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Mean Field Limit for Bosons and Infinite Dimensional Phase-Space Analysis

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Abstract. This article proposes the construction of Wigner measures in the infinite dimensional bosonic quantum field theory, with applications to the derivation of the mean field dynamics. Once these asymptotic objects are well defined, it is shown how they can be used to make connections between different kinds of results or to prove new ones.

1. Introduction

The bosonic quantum field theory relies on two different bases: On one side the quantization of a symplectic space, the approach followed for example by Berezin in [5], Kree-Raczka in [34]; on the other side the gaussian stochastic processes presentation also known as the integral functional point of view followed for example by Glimm-Jaffe in [25] and Simon in [43]. Both approaches have to be handled in order to tackle on the most basic problems in constructive quantum field theory (see [3, 15]). The interaction of constructive quantum field theory with other fields of mathematics like pseudodifferential calculus (see [6] or [35]) or stochastic processes (see [2, 38]) is often instructive.

In the recent years the mean field limit of N-body quantum dynamics has been reconsidered by various authors via a BBGKY-hierarchy approach (see [4, 16, 17, 19, 20, 45] and [21] for a short presentation) mainly motivated by the study of Bose–Einstein condensates (see [12]). Although this was present in earlier works around the so-called Hepp method (see [32] and [24]), the relationship with the microlocal or semiclassical analysis in infinite dimension has been neglected. Difficulties are known in this direction: 1) The gap between the inductive and projective construction of quantized observable in infinite dimension; 2) the difficulties to built algebras of pseudodifferential operators which contain the usual hamiltonians and preserve some properties of the finite dimensional calculus like a Calderon–Vaillancourt theorem, a good notion of ellipticity or the asymptotic positivity with a Gårding inequality; 3) even when step 2) is possible, no satisfactory Egorov theorem is available.

Recall the example of an N-body Schrödinger hamiltonian

$$H_N = -\Delta + \frac{1}{N} \sum_{1\leq i<j\leq N} V(x_i - x_j)\,, \quad \text{on} \quad \mathbb{R}^{dN}\,,$$

and consider the time-evolved wave function

$$\Psi_N(t) = e^{-itH_N}\, \psi^{\otimes N}\,, \quad \psi \in L^2(\mathbb{R}^d)\,.$$

The 1-particle marginal state, the quantum analogous of the one particle empirical distribution in the classical N-body problem, is given by

$$\mathrm{Tr}\big[A\varrho^1(t)\big] = \left\langle \Psi_N(t)\,, \frac{1}{N}\left[\sum_{i=1}^{N} I\otimes\cdots I\otimes I\otimes \underbrace{A}_{i} \otimes I\otimes\cdots\otimes I\right]\Psi_N(t)\right\rangle.$$

The mean field limit says that in the limit $N\to\infty$, the marginal state evolves according to a non-linear Hartree equation

$$\varrho^1(t) = |z(t)\rangle\langle z(t)| + o(1)\,, \quad \text{as} \quad N\to\infty\,,$$
$$\text{with} \quad \begin{cases} i\partial_t z = -\Delta z + (V * |z|^2)z & \text{on } \mathbb{R}_t\times\mathbb{R}^d \\ z(t=0) = \psi\,. \end{cases}$$

By setting $N = \frac{1}{\varepsilon}$ and in the Fock space framework with ε-dependent CCR (i.e.: $[a(g), a^*(f)] = \varepsilon\,\langle g,\, f\rangle$), the problem becomes

$$\begin{aligned} H_N &= \frac{1}{\varepsilon}\left[\int_{\mathbb{R}^d} \nabla a^*(x)\nabla a(x)\; dx + \int_{\mathbb{R}^{2d}} V(x-y)a^*(x)a^*(y)a(x)a(y)\; dxdy\right] \\ &= \frac{1}{\varepsilon} H^\varepsilon \\ e^{-itH_N} &= e^{-i\frac{t}{\varepsilon}H^\varepsilon}\,, \\ \mathrm{Tr}\left[A\varrho^1(t)\right] &= \left\langle \Psi_N(t)\,, d\Gamma(A)\Psi_N(t)\right\rangle = \left\langle \Psi_N(t)\,, p_A(z)^{Wick}\Psi_N(t)\right\rangle, \end{aligned}$$

where p_A is the polynomial $p_A(z) = \langle z\,, Az\rangle$. Higher order marginals, taking into accounts correlations, can be defined after using the polynomials $p_A(z) = \langle z^{\otimes k}\,, Az^{\otimes k}\rangle$ with $A\in\mathscr{L}(L^2(\mathbb{R}^{kd}))$.

On this example, the scaling of the hamiltonian, of the time scale and of the observables as Wick operators enters formally in the ε-dependent semiclassical analysis. The Hepp method concerns the evolution of squeezed coherent states [12, 24, 32], which amounts in the finite dimensional case to the phase-space evolution of a gaussian state according to the time dependent quadratic approximation of the non linear hamiltonian, centered on the solution to the classical hamiltonian equation. We refer the reader to [13] for accurate developments of such an approach in the finite dimensional case.

In the nineties and as a byproduct of the development of microlocal analysis, alternative and more flexible methods were introduced in order to study the semiclassical limit with the help of Wigner (or semiclassical) measures (see [10, 21, 29, 36, 46]). Such objects are defined by duality and rely on the asymptotic positivity of the ε-dependent quantizations. It gives a weak but more flexible form of the principal term of the semiclassical (here mean-field) approximation. Via the introduction of probability measures on the symplectic phase-space, it provides an interesting way to analyze the relationship between the two basic approaches to quantum field theory. Further in finite dimension, the Wick, anti-Wick and Weyl quantizations are asymptotically equivalent in the limit $\varepsilon \to 0$. This is not so obvious in infinite dimension.

Several attempts have been tried to develop an infinite dimensional Weyl pseudodifferential calculus with an inductive approach. Lascar in [35] introduced an algebra and a notion of ellipticity in this direction, making more effective the general presentation of [34]. The works of Helffer–Sjöstrand in [28, 31] and Amour–Kerdelhué–Nourrigat in [1] about the pseudodifferential calculus in large dimension motivated by the analysis of the thermodynamical limit enter in this category. With such an approach, it is not clear that the infinite dimensional phase-space is well explored and that no information is lost in the limit $\varepsilon \to 0$. Meanwhile this inductive approach is limited by Hilbert–Schmidt type restriction like in Shale's theorem about the quasi-equivalence of gaussian measures. It is known after [26] that the nonlinear transformations which preserve the quasi-equivalence with a given gaussian measure within the Schrödinger representation are very restricted and do not cover realistic models. Hence no Egorov theorem can be expected with Weyl observables.

Simple remarks suggests alternative point of views. The Wick calculus with polynomial symbols present encouraging specificities: It contains the standard hamiltonians, it makes an algebra under more general assumptions (the Hilbert–Schmidt condition can be relaxed) and allows some propagation results when tested on appropriate states (see [19, 20]). Meanwhile the Wigner measures in the limit $\varepsilon \to 0$ can be defined very easily via the separation of variables as weak distribution, in a projective way which fits with the stochastic processes point of view.

After reviewing and sometimes simplifying or improving known results and techniques about the mean field limit, our aim is to show the interests of the extension to the infinite dimensional case of Wigner measures:

- After the introduction of the small parameter $\varepsilon \to 0$ and the definition of Weyl operator $W(z)$, $z \in \mathscr{Z}$ the phase-space, choosing between the quantization of symplectic space and the stochastic processes point of view is no more a question of general principles nor of mathematical taste. It is a matter of scaling. The symplectic geometry arises when considering macroscopic phase-space translation $W(\frac{z}{\varepsilon})$, while the operator $W(z)$ is used with this scaling in the introduction of Wigner measures via their characteristic function. Corrections to the mean field limit considered for example in [11] with a stochastic processes point of view can be interpreted within this picture: They attempt to give a better information on the shape of the state in a small phase-space scale.

- Once the Wigner measures are well defined as Radon measures, it is possible to make explicit the relationship between different kinds of results and to extend them in a flexible way. It accounts for the propagation of chaos (result obtained via the BBGKY approach) according to the classical hamiltonian dynamics in the phase-space. Actually we shall prove in a very general framework that the propagation of squeezed coherent states as derived via the Hepp method implies a weak version of the mean field limit for product states. Further propagation results can be obtained for some non standard mixed states without reconsidering a rather heavy analysis process.
- The comparison between the Wick, Weyl and anti-Wick quantization can be analyzed accurately in the infinite dimensional case. With the Wick calculus, complete asymptotic expansions can be proved after testing with some specific states. The relationship of such results with the propagation of Wigner measures works in a rather general setting but has to be handled with care.
- The gap between the projective and inductive approaches can be formulated accurately in the limit $\varepsilon \to 0$. We shall explain in the examples the possibility of a dimensional defect of compactness.

This work is presented and illustrated with examples simpler than more realistic models considered in other works like [4, 16, 17, 24, 32] with more singular interaction potentials. That was our choice in order to make the correspondence between various approaches more straightforward and to pave the way for further improvements. We hope that this information will be valuable for other colleagues and useful for further developments.

The outline of this article is the following. In Section 2, standard notions about the symmetric Fock space are recalled and Wick calculus is specified. In Section 3 the Weyl and Anti-Wick calculus are introduced in a projective way after recalling accurately (most of all the scaling) of finite dimensional semiclassical calculus. The Section 4 recalls the distinction between coherent states and product or Hermite states, and their properties when measured with different kinds of observables. The two methods used to derive the mean field dynamics, the Hepp method and the analysis through truncated Dyson expansions, are reviewed within our formalism and with some variations in Section 5. The Wigner measures are introduced in Section 6 with the extension of some finite dimensional properties and specific infinite dimensional phenomena. Finally examples and applications are detailed in Section 7, in particular: 1) reconsidering a simple presentation of the Bose–Einstein condensation shows an interesting example of what we call the dimensional defect of compactness; 2) a general result says that the propagation of squeezed coherent states, which can be attacked via the Hepp method, implies a slightly weaker form of the propagation of chaos (formulated with product states and Wick observables); 3) the mean field dynamics can be easily derived for some states which present some asymptotically vanishing correlations.

2. Fock space and Wick quantization

After introducing the symmetric Fock space with ε-dependent CCR's, an algebra of observables resulting from the Wick quantization process is presented.

2.1. Fock space

Consider a separable Hilbert space $\mathscr{Z}$ endowed with a scalar product $\langle .,. \rangle$ which is anti-linear in the left argument and linear in the right one and with the associated norm $|z| = \sqrt{\langle z, z\rangle}$. Let $\sigma = \mathrm{Im}\langle .,. \rangle$ and $S = \mathrm{Re}\langle .,. \rangle$ respectively denote the canonical symplectic and the real scalar product over $\mathscr{Z}$. The symmetric Fock space on $\mathscr{Z}$ is the Hilbert space

$$\mathscr{H} = \bigoplus_{n=0}^{\infty} \bigvee^{n} \mathscr{Z} = \Gamma_s(\mathscr{Z})\,,$$

where $\bigvee^n \mathscr{Z}$ is the n-fold symmetric tensor product. Almost all the direct sums and tensor products are completed within the Hilbert framework. This is omitted in the notation. On the contrary, a specific $^{\mathrm{alg}}$ superscript will be used for the algebraic direct sums or tensor products.

For any $n \in \mathbb{N}$, the orthogonal projection of $\bigotimes^n \mathscr{Z}$ onto the closed subspace $\bigvee^n \mathscr{Z}$ will be denoted by $\mathscr{S}_n$. For any $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathscr{Z}^n$, the vector $\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \in \bigvee^n \mathscr{Z}$ will be

$$\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n = \mathscr{S}_n(\xi_1 \otimes \xi_2 \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \xi_{\sigma(1)} \otimes \xi_{\sigma(2)} \cdots \otimes \xi_{\sigma(n)}\,.$$

The family of vectors $(\xi_1 \vee \cdots \vee \xi_n)_{\xi_i \in \mathscr{Z}}$ is a generating family of $\bigvee^{n,alg} \mathscr{Z}$ and a total family of $\bigvee^n \mathscr{Z}$. Thanks to the polarization identity

$$\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left(\sum_{j=1}^{n} \varepsilon_j \xi_j \right)^{\otimes n}, \qquad (1)$$

the same property holds for the family $(z^{\otimes n})_{n \in \mathbb{N}, z \in \mathscr{Z}}$.

For two operators $A_k : \bigvee^{i_k} \mathscr{Z} \to \bigvee^{j_k} \mathscr{Z}$, $k = 1, 2$, the notation $A_1 \bigvee A_2$ stands for

$$A_1 \bigvee A_2 = \mathscr{S}_{j_1+j_2} \circ (A_1 \otimes A_2) \circ \mathscr{S}_{i_1+i_2} \in \mathscr{L}\left(\bigvee^{i_1+i_2} \mathscr{Z}, \bigvee^{j_1+j_2} \mathscr{Z} \right).$$

Any $z \in \mathscr{Z}$ is identified with the operator $|z\rangle : \bigvee^0 \mathscr{Z} = \mathbb{C} \ni \lambda \mapsto \lambda z \in \mathscr{Z} = \bigvee^1 \mathscr{Z}$ while $\langle z|$ denotes the linear form $\mathscr{Z} \ni \xi \mapsto \langle z\,,\, \xi\rangle \in \mathbb{C}$. The creation and annihilation operators $a^*(z)$ and $a(z)$, parameterized by $\varepsilon > 0$, are then defined by:

$$a(z)_{|\bigvee^n \mathscr{Z}} = \sqrt{\varepsilon n}\ \ \langle z| \otimes I_{\bigvee^{n-1} \mathscr{Z}}$$

$$a^*(z)_{|\bigvee^n \mathscr{Z}} = \sqrt{\varepsilon(n+1)}\ \ \mathscr{S}_{n+1} \circ (\,|z\rangle \otimes I_{\bigvee^n \mathscr{Z}}) = \sqrt{\varepsilon(n+1)}\, z \bigvee I_{\bigvee^n \mathscr{Z}}\,.$$

Each of $(a(z))_{z\in\mathscr{Z}}$ and $(a^*(z))_{z\in\mathscr{Z}}$ are commuting families of operators and they satisfy the canonical commutation relations (CCR):

$$\left[a(z_1), a^*(z_2)\right] = \varepsilon\langle z_1, z_2\rangle I\,. \tag{2}$$

We also consider the canonical quantization of the real variables $\Phi(z) = \frac{1}{\sqrt{2}}(a^*(z)+a(z))$ and $\Pi(z) = \Phi(iz) = \frac{1}{i\sqrt{2}}(a(z) - a^*(z))$. They are self-adjoint operators on $\mathscr{H}$ and satisfy the identities:

$$\left[\Phi(z_1), \Phi(z_2)\right] = i\varepsilon\sigma(z_1, z_2)I\,, \quad \left[\Phi(z_1), \Pi(z_2)\right] = i\varepsilon S(z_1, z_2)I\,.$$

The representation of the Weyl commutation relations in the Fock space

$$\begin{aligned} W(z_1)W(z_2) &= e^{-\frac{i\varepsilon}{2}\sigma(z_1,z_2)}W(z_1+z_2) \\ &= e^{-i\varepsilon\sigma(z_1,z_2)}W(z_2)W(z_1)\,, \end{aligned} \tag{3}$$

is obtained by setting $W(z) = e^{i\Phi(z)}$. The generating functional associated with this representation is given by

$$\left\langle \Omega, W(z)\Omega\right\rangle = e^{-\frac{\varepsilon}{4}|z|^2}\,,$$

where Ω is the vacuum vector $(1, 0, \ldots) \in \mathscr{H}$. The total family of vectors $E(z) = W\big(\frac{\sqrt{2}z}{i\varepsilon}\big)\Omega = e^{\frac{1}{\varepsilon}[a^*(z)-a(z)]}\Omega$, $z \in \mathscr{Z}$, have the explicit form

$$\begin{aligned} E(z) &= e^{-\frac{|z|^2}{2\varepsilon}}\sum_{n=0}^{\infty}\frac{1}{\varepsilon^n}\frac{a^*(z)^n}{n!}\Omega \\ &= e^{-\frac{|z|^2}{2\varepsilon}}\sum_{n=0}^{\infty}\varepsilon^{-n/2}\frac{z^{\otimes n}}{\sqrt{n!}}\,. \end{aligned} \tag{4}$$

The number operator is also parametrized by $\varepsilon > 0$,

$$N_{|\bigvee^n\mathscr{Z}} = \varepsilon n I_{|\bigvee^n\mathscr{Z}}\,.$$

It is convenient to introduce the subspace

$$\mathscr{H}_{fin} = \bigoplus_{n\in\mathbb{N}}^{\mathrm{alg}}\bigvee^n\mathscr{Z}$$

of $\mathscr{H}$, which is a set of analytic vectors for N.

For any contraction $S \in \mathscr{L}(\mathscr{Z})$, $|S|_{\mathscr{L}(\mathscr{H})} \leq 1$, $\Gamma(S)$ is the contraction in $\mathscr{H}$ defined by

$$\Gamma(S)_{|\bigvee^n\mathscr{Z}} = S\otimes S\cdots\otimes S\,.$$

More generally $\Gamma(B)$ can be defined by the same formula as an operator on $\mathscr{H}_{fin}$ for any $B \in \mathscr{L}(\mathscr{Z})$. Meanwhile, for any self-adjoint operator $A : \mathscr{Z} \supset \mathscr{D}(A) \to \mathscr{Z}$, the operator $d\Gamma(A)$ is the self-adjoint operator given by

$$e^{\frac{it}{\varepsilon}d\Gamma(A)} = \Gamma(e^{itA})$$
$$d\Gamma(A)_{|\bigvee^{n,\mathrm{alg}}\mathscr{D}(A)} = \varepsilon\left[\sum_{k=1}^{n} I\otimes\cdots\otimes\underbrace{A}_{k}\otimes\cdots\otimes I\right].$$

For example $N = d\Gamma(I)$.

2.2. Wick operators

In this subsection we consider the Wick symbolic calculus on (homogeneous) polynomials. We will show some product and commutation formulas useful later for the application. For example time evolved Wick observables can be expressed as ε-asymptotic expansion of quantized Wick symbols. For a detailed exposition on more general Wick polynomials we refer the reader to [15].

A (p,q)-homogeneous polynomial function of $z\in\mathscr{Z}$ is defined as $P_\ell(z) = \ell(z^{\otimes q}, z^{\otimes p})$, where ℓ is a sesquilinear form on $(\bigotimes^{q,alg}\mathscr{Z})\times(\bigotimes^{p,alg}\mathscr{Z})$, with $P_\ell(\lambda z)=\bar{\lambda}^q\lambda^p P_\ell(z)$. Owing to the polarization formula (1) and the identity

$$\ell(\eta^{\otimes q},\xi^{\otimes p}) = \int_0^1\int_0^1 \ell\big([e^{2i\pi\theta}\eta+e^{2i\pi\varphi}\xi]^{\otimes q},[e^{2i\pi\theta}\eta+e^{2i\pi\varphi}\xi]^{\otimes p}\big)\, e^{2i\pi(q\theta-p\varphi)}\, d\theta\, d\varphi$$

the correspondence $\ell\mapsto P_\ell$ is a bijection when the set of forms is restricted to the sesquilinear forms on $(\bigvee^{q,alg}\mathscr{Z})\times(\bigvee^{p,alg}\mathscr{Z})$. Any of the continuity properties of P_ℓ are thus encoded by the continuity properties of the sesquilinear form ℓ with the following hierarchy (from the weakest to the strongest)

$$|\ell(\eta_1\vee\cdots\vee\eta_q,\xi_1\vee\cdots\vee\xi_p)| \le C_\ell\, |\eta_1|_{\mathscr{Z}}\dots|\eta_q|_{\mathscr{Z}}\, |\xi_1|_{\mathscr{Z}}\dots|\xi_p|_{\mathscr{Z}}\,, \qquad \eta_i\in\mathscr{Z}\,,\quad \xi_j\in\mathscr{Z} \tag{5}$$

$$|\ell(\phi,\psi)| \le C_\ell\, |\phi|_{\bigvee^q\mathscr{Z}}\, |\psi|_{\bigvee^p\mathscr{Z}}\,,\quad \psi\in\bigvee^p\mathscr{Z}\,,\quad \phi\in\bigvee^q\mathscr{Z} \tag{6}$$

$$\left|\sum_{1\le i,j\le K} c_{i,j}\ell(\phi_i,\psi_j)\right| \le C_\ell\left|\sum_{1\le i,j\le K} c_{i,j}\langle\phi_i|\otimes\psi_j\right|_{(\bigvee^q\mathscr{Z})^*\otimes(\bigvee^p\mathscr{Z})}\,, \qquad K\in\mathbb{N}\,,\quad c_{ij}\in\mathbb{C}\,,\quad \phi_i\in\bigvee^q\mathscr{Z}\,,\quad \psi_j\in\bigvee^p\mathscr{Z}\,. \tag{7}$$

For example, when $p=q=1$ the two first ones define $\mathscr{L}(\mathscr{Z})$, while the third one defines the space of Hilbert–Schmidt operators. By Taylor expansion any (p,q)-homogeneous polynomial P admits Gâteaux differentials and we set

$$\partial_{\bar{z}}^k\partial_z^{k'}P(z)[u_1,\dots,u_k,v_1,\dots,v_{k'}] = \bar{\partial}_{u_1}\cdots\bar{\partial}_{u_k}\partial_{v_1}\cdots\partial_{v_{k'}}P(z)$$

where $\bar{\partial}_u,\partial_v$ are the complex directional derivatives relative to $u,v\in\mathscr{Z}$.

Definition 2.1. For $p, q \in \mathbb{N}$, the set of (p,q)-homogeneous polynomial functions on $\mathscr{Z}$ which satisfy the continuity condition (6) is denoted by $\mathscr{P}_{p,q}(\mathscr{Z})$:

$$\left(b(z) \in \mathscr{P}_{p,q}(\mathscr{Z})\right) \Leftrightarrow \left\{ \begin{array}{l} \tilde{b} = \frac{1}{p!}\frac{1}{q!}\partial_z^p \partial_{\overline{z}}^q b(z) \in \mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})\,, \\ b(z) = \left\langle z^{\otimes q}\,,\, \tilde{b} z^{\otimes p}\right\rangle . \end{array}\right.$$

The subspace of $\mathscr{P}_{p,q}(\mathscr{Z})$ made of polynomials b such that $\tilde{b}$ is a compact operator $\tilde{b} \in \mathscr{L}^\infty(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})$ (resp. $b \in \mathscr{L}^r(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})$) is denoted by $\mathscr{P}^\infty_{p,q}(\mathscr{Z})$ (resp. $\mathscr{P}^r_{p,q}(\mathscr{Z})$).

Remark 2.2. In the case of $\mathscr{Z} = \mathbb{C}^d$ the symbol is often written $b(z, \overline{z})$. Of course our polynomials have an holomorphic and antiholomorphic part but we prefer to keep the notation $b(z)$. The symbol b is simply considered as a function of the point $z \in \mathscr{Z}$. The writing $b(z,\overline{z})$ would suggest that $\mathscr{Z}$ is endowed with a complex conjugation operator, which is not necessary at this level.

It will be sometimes convenient to consider $\tilde{b}$ as an operator from $\bigotimes^p \mathscr{Z}$ into $\bigotimes^q \mathscr{Z}$ with the obvious convention for symmetric operators $\tilde{b} = \mathscr{S}_q \tilde{b} \mathscr{S}_p$. Owing to the condition $\tilde{b} \in \mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})$ for $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, this definition implies that any differential $\partial_{\overline{z}}^j \partial_z^k b(z)$ at the point $z \in \mathscr{Z}$ equals

$$\partial_{\overline{z}}^j \partial_z^k b(z) = \frac{p!}{(p-k)!}\frac{q!}{(q-j)!}\left(\langle z^{\otimes q-j}| \bigvee I_{\bigvee^j \mathscr{Z}}\right) \tilde{b}\left(z^{\otimes p-k} \bigvee I_{\bigvee^k \mathscr{Z}}\right) \in \mathscr{L}\left(\bigvee^k \mathscr{Z}, \bigvee^j \mathscr{Z}\right). \quad (8)$$

We will mainly work with fixed homogeneity degrees p, q but the key statement of this section (Proposition 2.7) says that $\oplus^{\mathrm{alg}}_{p,q\in\mathbb{N}} \mathscr{P}_{p,q}(\mathscr{Z})$ is an algebra of symbols with the same explicit product formula as in the finite dimensional case.

With any "symbol" $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, a Wick monomial b^{Wick} can be associated according to:

$$b^{Wick} : \mathscr{H}_{fin} \to \mathscr{H}_{fin}\,, \quad (9)$$
$$b^{Wick}_{|\bigvee^n \mathscr{Z}} = 1_{[p,+\infty)}(n) \frac{\sqrt{n!(n+q-p)!}}{(n-p)!}\, \varepsilon^{\frac{p+q}{2}} \left(\tilde{b} \bigvee I_{\bigvee^{n-p}\mathscr{Z}}\right) \in \mathscr{L}\left(\bigvee^n \mathscr{Z}, \bigvee^{n+q-p} \mathscr{Z}\right),$$

with $\tilde{b} = (p!)^{-1}(q!)^{-1}\partial_z^p \partial_{\overline{z}}^q b(z)$.

Here are the basic symbol-operator correspondence:

$$\begin{array}{ccccccc} \langle z, \xi\rangle & \longleftrightarrow & a^*(\xi)\,\sqrt{2}S(\xi,z) & \longleftrightarrow & \Phi(\xi)\,\langle z, Az\rangle & \longleftrightarrow & d\Gamma(A) \\ \langle \xi, z\rangle & \longleftrightarrow & a(\xi)\ \sqrt{2}\sigma(\xi,z) & \longleftrightarrow & \Pi(\xi)\quad |z|^2 & \longleftrightarrow & N\,. \end{array}$$

Other examples can be derived from the next propositions. The first one is a direct consequence of the definition (9).

Proposition 2.3. *The following identities hold true on $\mathscr{H}_{fin}$ for every $b \in \mathscr{P}_{p,q}(\mathscr{Z})$:*

(i) $\left(b^{Wick}\right)^* = \overline{b}^{Wick}$.

(ii) $\left(C(z)b(z)A(z)\right)^{Wick} = C^{Wick}b^{Wick}A^{Wick}$, *if* $A \in \mathscr{P}_{\alpha,0}(\mathscr{Z})$, $C \in \mathscr{P}_{0,\beta}(\mathscr{Z})$.

(iii) $e^{i\frac{t}{\varepsilon}\mathrm{d}\Gamma(A)}b^{Wick}e^{-i\frac{t}{\varepsilon}\mathrm{d}\Gamma(A)} = \left(b(e^{-itA}z)\right)^{Wick}$, *if* A *is a self-adjoint operator on* $\mathscr{Z}$.

Proposition 2.4.

(i) *The Wick operator associated with* $b(z) = \prod_{i=1}^{p}\langle z,\eta_i\rangle \times \prod_{j=1}^{q}\langle \xi_j, z\rangle$, $\eta_i,\xi_j \in \mathscr{Z}$, *equals*

$$b^{Wick} = a^*(\eta_1)\cdots a^*(\eta_p)a(\xi_1)\cdots a(\xi_q)\,.$$

(ii) *For* $b \in \mathscr{P}_{p,q}(\mathscr{Z})$ *and* $z \in \mathscr{Z}$ *the equality*

$$\langle z^{\otimes j}, b^{Wick}z^{\otimes k}\rangle = \delta^{+}_{k-p,j-q}\sqrt{\frac{k!j!}{(k-p)!(j-q)!}}\,\varepsilon^{\frac{p+q}{2}}\quad |z|^{k-p+j-q}\,b(z) \qquad (10)$$

holds for any $k,j \in \mathbb{N}$. *The symbol* $\delta^{+}_{\alpha,\beta}$ *denotes* $\delta_{\alpha,\beta}1_{[0,+\infty)}(\alpha)$ *where* $\delta_{\alpha,\beta}$ *is the standard Kronecker symbol.*

Proof. (i) is a direct consequence of Proposition 2.3 with $(\langle z\,,\,\xi\rangle)^{Wick} = a^*(\xi)$ and $(\langle \xi, z\rangle)^{Wick} = a(\xi)$.

(ii) This comes directly from the definition (9) of b^{Wick}. □

The next result specifies the boundedness properties of b^{Wick}.

Lemma 2.5. *For* $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, *the estimate*

$$\left|b^{Wick}\right|_{\mathscr{L}(\bigvee^k\mathscr{Z},\bigvee^j\mathscr{Z})} \le \delta^{+}_{k-p,j-q}\,(j\varepsilon)^{\frac{q}{2}}\,(k\varepsilon)^{\frac{p}{2}}\left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}\,,$$

$$\text{with}\quad \tilde{b} = \frac{1}{p!q!}\partial_z^p\partial_{\bar{z}}^q b\,, \qquad (11)$$

holds for any $k,j \in \mathbb{N}$.

This implies

$$\left|\langle N\rangle^{-\frac{q}{2}}\,b^{Wick}\,\langle N\rangle^{-\frac{p}{2}}\right|_{\mathscr{L}(\mathscr{H})} \le \left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}\,. \qquad (12)$$

Proof. A consequence of (10) is $b^{Wick}(\bigvee^k\mathscr{Z}) \subset \bigvee^j\mathscr{Z}$ with $j = k-p+q$. For $\psi \in \bigvee^k\mathscr{Z}$ and $j = k-p+q$, write

$$\begin{aligned}
&\left|b^{Wick}\psi\right|_{\bigvee^j\mathscr{Z}}\\
&\quad= \frac{\sqrt{j!k!}}{(k-p)!}\varepsilon^{\frac{p+q}{2}}\left|\mathscr{S}_j(b\otimes I_{\bigotimes^{k-p}\mathscr{Z}})\psi\right|_{\bigvee^j\mathscr{Z}}\\
&\quad\le (j\varepsilon)^{\frac{q}{2}}(k\varepsilon)^{\frac{p}{2}}\sqrt{\frac{j!}{(j-q)!j^q}}\sqrt{\frac{k!}{(k-p)!k^p}}\left|b\otimes I_{\bigotimes^{k-p}\mathscr{Z}}\right|_{\mathscr{L}(\bigotimes^k\mathscr{Z},\bigotimes^j\mathscr{Z})}|\psi|_{\bigvee^k\mathscr{Z}}\,.
\end{aligned}$$

□

An important property of our class of Wick polynomials is that a composition of $b_1^{Wick} \circ b_2^{Wick}$ with $b_1, b_2 \in \oplus_{p,q}^{\text{alg}} \mathscr{P}_{p,q}(\mathscr{Z})$ is a Wick polynomial with symbol in $\oplus_{p,q}^{\text{alg}} \mathscr{P}_{p,q}(\mathscr{Z})$. In the following we prove this result and specifies the Wick symbol of the product.

For $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, specific cases with $j = 0$ or $k = 0$ of (8) imply

$$\partial_z^k b(z) \in \left(\bigvee^k \mathscr{Z}\right)^* \quad \text{and} \quad \partial_{\bar z}^j b(z) \in \bigvee^j \mathscr{Z}\,,$$

for any fixed $z \in \mathscr{Z}$. For two symbols $b_i \in \mathscr{P}_{p_i,q_i}(\mathscr{Z})$, $i = 1, 2$, and any $k \in \mathbb{N}$, the new symbol $\partial_z^k b_1 . \partial_{\bar z}^k b_2$ is now defined by

$$\partial_z^k b_1 \,.\, \partial_{\bar z}^k b_2(z) = \left\langle \partial_z^k b_1(z), \partial_{\bar z}^k b_2(z)\right\rangle_{(\bigvee^k \mathscr{Z})^*, \bigvee^k \mathscr{Z}}\,. \tag{13}$$

We also use the following notation for multiple Poisson brackets:

$$\{b_1, b_2\}^{(k)} = \partial_z^k b_1 . \partial_{\bar z}^k b_2 - \partial_z^k b_2 . \partial_{\bar z}^k b_1\,,$$
$$\{b_1, b_2\} = \{b_1, b_2\}^{(1)}\,.$$

These operations with polynomials are easier to handle than there corresponding versions for the operators $\tilde b_i \in \mathscr{L}(\bigvee^{p_i} \mathscr{Z}, \bigvee^{q_i} \mathscr{Z})$. Nevertheless their explicit operator expressions as contracted products allow to check that $\oplus_{p,q}^{\text{alg}} \mathscr{P}_{p,q}(\mathscr{Z})$ is stable w.r.t these operations.

