

A Functional Integral Representation for Many Boson Systems

I: The Partition Function

Tadeusz Balaban, Joel Feldman, Horst Knörrer, and Eugene Trubowitz

Abstract. We derive a functional integral representation for the partition function of a many Boson system for which the configuration space consists of finitely many points.

1. Introduction

We are developing a set of tools and techniques for analyzing the large distance/infrared behaviour of a system of identical bosons, as the temperature tends to zero.

The total energy of the many boson systems considered in this paper has two sources. First, each particle in the system has a kinetic energy. We shall denote the corresponding quantum mechanical observable by h . The most common is $-\frac{1}{2m}\Delta$, but, in this paper, we allow any positive operator. Second, the particles interact with each other through a two-body potential, $v(\mathbf{x}, \mathbf{y})$. For stability, v is required to be repulsive. We assume that the system is in thermodynamic equilibrium and that expectations of observables are given by the grand canonical ensemble at temperature $T = \frac{1}{k\beta} > 0$ and chemical potential μ .

Functional integrals are an important source of intuition about the behaviour of quantum mechanical systems. They are also an important rigorous technical tool in the analysis of, for example, Euclidean quantum field theories. In this paper and its companion [2], we derive rigorous functional integral representations for the partition function and thermodynamic correlation functions of a many boson system.

There are many possible applications of our functional integral representations. However, we are motivated by the following potential specific application.

One may speculate (in agreement with the standard picture of condensed matter physics) that, at temperature zero and infinite volume, a weakly coupled, three dimensional many boson system will undergo a phase transition at some critical chemical potential μ_* .

- For $\mu < \mu_*$, the system is in a massive phase. That is, all correlation functions decay exponentially fast at large separation. The expected value of the field $\phi(\mathbf{x}) = \langle a^\dagger(\mathbf{x}) \rangle$, where $a^\dagger(\mathbf{x})$ is the particle creation operator at \mathbf{x} , is zero.
- For $\mu > \mu_*$, number symmetry is broken. In this phase, correlation functions fail to decay exponentially due to the presence of extended, collective excitations (the massless Goldstone bosons). The expected value of the field is nonzero. The presence of such anomalous nonzero amplitudes is used as a general criterion for a condensed quantum fluid.

The intuition behind this phase transition is easily obtained by using a formal coherent state functional integral [6, (2.66)] to express the grand canonical partition function as

$$Z = \int \dots \int_{\phi_\beta = \phi_0} \prod_{\substack{\mathbf{x} \in \mathbb{R}^3 \\ 0 \leq \tau \leq \beta}} \frac{d\phi_\tau^*(\mathbf{x}) d\phi_\tau(\mathbf{x})}{2\pi i} e^{\mathcal{A}(\phi^*, \phi)} \quad (1.1)$$

where

$$\mathcal{A}(\phi^*, \phi) = \int_0^\beta d\tau \int_{\mathbb{R}^3} d^3\mathbf{x} \phi_\tau^*(\mathbf{x}) \frac{\partial}{\partial \tau} \phi_\tau(\mathbf{x}) - \int_0^\beta d\tau K(\phi_\tau^*, \phi_\tau)$$

and

$$K(\alpha^*, \phi) = \iint d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) - \mu \int d\mathbf{x} \alpha(\mathbf{x})^* \phi(\mathbf{x}) + \frac{1}{2} \iint d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}).$$

Here $h(\mathbf{x}, \mathbf{y})$ is the kernel of the kinetic energy operator.

In the mean field approximation, that is, when $\phi_\tau(\mathbf{x})$ is independent of τ and \mathbf{x} , the action $\mathcal{A}(\phi^*, \phi)$ is minus the integral over τ and \mathbf{x} of the

$$\text{“naive effective potential”} = \frac{1}{2} \hat{v}(0) |\phi|^4 - \mu |\phi|^2$$

where $\hat{v}(0) = \int d\mathbf{y} v(\mathbf{x}, \mathbf{y})$. We have assumed that $v(\mathbf{x}, \mathbf{y})$ is translation invariant and that h annihilates constants. The minimum of the naive effective potential is

- nondegenerate at the point $\phi = 0$ when $\mu < 0$
and
- degenerate along the circle $|\phi| = \sqrt{\frac{\mu}{\hat{v}(0)}}$ when $\mu > 0$.

It is therefore reasonable that an attempt to rigorously justify the phase transition in the chemical potential discussed above would begin with the derivation of rigorous functional integral representations of the thermodynamic correlation functions in which the effective potential appears explicitly. We do so in this paper and the companion paper [2].

It is common practice in condensed matter physics, to discretize space, because the overall energy scale is low. On physical grounds, this does not affect the long range behaviour of the system. For this paper, space is an arbitrary, but fixed, finite set X , that we may imagine is a subset of a lattice. The second quantized Hamiltonian H and the number operator N act on the infinite dimensional Fock space

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} (\mathbb{C}^{|X|^n} / S_n).$$

The long distance behavior of the system is revealed in the thermodynamic limits of grand canonical correlation functions, such as derivatives of the partition function

$$Z = \text{Tr}_{\mathcal{F}} e^{-\beta(H-\mu N)}.$$

To implement the thermodynamic limit, one would take the usual family of finite spaces

$$X_L = \{x \in \mathbb{Z}^3 \mid |x_i| < L, i = 1, 2, 3\}$$

and send L to infinity. We shall not do so in this paper.

Our first result (Theorem 3.13), stated somewhat informally, is the representation

$$\text{Tr} e^{-\beta(H-\mu N)} = \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{T}_p} d\mu_{R(p)}(\phi_{\tau}^*, \phi_{\tau}) e^{\mathcal{F}(\frac{\beta}{p}, \phi^*, \phi)} \tag{1.2}$$

for the finite volume grand canonical partition function. Similar representations for general correlation functions are derived in [2]. Here, for each natural number p , the discrete time interval \mathcal{T}_p is given by

$$\mathcal{T}_p = \left\{ \tau = q \frac{\beta}{p} \mid q = 1, \dots, p \right\}.$$

