

Near-Equality of the Penrose Inequality for Rotationally Symmetric Riemannian Manifolds

Dan A. Lee and Christina Sormani

Abstract. This article is the sequel to Lee and Sormani (Stability of the Positive Mass Theorem for Rotationally Symmetric Riemannian Manifolds, 2011. Preprint), which dealt with the near-equality case of the Positive Mass Theorem. We study the near-equality case of the Penrose Inequality for the class of complete asymptotically flat rotationally symmetric Riemannian manifolds with nonnegative scalar curvature whose boundaries are outermost minimal hypersurfaces. Specifically, we prove that if the Penrose Inequality is sufficiently close to being an equality on one of these manifolds, then it must be close to a Schwarzschild space with an appended cylinder, in the sense of Lipschitz distance. Since the Lipschitz distance bounds the intrinsic flat distance on compact sets, we also obtain a result for intrinsic flat distance, which is a more appropriate distance for more general near-equality results, as discussed in Lee and Sormani (Stability of the Positive Mass Theorem for Rotationally Symmetric Riemannian Manifolds, 2011. Preprint).

1. Introduction

The (Riemannian) Penrose Inequality states that if (M^n, g) is a complete asymptotically flat manifold of nonnegative scalar curvature whose boundary is an outermost minimal hypersurface, then the ADM mass of (M, g) satisfies

$$m_{\text{ADM}} \geq \frac{1}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \quad (1)$$

where $|\partial M|$ denotes the (hyper-)area of ∂M , and ω_{n-1} is the (hyper-)area of the standard $(n-1)$ -sphere in \mathbb{R}^n . Furthermore, if equality holds, then (M, g)

Lee is partially supported by a PSC CUNY Research Grant and NSF DMS #0903467. Sormani is partially supported by a PSC CUNY Research Grant and NSF DMS #1006059.

must be isometric to a Riemannian Schwarzschild manifold (or more precisely, the part lying outside its outermost minimal hypersurface). The second statement may be thought of as a rigidity theorem, and it is natural to consider the *stability* of this rigidity statement. That is, if the ratio of the two sides of the inequality is close to one, then in what sense can we say that the manifold is “close” to a Schwarzschild manifold?

The ADM mass was defined by Arnowitt et al. [2]. The Penrose Inequality is a refinement of the positive mass theorem, which was proved by Schoen and Yau in dimensions less than eight [16, 17], and by Witten for spin manifolds [3, 19]. The 3-dimensional Penrose Inequality was first proved by Huisken and Ilmanen [12] (where $|\partial M|$ must be replaced by the area of the largest component of ∂M), and later proved in full by H. Bray using a different method [5]. Bray and the first author extended Bray’s proof to spin manifolds of dimension less than eight [6].

The problem of stability for the positive mass theorem has been studied by the first author in [14], by Finster with Bray and Kath in [4, 8, 9], and by Corvino [7]. Since the general problem of stability is a difficult one, we considered the special case of rotational symmetry (that is, $\text{SO}(n)$ symmetry), and in that case the authors were able to prove a comprehensive stability result [15]. Although there is no Lipschitz stability in that setting (in the sense of Lipschitz distance defined below), we formulated the stability in terms of intrinsic flat distance, a notion defined by the second author and Wenger in [18]. This article is a natural extension of that investigation to the problem of stability for the Penrose Inequality.

We find that stability fails for the Penrose Inequality in the sense that manifolds with almost equality in the Penrose Inequality are not necessarily close to Schwarzschild space in any reasonable topology. Specifically, they may instead be close to Schwarzschild spaces with a cylinder of arbitrary length appended to the boundary. See Example 5.2 depicted in Fig. 1 below. This behavior is to be expected because the rigidity in the Penrose Inequality only applies to manifolds whose boundaries are outermost minimal hypersurfaces. Since this class of Riemannian manifolds is not closed in any reasonable topology, one does not expect a true stability result. Before we state our result, we provide a few definitions.

Definition 1.1. Given $n \geq 3$, we say that a complete, smooth Riemannian n -manifold is in RotSym_n^∂ if and only if it is rotationally symmetric (that is, $\text{SO}(n)$ symmetry); it has nonnegative scalar curvature; its connected non-empty boundary is an outermost minimal hypersurface; and its symmetric $(n - 1)$ -spheres have unbounded (hyper-)area.

The class RotSym_n^∂ includes the Schwarzschild spaces. The nonnegative scalar curvature condition corresponds to the physical assumption of nonnegative mass density in the time-symmetric setting. The outermost condition is included here, just as it is in the Penrose Inequality, because complicated geometry can “hide” behind a minimal hypersurface without affecting the ADM mass cf. [10, 12]). The outermost minimal boundary is often called an

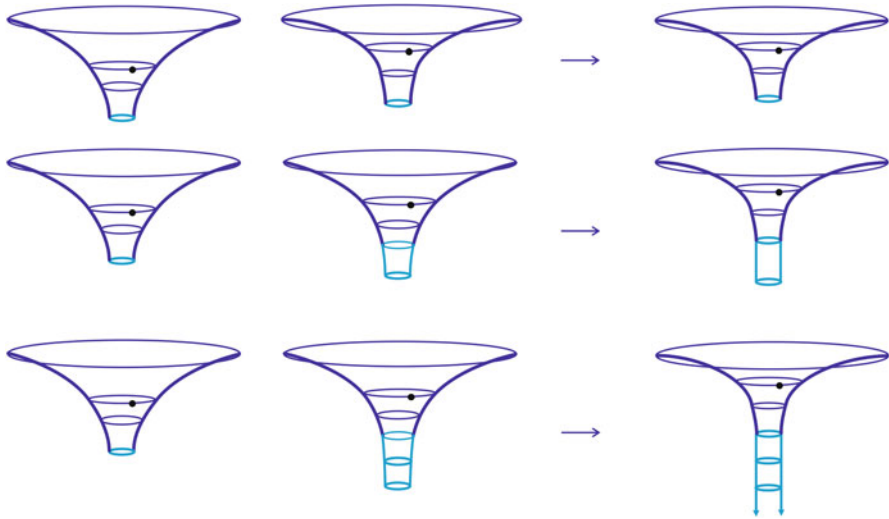


FIGURE 1. Sequences approaching various limits, $M_{\text{Sch}}(m, L)$

apparent horizon. The last condition is much weaker than asymptotic flatness but strong enough to exclude asymptotically cylindrical manifolds.

Recall the Lipschitz distance between metric spaces as defined in [11]:

Definition 1.2. Given two metric spaces (X, d_X) and (Y, d_Y) , the Lipschitz distance

$$d_{\mathcal{L}}(X, Y) := \inf \{ |\log \text{dil}(\varphi)| + |\log \text{dil}(\varphi^{-1})| : \varphi : X \longrightarrow Y \text{ is bi-Lipschitz} \} \tag{2}$$

where

$$\text{dil}(\varphi) = \sup_{x_i \in X} \frac{d_Y(\varphi(x_1), \varphi(x_2))}{d_X(x_1, x_2)}. \tag{3}$$

Definition 1.3. For any $m > 0$, define $M_{\text{Sch}}^n(m)$ to be the part of the n -dimensional Schwarzschild space of mass m that lies outside its outermost minimal hypersurface. For any $L \in [0, \infty)$, define $M_{\text{Sch}}^n(m, L)$ to be $M_{\text{Sch}}^n(m)$ with the cylinder $[-L, 0] \times S^{n-1}((2m)^{\frac{1}{n-2}})$ glued to its outermost minimal hypersurface boundary, where the sphere $S^{n-1}((2m)^{\frac{1}{n-2}}) \cong \partial M_{\text{Sch}}^n(m)$ denotes the standard $(n - 1)$ -sphere of radius $(2m)^{\frac{1}{n-2}}$.