Lemma 2.6. *Fix p_1, p_2, q_1 and q_2 in $\mathbb{N}$. For two polynomials $b_i \in \mathscr{P}_{p_i,q_i}(\mathscr{Z})$, $i = 1, 2$, set $\tilde b_i = (p_i! q_i!)^{-1}\ \partial_z^{p_i} \partial_{\bar z}^{q_i} b_i$ and for any $k \in \{0, \ldots, \min\{p_1, q_2\}\}$*

$$\tilde b_1 \overset{k}{\odot} \tilde b_2 = \frac{1}{(p_1 + p_2 - k)!(q_1 + q_2 - k)!} \partial_z^{p_1+p_2-k} \partial_{\bar z}^{q_1+q_2-k} \left[\partial_z^k b_1 . \partial_{\bar z}^k b_2\right].$$

Then

$$\tilde b_1 \overset{k}{\odot} \tilde b_2 = \frac{p_1!}{(p_1 - k)!} \frac{q_2!}{(q_2 - k)!}\ \mathscr{S}_{q_1+q_2-k}(\tilde b_1 \otimes I_{\bigotimes^{q_2-k} \mathscr{Z}})(I_{\bigotimes^{p_1-k}} \otimes \tilde b_2)$$
$$\in \mathscr{L}\left(\bigvee^{p_1+p_2-k} \mathscr{Z}, \bigvee^{q_1+q_2-k} \mathscr{Z}\right), \tag{14}$$

with the estimate

$$\left|\tilde b_1 \overset{k}{\odot} \tilde b_2\right|_{\mathscr{L}(\bigvee^{p_1+p_2-k} \mathscr{Z}, \bigvee^{q_1+q_2-k} \mathscr{Z})}$$
$$\leq \frac{p_1!}{(p_1 - k)!} \frac{q_2!}{(q_2 - k)!} \left|\tilde b_1\right|_{\mathscr{L}(\bigvee^{p_1} \mathscr{Z}, \bigvee^{q_1} \mathscr{Z})} \left|\tilde b_2\right|_{\mathscr{L}(\bigvee^{p_2} \mathscr{Z}, \bigvee^{q_2} \mathscr{Z})}\,. \tag{15}$$

Proof. For $\psi \in \bigvee^{p_1} \mathscr{Z}$ and $\phi \in \bigvee^{q_2} \mathscr{Z}$, introduce the vector

$$\langle z^{\otimes q_2-k}, \phi\rangle = \left(\langle z^{\otimes q_2-k}| \otimes I_{\bigotimes^k \mathscr{Z}}\right)\phi = \frac{(q_2 - k)!}{q_2!} \partial_{\bar z}^k b_\phi(z) \in \bigvee^k \mathscr{Z}$$

with $b_\phi(z) = \langle z^{q_2}\,,\,\phi\rangle$ and the form

$$\langle \psi, z^{\otimes p_1-k}\rangle := \frac{(p_1-k)!}{p_1!}\partial_z^k b_\psi(z) \in \left(\bigvee^k \mathscr{Z}\right)^*, \quad \text{with} \quad b_\psi(z) = \left\langle \psi\,,\,z^{\otimes p_1}\right\rangle .$$

The identity

$$\begin{aligned}\big\langle \langle \psi, z^{\otimes p_1-k}\rangle, \langle z^{\otimes q_2-k}, \phi\rangle\big\rangle&_{(\bigvee^k \mathscr{Z})^*, \bigvee^k \mathscr{Z}} \\ &= \langle \psi\otimes z^{\otimes q_2-k}, z^{\otimes p_1-k}\otimes\phi\rangle_{\bigotimes^{p_1+q_2-k}\mathscr{Z}}\end{aligned} \tag{16}$$

is obviously true when $\psi = \xi^{\otimes p_1}$ and $\phi = \eta^{\otimes q_2}$ with $\xi, \eta \in \mathscr{Z}$. Since $(\xi^{\otimes n})_{\xi\in\mathscr{Z}}$ is a total space of $\bigvee^n \mathscr{Z}$ with the polarization identity (1), the identity (16) holds for all $\phi \in \bigvee^{q_2}\mathscr{Z}$ and all $\psi \in \bigvee^{p_1}\mathscr{Z}$. After noticing the relations

$$\partial_z^k b_1(z) = \frac{p_1!}{(p_1-k)!}\langle \psi, z^{\otimes p_1-k}\rangle\,, \quad \partial_{\bar z}^k b_2(z) = \frac{q_2!}{(q_2-k)!}\langle z^{\otimes q_2-k}, \phi\rangle\,,$$

with $\psi = \tilde b_1^* z^{\otimes q_1}$ and $\phi = \tilde b_2 z^{\otimes p_2}$, the identity (16) leads to

$$\begin{aligned}&\partial_z^k b_1.\partial_{\bar z}^k b_2(z)\\ &= \frac{p_1!}{(p_1-k)!}\frac{q_2!}{(q_2-k)!}\left\langle z^{\otimes q_1+q_2-k}, (\tilde b_1\otimes I_{\bigotimes^{q_2-k}\mathscr{Z}})\,(I_{\bigotimes^{p_1-k}\mathscr{Z}}\otimes \tilde b_2) z^{\otimes p_2+p_1-k}\right\rangle .\end{aligned}$$

Therefore $\partial_z^k b_1.\partial_{\bar z}^k b_2$ is a continuous homogeneous polynomial in $\mathscr{P}_{p_1+p_2-k, q_1+q_2-k}(\mathscr{Z})$ with the associated operator given by (14). The estimate (15) follows immediately by (14). $\square$

Proposition 2.7. *The formulas*

(i)

$$b_1^{Wick}\, b_2^{Wick} = \left(\sum_{k=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^k}{k!}\ \partial_z^k b_1.\partial_{\bar z}^k b_2\right)^{Wick} = \left(e^{\varepsilon\langle\partial_z,\partial_{\bar\omega}\rangle} b_1(z) b_2(\omega)\,|_{z=\omega}\right)^{Wick}, \tag{17}$$

(ii)

$$[b_1^{Wick}, b_2^{Wick}] = \left(\sum_{k=1}^{\max\{\min\{p_1,q_2\}\,,\,\min\{p_2,q_1\}\}} \frac{\varepsilon^k}{k!}\ \{b_1, b_2\}^{(k)}\right)^{Wick}, \tag{18}$$

hold for any $b_i \in \mathscr{P}_{p_i,q_i}(\mathscr{Z})$, $i = 1, 2$ *as identities on* $\mathscr{H}_{fin}$.

Remark 2.8. This result has exactly the form of the finite dimensional formula. Lemma 2.6 gives the relation with the writing which can be found in [19].

Proof. The second statement (ii) is a straightforward consequence of the first one (i). Let us focus on (i) which will be proved in several steps.

Step 0: Before proving the identity, first notice that both sides are well defined. Actually, for any $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, the operator b^{Wick} sends $\mathscr{H}_{fin}$ into itself. Hence, the product $b_1^{Wick} \circ b_2^{Wick}$ is well defined as an operator $\mathscr{H}_{fin} \to \mathscr{H}_{fin}$. Finally we know from Lemma 2.6 that $e^{\varepsilon\langle\partial_z,\partial_{\overline{\omega}}\rangle} b_1(z)b_2(\omega)\big|_{z=\omega}$ belongs to $\oplus_{p,q}^{\text{alg}}\mathscr{P}_{p,q}(\mathscr{Z})$.

Step 1: Consider $b_1(z) = \langle \eta\,,\, z\rangle$ and $b_2(z) = \langle z\,,\, \xi\rangle^q$, $q \in \mathbb{N}$. The formula

$$a(\eta)a^*(\xi)^q = a^*(\xi)^q a(\eta) + \varepsilon q\langle \eta\,,\, \xi\rangle a^*(\xi)^{q-1}$$

is exactly

$$b_1^{Wick} b_2^{Wick} = (b_1 b_2)^{Wick} + \varepsilon(\partial_z b_1 . \partial_{\bar{z}} b_2)^{Wick}\,.$$

Step 2: Consider $b_1(z) = \beta_p(z) = \langle \eta\,,\, z\rangle^p$ and $b_2(z) = \langle z\,,\, \xi\rangle^q$, $p, q \in \mathbb{N}$. The induction is already initialized for $p = 1$ according to Step 1. Assume that the formula is true for $p-1$ and all $q \in \mathbb{N}$ and compute

$$\begin{aligned}
\beta_p^{Wick} b_2^{Wick} &= \beta_1^{Wick}\left[\beta_{p-1}^{Wick} b_2^{Wick}\right] = \beta_1^{Wick}\left[\sum_{k=0}^{\min\{p-1,q\}} \frac{\varepsilon^k}{k!}\left\langle \partial_z^k \beta_{p-1}\,,\, \partial_{\bar{z}}^k b_2\right\rangle^{Wick}\right] \\
&= a(\eta)\left[\sum_{k=0}^{\min\{p-1,q\}} \frac{\varepsilon^k}{k!}\langle \eta\,,\, \xi\rangle^k \frac{q!}{(q-k)!}\frac{(p-1)!}{(p-1-k)!} a^*(\xi)^{q-k} a(\eta)^{p-1-k}\right] \\
&= \sum_{k=0}^{\min\{p-1,q\}} \frac{\varepsilon^k}{k!}\langle \eta\,,\, \xi\rangle^k \frac{q!(p-1)!}{(q-k)!(p-1-k)!}\left[a^*(\xi)^{q-k} a(\eta)^{p-k}\right. \\
&\quad \left. + \varepsilon(q-k)\langle \eta\,,\, \xi\rangle a^*(\xi)^{q-k} a(\eta)^{p-(k+1)}\right] \\
&= \sum_{k=0}^{\min\{p,q\}} \frac{\varepsilon^k \langle \eta, \xi\rangle^k q!(p-1)!}{k!(q-k)!(p-1-k)!}\left[1_{[0,p-1]}(k) + \frac{k}{(p-k)} 1_{[1,p]}(k)\right] \\
&\quad \times a^*(\xi)^{q-k} a(\eta)^{p-k} \\
&= \sum_{k=0}^{\min\{p,q\}} \frac{\varepsilon^k}{k!}\left\langle \partial_z^k \beta_p\,,\, \partial_{\bar{z}}^k b_2\right\rangle^{Wick}\,.
\end{aligned}$$

We used several times the relation

$$\partial_z^j \beta_n(z) = \frac{n!}{(n-j)!}\langle \eta\,,\, z\rangle^{n-j} \langle \eta\,|^{\otimes j}$$

and its dual version for $\partial_{\bar{z}}^j b_2$.

Step 3: From Step 2, the statement (ii) of Proposition 2.3 leads to

$$\begin{aligned}
&a^*(\xi^1)^{q_1} a(\eta^1)^{p_1}\;\; a^*(\xi^2)^{q_2} a(\eta^2)^{p_2} \\
&\qquad = \sum_{k=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^k}{k!}\Big(\partial_z^k\big(\langle z, \xi^1\rangle^{q_1}\langle \eta^1, z\rangle^{p_1}\big).\partial_{\bar{z}}^k\big(\langle z, \xi^2\rangle^{q_2}\langle \eta^2, z\rangle^{p_2}\big)\Big)^{Wick}
\end{aligned}$$

for any $\xi^1, \xi^2, \eta^1, \eta^2 \in \mathscr{Z}$ and any $p_1, q_1, p_2, q_2 \in \mathbb{N}$. Again the polarization formula (1) in the form

$$\prod_{i=1}^{n} a^{\natural}(\xi_i) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left[a^{\natural}\left(\sum_{j=1}^{n} \varepsilon_j \xi_j \right) \right]^n ,$$

yields the result for any

$$b_\ell(z) = \prod_{i=1}^{p_\ell} \langle z \, , \, \xi_i^\ell \rangle \prod_{j=1}^{q_\ell} \langle \eta_j^\ell \, , \, z \rangle \, , \quad \ell = 1, 2 \, ,$$

that is for any $\tilde{b}_\ell$ in the form

$$\tilde{b}_\ell = |\xi_1^\ell \vee \cdots \vee \xi_{p_\ell}^\ell \rangle \langle \eta_1^\ell \vee \cdots \vee \eta_{q_\ell}^\ell | \, , \quad \ell = 1, 2 \, . \tag{19}$$

Step 4: We want to check the identity

$$\left\langle \psi_{n'} \, , \, b_1^{Wick} \circ b_2^{Wick} \psi_n \right\rangle = \sum_{p=0}^{\min\{p_1, q_2\}} \frac{\varepsilon^p}{p!} \langle \psi_{n'} \, , \, (\partial_z^p b_1 \partial_{\bar{z}}^p b_2)^{Wick} \psi_n \rangle$$

for any $\psi_n \in \bigvee^n \mathscr{Z}$ and any $\psi_{n'} \in \bigvee^{n'} \mathscr{Z}$, $n, n' \in \mathbb{N}$.

From the definition of b^{Wick}, the left-hand side equals

$$\begin{aligned} &\left\langle \psi_{n'} \, , \, b_1^{Wick} \circ b_2^{Wick} \psi_n \right\rangle \\ &\quad = C_{n,n',p_{1,2},q_{1,2},\varepsilon} \left\langle \psi_{n'} \, , \, \left(\tilde{b}_1 \bigvee I|_{\bigvee^{n+q_2-p_2-p_1} \mathscr{Z}} \right) \left(\tilde{b}_2 \bigvee I|_{\bigvee^{n-p_1} \mathscr{Z}} \right) \psi_n \right\rangle \\ &\quad = C_{n,n',p_{1,2},q_{1,2},\varepsilon} \left\langle \left(\tilde{b}_1^* \bigvee I|_{\bigvee^{n'-q_1} \mathscr{Z}} \right) \psi_{n'} \, , \, \left(\tilde{b}_2 \bigvee I|_{\bigvee^{n-p_1} \mathscr{Z}} \right) \psi_n \right\rangle . \end{aligned}$$

Similarly and owing to Lemma 2.6, every term of the right-hand side satisfies

$$\begin{aligned} &\left\langle \psi_{n'} \, , \, (\partial_z^p b_1 \partial_{\bar{z}}^p b_2)^{Wick} \psi_n \right\rangle \\ &\quad = C'_{n,n',p,p_{1,2},q_{1,2},\varepsilon} \left\langle \psi_{n'} \, , \left[\left(\tilde{b}_1 \otimes I_{\bigotimes^{q_2-p} \mathscr{Z}} \right) \left(I_{\bigotimes^{p_1-p} \mathscr{Z}} \otimes \tilde{b}_2 \right) \bigvee I_{\bigvee^{n-p_1-p_2+p} \mathscr{Z}} \right] \psi_n \right\rangle \\ &\quad = C'_{n,n',p,p_{1,2},q_{1,2},\varepsilon} \left\langle \left(\tilde{b}_1^* \otimes I_{\bigotimes^{n'-p_1} \mathscr{Z}} \right) \psi_{n'} \, , \left(I_{\bigotimes^{p_1-p} \mathscr{Z}} \otimes \tilde{b}_2 \otimes I_{\bigvee^{n-p_1-p_2+p} \mathscr{Z}} \right) \psi_n \right\rangle . \end{aligned}$$

Hence for fixed $\psi_n, \psi_{n'} \in \mathscr{H}_{fin}$, both side are sesquilinear continuous expression of $(\tilde{b}_1, \tilde{b}_2)$ when the first factor is considered with the $*$-strong topology of operators and the second one with the strong topology. The operators (19) for which the equality is true, form a total family for these topologies. This can be proved in two steps: approximate first any finite rank operators by linear combinations of the specific rank one operators (19) and then any bounded operators by finite rank operators. Thus the equality holds for any $b_\ell \in \mathscr{P}_{p_\ell, q_\ell}(\mathscr{Z})$, $\ell = 1, 2$. □

Remark 2.9. The formulas (17) and (18) make sense with ε-dependent symbols. One can work with polynomials in ε

$$b(z,\varepsilon)=\sum_{\alpha=0}^{n}\varepsilon^{\alpha}b_{\alpha}(z)\,,\quad b_{\alpha}\in\mathscr{P}_{p,q}(\mathscr{Z})$$

or with asymptotic sums

$$b(z,\varepsilon)\sim\sum_{\alpha=0}^{\infty}\varepsilon^{\alpha}b_{\alpha}(z)\quad b_{\alpha}\in\mathscr{P}_{p,q}(\mathscr{Z})\,.$$

The expression (17) and (18) take then the form

$$\begin{aligned}
b_1^{Wick}b_2^{Wick} &\sim \sum_{j=0}^{\infty}\varepsilon^{j}\left(\sum_{\alpha+\beta+k=j}\frac{1}{k!}\left(\partial_z^k b_{1,\alpha}.\partial_{\bar z}^k b_{2,\beta}\right)\right)^{Wick}\\
\left[b_1^{Wick},\,b_2^{Wick}\right] &\sim \sum_{j=1}^{\infty}\varepsilon^{j}\left(\sum_{\alpha+\beta+k=j}\frac{1}{k!}\left(\partial_z^k b_{1,\alpha}.\partial_{\bar z}^k b_{2,\beta}-\partial_z^k b_{2,\beta}.\partial_{\bar z}^k b_{1,\alpha}\right)\right)^{Wick}\,,
\end{aligned}$$

for $b_1\sim\sum_{\alpha}\varepsilon^{\alpha}b_{1,\alpha}\in\mathscr{P}_{p_1,q_1}(\mathscr{Z})$ and $b_2\sim\sum_{\beta}\varepsilon^{\beta}b_{2,\beta}\in\mathscr{P}_{p_2,q_2}(\mathscr{Z})$. Here (p_1,q_1) (resp. (p_2,q_2)) does not depend on α (resp. β).

We have the following useful result.

Proposition 2.10. *For any* $b\in\oplus_{p,q\in\mathbb{N}}^{\mathrm{alg}}\mathscr{P}_{p,q}(\mathscr{Z})$ *we have:*

(i) b^{Wick} *is closable and the domain of its closure contains*

$$\mathscr{H}_0=\mathrm{vect}\left\{W(z)\phi,\phi\in\mathscr{H}_{fin},z\in\mathscr{Z}\right\}.$$

(ii) *By setting* $E(z)=W(\frac{\sqrt{2}z}{i\varepsilon})\Omega$ *according to* (4), *the identity*

$$b(z)=\left\langle E(z)\,,b^{Wick}E(z)\right\rangle \tag{20}$$

holds for every $z\in\mathscr{Z}$ *.*

(iii) *For any* $z_0\in\mathscr{Z}$ *the identity*

$$W\left(\frac{\sqrt{2}}{i\varepsilon}z_0\right)^{*}b^{Wick}W\left(\frac{\sqrt{2}}{i\varepsilon}z_0\right)=\left(b(z+z_0)\right)^{Wick}$$

holds on $\mathscr{H}_0$ *where* $b(\,\cdot\,+z_0)\in\oplus_{p,q\in\mathbb{N}}^{\mathrm{alg}}\mathscr{P}_{p,q}(\mathscr{Z})$ *.*

Proof. (i) b^{Wick} is closable by Proposition 2.3 (i). It is enough to consider $b\in\mathscr{P}_{p,q}(\mathscr{Z})$ when we prove that $\mathscr{H}_0$ is a core for the closure of b^{Wick}. The last statement is deduced from the estimate

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left|b^{Wick}\,\Phi(z)^n\varphi^{(k)}\right|_{\mathscr{H}} \leq |\tilde{b}|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}\,|\varphi^{(k)}|_{\bigvee^k\mathscr{Z}} \times\sum_{n=0}^{\infty}\frac{(\sqrt{2\varepsilon})^n}{n!}\sqrt{\frac{(n+k)!}{k!}}\left[\varepsilon(n+k+q)\right]^{\frac{p+q}{2}}|z|^n < \infty \tag{21}$$

for any $\varphi^{(k)}\in\bigvee^k\mathscr{Z}$ and $z\in\mathscr{Z}$. In order to prove (21), use Lemma 2.5 and estimate the action of b^{Wick} on $\Phi(z)^n\varphi^{(k)}$ by $\max_{p\leq r\leq k+n}|b^{Wick}|_{\mathscr{L}(\bigvee^r\mathscr{Z},\bigvee^{r-p+q})}$ and bound the norm of $\Phi(z)^n\varphi^{(k)}$ by $|\varphi^{(k)}|\,|z|^n\sqrt{\frac{(2\varepsilon)^n(n+k)!}{k!}}$.

(ii) One writes for $b\in\mathscr{P}_{p,q}(\mathscr{Z})$ and $z\in\mathscr{Z}$

$$\begin{aligned}\left\langle E(z)\,,\,b^{Wick}E(z)\right\rangle &= e^{-\frac{|z|^2}{\varepsilon}}\sum_{n_1,n_2\in\mathbb{N}}\frac{\langle z^{\otimes n_1}\,,\,b^{Wick}z^{\otimes n_2}\rangle}{\sqrt{n_1!}\sqrt{n_2!}}\\ &= e^{-\frac{|z|^2}{\varepsilon}}\sum_{n_1,n_2\in\mathbb{N}}\delta^{+}_{n_1-q,n_2-p}\frac{\varepsilon^{\frac{p+q}{2}}|z|^{n_1-p+n_2-q}}{\sqrt{(n_1-q)!}\sqrt{(n_2-p)!}\varepsilon^{\frac{n_1+n_2}{2}}}b(z)\\ &= b(z)\,.\end{aligned}$$

(iii) The fact that $b(.+z_0)$ remains in the class $\oplus^{\mathrm{alg}}_{p,q\in\mathbb{N}}\mathscr{P}_{p,q}(\mathscr{Z})$ come from the Taylor expansion and (8). In order to prove the equality, differentiate $A(t)=\left[W(\frac{\sqrt{2}}{i\varepsilon}tz_0)b(z+tz_0)^{Wick}W(\frac{\sqrt{2}}{i\varepsilon}tz_0)^*\right]$ in a weak sense on $\mathscr{H}_0$. Proposition 2.7 implies

$$\begin{aligned}i\partial_t A(t) &= W\left(\frac{\sqrt{2}}{i\varepsilon}tz_0\right)\left[-\left[\Phi\left(\frac{\sqrt{2}}{i\varepsilon}z_0\right),b(z+tz_0)^{Wick}\right]\right.\\ &\qquad\left.+\,i\partial_t b(z+tz_0)^{Wick}\right]W\left(\frac{\sqrt{2}}{i\varepsilon}tz_0\right)^*\\ &= W\left(\frac{\sqrt{2}}{i\varepsilon}tz_0\right)\left[\left\langle iz_0,\partial_{\bar{z}}b(z+tz_0)\right\rangle-\left\langle\partial_z b(z+z_0)\,,\,iz_0\right\rangle+i\partial_t b(z+tz_0)\right]^{Wick}\\ &\qquad\times\,W\left(\frac{\sqrt{2}}{i\varepsilon}tz_0\right)^*\\ &= 0\,.\end{aligned}$$

□

Remark 2.11. The relation (20) allows to define easily the Wick symbol of an operator which is defined as a series, when it makes sense, instead of a Wick polynomial. For example the Wick symbol of the Weyl operator $W(\xi)$ equals

$$\left\langle E(z)\,,\,W(\xi)E(z)\right\rangle=\left\langle\Omega\,,\,e^{-i\varepsilon\sigma(\xi,\frac{\sqrt{2}z}{i\varepsilon})}W(\xi)\Omega\right\rangle=e^{i\sqrt{2}S(\xi,z)}e^{-\frac{\varepsilon|\xi|^2}{4}}\,. \tag{22}$$

A variation of Proposition 2.10 ensures that $b(Az+z_0)$ can be Wick quantized for any bounded complex affine transformation in $\mathscr{Z}$ when $b \in \mathscr{P}_{p,q}(\mathscr{Z})$. Actually real symplectic affine transformations of symbols in $\mathscr{P}_{p,q}(\mathscr{Z})$ may also be Wick quantized but only under a Hilbert–Schmidt condition on A which agrees with Shale's theorem or the presentation of general Bogoliubov transformations (see [5]). The following result will be useful in Subsection 5.1.

Proposition 2.12. *Let $B \in \mathscr{L}(\mathscr{Z})$ and let $B_2 \in \mathscr{L}^2(\mathscr{Z})$ be an Hilbert–Schmidt operator on $\mathscr{Z}$ and let $J : \mathscr{Z} \ni z \mapsto Jz =: \overline{z} \in \mathscr{Z}$ be any anti-unitary operator on $\mathscr{Z}$. Then for any $b \in \mathscr{P}_{p,q}(\mathscr{Z})$ the polynomial $b(Bz + B_2\overline{z})$ belongs to $\oplus_{p'+q'=p+q}\mathscr{P}_{p',q'}(\mathscr{Z})$ with the estimate*

$$\left|\partial_{\overline{z}}^{q'}\partial_z^{p'} b(Bz + B_2\overline{z})\right|_{\mathscr{L}(\bigvee^{p'}\mathscr{Z},\bigvee^{q'}\mathscr{Z})} \leq C_{p,q}\big(|B|_{\mathscr{L}(\mathscr{Z})} + |B_2|_{\mathscr{L}^2(\mathscr{Z})}\big)^{p+q}\left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}.$$

Proof. For $b \in \mathscr{P}_{p,q}(\mathscr{Z})$ write, after recalling $\tilde{b} = \mathscr{S}_q\tilde{b}\mathscr{S}_p$ in $\mathscr{L}(\bigotimes^p\mathscr{Z}, \bigotimes^q\mathscr{Z})$,

$$\begin{aligned} b(Bz + B_2\overline{z}) &= \left\langle (Bz + B_2\overline{z})^{\otimes q}\,,\, \tilde{b}(Bz + B_2\overline{z})^{\otimes p}\right\rangle \\ &= \sum_{j=0}^{q}\sum_{k=0}^{p} C_q^j C_p^k \left\langle (Bz)^{\otimes q-j}\otimes (B_2\overline{z})^{\otimes j}\,,\, \tilde{b}(B_2\overline{z})^{\otimes k}\otimes (Bz)^{\otimes p-k}\right\rangle \\ &= \sum_{j=0}^{q}\sum_{k=0}^{p} C_q^j C_p^k \ell_{j,k}(z^{\otimes q+k-j}\,,\, z^{\otimes p+j-k}). \end{aligned}$$

The sesquilinear form $\ell_{j,k}$ is defined on $(\bigotimes^{q-j}\mathscr{Z}\otimes^{alg}\bigotimes^k\mathscr{Z})\times(\bigotimes^j\mathscr{Z}\otimes^{alg}\bigotimes^{p-k}\mathscr{Z})$ by

$$\ell_{j,k}\left(\phi_1\otimes\phi_2, \psi_1\otimes\psi_2\right) = \left\langle (B^{\otimes q-j}\phi_1)\otimes(B_2^{\otimes j}\overline{\psi_2})\,,\, \tilde{b}(B_2^{\otimes k}\overline{\phi_2})\otimes(B^{\otimes p-k})\psi_1\right\rangle.$$

It satisfies for $\Phi = \sum_{\alpha=1}^N \phi_{1,\alpha}\otimes\phi_{2,\alpha}$ and $\Psi = \sum_{\beta=1}^N \psi_{1,\beta}\otimes\psi_{2,\beta}$

$$\begin{aligned} \ell_{j,k}\left(\Phi,\Psi\right) &= \sum_{\beta=1}^N \left\langle (B_2^{\otimes j}\overline{\psi_{2,\beta}})\,,\, C_\Phi(B^{\otimes p-k})\psi_{1,\beta}\right\rangle \\ &= \sum_{\beta=1}^N \langle\overline{\psi_{2,\beta}}\,,\, (B_2^*)^{\otimes j}C_\Phi(B^{\otimes p-k})\psi_{1,\beta}\rangle \end{aligned}$$

with

$$C_\Phi = \sum_{\alpha=1}^N \left(\langle B^{\otimes q-j}\phi_{1,\alpha}|\otimes I_{\bigotimes^j\mathscr{Z}}\right)\tilde{b}\left(|B_2^{\otimes k}\phi_{2,\alpha}\rangle\otimes I_{\bigotimes^{p-k}\mathscr{Z}}\right)\in\mathscr{L}\left(\bigotimes^{p-k}\mathscr{Z},\bigotimes^{j}\mathscr{Z}\right).$$

Since $B_2^{\otimes j}$ is a Hilbert–Schmidt operator the estimate

$$|\ell_{j,k}\left(\Phi,\Psi\right)| \leq |B_2|_{\mathscr{L}^2(\mathscr{Z})}^j\,|B|_{\mathscr{L}(\mathscr{Z})}^{p-k}\,|C_\Phi|_{\mathscr{L}(\bigotimes^{p-k}\mathscr{Z},\bigotimes^j\mathscr{Z})}\,|\Psi|_{\bigotimes^{p-k+j}(\mathscr{Z})}$$

holds for any $\Psi \in \bigotimes^j \mathscr{Z} \otimes^{alg} \bigotimes^{p-k} \mathscr{Z}$. In order to estimate $|C_\Phi|_{\mathscr{L}(\bigotimes^{p-k}\mathscr{Z},\bigotimes^j\mathscr{Z})}$ take any $U \in \bigotimes^j \mathscr{Z}$ and any $V \in \bigotimes^{p-k} \mathscr{Z}$ and compute

$$\begin{aligned}|\langle U\,,\,C_\Phi V\rangle| &= \left|\sum_{\alpha=1}^N \left\langle B^{\otimes q-j}\phi_{1,\alpha}\otimes U\,,\,\tilde{b}(B_2^{\otimes k}\phi_{2,\alpha}\otimes V)\right\rangle\right| \\ &= \left|\sum_{\alpha=1}^N \left\langle \phi_{1,\alpha}\,,\,(B^*)^{\otimes q-j}C_{UV}B_2^{\otimes k}\phi_{2,\alpha}\right\rangle\right| \\ \text{with}\quad C_{UV} &= \left(I_{\bigotimes^{q-j}\mathscr{Z}}\otimes\langle U|\right)\tilde{b}\left(I_{\bigotimes^k\mathscr{Z}}\otimes|V\rangle\right)\in\mathscr{L}\left(\bigotimes^k\mathscr{Z},\bigotimes^{q-j}\mathscr{Z}\right).\end{aligned}$$

Again the Hilbert–Schmidt condition implies

$$|\langle U\,,\,C_\Phi V\rangle| \le |B_2|^k_{\mathscr{L}^2(\mathscr{Z})}|B|^{q-j}_{\mathscr{L}(\mathscr{Z})}|U|_{\bigotimes^j\mathscr{Z}}\left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}|V|_{\bigotimes^{p-k}\mathscr{Z}}|\Phi|_{\bigotimes^{q-j+k}\mathscr{Z}}\,.$$

We have proved an estimate for $|C_\Phi|$ which implies that the estimate

$$|\ell_{j,k}(\Phi,\Psi)| \le |B_2|^{j+k}_{\mathscr{L}^2(\mathscr{Z})}|B|^{p+q-k-j}_{\mathscr{L}(\mathscr{Z})}\left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}|\Phi|_{\bigotimes^{q-j+k}\mathscr{Z}}|\Psi|_{\bigotimes^{p-k+j}}\,,$$

extends continuously to any $\Phi \in \bigotimes^{q-j+k}\mathscr{Z}$ and any $\Psi \in \bigotimes^{p-k+j}\mathscr{Z}$. It holds in particular when $\Phi \in \bigvee^{q-j+k}\mathscr{Z}$ and $\Psi \in \bigvee^{p-k+j}\mathscr{Z}$. Hence $\ell_{j,k}(z) \in \mathscr{P}_{p-k+j,q-j+k}(\mathscr{Z})$ holds for any (j,k), $j \le q$ and $k \le p$, with a norm estimate which yields the final result. □

3. Weyl and Anti-Wick quantization

Our extension of the Weyl and Anti-Wick pseudodifferential calculus to the infinite dimensional case is based on a separation of variables approach within a projective setting. This is slightly different than the one developed by B. Lascar in [35] where the inductive approach leads to a natural Hilbert–Schmidt condition and restricts the exploration of the infinite dimensional phase-space $\mathscr{Z}$.

3.1. Cylindrical functions and Weyl quantization

Let $\mathbb{P}$ denote the set of all finite rank orthogonal projections on $\mathscr{Z}$ and for a given $p \in \mathbb{P}$ let $L_p(dz)$ denote the Lebesgue measure on the finite dimensional subspace $p\mathscr{Z}$. A function $f : \mathscr{Z} \to \mathbb{C}$ is said cylindrical if there exists $p \in \mathbb{P}$ and a function g on $p\mathscr{Z}$ such that $f(z) = g(pz)$, for all $z \in \mathscr{Z}$. In this case we say that f is based on the subspace $p\mathscr{Z}$. We set $\mathscr{S}_{cyl}(\mathscr{Z})$ to be the cylindrical Schwartz space:

$$\left(f \in \mathscr{S}_{cyl}(\mathscr{Z})\right) \Leftrightarrow \left(\exists p \in \mathbb{P}, \exists g \in \mathscr{S}(p\mathscr{Z}),\quad f(z) = g(pz)\right).$$

It is well known that the Fourier–Wigner transform defined by the expression

$$z \mapsto \mathscr{V}[\phi,\psi](z) = \left\langle \psi, W(\sqrt{2}\pi z)\phi\right\rangle,$$

for any $\phi, \psi \in \mathscr{H}$, belongs to $L^2(p\mathscr{Z}, L_p(dz)) \cap C_0(p\mathscr{Z})$ for every $p \in \mathbb{P}$. Introduce the Fourier transform of a function $f \in \mathscr{S}_{cyl}(\mathscr{Z})$ based on the subspace $p\mathscr{Z}$ as

$$\mathscr{F}[f](z) = \int_{p\mathscr{Z}} f(\xi) \; e^{-2\pi i\, S(z,\xi)} \; L_p(d\xi)$$

and its inverse Fourier transform is

$$f(z) = \int_{p\mathscr{Z}} \mathscr{F}[f](z) \; e^{2\pi i\, S(z,\xi)} \; L_p(dz)\,.$$

Therefore the so-called Wigner transform can be written as $\mathscr{W}[\phi,\psi] = \mathscr{F}^{-1}[\mathscr{V}[\phi,\psi]]$. With any symbol $b \in \mathscr{S}_{cyl}(\mathscr{Z})$ based on $p\mathscr{Z}$, a *Weyl observable* can be associated according to

$$b^{Weyl} = \int_{p\mathscr{Z}} \mathscr{F}[b](z) \quad W(\sqrt{2}\pi z) \; L_p(dz)\,. \tag{23}$$

It can be expressed as a quadratic form in the following way

$$\begin{aligned}\langle \psi, b^{Weyl}\phi\rangle_{\mathscr{H}} &= \int_{p\mathscr{Z}} \mathscr{F}[b](z) \;\; \mathscr{V}[\phi,\psi](z) \; L_p(dz)\\ &= \int_{p\mathscr{Z}} b(z) \;\; \mathscr{W}[\phi,\psi](z) \; L_p(dz)\,.\end{aligned}$$

Note that b^{Weyl} is a well defined bounded operator on $\mathscr{H}$ for all $b \in \mathscr{S}_{cyl}(\mathscr{Z})$ since $\mathscr{V}[\phi,\psi](z)$ is a bounded function and $\mathscr{F}[b](z)$ is in $L^1(p\mathscr{Z}, L_p(dz))$. Remember also that this quantization of cylindrical symbols depends on the parameter ε like the Weyl operators $W(\sqrt{2}\pi z)$.

The next estimate will be useful. A similar inequality can be found in [15].

Lemma 3.1. *For any $\delta \in [0,1]$ there exists a constant $C_\delta > 0$ such that the estimate*

$$\left|\left[W(z_1) - W(z_2)\right](N+1)^{-\delta/2}\right| \leq C_\delta \;\; |z_1 - z_2|^\delta \;\; \left[\min(\varepsilon|z_1|, \varepsilon|z_2|)^\delta + \max(1,\varepsilon)^\delta\right],$$

holds for all $\varepsilon > 0$, and all $z_1, z_2 \in \mathscr{Z}$.

Proof. We have by Weyl's relation

$$\begin{aligned}&\left|\left[W(z_1) - W(z_2)\right](N+1)^{-\delta/2}\right|\\ &\qquad\qquad \leq \left|\left[W(z_1 - z_2) - I\right](N+1)^{-\delta/2}\right| + \left|e^{i\varepsilon\sigma(z_1,z_2)} - 1\right|.\end{aligned} \tag{24}$$

The estimate $|e^{is} - 1| \leq C_\delta\, |s|^\delta$, leads to

$$\begin{aligned}\left|e^{i\varepsilon\sigma(z_1,z_2)} - 1\right| &= \left|e^{i\varepsilon\sigma(z_1 - z_2, z_2)} - 1\right|\\ &= \left|e^{i\varepsilon\sigma(z_1, z_2 - z_1)} - 1\right| \leq C_\delta\, \varepsilon^\delta \;\; |z_1 - z_2|^\delta \;\; \min(|z_1|, |z_2|)^\delta\,.\end{aligned}$$

The first part of the r.h.s. in (24) is estimated via a complex interpolation argument. Indeed, for $\delta = 0$ notice that $|W(z_1 - z_2) - I| \leq 2$ and for $\delta = 1$ the estimate $\left|e^{is}-1\right| \leq C_1|s|$ combined with the spectral theorem yields

$$\begin{aligned}\left|\left[W(z_1-z_2)-I\right](N+1)^{-1/2}\psi\right| &\leq C_1\left||\Phi(z_1-z_2)|(N+1)^{-1/2}\psi\right| \\ &\leq C_1\left|\Phi(z_1-z_2)(N+1)^{-1/2}\psi\right|.\end{aligned}$$

Now by the number estimate (12) we obtain

$$\left|\left[W(z_1-z_2)-I\right](N+1)^{-1/2}\right| \leq C\ \max(1,\varepsilon)\ |z_1-z_2|.$$ □

3.2. Finite dimensional Weyl quantization

The finite dimensional Weyl calculus provides us a collection of results on the Weyl quantization. We specify here the relation between the Weyl quantization defined on $\mathscr{Z}$ via (23) and the usual semiclassical Weyl quantization within the Schrödinger representation on $\mathbb{R}^d$.