For each point (\mathbf{x}, τ) in the discrete space-time $X \times \mathcal{T}_p$, we have introduced the complex variable $\phi(\tau, \mathbf{x}) = \phi_{\tau}(\mathbf{x})$. For each $r > 0$, the measure

$$d\mu_{\tau}(\phi^*, \phi) = \prod_{\mathbf{x} \in X} \left[\frac{d\phi^*(\mathbf{x}) \wedge d\phi(\mathbf{x})}{2\pi i} \chi_r(|\phi(\mathbf{x})|) \right]$$

where, χ_r is the characteristic function of the closed interval $[0, r]$. The sequence $R(p) > 0$ in (1.2) tends to infinity at an appropriate rate as $p \rightarrow \infty$. The ‘‘action’’ $\mathcal{F}(\varepsilon, \phi^*, \phi)$, with $\varepsilon = \frac{\beta}{p}$, is given by

$$\begin{aligned} \mathcal{F}(\varepsilon, \phi^*, \phi) &= \iint d\tau d\mathbf{x} \phi_{\tau}^*(\mathbf{x})(\partial^{\varepsilon} \phi_{\tau})(\mathbf{x}) - \iint d\tau d\mathbf{x} \phi_{\tau}^*(\mathbf{x})(h\phi_{\tau})(\mathbf{x}) \\ &+ \mu \iint d\tau d\mathbf{x} \phi_{\tau}(\mathbf{x})^* \phi_{\tau}(\mathbf{x}) \\ &- \frac{1}{2} \iiint d\tau d\mathbf{x} d\mathbf{y} \phi_{\tau}(\mathbf{x})^* \phi_{\tau}(\mathbf{x}) v(\mathbf{x}, \mathbf{y}) \phi_{\tau}(\mathbf{y})^* \phi_{\tau}(\mathbf{y}) \end{aligned}$$

where

$$\iint d\tau d\mathbf{x} \psi(\tau, \mathbf{x}) = \varepsilon \sum_{\tau \in \mathcal{T}_p} \sum_{\mathbf{x} \in X} \psi(\tau, \mathbf{x})$$

$$\iiint d\tau d\mathbf{x} d\mathbf{y} \psi(\tau, \mathbf{x}, \mathbf{y}) = \varepsilon \sum_{\tau \in \mathcal{T}_p} \sum_{\mathbf{x} \in X} \sum_{\mathbf{y} \in X} \psi(\tau, \mathbf{x}, \mathbf{y})$$

and the difference operator ∂^ε acts by

$$\partial^\varepsilon \phi(\tau, \mathbf{x}) = \varepsilon^{-1}(\phi(\tau + \varepsilon, \mathbf{x}) - \phi(\tau, \mathbf{x})).$$

In (1.2), fields ϕ_τ with $\tau \notin \mathcal{T}_p$ are determined by the periodicity condition $\phi_\tau = \phi_{\tau-\beta}$. It is easy to check that the representation (1.2) generates the usual formal graphical perturbation series.

In the physics literature, coherent states¹ $|\phi\rangle$, $\phi \in \mathbb{C}^X$, the formal resolution of the identity

$$\mathbb{1} = \int \prod_{\mathbf{x} \in X} \left[\frac{d\phi^*(\mathbf{x})d\phi(\mathbf{x})}{2\pi i} e^{-|\phi_\tau(\mathbf{x})|^2} \right] |\phi\rangle \langle \phi| \tag{1.3}$$

and the formal trace formula

$$\text{Tr } B = \int \prod_{\mathbf{x} \in X} \left[\frac{d\phi^*(\mathbf{x})d\phi(\mathbf{x})}{2\pi i} e^{-|\phi_\tau(\mathbf{x})|^2} \right] \langle \phi | B | \phi \rangle \tag{1.4}$$

are used to justify (1.1) as follows. Formally,

$$\begin{aligned} & \text{Tr } e^{-\beta(H-\mu N)} \\ &= \lim_{p \rightarrow \infty} \text{Tr } e^{-\frac{\beta}{p}(H-\mu N)} \mathbb{1} e^{-\frac{\beta}{p}(H-\mu N)} \mathbb{1} \dots \mathbb{1} e^{-\frac{\beta}{p}(H-\mu N)} \\ &= \lim_{p \rightarrow \infty} \int \prod_{\substack{\mathbf{x} \in X \\ \tau \in \mathcal{T}_p \setminus \{\beta\}}} \left[\frac{d\phi_\tau^*(\mathbf{x})d\phi_\tau(\mathbf{x})}{2\pi i} e^{-|\phi_\tau(\mathbf{x})|^2} \right] \\ & \quad \times \text{Tr} \left[\prod_{\tau \in \mathcal{T}_p \setminus \{\beta\}} e^{-\frac{\beta}{p}(H-\mu N)} |\phi_\tau\rangle \langle \phi_\tau| \right] e^{-\frac{\beta}{p}(H-\mu N)} \\ &= \lim_{p \rightarrow \infty} \prod_{\substack{\mathbf{x} \in X \\ \tau \in \mathcal{T}_p}} \left[\int \frac{d\phi_\tau^*(\mathbf{x})d\phi_\tau(\mathbf{x})}{2\pi i} e^{-|\phi_\tau(\mathbf{x})|^2} \right] \prod_{\tau \in \mathcal{T}_p} \left\langle \phi_\tau \left| e^{-\frac{\beta}{p}(H-\mu N)} \right| \phi_{\tau+\frac{\beta}{p}} \right\rangle. \end{aligned} \tag{1.5}$$

¹We are using coherent states normalized so that $\langle \alpha | \gamma \rangle = e^{\int d\mathbf{y} \overline{\alpha(\mathbf{y})} \gamma(\mathbf{y})}$. We systematically use the convention that $\int d\mathbf{x} f(\mathbf{x}) = \sum_{\mathbf{x} \in X} f(\mathbf{x})$.

Then, one argues [6, (2.59)] that

$$\begin{aligned} \langle \alpha \mid e^{-\varepsilon(H-\mu N)} \mid \phi \rangle &= \langle \alpha \mid :e^{-\varepsilon(H-\mu N)}: \mid \phi \rangle + O(\varepsilon^2) \\ &= \exp \left\{ \int d\mathbf{y} \alpha(\mathbf{y})^* \phi(\mathbf{y}) - \varepsilon K(\alpha^*, \phi) + O(\varepsilon^2) \right\} \end{aligned} \tag{1.6}$$

where $: \cdot \cdot :$ denotes Wick ordering. If $\varepsilon = \frac{\beta}{p}$, one observes that there are $O(p)$ error terms of order $O(\varepsilon^2) = O(\frac{1}{p^2})$. “Off the cuff”, the error terms do not contribute when $p \rightarrow \infty$ and consequently,

$$\begin{aligned} \text{Tr} e^{-\beta(H-\mu N)} &= \lim_{p \rightarrow \infty} \prod_{\substack{\mathbf{x} \in X \\ \tau \in \mathcal{T}_p}} \left[\int \frac{d\phi_\tau^*(\mathbf{x}) d\phi_\tau(\mathbf{x})}{2\pi i} \right] \\ &\times \prod_{\tau \in \mathcal{T}_p} e^{\int d\mathbf{y} \phi_\tau(\mathbf{y})^* [\phi_{\tau+\beta/p}(\mathbf{y}) - \phi_\tau(\mathbf{y})] - \frac{\beta}{p} K(\phi_\tau^*, \phi_{\tau+\beta/p})}. \end{aligned}$$

It is the purpose of this paper, and its companion, to make this all precise.

