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Penrose Type Inequalities for Asymptotically Hyperbolic Graphs

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Abstract. In this paper, we study asymptotically hyperbolic manifolds given as graphs of asymptotically constant functions over hyperbolic space $\mathbb{H}^n$. The graphs are considered as unbounded hypersurfaces of $\mathbb{H}^{n+1}$ which carry the induced metric and have an interior boundary. For such manifolds, the scalar curvature appears in the divergence of a 1-form involving the integrand for the asymptotically hyperbolic mass. Integrating this divergence, we estimate the mass by an integral over the inner boundary. In case the inner boundary satisfies a convexity condition, this can in turn be estimated in terms of the area of the inner boundary. The resulting estimates are similar to the conjectured Penrose inequality for asymptotically hyperbolic manifolds. The work presented here is inspired by Lam's article (The graph cases of the Riemannian positive mass and Penrose inequalities in all dimensions. http://arxiv.org/abs/1010.4256, 2010) concerning the asymptotically Euclidean case. Using ideas developed by Huang and Wu (The equality case of the penrose inequality for asymptotically flat graphs. http://arxiv.org/abs/1205.2061, 2012), we can in certain cases prove that equality is only attained for the anti-de Sitter Schwarzschild metric.

1. Introduction

In 1973, R. Penrose conjectured that the total mass of a space–time containing black holes cannot be less than a certain function of the sum of the areas of the event horizons. Black holes are objects whose definition requires knowledge of the global space–time. Hence, given Cauchy data (which are the only data needed to define the total mass of a space–time), finding event horizons would require solving the Einstein equations. As a consequence, in the current formulation of the Penrose conjecture, event horizons are usually replaced by the weaker notion of apparent horizons. We refer the reader to [9, Chapter XIII] for further details.

The classical Penrose conjecture takes the following form: Let (M, g, k) be Cauchy data for the Einstein equations, that is a triple where (M, g) is a Riemannian 3-manifold and k is a symmetric 2-tensor on M. Assume that (M, g, k) satisfies the dominant energy condition

$$\mu \geq |J|,$$

where μ and J are defined through

$$\begin{cases} \mu := \frac{1}{2}\left(\mathrm{Scal}^g - |k|_g^2 + (\mathrm{tr}_g\, k)^2\right), \\ J := \mathrm{div}(k) - d(\mathrm{tr}_g\, k). \end{cases}$$

Assume further that (M, g, k) is asymptotically Euclidean. A compact oriented surface $\Sigma \subset M$ is called an apparent horizon if Σ satisfies

$$H^g + \mathrm{tr}^\Sigma k = 0,$$

where H^g is the trace of the second fundamental form S of Σ computed with respect to the outgoing normal ν of Σ, that is $S(X, Y) = \langle \nabla_X \nu, Y\rangle$ for any vectors X and Y tangent to Σ, and $\mathrm{tr}^\Sigma k$ is the trace of k restricted to the tangent space of Σ for the metric induced by g. Hence viewing (M, g, k) as immersed in a space–time, the expansion of Σ in the future outgoing light-like direction vanishes. We assume that Σ is outermost, that is Σ contains all other apparent horizons in its interior. Note that Σ may be disconnected. See [2] for further details. Then, the Penrose conjecture takes the form

$$m \geq \sqrt{\frac{|\Sigma|}{16\pi}},$$

where $|\Sigma|$ denotes the area of Σ and m is the mass of the manifold (M, g). Further, equality should hold only if the exterior of Σ is isometric to a hypersurface in the exterior region of a Schwarzschild black hole with k equal to the second fundamental form of this hypersurface.

This conjecture can be generalized to higher dimensional manifolds. All the statements are the same except for the inequality which in n dimensions reads

$$m \geq \frac{1}{2}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

where ω_{n-1} is the volume of the unit $(n-1)$-sphere.

Two major breakthroughs in the proof of this inequality were obtained almost simultaneously by Huisken and Ilmanen [21] and Bray [4] for 3-manifolds. They both deal with the time-symmetric case, i.e. when $k = 0$. The result of Bray was extended to higher dimensions in [6]. We refer the reader to the excellent reviews [23] and [5] for further details. Recently, Lam [22] gave a simple proof of the time-symmetric Penrose inequality for a manifold which is a graph of a smooth function over $\mathbb{R}^n$. His proof was extended by Huang and Wu [20] to give a proof of the positive mass theorem (including the rigidity statement) for asymptotically Euclidean manifolds which are submanifolds of $\mathbb{R}^{n+1}$. More general ambient spaces were considered by de Lima and Girão [11].

The Penrose conjecture can be generalized to spacetimes with negative cosmological constant. Up to rescaling, we can assume that the cosmological constant Λ equals $-\frac{n(n-1)}{2}$. Restricting ourselves to the time-symmetric case, the dominant energy condition then reads

$$\mathrm{Scal}^g \geq -n(n-1).$$

The lower bound for the mass (defined in Sect. 2.1) is then conjectured to be given by the mass of the anti-de Sitter Schwarzschild space–time (see Sect. 2.3),

$$m \geq \frac{1}{2}\left[\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right]. \tag{1}$$

In this paper, we prove weaker forms of this inequality for manifolds which are graphs over the hyperbolic space $\mathbb{H}^n$ when we endow the manifold $\mathbb{H}^n \times \mathbb{R}$ with a certain hyperbolic metric. See Theorem 2.1.

After the first version of this article appeared on arXiv, de Lima and Girão posted an article dealing with another case of the asymptotically hyperbolic Penrose inequality [13]. Rigidity was addressed by de Lima and Girão [14] and by Huang and Wu [19]. The approach used in [19] does not require any further assumption and we shall extend it to our context in Sect. 5.

The outline of this paper is as follows. In Sect. 2.1, we define the mass of a general asymptotically hyperbolic manifold. We explicit the anti-de Sitter Schwarzschild metric in Sect. 2.3. In Sect. 3, we prove that the scalar curvature of a graph has divergence form (Eq. (7)) and that its integral is related to the mass (Lemma 3.2). In Sect. 4, we prove the first part of Theorem 2.1. Rigidity is addressed in Sect. 5.

A note

After this paper was accepted for publication the articles by de Lima and Girão [12], and by Brendle et al. [7] appeared on arXiv. In the first of these papers, an Alexandrov–Fenchel type inequality for hypersurfaces in hyperbolic space is stated, which together with Proposition 4.1 implies the Penrose inequality (1) for graphs. Note that a special case of [7, Theorem 2] follows from our formula (13) in Sect. 4.2.

2. Preliminaries

2.1. Asymptotically Hyperbolic Manifolds and the Mass

We define the mass of an asymptotically hyperbolic manifold following Chruściel and Herzlich, see [10] and [17]. In the special case of conformally compact manifolds, this definition coincides with the definition given by Wang [31]. In what follows, n-dimensional hyperbolic space is denoted by $\mathbb{H}^n$ and its metric is denoted by b. In polar coordinates $b = dr^2 + \sinh^2 r\sigma$ where σ is the standard round metric on S^{n-1}.

Set $\mathcal{N} := \{V \in C^\infty(\mathbb{H}^n) \mid \operatorname{Hess}^b V = Vb\}$. A basis of this vector space consists of the functions

$$V_{(0)} = \cosh r,\ V_{(1)} = x^1 \sinh r, \dots,\ V_{(n)} = x^n \sinh r,$$

where $x^1, \dots, x^n$ are the coordinate functions on $\mathbb{R}^n$ restricted to S^{n-1}. If we consider $\mathbb{H}^n$ as the upper unit hyperboloid in Minkowski space $\mathbb{R}^{n,1}$ then the functions $V_{(i)}$ are the restrictions to $\mathbb{H}^n$ of the coordinate functions of $\mathbb{R}^{n,1}$. The vector space $\mathcal{N}$ is equipped with a Lorentzian inner product η characterized by the condition that the basis above is orthonormal, $\eta(V_{(0)}, V_{(0)}) = 1$, and $\eta(V_{(i)}, V_{(i)}) = -1$ for $i = 1, \dots, n$. We also give $\mathcal{N}$ a time orientation by specifying that $V_{(0)}$ is future directed. The subset $\mathcal{N}^+$ of positive functions then coincides with the interior of the future lightcone. Further, we denote by $\mathcal{N}^1$ the subset of $\mathcal{N}^+$ of functions V with $\eta(V, V) = 1$, this is the unit hyperboloid in the future lightcone of $\mathcal{N}$. All $V \in \mathcal{N}^1$ can be constructed as follows. Choose an arbitrary point $p_0 \in \mathbb{H}^n$. Then, the function

$$V := \cosh d_b(p_0, \cdot)$$

is in $\mathcal{N}^1$.

A Riemannian manifold (M, g) is said to be *asymptotically hyperbolic* if there exist a compact subset $K \subset M$ and a diffeomorphism at infinity $\Phi : M \backslash K \to \mathbb{H}^n \backslash B$, where B is a closed ball in $\mathbb{H}^n$, for which $\Phi_* g$ and b are uniformly equivalent on $\mathbb{H}^n \backslash B$ and

$$\int_{\mathbb{H}^n \backslash B} \left(|e|^2 + |\nabla^b e|^2\right) \cosh r \, \mathrm{d}\mu^b < \infty, \tag{2a}$$

$$\int_{\mathbb{H}^n \backslash B} |\mathrm{Scal}^g + n(n-1)| \cosh r \, \mathrm{d}\mu^b < \infty, \tag{2b}$$

where $e := \Phi_* g - b$ and r is the (hyperbolic) distance from an arbitrary given point in $\mathbb{H}^n$.

The *mass functional* of (M, g) with respect to Φ is the functional on $\mathcal{N}$ defined by

$$H_\Phi(V) = \lim_{r \to \infty} \int_{S_r} \Big(V(\operatorname{div}^b e - \mathrm{d} \operatorname{tr}^b e) + (\operatorname{tr}^b e) \mathrm{d}V - e(\nabla^b V, \cdot) \Big) (\nu_r) \, \mathrm{d}\mu^b$$

Proposition 2.2 of [10] tells us that this limit exists and is finite under the asymptotic decay conditions (2a)–(2b). If Φ is a chart at infinity as above and A is an isometry of the hyperbolic metric b then $A \circ \Phi$ is again such a chart and it is not complicated to verify that

$$H_{A \circ \Phi}(V) = H_\Phi(V \circ A^{-1}).$$

If Φ_1, Φ_2 are charts at infinity as above, then from [17, Theorem 2.3] we know that there is an isometry A of b so that $\Phi_2 = A \circ \Phi_1$ modulo lower order terms which do not affect the mass functional.

The mass functional H_Φ is timelike future directed if $H_\Phi(V) > 0$ for all $V \in \mathcal{N}^+$. In this case, the *mass* of the asymptotically hyperbolic manifold (M, g) is defined by

$$m := \frac{1}{2(n-1)\omega_{n-1}} \inf_{\mathcal{N}^1} H_\Phi(V).$$

Further if H_Φ is timelike future directed, we may replace Φ by $A \circ \Phi$ for a suitably chosen isometry A so that $m = H_\Phi(V_{(0)})$. Coordinates with this property are called *balanced*.

The positive mass theorem for asymptotically hyperbolic manifolds [10, Theorem 4.1] and [31, Theorem 1.1] states that the mass functional is timelike future directed or zero for complete asymptotically hyperbolic spin manifolds with scalar curvature $\mathrm{Scal} \geq -n(n-1)$. In [1, Theorem 1.3], the same result is proved with the spin assumption replaced by assumptions on the dimension and on the geometry at infinity.

2.2. Asymptotically Hyperbolic Graphs

The purpose of this paper is to prove versions of the Riemannian Penrose inequality for an asymptotically hyperbolic graph over the hyperbolic space $\mathbb{H}^n$. We consider such a graph as a submanifold of $\mathbb{H}^{n+1}$. In what follows we will denote tensors associated to $\mathbb{H}^{n+1}$ with a bar. In particular, $\bar{b}$ will denote the hyperbolic metric on $\mathbb{H}^{n+1}$.

To shorten notation, we now fix

$$V = V_{(0)} = \cosh r$$

for the rest of the paper. As a model of $\mathbb{H}^{n+1}$ we take $\mathbb{H}^n \times \mathbb{R}$ equipped with the metric

$$\bar{b} := b + V^2 \mathrm{d}s \otimes \mathrm{d}s.$$

Let Ω be a relatively compact open subset and let $f : \mathbb{H}^n \backslash \Omega \to \mathbb{R}$ be a continuous function which is smooth on $\mathbb{H}^n \backslash \overline{\Omega}$. We consider the graph

$$\Sigma := \{(x, s) \in \mathbb{H}^n \times \mathbb{R} \mid f(x) = s\}.$$

Define the diffeomorphism $\Phi : \Sigma \to \mathbb{H}^n \backslash \Omega$ by $\Phi^{-1}(p) = (p, f(p))$. The push-forward of the metric induced on Σ is

$$g := \Phi_* \bar{b} = (\Phi^{-1})^* \bar{b} = b + V^2 df \otimes df.$$

We will study the situation when the graph Σ is asymptotically hyperbolic with respect to the chart Φ, that is when

$$e = V^2 df \otimes df$$

satisfies (2a)–(2b) and

$$|e| = V^2 |df|^2 \to 0 \text{ at infinity.} \tag{3}$$

Note that Condition (2a) is a consequence of the following condition:

$$\int_{\mathbb{H}^n \setminus B} \left(|df|^4 + |\mathrm{Hess}\, f|^4 \right) \cosh^5 r \, \mathrm{d}\mu^b < \infty,$$

that is to say that df belongs to a certain weighted Sobolev space.