For $p \in \mathbb{P}$ the orthogonal projector $I-p$ is denoted by $p^\perp$. Let $\Gamma_s(p\mathscr{Z})$ denotes the symmetric Fock space over $p\mathscr{Z}$. The separation of variables in finite dimensions extends to general symmetric Fock spaces owing to the canonical isomorphism of Fock spaces

$$T_p : \mathscr{H} = \Gamma_s(\mathscr{Z}) \to \Gamma_s(p\mathscr{Z}) \otimes \Gamma_s(p^\perp\mathscr{Z}), \tag{25}$$

for any finite dimensional projector $p \in \mathbb{P}$, with $T_p\Omega = \Omega^{p\mathscr{Z}} \otimes \Omega^{p^\perp\mathscr{Z}}$ when $\Omega^{p\mathscr{Z}}$ and $\Omega^{p^\perp\mathscr{Z}}$ are the vacuum vectors of the corresponding Fock spaces. We will omit the notation T_p and identify directly the tensor products.

Fix $p \in \mathbb{P}$. The tensor decomposition of the Weyl quantization comes from the Weyl relation which implies

$$W(\xi+\xi') = W(\xi)W(\xi') = W_p(\xi) \otimes W_{p^\perp}(\xi')$$

for any $(\xi,\xi') \in p\mathscr{Z} \times p^\perp\mathscr{Z}$. The symbols W_p stands for the Weyl operator defined on the Fock space $\Gamma_s(p\mathscr{Z})$ and the Weyl quantization of $b \in \mathscr{S}(F)$, for any finite dimensional complex subspace F of $\mathscr{Z}$, is denoted by b_F^{Weyl}. Hence the Weyl quantization of $b \in \mathscr{S}_{cyl}(\mathscr{Z})$ based on $p\mathscr{Z}$ equals

$$b^{Weyl} = \int_{p\mathscr{Z}} \mathscr{F}[b](z)W(\sqrt{2}\pi z)\ L_p(dz) = b_{p\mathscr{Z}}^{Weyl} \otimes I_{\Gamma_s(p^\perp\mathscr{Z})}.$$

In order to apply directly the finite dimensional results on Weyl quantization, we need to specify the correspondence of representations.

On $\mathbb{R}^d$ the Weyl quantization is often introduced as

$$b^{Weyl}(x,hD_x)u(x) = \int_{\mathbb{R}^d} e^{i\frac{(x-y).\xi}{h}} b\left(\frac{x+y}{2},\xi\right) u(y)\ \frac{d\xi dy}{(2\pi h)^d}.$$

By a simple conjugation with a dilatation, it becomes $a^{Weyl}(\sqrt{h}x,\sqrt{h}D_x)$ where the position (x) and frequency (ξ) variables play the same role. An equivalent definition can be given with the help of the phase translations :

$$\tau_{(x_0,\xi_0)} = e^{i(\xi_0 x - x_0 D_x)} = \left(e^{i(\xi_0 x - x_0\xi)}\right)^{Weyl}, \quad [\tau_{x_0,\xi_0}u](x) = e^{i\xi_0(2x-x_0)/2}u(x-x_0).$$

It reads

$$\begin{aligned} b^{Weyl}(\sqrt{h}x, \sqrt{h}D_x) &= \int_{T^*\mathbb{R}^d} \mathscr{F}[b](y,\eta) e^{2i\pi(y.(\sqrt{h}x)+\eta.\sqrt{h}D_x)}\, dyd\eta \\ &= \int_{T^*\mathbb{R}^d} \mathscr{F}[b](y,\eta)\tau_{(-2\pi\sqrt{h}\eta, 2\pi\sqrt{h}y)}\, dyd\eta\,. \end{aligned}$$

The symplectic form $[\![\ ,\ ]\!]$ and the scalar product $(\ ,\ )$ on $T^*\mathbb{R}^d$ are defined according to

$$\begin{aligned} [\![(x,\xi),(y,\eta)]\!] &= \xi.y - x.\eta = -\mathrm{Im}\,\langle x+i\xi\,,\, y+i\eta\rangle = -\sigma(x+i\xi, y+i\eta) \\ \big((x,\xi),(y,\eta)\big) &= x.y + \xi.\eta = \mathrm{Re}\,\langle x+i\xi\,,\, y+i\eta\rangle = S(x+i\xi, y+i\eta)\,. \end{aligned}$$

After noting

$$\left[\sqrt{h}x + \sqrt{h}\partial_x, \sqrt{h}x - \sqrt{h}\partial_x\right] = 2h\,,$$

the correspondence with the definition (23) is summarized in the next table

$p\mathscr{Z} \sim \mathbb{C}^d$		$T^*\mathbb{R}^d$
$\Gamma_s(p\mathscr{Z}) \sim \Gamma_s(\mathbb{C}^d)\,,$		$L^2(\mathbb{R}^d)$
$\langle z_1, z_2\rangle = S(z_1,z_2) + i\sigma(z_1,z_2)$	$z = e^{i\theta}(x+i\xi)$	
		$\big((x_1,\xi_1)\,,\,(x_2,\xi_2)\big) = \xi_1.\xi_2 + x_1.x_2 = S(z_1,z_2)$
		$[\![(x_1,\xi_1),(x_2,\xi_2)]\!] = \xi_1.x_2 - x_1.\xi_2 = -\sigma(z_1,z_2)$
$a(z) = a\left(\sum_{j=1}^d \alpha_j e_j\right)$		$a(z) = \sum_{j=1}^d \overline{\alpha_j}(\sqrt{h}\partial_{x_j} + \sqrt{h}x_j)$
$a^*(z) = a^*\left(\sum_{j=1}^d \alpha_j e_j\right)$		$a^*(z) = \sum_{j=1}^d \alpha_j(-\sqrt{h}\partial_{x_j} + \sqrt{h}x_j)$
$\left[a(z_1), a^*(z_2)\right] = \varepsilon\,\langle z_1\,,\, z_2\rangle$	$\varepsilon = 2h$	$\left[a(z_1), a^*(z_2)\right] = 2h\,\langle z_1\,,\, z_2\rangle$
$\Phi(z_0) = \frac{1}{\sqrt{2}}\big(a(z_0) + a^*(z_0)\big)$	$z_0 = x_0 + i\xi_0$	$\sqrt{2h}(x_0.x + \xi_0.D_x)$
$W(z_0) = e^{i\Phi(z_0)}$	$\theta = 0$	$\tau_{(-\sqrt{2h}\xi_0, \sqrt{2h}x_0)}$
$E(z_0) = W\left(\frac{\sqrt{2}}{i\varepsilon}z_0\right)\Omega$	$\frac{z_0}{i} = \xi_0 - ix_0$	$\tau_{(\frac{x_0}{\sqrt{h}}, \frac{\xi_0}{\sqrt{h}})}(\pi^{-d/4}e^{-\frac{x^2}{2}})$
$z_0^{\otimes n}, \lvert z_0\rvert = 1$		Hermite function
		$(n!)^{-1/2}\left[z_0.(-\partial_x + x)\right]^n\big(\pi^{-d/4}e^{-\frac{x^2}{2}}\big)$
$\bigcap_{k\in\mathbb{N}} D\big(\langle N_{p\mathscr{Z}}\rangle^k\big)\,,\quad \bigcup_{k\in\mathbb{N}} D\big(\langle N_{p\mathscr{Z}}\rangle^k\big)^*$		$\mathscr{S}(\mathbb{R}^d)\,,\quad \mathscr{S}'(\mathbb{R}^d)$

Once this is fixed, the general results on the semiclassical Weyl–Hörmander pseudodifferential calculus (see for example [8, 33] for the general introduction and [37,39,41] for the small parameter version) can be applied for any fixed $p \in \mathbb{P}$.

The notion of slow and temperate metric and weight depend only on the symplectic structure which is given by $\sigma(z_1, z_2) = \mathrm{Im}\langle z_1\,,\, z_2\rangle$. With such a metric the gain function λ is given on $p\mathscr{Z}$ by

$$\lambda(z)^2 = \inf_{T\in p\mathscr{Z}\setminus\{0\}} \frac{g_z^\sigma(T)}{g_z(T)} \quad \text{with}$$
$$g_z^\sigma(T) = \sup_{S\in p\mathscr{Z}\setminus\{0\}} \frac{|[\![T,S]\!]|^2}{g(S)} = \sup_{S\in p\mathscr{Z}\setminus\{0\}} \frac{|\sigma(T,S)|^2}{g(S)}\,.$$

With a slow and temperate metric g and a slow and temperate weight m, is associated a symbol class usually denoted $S(m,g)$.

After writing $X = (x,\xi) \in T^*\mathbb{R}^d$ for the complete phase-space variable, the differential operator D_X is $(D_x, D_\xi) = (i^{-1}\partial_x, i^{-1}\partial_\xi)$. In the composition formula of symbols, the differential operator $\frac{ih}{2}[\![D_{X_1}, D_{X_2}]\!]$ appears. After recalling

$$\partial_{\overline{z}} = \frac{1}{2}(\nabla_x + i\nabla_\xi) \quad \text{and} \quad \partial_z = \frac{1}{2}(\nabla_x - i\nabla_\xi)$$

it equals

$$\frac{ih}{2}[\![D_{X_1}, D_{X_2}]\!] = \frac{\varepsilon}{2}\left(\partial_{z_1}.\partial_{\overline{z}_2} - \partial_{\overline{z}_1}.\partial_{z_2}\right).$$

We refer to [39] for an explicit semiclassical writing of the Weyl–Hörmander calculus within the Bony–Lerner presentation [8] and with a general version of the Beals criterion following Bony–Chemin [7].

Proposition 3.2. *Let g be a slow and temperate metric on $p\mathscr{Z}$, $dim_{\mathbb{C}}(p\mathscr{Z}) = d$ and let m_1 and m_2 be two slow and temperate weights for g. For $b_\ell \in S_{p\mathscr{Z}}(m_\ell, g), \ell = 1,2$, the operator $b^{Weyl}_{\ell,p\mathscr{Z}}$ acts continuously on $\cap_{k\in\mathbb{N}} D(\langle N_{p\mathscr{Z}}\rangle^k)$ and on $\cup_{k\in\mathbb{N}} D(\langle N_{p\mathscr{Z}}\rangle^k)^*$.*

The symbol $b_1\#^{\varepsilon/2}b_2$ of $b^{Weyl}_{1,p\mathscr{Z}} \circ b^{Weyl}_{2,p\mathscr{Z}}$ satisfies

$$\begin{aligned} b_1\#^{\varepsilon/2}b_2(z) &= e^{\frac{\varepsilon}{2}\left(\partial_{z_1}.\partial_{\overline{z}_2} - \partial_{\overline{z}_1}.\partial_{z_2}\right)} b_1(z_1)b_2(z_2)\Big|_{z_1=z_2=z} \\ &= \sum_{0\le j<\nu} \frac{1}{j!}\left(\frac{\varepsilon}{2}\left(\partial_{z_1}.\partial_{\overline{z}_2} - \partial_{\overline{z}_1}.\partial_{z_2}\right)\right)^j b_1(z_1)b_2(z_2)\Big|_{z_1=z_2=z} \\ &\quad + \varepsilon^\nu R_\nu(b_1, b_2;\varepsilon) \end{aligned}$$

where $R_\nu(b_1,b_2;\varepsilon)$ is uniformly bounded w.r.t ε in the Fréchet space $S_{p\mathscr{Z}}(\frac{m_1m_2}{\lambda^\nu}, g)$. The Calderon–Vaillancourt theorem

$$\left|b^{Weyl}_{p\mathscr{Z}}\right|_{\mathscr{L}(\Gamma_s(p\mathscr{Z}))} \le Cp_{k_d}(b)$$

and the Gårding inequality

$$(b \ge 0) \Rightarrow \left(b^{Weyl}_{p\mathscr{Z}} \ge -C'p'_{k_d}(b)\varepsilon\right)$$

respectively for $b \in S_{p\mathscr{Z}}(1,g)$ *and* $b \in S_{p\mathscr{Z}}(\lambda,g)$ *. The index* k_d *for the seminorms* p_{k_d} *and* p'_{k_d} *recalls the dimension dependent number of derivatives required in the estimates.*

The typical example Hörmander metrics, which will be used here, are $|dz|^2 = dx^2 + d\xi^2$ $(\lambda(z) = 1)$ and $\frac{|dz|^2}{\langle z\rangle^2} = \frac{dx^2}{\langle (x,\xi)\rangle^2} + \frac{d\xi^2}{\langle (x,\xi)\rangle^2}$ $(\lambda(z) = 1 + |z|^2)$. Both of them split up in the (x,ξ) coordinates and the Beals criterion of Bony–Chemin [7] translated in the semiclassical case in [39]-Appendix-A can be applied. Following the method recalled in [30]-Chapter-4, this allows to check that functions of fully elliptic self-adjoint pseudodifferential operators are pseudodifferential operators, with an explicit knowledge of their principal symbol. In particular, this can be applied with $1 + \frac{\varepsilon \mathrm{dim} p}{2} + N_{p\mathscr{Z}} = (1+|z|^2)^{Weyl}_{p\mathscr{Z}}$ while noticing that $1 + \frac{\varepsilon \mathrm{dim} p}{2} + N_{p\mathscr{Z}}$ is a fully elliptic operator in $S(\langle z\rangle^2, \frac{|dz|^2}{\langle z\rangle^2})$ (such a result with $\varepsilon = 1$ can be found also in [27]).

Proposition 3.3. *Fix* $p \in \mathbb{P}$*, fix the exponent* $s \in \mathbb{R}$ *and let* $N_{p\mathscr{Z}} = d\Gamma(I_{p\mathscr{Z}})$ *be the number operator on* $\Gamma_s(p\mathscr{Z})$*. For any* $s \in \mathbb{R}$*,* $(1 + \frac{\varepsilon dim p}{2} + N_{p\mathscr{Z}})^{s/2}$ *can be written* $(b(s,\varepsilon))^{Weyl}_{p\mathscr{Z}}$ *with* $\varepsilon^{-1}(b(z;s,\varepsilon) - \langle z\rangle^s)$ *uniformly bounded in* $S(\langle z\rangle^{s-2}, \frac{|dz|^2}{\langle z\rangle^2})$ *.*

3.3. Weyl quantization and Laguerre connection

In this paragraph, the relationship between the Wick and Weyl calculus is checked in the infinite dimensional setting. It specifies the relation between the representation of the Weyl algebra, generated by the $W(\xi)$, and the number representation which puts the stress on Wick symbols or Hermite states $z^{\otimes k}$. This relies on the introduction of Hermite and Laguerre polynomials, recalled below.

Let $h_n(x)$ denote, for any $n \in \mathbb{N}$, the n-th Hermite polynomial in $\mathbb{C}$:

$$h_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r} . \tag{26}$$

Those classical polynomials are also given by the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x) = e^{x^2} \left[\sum_{n=0}^{\infty} \frac{(-t\partial_x)^n}{n!} e^{-x^2} \right] = e^{x^2} e^{-t\partial_x}[e^{-x^2}] = e^{2tx - t^2} . \tag{27}$$

Lemma 3.4.

(i) *For any* $\xi \in \mathscr{Z}$*, the following identity holds in* $\mathscr{H}_{fin}$*:*

$$W(\xi) = \sum_{n=0}^{\infty} \frac{|\sqrt{\varepsilon}\xi|^n}{2^n n!} \; h_n \left(\frac{i\sqrt{2} S(\xi,z)}{|\sqrt{\varepsilon}\xi|} \right)^{Wick} .$$

(ii) *For any $n, j, k \in \mathbb{N}$ the estimate*

$$\left| 1_{\{j\varepsilon\}}(N) \circ h_n\left(i\sqrt{2}S(\xi,z)\right)^{Wick} \circ 1_{\{k\varepsilon\}}(N) \right|_{\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^j \mathscr{Z})} \leq \left(1 + 2\sqrt{2(k+j)\varepsilon}\ |\xi|\right)^n \frac{n!}{[n/2]!}\,,$$

holds for any $\xi \in \mathscr{Z}$.

Proof. Using the generating function (27) with $t = \frac{\sqrt{\varepsilon}|\xi|}{2}$ and $x = \frac{i\sqrt{2}S(\xi,z)}{\sqrt{|\varepsilon\xi|}}$ implies the equality of the Wick symbols

$$e^{i\sqrt{2}S(\xi,z)}e^{-\frac{\varepsilon|\xi|^2}{4}} = e^{i\frac{2\sqrt{2}S(\xi,z)}{\sqrt{\varepsilon}|\xi|}\frac{\sqrt{\varepsilon}|\xi|}{2}}e^{-\frac{\varepsilon|\xi|^2}{4}} = \sum_{n=0}^{\infty}\frac{(\sqrt{\varepsilon}|\xi|)^n}{2^n n!}h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right).$$

Nevertheless the equality of the the series of Wick quantized operators has to be checked.

Recall that elements of $\mathscr{H}_{fin}$ are analytic vectors with infinite radius of convergence for the field operators. Hence the sum

$$W(\xi)\psi = \sum_{n=0}^{\infty}\frac{i^n}{n!}\,\Phi(\xi)^n\psi\,,\quad \psi \in \mathscr{H}_{fin}\,,$$

is absolutely convergent for all $\xi \in \mathscr{Z}$. Therefore to prove (i) it is enough to compute the Wick symbol of $\Phi(\xi)^n$ for all n. Indeed using the Wick ordering rules, we have

$$\begin{aligned}
\Phi(\xi)^n &= \sum_{r=0}^{[n/2]}\frac{n!}{\sqrt{2^n}r!(n-2r)!}\frac{|\xi|^{2r}}{2^r}\,\varepsilon^r\sum_{s=0}^{n-2r}C_{n-2r}^s\ a^*(\xi)^s\ a(\xi)^{n-2r-s}\\
&= \frac{|\xi|^n}{2^n}\sum_{r=0}^{[n/2]}\frac{n!}{r!(n-2r)!}\,\varepsilon^r\left(\frac{\sqrt{2^{n-2r}}}{|\xi|^{n-2r}}\sum_{s=0}^{n-2r}C_{n-2r}^s\langle z,\xi\rangle^s\ \langle\xi,z\rangle^{n-2r-s}\right)^{Wick}\\
&= \frac{|\xi|^n}{2^n}\left(\sum_{r=0}^{[n/2]}\frac{n!}{r!(n-2r)!}\,\varepsilon^r\left(\frac{2\sqrt{2}S(\xi,z)}{|\xi|}\right)^{n-2r}\right)^{Wick}.
\end{aligned}$$

To prove the second statement (ii), take $\psi_k \in \bigvee^k \mathscr{Z}$ and $\psi_j \in \bigvee^j \mathscr{Z}$ and write

$$\left\langle \psi_j\,,\, h_n\left(i\sqrt{2}S(\xi,z)\right)^{Wick}\psi_k\right\rangle = \sum_{r=0}^{[n/2]}\frac{n!}{(n-2r)!r!}\left\langle\psi_j\,,\,\left(\left(2i\sqrt{2}S(\xi,z)\right)^{n-2r}\right)^{Wick}\psi_k\right\rangle.$$

Using Lemma 2.5 one obtains

$$\begin{aligned}
&\left|\left\langle \psi_j\,,\, h_n\left(i\sqrt{2}S(\xi,z)\right)^{Wick}\psi_k\right\rangle\right| \\
&\quad\le |\psi_j|_{\bigvee^j\mathscr{Z}}\,|\psi_k|_{\bigvee^k\mathscr{Z}}\sum_{r=0}^{[n/2]}\frac{n!}{(n-2r)!r!}\left(2\sqrt{2(k+j)\varepsilon}\,|\xi|\right)^{n-2r} \\
&\quad\le |\psi_j|_{\bigvee^j\mathscr{Z}}\,|\psi_k|_{\bigvee^k\mathscr{Z}}\sum_{s=0}^{n}\frac{n!}{(n-s)!s!}\left(2\sqrt{2(k+j)\varepsilon}\,|\xi|\right)^{n-s}\frac{s!}{[s/2]!} \\
&\quad\le |\psi_j|_{\bigvee^j\mathscr{Z}}\,|\psi_k|_{\bigvee^k\mathscr{Z}}\left(1+2\sqrt{2(k+j)\varepsilon}\,|\xi|\right)^{n}\frac{n!}{[n/2]!}\,.
\end{aligned}$$

□

The Laguerre polynomials are defined by the formula

$$L_k^{(j)}(t)=\sum_{m=0}^{k}(-1)^m\frac{(k+j)!}{(k-m)!(j+m)!m!}\,t^m\,,\quad t\in\mathbb{C}\,.$$

The following proposition gives the Laguerre connection (see [18], [40]).

Proposition 3.5. *For $z,\xi\in\mathscr{Z}$ with $|z|=1$, the next equalities hold according to the ordering of j and $k\in\mathbb{N}$,*

$$\mathscr{V}[z^{\otimes k},z^{\otimes j}]\left(\frac{\xi}{\pi\sqrt{2\varepsilon}}\right)=\begin{cases}(i)^{k-j}\sqrt{\frac{j!}{k!}}L_j^{(k-j)}(|\langle\xi,z\rangle|^2)\langle\xi,z\rangle^{k-j}e^{-|\xi|^2/2} & \text{if } k\ge j\,,\\ (i)^{j-k}\sqrt{\frac{k!}{j!}}L_k^{(j-k)}(|\langle\xi,z\rangle|^2)\langle z,\xi\rangle^{j-k}e^{-|\xi|^2/2} & \text{if } j\ge k\,.\end{cases}\tag{28}$$

Proof. Let us establish the expression of $\mathscr{V}[z^{\otimes k},z^{\otimes j}]$ in the case $k\ge j$. The case $j\le k$ is similar. Using Lemma 3.4 one obtains

$$\begin{aligned}
\mathscr{V}[z^{\otimes k},z^{\otimes j}]\left(\frac{\xi}{\pi\sqrt{2\varepsilon}}\right)&=\left\langle z^{\otimes j},W\left(\sqrt{\frac{2}{\varepsilon}}\xi\right)z^{\otimes k}\right\rangle\\
&=\sum_{n=0}^{\infty}\frac{|\xi|^n}{\sqrt{2^n}n!}\left\langle z^{\otimes j},\ h_n\left(\frac{iS(\sqrt{\frac{2}{\varepsilon}}\xi,.)}{|\xi|}\right)^{Wick}z^{\otimes k}\right\rangle.
\end{aligned}$$

Now let use the explicit form of h_n and Proposition 2.4. We obtain for $|z|=1$,

$$\begin{aligned}
&\mathscr{V}[z^{\otimes k},z^{\otimes j}]\left(\frac{\xi}{\pi\sqrt{2\varepsilon}}\right)\\
&=\sum_{n=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{s=0}^{n-2r}\frac{i^n|\xi|^{2r}}{2^r r!(n-2r)!}C_{n-2r}^{s}\ \varepsilon^{r-\frac{n}{2}}\left\langle z^{\otimes j},\left(\langle\xi,.\rangle^s\langle .,\xi\rangle^{n-2r-s}\right)^{Wick}z^{\otimes k}\right\rangle
\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{r=0}^{[n/2]}\sum_{s=0}^{n-2r} \frac{i^n|\xi|^{2r}}{2^r r!(k-j+s)!s!}\ |\langle\xi,z\rangle|^{2s}\ \langle\xi,z\rangle^{k-j}\frac{\sqrt{k!j!}}{(j-s)!}\ \delta^+_{k-n+2r+s,j-s}$$

$$= (i)^{k-j}\sqrt{\frac{j!}{k!}}\sum_{s=0}^{j}\sum_{r=0}^{\infty}\frac{(-1)^r|\xi|^{2r}}{2^r r!}\frac{(-1)^s k!}{s!(k-j+s)!(j-s)!}\ |\langle\xi,z\rangle|^{2s}\ \langle\xi,z\rangle^{k-j}\,.$$

The last term gives the claimed identity. □

3.4. Anti-Wick operators

The Anti-Wick quantization is introduced by a separation of variables process like the Weyl quantization. For a given $p\in\mathbb{P}$, set $p^\perp = 1-p$, and use the tensor decomposition (25). The Weyl operators on $p\mathscr{Z}$ and $p^\perp\mathscr{Z}$ are denoted by $W_p(\xi_1)$ and $W_{p^\perp}(\xi_2)$ with $W(\xi_1\oplus^\perp\xi_2) = W_p(\xi_1)\otimes W_{p^\perp}(\xi_2)$. For any $\xi\in p\mathscr{Z}$, the coherent state $E_p(\xi)$ is defined by $E_p(\xi) = W_p(\frac{\sqrt{2}\xi}{i\varepsilon})\Omega^{p\mathscr{Z}}$. Introduce the projector P_ξ on $\mathscr{H}$ after tensorization with $I_{\Gamma_s(p^\perp\mathscr{Z})}$:

$$p\mathscr{Z}\ni\xi\mapsto P^\varepsilon_\xi = \big(|E_p(\xi)\rangle\langle E_p(\xi)|\big)\otimes I_{\Gamma_s(p^\perp\mathscr{Z})}\,.$$

The Anti-Wick operator associated with a symbol $b\in\mathscr{S}_{cyl}(\mathscr{Z})$ based on $p\mathscr{Z}$ is then defined by

$$b^{A-Wick} = \int_{p\mathscr{Z}} b(\xi)\ \ P^\varepsilon_\xi\ \ \frac{L_p(d\xi)}{(\pi\varepsilon)^{\mathrm{dim} p\mathscr{Z}}} = b^{A-Wick}_{p\mathscr{Z}}\otimes I_{\Gamma_s(p^\perp\mathscr{Z})}\,.$$

The above formula can be first considered in a weak sense or as a Bochner integral when $b\in\mathscr{S}(p\mathscr{Z})$ and the bounded projector P^ε_ξ is continuous w.r.t. ξ. The finite dimensional identification of the Weyl symbol of $|W_p(\frac{\sqrt{2}\xi}{i\varepsilon})\Omega^{p\mathscr{Z}}\rangle\langle W_p(\frac{\sqrt{2}\xi}{i\varepsilon})\Omega^{p\mathscr{Z}}|$, can be deduced after completing the table of correspondences in Subsection 3.2:

$$\begin{array}{rcl} p\mathscr{Z}\sim\mathbb{C}^d & z = x+i\xi & T^*\mathbb{R}^d \\ \Gamma_s(p\mathscr{Z})\sim\Gamma_s(\mathbb{C}^d)\,, & \varepsilon = 2h & L^2(\mathbb{R}^d) \\ E_p(z_0) = W_p\left(\frac{\sqrt{2}}{i\varepsilon}z_0\right)\Omega^{p\mathscr{Z}} & \frac{z_0}{i} = \xi_0 - ix_0 & \tau_{(\frac{x_0}{\sqrt{h}},\frac{\xi_0}{\sqrt{h}})}(\pi^{-d/4}e^{-\frac{x^2}{2}}) \\ |\Omega^{p\mathscr{Z}}\rangle\langle\Omega^{p\mathscr{Z}}| = \gamma^{Weyl} & & (\pi)^{-d/2}e^{-\frac{x^2}{2}-\frac{y^2}{2}} = g^{Weyl}(\sqrt{h}x,\sqrt{h}D_x) \\ \gamma(z) = 2^d e^{-\frac{|z|^2_{p\mathscr{Z}}}{\varepsilon/2}} & \Leftarrow & \text{with}\quad g(x,\xi) = 2^d e^{-\frac{x^2+\xi^2}{h}} \end{array}$$

From the conjugation

$$\tau_{(\frac{x_0}{\sqrt{h}},\frac{\xi_0}{\sqrt{h}})}a^{Weyl}(\sqrt{h}x,\sqrt{h}D_x)\tau^*_{(\frac{x_0}{\sqrt{h}},\frac{\xi_0}{\sqrt{h}})} = a(.-x_0,.-\xi_0)^{Weyl}(\sqrt{h}x,\sqrt{h}D_x)$$

the above correspondence gives

$$|E_p(\xi)\rangle\langle E_p(\xi)| = \gamma^{Weyl}_\xi\quad\text{with}\quad\gamma_\xi(z) = 2^d e^{-\frac{|z-\xi|^2_{p\mathscr{Z}}}{\varepsilon/2}}\,.$$

Hence the usual finite dimensional relation between the Weyl and Anti-Wick quantization now reads (after tensorization with $I_{\Gamma_s(p^\perp \mathscr{Z})}$)

$$b^{A-Wick} = \left(b \underset{p\mathscr{Z}}{*} \frac{e^{-\frac{|z|^2_{p\mathscr{Z}}}{\varepsilon/2}}}{(\pi\varepsilon/2)^{\dim p\mathscr{Z}}} \right)^{Weyl} \tag{29}$$

$$= \int_{p\mathscr{Z}} \mathscr{F}[b](\xi)\ W(\sqrt{2}\pi\xi)\ e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2_{p\mathscr{Z}}}\ L_p(d\xi)\,, \tag{30}$$

for any $b \in \mathscr{S}(p\mathscr{Z})$ by setting

$$b \underset{p\mathscr{Z}}{*} \gamma(z) = \int_{p\mathscr{Z}} b(z)\gamma(z-z')\ L_p(dz')\,.$$

From (29), the Anti-Wick quantization can be extended to symbols in $S(1, |dz|^2)$ with the next properties (see [29]).

Proposition 3.6. *Fix $p \in \mathbb{P}$. Let $b \in S_{p\mathscr{Z}}(1, |dz|^2)$, the following statements hold true:*

(i) *If $b \geq 0$ then $b^{A-Wick} \geq 0$.*
(ii) $\left|b^{A-Wick}\right|_{\mathscr{L}(\mathscr{H})} \leq |b|_{L^\infty(p\mathscr{Z})}$.
(iii) *The comparison with the Weyl quantization is given by* (29) *with the estimate*

$$\left|b^{A-Wick} - b^{Weyl}\right|_{\mathscr{L}(\mathscr{H})} \leq C_d p_{k_d}(b)\varepsilon$$

where the constant $C_d > 0$ and the seminorm p_{k_d} depend essentially on the dimension $d = \dim p\mathscr{Z}$.

A variation of it holds when $b \in \mathscr{F}^{-1}(\mathscr{M}_b(p\mathscr{Z}))$, when $\mathscr{M}_b(p\mathscr{Z})$ denotes the set of bounded (Radon) measures on $p\mathscr{Z}$ and comes directly from (30).

Proposition 3.7. *For any $p \in \mathbb{P}$ and any $b \in \mathscr{F}^{-1}(\mathscr{M}_b(p\mathscr{Z}))$, the Anti-Wick and Weyl observables are asymptotically the same:*

$$\lim_{\varepsilon\to 0} \left|b^{A-Wick} - b^{Weyl}\right|_{\mathscr{L}(\mathscr{H})} = 0\,.$$

Proof. Recall that b^{Weyl} can be defined for any $b \in \mathscr{S}'(p\mathscr{Z})$ as a continuous operator from $\cap_{k\in\mathbb{N}} D(N^k_{p\mathscr{Z}}) \sim \mathscr{S}(\mathbb{R}^d)$ to $\cup_{k\in\mathbb{N}} D(N^k_{p\mathscr{Z}})^* \sim \mathscr{S}'(\mathbb{R}^d)$, with $d = \dim p\mathscr{Z}$ and (30) is still valid for such a symbol. Assume $\mathscr{F}b = \nu \in \mathscr{M}_b(p\mathscr{Z})$. The identity

$$\left\langle \psi\,, (b^{Weyl} - b^{A-Wick})\varphi \right\rangle = \int_{p\mathscr{Z}} \left\langle \psi\,, W(\sqrt{2}\pi\xi)\varphi \right\rangle \left(1 - e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2}\right) d\nu(\xi)$$

holds for any $\varphi, \psi \in \cap_{k\in\mathbb{N}} D(N^k_{p\mathscr{Z}})$. This implies

$$\left|b^{Weyl} - b^{A-Wick}\right|_{\mathscr{L}(\mathscr{H})} \leq \int_{p\mathscr{Z}} \left(1 - e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2}\right) d\,|\nu|\,(\xi) \overset{\varepsilon\to 0}{\to} 0\,. \qquad \square$$

3.5. Weyl quantization and specific Wick observables

In finite dimension, that is for any fixed $p \in \mathbb{P}$, polynomially bounded symbols can be introduced after considering the class of symbols $\cup_{s\in\mathbb{R}} S_{p\mathscr{Z}}(\langle z\rangle^s, g_p)$ where g_p is either the metric $|dz|^2$ or $\frac{|dz|^2}{\langle z\rangle^2}$ on $p\mathscr{Z}$. According to Proposition 3.2 it is an algebra with the Moyal product, $\#^{\varepsilon/2}$, associated with the composition of Weyl quantized observable with a complete asymptotic expansion of $b_1\#^{\varepsilon/2}b_2$. For any $m, q \in \mathbb{N}$, the finite dimensional space $\mathscr{P}_{m,q}(p\mathscr{Z})$ of (m,q)-homogeneous polynomials on $\mathscr{Z}$ is contained in $\mathscr{S}_{p\mathscr{Z}}(\langle z\rangle^{m+q}, g_p)$. The comparison between the Weyl and Wick quantizations is symmetric to (29) (see [6]):

$$\forall b \in \oplus^{\text{alg}}_{m,q}\mathscr{P}_{m,q}(p\mathscr{Z}), \quad b^{Weyl}_{p\mathscr{Z}} = \left(b \underset{p\mathscr{Z}}{*} \frac{e^{-\frac{|z|^2_{p\mathscr{Z}}}{\varepsilon/2}}}{(\pi\varepsilon/2)^{\dim p\mathscr{Z}}}\right)^{Wick}.$$

For polynomials the deconvolution is possible and we get for any $m, q \in \mathbb{N}$ and any $b \in \mathscr{P}_{m,q}(p\mathscr{Z})$

$$\varepsilon^{-1}(b^{Wick}_{p\mathscr{Z}} - b^{Weyl}_{p\mathscr{Z}}) = c_{p\mathscr{Z}}(\varepsilon)^{Weyl}$$

where the symbol $c(\varepsilon)$ equals

$$c(\varepsilon) = \varepsilon^{-1}\left[\left(b \underset{p\mathscr{Z}}{*} \frac{e^{\frac{|z|^2_{p\mathscr{Z}}}{\varepsilon/2}}}{(\pi\varepsilon/2)^{\dim p\mathscr{Z}}}\right) - b\right]$$

and is uniformly bounded in $S_{p\mathscr{Z}}(\langle z\rangle^{m+q-2}, g_p)$ w.r.t $\varepsilon \in (0, \bar{\varepsilon})$.