Theorem 1.4. Let $n \geq 3$. For any $\epsilon > 0$, there exists $\delta > 0$ such that if $M \in \text{RotSym}_n^\partial$ satisfies

$$m_{\text{ADM}} \leq \frac{1 + \delta}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{4}$$

then

$$d_{\mathcal{L}}(M, M_{\text{Sch}}^n(m_0, L)) < \epsilon, \tag{5}$$

where $m_0 = \frac{1}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$, and L is the depth of M , as defined in Sect. 3.1.

In other words, if the ratio of the two sides of the Penrose Inequality is close enough to one, then the space must be close to an appended Schwarzschild space in Lipschitz distance, see Theorem 3.4 for a more precise statement implicitly describing the dependence of δ on ϵ .

As alluded to above, the reason why we need the appended Schwarzschild spaces is that we may have a sequence of manifolds in RotSym_n^∂ whose limit has a boundary that is not outermost. This situation is described in Example 5.2 and depicted in Fig. 1 below. Example 5.4 demonstrates that one cannot improve the Lipschitz convergence to C^2 convergence.

Note that Theorem 1.4 does not hold without the assumption of rotational symmetry, as seen in Example 5.5. The basic reason why we obtain such a strong result in Theorem 1.4 is that rotational symmetry combined with the existence of a boundary of fixed size rules out the possibility of a thin, deep gravity well. These deep gravity wells are the reason why neither Lipschitz distance nor even Gromov–Hausdorff distance provide a useful topology for a more general theorem. This is discussed in our earlier work [15], in which we propose intrinsic flat distance as an appropriate topology.

The intrinsic flat distance was defined by the second author and Wenger in [18] based on the work of Ambrosio and Kirchheim [1]. It estimates the distances between compact Riemannian n -manifolds by filling in (in the sense of metric space isometric embeddings) the space between them with a countably \mathcal{H}^{n+1} rectifiable metric space and measuring the \mathcal{H}^{n+1} measure of the filling space and the \mathcal{H}^n measure of any excess boundary. It was proved in [18] that the intrinsic flat distance between two Riemannian manifolds with boundary can be bounded in terms of the Lipschitz distance between them, their diameters, their volumes and the (hyper-)areas of their boundaries. In recent work of the second author and Lakzian, an explicit filling manifold is constructed between two spaces which are close in the Lipschitz sense [13], see Sect. 4.1 for the precise statement. Using this bound we conclude the following:

Theorem 1.5. *Let $n \geq 3$, and let $0 < A_0 < A_1$. For any $D > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $M \in \text{RotSym}_n^\partial$ satisfies $|\partial M| = A_0$ and*

$$m_{\text{ADM}} \leq \frac{1 + \delta}{2} \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{6}$$

then

$$d_{\mathcal{F}}(T_D(\Sigma_1) \subset M, T_D(\bar{\Sigma}_1) \subset M_{\text{Sch}}^n(m_0, L)) < \epsilon, \tag{7}$$

where $m_0 = \frac{1}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$, L is the depth of M , Σ_1 and $\bar{\Sigma}$ are the respective symmetric spheres of area A_1 in M and $M_{\text{Sch}}(m_0, L)$, T_D denotes the tubular neighborhood of radius D , and $d_{\mathcal{F}}$ denotes intrinsic flat distance.

For more discussion of intrinsic flat distance and its application to the study of nonnegative scalar curvature, see [15]. Note that a scale-invariant version of this result would require a scalable version of intrinsic flat distance.

In the final section of the paper we provide examples and propose that some version of Theorem 1.5 holds without the restriction of rotational symmetry, see Conjecture 5.9 and subsequent remarks.

2. Basic Facts About RotSym_n^∂

2.1. Geodesic Coordinates

In this article we consider Riemannian manifolds (M, g) in RotSym_n^∂ , defined in Definition 1.1. Since (M, g) is rotationally symmetric we can write its metric in geodesic coordinates, as $g = ds^2 + r(s)^2 g_0$ for some function $r : [0, \infty) \rightarrow (0, \infty)$, where g_0 is the standard metric on the $(n - 1)$ -sphere and s is the distance from the boundary, ∂M .

Let Σ be the symmetric sphere that is a distance s from the boundary. We then have the following formulae for the (hyper-)area and mean curvature:

$$|\Sigma| = \omega_{n-1} r^{n-1} \tag{8}$$

$$H_\Sigma = \frac{n-1}{r} \frac{dr}{ds}. \tag{9}$$

At a point $p \in \Sigma$ the scalar curvature is

$$R = \frac{n-1}{r^2} \left((n-2) \left[1 - \left(\frac{dr}{ds} \right)^2 \right] - 2r \frac{d^2 r}{ds^2} \right). \tag{10}$$

Since ∂M is an outermost minimal hypersurface, $\frac{dr}{ds}|_{s=0} = 0$, and $\frac{dr}{ds}$ is either positive for all $s > 0$ or negative for all $s > 0$. The last condition in the definition of RotSym_n^∂ states that $r(s)$ is unbounded, therefore it must be the case that

$$\frac{dr}{ds} > 0 \quad \forall s \in (0, \infty). \tag{11}$$

Recall the definition of the Hawking mass of a surface Σ in a three-dimensional manifold:

$$m_H(\Sigma) := \frac{1}{2} \sqrt{\frac{|\Sigma|}{\omega_2}} \left(1 - \frac{1}{\omega_2} \int_\Sigma \left(\frac{H_\Sigma}{2} \right)^2 \right). \tag{12}$$

We define a natural Hawking mass function on symmetric spheres Σ in $M \in \text{RotSym}_n^\partial$ that agrees with the usual definition in dimension three.

$$m_H(\Sigma) := \frac{r^{n-2}}{2} \left(1 - \left(\frac{dr}{ds} \right)^2 \right). \tag{13}$$

Alternatively, we may view m_H as a function of s . Applying (10), one can compute

$$\frac{dm_H}{ds} = \frac{r^{n-1}}{2(n-1)} \frac{dr}{ds} R. \tag{14}$$

Since we are studying manifolds with $\frac{dr}{ds} > 0$ for $s > 0$, we observe that the monotonicity of the Hawking mass, usually called *Geroch monotonicity*.

$$\frac{dm_H}{ds} \geq 0, \tag{15}$$

is actually *equivalent* to nonnegativity of R .

We define the ADM mass of M to be the limit of the Hawking mass function:

$$m_{\text{ADM}}(M) := \lim_{s \rightarrow \infty} m_H \in [0, \infty]. \tag{16}$$

For asymptotically flat manifolds in RotSym_n^∂ , this agrees with the usual definition of the ADM mass. Note that the Penrose Inequality for manifolds in RotSym_n^∂ then follows immediately from Geroch monotonicity (15).