If this holds we say that f is an *asymptotically hyperbolic function* and Σ is an *asymptotically hyperbolic graph*. We define f to be *balanced at infinity* if Φ are balanced coordinates at infinity. In this case, the mass of Σ is given by $m = H_\Phi(V)$ with $V = V_{(0)}$.

In this paper, we will prove the following theorem which gives estimates similar to the Penrose inequality for asymptotically hyperbolic graphs. In certain cases this theorem also describes the situation when equality is attained. For exact formulations see Theorems 4.2, 4.4, and 5.13.

Theorem 2.1. *Let $\Omega \subset \mathbb{H}^n$ be a relatively compact open subset of $\mathbb{H}^n$ with smooth boundary. Assume that Ω contains an inner ball centered at the origin of radius r_0. Let $f : \mathbb{H}^n \setminus \Omega \to \mathbb{R}$ be an asymptotically hyperbolic function which is balanced at infinity. Assume that f is locally constant on $\partial\Omega$ and that $|df| \to \infty$ at $\partial\Omega$ so that $\partial\Omega$ is a horizon ($H^g = 0$). Further assume that the scalar curvature of the graph of f satisfies* $\mathrm{Scal} \geq -n(n-1)$. *Then, the mass m of the graph is bounded from below as follows.*

- *If $\partial\Omega$ has non-negative mean curvature with respect to the metric b, $H \geq 0$, we have*

$$m \geq \frac{n-2}{2^n (n-1) n^{\frac{n}{n-1}}} V(r_0) \left(\frac{|\partial\Omega|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$$

and

$$m \geq \frac{1}{2} V(r_0) \frac{|\partial\Omega|}{\omega_{n-1}}.$$

- *If Ω is an h-convex subset of $\mathbb{H}^n$ we have*

$$m \geq \frac{1}{2} \left[\left(\frac{|\partial\Omega|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \sinh r_0 \frac{|\partial\Omega|}{\omega_{n-1}} \right].$$

If equality holds and $df(\eta)(x) \to +\infty$ as $x \to \partial\Omega$ where η is the outward normal of the level sets of f then the graph of f is isometric to the $t = 0$ slice of the anti-de Sitter Schwarzschild space–time.

Note that since f is locally constant on $\partial\Omega$, the areas of $\partial\Omega$ computed using the metric b and using the metric induced on the graph are equal.

2.3. The Anti-de Sitter Schwarzschild Space–Time

We remind the reader that the metric outside the horizon of the anti-de Sitter-Schwarzschild space in (spatial) dimension $n \geq 3$ and of mass $m \geq 0$ is given by

$$\gamma_{\text{AdS-Schw}} = -\left(1 + \rho^2 - \frac{2m}{\rho^{n-2}}\right) \mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2 \sigma,$$

where σ is the standard round metric on the sphere S^{n-1}. See for example [23, Section 6]. The horizon is the surface $\rho = \rho_0(m)$, where $\rho_0 = \rho_0(m)$ is the unique solution of

$$1 + \rho^2 - \frac{2m}{\rho^{n-2}} = 0.$$

Its area is given by $A_m = \omega_{n-1}\rho_0^{n-1}$, hence multiplying the previous formula by ρ_0^{n-2}, we get

$$\begin{aligned} m &= \frac{1}{2}\left[\rho_0^{n-2} + \rho_0^n\right] \\ &= \frac{1}{2}\left[\left(\frac{A_m}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \left(\frac{A_m}{\omega_{n-1}}\right)^{\frac{n}{n-1}}\right]. \end{aligned}$$

This justifies the form of the right-hand side of (1).

Restricting to the slice $t = 0$, we get the following Riemannian metric:

$$g_{\text{AdS-Schw}} = \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2\sigma. \tag{4}$$

We want to express the spatial metric (4) as the induced metric of a graph $\Sigma_{\text{AdS-Schw}}$. By rotational symmetry, we choose the point $\rho = 0$ as the origin and $f = f(\rho)$. In this coordinate system, the reference hyperbolic metric b is given by

$$b = \frac{d\rho^2}{1 + \rho^2} + \rho^2\sigma.$$

The function V is given by $V = \sqrt{1 + \rho^2}$. Hence we seek a function f satisfying

$$V^2\left(\frac{\partial f}{\partial \rho}\right)^2 = \frac{1}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} - \frac{1}{1 + \rho^2}.$$

Note that when ρ is close to ρ_0, this forces $\frac{\partial f}{\partial \rho} = O((\rho - \rho_0)^{-\frac{1}{2}})$. Hence we can set

$$f(\rho) = \int_{\rho_0}^{\rho} \frac{1}{\sqrt{1 + s^2}}\sqrt{\frac{1}{1 + s^2 - \frac{2m}{s^{n-2}}} - \frac{1}{1 + s^2}}\, ds. \tag{5}$$

Similarly, when $\rho \to \infty, f$ converges to a constant. This contrasts with the Euclidean case where the Schwarzschild metric written as a graph is a half parabola in any radial direction, see [22].

3. Scalar Curvature of Graphs in $\mathbb{H}^{n+1}$

3.1. Computation of Scalar Curvature

Let $f : \mathbb{H}^n \backslash \overline{\Omega} \to \mathbb{R}$ be a smooth function. Recall that we defined its graph as

$$\Sigma := \{(x, s) \in \mathbb{H}^n \times \mathbb{R} \mid f(x) = s\} = F^{-1}(0),$$

where $F(x,s) := f(x) - s$. For vector fields X and Y on $\mathbb{H}^n$ the vector fields $\overline{X} = X + \nabla_X f \partial_0$ and $\overline{Y} = Y + \nabla_Y f \partial_0$ are tangent to Σ. We use coordinates on $\mathbb{H}^n$ to parametrize Σ.

Recall that we identify $\mathbb{H}^{n+1}$ with $\mathbb{H}^n \times \mathbb{R}$ with the metric $\overline{b} = b + V^2 \mathrm{d}s \otimes \mathrm{d}s$. When using coordinate notation, latin indices $i, j, \ldots \in \{1, \ldots, n\}$ denote (any) coordinates on $\mathbb{H}^n$ while a zero index denotes the s-coordinate on $\mathbb{R}$. Greek indices go from 0 to n, hence denote coordinates on $\mathbb{H}^{n+1}$. The Christoffel symbols of $\overline{b}$ are collected in the following Lemma.

Lemma 3.1.

$$\begin{cases} \overline{\Gamma}^0_{00} = 0 \\ \overline{\Gamma}^i_{00} = -V\nabla^i V \\ \overline{\Gamma}^0_{i0} = \dfrac{\nabla_i V}{V} \\ \overline{\Gamma}^i_{j0} = 0 \\ \overline{\Gamma}^0_{ij} = 0 \\ \overline{\Gamma}^k_{ij} = \Gamma^k_{ij} \quad (\textit{Christoffel symbols of}\ \mathbb{H}^n). \end{cases}$$

The induced metric on Σ is given by

$$g(X,Y) = \overline{b}(\overline{X}, \overline{Y}) = b(X,Y) + V^2 \nabla_X f \nabla_Y f.$$

The second fundamental form $\overline{S}$ of Σ is given by

$$\begin{aligned} \overline{S}(\overline{X}, \overline{Y}) &= \frac{1}{|\overline{\nabla} F|} \overline{\nabla}^2_{\overline{X}, \overline{Y}} F \\ &= \frac{1}{|\overline{\nabla} F|} \Big[\overline{\nabla}^2_{X,Y} F + \nabla_X f \overline{\nabla}^2_{\partial_0, Y} F + \nabla_Y f \overline{\nabla}^2_{X, \partial_0} F \\ &\qquad + \nabla_X f \nabla_Y f \overline{\nabla}^2_{\partial_0, \partial_0} F \Big] \\ &= \frac{1}{\sqrt{V^{-2} + |df|^2}} \Big[\nabla^2_{X,Y} f + \frac{\nabla_X f \nabla_Y V + \nabla_X V \nabla_Y f}{V} \\ &\qquad + V \langle df, \mathrm{d}V \rangle \nabla_X f \nabla_Y f \Big]. \end{aligned}$$

Using component notation, we get

$$\overline{S}_{ij} = \frac{V}{\sqrt{1 + V^2 |df|^2}} \left[\nabla^2_{i,j} f + \frac{\nabla_i f \nabla_j V + \nabla_i V \nabla_j f}{V} + V \langle df, \mathrm{d}V \rangle \nabla_i f \nabla_j f \right].$$

The metric g and its inverse are given by

$$\begin{aligned} g_{ij} &= b_{ij} + V^2 \nabla_i f \nabla_j f, \\ g^{ij} &= b^{ij} - \frac{V^2 \nabla^i f \nabla^j f}{1 + V^2 |df|^2}. \end{aligned}$$

We compute the mean curvature of Σ,

$$\begin{aligned}\overline{H} &= g^{ij}S_{ij}\\ &= \frac{1}{|\overline{\nabla}F|}\left(b^{ij}-\frac{V^2\nabla^i f\nabla^j f}{1+V^2|df|^2}\right)\\ &\quad\times\left[\nabla^2_{i,j}f+\frac{\nabla_i f\nabla_j V+\nabla_i V\nabla_j f}{V}+V\langle df,\mathrm{d}V\rangle\nabla_i f\nabla_j f\right]\\ &= \frac{1}{|\overline{\nabla}F|}\left[\Delta f+2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle+V\langle df,\mathrm{d}V\rangle|df|^2\right.\\ &\quad-\frac{V^2}{1+V^2|df|^2}\left(\langle \mathrm{Hess}\, f, df\otimes df\rangle+2|df|^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right.\\ &\quad\left.\left.+V^2|df|^4\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right)\right]\\ &= \frac{1}{|\overline{\nabla}F|}\left[\Delta f-\frac{V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle}{1+V^2|df|^2}+\frac{2+V^2|df|^2}{1+V^2|df|^2}\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right],\end{aligned}$$

or

$$\overline{H}=\frac{1}{|\overline{\nabla}F|}\left[\Delta f-\frac{V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle}{1+V^2|df|^2}+\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right].$$

The norm of the second fundamental form of Σ is given by

$$\begin{aligned}|\overline{S}|^2_g &= g^{ik}g^{jl}\overline{S}_{ij}\overline{S}_{kl}\\ &= \left(b^{ik}-\frac{V^2\nabla^i f\nabla^k f}{1+V^2|df|^2}\right)\left(b^{jl}-\frac{V^2\nabla^j f\nabla^l f}{1+V^2|df|^2}\right)\overline{S}_{ij}\overline{S}_{kl}\\ &= b^{ik}b^{jl}\overline{S}_{ij}\overline{S}_{kl}-2\frac{V^2b^{ik}\nabla^j f\nabla^l f}{1+V^2|df|^2}\overline{S}_{ij}\overline{S}_{kl}+\frac{V^4\nabla^i f\nabla^j f\nabla^k f\nabla^l f}{(1+V^2|df|^2)^2}\overline{S}_{ij}\overline{S}_{kl}\\ &= \underbrace{|\overline{S}|^2_b}_{(A)}\underbrace{-2\frac{V^2b^{ik}\nabla^j f\nabla^l f}{1+V^2|df|^2}\overline{S}_{ij}\overline{S}_{kl}}_{(B)}+\underbrace{\left(\frac{V^2\overline{S}(\nabla f,\nabla f)}{1+V^2|df|^2}\right)^2}_{(C)}.\end{aligned}$$

We compute each term separately. First

$$\begin{aligned}(A) &= |\overline{S}|^2_b\\ &= \frac{V^2}{1+V^2|df|^2}\left[|\mathrm{Hess}\, f|^2+2|df|^2\left|\frac{\mathrm{d}V}{V}\right|^2+2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle^2\right.\\ &\quad+V^4|df|^4\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle^2+4\left\langle \mathrm{Hess}\, f, df\otimes\frac{\mathrm{d}V}{V}\right\rangle\\ &\quad\left.+2V^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\langle \mathrm{Hess}\, f, df\otimes df\rangle+4V^2|df|^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle^2\right].\end{aligned}$$