The space $\mathscr{P}_{m,q}(p\mathscr{Z})$ is identified with a subspace of $\mathscr{P}_{m,q}(\mathscr{Z})$ and even of any $\mathscr{P}^r_{m,q}(\mathscr{Z})$ for any $r \in [1, +\infty]$ with

$$\forall b \in \mathscr{P}_{m,q}(p\mathscr{Z}), \quad \forall z \in \mathscr{Z}, \quad b(z) = b(pz + p^\perp z) = b(pz)$$
$$\tilde{b} = p^{\otimes q} \circ \tilde{b} \circ p^{\otimes m} = \Gamma_s(p)\tilde{b}\Gamma_s(p).$$

After tensoring the finite dimensional comparison with $I_{\Gamma_s(p^\perp\mathscr{Z})}$, we have proved

Proposition 3.8. *For any $p \in \mathbb{P}$, any $m, q \in \mathbb{N}$, the class of symbols $\mathscr{P}_{m,q}(p\mathscr{Z})$ is contained in $\mathscr{P}^1_{m,q}(\mathscr{Z}) \cap S_{p\mathscr{Z}}(\langle z\rangle^{m+q}, g_p)$. Moreover the operator $\varepsilon^{-1}(b^{Wick} - b^{Weyl})$ can be written c^{Weyl}_ε with c_ε uniformly bounded in $S_{p\mathscr{Z}}(\langle z\rangle^{m+q-2}, g_p)$ w.r.t $\varepsilon \in (0, \bar{\varepsilon})$. (The metric g_p can be either $|dz|^2$ or $\frac{|dz|^2}{\langle z\rangle^2}$ on $p\mathscr{Z}$.)*

4. Coherent and product states

We distinguish the coherent states $E(z) = W(\frac{\sqrt{2}}{i\varepsilon}z)\Omega$ (resp. the projector $|E(z)\rangle\langle E(z)|$) from the product or Hermite state $z^{\otimes k}$ (resp. the projector $|z^{\otimes k}\rangle\langle z^{\otimes k}|$). Although they are intimately related, the asymptotics of coherent state $E(z)$ tested on Wick, Weyl or Anti-Wick observables encoded exactly the geometry of the phase-space $\mathscr{Z}$, while the asymptotics of the product states $z^{\otimes k}$, $k\varepsilon \to 0$ keeps track of the gauge invariance

$$\forall\theta \in [0, 2\pi]\,, \quad |(e^{i\theta}z)^{\otimes k}\rangle\langle(e^{i\theta}z)^{\otimes k}| = |z^{\otimes k}\rangle\langle z^{\otimes k}|$$

with variations according to the quantization.

Proposition 4.1. *Fix $z, \xi \in \mathscr{Z}$ with $|z| = 1$.*

(i) *The convergence*

$$\lim_{\substack{\varepsilon\to 0\\ k\varepsilon\to 1}} \mathscr{V}[z^{\otimes k}, z^{\otimes k-m}](\xi) = \frac{1}{2\pi}\int_0^{2\pi} e^{2\pi i S(z^\theta,\xi)} e^{-im\theta}\, d\theta\,,$$

holds for any fixed $m \in \mathbb{N}$ by setting $z^\theta = e^{i\theta}z$.

(ii) *The coherent state $E(z) = W(\frac{\sqrt{2}}{i\varepsilon}z)\Omega$ satisfies*

$$\mathscr{V}\big[E(z), E(z)\big](\xi) = e^{2\pi i S(\xi,z)} e^{-\frac{\varepsilon|\xi|^2}{2}} \overset{\varepsilon\to 0}{\to} e^{2\pi i S(\xi,z)}\,.$$

Proof. i) Set $j = k - m$ and compute $\mathscr{V}[z^{\otimes k}, z^{\otimes j}](\xi)$ with $\xi = \frac{\xi'}{\sqrt{2\pi}}$ according to Proposition 3.5:

$$\begin{aligned}\mathscr{V}[z^{\otimes k}, z^{\otimes j}]\left(\frac{\xi'}{\sqrt{2\pi}}\right) &= (i)^m\sqrt{\frac{j!}{k!}}L_j^{(m)}\left(\frac{\varepsilon}{2}|\langle\xi', z\rangle|^2\right)\left(\frac{\varepsilon}{2}\right)^{m/2}\langle\xi', z\rangle^m e^{-\varepsilon|\xi'|^2/4}\\ &= (i)^m\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(s+m)!}1_{[0,j]}(s)\sqrt{\frac{j!}{(j-s)!k^s}}\sqrt{\frac{k!}{(j-s)!k^{m+s}}}\\ &\quad\left(\frac{\varepsilon k}{2}\right)^{\frac{2s+m}{2}}|\langle\xi', z\rangle|^{2s}\langle\xi', z\rangle^m e^{-\varepsilon|\xi'|^2/4}\,.\end{aligned}$$

The bounds $(\varepsilon k) \le C$ and $\sum_{s=0}^{\infty}\frac{C^s}{s!(s+m)!} < \infty$ imply

$$\lim_{\substack{\varepsilon\to 0\\ k\varepsilon\to 1}} \mathscr{V}[z^{\otimes k}, z^{\otimes j}]\left(\frac{\xi'}{\sqrt{2\pi}}\right) = (i)^m\sum_{s=0}^{\infty}\frac{(-1)^s}{2^{\frac{2s+m}{2}}s!(s+m)!}|\langle\xi', z\rangle|^{2s}\langle\xi', z\rangle^m\,,$$

owing to Lebesgue's theorem. A simple series expansion $e^t = \sum_{k=0}^{\infty}\frac{t^k}{k!}$ for $t = i\sqrt{2}S(z^\theta, \xi')$ gives

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i\sqrt{2}S(z^\theta,\xi')}e^{-im\theta}\, d\theta = (i)^m\sum_{s=0}^{\infty}\frac{(-1)^s}{2^{\frac{2s+m}{2}}s!(s+m)!}|\langle\xi', z\rangle|^{2s}\langle\xi', z\rangle^m.$$

ii) is a straightforward consequence of (22). □

The next result specifies the similarity and the differences between the product states and the coherent states in the mean-field or semiclassical limit.

Theorem 4.2. *Let $z \in \mathscr{Z}$ and $m \in \mathbb{N}$ be fixed with $|z| = 1$ and set $z^\theta = e^{i\theta}z$ for $\theta \in [0, 2\pi]$. The next limits exist as $\varepsilon \to 0$, $k\varepsilon \to 1$.*

(i) *For* $b \in \mathscr{S}_{cyl}(\mathscr{Z})$,

$$\lim_{\substack{\varepsilon\to 0\\ k\varepsilon\to 1}} \langle z^{\otimes k-m}, b^{Weyl}\, z^{\otimes k}\rangle = \lim_{\substack{\varepsilon\to 0\\ k\varepsilon\to 1}} \langle z^{\otimes k-m}, b^{A-Wick}\, z^{\otimes k}\rangle = \frac{1}{2\pi}\int_0^{2\pi} b(z^\theta)e^{-im\theta}d\theta\,.$$

Meanwhile the coherent state $E(z)$ *satisfies*

$$\lim_{\varepsilon\to 0}\big\langle E(z)\,,\, b^{Weyl}E(z)\big\rangle = \lim_{\varepsilon\to 0}\big\langle E(z)\,,\, b^{A-Wick}E(z)\big\rangle = b(z)\,.$$

(ii) *For* $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, *with* $p, q \in \mathbb{N}$ *fixed,*

$$\lim_{\substack{\varepsilon\to 0\\ k\varepsilon\to 1}} \langle z^{\otimes k-m}, b^{Wick}\, z^{\otimes k}\rangle = \delta_{p-q,m}\;\, b(z) = \frac{1}{2\pi}\int_0^{2\pi} b(z^\theta)e^{-im\theta}d\theta\,.$$

Meanwhile the coherent state $E(z)$ *satisfies*

$$\forall\varepsilon>0\,,\quad \big\langle E(z)\,,\, b^{Wick}E(z)\big\rangle = b(z)\,.$$

Proof. Set $j = k - m$, with $m \in \mathbb{N}$ fixed.

For (i), fix $b \in \mathscr{S}_{cyl}(\mathscr{Z})$. The definition of b^{Weyl} in (23), says

$$\begin{aligned}\langle z^{\otimes j}, b^{Weyl}\, z^{\otimes k}\rangle &= \int_{p\mathscr{Z}} \mathscr{F}[b](\xi)\,\big\langle z^{\otimes j}, W(\sqrt{2}\pi\xi)\, z^{\otimes k}\big\rangle\; L_p(d\xi)\\ &= \int_{p\mathscr{Z}} \mathscr{F}[b](\xi)\;\mathscr{V}[z^{\otimes k}, z^{\otimes j}](\xi)\; L_p(d\xi)\,.\end{aligned}$$

Since $\mathscr{F}[b] \in \mathscr{S}(p\mathscr{Z})$ and $\mathscr{V}[z^{\otimes k}, z^{\otimes j}](\xi)$ converges pointwise according to Proposition 4.1, Lebesgue's theorem yields

$$\begin{aligned}\lim_{\substack{\varepsilon\to 0\\ k\varepsilon\to 1}} \langle z^{\otimes j}, b^{Weyl}\, z^{\otimes k}\rangle &= \int_{p\mathscr{Z}} \mathscr{F}[b](\xi)\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i2\pi S(z^\theta,\xi)}e^{-im\theta}\,d\theta\right)\; L_p(d\xi)\\ &= \frac{1}{2\pi}\int_0^{2\pi} b(z^\theta)e^{-im\theta}\,d\theta\,.\end{aligned}$$

The limit with Anti-Wick observables is a consequence of the formula (30):

$$\langle z^{\otimes j}, b^{A-Wick}\, z^{\otimes k}\rangle = \int_{p\mathscr{Z}} \mathscr{F}[b](\xi)\;\;\big\langle z^{\otimes j}, W(\sqrt{2}\pi\xi) z^{\otimes k}\big\rangle\;\; e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2_{p\mathscr{Z}}}\; L_p(d\xi)\,.$$

The statement about the coherent state $E(z)$ is even simpler by referring to Proposition 4.1 (ii).

Let us prove (ii). The statement (ii) of Proposition 2.4 gives

$$\begin{aligned}\langle z^{\otimes j}, b^{Wick}\, z^{\otimes k}\rangle &= \delta^+_{k-p,j-q}\sqrt{\frac{k!j!}{(k-p)!(j-q)!}}\;\varepsilon^{\frac{p+q}{2}}\;\;\langle z^{\otimes q}, bz^{\otimes p}\rangle\\ &= \delta_{m,p-q}\sqrt{\frac{k!}{(k-p)!k^p}}\;\sqrt{\frac{j!}{(j-q)!k^q}}(\varepsilon k)^{p+q}\;\;\langle z^{\otimes q}, bz^{\otimes p}\rangle\,.\end{aligned}$$

We conclude again with $\sqrt{\frac{k!}{(k-p)!k^p}}\;\sqrt{\frac{j!}{(j-q)!k^q}} \to 1$ as $k\to\infty$. $\square$

5. An example of a dynamical mean-field limit

In order to illustrate the general presentation, the standard example of the mean field derivation of the Hartree equation from the non relativistic Hamiltonian of bosons with a quartic interaction is considered. Two standard methods are considered: The coherent state method (see [24, 32] or [12] for a rapid presentation) also known as Hepp method and the propagation of chaos approach with a truncated Dyson expansion according to [16, 17, 19, 20, 45].

Consider $\mathscr{Z} = L^2_{\mathbb{C}}(\mathbb{R}^d, dx)$ and take $V \in L^\infty_{\mathbb{R}}(\mathbb{R}^d, dx)$ such that $V(-x) = V(x)$. The polynomial $Q(z) = \langle z^{\otimes 2}, \tilde{Q} z^{\otimes 2}\rangle$ is associated with the operator $\tilde{Q} \in \mathscr{L}(\bigotimes^2 \mathscr{Z})$ defined by

$$\begin{aligned} \tilde{Q} : \otimes^2 \mathscr{Z} &\to \otimes^2 \mathscr{Z} , \\ u(x_1)w(x_2) &\mapsto \frac{1}{2} V(x_1 - x_2)\, u(x_1)w(x_2) . \end{aligned}$$

The Hamiltonian is defined as

$$H_\varepsilon = \mathrm{d}\Gamma(-\Delta) + Q^{Wick} ,$$

where $-\Delta$ is the Laplacian of $\mathbb{R}^d$, while H^0_ε denotes the free Hamiltonian $\mathrm{d}\Gamma(-\Delta)$. It is well known that H_ε is a self-adjoint operator on $\mathscr{H}$ (see [24]) and the quantum time-evolution group is denoted by $U_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}$ while $U^0_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_0} = \Gamma(e^{it\Delta})$ stands for the free dynamics. Although the Wick quantization of non continuous polynomials has not been introduced here, this Hamiltonian appears as the Wick quantization of the energy functional

$$h(z) = \int_{\mathbb{R}^d} |\nabla z|^2 \; dx + Q(z) . \tag{31}$$

It is also well known that the mean field limit is the nonlinear dynamics issued from the *Hartree equation*

$$i\partial_t z_t = -\Delta z_t + V * |z_t|^2 z_t = \partial_{\bar{z}} h(z_t) \tag{32}$$

with initial condition $z_0 = z \in \mathscr{Z}$.

An important property of the dynamical groups $U_\varepsilon(t)$ and $U^0_\varepsilon(t)$ is that they preserve the number

$$U_\varepsilon(t)^* N U_\varepsilon(t) = N , \quad [H_\varepsilon, N] = [H^0_\varepsilon, N] = [Q^{Wick}, N] = 0 .$$

Remark 5.1. All the results of this section can be easily adapted with a self-adjoint operator A on $\mathscr{Z}$ and a polynomial $Q(z) \in \oplus^{\mathrm{alg}}_{n\in\mathbb{N}} \mathscr{P}_{n,n}(\mathscr{Z})$. Nevertheless it is better to stick to this concrete presentation which fits better with a widely studied problem.

5.1. Propagation of squeezed coherent states (Hepp method)

In finite dimension it is nothing but checking the propagation of gaussian wave packets. Although it works only for some specific states it is a direct and very flexible method. Moreover it agrees very well with the phase-space geometric intuition. Extensions with more singular potentials or about the long time behaviour have been carried out in [24, 32].

Proposition 5.2. *For any $z_0 \in \mathscr{Z}$, the estimate*

$$\left| e^{-i\frac{t}{\varepsilon}H_\varepsilon} E(z_0) - e^{i\frac{\omega(t)}{\varepsilon}} W\left(\frac{\sqrt{2}}{i\varepsilon} z_t\right) U_2(t,0)\Omega \right|_{\mathscr{H}} \leq C\, e^{C|V|_{L^\infty}\langle z_0\rangle^2(|t|+1)}\, \varepsilon^{1/2}$$

holds with

$$i\partial_t z_t = -\Delta z_t + (V * |z_t|^2) z_t\,, \quad z_{t=0} = z_0 \tag{33}$$

$$\omega(t) = \int_0^t Q(z_s)\, ds \tag{34}$$

$$i\varepsilon\partial_t U_2(t,0) = \left[d\Gamma(-\Delta) + Q_2(t)^{Wick}\right] U_2(t,0)\,, \quad U_2(0,0) = I\,, \tag{35}$$

$$\begin{aligned} Q_2(t,z) = \frac{1}{2}\Big[&\left\langle \partial_z^2 Q(z_t)\,, z^{\otimes 2}\right\rangle \\ &+ \left\langle z^{\otimes 2}\,, \partial_{\bar z}^2 Q(z_t)\right\rangle + 2\left\langle z\,, \partial_{\bar z}\partial_z Q(z_t) z\right\rangle\Big]\,, \end{aligned} \tag{36}$$

$$\left\langle \partial_z^2 Q(z_t)\,, z^{\otimes 2}\right\rangle = 2\left\langle \tilde Q\, z_t^{\otimes 2}\,, z^{\otimes 2}\right\rangle \in \mathscr{P}_{2,0}(\mathscr{Z})\,,$$

$$\left\langle z\,, \partial_{\bar z}\partial_z Q(z_t) z\right\rangle = 4\left\langle z \vee z_t\,, \tilde Q\, z \vee z_t\right\rangle \in \mathscr{P}_{1,1}(\mathscr{Z})\,.$$

Proof. This proposition says that the evolution of a coherent state is well described after applying a time dependent (real) affine Bogoliubov transformation like the ones considered in Proposition 2.12.

It is sufficient that

$$\begin{aligned} &e^{i\frac{t}{\varepsilon}H_\varepsilon} e^{i\frac{\omega(t)}{\varepsilon}} W\left(\frac{\sqrt{2}}{i\varepsilon} z_t\right) U_2(t,0)\Omega \\ &\qquad = e^{i\frac{t}{\varepsilon}H_\varepsilon}\Gamma(e^{it\Delta}) e^{i\frac{\omega(t)}{\varepsilon}} W\left(\frac{\sqrt{2}}{i\varepsilon} e^{-it\Delta} z_t\right)\Gamma(e^{-it\Delta}) U_2(t,0)\Omega \end{aligned}$$

remains close enough to $E(z_0)$. The quantities $\hat U_\varepsilon(0,t) = e^{i\frac{t}{\varepsilon}H_\varepsilon}\Gamma(e^{it\Delta})$, $\hat U_2(t,0) = \Gamma(e^{-it\Delta})U_2(t,0)$ and $\hat z_t = e^{-it\Delta} z_t$ solve the differential equations

$$i\varepsilon\partial_t \hat U_\varepsilon(0,t) = -\hat U_\varepsilon(0,t)\Gamma(e^{-it\Delta}) Q^{Wick}\Gamma(e^{it\Delta}) = -\hat U_\varepsilon(t,0)\hat Q(t)^{Wick}\,, \tag{37}$$

$$i\varepsilon\partial_t \hat U_2(t,0) = \Gamma(e^{-it\Delta}) Q_2(t)^{Wick}\Gamma(e^{it\Delta})\hat U_2(t,0) = \hat Q_2(t)^{Wick}\hat U_2(t,0)\,, \tag{38}$$

$$i\partial_t \hat z_t = e^{-it\Delta}(V * |e^{it\Delta}\hat z_t|^2) e^{it\Delta}\hat z_t = \partial_{\bar z}\hat Q(t,\hat z_t)\,, \quad \hat z_0 = z_0\,, \tag{39}$$

after setting

$$\hat Q(t,z) = Q(e^{it\Delta} z) \quad \text{and} \quad \hat Q_2(t,z) = Q_2(t, e^{it\Delta} z)\,. \tag{40}$$

The main properties of $\hat{U}_2(t,0)$ are derived in [24, Proposition 4.1] and in particular we know that $\hat{U}_2(t,0)\Omega$ belongs to the domain of the closure of any b^{Wick} with $b \in \oplus^{\text{alg}}_{p,q\in\mathbb{N}}\mathscr{P}_{p,q}(\mathscr{Z})$.

The differentiation of the Weyl relation (3) on $\mathscr{H}_{fin}$ says

$$\begin{aligned}
i\varepsilon\partial_t W\left(\frac{\sqrt{2}}{i\varepsilon}\hat{z}_t\right) &= \left[-\mathrm{Re}\langle \hat{z}_t\,,\, i\partial_t\hat{z}_t\rangle + \sqrt{2}\Phi(i\partial_t\hat{z}_t)\right]W\left(\frac{\sqrt{2}}{i\varepsilon}\hat{z}_t\right)\\
&= \left[-\mathrm{Re}\left\langle \hat{z}_t\,,\,\partial_{\bar{z}}\hat{Q}(t,\hat{z}_t)\right\rangle + a^*\left(\partial_{\bar{z}}\hat{Q}_t(\hat{z}_t)\right) + a\left(\partial_{\bar{z}}\hat{Q}_t(\hat{z}_t)\right)\right]W\left(\frac{\sqrt{2}}{i\varepsilon}\hat{z}_t\right)\\
&= \left[-\mathrm{Re}\left\langle \hat{z}_t\,,\,\partial_{\bar{z}}\hat{Q}(t,\hat{z}_t)\right\rangle + 2\mathrm{Re}\left\langle z\,,\,\partial_{\bar{z}}\hat{Q}_t(\hat{z}_t)\right\rangle^{Wick}\right]W\left(\frac{\sqrt{2}}{i\varepsilon}\hat{z}_t\right).
\end{aligned}$$

The translation property (iii) of Proposition 2.10 then leads to

$$\begin{aligned}
&e^{i\frac{t}{\varepsilon}H_\varepsilon}e^{i\frac{\omega(t)}{\varepsilon}}W\left(\frac{\sqrt{2}}{i\varepsilon}z_t\right)U_2(t,0)\Omega - E(z_0)\\
&\qquad = \frac{1}{i\varepsilon}\int_0^t \hat{U}_\varepsilon(0,s)e^{i\frac{\omega(s)}{\varepsilon}}W\left(\frac{\sqrt{2}}{i\varepsilon}\hat{z}_s\right)\mathscr{A}(s)^{Wick}\hat{U}_2(s,0)\Omega\, ds
\end{aligned}$$

after testing both sides on $\mathscr{H}_{fin}$ and setting

$$\begin{aligned}
\mathscr{A}(s,z) &= -\hat{Q}(s,z+\hat{z}_s) - \omega'(s) + \mathrm{Re}\left\langle \hat{z}_s\,,\,\partial_{\bar{z}}\hat{Q}(s,\hat{z}_s)\right\rangle + 2\mathrm{Re}\left\langle z\,,\,\partial_{\bar{z}}\hat{Q}_s(\hat{z}_s)\right\rangle\\
&\quad + \hat{Q}_2(s,z)\\
&= -\hat{Q}(s,z+\hat{z}_s) + \hat{Q}(\hat{z}_s) + \left\langle z\,,\,\partial_{\bar{z}}\hat{Q}_s(\hat{z}_s)\right\rangle + \left\langle \partial_z\hat{Q}_s(\hat{z}_s)\,,\,z\right\rangle + \hat{Q}_2(s,z)\,.
\end{aligned}$$

The last equality is given by our choice of $\omega(t)$ in (34). It suffices to find a uniform estimate w.r.t $s\in[0,t]$ of the squared norm

$$\begin{aligned}
&\left|\varepsilon^{-1}\mathscr{A}(s)^{Wick}\hat{U}_2(s,0)\Omega\right|^2_{\mathscr{H}}\\
&\qquad = \varepsilon^{-2}\left\langle \Omega\,,\,\hat{U}_2(0,s)\mathscr{A}(s)^{Wick,*}\mathscr{A}(s)^{Wick}\hat{U}_2(s,0)\Omega\right\rangle. \quad (41)
\end{aligned}$$

The important point is that the symbol $\mathscr{A}(s)$ vanishes at the second order at $z=0$. More precisely it can be written

$$\begin{aligned}
\mathscr{A}(s) = \mathscr{A}_{1,2}(s) + \mathscr{A}_{2,1}(s) + \mathscr{A}_{2,2}(s) \quad \text{with}\\
\mathscr{A}_{p,q}(s) \in \mathscr{P}_{p,q}(\mathscr{Z}) \quad \text{and}\\
\left|\tilde{\mathscr{A}}_{p,q}(s)\right|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})} \le C_{p,q}\,|V|_{L^\infty}\,|z_0|^{4-p-q}\,.
\end{aligned}$$

Owing to Proposition 2.7 and Lemma 2.6 the operator $\mathscr{A}(s)^{Wick,*}\mathscr{A}^{Wick}(s)$ takes the form

$$\mathscr{A}(s)^{Wick,*}\mathscr{A}(s)^{Wick} = \sum_{k=0}^{2} \varepsilon^k \sum_{6-2k\le p+q\le 8} \mathscr{B}_{k,p,q}(s)^{Wick} \quad \text{with}$$

$$\left|\tilde{\mathscr{B}}_{k,p,q}(s)\right|_{\mathscr{L}(\bigvee^p \mathscr{Z},\bigvee^q \mathscr{Z})} \le C_{k,p,q}\,|V|_{L^\infty}^2 \,\langle z_0\rangle^2 .$$

The estimate of every term

$$\varepsilon^{k-2}\left\langle \Omega\,, \hat{U}_2(0,s)\mathscr{B}_{k,p,q}(s)^{Wick}\hat{U}_2(s,0)\Omega\right\rangle, \quad p+q\ge 6-2k$$

is given by the Lemma 5.3 below and yields the result. □

Lemma 5.3. *Consider the time dependent Wick operator* $\hat{Q}_2$ *defined by* (36) (40) *and parametrized by* $z_0 \in \mathscr{Z}$. *Consider the associated unitary operator* $\hat{U}_2(s,0)$ *defined by* (38). *For any* $p,q\in\mathbb{N}$, *there exists a constant* $C_{p,q}$ *such that the estimate*

$$\left|\left\langle \Omega\,, \hat{U}_2(0,s)b^{Wick}\hat{U}_2(s,0)\Omega\right\rangle\right| \le C_{p,q}\; e^{C_{p,q}|V|_{L^\infty}\langle z_0\rangle^2(|s|+1)}\left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p \mathscr{Z},\bigvee^q \mathscr{Z})}\varepsilon^{\frac{p+q}{2}}$$

holds for any $b\in\mathscr{P}_{p,q}(\mathscr{Z})$ *and any* $s\in\mathbb{R}$.

Proof. By introducing an anti-unitary operator $Jz=\overline{z}$, the $\mathbb{R}$-linear operator $\partial_{\overline{z}}\hat{Q}_2(t)$ can be written

$$\partial_{\overline{z}}\hat{Q}_2(t)z = R(t)z + R_2(t)\overline{z}\,.$$

The definitions (36) (40) ensure that $R(t)$ is a bounded operator strongly continuous with respect to $t\in\mathbb{R}$ and that $R_2(t)$ is a Hilbert–Schmidt operator which depends continuously on $t\in\mathbb{R}$ in the Hilbert–Schmidt norm. Moreover the following uniform estimates hold

$$|R(t)|_{\mathscr{L}(\mathscr{Z})} \le 2\,|V|_{L^\infty}\,|z_0|^2\,, \quad |R_2(t)|_{\mathscr{L}^2(\mathscr{Z})} \le 2\,|V|_{L^\infty}\,|z_0|^2\,.$$

Hence the equation

$$i\partial_t\Phi_2 = \partial_{\overline{z}}\hat{Q}_2(t)\Phi_2 = R(t)\Phi_2 + R_2(t)J\Phi_2$$

defines a dynamical system of bounded $\mathbb{R}$-linear operators with the estimate

$$|\Phi_2(t_2,t_1)|_{\mathscr{L}_{\mathbb{R}}(\mathscr{Z})} \le e^{4|t_2-t_1||V|_{L^\infty}|z_0|^2}\,.$$

More precisely the Duhamel formula

$$\Phi_2(t_2,t_1) = Te^{-i\int_{t_1}^{t_2}R(s)\;ds} - i\int_{t_1}^{t_2} Te^{-i\int_{t}^{t_2}R(s)\;ds}R_2(t)J\Phi_2(t,t_1)\;dt$$

implies that the $\mathbb{R}$-linear operator $\Phi_2(t_2,t_1)$ can be written

$$\Phi_2(t_2,t_1) = B(t_2,t_1) + B_2(t_2,t_1)J \quad \text{with}$$

$$|B(t_2,t_1)|_{\mathscr{L}(\mathscr{Z})} + |B_2(t_2,t_1)|_{\mathscr{L}^2(\mathscr{Z})} \le C\,|V|_{L^\infty}\,|z_0|^2\,(|t_2-t_1|+1)e^{C|t_2-t_1||V|_{L^\infty}|z_0|^2}\,.$$

According to Proposition 2.12, for any $c \in \oplus_{p+q=m}\mathscr{P}_{p,q}(\mathscr{Z})$ and any $t \in \mathbb{R}$, the polynomial $c(t,z) = c(\Phi_2(0,t)z)$ belongs to $\oplus_{p+q=m}\mathscr{P}_{p,q}(\mathscr{Z})$ with

$$\sum_{p+q=m} |\partial_{\bar{z}}^q \partial_z^p c(t,z)|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})} \leq C_m^1 e^{C_m^1 |V|_{L^\infty} \langle z_0\rangle^2 (|t|+1)} \sum_{p+q=m} |\partial_{\bar{z}}^q \partial_z^p c(z)|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})} .$$

Applying the characteristic method, that is differentiating $c(z) = c(t, \Phi_2(t,0)z)$, shows that $c(z,t)$ solves the equation

$$i\partial_t c(t,z) + \partial_z c(t,z).\partial_{\bar{z}} \hat{Q}_2(t,z) - \partial_z \hat{Q}_2(t,z)\partial_{\bar{z}} c(t,z) = 0 .$$

Thanks to the Wick calculus in Proposition 2.7 and the fact that $\hat{U}_2(t,0)\Omega \in \cap_{k\in\mathbb{N}}\mathscr{D}(N^k)$ (see [24, Proposition 4.1]), this leads to

$$\begin{aligned} &i\partial_t \hat{U}_2(0,t) c(t)^{Wick} \hat{U}_2(t,0)\Omega \\ &\quad = \hat{U}_2(0,t)\left(\varepsilon^{-1}\left[c^{Wick}(t), \hat{Q}_2(t)^{Wick}\right] + i\partial_t c(t)^{Wick}\right)\hat{U}_2(t,0)\Omega \\ &\quad = \hat{U}_2(0,t)\frac{\varepsilon}{2}\left(\left\{c(t), \hat{Q}_2(t)\right\}^{(2)}\right)^{Wick} \hat{U}_2(t,0)\Omega . \end{aligned}$$

Take $b \in \oplus_{p+q=m_0}\mathscr{P}_{p,q}(\mathscr{Z})$ and apply this result with c defined by $c(s,z) = b(z)$, which means

$$\begin{aligned} c\big(\Phi_2(0,s)z\big) = c(s,z) = b(z) \quad \text{or} \\ c(z) = b\big(\Phi_2(s,0)z\big) \in \oplus_{p+q=m_0}\mathscr{P}_{p,q}(\mathscr{Z}) \quad \text{with} \end{aligned}$$

$$\sum_{p+q=m_0} |\partial_{\bar{z}}^q \partial_z^p c(z)|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})} \leq C_{m_0}^1 e^{C_{m_0}^1 |V|_{L^\infty} \langle z_0\rangle^2 (|s|+1)} \times \sum_{p+q=m_0} |\partial_{\bar{z}}^q \partial_z^p b(z)|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})} .$$

This leads to

$$\begin{aligned} &\left\langle \Omega \,,\, \hat{U}_2(0,s) b^{Wick} \hat{U}_2(s,0)\Omega \right\rangle \\ &\quad = \left\langle \Omega \,, c^{Wick}\Omega \right\rangle + \int_0^s \left\langle \Omega \,,\, \partial_t \left(\hat{U}_2(0,t) c(t)^{Wick} \hat{U}_2(t,0)\right)\Omega \right\rangle dt \\ &\quad = -\frac{i\varepsilon}{2}\int_0^s \left\langle \Omega \,,\, \hat{U}_2(0,t)\left(\left\{c(t), \hat{Q}_2(t)\right\}^{(2)}\right)^{Wick} \hat{U}_2(t,0)\Omega \right\rangle dt . \end{aligned}$$

By noticing that the symbol $\{c(t), \hat{Q}_2(t)\}$ vanishes when $m_0 < 2$ or belongs to $\oplus_{p+q=m_0-2}\mathscr{P}_{p,q}(\mathscr{Z})$ with

$$\sum_{p+q=m_0-2} \left|\partial_{\bar z}^q \partial_z^p \{c(t), \hat{Q}_2(t)\}^{(2)}\right|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})}$$

$$\leq C\,|V|_{L^\infty}\,|z_0|^2 \sum_{p+q=m_0} |\partial_{\bar z}^q \partial_z^p c(t)|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})}$$

$$\leq C\,|V|_{L^\infty}\,|z_0|^2\, C^1_{m_0} e^{C^1_{m_0}|V|_{L^\infty}\langle z_0\rangle^2(2|s|+1)} \sum_{p+q=m_0} |\partial_{\bar z}^q \partial_z^p b|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})}$$

the result is proved by induction on m_0 and by using $x^n \leq n! e^x$ for $x > 0$. □

5.2. Truncated Dyson expansion

We focus now on the propagation of chaos point of view which has been considered by several authors in [4,16,17,20]. In the bosonic setting Hermite states tested on some Wick observable is exactly the BBGKY hierarchy. For example the reduced one particle density matrix can be defined as $\mathrm{Tr}[\varrho_1 A] = \mathrm{Tr}[\varrho d\Gamma(A)] = \mathrm{Tr}[\varrho \mathscr{A}^{Wick}]$ with $\mathscr{A}(z) = \langle z\,,\, Az\rangle$. While reproducing the Dyson expansion analysis of [20], we check here that a full asymptotic expansion can be written, when Wick observables are tested after the suitable number truncation.

The strategy of the proof in [20] relies on an analysis of the Schwinger–Dyson expansion of a time evolved observable $U_\varepsilon(t)^* \mathscr{O}\, U_\varepsilon(t)$ given by

$$U_\varepsilon(t)^* \mathscr{O}\, U_\varepsilon(t) = \mathscr{O}_t + \sum_{n=1}^{\infty} \left(\frac{i}{\varepsilon}\right)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \left[Q^{Wick}_{t_n}, \cdots [Q^{Wick}_{t_1}, \mathscr{O}_t] \cdots\right] \quad (42)$$

where $\mathscr{O}_t = U^0_\varepsilon(t)^* \mathscr{O}\ U^0_\varepsilon(t)$, $Q^{Wick}_s = U^0_\varepsilon(s)^* Q^{Wick}\ U^0_\varepsilon(s)$. The commutation relation in Proposition 2.3 (iii) yields

$$Q^{Wick}_s = \left(\left\langle (e^{is\Delta}z)^{\otimes 2}, Q(e^{is\Delta}z)^{\otimes 2}\right\rangle\right)^{Wick},$$

or shortly $Q_s(z) = Q(e^{is\Delta}z)$ and we shall set more generally for $b \in \mathscr{P}_{p,q}(\mathscr{Z})$ and $s \in \mathbb{R}$

$$b_s \in \mathscr{P}_{p,q}(\mathscr{Z}) : \qquad \forall z \in \mathscr{Z}\,, \quad b_s(z) = b(e^{is\Delta}z)\,.$$

Although the convergence of the series can be proved as an operator acting on $\bigvee^k \mathscr{Z}$, with $k \in \mathbb{N}$ fixed, the ε-asymptotic analysis is done with its truncated version

$$\begin{aligned} U_\varepsilon(t)^* \mathscr{O} U_\varepsilon(t) = \mathscr{O}_t &+ \sum_{n=1}^{\ell-1} \left(\frac{i}{\varepsilon}\right)^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \left[Q^{Wick}_{t_n}, \cdots [Q^{Wick}_{t_1}, \mathscr{O}_t] \cdots\right] \\ &+ \left(\frac{i}{\varepsilon}\right)^\ell \int_0^t dt_1 \cdots \int_0^{t_{\ell-1}} dt_\ell\ U_\varepsilon(t_\ell)^* U^0_\varepsilon(t_\ell) \left[Q^{Wick}_{t_\ell}, \cdots\right. \\ &\left. \cdots [Q^{Wick}_{t_1}, \mathscr{O}_t] \cdots\right] U^0_\varepsilon(t_\ell)^* U_\varepsilon(t_\ell)\,. \end{aligned} \quad (43)$$

The Poisson brackets analogue of the multicommutators will be necessary.