2.2. Graphical Coordinates

Since $\frac{dr}{ds} > 0$ for $s > 0$, the map $s \mapsto r(s)$ defines a smooth change of coordinates away from ∂M . In r coordinates, the metric takes the form

$$g = \left(\frac{ds}{dr}\right)^2 dr^2 + r^2 g_0. \tag{17}$$

By Geroch monotonicity (15), we see that $\frac{dr}{ds} < 1$ everywhere. We now choose $z(r)$ to be an increasing function (determined up to a constant) such that

$$1 + \left(\frac{dz}{dr}\right)^2 = \left(\frac{ds}{dr}\right)^2, \tag{18}$$

so that

$$g = \left(1 + \left(\frac{dz}{dr}\right)^2\right) dr^2 + r^2 g_0. \tag{19}$$

Note that this formula exhibits a Riemannian isometric embedding of (M, g) into \mathbb{R}^{n+1} as the graph of the radial function $z(r)$. That is, we may view

$$M = \{(x', x^{n+1}) \in \mathbb{R}^{n+1} : x^{n+1} = z(|x'|\}\}. \tag{20}$$

For this reason, we call this choice of coordinates *radial graphical coordinates*.

We also observe that $r \mapsto z(r)$ defines a smooth change of coordinates away from ∂M , and in z coordinates, the metric is

$$g = \left(\frac{ds}{dz}\right)^2 dz^2 + r(z)^2 g_0. \tag{21}$$

Also note that unlike the r -coordinates, these z -coordinates are nonsingular at ∂M . We call this choice of coordinates *vertical graphical coordinates*.

2.3. Appended Schwarzschild Spaces

We recall that the n -dimensional Schwarzschild space $M_{\text{Sch}}^n(m)$ of mass m may be described in radial graphical coordinates as the manifold $[(2m)^{\frac{1}{n-2}}, \infty) \times S^{n-1}$ with the metric

$$g_{\text{Sch}} = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 g_0. \tag{22}$$

In vertical graphical coordinates, the metric becomes

$$g_{\text{Sch}} = \frac{r(z)^{n-2}}{2m} dz^2 + r(z)^2 g_0, \tag{23}$$

where

$$\frac{dz}{dr} = \sqrt{\frac{2m}{r^{n-2} - 2m}}. \tag{24}$$

By convention, we choose vertical graphical coordinates so that $\partial M_{\text{Sch}}(m)$ has $z = 0$. For $L \geq 0$, the appended Schwarzschild space $M_{\text{Sch}}(m, L)$ is obtained by gluing $M_{\text{Sch}}(m)$ to the space $[-L, 0] \times S^{n-1}$ with the cylindrical metric

$$g_{\text{cyl}} = dz^2 + (2m)^{\frac{2}{n-2}} g_0 \tag{25}$$

along their common sphere at $z = 0$.

3. Lipschitz Estimates

In this section we define the depth and prove all Lipschitz estimates needed to prove our main results.

3.1. Depth

Let $M \in \text{RotSym}_n^\partial$, and assume that m_{ADM} is finite. For convenience, we express the important quantities m_{ADM} and $|\partial M|$ in different units, defining

$$m_0 := \frac{1}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}} > 0 \tag{26}$$

$$r_0 := r(0) = (2m_0)^{\frac{1}{n-2}} = \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{1}{n-1}} > 0 \tag{27}$$

$$r_1 := (2m_{\text{ADM}})^{\frac{1}{n-2}} > 0. \tag{28}$$

These quantities will be applied to define the depth and again in subsequent proofs.

We define δ so that

$$m_{\text{ADM}} = \frac{1 + \delta}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{29}$$

Using the definitions above, we can rewrite this as

$$r_1^{n-2} = (1 + \delta)r_0^{n-2}. \tag{30}$$

We now define r_δ and A_δ so that

$$r_\delta := (1 + \sqrt{\delta})^{\frac{1}{n-2}} r_0 \tag{31}$$

$$A_\delta := \omega_{n-1} r_\delta^{n-1}. \tag{32}$$

Recall that $M_{\text{Sch}}(m_0)$ may be isometrically embedded into \mathbb{R}^{n+1} with its boundary embedded at the level $z = 0$ and radius $r = r_0$, as described in Sect. 2.3. We can also isometrically embed M into \mathbb{R}^{n+1} via the function $z(r)$ introduced in Sect. 2.2, and this embedding is defined only up to constant. Let us choose this constant so that M intersects the standard embedding of $M_{\text{Sch}}(m_0)$ in \mathbb{R}^{n+1} at $r = r_\delta$, see Fig. 2 below.

Definition 3.1. We define the *depth* of $M \in \text{RotSym}_n^\partial$ to be L such that when M is embedded into \mathbb{R}^{n+1} as described above and depicted in Fig. 2, ∂M embeds into $r = r_0$ and $z = -L$.

Remark 3.2. We note that although the depth is an invariant of $M \in \text{RotSym}_n^\partial$ which could plausibly be generalized to non-rotationally symmetric spaces, the definition is not quite natural in the sense that it depends on an arbitrary choice for what r_δ should be. As one can see from the proof in the next section, if we were to replace $\sqrt{\delta}$ by $\delta^{1/3}$ in the definition of depth, Theorem 1.4 would still hold true. Nevertheless, the concept of depth may be worthy of further investigation.

Lemma 3.3. *The depth L is always nonnegative.*

Proof. Consider the functions $r(s)$ and $z(r)$ as defined for the manifold M . By Geroch monotonicity (15), we have

$$m_H(\partial M) \leq m_H(\Sigma) \leq m_{\text{ADM}} \tag{33}$$

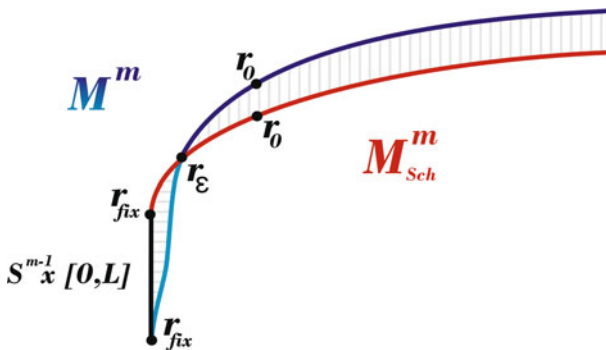


FIGURE 2. The depth L and the bi-Lipschitz map φ are defined by embedding M and $M_{\text{Sch}}(m_0)$ into \mathbb{R}^{n+1} so that they intersect at $r = r_\delta$

for any symmetric sphere Σ . By the definitions of Hawking mass (13) and r_0 and r_1 in (27) and (28), this becomes

$$r_0^{n-2} \leq r^{n-2} \left(1 - \left(\frac{dr}{ds} \right)^2 \right) \leq r_1^{n-2}. \tag{34}$$

Therefore

$$1 - \left(\frac{r_0}{r} \right)^{n-2} \geq \left(\frac{dr}{ds} \right)^2 \geq 1 - \left(\frac{r_1}{r} \right)^{n-2}. \tag{35}$$