Next,

$$
\begin{aligned}
(B) &= -2\frac{V^2 b^{ik}}{1+V^2|df|^2}\nabla^j f\overline{S}_{ij}\nabla^l f\overline{S}_{kl}\\
&= -2\frac{V^4}{1+V^2|df|^2}\left|\overline{S}(\nabla f,\cdot)\right|^2\\
&= -2\frac{V^4}{(1+V^2|df|^2)^2}\\
&\quad\times\left|\mathrm{Hess}\, f(\nabla f,\cdot)+(1+V^2|df|^2)\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle df+|df|^2\frac{\mathrm{d}V}{V}\right|^2\\
&= -2\frac{V^4}{(1+V^2|df|^2)^2}\\
&\quad\times\left[\left|\mathrm{Hess}\, f(\nabla f,\cdot)\right|^2+(1+V^2|df|^2)^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle^2|df|^2+|df|^4\left|\frac{\mathrm{d}V}{V}\right|^2\right.\\
&\quad+2(1+V^2|df|^2)\,\mathrm{Hess}\, f(\nabla f,\nabla f)\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\\
&\quad+2|df|^2\left\langle \mathrm{Hess}\, f,\nabla f\otimes\frac{\nabla V}{V}\right\rangle\\
&\quad\left.+\,2(1+V^2|df|^2)|df|^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle^2\right],
\end{aligned}
$$

and finally

$$
\begin{aligned}
(C) &= \left(\frac{V^2\overline{S}(\nabla f,\nabla f)}{1+V^2|df|^2}\right)^2\\
&= \frac{V^6}{(1+V^2|df|^2)^3}\\
&\quad\times\left[\nabla^i f\nabla^j f\nabla^2_{i,j}f+2|df|^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle+|df|^4V^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right]^2\\
&= \frac{V^6}{(1+V^2|df|^2)^3}\left[\nabla^i f\nabla^j f\nabla^2_{i,j}f+(2+V^2|df|^2)|df|^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right]^2\\
&= \frac{V^2}{1+V^2|df|^2}\left[\frac{V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle}{1+V^2|df|^2}\right.\\
&\quad\left.+\left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df,\frac{\mathrm{d}V}{V}\right\rangle\right]^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\overline{H}^2 - |\overline{S}|_g^2\\
&= \frac{V^2}{1+V^2|df|^2}\left(\left[\Delta f - \frac{V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle}{1+V^2|df|^2} + \left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\right]^2\right.\\
&\quad - \left[\frac{V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle}{1+V^2|df|^2} + \left(1+\frac{1}{1+V^2|df|^2}\right)V^2|df|^2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\right]^2\\
&\quad - |\mathrm{Hess}\, f|^2 - 2|df|^2\left|\frac{\mathrm{d}V}{V}\right|^2 - 2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2 - V^4|df|^4\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2\\
&\quad - 4\left\langle \mathrm{Hess}\, f, df\otimes\frac{\mathrm{d}V}{V}\right\rangle - 2V^2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\langle \mathrm{Hess}\, f, df\otimes df\rangle\\
&\quad - 4V^2|df|^2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2\\
&\quad + 2\frac{V^2}{1+V^2|df|^2}\left[|\mathrm{Hess}\, f(\nabla f,\cdot)|^2 + (1+V^2|df|^2)^2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2|df|^2\right.\\
&\quad + |df|^4\left|\frac{\mathrm{d}V}{V}\right|^2\\
&\quad + 2(1+V^2|df|^2)\,\mathrm{Hess}\, f(\nabla f,\nabla f)\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\\
&\quad + 2|df|^2\left\langle \mathrm{Hess}\, f, \nabla f\otimes\frac{\nabla V}{V}\right\rangle\\
&\quad \left.\left. + 2(1+V^2|df|^2)|df|^2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2\right]\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\overline{H}^2 - |\overline{S}|_g^2\\
&= \frac{V^2}{1+V^2|df|^2}\left(\left[\Delta f + (2+V^2|df|^2)\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\right]\right.\\
&\quad \times\left[\Delta f - \frac{2V^2}{1+V^2|df|^2}\langle \mathrm{Hess}\, f, df\otimes df\rangle\right.\\
&\quad \left. +(1-V^2|df|^2)\left(1+\frac{1}{1+V^2|df|^2}\right)\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\right]\\
&\quad - |\mathrm{Hess}\, f|^2 + 2\frac{V^2}{1+V^2|df|^2}|\mathrm{Hess}\, f(\nabla f,\cdot)|^2 - \frac{2}{1+V^2|df|^2}|df|^2\left|\frac{\mathrm{d}V}{V}\right|^2\\
&\quad + \left(-2+2V^2|df|^2+V^4|df|^4\right)\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2\\
&\quad \left. - \frac{4}{1+V^2|df|^2}\left\langle \mathrm{Hess}\, f, \nabla f\otimes\frac{\nabla V}{V}\right\rangle + 2V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{V^2}{1+V^2|df|^2}\Big[(\Delta f)^2 - |\mathrm{Hess}\, f|^2 \\
&\quad + 2\frac{V^2}{1+V^2|df|^2}\left(|\mathrm{Hess}\, f(\nabla f,\cdot)|^2 - \Delta f\, \langle \mathrm{Hess}\, f, df\otimes df\rangle\right) \\
&\quad + \left(2 + \frac{2}{1+V^2|df|^2}\right)\Delta f \left\langle df, \frac{\mathrm{d}V}{V}\right\rangle \\
&\quad - \frac{2V^2}{1+V^2|df|^2}\langle \mathrm{Hess}\, f, df\otimes df\rangle \left\langle df, \frac{\mathrm{d}V}{V}\right\rangle \\
&\quad + \frac{2}{1+V^2|df|^2}\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle^2 - \frac{2}{1+V^2|df|^2}|df|^2\left|\frac{\mathrm{d}V}{V}\right|^2 \\
&\quad - \frac{4}{1+V^2|df|^2}\left\langle \mathrm{Hess}\, f, df\otimes \frac{\mathrm{d}V}{V}\right\rangle\Big].
\end{aligned}
$$

By taking the trace of the Gauss equation for Σ, we finally arrive at the following formula for the scalar curvature Scal of Σ

$$
\begin{aligned}
&\mathrm{Scal} + n(n-1) \\
&\quad = \overline{H}^2 - |\overline{S}|_g^2 \\
&\quad = \frac{V^2}{1+V^2|df|^2}\Big[(\Delta f)^2 - |\mathrm{Hess}\, f|^2 \\
&\qquad +2\frac{V^2}{1+V^2|df|^2}\left(|\mathrm{Hess}\, f(\nabla f,\cdot)|^2 - \Delta f\, \langle \mathrm{Hess}\, f, df\otimes df\rangle\right) \\
&\qquad +\frac{2}{1+V^2|df|^2}\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\left(\Delta f - V^2\langle \mathrm{Hess}\, f, df\otimes df\rangle + \left\langle df, \frac{\mathrm{d}V}{V}\right\rangle\right) \\
&\qquad +2\left\langle df, \frac{\mathrm{d}V}{V}\right\rangle \Delta f \\
&\qquad -\frac{2}{1+V^2|df|^2}|df|^2\left|\frac{\mathrm{d}V}{V}\right|^2 - \frac{4}{1+V^2|df|^2}\left\langle \mathrm{Hess}\, f, df\otimes \frac{\mathrm{d}V}{V}\right\rangle\Big]. \qquad (6)
\end{aligned}
$$

In view of [22, Proof of Theorem 5] and [17, Definition 3.3], we compute

$$
\mathrm{div}^b\left[\frac{1}{1+V^2|df|^2}\left(V\,\mathrm{div}^b\, e - V\,\mathrm{d}\,\mathrm{tr}^b\, e - e(\nabla V,\cdot) + (\mathrm{tr}^b\, e)\mathrm{d}V\right)\right]
$$

with $e = V^2 df\otimes df$. We have

$$
\begin{aligned}
&V\,\mathrm{div}^b\, e - V\,\mathrm{d}\,\mathrm{tr}^b\, e - e(\nabla V,\cdot) + (\mathrm{tr}^b\, e)\mathrm{d}V \\
&\quad = 2V^2\langle df, \mathrm{d}V\rangle\, df + V^3\Delta f\, df + V^3\langle \mathrm{Hess}\, f, df\otimes\cdot\rangle \\
&\qquad - V\,\mathrm{d}\,\mathrm{tr}^b(V^2|df|^2) - V^2\langle df, \mathrm{d}V\rangle\, df + V^2|df|^2\mathrm{d}V \\
&\quad = V^3\Delta f\, df - V^3\langle \mathrm{Hess}\, f, df\otimes\cdot\rangle - V^2|df|^2\mathrm{d}V + V^2\langle df, \mathrm{d}V\rangle\, df
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{div}^b\left(V\operatorname{div}^b e - V\,\mathrm{d}\operatorname{tr}^b e - e(\nabla V,\cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right)\\
&= \operatorname{div}^b\left(V^3\Delta f df - V^3\langle \operatorname{Hess} f, df\otimes\cdot\rangle - V^2|df|^2\mathrm{d}V + V^2\left\langle df,\mathrm{d}V\right\rangle df\right)\\
&= 3V^2\Delta f\langle df,\mathrm{d}V\rangle + V^3\langle d\Delta f, df\rangle + V^3(\Delta f)^2\\
&\quad - 3V^2\langle \operatorname{Hess} f, df\otimes \mathrm{d}V\rangle - V^3\langle \operatorname{div}^b \operatorname{Hess} f, df\rangle - V^3|\operatorname{Hess} f|^2\\
&\quad - 2V|df|^2|\mathrm{d}V|^2 - 2V^2\langle \operatorname{Hess} f, df\otimes\mathrm{d}V\rangle - V^2|df|^2\Delta V\\
&\quad + 2V\langle df,\mathrm{d}V\rangle^2 + V^2\langle \operatorname{Hess} f, \mathrm{d}V\otimes df\rangle + V^2\langle df\otimes df, \operatorname{Hess} V\rangle\\
&\quad + V^2\langle df,\mathrm{d}V\rangle\Delta f\\
&= V^3\left[(\Delta f)^2 - |\operatorname{Hess} f|^2\right] - 4V^2\langle \operatorname{Hess} f, df\otimes\mathrm{d}V\rangle + 4V^2\langle df,\mathrm{d}V\rangle\Delta f\\
&\quad + V^3\langle d\Delta f - \operatorname{div}^b\operatorname{Hess} f, df\rangle - (n-1)V^3|df|^2\\
&\quad + 2V\langle df,\mathrm{d}V\rangle^2 - 2V|df|^2|\mathrm{d}V|^2\\
&= V^3\left[(\Delta f)^2 - |\operatorname{Hess} f|^2\right] - 4V^2\langle \operatorname{Hess} f, df\otimes\mathrm{d}V\rangle + 4V^2\langle df,\mathrm{d}V\rangle\Delta f\\
&\quad + 2V\langle df,\mathrm{d}V\rangle^2 - 2V|df|^2|\mathrm{d}V|^2.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\left\langle d\left(\frac{1}{1+V^2|df|^2}\right), V\operatorname{div}^b e - V\,\mathrm{d}\operatorname{tr}^b e - e(\nabla V,\cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right\rangle\\
&\quad = \Big\langle \frac{-2V|df|^2\mathrm{d}V - 2V^2\langle \operatorname{Hess} f, df\otimes\cdot\rangle}{(1+V^2|df|^2)^2}, V^3\Delta f df - V^3\langle \operatorname{Hess} s, df\otimes\cdot\rangle\\
&\qquad - V^2|df|^2\mathrm{d}V + V^2\left\langle df,\mathrm{d}V\right\rangle df\Big\rangle\\
&\quad = \frac{-2}{(1+V^2|df|^2)^2}\Big[V^4\Delta f\langle df,\mathrm{d}V\rangle - 2|df|^2V^4\langle \operatorname{Hess} f, df\otimes\mathrm{d}V\rangle\\
&\qquad - V^3|df|^4|\mathrm{d}V|^2 + V^3|df|^2\langle df,\mathrm{d}V\rangle^2\\
&\qquad + V^5\Delta f\langle \operatorname{Hess} f, df\otimes df\rangle - V^5|\langle \operatorname{Hess} f, df\otimes\cdot\rangle|^2\\
&\qquad + V^4\langle df,\mathrm{d}V\rangle\langle \operatorname{Hess} f, df\otimes df\rangle\Big],
\end{aligned}
$$

so

$$
\begin{aligned}
&\operatorname{div}^b\left[\frac{1}{1+V^2|df|^2}\left(V\operatorname{div}^b e - V\,\mathrm{d}\operatorname{tr}^b e - e(\nabla V,\cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right)\right]\\
&= \frac{1}{1+V^2|df|^2}\left[V^3\left((\Delta f)^2 - |\operatorname{Hess} f|^2\right) - 4V^2\langle \operatorname{Hess} f, df\otimes\mathrm{d}V\rangle\right.\\
&\quad \left.+4V^2\langle df,\mathrm{d}V\rangle\Delta f + 2V\langle df,\mathrm{d}V\rangle^2 - 2V|df|^2|\mathrm{d}V|^2\right]\\
&\quad -\frac{2}{(1+V^2|df|^2)^2}\left[V^4\Delta f\langle df,\mathrm{d}V\rangle - |df|^2V^4\langle \operatorname{Hess} f, df\otimes\mathrm{d}V\rangle\right.\\
&\quad -V^3|df|^4|\mathrm{d}V|^2 + V^3|df|^2\langle df,\mathrm{d}V\rangle^2\\
&\quad +V^5\Delta f\langle \operatorname{Hess} f, df\otimes df\rangle - V^5|\langle \operatorname{Hess} f, df\otimes\cdot\rangle|^2\\
&\quad \left.+V^4\langle df,\mathrm{d}V\rangle\langle \operatorname{Hess} f, df\otimes df\rangle\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{1+V^2|df|^2}\Bigg[V^3\left((\Delta f)^2 - |\operatorname{Hess} f|^2\right)\\
&\quad - \frac{2}{1+V^2|df|^2}\left(V^5\Delta f\langle \operatorname{Hess} f, df\otimes df\rangle - V^5|\langle \operatorname{Hess} f, df\otimes \cdot\rangle|^2\right)\\
&\quad - \frac{4V^2}{1+V^2|df|^2}\langle \operatorname{Hess} f, df\otimes \mathrm{d}V\rangle + \frac{2V}{1+V^2|df|^2}\left(\langle df, \mathrm{d}V\rangle^2 - |df|^2|\mathrm{d}V|^2\right)\\
&\quad - \frac{2V^4}{1+V^2|df|^2}\langle df, \mathrm{d}V\rangle\langle \operatorname{Hess} f, df\otimes df\rangle\\
&\quad + \left(2+\frac{1}{1+V^2|df|^2}\right)\Delta f|df|^2\langle df, \mathrm{d}V\rangle\Bigg].
\end{aligned}
$$

Comparing this formula with Eq. (6) we get

$$
\begin{aligned}
&V\left(\mathrm{Scal} + n(n-1)\right)\\
&\quad = \operatorname{div}^b\left[\frac{1}{1+V^2|df|^2}\left(V\operatorname{div}^b e - V\mathrm{d}\operatorname{tr}^b e - e(\nabla V,\cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right)\right], \qquad (7)
\end{aligned}
$$

where $e = V^2 df\otimes df$.