Definition 5.4. For $n, r \in \mathbb{N}$, $r \leq n$ and any fixed $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, the polynomial $C_r^{(n)}(t_1, \ldots, t_n)$ is defined by

$$\begin{aligned} & C_r^{(n)}(t_n, \ldots, t_1, t) \\ & = \frac{1}{2^r} \sum_{\sharp\{i:\, \varepsilon_i = 2\} = r} \underbrace{\left\{Q_{t_n}, \ldots, \{Q_{t_1}, b_t\}^{(\varepsilon_1)} \cdots\right\}^{(\varepsilon_n)}}_{\varepsilon_i \in \{1,2\}} \in \mathscr{P}_{p-r+n, q-r+n}(\mathscr{Z}), \quad (44) \end{aligned}$$

and $C_r^{(n)}(t_1, \ldots, t_n, t, z)$ denotes its values at $z \in \mathscr{Z}$ while $\widetilde{C_r^{(n)}}(t_1, \ldots, t_n, t)$ or simply $\widetilde{C_r^{(n)}}$ denotes the associated operator according to Definition 2.1.

We shall prove.

Theorem 5.5. *Fix $p, q \in \mathbb{N}$ and assume $b \in \mathscr{P}_{p,q}(\mathscr{Z})$. Then the asymptotic expansion*

$$\begin{aligned} U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t) = & \sum_{r=0}^{\ell-1} \varepsilon^r \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \, \left[C_r^{(n)}(t_n, \ldots, t_1, t)\right]^{Wick} \\ & + \varepsilon^\ell R_\ell(\varepsilon, t) \end{aligned}$$

holds for any $\ell \in \mathbb{N}$ and any $\delta > 0$ in $\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^{k-p+q} \mathscr{Z})$ with the uniform estimate

$$\begin{aligned} & |R_\ell(\varepsilon, t)|_{\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^{k-p+q} \mathscr{Z})} \leq C_{\ell,\delta} \\ & \qquad\qquad \text{when} \quad k\varepsilon \leq 1 + \delta/2 \quad \text{and} \quad 4(1+2\delta)|t|\,|V|_{L^\infty} \leq 1\,. \end{aligned}$$

A particular case takes a more explicit form.

Theorem 5.6. *Take $b \in \mathscr{P}_{p,q}(\mathscr{Z})$. Let $z \in \mathscr{Z}$ be such that $|z| = 1$ and call z_t the solution to (32) with $z_0 = z$.*

(i) *Then the expansion*

$$\left\langle z^{\otimes k-m}, U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\, z^{\otimes k}\right\rangle = \delta_{p-q,m} \left[\sum_{r=0}^{\ell-1} \varepsilon^r \;\beta^{(r)}(t, z, k, \varepsilon) + O_t(\varepsilon^\ell)\right], \quad (45)$$

holds as $\varepsilon \to 0$, $k\varepsilon \to 1$ by setting

$$\begin{aligned} \beta^{(0)}(t, z, k, \varepsilon) &= b(z_t), \\ \beta^{(r)}(t, z, k, \varepsilon) &= \sum_{n=r}^{k-p+r} i^n \frac{\sqrt{k!(k-m)!\, \varepsilon^{p+q+2(n-r)}}}{(k-(p+n-r))!} \int_0^t dt_1 \cdots \\ & \quad \cdots \int_0^{t_{n-1}} dt_n \; C_r^{(n)}(t_n, \ldots, t_1, t; z)\,, \qquad (46) \end{aligned}$$

and as soon as $4|t|\,|V|_{L^\infty} < 1$.

(ii) *More generally the limit*

$$\lim_{\substack{\varepsilon\to 0,\\ k\varepsilon\to 1}} \left\langle z^{\otimes k-m}, U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\, z^{\otimes k}\right\rangle = \delta_{p-q,m}\, b(z_t)$$

holds for all times $t\in\mathbb{R}$.

Corollary 5.7. *In the specific case* $m=0$, $q=p$, *the expansion* (45) *takes the form*

$$\left\langle z^{\otimes k}, U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\, z^{\otimes k}\right\rangle = \sum_{s=0}^{\ell-1}\varepsilon^s \sum_{n=0}^{\infty} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \left[\sum_{j=0}^{s} \alpha_j^{s-j,n}(k\varepsilon) C_{s-j}^{(n)}(t_n,\dots,t_1,t;z)\right] + O(\varepsilon^\ell)\,,$$

where the coefficients $\alpha_j^{r,n}(\kappa)$ *are polynomials in* κ *given by*

$$\sum_{j=0}^{p+n-r-1} \alpha_j^{r,n}(\kappa)\varepsilon^j = \kappa(\kappa-\varepsilon)(\kappa-2\varepsilon)\cdots\big(\kappa-(p+n-r-1)\varepsilon\big)\,,$$

and the convention that $\alpha_j^{r,n}=0$ *when* $j\geq(p+n-r)$ *or* $r>n$.

Proof. We are considering the particular case $p=q$, $m=0$. Setting $\kappa=k\varepsilon=(k-m)\varepsilon$ gives:

$$\frac{k!\varepsilon^{p+(n-r)}}{(k-(p+n-r))!} = \kappa(\kappa-\varepsilon)(\kappa-2\varepsilon)\cdots\big(\kappa-(p+n-r-1)\varepsilon\big)\,.$$

Putting together the terms of order ε^s, s less than $\ell-1$ in Theorem 5.5 (ii), yields the result. □

Before proving Theorem 5.5 and Theorem 5.6, let us collect some technical preliminaries.

Lemma 5.8. *For* $b\in\mathscr{P}_{p,q}(\mathscr{Z})$ *the identity*

$$\frac{1}{\varepsilon^n}\left[Q_{t_n}^{Wick},\cdots,[Q_{t_1}^{Wick},b_t^{Wick}]\right] = \sum_{r=0}^{n}\varepsilon^r\left(C_r^{(n)}(t_n,\dots,t_1,t)\right)^{Wick}\,,$$

holds with the symbols $C_r^{(n)}(t_1,\dots,t_n,t)$ *defined according to* (44) *in Definition* 5.4.

Proof. Proposition 2.7 provides the induction formula

$$C_r^{(n)} = \{Q_{t_n}, C_r^{(n-1)}\} + \frac{1}{2}\{Q_{t_n}, C_{r-1}^{(n-1)}\}^{(2)}\,, \tag{47}$$

with $C_r^{(l)}=0$ if $l<r$ or $r<0$. In particular, we get

$$C_0^{(n)} = \big\{Q_{t_n},\dots,\{Q_{t_1},b_t\}\big\}\,.$$

A simple iteration of (47) yields the result. □

Lemma 5.9. *Let b belong to $\mathscr{P}_{p,q}(\mathscr{Z})$.*

(i) *The estimate*

$$\left|\widetilde{\Xi_1}\right|_{\mathscr{L}(\bigvee^{p+1}\mathscr{Z},\bigvee^{q+1}\mathscr{Z})} \leq (p+q)\,|V|_{L^\infty}\,|b|_{\mathscr{L}(\bigvee^{p}\mathscr{Z},\bigvee^{q}\mathscr{Z})}\,,$$

holds by setting $\widetilde{\Xi_1}=\frac{1}{(p+1)!}\frac{1}{(q+1)!}\partial_z^{p+1}\partial_{\bar z}^{q+1}\{Q_s,b_t\}^{(1)}\in\mathscr{L}(\bigvee^{p+1}\mathscr{Z},\bigvee^{q+1}\mathscr{Z})$.

(ii) *Similarly, the inequality*

$$\left|\widetilde{\Xi_2}\right|_{\mathscr{L}(\bigvee^{p}\mathscr{Z},\bigvee^{q}\mathscr{Z})} \leq \big[p(p-1)+q(q-1)\big]\,|V|_{L^\infty}\,|b|_{\mathscr{L}(\bigvee^{p}\mathscr{Z},\bigvee^{q}\mathscr{Z})}$$

holds with $\widetilde{\Xi_2}=\frac{1}{p!}\frac{1}{q!}\partial_z^{p}\partial_{\bar z}^{q}\{Q_s,b_t\}^{(2)}$.

(iii) *For any $n\in\mathbb{N}$ and $r\in\{0,1,\dots,n\}$, the operator $\widetilde{C_r^{(n)}}$ associated with the symbol $C_r^{(n)}(t_n,\dots,t_1,t)\in\mathscr{P}_{p+n-r,q+n-r}(\mathscr{Z})$ according to Definition* 5.4 *satisfies*

$$\begin{aligned}&\left|\widetilde{C_r^{(n)}}\right|_{\mathscr{L}(\bigvee^{p+n-r}\mathscr{Z},\bigvee^{q+n-r}\mathscr{Z})}\\ &\qquad\leq 2^{n-r}C_n^r\,(p+n-r)^{2r}\,\frac{(p+n-r-1)!}{(p-1)!}\,|V|^n_{L^\infty}\,|b|_{\mathscr{L}(\bigvee^{p}\mathscr{Z},\bigvee^{q}\mathscr{Z})}\,,\end{aligned}$$

when $p\geq q$ with a similar expression when $q\geq p$ (replace $(p+n-r,p-1)$ with $(q+n-r,q-1)$).

Proof. The statements (i) and (ii) are particular cases of Lemma 2.6. The estimate in (iii) is a consequence of (i)(ii) and the definition (44). □

Proof of Theorem 5.5. Set $j=k-p+q$. Since $U_\varepsilon(t)$ and $U_\varepsilon^0(t)$ preserve the number like Q_t^{Wick} the equality

$$\begin{aligned}&U_\varepsilon(t)^*b^{Wick}U_\varepsilon(t)\\ &\quad=\sum_{n=0}^{\ell-1}\left(\frac{i}{\varepsilon}\right)^n\int_0^t dt_1\cdots\int_0^{t_{n-1}}dt_n\big[Q_{t_n}^{Wick},\cdots[Q_{t_1}^{Wick},b_t^{Wick}]\cdots\big]\\ &\quad\quad+\left(\frac{i}{\varepsilon}\right)^\ell\int_0^t dt_1\cdots\int_0^{t_{\ell-1}}dt_\ell\;U_\varepsilon(t_\ell)^*U_\varepsilon^0(t_\ell)\big[Q_{t_\ell}^{Wick},\cdots\\ &\quad\quad\cdots[Q_{t_1}^{Wick},b_t^{Wick}]\cdots\big]U_\varepsilon^0(t_\ell)^*U_\varepsilon(t_\ell)\,,\end{aligned}$$

derived from (43) holds in $\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^j \mathscr{Z})$. Then Lemma 5.8 implies

$$\begin{aligned}
&U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\\
&= \sum_{n=0}^{\ell-1} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{r=0}^{n} \varepsilon^r \left[C_r^{(n)}(t_n,\ldots,t_1,t)\right]^{Wick} &(48)\\
&\quad + i^\ell \int_0^t dt_1 \cdots \int_0^{t_{\ell-1}} dt_\ell\, U_\varepsilon(t_\ell)^* U_\varepsilon^0(t_\ell) \varepsilon^\ell \left[C_\ell^{(\ell)}(t_\ell,\ldots,t_1,t)\right]^{Wick} U_\varepsilon^0(t_\ell)^* U_\varepsilon(t_\ell) &(49)\\
&\quad + i^\ell \int_0^t dt_1 \cdots \int_0^{t_{\ell-1}} dt_\ell\, U_\varepsilon(t_\ell)^* U_\varepsilon^0(t_\ell) \sum_{r=0}^{\ell-1} \varepsilon^r \left[C_r^{(\ell)}(t_\ell,\ldots,t_1,t)\right]^{Wick} U_\varepsilon^0(t_\ell)^* U_\varepsilon(t_\ell)\,. &(50)
\end{aligned}$$

Keep untouched the part (48)–(49) and iterate the Dyson series on the third term (50). While doing so, use the formula

$$\begin{aligned}
&\left[\frac{Q_{t_{n+1}}^{Wick}}{\varepsilon}, \sum_{r=0}^{\ell-1} \varepsilon^r \left[C_r^{(n)}(t_n,\ldots,t_1,t)\right]^{Wick}\right]\\
&\quad = \sum_{r=0}^{\ell-1} \varepsilon^r \left[C_r^{(n+1)}(t_{n+1},\ldots,t_1,t)\right]^{Wick}\\
&\qquad + \frac{\varepsilon^\ell}{2} \left[\left\{Q_{t_{n+1}}, C_\ell^{(n)}(t_{n+1},\ldots,t_1,t)\right\}^{(2)}\right]^{Wick}, &(51)
\end{aligned}$$

inductively for $n = \ell, \ell+1, \ldots, M-1$. After $M-\ell$ steps, collecting the factors of ε^ℓ yields

$$\begin{aligned}
&U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\\
&= \sum_{n=0}^{M-1} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sum_{r=0}^{\min(\ell-1,n)} \varepsilon^r \left[C_r^{(n)}(t_n,\ldots,t_1,t)\right]^{Wick} &(52)\\
&\quad + \sum_{n=\ell}^{M} i^n \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, U_\varepsilon(t_n)^* U_\varepsilon^0(t_n) \frac{\varepsilon^\ell}{2} &(53)\\
&\quad \times \left[\left\{Q_{t_n}, C_{\ell-1}^{(n-1)}(t_{n-1},\ldots,t_1,t)\right\}^{(2)}\right]^{Wick} U_\varepsilon^0(t_n)^* U_\varepsilon(t_n)\\
&\quad + i^M \int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M\, U_\varepsilon(t_M)^* U_\varepsilon^0(t_M) &(54)\\
&\quad \times \sum_{r=0}^{\ell-1} \varepsilon^r \left[C_r^{(M)}(t_M,\ldots,t_1,t)\right]^{Wick} U_\varepsilon^0(t_M)^* U_\varepsilon(t_M)\,.
\end{aligned}$$

Assume that for $\delta > 0$ there exists a constant C_δ such that

$$\sum_{n=\ell}^{\infty}(1+\delta)^n \sum_{r=0}^{\ell} \int_0^t dt_1 \cdots \cdots \int_0^{t_{n-1}} dt_n \left|\widetilde{C_r^{(n)}}(t_n,\ldots,t_1,t)\right|_{\mathscr{L}(\bigvee^{p+n-r}\mathscr{Z},\bigvee^{q+n-r}\mathscr{Z})} < C_\delta\,. \quad (55)$$

According to Lemma 2.5, the first term (52) of (52) (53) (54) provides in $U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\big|_{\bigvee^k \mathscr{Z}}$ the partial sum of a convergent series in $\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^{k-p+q}\mathscr{Z})$ when $k\varepsilon \le 1+\frac{\delta}{2}$. With the same argument the remainder term (54) vanishes as $M \to \infty$ and $k\varepsilon \le 1+\frac{\delta}{2}$. By referring to Lemma 5.9 (ii) and again to Lemma 2.5 the factor of ε^ℓ in (53) is associated with a series which converges in $\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^{k-p+q}\mathscr{Z})$ as $M \to \infty$ uniformly w.r.t. (k,ε) when $k\varepsilon \le 1+\frac{\delta}{2}$. The sum of the series is simply denoted by $R_\ell(t,\varepsilon)$. Let us prove (55) to finish the proof of (ii). Lemma 2.5 and Lemma 5.9 say

$$\begin{aligned}
&\sum_{n=\ell}^{\infty}(1+\delta)^n \sum_{r=0}^{\ell} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \left|\widetilde{C_r^{(n)}}(t_n,\ldots,t_1,t)\right|_{\mathscr{L}(\bigvee^{p+n-r}\mathscr{Z},\bigvee^{q+n-r}\mathscr{Z})}\\
&\quad\le \sum_{n=\ell}^{\infty}(1+\delta)^n \sum_{r=0}^{\ell} \frac{|t^n|}{n!} \max_{t_n\le\cdots\le t_1\le t}\left|\widetilde{C_r^{(n)}}(t_n,\ldots,t_1,t)\right|_{\mathscr{L}(\bigvee^{p+n-r}\mathscr{Z},\bigvee^{q+n-r}\mathscr{Z})}\\
&\quad\le \sum_{n=\ell}^{\infty}(1+\delta)^n \sum_{r=0}^{\ell} \frac{2^{n-r}|t^n|}{n!} C_n^r \left[(p+n-r)(p+n-r-1)\right]^r \frac{(p+n-r-1)!}{(p-1)!}\\
&\quad\quad\times\ |V|_{L^\infty}^n\ |\tilde b|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}\\
&\quad\le \sum_{n=\ell}^{\infty}(1+\delta)^n |t|^n \sum_{r=0}^{\ell} \frac{2^{n-r}}{r!}(p+n)^{2r} C_{n-r+p-1}^{p-1}\ |V|_{L^\infty}^n\ |\tilde b|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}\\
&\quad\le 2^p \sum_{n=\ell}^{\infty}(1+\delta)^n 4^n |t|^n (n+p)^{2\ell}\ |V|_{L^\infty}^n\ |\tilde b|_{\mathscr{L}(\bigvee^p\mathscr{Z},\bigvee^q\mathscr{Z})}\,.
\end{aligned}$$

The last r.h.s. is finite whenever $4|t||V|_{L^\infty} < (1+\delta)^{-1}$. The condition $(1+2\delta)4|t||V|_{L^\infty} \le 1$ is sufficient and provides the uniform bound C_δ in (55). □

Proof of Theorem 5.6. Set $j = k-m$. By Theorem 5.5, the right-hand side of (45) vanishes when $m \ne p-q$ and the convergence of the series in $\mathscr{L}(\bigvee^k \mathscr{Z}, \bigvee^{k-p+q}\mathscr{Z})$

combined with Proposition 2.4-ii) implies

$$\begin{aligned}\left\langle z^{\otimes j}, U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t)\, z^{\otimes k}\right\rangle &= \sum_{r=0}^{\ell-1} \varepsilon^r \sum_{n=0}^{\infty} i^n \sqrt{\frac{k!j!\,\varepsilon^{p+q+2(n-r)}}{(k-(p+n-r))!(j-(q+n-r))!}}\,\delta^+_{k-(p+n-r),\,j-(q+n-r)} \\ &\quad\times \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n\, C_r^{(n)}(t_n,\ldots,t_1,t;z) + O_\delta(\varepsilon^\ell)\,,\end{aligned}$$

when $k\varepsilon \leq 1+\frac{\delta}{2}$, for any $\delta > 0$. By considering the limit $\varepsilon \to 0$, $k\varepsilon \to 1$ every factor

$$\sqrt{\frac{k!j!\,\varepsilon^{p+q+2(n-r)}}{(k-(p+n-r))!(j-(q+n-r))!}}$$

converges to 1. Therefore this proves (ii) for small times t such that $4|t||v|_{L^\infty} < 1$ up to the identification of the first term as $b(z_t)$. From our definitions we know

$$b(z_t) = \left\langle z_t^{\otimes q}\,,\, \tilde{b} z_t^{\otimes p}\right\rangle = b_t(e^{-is\Delta} z_s)\big|_{s=t}\,.$$

By setting $w_s = e^{-is\Delta} z_s$, the quantity $b(z_t)$ equals

$$b(z_t) = b_t(w_0) + \int_0^t \partial_s\big[b_t(w_s)\big]\, ds = b_t(w_0) + \int_0^t \overline{\partial_s w_s}.\partial_{\bar z} b_t(w_s) + \partial_z b_t(w_s).\partial_s w_s\, ds\,.$$

Moreover the equation (32) has the equivalent form with the vector $w_s = e^{-is\Delta} z_s$ and $\overline{w_s}$

$$i\partial_s w_s = e^{-is\Delta}\partial_{\bar z} Q(z_s) = \partial_{\bar z} Q_s(w_s) \qquad -i\partial_s \overline{w_s} = \partial_z Q_s(w_s)\,.$$

Hence we get

$$b(z_t) = b_t(w_0) + i\int_0^t \left\{Q_{t_1}, b_t\right\}(w_{t_1})\, dt_1\,.$$

An induction with $w_0 = z$ and the convergence of the series already checked yields

$$b(z_t) = \sum_{n=0}^{\infty} i^n \int_0^t d_{t_1} \cdots \int_0^{t_{n-1}} dt_n\;\; C_0^{(n)}(t_n,\ldots,t_1,t;z)\,.$$

Now let us prove the limit (i) for all times by following the argument in [20,45]. Assume that the result is true for $|t| \leq \frac{K}{4|V|_{L^\infty}}$. Let s be such that $|s| < 1/4|V|_{L^\infty}$. The convergence of the series given in Theorem 5.5 and the fact that $U_\varepsilon(t)$ preserves the number gives

$$\begin{aligned}
&\left\langle z^{\otimes j}\,,\,U_\varepsilon(t+s)^* b^{Wick} U_\varepsilon(t+s)\,z^{\otimes k}\right\rangle \\
&= \sum_{n=0}^{\infty} i^n \sum_{r=0}^{n} \varepsilon^r \int_0^s ds_1 \cdots \int_0^{s_{n-1}} ds_n \left\langle z^{\otimes j}, U_\varepsilon(t)^* \left[C_r^{(n)}(s_n,\ldots,s_1,s)\right]^{Wick} U_\varepsilon(t)\,z^{\otimes k}\right\rangle \\
&= \sum_{n=0}^{\infty} i^n \int_0^s ds_1 \cdots \int_0^{s_{n-1}} ds_n \left\langle z^{\otimes j}, U_\varepsilon(t)^* \left[C_0^{(n)}(s_n,\ldots,s_1,s)\right]^{Wick} U_\varepsilon(t)\,z^{\otimes k}\right\rangle \\
&\quad + O_s(\varepsilon)
\end{aligned} \tag{56}$$

with an absolutely and uniformly convergent series in the (56) when $k\varepsilon$ is close to 1. Hence the limit $\varepsilon \to 0$, $\varepsilon k \to 1$ and the sum $\sum_{n=0}^{\infty}$ in (56) can be interchanged when $4|s||V|_{L^\infty} < 1$. An induction on $K = 0, 1, 2\ldots$ finishes the proof. □

5.3. Coherent states and Wick observables

We show here that information on the propagation of coherent states can be directly deduced from the results about Hermite states.

Proposition 5.10. *For any $z_0 \in \mathscr{Z}$ and any $b \in \mathscr{P}_{p,q}(\mathscr{Z})$, the limit*

$$\lim_{\varepsilon\to 0} \left\langle U_\varepsilon(t)E(z_0)\,,\, b^{Wick} U_\varepsilon(t)E(z_0)\right\rangle = b(z_t)$$

holds for any $t \in \mathbb{R}$ when z_t denotes the solution to the Hartree equation (32).

Proof. By symmetry, one can assume $m = p - q \geq 0$. Recall that $E(z_0) = e^{-\frac{|z_0|^2}{2\varepsilon}} \sum_{n=0}^{\infty} \frac{\varepsilon^{-n/2}}{\sqrt{n!}} z_0^{\otimes n}$ and start first with $|z_0| = 1$. Since $U_\varepsilon(t)$ preserves the number, one gets

$$\left\langle U_\varepsilon(t)E(z_0)\,,\, b^{Wick} U_\varepsilon(t)E(z_0)\right\rangle = \sum_{n=m}^{\infty} e^{-\varepsilon^{-1}} \frac{\varepsilon^{-n}}{n!} a_n\left(\varepsilon^{-1}\right) \quad \text{with}$$

$$a_n\left(\varepsilon^{-1}\right) = \varepsilon^{m/2}\sqrt{n(n-1)\ldots(n-m+1)}\left\langle z_0^{\otimes n-m}\,,\, U_\varepsilon(t)^* b^{Wick} U_\varepsilon(t) z_0^{\otimes n}\right\rangle .$$

By Lemma 2.5 the quantity $a_n\left(\varepsilon^{-1}\right)$ satisfies

$$|a_n\left(\varepsilon^{-1}\right)| \leq (n\varepsilon)^{\frac{p+q+m}{2}} \left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})} \leq \langle n\varepsilon\rangle^p \left|\tilde{b}\right|_{\mathscr{L}(\bigvee^p \mathscr{Z}, \bigvee^q \mathscr{Z})} .$$

Hence Lemma A.1 applied here with $\lambda = \varepsilon^{-1}$ and $\mu = p$ reduces the problem to the proof of

$$\lim_{\lambda\to\infty} \int_{\mathbb{R}} a_{[\sqrt{\lambda}s+\lambda]}(\lambda) \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\, ds\,.$$

The uniform estimate

$$\left|a_{[\sqrt{\lambda}s+\lambda]}(\lambda)\right| \leq C_p \left\langle 1 + \frac{|s|}{\sqrt{\lambda}}\right\rangle^p \leq C'_p \langle s\rangle^p$$

and the pointwise convergence induced by Theorem 5.6 with $z = z_0$, $k = [\sqrt{\lambda}s+\lambda]$ and $\varepsilon = \lambda^{-1}$ yields the result.

For a general $|z_0| > 0$, write

$$E(z_0) = e^{-\frac{1}{2\varepsilon'}} \sum_{n=0}^{\infty} \frac{(\varepsilon')^{-n/2}}{\sqrt{n!}} (z_0')^{\otimes n} = E'(z_0')$$

with $z_0' = \frac{z_0}{|z_0|}$ and $\varepsilon' = \frac{\varepsilon}{|z_0|^2}$. By replacing the ε-quantization by the ε'-quantization, with

$$b^{Wick,\varepsilon'} = |z_0|^{-p-q}\, b^{Wick} \quad \text{for} \quad b \in \mathscr{P}_{p,q}(\mathscr{Z})$$

$$H_\varepsilon = |z_0|^2 d\Gamma_{\varepsilon'}(-\Delta) + |z_0|^4 Q^{Wick,\varepsilon'} \quad \text{and}$$

$$(i\varepsilon\partial_t u = H_\varepsilon u) \Leftrightarrow \left(i\varepsilon'\partial_t u = d\Gamma_{\varepsilon'}(-\Delta)u + |z_0|^2\, Q^{Wick,\varepsilon'} u\right).$$

Hence the previous result applied with $E'(z_0')$, $|z_0'| = 1$ and the ε'-quantization implies

$$\lim_{\varepsilon\to 0} \left\langle U_\varepsilon(t)E(z_0)\,,\, b^{Wick} U_\varepsilon(t)E(z_0)\right\rangle = |z_0|^{p+q}\, b(z_t')$$

where z_t' solves

$$i\partial_t z_t' = -\Delta z_t' + |z_0|^2 \,(V * |z_t'|^2) z_t'\,, \quad z_{t=0}' = z_0' = \frac{z_0}{|z_0|}.$$

Since this mean field equation preserves the norm $|z_t'|$ like (32) does for $|z_t|$, this implies

$$z_t' = |z_0|^{-1} z_t = |z_t|^{-1} z_t \quad \text{and} \quad |z_0|^{p+q}\, b(z_t') = b(z_t)\,. \qquad \square$$

Remark 5.11. Another proof can be obtained directly from Proposition 5.2 after checking uniform number estimates for $U_2(t,0)\Omega$. But working in this direction is more efficient with the help of Wigner measures.

6. Wigner measures: Definition and first properties

The notion of Wigner (or semiclassical) measures is well established in the finite dimensional case. We refer the reader to [10, 22, 23, 29, 36, 46] for details. The extension that we propose here to the infinite dimensional case follows a projective approach.

6.1. Wigner measures of normal states

Consider the algebra of cylindrical sets $\mathscr{B}_{cyl}(\mathscr{Z}) = \{X(p,E) = p^{-1}(E),\, p \in \mathbb{P}, E \in \mathscr{B}(p\mathscr{Z})\}$ where $\mathscr{B}(p\mathscr{Z})$ denotes for any $p \in \mathbb{P}$ the set of Borel subsets of $p\mathscr{Z}$. A cylindrical measure μ is a mapping defined on $\mathscr{B}_{cyl}(\mathscr{Z})$ such that:

- $\mu(\mathscr{Z}) = 1$,
- For any $p \in \mathbb{P}$, $\mu_p(A) = \mu(p^{-1}(A))$ for $A \in \mathscr{B}(p\mathscr{Z})$ defines a probability measure μ_p on $\mathscr{B}(p\mathscr{Z})$.

The family of measures $\{\mu_p\}_{p\in\mathbb{P}}$ is often called a weak distribution.

This notion is often introduced within the framework of real Hilbert spaces (or more generally real topological vector spaces). This makes no difference at this level. The real structure on $\mathscr{Z}$, namely the real scalar product S, is useful for the application of Bochner's theorem. For any $\xi \in \mathscr{Z}$ the function $z \mapsto e^{-2\pi i\, S(z,\xi)}$ is a cylindrical measurable function and the Fourier transform of μ is well defined by

$$\mathscr{F}[\mu](\xi) = \int_{\mathscr{Z}} e^{-2\pi i\, S(z,\xi)}\, d\mu\,.$$

Bochner's theorem characterizes the Fourier transform of a weak distribution. It says (see for example [3]) that a function G is the Fourier transform of a weak distribution if and only if

- G is normalized: $G(0) = 1$,
- G is of positive type: $\sum_{i,j=1}^{N} \lambda_i \overline{\lambda_j} G(\xi_i - \xi_j) \geq 0$,
- For any $p \in \mathbb{P}$, the restricted function $G|_{p\mathscr{Z}}$ is continuous.

An important point is that $\mathscr{Z}$ is a separable Hilbert space. Hence the σ-algebra generated by the cylindrical sets, that is containing $\mathscr{B}_{cyl}(\mathscr{Z})$, is nothing but the Borel σ-algebra, $\mathscr{B}(\mathscr{Z})$, associated with the norm topology on $\mathscr{Z}$. A probability measure well defined on $\mathscr{B}(\mathscr{Z})$ will be shortly called a probability measure on $\mathscr{Z}$. The tightness Prokhorov's criterion (see [42]) has within this setting the next simple form.

Lemma 6.1 (See [44]). *A cylindrical measure μ on $\mathscr{Z}$ extends to a probability measure on $\mathscr{Z}$ if and only if for any $\eta > 0$ there exists $R_\eta > 0$ such that*

$$\forall p \in \mathbb{P}, \quad \mu\left(\{z \in \mathscr{Z},\ |pz| \leq R_\eta\}\right) \geq 1 - \eta\,.$$

By recalling that for any $R > 0$ the ball $\{z \in \mathscr{Z} : |z| \leq R\}$ is weakly compact, this can be reinterpreted by saying that a weak distribution μ extends as a Borel probability measure if and only if its outer extension is a Radon measure on $\mathscr{Z}$ endowed with the weak topology (see [42]).

Consider a family $(\rho^\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}$ of non negative trace class operators on $\mathscr{H}$ such that $\mathrm{Tr}[\rho^\varepsilon] = 1$, or equivalently normal states $\mathscr{O} \mapsto \mathrm{Tr}[\rho^\varepsilon \mathscr{O}]$ on the space of all bounded operators $\mathscr{L}(\mathscr{H})$. An additional number estimate assumption allows to associate with such a family, Wigner probability measures on $\mathscr{Z}$.

Theorem 6.2. *Let $(\varrho^\varepsilon)_{\varepsilon \in (0,\bar{\varepsilon})}$ be a family of normal states on $\mathscr{L}(\mathscr{H})$ parametrized by ε. Assume $\mathrm{Tr}[N^\delta \rho^\varepsilon] \leq C_\delta$ uniformly w.r.t. $\varepsilon \in (0, \bar{\varepsilon})$ for some fixed $\delta > 0$ and $C_\delta \in (0, +\infty)$. Then for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ the exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a Borel probability measure μ on $\mathscr{Z}$ such that*

$$\lim_{k\to\infty} Tr[\rho^{\varepsilon_{n_k}} b^{Weyl}] = \lim_{k\to\infty} Tr[\rho^{\varepsilon_{n_k}} b^{A-Wick}] = \int_{\mathscr{Z}} b(z)\, d\mu(z)\,,$$

for all $b \in \cup_{p\in\mathbb{P}} \mathscr{F}^{-1}\left(\mathscr{M}_b(p\mathscr{Z})\right)$.

Moreover this probability measure μ satisfies $\int_{\mathscr{Z}} |z|^{2\delta}\, d\mu(z) < \infty$.

Remark 6.3.

a) By introducing the reduced density matrix $\varrho_p^\varepsilon \in \mathscr{L}^1(\Gamma_s(p\mathscr{Z}))$ defined for $p \in \mathbb{P}$ as a partially traced operator $\mathrm{Tr}[\varrho_p^\varepsilon A] = \mathrm{Tr}[\varrho^\varepsilon(A \otimes I_{\Gamma_s(p^\perp\mathscr{Z})})]$, one could consider the Husimi function μ_p^ε of ϱ_p^ε which is its finite dimensional

Wick symbol. It is known that this makes a weak probability distribution which admits weak limits after extracting subsequences $\varepsilon_{n_k} \to \infty$. The number estimate implies in finite dimension that such a limit is a probability measure. Our results say essentially two things: First after a proper extraction of subsequences, the family $(\mu_p)_{p\in\mathbb{P}}$ makes a weak distribution, i.e. the convergence can hold simultaneously for all the non countable family $p \in \mathbb{P}$. Secondly the weak distribution is a Borel probability measure.

b) The estimate $\int_{\mathscr{Z}} |z|^{2\delta} \; d\mu(z) < +\infty$ will be proved in the more precise form

$$\int_{\mathscr{Z}} \left(1+|z|^2\right)^{\delta} \, d\mu(z) \le \liminf_{\varepsilon_{n_k}\to\infty} \, \mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}}(1+N)^{\delta}\right] \le C'_{\delta} < +\infty \, .$$

Contrary to the finite dimensional case, the first inequality is not an equality even when the right-hand side converges. Examples are given in Section 7.4.

c) For a non negative trace-class operator ϱ, the assumption

$$\begin{aligned} C \ge \mathrm{Tr}[N^{\delta}\varrho] = \sup_{\substack{A\in\mathscr{L}(\mathscr{H})\\ 0\le A\le N^{\delta}}} \mathrm{Tr}[A\varrho] &= \sup_{k\in\mathbb{N}} \, \mathrm{Tr}\left[N^{\delta/2}1_{[0,k]}(N)\varrho 1_{[0,k]}(N)N^{\delta/2}\right] \\ &= \sup_{k\in\mathbb{N}} \, \mathrm{Tr}\left[\varrho^{1/2}1_{[0,k]}(N)N^{\delta}\varrho^{1/2}\right] \end{aligned}$$

implies $(1+N)^{\delta/2}\varrho(1+N)^{\delta/2} \in \mathscr{L}^1(\mathscr{H})$ with a norm estimate.