In §II, we review the basic formalism of bosonic quantum statistical mechanics and the formalism of coherent states in the context of a finite configuration space X . In particular, we prove, in Theorem 2.26, a rigorous version of the formal resolution of the identity (1.3) and we prove, in Proposition 2.28, that the trace formula (1.4) applies rigorously to a certain class of operators. In this way, we obtain a rigorous variant of (1.5). See Theorem 3.1.

It is by no means clear that dropping, “off the cuff”, the $O(p)$ error terms of order $O(\varepsilon^2)$ is justified, because the error terms are unbounded functions of the fields ϕ_τ . We circumvent this part of the formal argument by directly constructing the logarithm $F(\varepsilon, \alpha^*, \phi) = \ln \langle \alpha \mid e^{-\varepsilon K} \mid \phi \rangle$, at least for α and ϕ not too large. See Proposition 3.6. To this end, we derive and then solve an evolution equation in ε for $F(\varepsilon, \alpha^*, \phi)$. It follows that

$$F(\varepsilon, \alpha^*, \phi) = \int_X d\mathbf{x} \alpha(\mathbf{x})^* \phi(\mathbf{x}) - \varepsilon K(\alpha^*, \phi) + O(\varepsilon^2).$$

We then show, in Theorem 3.13, that the “matrix element” $\langle \alpha \mid e^{-\varepsilon(H-\mu N)} \mid \phi \rangle$ can be replaced by $e^{\int d\mathbf{y} \alpha(\mathbf{y})^* \phi(\mathbf{y}) - \varepsilon K(\alpha^*, \phi)}$ in the formula for $\text{Tr} e^{-\beta(H-\mu N)}$ of Theorem 3.1, provided the integration radius $R(p)$ of (1.2) is chosen appropriately.

In the physics literature, one simply “evaluates” the limit

$$\lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{T}_p} d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{\mathcal{F}(\frac{\beta}{p}, \phi^*, \phi)} = \int \dots \int \prod_{\substack{\mathbf{x} \in X \\ 0 \leq \tau \leq \beta}} \frac{d\phi_\tau^*(\mathbf{x}) d\phi_\tau(\mathbf{x})}{2\pi i} e^{\mathcal{A}_X(\phi^*, \phi)}$$

where

$$\begin{aligned} \mathcal{A}_X(\phi^*, \phi) &= \int_0^\beta d\tau \int_X d\mathbf{x} \left\{ \phi_\tau^*(\mathbf{x}) \frac{\partial}{\partial \tau} \phi_\tau(\mathbf{x}) - \phi_\tau^*(\mathbf{x})(h - \mu)\phi_\tau(\mathbf{x}) \right\} \\ &\quad - \frac{1}{2} \int_0^\beta d\tau \iint_{X^2} d\mathbf{x}d\mathbf{y} \phi_\tau(\mathbf{x})^* \phi_\tau(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \phi_\tau(\mathbf{x}) \phi_\tau(\mathbf{y}). \end{aligned}$$

The first impulse of a mathematical physicist determined to ascribe a rigorous meaning to this formal functional integral representation for the partition function, would be to construct a “complex Gaussian measure” $d\mu_C$, with covariance

$$C = \left(-\frac{\partial}{\partial \tau} + (h - \mu) \right)^{-1}$$

out of the formal measure

$$\exp \left\{ \int_0^\beta d\tau \int_X d\mathbf{x} \left\{ \phi_\tau^*(\mathbf{x}) \frac{\partial}{\partial \tau} \phi_\tau(\mathbf{x}) - \phi_\tau^*(\mathbf{x})(h - \mu)\phi_\tau(\mathbf{x}) \right\} \right\} \prod_{\substack{\mathbf{x} \in X \\ 0 \leq \tau \leq \beta}} \frac{d\phi_\tau^*(\mathbf{x}) d\phi_\tau(\mathbf{x})}{2\pi i}.$$

Normally, one starts by defining the integral of any polynomial in the complex fields $\phi_\tau^*(\mathbf{x}), \phi_\tau(\mathbf{x}), \tau \in [0, \beta), \mathbf{x} \in X$, against $d\mu_C$ as cumulants of “matrix elements” of the covariance C . Then one constructs the characteristic function of $d\mu_C$, as a limit of integrals of polynomials, and the corresponding measure. However, the explicit calculations in Appendix A, modelled on those of Cameron [3], show that the purely imaginary term

$$\int_0^\beta d\tau \int_X d\mathbf{x} \phi_\tau^*(\mathbf{x}) \frac{\partial}{\partial \tau} \phi_\tau(\mathbf{x})$$

in the exponential generates oscillations that are so severe that there is no complex Gaussian measure.

To work with the ultraviolet limit

$$\lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{T}_p} d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{\mathcal{F}(\frac{\beta}{p}, \phi^*, \phi)}$$

to, for example, construct the thermodynamic limit and justify the phase transition in the chemical potential, one must exploit the cancellations arising from the oscillations generated by the purely imaginary term in the action.

One methodology for the explicit control of cancellations of this kind, when the coupling constant $\lambda = \hat{v}(0)$ is small, is known as “multiscale analysis”. In the present case “scales” refer to blocks of frequencies in space \mathbf{x} and inverse temperature τ . There are infinitely many scales. Cancellations are implemented at each scale.

Typically, the total contribution of “large fields”, for example field configurations with $\phi_\tau(\mathbf{x})$ or appropriate derivatives large, is smaller than any power of λ , reminiscent of large deviations in probability theory. This can be proven without attention to cancellations. On the other hand, oscillations are fully exploited in

the complementary “small field regions” by high dimensional steepest descent calculations around the complex critical points of the effective actions. In the end the functional integral becomes an infinite sum over small and large field regions. The physicists formal functional integral is morally “the dominant term”.

2. Finite systems of bosons

In this section we carefully review the basic formalism of bosonic quantum statistical mechanics and introduce the notation that we will systematically use.

Fock space

Fix a finite set X .

