions in the sense that

$$f(q+\xi) = f(q) + \mathrm{d}f(q)(\xi) + \frac{1}{2} D^2 f(q)(\xi, \xi) + o(|\xi|^2),$$
$$g(q+\xi) = g(q) + \mathrm{d}g(q)(\xi) + \frac{1}{2} D^2 g(q)(\xi, \xi) + o(|\xi|^2).$$

Then, since $f(q) = g(q)$ and $\mathrm{d}f(q) = \mathrm{d}g(q)$,

$$f(q+\xi) - g(q+\xi) = \frac{1}{2}\left(D^2 f(q) - D^2 g(q)\right)(\xi,\xi) + o(|\xi|^2).$$

If $\left(D^2 f(q) - D^2 g(q)\right)(\xi,\xi) = 0$ for every vector ξ then $D^2 f(q) - D^2 g(q) = 0$ and we are done (since $D^2 f(q) - D^2 g(q)$ is symmetric), so suppose that there is some ξ with $\left(D^2 f(q) - D^2 g(q)\right)(\xi,\xi) \neq 0$. By continuity, this then holds for all ν in a neighborhood of ξ, and this means that $f - g$ is nonzero on a small open cone from q. This means that $\Theta(A,q)$ cannot be equal to 1 contradicting the fact that q is a full-density point of A, showing that indeed $D^2 f(q) = D^2 g(q)$. □

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