Reciprocally, assuming $\left|(1+N)^{\delta/2}\varrho(1+N)^{\delta/2}\right|_{\mathscr{L}^1} \le C$ implies that the quantity $\mathrm{Tr}[N^{\delta}\varrho]$ defined as the above supremum is bounded by C. Such an equivalence is no more true when $\varrho \ge 0$ is not assumed and the second version has to be considered (see Proposition 6.4).

Proof. i) The Proposition 3.7 implies

$$\left|\mathrm{Tr}\left[\varrho^{\varepsilon}b^{Weyl}\right] - \mathrm{Tr}\left[\varrho^{\varepsilon}b^{A-Wick}\right]\right| \le \left|b^{Weyl} - b^{A-Wick}\right| \overset{\varepsilon\to 0}{\to} 0 \, ,$$

for fixed $b \in \cup_{p\in\mathbb{P}} \mathscr{F}^{-1}\left(\mathscr{M}_b(p\mathscr{Z})\right)$. Hence the result is true when it is proved after considering simply the Anti-Wick observables.

ii) Consider for $\varepsilon > 0$ the function

$$G_{\varepsilon}(\xi) = \mathrm{Tr}\left[\varrho^{\varepsilon}W(\sqrt{2}\pi\xi)\right] e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2} = \mathrm{Tr}\left[\varrho^{\varepsilon}(e^{2i\pi S(\xi,.)})^{A-Wick}\right] .$$

The positive type property and the normalization come from

$$G_{\varepsilon}(0) = \mathrm{Tr}\left[\varrho^{\varepsilon}\right] = 1$$

$$\sum_{i,j=1}^{N} \lambda_i\overline{\lambda_j}G_{\varepsilon}(\xi_i-\xi_j) = \mathrm{Tr}\left[\varrho^{\varepsilon}\left(\left|\sum_{k=1}^{N}\lambda_k e^{2i\pi S(\xi_k,.)}\right|^2\right)^{A-Wick}\right] \ge 0 \, .$$

The continuity when ξ is restricted to any fixed finite dimensional $p\mathscr{Z}$ can be written with uniform estimates w.r.t $\varepsilon \in (0,\bar{\varepsilon})$. Consider the estimate $\mathrm{Tr}\left[\varrho^{\varepsilon}(1+N)^{\delta_1}\right]$

$\leq C_{\delta_1}$ with $\delta_1 \in (0, \min(1, 2\delta))$. Write for any $\xi, \eta \in \mathscr{Z}$

$$\begin{aligned}|G_\varepsilon(\eta) - G_\varepsilon(\xi)| &= \left|\mathrm{Tr}\left[\rho^\varepsilon \frac{(N+1)^{\delta_1/2}}{(N+1)^{\delta_1/2}}\left[W(\sqrt{2}\pi\eta) - W(\sqrt{2}\pi\xi)\right]\frac{(N+1)^{\delta_1/2}}{(N+1)^{\delta_1/2}}\right]\right| \\ &\quad + \left|e^{-\frac{\varepsilon\pi^2}{2}|\eta|^2} - e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2}\right| \\ &\leq \left|\left[W(\sqrt{2}\pi\eta) - W(\sqrt{2}\pi\xi)\right](N+1)^{-\delta_1/2}\right|_{\mathscr{L}(\mathscr{H})} \mathrm{Tr}\left[(N+1)^{\delta_1}\rho^\varepsilon\right] \\ &\quad + \left|e^{-\frac{\varepsilon\pi^2}{2}|\eta|^2} - e^{-\frac{\varepsilon\pi^2}{2}|\xi|^2}\right|.\end{aligned}$$

We have found by Lemma 3.1 two constants $\delta_1 \in (0,1)$ and $C'_{\delta_1} > 0$ such that

$$\forall \xi, \eta \in \mathscr{Z}, \quad |G_\varepsilon(\eta) - G_\varepsilon(\xi)| \leq C'_{\delta_1} |\eta - \xi|^{\delta_1} \left[(|\eta|^2 + |\xi|^2)^{\delta_1/2} + 1\right], \tag{57}$$

holds uniformly w.r.t. $\varepsilon \in (0, \bar{\varepsilon})$ and we recall the uniform estimate $|G_\varepsilon(\xi)| \leq 1$. Hence for any $\varepsilon \in (0, \bar{\varepsilon})$, G_ε is the Fourier transform of a weak distribution μ^ε such that

$$\mathrm{Tr}\left[\varrho^\varepsilon b^{A-Wick}\right] = \int_{\mathscr{Z}} b(z)\, d\mu^\varepsilon(z)$$

holds for all $b \in \cup_{p \in \mathbb{P}} \mathscr{F}^{-1}(\mathscr{M}_b(p\mathscr{Z}))$.

iii) Actually the uniform estimate (57) allows to apply an Ascoli type argument after considering sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$:

- Since $\mathscr{Z}$ is separable, it admits a countable dense set $\mathscr{N} = \{\xi_\ell,\ \ell \in \mathbb{N}\}$. For any $\ell \in \mathbb{N}$ the sequence $G_{\varepsilon_n}(\xi_\ell)$ remains in $\{\sigma \in \mathbb{C}, |\sigma| \leq 1\}$. Hence by a diagonal extraction process there exists a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that for all $\ell \in \mathbb{N}$, $G_{\varepsilon_{n_k}}(\xi_\ell)$ converges in $\{\sigma \in \mathbb{C}, |\sigma| \leq 1\}$ as $k \to \infty$. Set
$$G(\xi_\ell) = \lim_{k\to\infty} G_{\varepsilon_{n_k}}(\xi_\ell)$$
for all $\ell \in \mathbb{N}$.
- The uniform estimate (57) implies that the limit G is uniformly continuous on any set $\mathscr{N} \cap \{z \in \mathscr{Z} : |z| \leq R\}$. Hence it admits a continuous extension still denoted G in $(\mathscr{Z}, |\ |_{\mathscr{Z}})$. An "epsilon/3"-argument shows that for any $\xi \in \mathscr{Z}$ $\lim_{k\to\infty} G_{\varepsilon_{n_k}}(\xi)$ exists and equals $G(\xi)$.
- Finally G is a normalized function of positive type as a limit of such functions.

Finally the uniform estimates $|G_\varepsilon(\xi)| \leq 1$ and $|G(\xi)| \leq 1$ allow to test the convergence against any $\nu \in \mathscr{M}_b(p\mathscr{Z})$ and to apply the Parseval identity with $b = \mathscr{F}^{-1}(\nu)$. From any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \varepsilon_n = 0$, one can extract a subsequence $(\varepsilon_{n_k})_{k\to\infty}$ and find a weak distribution such that the limit

$$\lim_{n_k\to\infty} \mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}} b^{Weyl}\right] = \lim_{n_k\to\infty} \mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}} b^{A-Wick}\right] = \int_{\mathscr{Z}} b(z)\, d\mu(z)$$

holds for any $b \in \mathscr{F}\left(L^1(p\mathscr{Z}, L_p(dz))\right)$ and therefore for any $b \in \mathscr{S}_{cyl}(\mathscr{Z})$.

iv) The Prokhorov's criterion for μ in the form stated in Lemma 6.1 is again a consequence of the uniform number estimate $\mathrm{Tr}\left[N^\delta \varrho^\varepsilon\right] \leq C_\delta$. Fix any $p \in \mathbb{P}$ and set $d = \dim p$. The operators $N_p = N_{p\mathscr{Z}} \otimes I_{\Gamma_s(p^\perp\mathscr{Z})} = \left(d\Gamma(I_{p\mathscr{Z}}) \otimes I_{\Gamma_s(p^\perp\mathscr{Z})}\right) =$

$d\Gamma(p)$, $N_{p^\perp} = \big(I_{p\mathscr{Z}} \otimes d\Gamma(I_{p^\perp\mathscr{Z}})\big) = d\Gamma(p^\perp)$ and $N = d\Gamma(I)$ make a commuting family of non negative operators such that $N = N_p + N_{p^\perp}$. Thus the inequality

$$\left(1 + \frac{d\varepsilon}{2} + N\right)^s \geq \left(1 + \frac{d\varepsilon}{2} + N_p\right)^s$$

holds for any $s \geq 0$. Hence the estimate $\mathrm{Tr}\left[\varrho^\varepsilon N^\delta\right] \leq C_\delta$ implies

$$\mathrm{Tr}\left[\varrho^\varepsilon \left(1 + \frac{d\varepsilon}{2} + N_p\right)^\delta\right] \leq \mathrm{Tr}\left[\varrho^\varepsilon \left(1 + \frac{d\varepsilon}{2} + N\right)^\delta\right] \leq \mathrm{Tr}\left[\varrho^\varepsilon (2+N)^\delta\right] \leq C'_\delta\,,$$

with $C'_\delta > 0$ independent of ε and p as soon as $\varepsilon \leq \frac{1}{d}$.

Let $\chi \in \mathscr{C}^\infty(p\mathscr{Z})$ be a non negative function on $p\mathscr{Z}$, such that $\chi \equiv 0$ in a neighborhood of $\{|z| \leq 1\}$. For any $R \geq 1$ the estimates

$$\frac{(1+R^2)^\delta}{(1+|z|^2)^\delta}\chi(R^{-1}z) \leq 1$$

holds with uniform estimates of the left-hand side in $S_{p\mathscr{Z}}(1, \frac{|dz|^2}{\langle z\rangle^2})$. The pseudodifferential calculus in $p\mathscr{Z}$ with the metric $\frac{|dz|^2}{\langle z\rangle^2}$, provides the inequality of bounded operators on $\Gamma_s(p\mathscr{Z})$

$$(1+R^2)^\delta A \circ B_R \circ A - C\varepsilon \leq \left[\frac{(1+R^2)^\delta}{(1+|z|^2)^\delta}\chi(R^{-1}z)\right]^{Weyl} \leq 1 + C\varepsilon$$

with $A = \left[(1+|z|^2)^{-\delta/2}\right]^{Weyl}$, $B_R = \left[\chi(R^{-1}z)\right]^{Weyl}$ and $|B_R|_{\mathscr{L}(\Gamma_s(p\mathscr{Z}))} \leq C$,

with a constant $C > 0$ independent of $\varepsilon \in (0, \frac{1}{d})$ and $R \geq 1$. By Proposition 3.3, there exists a constant $C' > 0$ independent of $\varepsilon \in (0, \frac{1}{d})$ (and $R \geq 1$) such that

$$\left| A^2 \circ \left(1 + \frac{d\varepsilon}{2} + N_{p\mathscr{Z}}\right)^\delta - I_{\Gamma_s(p\mathscr{Z})}\right|_{\mathscr{L}(\Gamma_s(p\mathscr{Z}))} \leq C'\varepsilon\,.$$

Hence the inequality

$$(1+R^2)^\delta \chi(R^{-1}pz)^{Weyl} \leq (1 + 2C\varepsilon)A^{-\delta}$$

after tensorization with $I_{\Gamma_s(p^\perp\mathscr{Z})}$ and testing on the normal state ϱ^ε yields

$$(1+R^2)^\delta \mathrm{Tr}\left[\varrho^\varepsilon \chi(R^{-1}pz)^{Weyl}\right] \leq C''_\delta$$

with a uniform constant C''_δ with respect to $\varepsilon \in (0, \frac{1}{d})$ and $R \geq 1$. After taking the limit $n_k \to \infty$, $\varepsilon_{n_k} \to 0$, we get

$$\begin{aligned}\int_{\mathscr{Z}} 1_{\{|pz|\geq R\}}(z)\ d\mu(z) \leq \int_{\mathscr{Z}} \chi(R^{-1}pz)\ d\mu(z) &= \lim_{n_k\to\infty} \mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}} \chi(R^{-1}pz)^{Weyl}\right] \\ &\leq C''_\delta (1+R^2)^{-\delta}\,.\end{aligned}$$

This inequality is valid for any $p \in \mathbb{P}$ and the Prokhorov's criterion of Lemma 6.1 is satisfied. The weak distribution μ is a probability measure on $\mathscr{Z}$.

v) First the function $\langle z\rangle^{2\delta}$ is Borel measurable in $\mathscr{Z}$. Take $p \in \mathbb{P}$ and $R \geq 1$ and take now $\chi_0 \in \mathscr{C}_0^\infty(p\mathscr{Z})$, such that $0 \leq \chi_0 \leq 1$ and $\chi_0 \equiv 1$ in a neighborhood of 0. Consider the estimates

$$(1+N)^\delta \geq (1+N_p)^\delta \geq (1+N_p)^{\delta/2}\chi_0(R^{-1}pz)^{Weyl}(1+N_p)^{\delta/2} - C_p\varepsilon(1+N_p)^\delta$$
$$\geq \left[\left((1+|pz|^2)\right)^\delta \chi_0(R^{-1}pz)\right]^{Weyl} - C_p'\varepsilon(1+N)^\delta$$

where the two last inequalities are again derived from the finite dimensional Weyl calculus (with a uniform control w.r.t. $R \geq 1$). After taking the limit $n_k \to \infty$, $\varepsilon_{n_k} \to 0$, this implies

$$\int_{\mathscr{Z}} \left(1+|pz|^2\right)^\delta \chi_0(R^{-1}pz)\, d\mu(z) = \lim_{n_k\to\infty} \mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}}\left[\left((1+|pz|^2)\right)^\delta \chi_0(R^{-1}pz)\right]^{Weyl}\right]$$
$$\leq \liminf_{n_k\to\infty} \mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}}(1+N)^\delta\right] \leq C_\delta'\,.$$

Taking the supremum w.r.t $R \geq 1$ and then w.r.t a countable increasing sequence $(p_n)_{n\in\mathbb{N}}$, $p_n \in \mathbb{P}$, such that $\sup_{n\in\mathbb{N}} p_n = I_{\mathscr{Z}}$, yields

$$\int_{\mathscr{Z}} (1+|z|^2)^\delta d\mu(z) \leq C_\delta' < +\infty\,. \qquad \square$$

6.2. Complex Wigner measures, pure sequences

More general families of trace class operators can be considered by linear decomposition

$$\varrho^\varepsilon = \lambda_{R+}^\varepsilon \varrho_{R+}^\varepsilon - \lambda_{R-}^\varepsilon \varrho_{R-}^\varepsilon + i\lambda_{I+}^\varepsilon \varrho_{I+}^\varepsilon - i\lambda_{I-}^\varepsilon \varrho_{I-}^\varepsilon\,, \tag{58}$$

with

$$\lambda_{R+}^\varepsilon \varrho_{R+}^\varepsilon = \frac{1}{4}\left[|\varrho^\varepsilon + (\varrho^\varepsilon)^*| + \varrho^\varepsilon + (\varrho^\varepsilon)^*\right], \qquad \lambda_{R-}^\varepsilon \varrho_{R-}^\varepsilon = \frac{1}{4}\left[|\varrho^\varepsilon + (\varrho^\varepsilon)^*| - \varrho^\varepsilon - (\varrho^\varepsilon)^*\right]$$
$$\lambda_{I+}^\varepsilon \varrho_{I+}^\varepsilon = \frac{1}{4}\left[|\varrho^\varepsilon - (\varrho^\varepsilon)^*| - i\varrho^\varepsilon + i(\varrho^\varepsilon)^*\right], \qquad \lambda_{I-}^\varepsilon \varrho_{I-}^\varepsilon = \frac{1}{4}\left[|\rho^\varepsilon - (\varrho^\varepsilon)^*| + i\varrho^\varepsilon - i(\varrho^\varepsilon)^*\right],$$

such that $\lambda_\bullet^\varepsilon \geq 0$, $\varrho_\bullet^\varepsilon \geq 0$, $\mathrm{Tr}\left[\varrho_\bullet^\varepsilon\right] = 1$ and

$$\lambda_{R+}^\varepsilon + \lambda_{R-}^\varepsilon + \lambda_{I+}^\varepsilon + \lambda_{I-}^\varepsilon \leq 4\left|\varrho^\varepsilon\right|_{\mathscr{L}^1(\mathscr{H})}\,.$$

Proposition 6.4. *Let $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of trace class operators such that*

$$\left|(1+N)^{\delta/2}\varrho^\varepsilon(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})} \leq C_\delta \tag{59}$$

uniformly for some $\delta > 0$ and some $C_\delta < +\infty$. Then for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$, one can extract a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and find a (complex) Borel measure μ on $\mathscr{Z}$ such that

$$\lim_{k\to\infty} Tr[\rho^{\varepsilon_{n_k}} b^{Weyl}] = \lim_{k\to\infty} Tr[\rho^{\varepsilon_{n_k}} b^{A-Wick}] = \int_{\mathscr{Z}} b(z)\, d\mu(z)\,, \tag{60}$$

for all $b \in \cup_{p\in\mathbb{P}} \mathscr{F}^{-1}\left(\mathscr{M}_b(p\mathscr{Z})\right)$.

This measure satisfies $\int_{\mathscr{Z}} \langle z\rangle^\delta \; d\left|\mu\right|(z) < +\infty$.

After assuming additionally the stronger uniform estimate

$$Tr\big[(1+N)^{\delta}|\varrho^{\varepsilon}+(\varrho^{\varepsilon})^*|\big]+Tr\big[(1+N)^{\delta}|\varrho^{\varepsilon}-(\varrho^{\varepsilon})^*|\big]\le C'_{\delta}\,, \tag{61}$$

this measure satisfies $\int_{\mathscr{Z}}\langle z\rangle^{2\delta}\;d\,|\mu|\,(z)<+\infty$.

Proof. Owing to the estimate

$$\lambda_{R+}^{\varepsilon}+\lambda_{R-}^{\varepsilon}+\lambda_{I+}^{\varepsilon}+\lambda_{I-}^{\varepsilon}\le 4\,|\varrho^{\varepsilon}|_{\mathscr{L}^1(\mathscr{H})}\le 4C_{\delta}\,,$$

with all the $\lambda_{\bullet}^{\varepsilon}$ non negative, the extraction of a subsequence allows to reduce the analysis to the case when all the $\lambda_{\bullet}^{\varepsilon_n}$ converge: $\lim_{n\to\infty}\lambda_{\bullet}^{\varepsilon_n}=\lambda_{\bullet}^{0}\in[0,+\infty)$.

If one $\lambda_{\bullet}^{0}$ equals 0 then $|\lambda_{\bullet}^{\varepsilon_n}\varrho_{\bullet}^{\varepsilon_n}|_{\mathscr{L}^1(\mathscr{H})}=\lambda_{\bullet}^{\varepsilon_n}\overset{n\to\infty}{\to}0$ and $\lim_{\varepsilon_n\to 0}\mathrm{Tr}\left[b^{Weyl}(\lambda_{\bullet}^{\varepsilon_n}\varrho_{\bullet}^{\varepsilon_n})\right]=0$, for all $b\in\mathscr{S}_{cyl}(\mathscr{Z})$, and the corresponding term does not contribute to the asymptotic measure μ.

Hence the problem is now reduced to the case when all the $\lambda_{\bullet}^{0}$ are positive, and therefore for $N_0>0$ large enough, all the $(\lambda_{\bullet}^{\varepsilon_n})_{n>N_0}$ are uniformly positive. Set in this case

$$c=\min\left\{\lambda_{R+}^{\varepsilon_n},\lambda_{R-}^{\varepsilon_n},\lambda_{I+}^{\varepsilon_n},\lambda_{I-}^{\varepsilon_n};\ n>N_0\right\}>0$$

for $N_0>0$ large enough. The decomposition (58) implies

$$(1+N)^{\delta/4}\varrho^{\varepsilon}(1+N)^{\delta/4}=\lambda_{R+}^{\varepsilon}r_{R+,\varepsilon}^{\varepsilon}-\lambda_{R-}^{\varepsilon}r_{R-,\delta}+i\lambda_{I+}^{\varepsilon}r_{I+,\delta}^{\varepsilon}-i\lambda_{I-}r_{I-,\delta}^{\varepsilon}$$
$$\text{with}\quad r_{\bullet,\delta}^{\varepsilon}=(1+N)^{\delta/4}\varrho_{\bullet}^{\varepsilon}(1+N)^{\delta/4}\ge 0\,.$$

All the terms $r_{\bullet,\delta}^{\varepsilon_n}$ are estimated in the same way as follows. For $k\in\mathbb{N}$, consider the quantity:

$$\begin{aligned}
&\mathrm{Tr}\big[1_{[0,k]}(N)\;r_{R+,\delta}^{\varepsilon_n}\;1_{[0,k]}(N)\big]\\
&\qquad=\frac{1}{4\lambda_{R+}^{\varepsilon_n}}\mathrm{Tr}\Big[\big(|\varrho^{\varepsilon_n}+(\varrho^{\varepsilon_n})^*|+\varrho^{\varepsilon_n}+(\varrho^{\varepsilon_n})^*\big)(1+N)^{\delta/2}1_{[0,k]}(N)\Big]\\
&\qquad\le\frac{1}{4c}\left||\varrho^{\varepsilon_n}+(\varrho^{\varepsilon_n})^*|(1+N)^{\delta/2}1_{[0,k]}(N)\right|_{\mathscr{L}^1(\mathscr{H})}\\
&\qquad\quad+\frac{1}{2c}\left|(1+N)^{\delta/2}\varrho^{\varepsilon_n}(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\,.
\end{aligned}$$

The polar decomposition $\varrho^{\varepsilon_n}+(\varrho^{\varepsilon_n})^*=U_{\varepsilon_n}\,|\varrho^{\varepsilon_n}+(\varrho^{\varepsilon_n})^*|$ provides the inequality

$$\begin{aligned}
&\left||\varrho^{\varepsilon_n}+(\varrho^{\varepsilon_n})^*|(1+N)^{\delta/2}1_{[0,k]}(N)\right|_{\mathscr{L}^1(\mathscr{H})}\\
&\qquad\le 2\left|U_{\varepsilon_n}^*(1+N)^{-\delta/2}\right|_{\mathscr{L}(\mathscr{H})}\left|(1+N)^{\delta/2}\varrho^{\varepsilon_n}(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\,.
\end{aligned}$$

Therefore, this yields

$$\mathrm{Tr}\left[r_{\bullet,\delta}^{\varepsilon_n}\right]=\sup_{k\in\mathbb{N}}\;\mathrm{Tr}\left[1_{[0,k]}(N)\,r_{\bullet,\delta}^{\varepsilon_n}\,1_{[0,k]}(N)\right]\le\frac{1}{c}\;\left|(1+N)^{\delta/2}\varrho^{\varepsilon_n}(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\,.$$

Hence the four families of normal state $(\varrho_{\bullet}^{\varepsilon_n})_{n>N_0}$ fulfill the assumptions of Theorem 6.2, with δ replaced with $\delta/2$ and in the symmetric writing of Remark 6.3 c).

Hence four Borel probability measures, μ_{R+}, μ_{R-}, μ_{I+} and μ_{I-} exist and a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ can be extracted so that

$$\lim_{k\to\infty} \mathrm{Tr}\left[b^{Weyl}\varrho_{\bullet}^{\varepsilon_{n_k}}\right] = \int_{\mathscr{Z}} b\, d\mu_{\bullet}\,,$$

with the estimates $\int_{\mathscr{Z}} \langle z\rangle^{2(\delta/2)}\, d\mu_{\bullet} < +\infty$. We conclude by taking

$$\mu = \lambda_{R+}^0\mu_{R+} + \lambda_{R-}^0\mu_{R-} + i\lambda_{I+}^0\mu_{I+} - i\lambda_{I-}^0\mu_{I-}\,.$$

Finally the last statement with the exponent 2δ comes from the operator inequalities

$$(1+N)^{\delta/2}\varrho_{R\pm}^{\varepsilon_n}(1+N)^{\delta/2} \leq \frac{1}{2c}(1+N)^{\delta/2}\left|\varrho^{\varepsilon_n} + (\varrho^{\varepsilon_n})^*\right|(1+N)^{\delta/2}\,, \quad \text{and}$$

$$(1+N)^{\delta/2}\varrho_{I\pm}^{\varepsilon_n}(1+N)^{\delta/2} \leq \frac{1}{2c}(1+N)^{\delta/2}\left|\varrho^{\varepsilon_n} - (\varrho^{\varepsilon_n})^*\right|(1+N)^{\delta/2}\,,$$

while considering the case when all the $\lambda_{\bullet}^0$ are positive. □

Definition 6.5. For a family $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$, satisfying (59), the set of Borel measures μ which satisfy (60) is denoted $\mathscr{M}(\varrho^\varepsilon, \varepsilon\in(0,\bar\varepsilon))$ or simply $\mathscr{M}(\varrho^\varepsilon)$.

Such a family $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ (resp. a sequence $(\varrho^{\varepsilon_n})_{n\in\mathbb{N}}$) is said pure if $\mathscr{M}(\varrho^\varepsilon, \varepsilon\in(0,\bar\varepsilon))$ (resp. $\mathscr{M}(\varrho^{\varepsilon_n}, n\in\mathbb{N})$) has a single element μ.

When the family $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ is pure the limit in (60) can be written with $\lim_{\varepsilon\to 0}$ instead of $\lim_{n_k\to\infty}$. This provides a characterization of $\mathscr{M}(\varrho^\varepsilon) = \{\mu\}$. For simplicity, we shall often assume that the family $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ is pure, when the reduction to such a case can be done after extracting a suitable sequence.

6.3. Countably separating sets of observables

In order to identify a Wigner measure of $\mu\in\mathscr{M}(\varrho^\varepsilon)$ it is sufficient to test on a "dense set" of observables. The good notion is given by the Stone–Weierstrass theorem for L^1 spaces. It can be recovered from the standard Stone–Weierstrass theorem for continuous functions in our case.

Lemma 6.6 (cf [14]). *Let ν be a Borel probability measure on a separable Banach space X and let $\{f_n, n\in\mathbb{N}\}$ be a countable set of bounded ν-measurable functions which separates the points*

$$\forall x,y\in X\,,\quad \exists n\in\mathbb{N}\,,\quad f_n(x)\neq f_n(y)\,.$$

Then for any $p\in[0,\infty)$, the algebra generated by $\{f_n, n\in\mathbb{N}\}$ is dense in $L^p(X,d\nu)$.

Since "the" Wigner measure is not known a priori, the good notion of "dense set" that we shall use is the following.

Definition 6.7. A subset $\mathscr{D}\subset\cup_{p\in\mathbb{P}}\mathscr{F}^{-1}(\mathscr{M}_b(p\mathscr{Z}))$ is said countably separating whenever it contains a countable subset, $\mathscr{D}\supset\mathscr{D}_0\sim\mathbb{N}$, which separates the point of $\mathscr{Z}$:

$$\forall x,y\in\mathscr{Z}\,,\quad \exists f\in\mathscr{D}_0\,,\quad f(x)\neq f(y)\,.$$

Proposition 6.8. *Let μ_1 be a bounded Borel measure on $\mathscr{Z}$ and let $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of operators which fulfills the assumptions of Definition 6.5. The two next statements are equivalent:*

1. $\mathscr{M}(\varrho^\varepsilon)=\{\mu_1\}$.
2. *There exists a countably separating subset $\mathscr{D}\subset\cup_{p\in\mathbb{P}}\mathscr{F}^{-1}(\mathscr{M}_b(p\mathscr{Z}))$ such that*

$$\forall b\in\mathscr{D}\,,\quad \lim_{\varepsilon\to 0} Tr\left[\varrho^\varepsilon b^{Weyl}\right]=\lim_{\varepsilon\to 0} Tr\left[\varrho^\varepsilon b^{A-Wick}\right]=\int_{\mathscr{Z}} b(z)\;d\mu_1(z)\,.$$

Remark 6.9. A similar equivalence is obtained for $\mu_1\in\mathscr{M}(\varrho^\varepsilon)$ after a subsequence extraction.

Proof. Assume $\mu\in\mathscr{M}(\varrho^\varepsilon)$. There exists a sequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ and a Borel measure μ such that (60) holds for any $b\in\cup_{p\in\mathbb{P}}\mathscr{F}^{-1}\mathscr{M}_b(p\mathscr{Z})$. In particular this holds for any $b\in\mathscr{D}$:

$$\int_{\mathscr{Z}} b(z)\;d\mu(z)=\lim_{k\to\infty}\mathrm{Tr}\left[\varrho^{\varepsilon_{n_k}}b^{Weyl}\right]=\int_{\mathscr{Z}} b(z)\;d\mu_1(z)\,.$$

The set $\mathscr{D}$ is dense in $L^1(\mathscr{Z},d|\mu_1|)$ and in $L^1(\mathscr{Z},d|\mu|)$ so that the above equality of the extreme sides extend to any bounded Borel function. This implies $\mu=\mu_1$. □

The next examples will be useful in the application and allow to reconsider an inductive point of view.

Proposition 6.10. *Let $(p_\ell)_{\ell\in\mathbb{N}}$ be an increasing sequence of projectors in $\mathbb{P}$ such that $\sup_\ell p_\ell=I_{\mathscr{Z}}$ and let the family of operators $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfy the assumptions of Definition 6.5. Then the identity $\mathscr{M}(\varrho^\varepsilon)=\{\mu\}$ is equivalent to any of the next statement*

1. *For all $b\in\cup_{\ell\in\mathbb{N}}\mathscr{S}(p_\ell\mathscr{Z})$, the quantity $Tr[\varrho^\varepsilon b^{Weyl}]$ converges to $\int_{\mathscr{Z}} b(z)\;d\mu(z)$ as $\varepsilon\to 0$.*
2. *For all $b\in\mathscr{S}_{cyl}(\mathscr{Z})$, the quantity $Tr[\varrho^\varepsilon b^{Weyl}]$ converges to $\int_{\mathscr{Z}} b(z)\;d\mu(z)$ as $\varepsilon\to 0$.*

Proof. It suffices to notice that $\cup_{\ell\in\mathbb{N}}\mathscr{S}(p_\ell\mathscr{Z})$, and therefore $\mathscr{S}_{cyl}(\mathscr{Z})$, is countably separating because the weak topology separates the points. □

6.4. Orthogonality argument

Complex Wigner measures are especially interesting while considering the joint measure associated with two families of vectors $(u^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ and $(v^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$. Introduce the notation

$$\varrho^\varepsilon_{uv}=|u^\varepsilon\rangle\langle v^\varepsilon|\,.$$

Proposition 6.11. *Assume that the family of vectors $(u^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ and $(v^\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfy the uniform estimates*

$$\left|(1+N)^{\delta/2}u^\varepsilon\right|_{\mathscr{H}}+\left|(1+N)^{\delta/2}v^\varepsilon\right|_{\mathscr{H}}\le C\,,\quad |u^\varepsilon|_{\mathscr{H}}=|v^\varepsilon|_{\mathscr{H}}=1$$

for some fixed $\delta > 0$ *and* $C > 0$. *Assume further that any* $\mu \in \mathscr{M}(\varrho^{\varepsilon}_{uu})$ *and any* $\nu \in \mathscr{M}(\varrho^{\varepsilon}_{vv})$ *are mutually orthogonal. Then the family* $(\varrho^{\varepsilon}_{uv})_{\varepsilon\in(0,\bar{\varepsilon})}$ *is pure with*

$$\mathscr{M}\left(\varrho^{\varepsilon}_{uv}, \varepsilon \in (0,\bar{\varepsilon})\right) = \{0\}$$

$$i.e. \quad \lim_{\varepsilon\to 0} \left\langle u^{\varepsilon}\,,\, b^{Weyl} v^{\varepsilon}\right\rangle = \lim_{\varepsilon\to 0} \left\langle u^{\varepsilon}\,,\, b^{A-Wick} v^{\varepsilon}\right\rangle = 0$$

for any $b \in \mathscr{F}^{-1}(\mathscr{M}_b(p\mathscr{Z}))$ *and any* $p \in \mathbb{P}$.

Proof. Assume $\mathscr{M}(\varrho_{uu}) = \{\mu\}$ and $\mathscr{M}(\varrho^{\varepsilon}_{vv}) = \{\nu\}$ with $\mu \perp \nu$. Take $\eta > 0$. There exist two bounded closed subsets K_1 and K_2 such that

$$\mu(K_1) \geq 1-\eta\,, \quad \nu(K_2) \geq 1-\eta\,, \quad K_1 \cap K_2 = \emptyset\,.$$

Since K_1 and K_2 are compact in the weak topology, $K_1 \subset \complement K_2$, $\complement K_2$ open in the weak topology, there exists a finite covering of K_1 of the form

$$K_1 \subset \bigcup_{k=1}^{K} \left\{|p_k(z-z_k)| \leq r_k\right\}, \quad \bigcup_{k=1}^{K} \left\{|p_k(z-z_k)| \leq 2r_k\right\} \cap K_2 = \emptyset$$

with $p_k \in \mathbb{P}$, $z_k \in \mathscr{Z}$ and $r_k > 0$ for all $k \in \{1,\ldots,K\}$. By choosing for any k a function $\chi_k \in \mathscr{C}^{\infty}_0(p_k\mathscr{Z})$ such that $\chi_k(p_k(z)) \equiv 1$ when $|p_k(z-z_k)| \leq r_k$ and $\chi_k(p_k z) = 0$ when $|p_k(z-z_k)| \geq 2r_k$ the sum $\chi(z) = \sum_{k=1}^{N} \frac{\chi_k(p_k z)}{\sum_{k'} \chi_{k'}(p_{k'} z)}$ defines a cylindrical function $\chi \in \mathscr{S}_{cyl}(\mathscr{Z})$ such that $\chi \equiv 1$ on K_1 and $\chi \equiv 0$ on K_2.

Take now any $b \in \mathscr{S}_{cyl}(\mathscr{Z})$ and write

$$\begin{aligned} \left|\left\langle u^{\varepsilon}\,, b^{Weyl} v^{\varepsilon}\right\rangle\right| &= \left|\left\langle u^{\varepsilon}\,, (b\chi)^{Weyl} v^{\varepsilon}\right\rangle\right| + \left|\left\langle u^{\varepsilon}\,, \left(b(1-\chi)\right)^{Weyl} v^{\varepsilon}\right\rangle\right| \\ &\leq \left|\left(\bar{b}(1-\chi)\right)^{Weyl} u^{\varepsilon}\right|_{\mathscr{H}} + \left|(b\chi)^{Weyl} v^{\varepsilon}\right|_{\mathscr{H}}\,. \end{aligned}$$

From the Weyl pseudodifferential calculus we get

$$\left|\left(\bar{b}(1-\chi)\right)^{Weyl} u^{\varepsilon}\right|^2_{\mathscr{H}} \leq \mathrm{Tr}\left[\varrho^{\varepsilon}_{uu}\left((1-\chi)^2|b|^2\right)^{Weyl}\right] + C_{b\chi}$$

where the right-hand side converges to $\int_{\mathscr{Z}} |b|^2(1-\chi)^2(z)\; d\mu(z)$ as $\varepsilon \to 0$. The property $\chi \equiv 1$ on K_1 with $\mu(K_1) \geq 1-\eta$ implies

$$\limsup_{\varepsilon\to 0} \left|\left(\bar{b}(1-\chi)\right)^{Weyl} u^{\varepsilon}\right|^2_{\mathscr{H}} \leq \eta\, |b|^2_{L^\infty}$$

and with the symmetric argument $\limsup_{\varepsilon\to 0} \left|(b\chi)^{Weyl} v^{\varepsilon}\right|^2_{\mathscr{H}} \leq \eta\, |b|^2_{L^\infty}$. Hence we get

$$\forall \eta > 0\,, \quad \limsup_{\varepsilon\to 0} \left|\left\langle u^{\varepsilon}\,, b^{Weyl} v^{\varepsilon}\right\rangle\right| \leq 2\, |b|_{L^\infty} \sqrt{\eta}$$

for any $b \in \mathscr{S}_{cyl}(\mathscr{Z})$. This implies $\mathscr{M}(\varrho^{\varepsilon}_{uv}, \varepsilon \in (0,\bar{\varepsilon})) = \{0\}$. □

A straightforward consequence is the next proposition.