Definition 2.1. (i) Let $n \in \mathbb{N} \cup \{0\}$. The action $\pi \cdot f$ of a permutation $\pi \in S_n$ on f in $L^2(X^n)$ is given by

$$\pi \cdot f(\mathbf{x}_1, \dots, \mathbf{x}_n) = f(\mathbf{x}_{\pi^{-1}(1)}, \dots, \mathbf{x}_{\pi^{-1}(n)}).$$

The bosonic n -particle space

$$\mathcal{B}_n(X) = \left\{ f \in L^2(X^n) \mid \pi \cdot f = f \text{ for all } \pi \in S_n \right\}$$

is the $\binom{n+|X|-1}{n}$ dimensional complex Hilbert space of all symmetric functions on X^n with inner product

$$\langle f, g \rangle_{\mathcal{B}_n} = \int_{X^n} d\mathbf{x}_1 \dots d\mathbf{x}_n \overline{f(\mathbf{x}_1, \dots, \mathbf{x}_n)} g(\mathbf{x}_1, \dots, \mathbf{x}_n).$$

In particular, $\mathcal{B}_0(X) = \mathbb{C}$ and $\mathcal{B}_1(X) = L^2(X)$.

(ii) The bosonic Fock space $\mathcal{B}(X)$ over X is the orthogonal direct sum $\mathcal{B}(X) = \bigoplus_{n \geq 0} \mathcal{B}_n(X)$. It is an infinite dimensional complex Hilbert space. The inner product between \mathbf{f} and \mathbf{g} in $\mathcal{B}(X)$ is

$$\langle \mathbf{f}, \mathbf{g} \rangle_{\mathcal{B}} = \sum_{n \geq 0} \langle f_n, g_n \rangle_{\mathcal{B}_n}.$$

Definition 2.2. (i) Let h be a (single particle) operator on $L^2(X)$, with kernel $h(\mathbf{x}, \mathbf{y})$. Assume that $h(\mathbf{y}, \mathbf{x}) = h(\mathbf{x}, \mathbf{y})^*$ so that h is self-adjoint. The corresponding independent particle operator on $\mathcal{B}_n(X)$ is

$$H_0(h, n, X) = \sum_{i=1}^n h^{(i)}.$$

The superscript on $h^{(i)}$, $i = 1, \dots, n$, indicates that the single particle operator h acts on the variable \mathbf{x}_i appearing in a function $g(\mathbf{x}_1, \dots, \mathbf{x}_n)$. That is,

$$(h^{(i)}g)(\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_X d\mathbf{x}'_i h(\mathbf{x}_i, \mathbf{x}'_i) g(\mathbf{x}_1, \dots, \mathbf{x}'_i, \dots, \mathbf{x}_n).$$

By convention, $H_0(h, 0, X) = 0$. The kernel of $H_0(h, n, X)$ is

$$H_0(h, n, X)(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_1, \dots, \mathbf{y}_n) = \sum_{i=1}^n \delta_{\mathbf{x}_1}(\mathbf{y}_1) \dots \delta_{\mathbf{x}_{i-1}}(\mathbf{y}_{i-1}) h(\mathbf{x}_i, \mathbf{y}_i) \delta_{\mathbf{x}_{i+1}}(\mathbf{y}_{i+1}) \dots \delta_{\mathbf{x}_n}(\mathbf{y}_n).$$

Here $\delta_{\mathbf{x}}(\mathbf{y}) = \delta_{\mathbf{x}, \mathbf{y}} = \begin{cases} 1 & \text{if } \mathbf{x}=\mathbf{y} \\ 0 & \text{if } \mathbf{x} \neq \mathbf{y} \end{cases}$ is the delta function on X concentrated at the point \mathbf{x} .

The second quantization of h is the direct sum

$$H_0(h, X) = \bigoplus_{n \geq 0} H_0(h, n, X)$$

acting on the domain $\{ \mathbf{f} \in \mathcal{B}(X) \mid \sum_{n \geq 0} \| H_0(h, n, X) f_n \|_{\mathcal{B}_n}^2 < \infty \}$.

- (ii) Let $v(\mathbf{x}, \mathbf{y})$ be a real valued (two body) potential on X^2 satisfying $v(\mathbf{x}, \mathbf{y}) = v(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in X$. Multiplication by v determines a (two particle) operator on $\mathcal{B}_2(X)$. Its kernel is $v(\mathbf{x}_1, \mathbf{x}_2) \delta_{\mathbf{x}_1}(\mathbf{y}_1) \delta_{\mathbf{x}_2}(\mathbf{y}_2)$. The corresponding 2 particle interaction operator on $\mathcal{B}_n(X)$ is

$$V(v, n, X) = \sum_{i_1 < i_2} v^{(i_1, i_2)}.$$

Here, for each pair of indices $1 \leq i_1 < i_2 \leq n$, the superscript on $v^{(i_1, i_2)}$ indicates that the operator v acts on the variables $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}$ appearing in a function $g(\mathbf{x}_1, \dots, \mathbf{x}_n)$. That is,

$$(v^{(i_1, i_2)}g)(\mathbf{x}_1, \dots, \mathbf{x}_n) = v(\mathbf{x}_{i_1}, \mathbf{x}_{i_2}) g(\mathbf{x}_1, \dots, \mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_2}, \dots, \mathbf{x}_n).$$

By convention, $V(v, n, X) = 0$ for $n = 0, 1$.

The second quantization of the two particle interaction v is the direct sum

$$V(v, X) = \bigoplus_{n \geq 0} V(v, n, X)$$

acting on the domain $\{ \mathbf{f} \in \mathcal{B}(X) \mid \sum_{n \geq 0} \| V(v, n, X) f_n \|_{\mathcal{B}_n}^2 < \infty \}$.

Example 2.3. Let $\mathbb{1}_X$ be the identity operator on $L^2(X)$. Then,

$$H_0(\mathbb{1}_X, n, X) = n \mathbb{1}_{X^n}$$

where, $\mathbb{1}_{X^n}$ is the identity operator on $L^2(X^n)$. By definition, the number operator N is $H_0(\mathbb{1}_X, X)$.