By combining (35) and (18), we see that for all $r \geq r_0$,

$$1 - \left(\frac{r_0}{r} \right)^{n-2} \geq \left(1 + \left(\frac{dz}{dr} \right)^2 \right)^{-1} \tag{36}$$

$$\frac{r^{n-2}}{r^{n-2} - r_0^{n-2}} \leq 1 + \left(\frac{dz}{dr} \right)^2 \tag{37}$$

$$\frac{dz}{dr} \geq \sqrt{\frac{r_0^{n-2}}{r^{n-2} - r_0^{n-2}}} = \sqrt{\frac{2m_0}{r^{n-2} - 2m_0}}. \tag{38}$$

Using (24) and the fact that the graphs of M and $M_{\text{Sch}}(m_0)$ intersect at $r = r_\delta$, we can integrate the above inequality to see that M cannot lie above $M_{\text{Sch}}(m_0)$ in \mathbb{R}^{n+1} in the region where $r \leq r_\delta$. In particular, this means that ∂M cannot lie above $\partial M_{\text{Sch}}(m_0)$, so that $-L = z(r_0) \leq 0$, as desired. \square

3.2. Proof of Theorem 1.4

Theorem 1.4 is an easy consequence of the following theorem.

Theorem 3.4. *Let $n \geq 3$, and let $M \in \text{RotSym}_n^\partial$. Define δ so that*

$$m_{\text{ADM}} = \frac{1 + \delta}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{39}$$

and assume $0 \leq \delta < 1$. Define $m_0 = \frac{1}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$, $A_\delta = (1 + \sqrt{\delta})^{\frac{n-1}{n-2}} |\partial M|$, and let L be depth of M . Then there exists a rotationally symmetric bi-Lipschitz map $\varphi : M \rightarrow M_{\text{Sch}}(m_0, L)$ that maps symmetric spheres of area $A \geq A_\delta$ to symmetric spheres of area A such that for any tangent vector $v \in TM$,

$$h_\delta^{-1} \leq \frac{g(v, v)}{g_{\text{Sch}, L}(\varphi_* v, \varphi_* v)} \leq h_\delta \tag{40}$$

where

$$h_\delta = \max \left\{ (1 + \sqrt{\delta})^{\frac{2}{n-2}}, (1 + \sqrt{\delta})(1 + \delta), (1 - \sqrt{\delta})^{-1} \right\}. \tag{41}$$

Theorem 1.4 follows from this theorem because $\sqrt{h_\delta}$ is then a bound on $\text{dil}(\varphi)$ and $\text{dil}(\varphi^{-1})$, so that

$$d_{\mathcal{L}}(M, M_{\text{Sch}}(m_0, L)) \leq 2 \log \sqrt{h_\delta} = \log(h_\delta), \tag{42}$$

which converges to 0 as $\delta \rightarrow 0$.

Proof. We will prove Theorem 3.4 by explicitly constructing the bi-Lipschitz map. Let (M, g) , δ , m_0 , A_δ , and L be as described in the statement of the theorem. We define r_0 , r_1 , and r_δ as in Sect. 3.1, and we isometrically embed M and $M_{\text{Sch}}(m_0)$ into \mathbb{R}^{n+1} as in Sect. 3.1. Then we embed $M_{\text{Sch}}(m_0, L)$ so that $\partial M = \partial M_{\text{Sch}}(m_0, L)$ and such that the Schwarzschild domain within $M_{\text{Sch}}(m_0, L)$ agrees with the embedding of $M_{\text{Sch}}(m_0)$, see Fig. 2 above.

We define the bi-Lipschitz map $\varphi : M \rightarrow M_{\text{Sch}}(m_0, L)$ as follows. In the region where $r \geq r_\delta$, φ projects M vertically in \mathbb{R}^{n+1} to $M_{\text{Sch}}(m_0, L)$. That is, under this map, the two manifolds have the same r coordinates and sphere coordinates, but different z coordinates. In particular φ preserves the areas of symmetric spheres of area $A \geq A_\delta$. In the region where $r \leq r_\delta$, φ projects M horizontally in \mathbb{R}^{n+1} to $M_{\text{Sch}}(m_0, L)$. That is, under this map, the two manifolds have the z coordinates and sphere coordinates, but different r coordinates. The vertical and horizontal lines in Fig. 2 depict the map φ . Since the two definitions match up at $r = r_\delta$, this map is clearly bi-Lipschitz.

We now prove (40) for the region where $r \geq r_\delta$. Recall that in this region, φ preserves both the r coordinate and the sphere coordinates, so it suffices to assume v is purely radial. Note that the assumption that $\delta < 1$ implies that $r_0 \leq r_1 < r_\delta$, so that when $r \geq r_\delta$, (35) tells us that

$$1 - \left(\frac{r_0}{r}\right)^{n-2} \geq \left(\frac{dr}{ds}\right)^2 \geq 1 - \left(\frac{r_1}{r}\right)^{n-2} > 0. \tag{43}$$

By comparing (17) to (22), the inequalities in (43) tell us that for any radial tangent vector $v \in TM$ based at a point with $r \geq r_\delta$,

$$\frac{1 - \frac{2m_0}{r^{n-2}}}{1 - \left(\frac{r_0}{r}\right)^{n-2}} \leq \frac{g(v, v)}{g_{\text{Sch}, L}(\varphi_*v, \varphi_*v)} \leq \frac{1 - \frac{2m_0}{r^{n-2}}}{1 - \left(\frac{r_1}{r}\right)^{n-2}}. \tag{44}$$

By the definition of r_0 , we have

$$1 \leq \frac{g(v, v)}{g_{\text{Sch}, L}(\varphi_*v, \varphi_*v)} \leq \frac{1 - \left(\frac{r_0}{r}\right)^{n-2}}{1 - \left(\frac{r_1}{r}\right)^{n-2}}. \tag{45}$$

One can see that over $r \in [r_\delta, \infty)$, the right hand side is maximized at $r = r_\delta$, so that

$$\frac{g(v, v)}{g_{\text{Sch}, L}(\varphi_*v, \varphi_*v)} \leq \frac{r_\delta^{n-2} - r_0^{n-2}}{r_\delta^{n-2} - r_1^{n-2}} = \frac{1}{1 - \sqrt{\delta}}. \tag{46}$$

We now prove (40) for the region where $r_0 \leq r \leq r_\delta$. Since in this region, φ changes the r -coordinate, we must separately consider vectors that are tangent to the symmetric spheres, as well as ones that are orthogonal to the symmetric spheres. However, the image of a symmetric sphere in M with radius $r_0 \leq r \leq r_\delta$ under φ is another symmetric sphere with radius $r_0 \leq r \leq r_\delta$. Therefore it is clear that for any $v \in TM$ that is tangent to a symmetric sphere

and based at a point with $r_0 \leq r \leq r_\delta$,

$$\left(\frac{r_0}{r_\delta}\right)^2 \leq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \leq \left(\frac{r_\delta}{r_0}\right)^2 \tag{47}$$

$$(1 + \sqrt{\delta})^{-\frac{2}{n-2}} \leq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \leq (1 + \sqrt{\delta})^{\frac{2}{n-2}}. \tag{48}$$