3.2. A Mass Formula

We now integrate Formula (7) from the previous section over an outer domain under the additional condition that f is locally constant on the boundary.

Lemma 3.2. *Let $\Omega\subset\mathbb{H}^n$ be a relatively compact open subset of $\mathbb{H}^n$ with smooth boundary. Let $f:\mathbb{H}^n\backslash\Omega\to\mathbb{R}$ be an asymptotically hyperbolic function which is locally constant on $\partial\Omega$ and such that $df\neq 0$ at every point of $\partial\Omega$. Then*

$$
H_\Phi(V) = \int\limits_{\mathbb{H}^n\backslash\Omega}\frac{V[\mathrm{Scal}+n(n-1)]}{\sqrt{1+V^2|df|^2}}\,\mathrm{d}\mu^g + \int\limits_{\partial\Omega} HV\frac{V^2|df|^2}{1+V^2|df|^2}\,\mathrm{d}\mu^b. \qquad (8)
$$

Here H is the mean curvature of $\partial\Omega$ with respect to the metric b.

Proof. Let ν denote the outgoing unit normal to $\partial\Omega$ and let $\nu_r = \partial_r$ be the normal to the spheres of constant r. From Formula (7), we get

$$
\begin{aligned}
&\int\limits_{\mathbb{H}^n\backslash\Omega} V\left(\mathrm{Scal}+n(n-1)\right)\,\mathrm{d}\mu^b\\
&\quad = \lim_{r\to\infty}\int\limits_{B_r(0)\backslash\Omega} V\left(\mathrm{Scal}+n(n-1)\right)\mathrm{d}\mu^b\\
&\quad = \lim_{r\to\infty}\int\limits_{S_r(0)}\frac{1}{1+V^2|df|^2}\left(V\operatorname{div}^b e - V\mathrm{d}\operatorname{tr}^b e - e(\nabla V,\cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right)(\nu_r)\mathrm{d}\mu^b\\
&\qquad - \int\limits_{\partial\Omega}\frac{1}{1+V^2|df|^2}\left(V\operatorname{div}^b e - V\mathrm{d}\operatorname{tr}^b e - e(\nabla V,\cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right)(\nu)\mathrm{d}\mu^b
\end{aligned}
$$

$$= H_\Phi(V)$$
$$-\int_{\partial\Omega} \frac{1}{1+V^2|df|^2}\left(V \operatorname{div}^b e - V \mathrm{d}\operatorname{tr}^b e - e(\nabla V, \cdot) + (\operatorname{tr}^b e)\mathrm{d}V\right)(\nu)\mathrm{d}\mu^b.$$

Here we used that $e = V^2 df \otimes df$ satisfies (3) to replace the factor $\frac{1}{1+V^2|df|^2}$ by 1 in the limit of the outer boundary integral. We next compute the integral over $\partial\Omega$. We will do the calculations assuming that $\nu = \frac{\nabla f}{|\nabla f|}$, the case $\nu = -\frac{\nabla f}{|\nabla f|}$ is similar. The last two terms are

$$-e(\nabla V, \nu) + (\operatorname{tr}^b e)\mathrm{d}V(\nu) = -V^2\langle df, \mathrm{d}V\rangle\langle df, \nu\rangle + V^2|df|^2\langle \mathrm{d}V, \nu\rangle = 0,$$

and the first two give

$$\begin{aligned}
&V \operatorname{div}^b e(\nu) - V d \operatorname{tr}^b e(\nu)\\
&\quad = 2V^2\langle df, \mathrm{d}V\rangle df(\nu) + V^3(\Delta f) df(\nu) + V^3 \operatorname{Hess} f(\nabla f, \nu)\\
&\qquad - 2V^2|df|^2 \mathrm{d}V(\nu) - 2V^3 \operatorname{Hess} f(\nabla f, \nu)\\
&\quad = V^3(\Delta f) df(\nu) - V^3 \operatorname{Hess} f(\nabla f, \nu).
\end{aligned}$$

We next use the following formula for the Laplacian of f on $\partial\Omega$,

$$\Delta f = \Delta^{\partial\Omega} f + \operatorname{Hess} f(\nu, \nu) + H df(\nu).$$

Since f is locally constant on $\partial\Omega$ we obtain

$$V \operatorname{div}^b e(\nu) - V d \operatorname{tr}^b e(\nu) = V^3 H df(\nu)^2 = V^3 H |df|^2.$$

Hence,

$$\int_{\mathbb{H}^n\setminus\Omega} V\left(\operatorname{Scal} + n(n-1)\right) \mathrm{d}\mu^b = H_\Phi(V) - \int_{\partial\Omega} VH \frac{V^2|df|^2}{1+V^2|df|^2}\,\mathrm{d}\mu^b.$$

It then suffices to note that $\mathrm{d}\mu^g = \sqrt{1+V^2|df|^2}\mathrm{d}\mu^b$ to prove Formula (8). □

4. Penrose Type Inequalities

4.1. Horizon Boundary

From now on, we assume that $|df| \to \infty$ at $\partial\Omega$, it then follows that the boundary is a minimal hypersurface, or a horizon. This can be seen by taking the double over the boundary of the graph of f. The double is then a C^1 Riemannian manifold for which the original boundary is the fixed point set of a reflection, and thus the boundary is minimal.

From Lemma 3.2, we conclude the following proposition.

Proposition 4.1. *Let $\Omega \subset \mathbb{H}^n$ be a relatively compact open subset of $\mathbb{H}^n$ with smooth boundary. Let $f : \mathbb{H}^n\setminus\Omega \to \mathbb{R}$ be an asymptotically hyperbolic function which is locally constant on $\partial\Omega$ and such that $|df| \to \infty$ at $\partial\Omega$. Further assume that* $\operatorname{Scal} \geq -n(n-1)$. *Then*

$$H_\Phi(V) \geq \int_{\partial\Omega} VH\,\mathrm{d}\mu^b. \tag{9}$$

Applying the Hoffman–Spruck inequality or the Minkowski formula we get estimates of the boundary term in (9) and conclude the following Theorem.

Theorem 4.2. *Let $\Omega \subset \mathbb{H}^n$ be a relatively compact open subset of $\mathbb{H}^n$ with smooth boundary. Assume that Ω contains an inner ball centered at the origin of radius r_0. Let $f : \mathbb{H}^n \backslash \Omega \to \mathbb{R}$ be an asymptotically hyperbolic function which is locally constant on $\partial\Omega$ and such that $|df| \to \infty$ at $\partial\Omega$. Further assume that* $\mathrm{Scal} \geq -n(n-1)$ *and that $\partial\Omega$ has non-negative mean curvature $H \geq 0$. Then*

$$H_\Phi(V) \geq \frac{n-2}{2^{n-1} n^{\frac{n}{n-1}}} V(r_0)\omega_{n-1} \left(\frac{|\partial\Omega|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \tag{10}$$

and

$$H_\Phi(V) \geq (n-1)V(r_0)|\partial\Omega|. \tag{11}$$

Proof. The Hoffman–Spruck inequality, [18,26,30], applied to a compact hypersurface M of hyperbolic space $\mathbb{H}^n$ tells us that

$$\left(\int\limits_M h^{\frac{n-1}{n-2}} \, \mathrm{d}\mu^b \right)^{\frac{n-2}{n-1}} \leq C_n \int\limits_M (|dh| + h|H|) \, \mathrm{d}\mu^b \tag{12}$$

for any smooth non-negative function h on M. Here

$$C_n = 2^{n-1} \frac{n}{n-2} \left(\frac{n}{\omega_{n-1}} \right)^{\frac{1}{n-1}}.$$

Setting $h \equiv 1$ and $M = \partial\Omega$ in (12) yields (10).

The estimate (11) follows from the Minkowski formula in hyperbolic space, see [24, Equation (4')] with the point $a = (1, 0, \ldots, 0)$ (note that in the cited article the mean curvature is defined as an average and not a sum). $\square$

Neither of the inequalities (10) and (11) is optimal, so we do not get a characterization of the case of equality in the corresponding Penrose type inequalities.

4.2. Changing to the Euclidean Metric

We will now find an estimate of the boundary term in (9) by changing to the Euclidean metric $\widetilde{b} := b + \mathrm{d}V \otimes \mathrm{d}V$. In the hyperboloid model of hyperbolic space, this transformation can be viewed as the vertical projection of $\mathbb{H}^n$ onto $\mathbb{R}^n \subset \mathbb{R}^{n,1}$.

Lemma 4.3. *Let ν be the outgoing unit normal to $\partial\Omega$. The second fundamental form of $\partial\Omega$ with respect to the metric $\widetilde{b}$ is given by*

$$\widetilde{S}_{ij} = \frac{V}{\sqrt{V^2 - \langle \mathrm{d}V, \nu \rangle^2}} \left(S_{ij} - \frac{\nabla^k V \nabla_k \psi}{V} b_{ij} \right),$$

where ψ is a defining function for $\partial\Omega$ such that $\nabla\psi = \nu$. Further, we have

$$\int_{\partial\Omega} HV \,\mathrm{d}\mu = \int_{\partial\Omega} \widetilde{H}\,\mathrm{d}\widetilde{\mu} + (n-1)\int_{\partial\Omega} \langle \mathrm{d}V, \nu\rangle \,\mathrm{d}\mu$$
$$+ \int_{\partial\Omega} \frac{1}{1+|\nabla^T V|^2}\left(S(\nabla^T V, \nabla^T V)V - |\nabla^T V|^2 \langle \mathrm{d}V, \nu\rangle\right) \mathrm{d}\mu, \quad (13)$$

where $\nabla^T V$ is the gradient of V for the metric induced by b on $\partial\Omega$.