Proposition 2.4. *For any single particle operator h on X , any two particle potential v on X and any $\mu \in \mathbb{R}$, the operator $H_0(h, X) + V(v, X) - \mu N$ is essentially self adjoint on $(f_n)_{n \geq 0} f_n \in \mathcal{B}_n(X)$, for all $n \geq 0$, $f_n = 0$ for all but finitely many n . It commutes with the number operator N .*

Proof. The proof is routine. □

Creation and annihilation operators

Let $u \in \mathcal{B}_m(X)$ and let $v \in \mathcal{B}_n(X)$ and write $u \otimes v(\mathbf{x}_1, \dots, \mathbf{x}_{m+n})$ for the tensor product

$$u(\mathbf{x}_1, \dots, \mathbf{x}_m) v(\mathbf{x}_{m+1}, \dots, \mathbf{x}_{m+n}).$$

The symmetric tensor product $u \otimes_s v$ of u and v is defined by

$$u \otimes_s v = \frac{1}{\sqrt{m!(m+n)!n!}} \sum_{\pi \in S_{m+n}} \pi \cdot (u \otimes v)$$

and belongs to $\mathcal{B}_{m+n}(X)$. The symmetric tensor product extends by linearity to a commutative, associative and distributive multiplication on $\mathcal{B}(X)$. If $u \in \mathcal{B}_1(X)$ and $v \in \mathcal{B}_{n-1}(X)$, then

$$u \otimes_s v(\mathbf{x}_1, \dots, \mathbf{x}_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^n u(\mathbf{x}_i) v(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_i, \dots, \mathbf{x}_n).$$

Successively multiplying $u_1, \dots, u_n \in \mathcal{B}_1(X)$, one obtains by induction the element

$$\begin{aligned} u_1 \otimes_s \dots \otimes_s u_n(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} u_1(\mathbf{x}_{\pi(1)}) \dots u_n(\mathbf{x}_{\pi(n)}) = \frac{1}{\sqrt{n!}} \text{perm}(u_i(\mathbf{x}_j)) \end{aligned}$$

of $\mathcal{B}_n(X)$. Here $\text{perm}(u_i(\mathbf{x}_j))$ denotes the permanent of the matrix $[u_i(\mathbf{x}_j)]_{1 \leq i, j \leq n}$. By direct calculation,

$$\langle u_1 \otimes_s \dots \otimes_s u_n, v_1 \otimes_s \dots \otimes_s v_n \rangle_{\mathcal{B}_n} = \text{perm}(\langle u_i, v_j \rangle_{\mathcal{B}_1})$$

for all u_1, \dots, u_n and v_1, \dots, v_n in $\mathcal{B}_1(X)$.

Remark 2.5. It is common to define the symmetric tensor product with the alternative combinatorial factors $\frac{1}{(m+n)!}$ or $\frac{1}{m!n!}$ rather than $\frac{1}{\sqrt{m!(m+n)!n!}}$. We make the present choice because the annihilation operators become derivations on the algebra $\mathcal{B}(X)$. See, Lemma 2.11.

Proposition 2.6. *Suppose $u_i, i = 1, \dots, |X|$, is an orthonormal basis for $\mathcal{B}_1(X)$. Then, the family of functions*

$$\left\{ \frac{1}{\sqrt{\alpha!}} u^{\otimes_s \alpha} \mid \alpha \in (\mathbb{N} \cup \{0\})^{|X|}, |\alpha| = n \right\}$$

where

$$u^{\otimes_s \alpha}(\mathbf{x}_1, \dots, \mathbf{x}_n) = u_1^{\otimes_s \alpha_1} \otimes_s \dots \otimes_s u_{|X|}^{\otimes_s \alpha_{|X|}}(\mathbf{x}_1, \dots, \mathbf{x}_n)$$

is an orthonormal basis for $\mathcal{B}_n(X)$.

If Y is a subset of X , then $\mathcal{B}(Y)$ can be identified with the subalgebra

$$\left\{ \mathbf{f} = (f_0, f_1, \dots) \in \mathcal{B}(X) \mid f_n(\mathbf{x}_1, \dots, \mathbf{x}_n) \text{ is supported in } Y^n \text{ for each } n \geq 0 \right\}$$

of $\mathcal{B}(X)$. Furthermore,

Lemma 2.7. *Let Y and Z be disjoint subsets of X . For all $u, u' \in \mathcal{B}(Y)$ and $v, v' \in \mathcal{B}(Z)$*

$$\langle u \otimes_s v, u' \otimes_s v' \rangle_{\mathcal{B}(X)} = \langle u, u' \rangle_{\mathcal{B}(Y)} \langle v, v' \rangle_{\mathcal{B}(Z)}.$$

Remark 2.8. For each $\mathbf{x} \in X$, the bosonic Fock space $\mathcal{B}(\{\mathbf{x}\})$ is canonically isomorphic to a subalgebra of $\mathcal{B}(X)$. The abstract tensor product of all these subalgebras is isomorphic to $\mathcal{B}(X)$. By the last lemma, the global inner product completely factors.

Definition 2.9. For each $\mathbf{x} \in X$ the annihilation operator $\psi(\mathbf{x})$ acts on $\mathcal{B}(X)$ by

$$(\psi(\mathbf{x})u)(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \sqrt{n} u(\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_{n-1})$$

for all $u \in \mathcal{B}_n(X), n \geq 0$. The adjoint $\psi^\dagger(\mathbf{x})$ of $\psi(\mathbf{x})$ is the creation operator on $\mathcal{B}(X)$ that acts by

$$\begin{aligned} (\psi^\dagger(\mathbf{x})u)(\mathbf{x}_1, \dots, \mathbf{x}_{n+1}) &= (\delta_{\mathbf{x}} \otimes_s u)(\mathbf{x}_1, \dots, \mathbf{x}_{n+1}) \\ &= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \delta_{\mathbf{x}}(\mathbf{x}_i) u(\mathbf{x}_1, \dots, \widehat{\mathbf{x}}_i, \dots, \mathbf{x}_{n+1}) \end{aligned}$$

for all $u \in \mathcal{B}_n(X), n \geq 0$. The utility of these operators is due to the commutation relations

$$\begin{aligned} [\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')] &= \delta_{\mathbf{x}}(\mathbf{x}') \\ [\psi(\mathbf{x}), \psi(\mathbf{x}')] &= 0 \\ [\psi^\dagger(\mathbf{x}), \psi^\dagger(\mathbf{x}')] &= 0 \end{aligned} \tag{2.1}$$

for all $\mathbf{x}, \mathbf{x}' \in X$.