Now let us consider tangent vectors that are radial, that is, orthogonal to the symmetric spheres. Using (34), we see that for any $r \geq r_0$,

$$\left(\frac{r_0}{r}\right)^{n-2} \leq \left(1 - \left(\frac{dr}{ds}\right)^2\right) \leq \left(\frac{r_1}{r}\right)^{n-2}. \tag{49}$$

By (18), this may be written

$$\left(\frac{r_0}{r}\right)^{n-2} \leq \left(\frac{dz}{ds}\right)^2 \leq \left(\frac{r_1}{r}\right)^{n-2}. \tag{50}$$

Consider a radial tangent vector $v \in TM$ based at a point with $r_0 \leq r \leq r_\delta$. We consider two cases. In the first case, $z(r) \geq 0$, so that φ projects the point to a point on $M_{\text{Sch}}(m_0)$. Recall that in this region, φ does not change z . So by comparing (21) to (23), the inequalities in (50) tell us that

$$\left(\frac{r(z)}{r_0}\right)^{n-2} \frac{2m_0}{r_{\text{Sch}}(z)^{n-2}} \geq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \geq \left(\frac{r(z)}{r_1}\right)^{n-2} \frac{2m_0}{r_{\text{Sch}}(z)^{n-2}} \tag{51}$$

$$\left(\frac{r_\delta}{r_0}\right)^{n-2} \geq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \geq \left(\frac{r_0}{r_1}\right)^{n-2} \left(\frac{r_0}{r_\delta}\right)^{n-2} \tag{52}$$

$$1 + \sqrt{\delta} \geq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \geq (1 + \delta)^{-1}(1 + \sqrt{\delta})^{-1}, \tag{53}$$

where $r_{\text{Sch}}(z)$ describes how the r coordinate on $M_{\text{Sch}}(m_0)$ depends on z . In the second case, v is based at a point with $z(r) < 0$, so that it projects to the cylindrical part of $M_{\text{Sch}}(m_0, L)$. By comparing (21) to (25), then inequalities in (50) tell us that

$$\left(\frac{r(z)}{r_0}\right)^{n-2} \geq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \geq \left(\frac{r(z)}{r_1}\right)^{n-2} \tag{54}$$

$$1 + \sqrt{\delta} \geq \frac{g(v, v)}{g_{\text{Sch},L}(\varphi_*v, \varphi_*v)} \geq (1 + \delta)^{-1}. \tag{55}$$

So we have (40) for vectors tangent to symmetric spheres or perpendicular to them. In this situation, one can also see that it holds for their linear combinations. □

3.3. Lipschitz Estimates for Tubular Neighborhoods

The following proposition is useful for the proof of Theorem 1.5.

Proposition 3.5. *Let $n \geq 3$, and let $0 < A_0 < A_1$. For any $D > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $M \in \text{RotSym}_n^\delta$ satisfying $|\partial M| = A_0$ and*

$$m_{\text{ADM}} < \frac{1 + \delta}{2} \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{56}$$

we have

$$d_{\mathcal{L}}(T_D(\Sigma_1) \subset M, T_D(\bar{\Sigma}_1) \subset M_{\text{Sch}}^n(m_0, L)) < \epsilon, \tag{57}$$

where $m_0 = \frac{1}{2} \left(\frac{|\partial M|}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$, L is the depth of M , Σ_1 and $\bar{\Sigma}$ are the respective symmetric spheres of area A_1 in M and $M_{\text{Sch}}(m_0, L)$, T_D denotes the tubular neighborhood of radius D , and $d_{\mathcal{L}}$ denotes Lipschitz distance.

Proof. Following the proof of Theorem 1.4, for small enough δ there exist $L \geq 0$ and a bi-Lipschitz map $\varphi : M \rightarrow M_{\text{Sch}}(m_0, L)$ such that

$$d_{\mathcal{L}}(T_D(\Sigma_1), \varphi(T_D(\Sigma_1))) < \epsilon/2. \tag{58}$$

While $\varphi(\Sigma_1) = \bar{\Sigma}_1$, it is not true that $\varphi(T_D(\Sigma_1)) = T_D(\bar{\Sigma}_1)$. We can define a bi-Lipschitz map $\psi : \varphi(T_D(\Sigma_1)) \rightarrow T_D(\bar{\Sigma}_1)$ that preserves $\bar{\Sigma}_1$ while scaling the distance to $\bar{\Sigma}_1$ by a constant factor on the part inside $\bar{\Sigma}_1$ and another constant factor on the outside part. Since we know that the ratios between distances in M and $M_{\text{Sch}}(m_0, L)$ tend to 1 as $\delta \rightarrow 0$, we can arrange for these constant factors to be close to 1. This controls $g_{\text{Sch},L}(v, v)/g_{\text{Sch},L}(\psi_*v, \psi_*v)$ for radial tangent vectors v . And since $\frac{dr_{\text{Sch}}}{ds} \leq 1$, we can also control $g_{\text{Sch},L}(v, v)/g_{\text{Sch},L}(\psi_*v, \psi_*v)$ for v that are tangent to the symmetric spheres. So we can choose δ small enough so that

$$d_{\mathcal{L}}(\varphi(T_D(\Sigma_1)), T_D(\bar{\Sigma}_1)) < \epsilon/2, \tag{59}$$

completing the proof of (57). □

4. Intrinsic Flat Distance

4.1. Review of the Intrinsic Flat Distance

The intrinsic flat distance measures the distances between compact oriented Riemannian manifolds by filling in the space between them. This notion was first defined in work of the second author and Wenger in [18] applying the theory of integral currents on metric spaces developed by Ambrosio and Kirchheim [1]. However, here we estimate the intrinsic flat distance using only an understanding of Riemannian geometry.

Given two compact oriented Riemannian manifolds M_1^n and M_2^n with boundary, and metric isometric embeddings $\psi_i : M_i \rightarrow Z$ into some Riemannian manifold (possibly piecewise smooth with corners) an upper bound for the intrinsic flat distance is attained as follows:

$$d_{\mathcal{F}}(M_1^n, M_2^n) \leq \text{vol}_{n+1}(B^{n+1}) + \text{vol}_n(A^n) \tag{60}$$

where B^{n+1} is an oriented region in Z and A^n is defined so that the oriented integrals satisfy

$$\int_{\psi_1(M_1)} \omega - \int_{\psi_2(M_2)} \omega = \int_{\partial B} \omega + \int_A \omega \tag{61}$$

for any differential n -form ω on Z . We call B^{n+1} a *filling manifold* between M_1 and M_2 and A^n the *excess boundary*.

Recall that a metric isometric embedding, $\psi : M \rightarrow Z$ is a map such that

$$d_Z(\psi(x), \psi(y)) = d_M(x, y) \quad \forall x, y \in M. \tag{62}$$

This is significantly stronger than a Riemannian isometric embedding which preserves only the Riemannian structure and thus lengths of curves but not distances between points as in (62). For example, S^1 has a metric isometric embedding as the equator of a hemisphere but only a Riemannian isometric embedding as a circle in a plane.