Proof. The second fundamental form of $\partial\Omega$ with respect to the metric $\widetilde{b}$ is given by

$$\widetilde{S}_{ij} = \frac{1}{|\mathrm{d}\psi|_{\widetilde{b}}} \widetilde{\nabla}^2_{i,j}\psi.$$

We compute

$$\widetilde{\nabla}^2_{i,j}\psi - \nabla^2_{i,j}\psi = \left(\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}\right)\partial_k\psi.$$

At the center point p_0 of normal coordinates for the metric b, the difference between the two Christoffel symbols is given by

$$\begin{aligned}
\widetilde{\Gamma}^k_{ij} &- \Gamma^k_{ij} \\
&= \frac{1}{2}\widetilde{b}^{kl}\left(\nabla_i \widetilde{b}_{lj} + \nabla_j \widetilde{b}_{il} - \nabla_l \widetilde{b}_{ij}\right) \\
&= \frac{1}{2}\left(b^{kl} - \frac{\nabla^k V \nabla^l V}{1+|\mathrm{d}V|^2}\right)\left[\nabla_i(\nabla_l V \nabla_j V) + \nabla_j(\nabla_i V \nabla_l V) - \nabla_l(\nabla_i V \nabla_j V)\right] \\
&= \left(b^{kl} - \frac{\nabla^k V \nabla^l V}{1+|\mathrm{d}V|^2}\right)\nabla_l V \nabla^2_{i,j} V \\
&= \frac{\nabla^k V}{1+|\mathrm{d}V|^2}\nabla^2_{i,j} V \\
&= \frac{\nabla^k V}{V} b_{ij},
\end{aligned}$$

where we used that $\mathrm{Hess}^b V = Vb$ and $1 + |\mathrm{d}V|^2 = V^2$ in the last line. Further, we have

$$|\mathrm{d}\psi|_{\widetilde{b}} = \sqrt{1 - \frac{\langle \mathrm{d}V, \nu\rangle^2}{1+|\mathrm{d}V|^2}} = \frac{1}{V}\sqrt{V^2 - \langle \mathrm{d}V, \nu\rangle^2}.$$

Hence,

$$\begin{aligned}
\widetilde{S}_{ij} &= \frac{V}{\sqrt{V^2 - \langle \mathrm{d}V, \nu\rangle^2}}\left(\nabla^2_{i,j}\psi - \frac{\nabla^k V \nabla_k \psi}{V} b_{ij}\right) \\
&= \frac{V}{\sqrt{V^2 - \langle \mathrm{d}V, \nu\rangle^2}}\left(S_{ij} - \frac{\nabla^k V \nabla_k \psi}{V} b_{ij}\right).
\end{aligned}$$

We take the trace of this formula with respect to the metric $\widetilde{b}$. For this, we select an orthogonal basis $(e_1, \ldots, e_{n-1})$ of $T_{p_0}\partial\Omega$ for the metric b such that

$e_k \in \ker \mathrm{d}V$ for $k \geq 2$. An orthogonal basis for the metric $\widetilde{b}$ is then given by

$$\begin{aligned}\widetilde{e}_1 &= \frac{1}{\sqrt{1+(\nabla_{e_1}V)^2}} e_1,\\ \widetilde{e}_k &= e_k \qquad \text{for } k \geq 2.\end{aligned}$$

Thus, we find

$$\begin{aligned}\widetilde{H} &= \sum_{k=1}^{n-1} \widetilde{S}(\widetilde{e}_k, \widetilde{e}_k)\\ &= \sum_{k=1}^{n-1} \widetilde{S}(e_k, e_k) - \left(1 - \frac{1}{1+(\nabla_{e_1}V)^2}\right) \widetilde{S}(e_1, e_1)\\ &= \frac{V}{\sqrt{V^2 - \langle \mathrm{d}V, \nu\rangle^2}} \left[\left(H - (n-1)\frac{\langle \mathrm{d}V, \mathrm{d}\psi\rangle}{V}\right)\right.\\ &\quad \left. - \frac{(\nabla_{e_1}V)^2}{1+(\nabla_{e_1}V)^2}\left(S(e_1, e_1) - \frac{\langle \mathrm{d}V, \mathrm{d}\psi\rangle}{V}\right)\right]\\ &= \frac{1}{\sqrt{V^2 - \langle \mathrm{d}V, \nu\rangle^2}} \left[(HV - (n-1)\langle \mathrm{d}V, \mathrm{d}\psi\rangle)\right.\\ &\quad \left. - \frac{(\nabla_{e_1}V)^2}{1+(\nabla_{e_1}V)^2}\left(S(e_1, e_1)V - \langle \mathrm{d}V, \mathrm{d}\psi\rangle\right)\right].\end{aligned}$$

Next we note that $(\nabla_{e_1}V)^2 = |\mathrm{d}V|^2 - \langle \mathrm{d}V, \nu\rangle^2$ is the norm of $\mathrm{d}V$ restricted to the tangent space of $\partial\Omega$. Hence the measure $\mathrm{d}\widetilde{\mu}$ induced on $\partial\Omega$ by $\widetilde{b}$ is given by

$$\mathrm{d}\widetilde{\mu} = \sqrt{1 + |\mathrm{d}V|^2 - \langle \mathrm{d}V, \nu\rangle^2}\, \mathrm{d}\mu = \sqrt{V^2 - \langle \mathrm{d}V, \nu\rangle^2}\, \mathrm{d}\mu$$

where $\mathrm{d}\mu$ is the measure induced on $\partial\Omega$ by b. Finally, we conclude

$$\begin{aligned}\int_{\partial\Omega} \widetilde{H}\, \mathrm{d}\widetilde{\mu} &= \int_{\partial\Omega} (HV - (n-1)\langle \mathrm{d}V, \mathrm{d}\psi\rangle)\, \mathrm{d}\mu\\ &\quad - \int_{\partial\Omega} \frac{|\nabla_{e_1}V|^2}{1+|\nabla_{e_1}V|^2}\left(S(e_1, e_1)V - \langle \mathrm{d}V, \mathrm{d}\psi\rangle\right) \mathrm{d}\mu.\end{aligned}$$

□

The assumption $\widetilde{S} > 0$ is equivalent to

$$S > \frac{\nabla^k V \nabla_k \psi}{V} b,$$

where this inequality is to be understood as an inequality between quadratic forms on $T\partial\Omega$. This notion of convexity is not invariant under the action of isometries of the hyperbolic space. Since $|\mathrm{d}V| < V$, it is natural to replace this assumption by

$$S \geq b.$$

This new assumption is equivalent to the definition of h-convexity (see for example [3]). Assuming that Ω is h-convex, we get the following inequality from (13):

$$\int_{\partial\Omega} HV \,\mathrm{d}\mu \geq \int_{\partial\Omega} \widetilde{H} \,\mathrm{d}\widetilde{\mu} + (n-1) \int_{\partial\Omega} \langle \mathrm{d}V, \nu \rangle \,\mathrm{d}\mu. \tag{14}$$

We estimate the first term of the right-hand side by the Aleksandrov–Fenchel inequality, see [16, Theorem 2], [22, Lemma 12], [28] or [8].

$$\int_{\partial\Omega} \widetilde{H} \,\mathrm{d}\widetilde{\mu} \geq (n-1)\omega_{n-1} \left(\frac{|\partial\Omega|_{\widetilde{b}}}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} \geq (n-1)\omega_{n-1} \left(\frac{|\partial\Omega|_{b}}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}.$$

Equality in the first inequality here implies that $\partial\Omega$ is a round sphere in the Euclidean metric $\widetilde{b}$, equality in the second inequality tells us that it must be centered at the origin.

To estimate the second term of (14), we rely on [3, Theorem 2]. Assuming that the origin is the center of an inner ball of Ω and denoting by l the distance from the origin, we have, for any point $p \in \partial\Omega$,

$$\langle \nu, \nabla l \rangle \geq \frac{\tanh^2 \frac{l}{2}(p) + \tau}{\tanh \frac{l}{2}(p)(1+\tau)},$$

where $\tau = \tanh \frac{r_0}{2}$ and r_0 is the radius of an inner ball of Ω. Hence, setting $t = \tanh \frac{l}{2}(p)$, we have

$$\begin{aligned}
\int_{\partial\Omega} \langle \mathrm{d}V, \nu \rangle \,\mathrm{d}\mu &= \int_{\partial\Omega} \sinh l \langle \nabla l, \nu \rangle \,\mathrm{d}\mu \\
&\geq \int_{\partial\Omega} \sinh l \frac{t^2+\tau}{t(1+\tau)} \,\mathrm{d}\mu \\
&= \int_{\partial\Omega} \frac{2t}{1-t^2} \frac{t^2+\tau}{t(1+\tau)} \,\mathrm{d}\mu \\
&= \frac{2}{1+\tau} \int_{\partial\Omega} \frac{t^2+\tau}{1-t^2} \,\mathrm{d}\mu \\
&\geq \frac{2}{1+\tau} \frac{\tau^2+\tau}{1-\tau^2} |\partial\Omega|_b \\
&\geq \sinh r_0 |\partial\Omega|_b.
\end{aligned}$$

It is also easy to check that the equality

$$\int_{\partial\Omega} \langle \mathrm{d}V, \nu \rangle \,\mathrm{d}\mu = \sinh r_0 |\partial\Omega|_b$$

holds if and only if Ω is the ball of radius r_0 centered at the origin.

Combining the last two estimates, we get the following inequality:

$$\int_{\partial\Omega} HV \,\mathrm{d}\mu \geq (n-1)\omega_{n-1}\left[\left(\frac{|\partial\Omega|_b}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \sinh r_0 \frac{|\partial\Omega|_b}{\omega_{n-1}}\right]. \tag{15}$$

From Proposition 4.1 and Inequality (15), we immediately get the following theorem.

Theorem 4.4. *Let Ω be a non-empty h-convex subset of $\mathbb{H}^n$ admitting an inner ball centered at the origin of radius r_0. Let $f : \mathbb{H}^n\backslash\Omega \to \mathbb{R}$ be an asymptotically hyperbolic function such that f is locally constant on $\partial\Omega, |df| \to \infty$ at $\partial\Omega$. Assume that the scalar curvature* Scal *of its graph is greater than or equal to $-n(n-1)$. Then*

$$H_\Phi(V) \geq (n-1)\omega_{n-1}\left[\left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}} + \sinh r_0 \frac{|\partial\Omega|}{\omega_{n-1}}\right]. \tag{16}$$

Moreover, equality holds in (16) if and only if Scal $= -n(n-1)$ *and $\partial\Omega$ is round sphere centered at the origin.*

We make a couple of remarks concerning this theorem.

Remark 4.5.
1. If Ω is a ball of radius r then $r_0 = r$ and

$$|\partial\Omega| = \omega_{n-1} \sinh^{n-1} r_0,$$

so (16) coincides with the standard Penrose inequality (1) in this case.

2. The second term of (14) can be written as follows,

$$\int_{\partial\Omega} \langle \mathrm{d}V, \nu\rangle \,\mathrm{d}\mu = \int_{\Omega} \Delta V \,\mathrm{d}\mu = n \int_{\Omega} V \,\mathrm{d}\mu.$$

Thus, this term may be thought of as a volume integral (compare with [29]). Let $V_p := \cosh d_b(p, \cdot)$. Changing the origin p of hyperbolic space leads to considering the function

$$p \mapsto \int_{\Omega} V_p \,\mathrm{d}\mu.$$

It is fairly straightforward to see that this function is proper and strictly convex. So there exists a unique point p_0 such that, choosing p_0 as the origin, this integral is minimal. Obviously, $p_0 \in \Omega$. From symmetry considerations this point can be seen to coincide with the center of an inner ball for many Ω's.

3. It follows from the previous remark that it is possible to prove a Penrose inequality when Ω has several (h-convex) components assuming for example that if one component contains the origin then it is the center of one of its inner balls. For each of the other components, note that translating them using an isometry of the hyperbolic space so that the origin

becomes the center of one of its inner balls makes the integral $\int HV\,\mathrm{d}\mu$ smaller. Hence we get the following inequality:

$$H_\Phi(V) \geq (n-1)\omega_{n-1} \sum_i \left[\left(\frac{|\partial\Omega_i|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} + \sinh r_i \frac{|\partial\Omega_i|}{\omega_{n-1}} \right],$$

where Ω_i are the connected components of Ω and r_i is the inner radius of Ω_i.

5. Rigidity

In this section we will prove the rigidity statement concluding Theorem 2.1. The scheme of the proof we give differs very little from [19]. As a first step, we prove the following proposition which is similar to [19, Theorem 3].

Proposition 5.1. *Let $f : \mathbb{H}^n \backslash \overline{\Omega} \to \mathbb{R}$ be a function satisfying the assumptions of Theorem 2.1 and let Σ be its graph. Assume further that Ω is convex. Then, the mean curvature $\overline{H}$ of Σ does not change sign.*

The proof of this proposition requires several preliminary results. The main observation is the fact that the assumption $\mathrm{Scal} \geq -n(n-1)$ is equivalent to $\left|\overline{S}\right|^2 \leq \overline{H}^2$. This follows at once from the Gauss equation. In particular, any point $p \in \Sigma$ such that $\overline{H}(p) = 0$ has $\overline{S}(p) = 0$. We denote by Σ_0 the set of such points,

$$\Sigma_0 := \{p \in \mathrm{int}(\Sigma) \mid \overline{H}(p) = 0\},$$

where $\mathrm{int}(\Sigma) = \Sigma \backslash (\partial\Omega \times \mathbb{R})$.

Lemma 5.2. *Let Σ_0' be a connected component of Σ_0. Then Σ_0' lies in a codimension 1 hyperbolic subspace tangent to Σ at every point of Σ_0'.*

Proof. Let $V_{(0)}, \ldots, V_{(n)}$ be as in Sect. 2.1 and let ν be the unit normal vector field of Σ in $\mathbb{H}^{n+1}$. For any vector $X \in T\Sigma$ at a point of Σ_0', we have

$$\begin{aligned} \overline{\nabla}_X(dV_{(i)}(\nu)) &= \overline{\nabla}^2_{X,\nu} V_{(i)} + \mathrm{d}V_{(i)}(\overline{\nabla}_X \nu) \\ &= V_{(i)}\overline{b}(X,\nu) + \mathrm{d}V_{(i)}(S(X)) = 0, \end{aligned}$$

where $S(X)$ denotes the Weingarten operator which is zero by assumption. From [25, Theorem 4.4], we conclude that $\mathrm{d}V_{(i)}(\nu)$ is constant on Σ_0'. If we consider $\mathbb{H}^{n+1}$ as the unit hyperboloid in Minkowski space $\mathbb{R}^{n+1,1}$, then the $V_{(i)}$ are the coordinate functions of $\mathbb{R}^{n+1,1}$ restricted to $\mathbb{H}^{n+1}$ so ν is a constant vector in $\mathbb{R}^{n+1,1}$. Further, ν is tangent to $\mathbb{H}^{n+1}$ so it is orthogonal to the position vector in $\mathbb{R}^{n+1,1}$. This means that ν is everywhere orthogonal to a linear subspace $W \subset \mathbb{R}^{n+1,1}$. We conclude that $\Sigma_0' \subset W \cap \mathbb{H}^{n+1} \simeq \mathbb{H}^n$. □

The next result is taken from [20, Proposition 2.1].