Remark 2.10. It is easy to see that

$$\begin{aligned} (\psi^\dagger(\mathbf{x})\psi(\mathbf{x})u)(\mathbf{x}_1, \dots, \mathbf{x}_n) &= \sum_{i=1}^n \delta_{\mathbf{x}}(\mathbf{x}_i) u(\mathbf{x}_1, \dots, \mathbf{x}, \dots, \mathbf{x}_n) \\ &= \left(\sum_{i=1}^n \delta_{\mathbf{x}}(\mathbf{x}_i) \right) u(\mathbf{x}_1, \dots, \mathbf{x}_i, \dots, \mathbf{x}_n). \end{aligned}$$

In other words, the restriction of the product $\psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ to $\mathcal{B}_n(X)$ is multiplication by the function $\sum_{i=1}^n \delta_{\mathbf{x}}(\mathbf{x}_i)$. By definition the density operator $n(\mathbf{x})$ at $\mathbf{x} \in X$ acts by

$$n(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}).$$

Lemma 2.11. *For all $\mathbf{x} \in X, u \in \mathcal{B}_m(X)$ and $v \in \mathcal{B}_n(X)$,*

$$\psi(\mathbf{x}) (u \otimes_s v) = (\psi(\mathbf{x})u) \otimes_s v + u \otimes_s (\psi(\mathbf{x})v).$$

Proof. The proof is by induction on m . □

Definition 2.12. Fix a natural number $n \geq 0$. For each Y in the quotient space X^n/S_n set

$$\delta_Y = c_Y \delta_{\mathbf{y}_1} \otimes_s \cdots \otimes_s \delta_{\mathbf{y}_n}$$

where $(\mathbf{y}_1, \dots, \mathbf{y}_n)$ in X^n is a representative for Y . The constant

$$c_Y = \frac{1}{\prod_{\mathbf{x} \in X} \sqrt{\mu_Y(\mathbf{x})!}} \quad \text{where} \quad \mu_Y(\mathbf{x}) = \sum_{i=1}^n \delta_{\mathbf{x}, \mathbf{y}_i}$$

is the multiplicity of \mathbf{x} in Y . By construction, $\delta_{\mathbf{y}}$, $\mathbf{y} \in X$, is an orthonormal basis for $L^2(X)$. By Proposition 2.6, the family $\delta_Y, Y \in X^n/S_n$, is an orthonormal basis for \mathcal{B}_n .

Lemma 2.13. Fix the point $\mathbf{x} \in X$ and the set $Y \in X^n/S_n$. Then,

$$\begin{aligned} \psi(\mathbf{x}) \delta_Y &= \begin{cases} c_Y^{-1} c_{Y \setminus \{\mathbf{x}\}} \delta_{Y \setminus \{\mathbf{x}\}} & \text{if } \mathbf{x} \in Y \\ 0 & \text{if } \mathbf{x} \notin Y \end{cases} \\ \psi^\dagger(\mathbf{x}) \delta_Y &= c_Y c_{Y \sqcup \{\mathbf{x}\}}^{-1} \delta_{Y \sqcup \{\mathbf{x}\}} \\ n(\mathbf{x}) \delta_Y &= \mu_Y(\mathbf{x}) \delta_Y. \end{aligned}$$

Here, the ‘disjoint union’ $Y \sqcup \{\mathbf{x}\}$ is the element of X^{n+1}/S_{n+1} with

$$\mu_{Y \sqcup \{\mathbf{x}\}}(\mathbf{y}) = \begin{cases} \mu_Y(\mathbf{y}) & \text{if } \mathbf{y} \neq \mathbf{x} \\ \mu_Y(\mathbf{y}) + 1 & \text{if } \mathbf{y} = \mathbf{x}. \end{cases}$$

Proposition 2.14. For any single particle operator h and any 2 particle potential v ,

$$\begin{aligned} H_0(h, X) &= \int d\mathbf{x} d\mathbf{y} \psi^\dagger(\mathbf{x}) h(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \\ V(v, X) &= \frac{1}{2} \int d\mathbf{x}_1 d\mathbf{x}_2 \psi^\dagger(\mathbf{x}_1) \psi^\dagger(\mathbf{x}_2) v(\mathbf{x}_1, \mathbf{x}_2) \psi(\mathbf{x}_1) \psi(\mathbf{x}_2) \\ &= \frac{1}{2} \int d\mathbf{x}_1 d\mathbf{x}_2 \psi^\dagger(\mathbf{x}_1) \psi(\mathbf{x}_1) v(\mathbf{x}_1, \mathbf{x}_2) \psi^\dagger(\mathbf{x}_2) \psi(\mathbf{x}_2) \\ &\quad - \frac{1}{2} \int d\mathbf{x} \psi^\dagger(\mathbf{x}) v(\mathbf{x}, \mathbf{x}) \psi(\mathbf{x}). \end{aligned}$$

Remark 2.15. We have

$$N = H_0(\mathbb{1}, X) = \int d\mathbf{x} \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) = \int d\mathbf{x} n(\mathbf{x})$$

and

$$V(v, X) = \frac{1}{2} \int d\mathbf{x}_1 d\mathbf{x}_2 n(\mathbf{x}_1) v(\mathbf{x}_1, \mathbf{x}_2) n(\mathbf{x}_2) - \frac{1}{2} \int d\mathbf{x} v(\mathbf{x}, \mathbf{x}) n(\mathbf{x}).$$

To ensure that the Hamiltonian $H_0(h, X) + V(v, X)$ is stable, we shall assume that the interaction potential v is repulsive in the sense of the following definition.

Definition 2.16. Define, for any real, symmetric, 2 particle potential $v(\mathbf{x}, \mathbf{y})$,

$$\lambda_0(v) = \inf \left\{ \int d\mathbf{x}d\mathbf{y} \rho(\mathbf{x})v(\mathbf{x}, \mathbf{y})\rho(\mathbf{y}) \mid \int d\mathbf{x} \rho(\mathbf{x})^2 = 1, \rho(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in X \right\}.$$

We call the potential v repulsive if $\lambda_0(v) > 0$.

If $v(\mathbf{x}, \mathbf{y})$ is the kernel of a strictly positive definite operator acting on $L^2(X)$, then v is repulsive with $\lambda_0(v)$ at least as large as the smallest eigenvalue of the operator. If $v(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in X$ and $v(\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \in X$, then v is repulsive with $\lambda_0(v) \geq \min_{\mathbf{x} \in X} v(\mathbf{x}, \mathbf{x})$.

Proposition 2.17. Let h be a single particle operator and $v(\mathbf{x}_1, \mathbf{x}_2)$ be a real, symmetric, pair potential. Assume that the self adjoint operator v acting on $L^2(X)$ with kernel $v(\mathbf{x}, \mathbf{y})$ is strictly positive definite. Then

$$\begin{aligned} -\frac{v_0^2}{8\lambda_0} |X| &\leq \frac{1}{2} \left(\lambda_0 \frac{N}{|X|} - v_0 \right) N \leq V(v, X) \leq \frac{1}{2} (\Lambda N - \lambda_0) N \\ \lambda'_0 N &\leq H_0(h, X) \leq \Lambda' N \end{aligned}$$

on the domain $\mathcal{D}(V) = \mathcal{D}(N^2)$. Here, $\lambda_0 = \lambda_0(v)$, Λ is the largest eigenvalues of the operator v , λ'_0 and Λ' are the smallest and largest eigenvalues of the operator h and $v_0 = \max_{\mathbf{x} \in X} v(\mathbf{x}, \mathbf{x})$. The leftmost bound on V is called the ‘linear lower bound’.