Proposition 6.12. *Make the same assumptions as in Proposition* 6.11 *with the additional condition* $\mathscr{M}(\varrho^{\varepsilon}_{uu}) = \{\mu_u\}$ *and* $\mathscr{M}(\varrho^{\varepsilon}_{vv}) = \{\mu_v\}$. *Then the family of trace class operators* $(\varrho^{\varepsilon}_{u+v,u+v})_{\varepsilon\in(0,\bar{\varepsilon})}$ *satisfies*

$$\mathscr{M}(\varrho^{\varepsilon}_{u+v,u+v}) = \{\mu_u + \mu_v\}\,.$$

Proof. Write simply

$$\left\langle u^{\varepsilon}+v^{\varepsilon}\,,\,b^{Weyl}(u^{\varepsilon}+v^{\varepsilon})\right\rangle=\left\langle u^{\varepsilon}\,,\,b^{Weyl}u^{\varepsilon}\right\rangle+\left\langle v^{\varepsilon}\,,\,b^{Weyl}v^{\varepsilon}\right\rangle$$
$$+\left\langle u^{\varepsilon}\,,\,b^{Weyl}v^{\varepsilon}\right\rangle+\left\langle v^{\varepsilon}\,,\,b^{Weyl}u^{\varepsilon}\right\rangle\,,$$

and take the limit of every term as $\varepsilon\to 0$. □

6.5. Wigner measure and Wick observables

Up to some additional assumption on the state and by restricting the class of Wick observables, we check in this subsection that testing with Weyl, (or Anti-Wick) and Wick observables provides the same asymptotic information as $\varepsilon\to 0$.

Fix once and for all $p\in\mathbb{P}$, the choice of the metric $g_p=|dz|^2$ or $g_p=\frac{|dz|^2}{\langle z\rangle^2}$. From Proposition 3.8 we know that the class of symbols $\cup_{p\in\mathbb{P},s\in\mathbb{R}}S_{p\mathscr{Z}}(\langle z\rangle^s,g_p)$ and $\oplus^{\mathrm{alg}}_{m,q\in\mathbb{N}}\mathscr{P}_{m,q}(\mathscr{Z})$ both contain all the classes $\mathscr{P}_{m,q}(p\mathscr{Z})$, with a good comparison of Weyl and Wick quantizations on these smaller sets. In the limit $\varepsilon\to 0$, this comparison can be carried out to any $b\in\oplus^{\mathrm{alg}}_{m,q\in\mathbb{N}}\mathscr{P}^{\infty}_{m,q}(\mathscr{Z})$.

Theorem 6.13. *Assume that the family of operators $(\varrho^{\varepsilon})_{\varepsilon\in(0,\overline{\varepsilon})}$ satisfies*

$$\left|(1+N)^{\delta/2}\varrho^{\varepsilon}(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\leq C_{\delta}$$

uniformly w.r.t $\varepsilon\in(0,\overline{\varepsilon})$ for any $\delta>0$.

1. *For any fixed $\beta\in\cup_{p\in\mathbb{P},s\in\mathbb{R}}S_{p\mathscr{Z}}(\langle z\rangle^s\,,g_p)$, the families $(\beta^{Weyl}\varrho^{\varepsilon})_{\varepsilon\in(0,\overline{\varepsilon})}$ and $(\beta^{A-Wick}\varrho^{\varepsilon})_{\varepsilon\in(0,\overline{\varepsilon})}$ satisfy the assumptions of Definition* 6.5 *and*
$$\mathscr{M}(\beta^{Weyl}\varrho^{\varepsilon})=\mathscr{M}(\beta^{A-Wick}\varrho^{\varepsilon})=\left\{\beta\mu\,,\ \mu\in\mathscr{M}(\varrho^{\varepsilon})\right\}\tag{62}$$
2. *For any fixed $\beta\in\oplus^{\mathrm{alg}}_{m,q\in\mathbb{N}}\mathscr{P}^{\infty}_{m,q}(\mathscr{Z})$ the family $(\beta^{Wick}\varrho^{\varepsilon})_{\varepsilon\in(0,\overline{\varepsilon})}$ satisfies the assumptions of Definition* 6.5 *and*
$$\mathscr{M}(\beta^{Wick}\varrho^{\varepsilon})=\left\{\beta\mu\,,\ \mu\in\mathscr{M}(\varrho^{\varepsilon})\right\}.\tag{63}$$

A particular case holds when the measure is tested with $b=1$.

Corollary 6.14. *Assume the uniform estimate $\left|(1+N)^{\delta/2}\varrho^{\varepsilon}(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\leq C_{\delta}$ for all $\delta>0$ and further $\mathscr{M}(\varrho^{\varepsilon})=\{\mu\}$.*

1. *The equality*
$$\lim_{\varepsilon\to 0}Tr\left[\beta^{Weyl}\varrho^{\varepsilon}\right]=\lim_{\varepsilon\to 0}Tr\left[\beta^{A-Wick}\varrho^{\varepsilon}\right]=\int_{\mathscr{Z}}\beta(z)\ d\mu(z)$$
holds when $\beta\in\cup_{p\in\mathbb{P},s\in\mathbb{R}}S_{p\mathscr{Z}}(\langle z\rangle^s,g_p)$
2. *The limit*
$$\lim_{\varepsilon\to 0}Tr\left[\beta^{Wick}\varrho^{\varepsilon}\right]=\int_{\mathscr{Z}}\beta(z)\ d\mu(z)$$

holds for any $\beta\in\oplus^{\mathrm{alg}}_{m,q\in\mathbb{N}}\mathscr{P}^{\infty}_{m,q}(\mathscr{Z})$.

Proof of Theorem 6.13. 1) The relation (29) extends to any $b \in S_{p\mathscr{Z}}(\langle z\rangle^s, g_p)$ and implies $\varepsilon^{-1}(b^{Weyl} - b^{A-Wick}) = c(\varepsilon)^{Weyl}$ with $c(\varepsilon)$ uniformly bounded in $S_{p\mathscr{Z}}(\langle z\rangle^{s-2}, g_p)$. The result for β^{A-Wick} can be deduced from the one for β^{Weyl}.

Take $p \in \mathbb{P}$, $s \geq 0$ (this contains the case $s < 0$) and $\beta \in S_{p\mathscr{Z}}(\langle z\rangle^s, g_p)$. Let $N_p = N_{p\mathscr{Z}} \otimes I_{\Gamma_s(p^\perp \mathscr{Z})}$ and $N_{p^\perp} = I_{\Gamma_s(p\mathscr{Z})} \otimes N_{p^\perp \mathscr{Z}}$. Our assumption on $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ and the commutations $[N_{p^\perp}, N_p] = [N_{p^\perp}, \beta^{Weyl}] = 0$ imply for any $\delta > 0$

$$(1+N)^{\delta/2}\beta^{Weyl}\varrho^\varepsilon(1+N)^{\delta/2} = ABA'RC \quad \text{with}$$

$$\begin{aligned}
A &= (1+N)^{\delta/2}(1+N_p)^{-\delta/2}(1+N_{p^\perp})^{-\delta/2}\\
B &= (1+N_p)^{\delta/2}\beta^{Weyl}(1+N_p)^{-\delta/2-s}\\
A' &= (1+N_p)^{\delta/2+s}(1+N_{p^\perp})^{\delta/2}(1+N)^{-\delta-s}\\
R &= (1+N)^{\delta+s}\varrho^\varepsilon(1+N)^{\delta+s} \quad \text{and}\\
C &= (1+N)^{-\delta/2-s}\,.
\end{aligned}$$

The factors A, A' and C are uniformly bounded operators when $\delta > 0$ (and s) is fixed. The trace class norm of the factor R is uniformly bounded by $C_{\delta+s}$. Finally the Weyl pseudodifferential calculus on $p\mathscr{Z}$ implies that $B = \gamma^{Weyl}$ with $\gamma(\varepsilon)$ uniformly bounded in $S_{p\mathscr{Z}}(1, g_p)$ and therefore $|B|_{\mathscr{L}(\mathscr{H})} \leq C'_{\delta,s}$ uniformly w.r.t $\varepsilon \in (0, \bar{\varepsilon})$.

Hence the family $(\beta^{Weyl}\varrho^\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies the assumptions of Definition 6.5. Let μ_1 belong to $\mathscr{M}(\beta^{Weyl}\varrho^\varepsilon)$. After extracting the proper sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\varepsilon_n = 0$, one can assume

$$\lim_{n\to\infty} \mathrm{Tr}\left[b^{Weyl}\beta^{Weyl}\varrho^{\varepsilon_n}\right] = \int_{\mathscr{Z}} b(z)\ d\mu_1(z) \quad \text{and}$$

$$\lim_{n\to\infty} \mathrm{Tr}\left[b^{Weyl}\varrho^{\varepsilon_n}\right] = \int_{\mathscr{Z}} b(z)\ d\mu(z)$$

for any $b \in \mathscr{S}_{cyl}(\mathscr{Z})$. But the finite dimensional pseudodifferential calculus implies $b^{Weyl}\beta^{Weyl} = (b\beta)^{Weyl} + O_{\mathscr{L}(\mathscr{H})}(\varepsilon_n)$ with $b\beta \in \mathscr{S}_{cyl}(\mathscr{Z})$. This implies

$$\int_{\mathscr{Z}} b(z)\ d\mu_1(z) = \int_{\mathscr{Z}} b(z)\beta(z)\ d\mu(z)$$

for all $b \in \mathscr{S}_{cyl}(\mathscr{Z})$. According to Proposition 6.10 this implies $\mu_1 = \beta\mu$.

2) Since the $\cup_{p\in\mathbb{P},s\in\mathbb{R}} S_{p\mathscr{Z}}(\langle z\rangle^s, g_p)$ contains $\cup_{p\in\mathscr{P}}\left(\oplus^{\mathrm{alg}}_{m,q\in\mathbb{N}}\mathscr{P}_{m,q}(p\mathscr{Z})\right)$, the result is proved for any polynomial symbol $b \in \mathscr{P}^\infty_{m,q}(\mathscr{Z})$ such that $\tilde{b} = \Gamma(p)\tilde{b}\Gamma(p)$ for some finite dimensional projector $p \in \mathbb{P}$. Consider now a general $b \in \mathscr{P}^\infty_{m,q}(\mathscr{Z})$ with $m, q \in \mathbb{N}$. By Lemma 2.5, the operator

$$(1+N)^{\delta/2}b^{Wick}(1+N)^{-\delta/2-m/2-q/2}$$

is uniformly bounded for any $\delta > 0$. Since the trace class norm of $(1+N)^{\frac{\delta+m+q}{2}}\varrho^\varepsilon(1+N)^{\frac{\delta+m+q}{2}}$ is uniformly bounded w.r.t $\varepsilon \in (0, \bar{\varepsilon})$, the family $(\beta^{Wick}\varrho^\varepsilon)$ satisfies the assumptions of Definition 6.5. Introduce now an increasing sequence $(p_\ell)_{\ell\in\mathbb{N}}$ of $\mathbb{P}$ such that $\sup_{\ell\in\mathbb{N}} p_\ell = I$ and consider for $\ell \in \mathbb{N}$

$$\beta_\ell(z) = \beta(p_\ell z)\,, \quad \tilde{\beta}_\ell = p_\ell^{\otimes q} \circ \tilde{\beta} \circ p_\ell^{\otimes m}\,.$$

Since $\tilde{\beta}$ is a compact operator, the finite rank operator $\tilde{\beta}_\ell$ converges to $\tilde{\beta}$ in the norm topology in $\mathscr{L}(\bigvee^m \mathscr{Z}, \bigvee^q \mathscr{Z})$. The uniform estimates

$$\left|(\beta - \beta_\ell)^{Wick}(1+N)^{-m/2-q/2}\right|_{\mathscr{L}(\mathscr{H})} \leq C\left|\tilde{\beta} - \tilde{\ell}\right|_{\mathscr{L}(\bigvee^m \mathscr{Z}, \bigvee^q \mathscr{Z})}\,,$$
$$\left(1+|z|^2\right)^{m/2+q/2}\left(\,|\beta(z)| + |\beta_\ell(z)|\,\right) \leq C \quad \text{with} \quad \lim_{\ell\to\infty} \beta_\ell(z) = \beta(z)\,,$$

and the convergence

$$\forall b \in \mathscr{S}_{cyl}(\mathscr{Z})\,, \quad \lim_{n\to\infty} \mathrm{Tr}\left[b^{Weyl}\beta_\ell^{Wick}\varrho^{\varepsilon_n}\right] = \int_{\mathscr{Z}} b(z)\beta_\ell(z)\ d\mu(z)$$

after extracting a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, $\lim_{n\to\infty}\varepsilon_n = 0$, with $\int_{\mathscr{Z}}(1+|z|^2)^{m/2+q/2}\ d\mu(z) < +\infty$, lead to

$$\forall b \in \mathscr{S}_{cyl}(\mathscr{Z})\,, \quad \lim_{n\to\infty} \mathrm{Tr}\left[b^{Weyl}\beta^{Wick}\varrho^{\varepsilon_n}\right] = \int_{\mathscr{Z}} b(z)\beta(z)\ d\mu(z)\,. \qquad \square$$

The previous results provide the behaviour of $\lim_{\varepsilon\to 0}\mathrm{Tr}\left[\beta^{Wick}\varrho^\varepsilon\right]$ for $\beta \in \oplus^{\mathrm{alg}}_{m,q\in\mathbb{N}}\mathscr{P}^\infty_{m,q}(\mathscr{Z})$ when $\mathscr{M}(\varrho^\varepsilon) = \{\mu\}$. The next result checks the other way.

Proposition 6.15. *Assume that $(\rho^\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ is a family of normal states satisfying for any $C > 0$ there exist $K_C > 0$ such that*

$$\sum_{k=0}^{\infty} \frac{C^k}{[k/2]!}\ Tr[N^k\rho^\varepsilon] \leq K_C < \infty$$

holds uniformly w.r.t $\varepsilon \in (0,\overline{\varepsilon})$. Assume that there exists a Borel measure μ such that

$$\lim_{\varepsilon\to 0} Tr\left[b^{Wick}\varrho^\varepsilon\right] = \int_{\mathscr{Z}} b(z)\ d\mu(z)$$

holds for any $b \in \oplus^{\mathrm{alg}}_{m,q}\mathscr{P}^\infty_{m,q}(\mathscr{Z})$. This implies

$$\mathscr{M}(\varrho^\varepsilon) = \{\mu\}\,.$$

Remark 6.16. A similar result for non self-adjoint trace-class operators with complex valued measure can be obtained by replacing the quantities $\mathrm{Tr}[N^k\varrho^\varepsilon]$ with $\mathrm{Tr}[N^k|\varrho^\varepsilon + (\varrho^\varepsilon)^*|] + \mathrm{Tr}[N^k|\varrho^\varepsilon - (\varrho^\varepsilon)^*|]$ like in (61).

Proof. It is enough to prove the following statement:

$$\lim_{\varepsilon\to 0}\mathrm{Tr}\big[W(\xi)\rho^\varepsilon\big] = \int_{\mathscr{Z}} e^{\sqrt{2}iS(\xi,z)}d\mu\,.$$

It is done when the right-hand side of

$$\operatorname{Tr}\big[W(\xi)\rho^{\varepsilon}\big] = \sum_{n=0}^{\infty} \frac{|\sqrt{\varepsilon}\xi|^n}{2^n n!} \ \operatorname{Tr}\left[h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} \rho^{\varepsilon}\right] \tag{64}$$

is proved to be an absolutely convergent series, uniformly w.r.t. $\varepsilon \in (0,\bar{\varepsilon})$. With

$$\begin{aligned}\operatorname{Tr}\big[W(\xi)\rho^{\varepsilon}\big] &= \lim_{M\to\infty} \operatorname{Tr}\big[W(\xi)1_{[0,M]}(N)\,\rho^{\varepsilon}\big] \\ &= \lim_{M\to\infty} \sum_{n=0}^{\infty} \frac{|\sqrt{\varepsilon}\xi|^n}{2^n n!} \ \operatorname{Tr}\left[h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} 1_{[0,M]}(N)\,\rho^{\varepsilon}\right]\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\left|\operatorname{Tr}\left[h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} 1_{[0,M]}(N)\,\rho^{\varepsilon}\right]\right| \\ &\qquad \le M_n \left|(N+1)^{-n/2}\, h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} (N+1)^{-n/2}\right|_{\mathscr{L}(\mathscr{H})},\end{aligned}$$

with $M_n = \operatorname{Tr}\left[(1+N)^n \varrho^{\varepsilon}\right]$, Lemma 3.4 implies

$$\begin{aligned}&\left|(N+1)^{-n/2}\, h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} (N+1)^{-n/2}\right|_{\mathscr{L}(\mathscr{H})} \\ &\qquad \le \sup_{k,j\in\mathbb{N}} \frac{(1+2\sqrt{2(k+j)\varepsilon})^n}{(k\varepsilon+1)^{n/2}(j\varepsilon+1)^{n/2}} \ \frac{n!}{[n/2]!} \\ &\qquad \le 8^n \frac{n!}{[n/2]!}\,.\end{aligned}$$

This leads to

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{|\sqrt{\varepsilon}\xi|^n}{2^n n!} \left|\operatorname{Tr}\left[h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} 1_{[0,M]}(N)\,\rho^{\varepsilon}\right]\right| &\le \sum_{n=0}^{\infty} \frac{(4\sqrt{\bar{\varepsilon}}|\xi|)^n}{[n/2]!} M_n \\ &< \infty\end{aligned} \tag{66}$$

uniformly w.r.t. $\varepsilon \in (0,\bar{\varepsilon})$ and $M > 0$. Hence we can take the limit $M \to \infty$ inside in all the terms of (65). This leads to (64) with a uniformly absolutely convergent series in the right-hand side according to (66) and our initial assumption.

Thus the sum and the limit as $\varepsilon \to 0$ can be interchanged in (64):

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathrm{Tr}\big[W(\xi)\rho^\varepsilon\big] &= \sum_{n=0}^{\infty} \frac{|\xi|^n}{2^n n!} \lim_{\varepsilon\to 0} \mathrm{Tr}\left[\sqrt{\varepsilon^n}\, h_n\left(\frac{i\sqrt{2}S(\xi,z)}{|\sqrt{\varepsilon}\xi|}\right)^{Wick} \rho^\varepsilon\right] \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathscr{Z}} \big(i\sqrt{2}S(\xi,z)\big)^n \, d\mu \\
&= \int_{\mathscr{Z}} e^{\sqrt{2}iS(\xi,z)} d\mu\,.
\end{aligned}$$

The last equality follows owing to the dominated convergence theorem and

$$\int_{\mathscr{Z}} e^{\delta|1_{p\mathscr{Z}} z|^2} d\mu = \lim_{\varepsilon\to 0} \sum_{k=0}^{\infty} \frac{\delta^k}{k!} \mathrm{Tr}\big[\rho^\varepsilon\, d\Gamma(1_{p\mathscr{Z}})^k\big] < \infty\,,$$

for any $\delta > 0$ and any $p \in \mathbb{P}$. This completes the proof. □

7. Examples and applications of Wigner measures

7.1. Finite dimensional cases

The first examples are given by Theorem 4.2

1. For any $z \in \mathscr{Z}$ the family of operators $\varrho^\varepsilon = |E(z)\rangle\langle E(z)|$ has a unique Wigner measure
$$\mathscr{M}\big(|E(z)\rangle\langle E(z)|\,, \quad \varepsilon \in (0,\bar{\varepsilon})\big) = \{\delta_z\}\,.$$
2. For any $z \in \mathscr{Z}$ and any $m \in \mathscr{Z}$ the family of operators $\varrho^\varepsilon = |z^{\otimes k_\varepsilon - m}\rangle\langle z^{\otimes k_\varepsilon}|$ with $|z| = 1$ and $\lim_{\varepsilon\to 0} \varepsilon k_\varepsilon = 1$ has a unique Wigner measure
$$\mathscr{M}\big(|z^{\otimes k_\varepsilon - m}\rangle\langle z^{\otimes k_\varepsilon}|\,, \quad \varepsilon \in (0,\bar{\varepsilon})\big) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\theta}\delta_{e^{i\theta}z}\, d\theta\,.$$
3. In case 1) and 2) the convergence can be tested with Weyl, Anti-Wick or Wick observables according to Proposition 6.4 and Theorem 6.13.

Beside the explicit calculation of Theorem 4.2 these results can be considered through an inductive approach since $E(z)$ or $z^{\otimes n}$ lie in $\Gamma_s(\mathbb{C}z)$. The natural extension comes from Proposition 6.10-1) with a proper choice of the first term in the increasing sequence $(p_\ell)_{\ell\in\mathbb{N}}$.

Proposition 7.1. *Assume that the family $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies the assumptions of Definition 6.5. Assume further that there exists a finite dimensional space $p_0 \in \mathbb{P}$ such that*

$$\varrho^\varepsilon = \Gamma(p_0)\varrho\Gamma(p_0) = \varrho^\varepsilon_{p_0} \otimes |\Omega\rangle\langle\Omega|$$

for all $\varepsilon \in (0,\bar{\varepsilon})$ with $\varrho^\varepsilon_{p_0} \in \mathscr{L}^1(\Gamma_s(p_0\mathscr{Z}))$. Then the Wigner measures of $(\varrho^\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ are given by

$$\mathscr{M}\left(\varrho^\varepsilon\right) = \left\{\mu_1 \otimes \delta_{0,p_0^\perp\mathscr{Z}}\,, \quad \mu_1 \in \mathscr{M}(\varrho^\varepsilon_{p_0})\right\}.$$

7.2. Superpositions

Two kinds of superpositions can be considered: 1) convex or linear combination of trace class operators; 2) convex or linear combination of wave functions. The first one is the simplest.

Proposition 7.2.

1. *Let (M,π) be a probability space. Let $(\varrho^\varepsilon(m))_{\varepsilon\in(0,\bar\varepsilon),m\in M}$ be a family of operators such that*
$$\left|(1+N)^{\delta/2}\varrho^\varepsilon(m)(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\leq C_\delta(m)$$
for π-almost every $m\in M$ with $C_\delta\in L^1(M,d\pi)$ for some $\delta>0$. Assume further $\mathscr{M}(\varrho^\varepsilon(m),\ \varepsilon\in(0,\bar\varepsilon))=\{\mu(m)\}$ for π-almost every $m\in M$, then the family $(\int_M\varrho^\varepsilon(m)\ d\pi(m))_{\varepsilon\in\in(0,\bar\varepsilon)}$ satisfies the assumptions of Definition 6.5 and
$$\mathscr{M}\left(\int_M\varrho^\varepsilon(m)\ d\pi(m)\,,\quad \varepsilon\in(0,\bar\varepsilon)\right)=\left\{\int_M\mu(m)\ d\pi(m)\right\}.$$
2. *Any bounded Borel measure on $\mathscr{Z}$ can be achieved as a Wigner measure.*

Proof. 1) Set $\varrho^\varepsilon=\int_M\varrho^\varepsilon(m)\ d\pi(m)$ and write
$$\left|(1+N)^{\delta/2}\varrho^\varepsilon(1+N)^{\delta/2}\right|_{\mathscr{L}^1(\mathscr{H})}\leq\int_M C_\delta(m)\ d\pi(m)\,.$$
Then apply Lebesgue's convergence theorem to
$$\mathrm{Tr}\left[b^{Weyl}\varrho^\varepsilon\right]=\int_M\mathrm{Tr}\left[b^{Weyl}\varrho^\varepsilon(m)\right]\ d\pi(m)\,.$$

2) After reducing the problem to the case when μ is a Borel probability measure on $\mathscr{Z}$, apply 1) with $M=\mathscr{Z}$, $\pi=\mu$, $m=z$ and $\varrho^\varepsilon(z)=|E(z)\rangle\langle E(z)|$. □

The second type of superposition requires an orthogonality property. It is given by Proposition 6.12. Here are a few examples

1. Take $u_\ell^\varepsilon=E(z_\ell)$ for $\ell=1,\ldots,L$, with $L\in\mathbb{N}$ fixed, and set $u^\varepsilon=L^{-1/2}\sum_{\ell=1}^L u_\ell^\varepsilon$. When the z_ℓ are distinct, the family $(|u^\varepsilon\rangle\langle u^\varepsilon|)_{\varepsilon\in(0,\bar\varepsilon)}$ has a unique Wigner measure
$$\mathscr{M}(|u^\varepsilon\rangle\langle u^\varepsilon|)=\left\{L^{-1}\sum_{\ell=1}^L\delta_{z_\ell}\right\}.$$
2. Take for any $\ell\in\{1,\ldots,L\}$, $u_\ell^\varepsilon=z_\ell^{\otimes k_\varepsilon}$ with $|z_\ell|=1$ and $\lim_{\varepsilon\to0}\varepsilon k_\varepsilon=1$. The family $(|u^\varepsilon\rangle\langle u^\varepsilon|)_{\varepsilon\in(0,\bar\varepsilon)}$ has a unique Wigner measure:
$$\mathscr{M}(|u^\varepsilon\rangle\langle u^\varepsilon|)=\left\{(2\pi L)^{-1}\sum_{\ell=1}^L\int_0^{2\pi}\delta_{e^{i\theta}z_\ell}\ d\theta\right\}.$$

3. For $z \in \mathscr{Z}$ and $u^\varepsilon = \frac{E(z)+|z^{\otimes k_\varepsilon}\rangle}{\sqrt{2}}$ with $|z|=1$ and $\lim_{\varepsilon\to 0}\varepsilon k_\varepsilon = 1$, the family $(|u^\varepsilon\rangle\langle u^\varepsilon|)_{\varepsilon\in(0,\bar\varepsilon)}$ has a unique Wigner measure:
$$\mathscr{M}(|u^\varepsilon\rangle\langle u^\varepsilon|) = \left\{\frac{1}{2}\delta_z + \frac{1}{4\pi}\int_0^{2\pi}\delta_{e^{i\theta}z}\,d\theta\right\}.$$
4. All this examples can be tested with Weyl, Anti-Wick or Wick observables according to Proposition 6.4 and Theorem 6.13.

7.3. Propagation of chaos and propagation of (squeezed) coherent states

Let us go back to the example of Section 5 where $U_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon}$ with $H_\varepsilon = d\Gamma(-\Delta)+Q^{Wick}$, $\tilde{Q} = \frac{1}{2}V(x_1-x_2)$ and z_t solution to $i\partial_t z_t = -\Delta z + (V*|z_t|^2)z_t$. Theorem 5.6, Proposition 5.10 and Proposition 6.15 imply:

1. For any $z_0\in\mathscr{Z}$ with $|z_0|=1$, the family $(|U_\varepsilon(t)z_0^{\otimes k_\varepsilon}\rangle\langle U_\varepsilon(t)z_0^{\otimes k_\varepsilon}|)_{\varepsilon\in(0,\bar\varepsilon)}$ with $\lim_{\varepsilon\to 0}\varepsilon k_\varepsilon = 1$ is pure with
$$\mathscr{M}\left(|U_\varepsilon(t)z_0^{\otimes k_\varepsilon}\rangle\langle U_\varepsilon(t)z_0^{\otimes k_\varepsilon}|\right) = \left\{\frac{1}{2\pi}\int_0^{2\pi}\delta_{e^{i\theta}z_t}\,d\theta\right\} = \mathscr{M}\left(|z_t^{\otimes k_\varepsilon}\rangle\langle z_t^{\otimes k_\varepsilon}|\right)$$
2. For any $z_0\in\mathscr{Z}$, the family $(|U_\varepsilon(t)E(z_0)\rangle\langle U_\varepsilon(t)E(z_0)|)_{\varepsilon\in(0,\bar\varepsilon)}$ is pure with
$$\mathscr{M}\big(|U_\varepsilon(t)E(z_0)\rangle\langle U_\varepsilon(t)E(z_0)|\big) = \{\delta_{z_t}\} = \mathscr{M}\big(|E(z_t)\rangle\langle E(z_t)|\big)\,.$$

These results are derived from the results for product states after testing with Wick observable (any $b\in\oplus^{\mathrm{alg}}_{m,q}\mathscr{P}_{m,q}(\mathscr{Z})$). Actually it is possible to recover the second one directly from the Hepp method. For any $b\in\mathscr{S}_{cyl}(\mathscr{Z})$, Proposition 5.2 implies
$$\lim_{\varepsilon\to 0}\mathrm{Tr}\Bigg[b^{Weyl}\Bigg(|U_\varepsilon(t)E(z_0)\rangle\langle U_\varepsilon(t)E(z_0)| - \left|W\left(\frac{\sqrt{2}}{i\varepsilon}z_t\right)U_2(t,0)\Omega\right\rangle\left\langle W\left(\frac{\sqrt{2}}{i\varepsilon}z_t\right)U_2(t,0)\Omega\right|\Bigg)\Bigg] = 0\,.$$
By the finite dimensional Weyl quantization, the second term equals
$$\left\langle U_2(t,0)\Omega\,, b(.-z_t)^{Weyl}U_2(t,0)\Omega\right\rangle$$
and it suffices to check that the family $(|U_2(t,0)\Omega\rangle\langle U_2(t,0)\Omega|)_{\varepsilon\in(0,\bar\varepsilon)}$ admits the unique Wigner measure δ_0. This is a consequence of Lemma 5.3 which first says $|N^k U_2(t,0)\Omega|_{\mathscr{H}}\le C_k$ for any $k\ge 0$ and then $\lim_{\varepsilon\to 0}\langle U_2(t,0)\Omega\,, b^{Wick}U_2(t,0)\Omega\rangle = 0$ when $b(0)=0$.

7.4. Dimensional defect of compactness

In the last example the mean field propagation of Wigner measure attached with $U_\varepsilon(t)E(z_0)$ can be proved directly without using the result on Wick observables. As a corollary, this provides the result for Wick observables b^{Wick} when $b\in\oplus^{\mathrm{alg}}_{m,q}\mathscr{P}^\infty_{m,q}(\mathscr{Z})$ according to Theorem 6.13. The result for a general $b\in\oplus^{\mathrm{alg}}_{m,q}\mathscr{P}_{m,q}(\mathscr{Z})$ is still true but comes from a direct proof or from Proposition 5.10.

A natural question is whether the result of Theorem 6.13 can be extended to any observable b^{Wick} with $b \in \oplus^{\mathrm{alg}}_{m,q}\mathscr{P}_{m,q}(\mathscr{Z})$. The answer is no, because in the infinite dimensional case there can be some defect of compactness w.r.t to the dimension variable.

Here is a typical example. Consider a family $(z_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ such that z_ε converges weakly to 0. There exists a constant $C>0$ such that $|z_\varepsilon|\le C$ for all $\varepsilon\in(0,\bar\varepsilon)$ and the family $(E(z_\varepsilon))_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies the assumptions of Proposition 6.15. The Wigner measures $\mu\in\mathscr{M}(|E(z_\varepsilon\rangle\langle E(z_\varepsilon)|))$ are determined by testing on any $b\in\mathscr{P}^\infty_{m,q}(\mathscr{Z})$. But Theorem 4.2 says

$$\left\langle E(z_\varepsilon)\,, b^{Wick}E(z_\varepsilon)\right\rangle = b(z_\varepsilon) = \langle z_\varepsilon^{\otimes q}\,, \tilde b z_\varepsilon^{\otimes m}\rangle\,.$$

When $m+q\ge 1$ the operator $\tilde b$ is compact, the right-hand side converges to 0 as $\varepsilon\to 0$. According to Proposition 6.15 this implies

$$\mathscr{M}\left(|E(z_\varepsilon)\rangle\langle E(z_\varepsilon)|\right) = \{\delta_0\}\,.$$

Meanwhile testing with $N = d\Gamma(I) = \left(|z|^2\right)^{Wick}$ implies

$$\left\langle E(z_\varepsilon)\,, NE(z_\varepsilon)\right\rangle = |z_\varepsilon|^2$$

where the right-hand side can reach any possible limit in $[0,C]$.

7.5. Bose–Einstein condensates

The thermodynamic limit of the ideal Bose Gas presented within a local algebra presentation in [9] can be reconsidered by introducing a small parameter $\varepsilon\to 0$. Namely, the large domain limit where bosonic particles are moving freely in a domain Λ, with volume $|\Lambda|\to\infty$, can be formulated with $|\Lambda| = \frac{1}{\varepsilon}$ and $\varepsilon\to 0$. For a fixed particle density the total number of particle is $O(\frac{1}{\varepsilon})$ coherent with a mean field approach. Before considering any dynamical problem, Wigner measures of ε-dependent Gibbs states bring some interesting presentation of the Bose–Einstein condensation.