Theorem 5.6 of [18] states that the intrinsic flat distance between oriented Riemannian manifolds may be bounded as follows:

$$d_{\mathcal{F}}(M_1, M_2) \leq \left(\frac{n+1}{2}\right) \lambda^{n-1}(\lambda-1) \max\{D_1, D_2\}(V_1 + A_1) \tag{63}$$

where $\lambda = e^{d_{\mathcal{L}}(M_1, M_2)}$, $D_i = \text{diam}(M_i)$, $V_i = \text{vol}_n(M_i)$ and $A_i = \text{vol}_{n-1}(\partial M_i)$. This is proved using the theory of currents.

In work of the second author and Lakzian, the following more constructive statement is proved using Riemannian geometry [13]. Here the statement provides the explicit Riemannian manifold Z which in this case is the filling manifold, B , and

$$A = \partial B \setminus (\psi_1(M_1) \cup \psi_2(M_2)). \tag{64}$$

Suppose M_1 and M_2 are oriented precompact Riemannian manifolds with a bi-Lipschitz map $\varphi : M_1 \rightarrow M_2$ and suppose there exists $\epsilon > 0$ such that

$$(1 + \epsilon)^{-2} g_1(V, V) < g_2(\varphi_* V, \varphi_* V) < (1 + \epsilon)^2 g_1(V, V) \quad \forall V \in TM. \tag{65}$$

Then for any

$$t_2 - t_1 > \frac{\arccos(1 + \epsilon)^{-1}}{\pi} \max\{\text{diam}(M_1), \text{diam}(M_2)\} \tag{66}$$

there is a pair of metric isometric embeddings $\psi_i : M_i \rightarrow Z = \bar{M} \times [t_1, t_2]$ with a metric

$$g' \geq dt^2 + \max_{i=1,2} \cos^2((t - t_i)\pi/D_i) g_i \text{ on } Z \tag{67}$$

and

$$g' = dt^2 + g_i \text{ on } \psi_i(M) = M \times \{t_i\}. \tag{68}$$

Thus the intrinsic flat distance between the manifolds

$$d_{\mathcal{F}}(M_1, M_2) \leq 2|t_2 - t_1|(V_1 + V_2 + A_1 + A_2), \tag{69}$$

where $D_i = \text{diam}(M_i)$, $V_i = \text{vol}_n(M_i)$ and $A_i = \text{vol}_{n-1}(\partial M_i)$. In the proof of this theorem, the cosine term in (67) arises from a comparison to the equator isometrically embedded in a sphere of diameter D_i .

Constructive estimates on the intrinsic flat distance are also provided in previous work of the authors studying the stability of the positive mass theorem [15]. In that paper, no bi-Lipschitz maps are constructed. Instead metric isometric embeddings are created from Riemannian isometric embeddings using strips whose widths are bounded by controlling the embedding constant.

4.2. Intrinsic Flat Estimates

Theorem 1.5 now follows from Proposition 3.5 combined with the bound on intrinsic flat distance in terms of Lipschitz distance.

Proof of Theorem 1.5. Assume the hypotheses of Theorem 1.5. Note that the r -coordinate of ∂M is $r_0 := \left(\frac{A_0}{\omega_{n-1}}\right)^{\frac{1}{n-1}}$, while the r -coordinate of both Σ_1 and $\bar{\Sigma}_1$ is $r_1 := \left(\frac{A_1}{\omega_{n-1}}\right)^{\frac{1}{n-1}}$. (Note that this is not the same r_1 from Sect. 3.)

Choose M_1 to be $T_D(\bar{\Sigma}_1) \subset M_{\text{Sch}}(m_0, L)$ and M_2 to be $T_D(\Sigma_1) \subset M$, and apply the bound (63) together with Proposition 3.5 to see that for any $\epsilon > 0$, we can find $\delta > 0$ so that

$$d_{\mathcal{F}}(T_D(\bar{\Sigma}_1), T_D(\Sigma_1)) \leq \left(\frac{n+1}{2}\right) e^{(n-1)\epsilon} (e^\epsilon - 1) \max\{D_1, D_2\} (V_1 + A'_1), \tag{70}$$

where D_1, D_2, V_1 , and A'_1 are defined as described just below (63). (Here we use A'_1 instead of A_1 because A_1 is already in use in this proof.)

Using monotonicity of the areas of the level sets and the triangle inequality, we have

$$D_1 = \text{diam}(T_D(\bar{\Sigma}_1)) \leq 2D + \pi(r_1 + D) \tag{71}$$

$$D_2 = \text{diam}(T_D(\Sigma_1)) \leq 2D + \pi(r_1 + D) \tag{72}$$

$$V_1 = \text{vol}_n(T_D(\bar{\Sigma}_1)) \tag{73}$$

$$\leq DA_0 + \text{vol}_n(r_{\text{Sch}}^{-1}[r_0, r_1 + D]) \subset M_{\text{Sch}}(m_0) \tag{74}$$

$$A'_1 = \text{vol}_{n-1}(\partial T_D(\bar{\Sigma}_1)) \tag{75}$$

$$\leq 2 \text{vol}_{n-1}(r_{\text{Sch}}^{-1}(r_1 + D)) \subset M_{\text{Sch}}(m_0). \tag{76}$$

Since all of these terms are bounded uniformly depending only on r_0, r_1, m_0 , and D , which in turn are bounded in terms of A_0, A_1 , and D , the result follows. □

5. Further Remarks

In the first subsection we explain why it is necessary to introduce a notion of *depth* and use *appended* Schwarzschild manifolds in the statements of our main theorems. We also explain why Theorem 1.4 cannot be strengthened to C^2 closeness of the metrics. In the second subsection, we consider the stability

of the positive mass theorem in [15] and contrast it with the situation for the Penrose Inequality. In the third subsection, we present Example 5.5 demonstrating the necessity of rotational symmetry to obtain the Lipschitz bounds in Theorem 1.4, followed by Remark 5.7 explaining heuristically why these examples may be controlled using the intrinsic flat distance. We close with a subsection describing open problems and conjectures.

5.1. Rotationally Symmetric Examples

First we describe the examples depicted in Fig. 1. Recall from Sect. 2 that for rotationally symmetric manifolds with outermost minimal boundaries, monotonicity of the Hawking mass of the symmetric spheres is *equivalent* to non-negativity of scalar curvature. Therefore, we have the following lemma which will be useful for constructing examples:

Lemma 5.1. *There is a bijection between elements of RotSym_n^∂ and smooth increasing functions $m_H : [r_0, \infty) \rightarrow \mathbb{R}$ such that*

$$m_H(r_0) = \frac{1}{2}r_0^{n-2} \tag{77}$$

and

$$m_H(r) < \frac{1}{2}r^{n-2} \tag{78}$$

for $r > r_0$. In this section we will call these functions *admissible Hawking mass functions*.

This lemma was proved in [15].

Example 5.2. Let $n \geq 3$. Given any $0 < A_0 < A_1$, and any large $L > 0$ and small $\delta > 0$, there exists $M \in \text{RotSym}_n^\partial$ with $|\partial M| = A_0$ such that

$$m_{\text{ADM}} \leq \frac{1 + \delta}{2} \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{79}$$

and also

$$d(\partial M, \Sigma_1) > L, \tag{80}$$

where Σ_1 is the symmetric sphere of area A_1 .

Remark 5.3. This example shows that the appended Schwarzschild manifolds must be included in Theorem 1.4.