Lemma 5.3 (A matrix inequality). *Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix. Set*

$$\sigma_1(A) := \sum_{i=1}^{n} a_{ii},$$
$$\sigma_1(A|k) := \left(\sum_{i=1}^{n} a_{ii}\right) - a_{kk},$$
$$\sigma_2(A) := \sum_{1 \le i < j \le n} \left(a_{ii}a_{jj} - a_{ij}^2\right).$$

Then we have

$$\sigma_1(A)\sigma_1(A|k) = \sigma_2(A) + \frac{n}{2(n-1)}\sigma_1(A|k)^2 + \sum_{1 \le i < j \le n} a_{ij}^2 + \frac{1}{2(n-1)} \sum_{\substack{1 \le i < j \le n \\ i \ne k, j \ne k}} (a_{ii} - a_{jj})^2$$

for each $1 \le k \le n$. In particular,

$$\sigma_1(A)\sigma_1(A|k) \ge \sigma_2(A) + \frac{n}{2(n-1)}\sigma_1(A|k)^2,$$

where equality holds if and only if A is diagonal and all a_{ii} are equal for $i = 1, \ldots, n, i \ne k$.

Proposition 5.4. *Let Σ and s_0 be given. Assume that s_0 is a regular value for f on Σ. Set $\Sigma(s_0) = \Sigma \cap f^{-1}(s_0)$. Let ν be the unit normal vector field of Σ in $\mathbb{H}^{n+1}$, let η be the unit normal vector field to $\Sigma(s_0)$ in $\mathbb{H}^n \times \{s_0\}$ and let $H(s_0)$ be the mean curvature of $\Sigma(s_0)$ in $\mathbb{H}^n \times \{s_0\}$ computed with respect to η. Then*

$$\langle \nu, \eta \rangle \overline{H} H(s_0) \ge \frac{\mathrm{Scal} + n(n-1)}{2} + \frac{n}{2(n-1)} \langle \nu, \eta \rangle^2 H(s_0)^2.$$

Equality holds at a point in $\Sigma(s_0)$ if and only if

- *$\Sigma(s_0) \subset \mathbb{H}^n \times \{s_0\}$ is umbilic with principal curvature κ, and*
- *$\langle \nu, \eta \rangle \kappa$ is a principal curvature of Σ with multiplicity at least $(n-1)$.*

Proof. Let p be a point in $\Sigma(s_0)$. We compute the second fundamental form of $\Sigma(s_0)$ in $\mathbb{H}^{n+1}$ at p in two different ways. Let $e_1 \in T_p\Sigma$ be a unit vector field orthogonal to $T_p\Sigma(s_0)$. We denote by $\overline{S}_0$ the second fundamental form of $\Sigma(s_0)$ in $\mathbb{H}^{n+1}$. This is a symmetric bilinear form on $T_p\Sigma(s_0)$ taking values in the normal bundle $N_p\Sigma(s_0) \subset T_p\mathbb{H}^{n+1}$. Further, we denote by S_1 the second fundamental form of $\Sigma(s_0)$ in Σ computed with respect to the vector e_1. Since $\mathbb{H}^n \times \{s_0\}$ is totally geodesic in $\mathbb{H}^{n+1}$, we have

$$\overline{S}_0 = S_0 \eta.$$

Similarly,

$$\overline{S}_0 = \overline{S}\nu + S_1 e_1.$$

Hence, taking the scalar product of the last two equalities with ν, we get

$$\langle \eta, \nu \rangle S_0 = \overline{S}.$$

Let $\{e_2, \ldots, e_n\}$ be an orthonormal basis of $T\Sigma(s_0)$, then $\{e_1, \ldots, e_n\}$ is an orthonormal basis of $T_p\Sigma$. Set

$$\overline{S}_{ij} := \overline{S}(e_i, e_j).$$

Then, using the notation of Lemma 5.3, we have

$$\begin{aligned}
\sigma_1(\overline{S}) &= \overline{H},\\
\sigma_1(\overline{S}|1) &= \sum_{i=2}^{n} \overline{S}(e_i, e_i)\\
&= \langle \eta, \nu \rangle \sum_{i=2}^{n} S_0(e_i, e_i)\\
&= \langle \eta, \nu \rangle H(s_0),\\
\sigma_2(\overline{S}) &= \frac{1}{2}\left(\overline{H}^2 - \left|\overline{S}\right|^2\right)\\
&= \frac{\mathrm{Scal} + n(n-1)}{2}.
\end{aligned}$$

Proposition 5.4 now follows from Lemma 5.3. □

The proof of Proposition 5.1 will also require the following two lemmas, analogous to [19, Lemma 3.3 and Lemma 3.4].

Lemma 5.5. *Let W be an open subset of $\mathbb{H}^n$, possibly unbounded. Let $p \in \partial W$, and let $B(p)$ be a geodesic open ball in $\mathbb{H}^n$ centered at p. Consider $f \in C^2(W \cap B(p)) \cap C^1(\overline{W} \cap B(p))$ and let $\overline{H}$ denote the mean curvature of its graph. If $f = C$ and $|df| = 0$ on $\partial W \cap B(p)$, where C is a constant, and $\overline{H} \geq 0$ on $W \cap B(p)$ then either $f \equiv C$ in $W \cap B(p)$, or*

$$\{x \in W \cap B(p) \mid f(x) > C\} \neq \emptyset.$$

Proof. If $f \equiv C$ then there is nothing to prove. Suppose, therefore, that $f \not\equiv C$ and assume to get a contradiction that $f(x) \leq C$ everywhere in $W \cap B(p)$.

We first note that in fact $f < C$ everywhere in $W \cap B(p)$. Indeed, let $q \in W \cap B(p)$ be such that $f(q) = C$. Then q is an interior maximum point of f in $W \cap B(p)$, whereas

$$\begin{aligned}
\overline{H} = &\frac{V}{1 + V^2\,|df|^2}\left(b^{ij} - \frac{V^2 \nabla^i f \nabla^j f}{1 + V^2 |df|^2}\right)\\
&\times \left[\nabla^2_{i,j} f + \frac{\nabla_i f \nabla_j V + \nabla_i V \nabla_j f}{V} + V \langle df, \mathrm{d}V \rangle \nabla_i f \nabla_j f\right] \geq 0
\end{aligned}$$

in $W \cap B(p)$, see Sect. 3.1. By the Hopf strong maximum principle it follows that $f \equiv C$ in $W \cap B(p)$, which is a contradiction.

Now suppose that $B(p) = B_r(p)$ is the ball of radius r around p. Fix a point $q \in B_{r/2}(p)$ and define $r' := \sup\{r \mid B_r(q) \subset W\}$. It is clear that

$B_{r'}(q) \subset W \cap B(p)$ and $\overline{B}_{r'}(q) \cap \partial W \neq \emptyset$. Consequently, there is a point $s \in \partial W$ such that the interior sphere condition holds at s. Then by the Hopf boundary lemma [15, Lemma 3.4], we have $|df| > 0$ at s, which is a contradiction. We conclude that $f > C$ holds somewhere in $W \cap B(p)$. □

Definition 5.6. Let W be a bounded subset of $\mathbb{H}^n$ and let $\overline{W}$ be its closure. A point $p \in \partial W$ is called convex if there is a geodesic $(n-1)$-sphere S in $\mathbb{H}^n$ passing through p such that $\overline{W}\backslash\{p\}$ is contained in the open geodesic ball enclosed by S.

Note that every bounded set in $\mathbb{H}^n\backslash\Omega$ has at least one convex point. This follows from the assumption that Ω is convex. We only sketch the proof of this fact leaving the details to the reader. Choose a point $p \in W$ and let q be the projection of p onto $\partial\Omega$. Then, the hyperbolic subspace passing through q and orthogonal to the geodesic joining p to q cuts $\mathbb{H}^n$ in two half-spaces, a "left" one containing Ω and a "right" one containing p. Then if O' is located very far on the left side of the geodesic (qp), it is clear that the smallest sphere S centered at O' containing $\Omega \cup W$ has a non-trivial intersection with ∂W. Any point in $S \cap \partial W$ is then a convex point.

Lemma 5.7. *Let W be an open bounded subset of $\mathbb{H}^n$ and let $p \in \partial W$ be a convex point. Suppose that $f \in C^n(W \cap B(p)) \cap C^1(\overline{W} \cap B(p))$ is such that $f = C$ and $|df| = 0$ on $\partial W \cap B(p)$ for some constant C. If the graph of f has scalar curvature* $\mathrm{Scal} \geq -n(n-1)$, *then its mean curvature $\overline{H}$ must change sign in $W \cap B(p)$, unless $f \equiv C$ in $W \cap B(p)$.*

Proof. Suppose on the contrary that $\overline{H}$ does not change sign and $f \not\equiv 0$. By possibly reversing sign and adding a constant to f we may assume that $\overline{H} \geq 0$ and that $C = 0$.

Let S_r be a geodesic $(n-1)$-sphere of radius r as in Definition 5.6, centered at a point $O' \in \mathbb{H}^n$, and such that $S_r \cap \overline{W} = \{p\}$. Let μ be a positive number strictly less than the distance from $W\backslash B(p)$ to S_r. Then for every sphere $S_{r'}$ of radius $r' \in (r-\mu, r)$ and centered at O' we obviously have $S_{r'} \cap W \subset B(p)$. Let f_0 be a continuous function on $B(p)$ such that $f_0 = f$ on $W \cap B(p)$ and $f_0 = 0$ on $B(p)\backslash W$. Define the function

$$g(r') := \sup_{q \in S_{r'} \cap B(p)} f_0(q)$$

for $r' \in [r-\mu, r]$. It is easy to check that g is continuous and satisfies $g(r) = 0$. Next, we observe that by Lemma 5.5 the ball $B_\mu(p)$ contains a point q such that $f_0(q) = \varepsilon > 0$. By the Morse-Sard theorem [27, Theorem 7.2] we may assume that each connected component of the corresponding level set

$$\Sigma(\epsilon) = \{x \in W \cap B(p) \mid f_0(x) = \epsilon\}$$

of f_0 inside $W \cap B(p)$ is a C^n hypersurface. It is clear that $g([r-\mu, r]) = [0, \epsilon']$, where $\epsilon \leq \epsilon'$, and hence

$$r_0 := \max_{r' \in [r-\mu, r]} \{r' \mid g(r') = \epsilon\}$$

is well-defined. Then $S_{r_0} \cap \Sigma_\epsilon \neq \emptyset$, whereas $S_{r'} \cap \Sigma_\epsilon = \emptyset$ for $r_0 < r' \leq r$, thus S_{r_0} is tangent to $\Sigma(\epsilon)$ at some interior point q. Let U be the open subset of $W \cap B(p)$ bounded by S_{r_0} and ∂W,

$$U = \{x \in W \cap B(p) | d(O', x) > r_0\},$$

then $q \in \partial U$. We have $f(q) = \epsilon > f(x)$ for any $x \in U, \overline{H} \geq 0$ holds in U, and the interior sphere condition is obviously satisfied at $q \in S_{r_0}$. Since $\eta = -\frac{\nabla f}{|df|}$ is orthogonal to ∂U at q, it is easy to conclude by the Hopf boundary lemma that η is the inward pointing normal to ∂U. Hence η is the outward pointing normal for both S_{r_0} and $\Sigma(\epsilon)$ at q. By the comparison principle, the mean curvature $H(\epsilon)$ of $\Sigma(\epsilon)$ satisfies $H(\epsilon) > 0$ at q. On the other hand, since the scalar curvature of the graph of f is greater than or equal to $-n(n-1)$ by Proposition 5.4 at q we have

$$\langle \nu, \eta \rangle \overline{H} H(\epsilon) \geq 0.$$

Here $\langle \nu, \eta \rangle < 0$ since $\nu = \frac{(\nabla f, -V^{-2})}{\sqrt{V^{-2} + |df|^2}}, \overline{H} \geq 0$, and if $\overline{H} = 0$ then $H(\epsilon) = 0$. This means that $H(\epsilon) \leq 0$ at q, which is a contradiction. Hence $\overline{H}$ must change sign in $W \cap B(p)$. $\square$

Proof of Proposition 5.1. To prove that $\overline{H}$ does not change sign, we argue by contradiction assuming that both the sets $\{\overline{H} > 0\}$ and $\{\overline{H} < 0\}$ are nonempty. Our first observation is that each connected component of these two sets is unbounded. Indeed, let Σ_+ be a bounded connected component of $\{\overline{H} > 0\}$ and let $\partial_0 \Sigma_+$ be its outer boundary component. By Lemma 5.2, we know that $\partial_0 \Sigma_+$ lies in an n-dimensional hyperbolic subspace Π. We view $\mathbb{H}^{n+1}$ as $\Pi \times \mathbb{R}$ with the metric $b + V^2 \mathrm{d}\widetilde{s} \otimes \mathrm{d}\widetilde{s}$, and we let W be a subset of $\{\widetilde{s} = 0\}$ bounded by $\partial_0 \Sigma_+$. Then in some neighborhood of ∂W we can write Σ_+ as the graph of a function u such that $u = 0$ and $|du| = 0$ on ∂W. Now, considering a sufficiently small ball $B(p)$ around $p \in \partial W$, we immediately arrive at the contradiction, since $\overline{H}$ must change sign in $W \cap B(p)$ by Lemma 5.7.