Proof. We have

$$\begin{aligned} \left\langle \delta_Y, \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 n(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)n(\mathbf{x}_2) \delta_Y \right\rangle &= \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 \mu_Y(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)\mu_Y(\mathbf{x}_2) \\ &\geq \lambda_0 \int_X d\mathbf{x} \mu_Y^2(\mathbf{x}) \\ &\geq \frac{\lambda_0}{|X|} \left(\int_X d\mathbf{x} \mu_Y(\mathbf{x}) \right)^2 \\ &= \frac{\lambda_0}{|X|} n^2 \end{aligned}$$

where Schwarz’s inequality was used in the third line. Similarly,

$$\left\langle \delta_Y, \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 n(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)n(\mathbf{x}_2) \delta_Y \right\rangle \leq \Lambda n^2.$$

Let $u = \sum_{Y \in X^n/S_n} \varphi_Y \delta_Y$ be an arbitrary element of $\mathcal{B}_n(X)$. We have

$$\begin{aligned} & \left\langle u, \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 n(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)n(\mathbf{x}_2) u \right\rangle \\ &= \sum_{Y_1, Y_2 \in X^n/S_n} \varphi_{Y_1}^* \varphi_{Y_2} \left\langle \delta_{Y_1}, \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 n(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)n(\mathbf{x}_2) \delta_{Y_2} \right\rangle \\ &= \sum_{Y \in X^n/S_n} |\varphi_Y|^2 \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 \mu_Y(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)\mu_Y(\mathbf{x}_2) \\ &\geq \frac{\lambda_0}{|X|} n^2 \sum_{Y \in X^n/S_n} |\varphi_Y|^2 \\ &= \frac{\lambda_0}{|X|} n^2 \|u\|_{\mathcal{B}_n}^2. \end{aligned}$$

In the same way,

$$\left\langle u, \int_{X^2} d\mathbf{x}_1 d\mathbf{x}_2 n(\mathbf{x}_1)v(\mathbf{x}_1, \mathbf{x}_2)n(\mathbf{x}_2) u \right\rangle \leq \Lambda n^2 \|u\|_{\mathcal{B}_n}^2.$$

Since $a \leq v(x, x) \leq b$, a similar, but simpler, argument gives

$$\lambda_0 n \|u\|_{\mathcal{B}_n}^2 \leq \left\langle u, \int_X d\mathbf{x} v(\mathbf{x}, \mathbf{x})n(\mathbf{x}) u \right\rangle \leq v_0 n \|u\|_{\mathcal{B}_n}^2.$$

The bound on $H_0(h, X)$ follows directly from Definition 2.2.i. □

Coherent states

We now review the formalism of coherent states [6, §1.5].

Definition 2.18. (i) The family $|z\delta_{\mathbf{x}}\rangle$, $z \in \mathbb{C}$, of coherent states concentrated at $\mathbf{x} \in X$ is given by

$$|z\delta_{\mathbf{x}}\rangle = \sum_{n \geq 0} \frac{1}{n!} z^n \psi^\dagger(\mathbf{x})^n \mathbf{1} = \sum_{n \geq 0} \frac{1}{n!} z^n \delta_{\mathbf{x}}^{\otimes n}$$

where $\mathbf{1}$ is the ‘vacuum’ $1 \in \mathbb{C} = \mathcal{B}_0(X)$. That $|z\delta_{\mathbf{x}}\rangle \in \mathcal{B}(X)$ is a consequence of Proposition 2.22, below.

(ii) If $\phi(\mathbf{y}) \in L^2(X)$, the coherent state $|\phi\rangle \in \mathcal{B}(X)$ is

$$|\phi\rangle = \bigotimes_{\mathbf{y} \in X} |\phi(\mathbf{y})\delta_{\mathbf{y}}\rangle = e^{\int d\mathbf{y} \phi(\mathbf{y})\psi^\dagger(\mathbf{y})} \mathbf{1}.$$

Lemma 2.19. For all ϕ in $L^2(X)$,

$$|\phi\rangle = \sum_{n \geq 0} \sum_{Y \in X^n/S_n} \phi(Y) c_Y \delta_Y.$$

Here, $\phi(Y) = \prod_{\mathbf{y} \in X} \phi^{\mu_Y(\mathbf{y})}(\mathbf{y})$ for each Y in the quotient space X^n/S_n .

Proof.

$$\begin{aligned}
 |\phi\rangle &= \bigotimes_{\mathbf{y} \in X} \sum_{n \geq 0} \frac{1}{n!} \phi^n(\mathbf{y}) \delta_{\mathbf{y}}^{\otimes n} = \sum_{\substack{n_{\mathbf{y}} \geq 0 \\ \mathbf{y} \in X}} \left(\prod_{\mathbf{y} \in X} \frac{1}{n_{\mathbf{y}}!} \phi^{n_{\mathbf{y}}}(\mathbf{y}) \right) \bigotimes_{\mathbf{y} \in X} \delta_{\mathbf{y}}^{\otimes n_{\mathbf{y}}} \\
 &= \sum_{n \geq 0} \sum_{\substack{n_{\mathbf{y}} \geq 0 \\ \mathbf{y} \in X}} \chi \left(n = \sum_{\mathbf{y} \in X} n_{\mathbf{y}} \right) \left(\prod_{\mathbf{y} \in X} \frac{1}{n_{\mathbf{y}}!} \phi^{n_{\mathbf{y}}}(\mathbf{y}) \right) \bigotimes_{\mathbf{y} \in X} \delta_{\mathbf{y}}^{\otimes n_{\mathbf{y}}} \\
 &= \sum_{n \geq 0} \sum_{Y \in X^n / S_n} \left(\prod_{\mathbf{y} \in X} \phi^{\mu_Y(\mathbf{y})}(\mathbf{y}) \right) c_Y^2 \bigotimes_{\mathbf{y} \in X} \delta_{\mathbf{y}}^{\otimes \mu_Y(\mathbf{y})} \\
 &= \sum_{n \geq 0} \sum_{Y \in X^n / S_n} \phi(Y) c_Y \delta_Y. \quad \square
 \end{aligned}$$