Proof. Let $r_i := \left(\frac{A_i}{\omega_{n-1}} \right)^{\frac{1}{n-1}}$. (Again, this r_1 is not the same r_1 from Sect. 3.) Then

$$d(\partial M, \Sigma_1) = \int_{r_0}^{r_1} \frac{ds}{dr} dr. \tag{81}$$

Let $\epsilon > 0$ and define \underline{r}_ϵ and \bar{r}_ϵ so that

$$\underline{r}_\epsilon^{n-2}(1 - \epsilon) = r_0^{n-2} \tag{82}$$

$$\bar{r}_\epsilon^{n-2}(1 - \epsilon) = (1 + \delta)r_0^{n-2}. \tag{83}$$

We define a continuous admissible Hawking function $m_H : [r_0, \infty) \rightarrow \infty$ as follows:

$$m_H(r) = \begin{cases} \frac{1}{2}r_0^{n-2} & \text{for } r_0 \leq r \leq \underline{r}_\epsilon \\ \frac{1}{2}r^{n-2}(1 - \epsilon) & \text{for } \underline{r}_\epsilon \leq r \leq \bar{r}_\epsilon \\ \frac{1}{2}r^{n-2}(1 + \delta) & \text{for } \bar{r}_\epsilon \leq r \end{cases} \tag{84}$$

By Lemma 5.1, this admissible Hawking function defines an element of RotSym_n^∂ , and one can easily see that it has all of the desired properties except for $d(\partial M, \Sigma_1) > L$. To see that we can choose ϵ small enough to make this true, observe that by the definition of m_H (13), we have $\frac{dr}{ds} = \sqrt{\epsilon}$ over the region $\underline{r}_\epsilon \leq r \leq \bar{r}_\epsilon$. Assuming that δ is small enough so that $\bar{r}_\epsilon \leq r_1$, we have

$$d(\partial M, \Sigma_1) \geq \int_{\underline{r}_\epsilon}^{\bar{r}_\epsilon} \frac{1}{\sqrt{\epsilon}} dr \tag{85}$$

$$= \frac{r_0}{\sqrt{\epsilon}} \frac{(1 + \delta)^{\frac{1}{n-2}} - 1}{(1 - \epsilon)^{\frac{1}{n-2}}}, \tag{86}$$

which tends to infinity as $\epsilon \rightarrow 0$. Note that this example can be modified in a straightforward manner to obtain a smooth example. \square

The next example proves that the Lipschitz convergence cannot be improved to C^2 convergence in Theorem 1.4.

Example 5.4. There exists a sequence $M_j \in \text{RotSym}_n^\partial$ such that each $|\partial M_j| = A_0$,

$$\lim_{j \rightarrow \infty} m_{\text{ADM}}(M_j) = \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}, \tag{87}$$

and M_j converges to $M_{\text{Sch}}(m_0)$ in the Lipschitz sense, but not in the C^2 sense, where $m_0 = \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}$.

Proof. Choose a sequence $m_j \searrow m_0$. For each j we construct an admissible Hawking function $m_H^{(j)}$ such that $m_H^{(j)}(r) = m_0$ for $r \in [r_0, 2r_0]$, and then $m_H^{(j)}(r)$ takes a very brief sharp turn upward so that $\frac{m_H^{(j)}}{dr} > \frac{1}{j}$ for some $r \in (2r_0, 3r_0)$, and then $m_H^{(j)}(r) = m_j$ for $r \in [3r_0, \infty)$. By our proof Theorem 1.4, the corresponding manifolds must converge to $M_{\text{Sch}}(m_0)$ in the Lipschitz sense. (We know that the depths are zero, because each M_j is exactly Schwarzschild in a fixed neighborhood of ∂M_j .) But (13) tells us that

$$\frac{m_H^{(j)}}{dr} = \frac{r^{n-2}}{2} R, \tag{88}$$

and therefore R can be arbitrarily large as $j \rightarrow \infty$, so C^2 convergence is impossible. \square

5.2. Contrasting with Positive Mass Stability

Theorem 1.5 is not a stability statement, because the appended Schwarzschild space depends on the depth of M . In particular, this means that if we have a sequence of manifolds M_j with fixed $|\partial M_j|$ whose δ_j 's approach zero, the sequence need not converge to anything, since their depths need not converge (which is intuitively clear from 5.2). In this sense Theorem 1.5 is weaker than the positive mass stability theorem proved by the authors in [15]. There we proved that if $M \in \text{RotSym}_n^\partial$ has ADM mass close to zero, then tubular neighborhoods in M are close in the intrinsic flat sense to tubular neighborhoods in Euclidean space. In particular, a sequence M_j whose masses approach zero must converge to Euclidean space in the pointed intrinsic flat, with appropriately chosen basepoints.

On the other hand Theorem 1.4 is far stronger than the positive mass stability theorem in [15] in the sense that we are able to obtain Lipschitz estimates rather than just intrinsic flat estimates. The reason one cannot obtain Lipschitz estimates for the near-equality case of the positive mass theorem is that there ∂M can be both arbitrarily small and arbitrarily deep. If we look at a limit of such examples with masses approaching zero, we see that the deep regions are not becoming close to a cylinder but rather a line segment. There cannot be a bi-Lipschitz map from a region of dimension ≥ 3 , to a region of dimension 1. However, this interior region can be controlled by constructing a filling manifold and estimating its volume, leading to a stability result formulated in terms of intrinsic flat distance [15].

Note that in [15], it was possible to obtain a Lipschitz estimate *away* from the thin, deep well, which could have been proved in a manner similar to Sect. 3 and such a proof could have been used as a step in proving the main results of [15], but instead we chose a proof that emphasized features of intrinsic flat distance which are better suited to more general problems. Even in the Penrose setting, the Lipschitz estimate in Theorem 1.4 is special to the rotationally symmetric case, as seen below.

5.3. Many Wells

In this subsection we present Example 5.5 demonstrating the necessity of rotational symmetry in Theorem 1.4, followed by Remark 5.7 explaining heuristically why these example with many wells can be controlled with the intrinsic flat distance.

Example 5.5. There exist sequences of Riemannian manifolds satisfying all the hypotheses of Theorem 1.4 except the rotational symmetry, with $\delta_j \rightarrow 0$, such that the sequence does not become Lipschitz close (or even Gromov–Hausdorff close) to $M_{\text{Sch}}(m_0, L)$ for any L .

Proof. One may construct a Riemannian manifold with many wells similar to the example constructed in [15]. As in that example, one begins with a rotationally symmetric manifold with thin regions of constant sectional curvature. The only difference is that here we start with manifolds $M'_j \in \text{RotSym}_n^\partial$ that

have fixed boundary of (hyper-)area A_0 and edit in thin strips of constant curvature while ensuring that

$$m_{\text{ADM}}(M'_j) \rightarrow m_0 = \frac{1}{2} \left(\frac{A_0}{\omega_{n-1}} \right)^{\frac{n-2}{n-1}}. \tag{89}$$

We can do this in a way such that M'_j converge in the Lipschitz sense to some $M_{\text{Sch}}(m_0)$.