We have just seen that if Σ_+ is a connected component of $\{\overline{H} > 0\}$ then it must be unbounded, and the same is clearly true for a connected component Σ_- of $\{\overline{H} < 0\}$. Moreover, it follows by Proposition A.1 in Appendix A that one of the connected components of its boundary $\partial \Sigma_+$ is unbounded, and the same holds for $\partial \Sigma_-$. Let us denote such an unbounded component by $\partial_0 \Sigma_+$. By Lemma 5.2 we know that $\partial_0 \Sigma_+$ lies in an n-dimensional hyperbolic subspace Π tangent to Σ at every point of $\partial_0 \Sigma_+$. Since Σ is asymptotically hyperbolic, f tends to a constant value C at infinity, so the fact that $\partial_0 \Sigma_+$ is unbounded forces Π to coincide with the plane $\{s = C\}$.

The component Σ_+ is the graph of f over some open subset W of $\mathbb{H}^n$. Moreover, there is an unbounded component $\partial_0 W$ of the boundary ∂W such that $f = C$ and $|df| = 0$ on $\partial_0 W$. By Lemma 5.5, there exists $q \in W$ such that $f(q) = C + \varepsilon$ for some $\varepsilon > 0$. By the Morse–Sard theorem, we know that there is an ε such that $C + \varepsilon$ is a regular value of f, so that the corresponding level set $f^{-1}(C + \varepsilon) = \{p \mid f(p) = C + \varepsilon\}$ is a C^n hypersurface with $|df| > 0$ at each point. Suppose that U is a connected component of $\{H \geq 0\}$ in $\mathbb{H}^n$ which contains W. Then, using Proposition A.1 and the fact that f tends to C at infinity, it is easy to check that if some connected component of $f^{-1}(C + \epsilon)$ intersects U, then it is contained in U. It is also obvious that $f^{-1}(C + \epsilon) \cap U$ is nonempty and bounded, so we can find a point p in this set which is at the largest distance d from the origin O of $\mathbb{H}^n$. Let $\Sigma(C + \epsilon)$ be the connected component of $f^{-1}(C + \epsilon)$ which contains p. Then, the geodesic sphere of radius d centered at O touches $\Sigma(C + \epsilon)$ at p, and there are no points x such that $f(x) \geq C + \epsilon$ in $\{r > d\} \cap U$. Arguing as in the proof of Lemma 5.7, we can show that $\eta := -\frac{\nabla f}{|df|} = \partial_r$ at p, that is, ν is an outgoing normal to $\Sigma(C + \epsilon)$. The mean curvature $H(C + \epsilon)$ is then positive at p, whereas Proposition 5.4 tells us that $H(C + \epsilon) \leq 0$ at p, which is a contradiction. □

Let f be as in Theorem 2.1. We recall the expressions for $g, \overline{S}, \overline{H}$, and Scal obtained in Sect. 2.2, and rewrite them as functions of the arguments Df and $D^2 f$, where Df and $D^2 f$ denote the Euclidean gradient and the Euclidean Hessian respectively:

$$\begin{aligned}
g^{ij}(Df) &= b^{ij} - \frac{V^2 f^i f^j}{1 + V^2 |df|^2},\\
\overline{S}_{ij}(Df, D^2 f) &= \frac{V}{\sqrt{1 + V^2 |df|^2}} \left[f_{ij} - \Gamma^l_{ij} f_l + \frac{f_i V_j + V_i f_j}{V} + V \langle df, \mathrm{d}V \rangle f_i f_j \right],\\
\overline{S}^i_j(Df, D^2 f) &= \frac{V}{\sqrt{1 + V^2 |df|^2}} \left(b^{ik} - \frac{V^2 f^i f^k}{1 + V^2 |df|^2} \right)\\
&\quad \left(f_{kj} - \Gamma^l_{kj} f_l + \frac{f_k V_j + V_k f_j}{V} + V \langle df, \mathrm{d}V \rangle f_k f_j \right),\\
\overline{H}(Df, D^2 f) &= \frac{V}{\sqrt{1 + V^2 |df|^2}} \left(b^{ij} - \frac{V^2 f^i f^j}{1 + V^2 |df|^2} \right)\\
&\quad \left(f_{ij} - \Gamma^l_{ij} f_l + \frac{f_i V_j + V_i f_j}{V} + V \langle df, \mathrm{d}V \rangle f_i f_j \right),\\
\mathrm{Scal}(Df, D^2 f) &= -n(n-1) + \overline{H}^2(Df, D^2 f) - \overline{S}^j_i(Df, D^2 f) \overline{S}^i_j(Df, D^2 f).
\end{aligned}$$

Following [19, Section 4], we will now prove maximum principles for the scalar curvature equation $\mathrm{Scal}(Df, D^2 f) + n(n-1) = 0$. The lemma below concerns ellipticity of this equation.

Lemma 5.8.

$$\frac{\partial \mathrm{Scal}}{\partial f_{ij}} = \frac{2V}{\sqrt{1 + V^2 |df|^2}} \left(\overline{H} g^{ij} - \overline{S}^{ij} \right).$$

Proof. A straightforward computation gives

$$\begin{aligned}\frac{\partial \mathrm{Scal}}{\partial f_{ij}} &= 2\overline{H}\frac{\partial \overline{H}}{\partial f_{ij}} - 2\overline{S}_l^k \frac{\partial \overline{S}_k^l}{\partial f_{ij}} \\ &= \frac{2V}{\sqrt{1+V^2|df|^2}}\left(\overline{H}g^{ij} - \overline{S}_l^k g^{lm}\frac{\partial f_{mk}}{\partial f_{ij}}\right) \\ &= \frac{2V}{\sqrt{1+V^2|df|^2}}\left(\overline{H}g^{ij} - \overline{S}^{ij}\right).\end{aligned}$$

□

Proposition 5.9. *Let f be as in Theorem 2.1. Suppose that the scalar curvature* Scal *and the mean curvature $\overline{H}$ of its graph satisfy* $\mathrm{Scal} \geq -n(n-1)$ *and* $\overline{H} \geq 0$*. Then, the matrix $\left(\overline{H}g^{ij} - \overline{S}^{ij}\right)$ is positive semi-definite everywhere in* $\mathbb{H}^n\backslash\overline{\Omega}$*.*

Proof. We work at a point $p \in \mathbb{H}^n\backslash\overline{\Omega}$. Since $\overline{H}g^{ij} - \overline{S}^{ij} = \sum_k(\overline{H}\delta_k^j - \overline{S}_k^j)g^{ik}$, where g^{ik} is positive definite, we only need to show that $(\overline{H}\delta_k^j - \overline{S}_k^j)$ is positive semi-definite. After possibly rotating the coordinates, we may assume that $\overline{S} = (\overline{S}_k^j) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then, in the notation of Lemma 5.3, we have

$$\left(\overline{H}\delta_k^j - \overline{S}_k^j\right) = \mathrm{diag}\left(\sigma_1(\overline{S}|1), \ldots, \sigma_1(\overline{S}|n)\right).$$

By Lemma 5.3 it follows that

$$\sigma_1(\overline{S})\sigma_1(\overline{S}|k) \geq \sigma_2(\overline{S}) + \frac{n}{2(n-1)}\left(\sigma_1(\overline{S}|k)\right)^2,$$

for $k = 1, \ldots, n$. If $\sigma_1(\overline{S}) = \overline{H} > 0$, since $\sigma_2(\overline{S}) = \frac{1}{2}(\mathrm{Scal} + n(n-1)) \geq 0$, it is obvious that $\sigma_1(\overline{S}|k) \geq 0$ for every $k = 1, \ldots, n$. Otherwise if $\overline{H} = 0$ then $\overline{S} = 0$ and hence $\sigma_1(\overline{S}|k) = 0$. This proves that $\sigma_1(\overline{S}|k) \geq 0$. □

In the next two propositions, we prove versions of the maximum principle for the scalar curvature equation, the first one for points in the interior and the second one for points on the boundary.

Proposition 5.10. *Let $f_i : \mathbb{H}^n\backslash\overline{\Omega} \to \mathbb{R}, i = 1, 2$, be two functions satisfying the assumptions of Theorem 2.1. Suppose that $f_1 \geq f_2$ in $\mathbb{H}^n\backslash\overline{\Omega}$, and that $f_i, i = 1, 2$, satisfy the inequalities*

$$\begin{aligned}\mathrm{Scal}(Df_1, D^2f_1) = -n(n-1), \quad \overline{H}(Df_1, D^2f_1) \geq 0, \\ \mathrm{Scal}(Df_2, D^2f_2) \geq -n(n-1), \quad \overline{H}(Df_2, D^2f_2) \geq 0\end{aligned}$$

in $\mathbb{H}^n\backslash\overline{\Omega}$. If the matrix $\left(\overline{H}g^{ij} - \overline{S}^{ij}\right)$ is positive definite in $\mathbb{H}^n\backslash\overline{\Omega}$ for either f_1 or f_2, and if $f_1 = f_2$ at some point of $\mathbb{H}^n\backslash\overline{\Omega}$, then $f_1 \equiv f_2$ in $\mathbb{H}^n\backslash\overline{\Omega}$.

Proof. We consider the scalar curvature operator as $\mathrm{Scal}(p,\xi) \in C^1(\mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n)$. Then

$$\begin{aligned} 0 &\geq \mathrm{Scal}(Df_1, D^2 f_1) - \mathrm{Scal}(Df_2, D^2 f_2) \\ &= \mathrm{Scal}(Df_1, D^2 f_1) - \mathrm{Scal}(Df_1, D^2 f_2) + \mathrm{Scal}(Df_1, D^2 f_2) - \mathrm{Scal}(Df_2, D^2 f_2) \\ &= \sum_{i,j} a^{ij}((f_1)_{ij} - (f_2)_{ij}) + \sum_i b^i((f_1)_i - (f_2)_i), \end{aligned}$$

where

$$b^i = \int_0^1 \frac{\partial \mathrm{Scal}}{\partial p_i}(tDf_1 + (1-t)Df_2, D^2 f_2)\,\mathrm{d}t,$$

and

$$a^{ij} = \int_0^1 \frac{\partial \mathrm{Scal}}{\partial \xi_{ij}}(Df_1, tD^2 f_1 + (1-t)D^2 f_2)\,\mathrm{d}t.$$

Note that by Lemma 5.8 we have

$$\begin{aligned} a^{ij} &= \int_0^1 \frac{\partial \mathrm{Scal}}{\partial \xi_{ij}}(Df_1, tD^2 f_1 + (1-t)D^2 f_2)\,\mathrm{d}t \\ &= \frac{2V}{\sqrt{1+V^2|df|^2}} \int_0^1 \left(\overline{H} g^{ij} - \overline{S}^j_k g^{ik}\right)(Df_1, tD^2 f_1 + (1-t)D^2 f_2)\,\mathrm{d}t \\ &= \frac{2V}{\sqrt{1+V^2|df|^2}} \left[\int_0^1 t\Big(\overline{H}(Df_1, D^2 f_1) g^{ij}(Df_1) \right. \\ &\quad - \overline{S}^j_k(Df_1, D^2 f_1) g^{ik}(Df_1)\Big)\,\mathrm{d}t \\ &\quad \left. + \int_0^1 (1-t)\left(\overline{H}(Df_1, D^2 f_2) g^{ij}(Df_1) - \overline{S}^j_k(Df_1, D^2 f_2) g^{ik}(Df_1)\right)\mathrm{d}t\right] \\ &= \frac{V}{\sqrt{1+V^2|df|^2}} \Big[\Big(\overline{H}(Df_1, D^2 f_1) g^{ij}(Df_1) - \overline{S}^j_k(Df_1, D^2 f_1) g^{ik}(Df_1)\Big) \\ &\quad + \Big(\overline{H}(Df_1, D^2 f_2) g^{ij}(Df_1) - \overline{S}^j_k(Df_1, D^2 f_2) g^{ik}(Df_1)\Big)\Big]. \end{aligned}$$

If $f_1 = f_2$ at $p \in \mathbb{H}^n \setminus \overline{\Omega}$, then p is a local minimum point of $f_1 - f_2$, hence $Df_1 = Df_2$ at p. Consequently, a^{ij} is positive definite at p. By continuity, a^{ij} is positive definite in some open neighborhood U of p in $\mathbb{H}^n \setminus \overline{\Omega}$. Then $f_1 \equiv f_2$ in U by the Hopf strong maximum principle. It follows that the set $\{p \in \mathbb{H}^n \setminus \overline{\Omega} \mid f_1(p) = f_2(p)\}$ is both open and closed in $\mathbb{H}^n \setminus \overline{\Omega}$. Since $\mathbb{H}^n \setminus \overline{\Omega}$ is connected, we conclude that $f_1 \equiv f_2$ everywhere $\mathbb{H}^n \setminus \overline{\Omega}$. $\square$

Proposition 5.11. *Let $f_i : \mathbb{H}^n \backslash \overline{\Omega} \to \mathbb{R}, i = 1, 2$, be functions satisfying the assumptions of Theorem 2.1. Suppose that $f_1 \geq f_2 \geq C$ in $\mathbb{H}^n \backslash \overline{\Omega}$, and that $f_i, i = 1, 2$, satisfy the inequalities*

$$\mathrm{Scal}(Df_1, D^2 f_1) = -n(n-1), \quad \overline{H}(Df_1, D^2 f_1) \geq 0,$$
$$\mathrm{Scal}(Df_2, D^2 f_2) \geq -n(n-1), \quad \overline{H}(Df_2, D^2 f_2) \geq 0$$

in $\mathbb{H}^n \backslash \overline{\Omega}$. If the matrix $\left(\overline{H} g^{ij} - \overline{S}^{ij}\right)$ is positive definite in $\mathbb{H}^n \backslash \Omega$ for either f_1 or f_2, and if $f_1 = f_2 = C$ on $\partial\Omega$, then $f_1 \equiv f_2$ in $\mathbb{H}^n \backslash \overline{\Omega}$.