Consider the Laplace operator $H_0 = -\Delta_x$ on the ε-dependent torus $\mathbb{R}^d/(\varepsilon^{-1/d}\mathbb{Z})^d$ with spectrum $\sigma(H_0) = \left\{\varepsilon^{2/d}|2\pi n|, n\in\mathbb{Z}^d\right\}$. The one particle space is $\mathscr{Z}^\varepsilon = L^2(\mathbb{R}^d/(\varepsilon^{-1/d}\mathbb{Z})^d)$ and the bosonic Fock space is $\mathscr{H}^\varepsilon = \Gamma_s(\mathscr{Z}^\varepsilon)$. For the inverse temperature $\beta = \frac{1}{k_BT} > 0$ and a chemical potential μ, the Gibbs grand canonical equilibrium state is associated with the operator $e^{-\beta d\Gamma(H_0-\mu I)} = \Gamma(e^{-\beta(H_0-\mu I)})$, which is trace class if and only if $\mu<0$ (see [9, Proposition 5.2.27]). This Gibbs state on $\mathscr{L}(\mathscr{H}^\varepsilon)$ is given by

$$\omega_\varepsilon(A) = \mathrm{Tr}\,[\varrho_\varepsilon A] \quad \text{with} \quad \varrho_\varepsilon = \frac{1}{\mathrm{Tr}[\Gamma(e^{-\beta(H_0-\mu)})]}\Gamma(e^{-\beta(H_0-\mu)})\,, \quad \mu<0\,.$$

It is convenient to introduce the parameter $z = e^{\beta\mu}$ and this Gibbs state restricted to the CCR-algebra (the C^*-algebra generated by the Weyl operators $W_1(f)$, $f\in\mathscr{Z}^\varepsilon$) is the gauge-invariant quasi-free state given by the two-point function: $\omega_\varepsilon(a_1^*(f)a_1(g)) = \left\langle g\,, ze^{-\beta H_0}(1-ze^{-\beta H_0})^{-1}f\right\rangle$. The index $_1$ means that the CCR are written at this level in their initial form: $[a_1(g), a_1^*(f)] = \langle g\,,\, f\rangle$. This is proved in [9, Proposition 5.2.28] with the straightforward rewriting

$$\omega_\varepsilon\big(W_1(f)\big) = \exp\Big[-\big\langle f\,,\,(1+ze^{-\beta H_0})(1-ze^{-\beta H_0})^{-1}f\big\rangle/4\Big]\,.$$

The mean field analysis consists here in introducing $a(f) = \varepsilon^{1/2}a_1(f)$ and $W(f) = W_1(\varepsilon^{1/2}f)$:

$$\omega_\varepsilon\big(a^*(f)a(g)\big) = \varepsilon\big\langle g\,,\, ze^{-\beta H_0}(1-ze^{-\beta H_0})^{-1}f\big\rangle$$
$$\omega_\varepsilon\big(W(f)\big) = \exp\Big[-\varepsilon\big\langle f\,,\,(1+ze^{-\beta H_0})(1-ze^{-\beta H_0})^{-1}f\big\rangle/4\Big]\,.$$

Further a rescaling motivated by the observation of the phenomena on a large scale, is implemented with $f(x) = \varepsilon^{1/2}\varphi(\varepsilon^{1/d}x) = D_\varepsilon\varphi$. After conjugating with the unitary transform $\Gamma(D_\varepsilon) : \mathscr{H} = \Gamma_s(\mathscr{Z}) \to \mathscr{H}^\varepsilon = \Gamma_s(\mathscr{Z}^\varepsilon)$, with $\mathscr{Z} = L^2(\mathbb{R}^d/\mathbb{Z}^d)$ we are led to consider the asymptotic behaviour as $\varepsilon \to 0$ of the normal state

$$\varrho^\varepsilon = \Gamma(D_\varepsilon)^*\varrho_\varepsilon\Gamma(D_\varepsilon) = \frac{1}{\mathrm{Tr}[\Gamma(e^{-\beta(-\varepsilon^{2/d}\Delta-\mu)})]}\Gamma(e^{-\beta(-\varepsilon^{2/d}\Delta-\mu)})$$

which satisfies

$$\begin{aligned}\mathrm{Tr}\big[\varrho^\varepsilon W(f)\big] &= \exp\Big[-\frac{\varepsilon}{4}\big\langle f\,,\,(1+ze^{\beta\varepsilon^{2/d}\Delta})(1-ze^{\beta\varepsilon^{2/d}\Delta})^{-1}f\big\rangle_{\mathscr{Z}}\Big]\\ &= e^{-\frac{\varepsilon}{4}|f|^2_{\mathscr{Z}}}\exp\Big[-\frac{\varepsilon}{2}\big\langle f\,,\, ze^{\beta\varepsilon^{2/d}\Delta}(1-ze^{\beta\varepsilon^{2/d}\Delta})^{-1}f\big\rangle_{\mathscr{Z}}\Big]\\ \mathrm{Tr}\big[\varrho^\varepsilon a^*(f)a(g)\big] &= \varepsilon\big\langle g\,,\, ze^{\beta\varepsilon^{2/d}\Delta}(1-ze^{\beta\varepsilon^{2/d}\Delta})^{-1}f\big\rangle_{\mathscr{Z}}\,.\end{aligned}$$

The above expressions are explicit after the decomposition in the Fourier basis $f = \sum_{n\in\mathbb{Z}^d} f_n e^{2i\pi n.z}$ of any element $f \in \mathscr{Z}$. For a given $z < 1$ and $\beta > 0$ the rescaled particle density is given by

$$\frac{\varepsilon z}{1-z} + \varepsilon\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\frac{ze^{-\beta\varepsilon^{2/d}|2\pi n|^2}}{(1-ze^{-\beta\varepsilon^{2/d}|2\pi n|^2})} = \frac{\varepsilon z}{1-z} + \nu_\varepsilon(\beta, z)\,. \tag{67}$$

One checks easily for $\varepsilon' \geq \varepsilon$ and $z' \leq z < 1$

$$\nu_{\varepsilon'}(\beta,z) \leq \nu_\varepsilon(\beta,z) \overset{\varepsilon\to 0}{\to} \nu_0(\beta,z) = \int_{\mathbb{R}^d}\frac{ze^{-\beta|2\pi u|^2}}{1-ze^{-\beta|2\pi u|^2}}\,du$$
$$\text{and}\quad \forall\varepsilon\in[0,1)\,,\quad \nu_\varepsilon(\beta,z) \geq \nu_\varepsilon(\beta,z')\,.$$

Here comes the discussion about the Bose–Einstein condensation. In dimension $d \geq 3$ (this restriction may change with an alternative Hamiltonian $H_0 = \lambda(D_x)$), the quantity

$$\nu_0(\beta,1) = \int_{\mathbb{R}^d}\frac{e^{-\beta|2\pi u|^2}}{1-e^{-\beta|2\pi u|^2}}\,du < +\infty\,.$$

is well defined.

We focus on the case $d \geq 3$.

The previous discussion imply

$$\forall \varepsilon > 0\,, \quad \forall z \in (0,1)\,, \quad \nu_\varepsilon(\beta, z) \le \nu_0(\beta, 1)$$

while any total density can be achieved by (67). The Bose–Einstein condensation occurs while considering the limit $\varepsilon \to 0$ with the constraint $\frac{z_\varepsilon \varepsilon}{1-z_\varepsilon} + \nu_\varepsilon(z_\varepsilon, \beta) = \nu$ with $\beta > 0$ and $\nu > 0$ fixed. There are two possible cases:

- $\nu \le \nu_0(\beta, 1)$: Then $\lim_{\varepsilon\to 0} z_\varepsilon = z < 1$ and $\lim_{\varepsilon \to 0} \frac{\varepsilon z_\varepsilon}{1-z_\varepsilon} = 0$.
- $\nu > \nu_0(\beta, 1)$: The inequality $\nu - \nu_0(\beta, 1) \le \frac{\varepsilon z_\varepsilon}{1-\varepsilon} \le \nu$ leads to $z_\varepsilon = 1 - \frac{\varepsilon}{\nu - \nu_0(\beta,1)} + o(\varepsilon)$. The proportion $1 - \nu_0(\beta, 1)/\nu$ of the gas lies in the ground state $n = 0$ of the one-body Hamiltonian. This is the Bose–Einstein condensation phenomenon.

It is interesting to reconsider this limit $\varepsilon \to 0$ with $\beta > 0$ and $\nu > 0$ fixed ($d \ge 3$) within the Wigner measure point of view. This is possible owing to the explicit formula

$$\mathrm{Tr}\left[\varrho^\varepsilon W(\sqrt{2}\pi f)\right] = e^{-\varepsilon\pi^2 |f|^2_{\mathscr{Z}}} \exp\left[-\varepsilon\pi^2 \sum_{n\in\mathbb{Z}^d} |f_n|^2 \frac{z_\varepsilon e^{-\beta\varepsilon^{2/d}|2\pi n|^2}}{(1 - z_\varepsilon e^{-\beta\varepsilon^{2/d}|2\pi n|^2})}\right], \quad (68)$$

where $f = \sum_{n\in\mathbb{Z}^d} f_n e^{2i\pi n.x}$. Remember that the characteristic function of Wigner measures are determined after considering the limit $\varepsilon \to 0$ of the above expression for any <u>fixed</u> $f \in \mathscr{Z}$. Hence the problem is reduced to the application of Lebesgue's theorem in the argument of the exponential.

For any $n \ne 0$ the quantity $\frac{z_\varepsilon e^{-\beta\varepsilon^{2/d}|2\pi n|^2}}{(1-z_\varepsilon e^{-\beta\varepsilon^{2/d}|2\pi n|^2})}$ converges to 0 as $\varepsilon \to 0$ because $d/2 < 1$ and $z_\varepsilon \le 1$. Hence we get

$$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho^\varepsilon W(\sqrt{2}\pi f)\right] = \lim_{\varepsilon\to 0} \exp\left[-\frac{\varepsilon\pi^2 z_\varepsilon}{1 - z_\varepsilon} |f_0|^2\right].$$

With the constraint $\frac{\varepsilon z_\varepsilon}{1-z_\varepsilon} \le \nu < +\infty$, there are two possibilities

- First $\lim_{\varepsilon\to 0} \frac{\varepsilon z_\varepsilon}{1-z_\varepsilon} = 0$ implies $\nu \le \nu_0(\beta, 1)$ and $\mathscr{M}(\varrho^\varepsilon) = \{\delta_0\}$.
- The second case $\lim_{\varepsilon\to 0} \frac{\varepsilon z_\varepsilon}{1-z_\varepsilon} = \nu - \nu_0(\beta, 1) > 0$ implies

  $$\lim_{\varepsilon\to 0} \mathrm{Tr}\left[\varrho^\varepsilon W(\sqrt{2}\pi f)\right] = e^{-\pi^2(\nu-\nu_0(\beta,1))|f_0|^2} = e^{-\pi^2(\nu-\nu_0(\beta,1))|\langle f,1\rangle|^2}.$$

  Hence the Wigner measure of the family $(\varrho^\varepsilon)_{\varepsilon>0}$ equals $\gamma_\nu \otimes \delta_0$ on $\mathscr{Z} = \mathbb{C}1 \times \{1\}^\perp$ where γ_ν is the gaussian measure

  $$\gamma_\nu(z_1) = \frac{e^{-\frac{|z_1|^2}{\nu-\nu_0(\beta,1)}}}{(\pi(\nu - \nu_0(\beta, 1))^{d/2}}, \quad z_1 \in \mathbb{C}.$$

Our scaled observables can measure asymptotically only the Bose–Einstein phase in a non trivial way. The rest of the state provides the factor δ_0. While testing with the observable $(|z|^2)^{Wick} = N$, the dimensional defect of compactness phenomenon already illustrated in Subsection 7.4 occurs again: only the density of the condensate remains.

Remark 7.3.

i) It is possible to consider various dispersion relations $H_0 = \lambda(D_x)$ and the discussion about the dimension may change. Other boundary conditions (here periodic boundary conditions are considered) and the discussion about the convergence of $\lim_{\varepsilon\to 0} z_\varepsilon = 1$ may change a little bit. We refer the reader to [9] for the case of Dirichlet boundary conditions.
ii) From (68) it is possible to consider the limit for any fixed $f \in \mathscr{Z}$ as $\varepsilon \to 0$ with various behaviours of z_ε. This provides asymptotically a weak distribution. But the uniform tightness assumption $\mathrm{Tr}\left[\varrho^\varepsilon(1+N)^\delta\right] \leq C$ is not satisfied. The scaling has to be adapted differently to the dimension $d = 2$ or $d = 1$ by taking care of the singularity at the momentum 0, in order to allow a non trivial Wigner measure in the thermodynamic and mean field limit.

7.6. Application 1: From the propagation of coherent states to the propagation of chaos via Wigner measures

In the previous sections we showed how the propagation of (squeezed) coherent states can be derived from the propagation of Hermite states or directly via the Hepp method. The Hepp method is very flexible (see [24] for example) and therefore it is interesting to know whether a result for coherent states provides an information for product states or more general states. Here is a simple and abstract result which relies on some gauge invariance argument.

Theorem 7.4. *Let U_ε be a unitary operator on $\mathscr{H}$ possibly depending on $\varepsilon \in (0, \bar{\varepsilon})$ which commutes with the number operator $[N, U_\varepsilon] = 0$. Assume that for a given $z \in \mathscr{Z}$ such that $|z| = 1$, there exists $z_U \in \mathscr{Z}$ such that*

$$\mathscr{M}\left(|U_\varepsilon E(z)\rangle\langle U_\varepsilon E(z)|\right) = \{\delta_{z_U}\}\,.$$

Then for any non negative function $\varphi \in L^1(\mathbb{R}, ds)$ such that $\int_{\mathbb{R}} \varphi(s)(1+|s|)^\delta\, ds < \infty$ for some $\delta > 0$ and $\int_{\mathbb{R}} \varphi(s)\, ds = 1$, the state

$$\varrho_\varphi^\varepsilon = \sum_{n=0}^{\infty} \varepsilon^{1/2} \varphi\big(\varepsilon^{1/2}(n-\varepsilon^{-1})\big) |U_\varepsilon z^{\otimes n}\rangle\langle U_\varepsilon z^{\otimes n}|$$

satisfies the conditions of Definition 6.5 *and*

$$\mathscr{M}\left(\varrho_\varphi^\varepsilon\right) = \frac{1}{2\pi}\int_0^{2\pi} \delta_{e^{i\theta} z_U}\, d\theta\,.$$

Proof. Owing to the relation

$$\Gamma(e^{-i\theta}) b^{Weyl} \Gamma(e^{i\theta}) = e^{-i\theta N} b^{Weyl} e^{i\theta N} = b(e^{-i\theta}.)^{Weyl}\,.$$

Our assumptions imply

$$\mathscr{M}\left(\Gamma(e^{i\theta})|U_\varepsilon E(z)\rangle\langle U_\varepsilon E(z)|\Gamma(e^{-i\theta})\right) = \delta_{e^{i\theta} z_U}$$

for any $\theta \in \mathbb{R}$. The assumptions of Definition 6.5 are satisfied because U_ε preserves the number. After taking the average w.r.t $\theta \in [0, 2\pi]$:

$$\sigma^\varepsilon = \frac{1}{2\pi} \int_0^{2\pi} \Gamma(e^{i\theta}) |U_\varepsilon E(z)\rangle\langle U_\varepsilon E(z)| \Gamma(e^{-i\theta})\ d\theta$$

this implies

$$\mathscr{M}(\sigma^\varepsilon) = \frac{1}{2\pi} \int_0^{2\pi} \delta_{e^{i\theta} z_U}\ d\theta$$

where the right-side is an extremal point of the convex set of Borel probability measure which are invariant after the natural action of S^1 on $\mathscr{Z}$: $S^1 \times \mathscr{Z} \ni (\gamma, z) \to \gamma z \in \mathscr{Z}$.

Again the commutation $[U_\varepsilon, N] = 0$ and the expression (4) for $E(z)$ imply

$$\begin{aligned}
\sigma^\varepsilon &= (2\pi)^{-1} \int_0^{2\pi} U_\varepsilon |\Gamma(e^{i\theta}) E(z)\rangle\langle \Gamma(e^{i\theta}) E(z)| U_\varepsilon^*\ d\theta \\
&= (2\pi)^{-1} \int_0^{2\pi} U_\varepsilon |E(e^{i\theta} z)\rangle\langle E(e^{i\theta} z)| U_\varepsilon^*\ d\theta \\
&= \sum_{n=0}^{\infty} \frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon^n n!} |U_\varepsilon z^{\otimes n}\rangle\langle U_\varepsilon z^{\otimes n}|\,.
\end{aligned}$$

For any $b \in \mathscr{S}_{cyl}(\mathscr{Z})$, the quantity

$$\sum_{n=0}^{\infty} \frac{e^{-\frac{1}{\varepsilon}}}{\varepsilon^n n!} \langle U_\varepsilon z^{\otimes n}\,,\, b^{Weyl} U_\varepsilon z^{\otimes}\rangle = \mathrm{Tr}\left[b^{Weyl} \sigma^\varepsilon\right]$$

converges as $\varepsilon \to 0$ to $(2\pi)^{-1} \int_0^{2\pi} b(e^{i\theta} z_U)\ d\theta$. By Lemma A.1 this implies

$$\forall b \in \mathscr{S}_{cyl}(\mathscr{Z})\,, \quad \lim_{\varepsilon \to 0} \int_{\mathbb{R}} a_{[\varepsilon^{-1/2} s + \varepsilon^{-1}]}(\varepsilon^{-1})\ \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\ ds = (2\pi)^{-1} \int_0^{2\pi} b(e^{i\theta} z_U)\ d\theta\,,$$

where $[t]$ is the integer part of $t \in \mathbb{R}$ and

$$a_n(\varepsilon^{-1}) = \langle U_\varepsilon z^{\otimes n}\,, b^{Weyl} U_\varepsilon z^{\otimes n}\rangle\,.$$

Call γ the Gaussian measure $e^{-\frac{s^2}{2}} \frac{ds}{\sqrt{2\pi}}$ on $\mathbb{R}$. For any finite subdivision $\mathscr{I} = \{I_1\ \ldots, I_L\}$ of $\mathbb{R} = I_1 \sqcup \ldots \sqcup I_L$ with intervals, the states

$$\sigma^\varepsilon_{I_\ell} = \big(\gamma(I_\ell)\big)^{-1} \int_{I_\ell} |U_\varepsilon z^{\otimes [\varepsilon^{-1/2} s + \varepsilon^{-1}]}\rangle\langle U_\varepsilon z^{\otimes [\varepsilon^{-1/2} s + \varepsilon^{-1}]}|\ d\gamma(s)$$

satisfy the assumptions of Definition 6.5 with the gauge invariance

$$\Gamma(e^{i\theta}) \sigma^\varepsilon_{I_\ell} \Gamma(e^{-i\theta}) = \sigma^\varepsilon_{I_\ell}\,.$$

Moreover the state

$$\underline{\sigma}^\varepsilon = \int_{\mathbb{R}} |U_\varepsilon z^{\otimes [\varepsilon^{-1/2} s + \varepsilon^{-1}]}\rangle\langle U_\varepsilon z^{\otimes [\varepsilon^{-1/2} s + \varepsilon^{-1}]}|\ d\gamma(s) = \sum_{\ell=1}^{L} \gamma(I_\ell) \sigma^\varepsilon_{I_\ell}$$

is a finite barycenter of the $\sigma^{\varepsilon}_{I_\ell}$ with a unique Wigner measure $(2\pi)\int_0^{2\pi}\delta_{e^{i\theta}z_U}\,d\theta$. Since $\mathscr{I}$ is finite (or countable), from any sequence $(\sigma^{\varepsilon_n}_{I_\ell})$ with $\lim_{n\to\infty}\varepsilon_n=0$, one can extract a subsequence $(\varepsilon_{n_k})_{k\in\mathbb{N}}$ such that

$$\mathscr{M}(\sigma^{\varepsilon_{n_k}}_{I_\ell}, k\in\mathbb{N})=\{\nu_\ell\}\,.$$

Since the measure μ_U is an extremal point in the convex set of gauge invariant probability measures, all the ν_ℓ have to be identical to μ_U. Since this holds for any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, we have proved for any interval $I=(\alpha,\beta)$ with $\alpha<\beta$, $\mathscr{M}\left(\sigma^{\varepsilon}_{I},\varepsilon\in(0,\overline{\varepsilon})\right)=\{\mu_U\}$.

Now take $\psi\in L^1(\mathbb{R},\gamma)$ and consider the state

$$\underline{\sigma}^{\varepsilon}_{\psi}=\int_{\mathbb{R}}|U_\varepsilon z^{\otimes[\varepsilon^{-1/2}s+\varepsilon^{-1}]}\rangle\langle U_\varepsilon z^{\otimes[\varepsilon^{-1/2}s+\varepsilon^{-1}]}|\;d\gamma(s)=\sum_{\ell=1}^{L}\gamma(I_\ell)\sigma^{\varepsilon}_{I_\ell}\,.$$

If there exists $\delta>0$ such that $\int_{\mathbb{R}}(1+|s|)^\delta\psi(s)\;d\gamma(s)<+\infty$, the family $(\underline{\sigma}^{\varepsilon}_{\psi})_{\varepsilon\in(0,\overline{\varepsilon})}$ satisfy the assumption of Definition 6.5. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence such that $\mathscr{M}(\underline{\sigma}^{\varepsilon_n}_{\psi},n\in\mathbb{N})=\{\nu\}$. Fix $b\in\mathscr{S}_{cyl}(\mathscr{Z})$. The function ψ can be approximated in $L^1(\mathbb{R},d\gamma)$ by $\psi_c\in\mathscr{C}^0_c(\mathbb{R})$. After choosing a finite subdivision $\mathscr{I}$ such that the diameter of any I_ℓ intersecting the support of ψ_c is bounded by Δ one gets

$$\left|\mathrm{Tr}\left[b^{Weyl}\underline{\sigma}^{\varepsilon_n}_{\psi}\right]-\mathrm{Tr}\left[b^{Weyl}\sum_{\ell=0}^{L}\frac{\int_{I_\ell}\psi_c(t)\;dt}{\gamma(I_\ell)}\sigma^{\varepsilon}_{I_\ell}\right]\right|\le C_b\left[\omega(\psi_c)\Delta+|\psi-\psi_c|_{L^1(\mathbb{R},\gamma)}\right]$$

where $\omega(\psi_c)$ is the continuity modulus of ψ_c. Hence the right-hand side can be made arbitrarily small, uniformly with respect to ε_n, while we know that the second term of the left-hand side converges when ψ_c and $\mathscr{I}$ are fixed. We have proved

$$\int_{\mathscr{Z}}b(z)\;d\nu(z)=\lim_{n\to\infty}\mathrm{Tr}\left[b^{Weyl}\varrho^{\varepsilon_n}\right]=\int_{\mathscr{Z}}b(z)\;d\mu_U(z)$$

for any $b\in\mathscr{S}_{cyl}(\mathscr{Z})$ and this proves $\nu=\mu_U$. Since this holds for any $\nu\in\mathscr{M}(\underline{\sigma}^{\varepsilon}_{\psi})$, we obtain

$$\mathscr{M}(\underline{\sigma}^{\varepsilon}_{\psi})=\{\mu_U\}\,.$$

The result for $\varrho^{\varepsilon}_{\varphi}$ comes from

$$\left|\varrho^{\varepsilon}_{\varphi}-\underline{\sigma}^{\varepsilon}_{\psi}\right|_{\mathscr{L}^1(\mathscr{H})}\le\left|\varphi-\sum_{k\in\mathbb{Z}}\varepsilon^{-1/2}\left(\int_{I^\varepsilon_k}\varphi(t)\;dt\right)1_{I^\varepsilon_k}\right|_{L^1(\mathbb{R},ds)}\overset{\varepsilon\to0}{\to}0$$

with $I^\varepsilon_k=[\varepsilon^{1/2}k-\varepsilon^{-1/2},\varepsilon^{1/2}(k+1)-\varepsilon^{-1/2}]$ and $\psi(s)=\varphi(s)\sqrt{2\pi}e^{\frac{s^2}{2}}$. The condition $\int_{\mathbb{R}}(1+|s|)^\delta\;\varphi(s)ds<+\infty$ ensures that $\mathscr{M}(\varrho^{\varepsilon}_{\varphi})$ is well defined. $\square$

7.7. Application 2: Propagation of correlated states

This a simple application of the orthogonality of Wigner measures combined with the results of Subsection 7.3.

Let $H_\varepsilon = d\Gamma(-\Delta) + Q^{Wick}$ be the Hamiltonian studied in Section 5 and let z_t denote the solution to $i\partial_t z_t = -\Delta z_t + (V * |z_t|^2) z_t$. The family of integers $(k_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ is assumed to satisfy $\lim_{\varepsilon\to 0} \varepsilon k_\varepsilon = 1$.

1. Let $z_{0,\ell} \in \mathscr{Z}$, $\ell = 1, \ldots, L$, satisfy $|z_{0,\ell}| = 1$ and set $u^\varepsilon = L^{-1/2} \sum_{\ell=1}^{L} z_{0,\ell}^{\otimes k_\varepsilon}$, $u^\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H_\varepsilon} u_\varepsilon$. At any time $t \in \mathbb{R}$ the identity
$$\mathscr{M}\big(|u_\varepsilon(t)\rangle\langle u_\varepsilon(t)|\big) = \left\{ (2\pi L)^{-1} \sum_{\ell=1}^{L} \int_0^{2\pi} \delta_{e^{i\theta} z_{t,\ell}} \, d\theta \right\}$$
as soon as $z_{1,t}, \ldots, z_{\ell,t}$ are linearly independent. In particular this holds for any $t \in \mathbb{R}$ when $L = 2$ and $z_{0,1}$ and $z_{0,2}$ are linearly independent.
2. Let $z_0 \in \mathscr{Z}$ satisfy $|z_0| = 1$ and set $u^\varepsilon = 2^{-1/2} z_0^{\otimes k_\varepsilon} + 2^{-1/2} E(z_0)$ and $u^\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H^\varepsilon} u_\varepsilon$. Then
$$\mathscr{M}\big(|u^\varepsilon(t)\rangle\langle u^\varepsilon(t)|\big) = \left\{ \frac{1}{2}\delta_{z_t} + \frac{1}{4\pi} \int_0^{2\pi} \delta_{e^{i\theta} z_t} \, d\theta \right\}.$$
3. Moreover the convergence can be tested with Weyl, Anti-Wick and Wick operators according to Theorem 6.2 and Theorem 6.13.

Appendix A. Normal approximation

We prove a technical lemma which is a slight adaptation of the normal approximation to the Poisson distribution. Recall that for all $-\infty \le \alpha < \beta \le \infty$ we have the well known fact:
$$\lim_{\lambda\to\infty} \sum_{1+\frac{\alpha}{\sqrt{\lambda}} \le \frac{n}{\lambda} \le 1+\frac{\beta}{\sqrt{\lambda}}} \frac{\lambda^n}{n!} \, e^{-\lambda} = \int_\alpha^\beta \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \, ds \,. \tag{69}$$

Lemma A.1. *Let $\{a_n(\lambda)\}_{n\in\mathbb{Z},\lambda>0}$ be a family of complex numbers with $a_n(\lambda) = 0$ if $n < 0$. Assume that there exist $\mu \in \mathbb{N}$ and $C_\mu > 0$ such that:*
$$\sup_{n\in\mathbb{N},\lambda>0} |a_n(\lambda)| \left\langle \frac{n}{\lambda} \right\rangle^{-\mu} \le C_\mu \,.$$
Then the equality
$$\lim_{\lambda\to\infty} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} \, a_n(\lambda) = \lim_{\lambda\to\infty} \int_{\mathbb{R}} a_{[\sqrt{\lambda}s+\lambda]}(\lambda) \, \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} \, ds \tag{70}$$
holds whenever one of the two limits exists.

Proof. Notice that both the series and the integral in (70) are absolutely convergent for finite values of λ. By hypothesis $\tilde{a}_n(\lambda) = a_n(\lambda)\langle\frac{n}{\lambda}\rangle^{-\mu}$ are bounded and moreover they satisfy

$$\lim_{\lambda\to\infty}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda}\tilde{a}_n(\lambda)\ \left(1-\left\langle\frac{n}{\lambda}\right\rangle^{\mu}\right) = 0\,, \tag{71}$$

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}}\tilde{a}_{[\sqrt{\lambda}s+\lambda]}(\lambda)\ \left(1-\left\langle\frac{[\sqrt{\lambda}s+\lambda]}{\lambda}\right\rangle^{\mu}\right)\ \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds = 0 \tag{72}$$

since we may bound uniformly for λ large each of the terms inside the sum and the integral respectively by

$$C^1_\mu\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda}\ n^\mu < C^0_\mu\,,\quad \text{and}\quad C^2_\mu\int_{\mathbb{R}}|s|^\mu\ \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds < C^0_\mu\,,\quad \forall\lambda>1\,.$$

Therefore there is no restriction if we assume all $a_n(\lambda)$ bounded by 1 since if we prove (70) for $\tilde{a}_n(\lambda)$ then it holds for $a_n(\lambda)$ by the limits (71)–(72).

For all $h>0$ there exists $\alpha<\beta$ such that

$$\int_\beta^\infty\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds < h/7\,,\quad \int_{-\infty}^{\alpha}\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds < h/7\,.$$

Now by (69) we have

$$\lim_{\lambda\to\infty}\sum_{1+\frac{\beta}{\sqrt{\lambda}}\le\frac{n}{\lambda}}\frac{\lambda^n}{n!}e^{-\lambda} = \int_\beta^\infty\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds\,,\quad \lim_{\lambda\to\infty}\sum_{\frac{n}{\lambda}\le 1+\frac{\alpha}{\sqrt{\lambda}}}\frac{\lambda^n}{n!}e^{-\lambda} = \int_{-\infty}^{\alpha}\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds$$

Therefore there exists λ_1 such that for all $\lambda>\lambda_1$ we have

$$\sum_{1+\frac{\beta}{\sqrt{\lambda}}\le\frac{n}{\lambda}}\frac{\lambda^n}{n!}e^{-\lambda}\le h/6\,,\qquad \sum_{\frac{n}{\lambda}\le 1+\frac{\alpha}{\sqrt{\lambda}}}\frac{\lambda^n}{n!}e^{-\lambda}\le h/6\,.$$

Let denote $I_{\alpha,\beta}(\lambda) = \int_\alpha^\beta a_{[\sqrt{\lambda}s+\lambda]}(\lambda)\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds$. We obtain for all $\lambda>\lambda_1$:

$$\begin{aligned}&\left|\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda}a_n(\lambda) - \int_{-\infty}^{\infty}a_{[\sqrt{\lambda}s+\lambda]}(\lambda)\frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}}\,ds\right| \\ &\qquad\le \underbrace{\left|\sum_{\alpha<\frac{n-\lambda}{\sqrt{\lambda}}<\beta}\frac{\lambda^n}{n!}e^{-\lambda}a_n(\lambda) - I_{\alpha,\beta}(\lambda)\right|}_{J_{\alpha,\beta}(\lambda)} + 2h/3\,.\end{aligned} \tag{73}$$

Using the Stirling formula there exists λ_2 such that for all $\lambda > \lambda_2$ we have

$$\left| \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{\lambda^n}{n!} e^{-\lambda} \left[1 - \frac{n!}{\sqrt{2\pi n}(n/e)^n} \right] \, a_n(\lambda) \right| \leq h/9 \, .$$

This yields the following estimate

$$\begin{aligned} J_{\alpha,\beta}(\lambda) \leq & \left| \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{1}{\sqrt{2\pi n}} \left[e^{\lambda \varphi(\frac{n}{\lambda})} - e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2} \right] \right| \\ & + \underbrace{\left| \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi n}} a_n(\lambda) - I_{\alpha,\beta}(\lambda) \right|}_{L_{\alpha,\beta}(\lambda)} + h/12 \, , \end{aligned} \tag{74}$$

where $\varphi(x) = x - 1 - x \ln(x)$. To complete the proof one needs to estimate infinitesimally the two terms in the r.h.s. of the above inequality. Notice that by means of Riemann sums we have

$$\lim_{\lambda \to \infty} \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi n}} = \lim_{\lambda \to \infty} \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi \lambda}} = \int_\alpha^\beta \frac{e^{-s^2/2}}{\sqrt{2\pi}} \, ds \, . \tag{75}$$

We have

$$\sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{1}{\sqrt{2\pi n}} \left[e^{\lambda \varphi(\frac{n}{\lambda})} - e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2} \right] = \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi n}} \left[e^{\lambda \tilde{\varphi}(\frac{n}{\lambda})} - 1 \right] ,$$

where $\tilde{\varphi}(x) = x - 1 - x \ln(x) + (x-1)^2/2$ which is an increasing function null at 1. Therefore one obtains

$$\begin{aligned} & \left| \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{1}{\sqrt{2\pi n}} \left[e^{\lambda \varphi(\frac{n}{\lambda})} - e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2} \right] \right| \\ & \qquad \leq \int_\alpha^\beta \frac{e^{-s^2/2}}{\sqrt{2\pi}} \, ds \quad \left[e^{\lambda \tilde{\varphi}(\frac{\beta}{\sqrt{\lambda}} + 1)} - 1 \right] , \end{aligned} \tag{76}$$

with a r.h.s. converging to 0 when $\lambda \to \infty$ since $\lim_{\lambda \to \infty} e^{\lambda \tilde{\varphi}(\frac{\beta}{\sqrt{\lambda}} + 1)} = 1$, which we bound by $h/12$ for λ larger than a given λ_3. One can obtain the estimate

$$L_{\alpha,\beta}(\lambda) \leq \left| \sum_{\alpha < \frac{n-\lambda}{\sqrt{\lambda}} < \beta} \frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi \lambda}} \, a_n(\lambda) - I_{\alpha,\beta}(\lambda) \right| + h/18 \, ,$$

using the fact that

$$\sum_{\alpha<\frac{n-\lambda}{\sqrt{\lambda}}<\beta} \frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi\lambda}} \underbrace{\left| \frac{1}{\sqrt{(\frac{n-\lambda}{\sqrt{\lambda}})\frac{1}{\sqrt{\lambda}}+1}} - 1 \right|}_{(1)} \le h/18\,,$$

since $\lim_{\lambda\to\infty}(1) = 0$ and the sum is uniformly bounded by (Equ. 75). By splitting the integral in $I_{\alpha,\beta}(\lambda)$ over the intervals $[\frac{n-\lambda}{\sqrt{\lambda}}, \frac{n+1-\lambda}{\sqrt{\lambda}})$ one can show that

$$\left| I_{\alpha,\beta}(\lambda) - \sum_{\alpha<\frac{n-\lambda}{\sqrt{\lambda}}<\beta} a_n(\lambda) \int_{\frac{n-\lambda}{\sqrt{\lambda}}}^{\frac{n+1-\lambda}{\sqrt{\lambda}}} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds \right| \le h/18\,.$$

This yields

$$L_{\alpha,\beta}(\lambda) \le h/9 + \sum_{\alpha<\frac{n-\lambda}{\sqrt{\lambda}}<\beta} \left[\frac{e^{-(\frac{n-\lambda}{\sqrt{\lambda}})^2/2}}{\sqrt{2\pi\lambda}} - \int_{\frac{n-\lambda}{\sqrt{\lambda}}}^{\frac{n+1-\lambda}{\sqrt{\lambda}}} \frac{e^{-s^2/2}}{\sqrt{2\pi}} ds \right] \tag{77}$$

with a r.h.s. converging to 0 when $\lambda \to \infty$ which we bound by $h/18$ for λ larger than λ_4. Combining the estimates (74), (76) and (77) with (73) we obtain that for all $h > 0$, there exists λ_0 such that for all $\lambda > \lambda_0$ we have

$$\left| \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} a_n(\lambda) - \int_{-\infty}^{\infty} a_{[\sqrt{\lambda}s+\lambda]}(\lambda) \frac{e^{-\frac{s^2}{2}}}{\sqrt{2\pi}} ds \right| \le h\,.$$

This gives the claimed result. □

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