Next we edit in rotationally symmetric wells of some fixed depth $h > 0$ replacing small balls in the strips of constant sectional curvature. We can edit in arbitrarily many such wells just as in the example constructed in [15]. This creates new manifolds M_j which satisfy all the conditions of Theorem 1.4 except the rotational symmetry. As they have increasingly many wells of depth h , they have no Gromov–Hausdorff limit and do not converge in the Lipschitz sense. \square

Remark 5.6. Suppose that M^n is asymptotically flat with positive scalar curvature and that ∂M is outermost minimizing. Suppose further that M contains many rotationally symmetric wells, W_i , as in Example 5.5. If $n < 8$, the Penrose Inequality guarantees that

$$\sum_i m_H(\partial W_i)^{\frac{n-1}{n-2}} \leq m_{\text{ADM}}(M)^{\frac{n-1}{n-2}} - m_H(\partial M)^{\frac{n-1}{n-2}} \tag{90}$$

To see this, replace each well W_i by a region of $m_{\text{Sch}}(m_H(W_i))$ to create a new manifold N , and then apply the Penrose Inequality to N [6]. (Even though we are in higher dimensions, we can make sense of the Hawking mass here because W_i is rotationally symmetric, and ∂M is minimal.)

Remark 5.7. Example 5.5 does not appear to contradict an extension of Theorem 1.5 to the setting where there is no rotational symmetry. One can imagine constructing fillings for each well and controlling the volume of each filling in terms of the Hawking mass at the top of each well using techniques similar to those developed in [15].

5.4. Conjectures

Although Theorem 1.4 does not generalize, we hope that Theorem 1.5 does. We propose a conjecture along the lines of what we conjectured for stability of the positive mass theorem in [15].

Definition 5.8. Let \mathcal{M} be a subclass of asymptotically flat three dimensional Riemannian manifolds of nonnegative scalar curvature, whose boundaries are outermost minimal surfaces.

Conjecture 5.9. *Let $0 < A_0 < A_1$. For any $D > 0$ and $\epsilon > 0$, there exists $\delta > 0$ depending on these constants, such that for any $M^3 \in \mathcal{M}$ satisfying*

$$m_{\text{ADM}} \leq (1 + \delta) \sqrt{\frac{A_0}{16\pi}}, \tag{91}$$

where A_0 is the area of one connected component of ∂M , then

$$d_{\mathcal{F}}(T_D(\Sigma_1) \subset M, T_D(\bar{\Sigma}_1) \subset M_{\text{Sch}}^3(m_0, L)) < \epsilon, \tag{92}$$

where Σ_1 is a special surface of area A_1 , L is the depth of M , $\bar{\Sigma}_1$ is the symmetric sphere of area A_1 in $M_{\text{Sch}}(m_0, L)$ and $m_0 = \sqrt{\frac{A_0}{16\pi}}$.

We are deliberately vague as to how restrictive the class \mathcal{M} needs to be. The conjecture may require uniform conditions at infinity. We have also been vague as to what the *special surface*, Σ_1 , should be. The main point about Σ_1 is that it should avoid the wells and also not escape to infinity. Please see [15] for more discussion about how Σ_1 might be defined. We have also been vague as to what the appropriate definition of depth should be in the general setting; see Remark 3.2 for further discussion regarding this notion of depth.

Acknowledgements

The authors would like to thank Lars Andersson, Hubert Bray, Piotr Chruściel, Lan-Hsuan Huang, Gerhard Huisken, Tom Ilmanen, James Isenberg, John Lee, and Shing-Tung Yau for their interest in this work. They also thank the referee for a careful reading of the article.

References

- [1] Ambrosio, L., Kirchheim, B.: Currents in metric spaces. *Acta Math.* **185**(1), 1–80 (2000)
- [2] Arnowitt, R., Deser, S., Misner, C.W.: Coordinate invariance and energy expressions in general relativity. *Phys. Rev. (2)* **122**, 997–1006 (1961)
- [3] Bartnik, R.: The mass of an asymptotically flat manifold. *Commun. Pure Appl. Math.* **39**(5), 661–693 (1986)
- [4] Bray, H., Finster, F.: Curvature estimates and the positive mass theorem. *Commun. Anal. Geom.* **10**(2), 291–306 (2002)
- [5] Bray, H.L.: Proof of the Riemannian Penrose Inequality using the positive mass theorem. *J. Differ. Geom.* **59**(2), 177–267 (2001)
- [6] Bray, H.L., Lee, D.A.: On the Riemannian Penrose Inequality in dimensions less than eight. *Duke Math. J.* **148**(1), 81–106 (2009)
- [7] Corvino, J.: A note on asymptotically flat metrics on \mathbb{R}^3 which are scalar-flat and admit minimal spheres. *Proc. Am. Math. Soc.* **133**(12), 3669–3678 (electronic) (2005)
- [8] Finster, F.: A level set analysis of the Witten spinor with applications to curvature estimates. *Math. Res. Lett.* **16**(1), 41–55 (2009)
- [9] Finster, F., Kath, I.: Curvature estimates in asymptotically flat manifolds of positive scalar curvature. *Commun. Anal. Geom.* **10**(5), 1017–1031 (2002)
- [10] Gibbons, G.W.: The isoperimetric and Bogomolny inequalities for black holes. In: *Global Riemannian Geometry* (Durham, 1983). Ellis Horwood Series in Mathematics and its Applications, pp. 194–202. Horwood, Chichester (1984)
- [11] Gromov, M.: *Metric Structures for Riemannian and Non-Riemannian Spaces*. Progress in Mathematics, vol. 152. Birkhäuser Boston Inc., Boston (1999) (Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by

- M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates)
- [12] Huisken, G., Ilmanen, T.: The inverse mean curvature flow and the Riemannian Penrose Inequality. *J. Differ. Geom.* **59**(3), 353–437 (2001)
 - [13] Lakzian, S., Sormani, C.: Smooth convergence away from singular sets. Preprint
 - [14] Lee, D.A.: On the near-equality case of the positive mass theorem. *Duke Math. J.* **148**(1), 63–80 (2009)
 - [15] Lee, D.A., Sormani, C.: Stability of the positive mass theorem for rotationally symmetric riemannian manifolds (2011). Preprint
 - [16] Schoen, R., Yau, S.T.: On the proof of the positive mass conjecture in general relativity. *Commun. Math. Phys.* **65**(1), 45–76 (1979)
 - [17] Schoen, R.M.: Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In: *Topics in Calculus of Variations* (Montecatini Terme, 1987). *Lecture Notes in Mathematics*, vol. 1365, pp. 120–154. Springer, Berlin (1989)
 - [18] Sormani, C., Wenger, S.: Intrinsic flat convergence of manifolds and other integral current spaces. *J. Differ. Geom.* **87**, (2011)
 - [19] Witten, E.: A new proof of the positive energy theorem. *Commun. Math. Phys.* **80**(3), 381–402 (1981)

Dan A. Lee
CUNY Graduate Center and Queens College
365 Fifth Avenue
New York, NY 10016, USA
e-mail: dan.lee@qc.cuny.edu

Christina Sormani
CUNY Graduate Center and Lehman College
365 Fifth Avenue
New York, NY 10016, USA
e-mail: sormanic@member.ams.org

Communicated by Piotr T. Chrusciel.

Received: September 22, 2011.

Accepted: February 6, 2012.