Proof. Let Σ_i denote the graph of $f_i, i = 1, 2$. Take $p \in \partial\Sigma_1 = \partial\Sigma_2 \subset \{s = C\}$, and let $\nu(p)$ be the common normal to $\Sigma_i, i = 1, 2$, at this boundary point. Suppose that Π is the hyperbolic subspace orthogonal to $\nu(p)$, then Π is isometric to $\mathbb{H}^n$. Let $B_r(p)$ be a geodesic ball of radius r in Π centered at p, and let $U = B_r(p) \cap \{s > C\}$. If r is sufficiently small, we can write Σ_i near p as the graph of $\widetilde{f}_i : U \to \mathbb{R}, i = 1, 2$, in $U \times \mathbb{R}$ with the metric $b + V^2 \mathrm{d}\widetilde{s} \otimes \mathrm{d}\widetilde{s}$, where b is the hyperbolic metric on U, and $\widetilde{s}$ is the coordinate along the $\mathbb{R}$-factor. It is obvious that $\nabla \widetilde{f}_i = 0$ at p for $i = 1, 2$. We also have $\widetilde{f}_1 \geq \widetilde{f}_2$ in U, and

$$\mathrm{Scal}(D\widetilde{f}_1, D^2 \widetilde{f}_1) = -n(n-1), \quad \overline{H}(D\widetilde{f}_1, D^2 \widetilde{f}_1) \geq 0,$$
$$\mathrm{Scal}(D\widetilde{f}_2, D^2 \widetilde{f}_2) \geq -n(n-1), \quad \overline{H}(D\widetilde{f}_2, D^2 \widetilde{f}_2) \geq 0.$$

Moreover, either $\widetilde{f}_1$ or $\widetilde{f}_2$ has positive definite matrix $(\overline{H} g^{ij} - \overline{S}^{ij})$ at p. Arguing as in the proof of Proposition 5.10, we see that $(\widetilde{f}_1 - \widetilde{f}_2)$ satisfies

$$0 \geq \sum_{i,j} a^{ij}((\widetilde{f}_1)_{ij} - (\widetilde{f}_2)_{ij}) + \sum_i b^i((\widetilde{f}_1)_i - (\widetilde{f}_2)_i),$$

where we may assume (after decreasing r) that a^{ij} is positive definite on U. If we assume that $\widetilde{f}_1 > \widetilde{f}_2$ in U then by the Hopf boundary lemma we have $\nabla(\widetilde{f}_1 - \widetilde{f}_2)(p) \neq 0$, a contradiction. Consequently, $\widetilde{f}_1(q) = \widetilde{f}_2(q)$ at some interior point $q \in \mathbb{H}^n \backslash \overline{\Omega}$. Application of Proposition 5.10 completes the proof. $\square$

We recall that $\rho := \sinh(r)$. The hyperbolic metric b takes the form

$$b = \frac{(\mathrm{d}\rho)^2}{1 + \rho^2} + \rho^2 \sigma,$$

and the function $V = \cosh(r) = \sqrt{1 + \rho^2}$.

Proposition 5.12. *The second fundamental form of the graph given by (5) is given by*

$$\overline{S} = -\frac{n-2}{2} \frac{\sqrt{2m}\rho^{-\frac{n}{2}}}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} \mathrm{d}\rho^2 + \sqrt{2m}\rho^{-\frac{n}{2}+2}\sigma.$$

In particular, the principal curvatures of the graph Σ are $-\frac{n-2}{2}\sqrt{2m}\rho^{-\frac{n}{2}}$ with multiplicity 1 and $\sqrt{2m}\rho^{-\frac{n}{2}}$ with multiplicity $n-1$. The mean curvature $\overline{H}$ is given by

$$\overline{H} = \frac{n}{2}\sqrt{2m}\rho^{-\frac{n}{2}}.$$

In particular, the quadratic form

$$\overline{H}g - \overline{S} = (n-1)\frac{\sqrt{2m}\rho^{-\frac{n}{2}}}{1+\rho^2-\frac{2m}{\rho^{n-2}}}\mathrm{d}\rho^2 + \frac{n-2}{2}\sqrt{2m}\rho^{-\frac{n}{2}+2}\sigma$$

is positive definite.

Proof. Straightforward calculations. □

We are now ready to prove the result on rigidity for the case of equality in the last inequality of Theorem 2.1. From Theorem 4.4, we know that in this case $\mathrm{Scal} = -n(n-1)$ and $\partial\Omega \subset \mathbb{H}^n$ is a round sphere centered at the origin. The result thus follows from the next theorem.

Theorem 5.13. *Let* $f : \mathbb{H}^n\backslash\Omega \to \mathbb{R}$ *be an asymptotically hyperbolic function which satisfies the assumptions of Theorem 2.1 and such that the graph of* f *has constant scalar curvature* $\mathrm{Scal} = -n(n-1)$*. Also assume that* $\partial\Omega$ *is a round sphere centered at the origin and that* $df(\eta)(x) \to \infty$ *as* $x \to \partial\Omega$ *where* η *is the outward normal of the level sets of* f*. Then, the graph of* f *is isometric to the* $t = 0$ *slice of the anti-de Sitter Schwarzschild space–time, as described in Sect. 2.3.*

Proof. By adding a constant to f we assume that $f = 0$ on $\partial\Omega$. From Proposition 5.1 we know that $\overline{H}$ does not change sign. Proposition 5.4 together with the fact that H is positive on $\partial\Omega$ tells us that $\overline{H} \geq 0$ on the boundary, and thus $\overline{H} \geq 0$ everywhere. The maximum principle applied to $\overline{H}$ together with $df(\eta) \to +\infty$ at $\partial\Omega$ tells us that $\limsup_{x\to\infty} f(x) > 0$. Since f is an asymptotically hyperbolic function we conclude that $\lim_{x\to\infty} f(x) = C$ where $0 < C < \infty$.

Let $f_{\text{AdS-Schw}}$ be the asymptotically hyperbolic function whose graph is isometric to the $t = 0$ slice of anti-de Sitter Schwarzschild space–time, with mass parameter m such that its horizon is exactly the sphere $\partial\Omega$. This function vanishes on $\partial\Omega$ and has $\lim_{x\to\infty} f_{\text{AdS-Schw}} = C_0$ where $0 < C_0 < \infty$.

If $C \leq C_0$ we set $u_\lambda = f_{\text{AdS-Schw}} + \lambda$ for $\lambda \geq 0$. If λ is large enough then $u_\lambda > f$. We decrease λ until finally $u_\lambda(p) = f(p)$ at a point p, possibly $p = \infty$. If p is an interior point then Proposition 5.10 tells us that $u_\lambda \equiv f$, if p is a boundary point then Proposition 5.11 tells us that $u_\lambda \equiv f$. There is, however, one more situation to consider, namely when $u_\lambda > f$ and $\lim_{x\to\infty}(u_\lambda - f) = 0$. Since both the graph of u_λ and the graph of f have $\mathrm{Scal} = -n(n-1)$, arguing as in the proof of Proposition 5.10 we conclude that $u_\lambda - f$ satisfies the equation

$$\sum_{i,j} a^{ij}(u_\lambda - f)_{ij} + \sum_i b^i(u_\lambda - f)_i = 0.$$

In this case, the Hopf strong maximum principle tells us that $u_\lambda - f$ attains its positive maximum either at an interior point or at a point of $\partial\Omega$. Let us denote this point by q and suppose that $(u_\lambda - f)(q) = \beta > 0$. Clearly, $f \geq u_\lambda - \beta$, and $f(q) = (u_\lambda - \beta)(q)$. By either Proposition 5.10 or Proposition 5.11 we conclude that $u_\lambda - \beta \equiv f$.

If $C > C_0$ we set $v_\lambda = f_{\text{AdS-Schw}} - \lambda$ for $\lambda \geq 0$. For λ large enough we have $v_\lambda < f$ and we decrease λ until v_λ hits f. Arguing as above it is easy to show that $v_\lambda \equiv f$.

In any case, we have found that f and $f_{\text{AdS-Schw}}$ differ by a constant, which is the conclusion of the theorem. □

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Appendix A. A Property of Unbounded Open Subsets of $\mathbb{R}^n$

In this appendix we will prove the following result on the boundary components of an unbounded open subset of $\mathbb{R}^n$.

Proposition A.1. *Let $H : \mathbb{R}^n \to \mathbb{R}, n \geq 2$, be a continuous function which takes both positive and negative values. Assume that each connected component of $H^{-1}((0,\infty))$ and $H^{-1}((-\infty,0))$ is unbounded. Then, there is a connected component of $H^{-1}(0)$ which is unbounded.*

To prove the proposition we use the following lemma.

Lemma A.2. *Let $K \subset \mathbb{R}^n, n \geq 2$, be compact and connected. Let U be the unbounded connected component of $\mathbb{R}^n \setminus K$. Then $U_\epsilon := \{x \in U \mid d(x,K) < \epsilon\}$ is connected.*

Proof. Let $F := \mathbb{R}^n \setminus U$. This set is closed and bounded and, therefore, compact. We show that F is connected. Let $f : F \to \{0,1\}$ be continuous. Then f is constant on K. Take $x \in F\setminus K$. For $0 \neq a \in \mathbb{R}^n$ consider the half-line $\{x + ta \mid 0 \leq t\}$. Let t_0 be the smallest number so that $x + t_0 a \in K$. Then, the line segment $\{x+ta \mid 0 \leq t \leq t_0\}$ is a subset of F, and we conclude that f must be constant on F so F is connected. Next define $F_\epsilon := \{x \in \mathbb{R}^n \mid d(x,F) < \epsilon\}$. Since $F_\epsilon = \cup_{p\in F} B_\epsilon(p)$ this is a connected set. Note that $F_\epsilon = U_\epsilon \cup F$. The Mayer–Vietoris sequence for homology tells us that

$$\cdots \to H_1(\mathbb{R}^n) \to H_0(U_\epsilon) \to H_0(U) \oplus H_0(F_\epsilon) \to H_0(\mathbb{R}^n) \to 0,$$

from which we conclude that U_ϵ is connected. □

Proof of Proposition A.1. Let V be a connected component of $H^{-1}((0,\infty))$. Let $V' \subset \mathbb{R}^n$ be the image of V when compactifying $\mathbb{R}^n$ with a point at infinity and then removing a point p lying in an unbounded component of $\mathbb{R}^n \setminus V$. The set V' is open, bounded and connected, so the closure $K := \overline{V'}$ is compact and connected. Let $\partial^\infty K$ be the part of the boundary of K facing the unbounded component of $\mathbb{R}^n \setminus K$. Since the intersection of a nested sequence of compact connected sets is connected we conclude from the Lemma that $\partial^\infty K$ is connected. Going back to V this means that the union $\partial^\infty V \cup \{\infty\}$ is connected, where $\partial^\infty V$ is the part of the boundary facing the component of $\mathbb{R}^n \backslash V$ containing p. From this, we see that all components of $\partial^\infty V$ must be unbounded, since if there was a bounded component this would remain disconnected from the others when adding the point at infinity. Finally, every component of $\partial^\infty V$ is contained in some connected component of $H^{-1}(0)$, and those components of $H^{-1}(0)$ are, therefore, unbounded. □

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