Coherent states have been defined to give

Proposition 2.20. *For all $\mathbf{x} \in X$ and $\phi \in L^2(X)$,*

$$\begin{aligned}
 \psi(\mathbf{x}) |\phi\rangle &= \phi(\mathbf{x}) |\phi\rangle \\
 \psi^\dagger(\mathbf{x}) |\phi\rangle &= \frac{\partial}{\partial \phi(\mathbf{x})} |\phi\rangle.
 \end{aligned}$$

Convention 2.21. *For any ϕ in $L^2(X)$ and any state \mathbf{f} in the Fock space $\mathcal{B}(X)$, abusing notation, the inner product between the coherent state $|\phi\rangle$ and \mathbf{f} is written as $\langle \phi | \mathbf{f} \rangle$. That is, $\langle \phi | \mathbf{f} \rangle = \langle |\phi\rangle, \mathbf{f} \rangle_{\mathcal{B}}$. Similarly, the inner product between the coherent states $|\phi\rangle$ and $|\phi'\rangle$ is written $\langle \phi | \phi' \rangle$.*

Proposition 2.22. *For all α and γ in $L^2(X)$, we have $\|P^{(m)} |\alpha\rangle\|_{\mathcal{B}_m} = \frac{\|\alpha\|^m}{\sqrt{m!}}$ and*

$$\langle \alpha | \gamma \rangle = e^{\int d\mathbf{y} \overline{\alpha(\mathbf{y})} \gamma(\mathbf{y})}$$

where $P^{(m)}$ is the orthogonal projection from $\mathcal{B}(X)$ onto the m particle subspace $\mathcal{B}_m(X)$.

Lemma 2.23. *For any single particle operator h and any ϕ in $L^2(X)$,*

$$e^{-\tau H_0(h, X)} |\phi\rangle = |e^{-\tau h} \phi\rangle.$$

Proof. By Proposition 2.14,

$$\frac{d}{d\tau} e^{-\tau H_0} |\phi\rangle = - \int d\mathbf{x}_1 d\mathbf{x}_2 \psi^\dagger(\mathbf{x}_1) h(\mathbf{x}_1, \mathbf{x}_2) \psi(\mathbf{x}_2) e^{-\tau H_0} |\phi\rangle$$

and, by Definition 2.18.ii,

$$\begin{aligned} \frac{d}{d\tau} | e^{-\tau h} \phi \rangle &= \frac{\partial}{\partial \tau} e^{\int d\mathbf{y} (e^{-\tau h} \phi)(\mathbf{y}) \psi^\dagger(\mathbf{y})} \mathbf{1} \\ &= \int d\mathbf{y} (-h e^{-\tau h} \phi)(\mathbf{y}) \psi^\dagger(\mathbf{y}) e^{\int d\mathbf{y} (e^{-\tau h} \phi)(\mathbf{y}) \psi^\dagger(\mathbf{y})} \mathbf{1} \\ &= - \int d\mathbf{y} \psi^\dagger(\mathbf{y}) (h e^{-\tau h} \phi)(\mathbf{y}) | e^{-\tau h} \phi \rangle \\ &= - \int d\mathbf{x}_1 d\mathbf{x}_2 \psi^\dagger(\mathbf{x}_1) h(\mathbf{x}_1, \mathbf{x}_2) \psi(\mathbf{x}_2) | e^{-\tau h} \phi \rangle. \end{aligned}$$

As $\frac{d}{d\tau} e^{-\tau H_0} | \phi \rangle$ and $\frac{d}{d\tau} | e^{-\tau h} \phi \rangle$ satisfy the same first order ordinary differential equation and initial condition, they coincide. \square

Lemma 2.24. *Suppose both \mathbf{f} and $N^{2|X|}\mathbf{f}$ belong to the Fock space \mathcal{B} . Then,*

$$|\langle \phi | \mathbf{f} \rangle| \leq \left(\prod_{\mathbf{y} \in X} \frac{1}{1 + |\phi(\mathbf{y})|^4} \right)^{\frac{1}{2}} 2^{|X|} e^{\frac{1}{2} \|\phi\|^2} \|(N + 4|X|)^{2|X|} \mathbf{f}\|_{\mathcal{B}}.$$

Proof. Fix P in X^n/S_n . Then,

$$\begin{aligned} \phi(P) \langle \phi | \mathbf{f} \rangle &= \sum_Y C_Y \phi(P) \phi(Y) \langle \delta_Y, \mathbf{f} \rangle_{\mathcal{B}} \\ &= \sum_Y C_{P \sqcup Y} \phi(P \sqcup Y) \left\langle \frac{C_Y}{C_{P \sqcup Y}} \delta_Y, \mathbf{f} \right\rangle_{\mathcal{B}} \\ &= \sum_Y C_{P \sqcup Y} \phi(P \sqcup Y) \left\langle \left[\prod_{\mathbf{x} \in X} \frac{\mu_{P \sqcup Y}(\mathbf{x})!}{\mu_Y(\mathbf{x})!} \right]^{\frac{1}{2}} \delta_Y, \mathbf{f} \right\rangle_{\mathcal{B}} \\ &= \sum_Y C_{P \sqcup Y} \phi(P \sqcup Y) \left\langle \prod_{\mathbf{x} \in X} \prod_{k=1}^{\mu_P(\mathbf{x})} (n(\mathbf{x}) + k)^{\frac{1}{2}} \delta_Y, \mathbf{f} \right\rangle_{\mathcal{B}} \\ &= \sum_Y C_{P \sqcup Y} \phi(P \sqcup Y) \left\langle \delta_Y, \prod_{\mathbf{x} \in X} \prod_{k=1}^{\mu_P(\mathbf{x})} (n(\mathbf{x}) + k)^{\frac{1}{2}} \mathbf{f} \right\rangle_{\mathcal{B}}. \end{aligned}$$

It follows from Schwarz's inequality and Parseval's identity that

$$\begin{aligned} |\phi(P) \langle \phi | \mathbf{f} \rangle| &\leq \left(\sum_Y |C_{P \sqcup Y} \phi(P \sqcup Y)|^2 \right)^{\frac{1}{2}} \left\| \prod_{\mathbf{x} \in X} \prod_{k=1}^{\mu_P(\mathbf{x})} (n(\mathbf{x}) + k)^{\frac{1}{2}} \mathbf{f} \right\|_{\mathcal{B}} \\ &\leq \left(\sum_Y |C_Y \phi(Y)|^2 \right)^{\frac{1}{2}} \left\| \prod_{\mathbf{x} \in X} \prod_{k=1}^{\mu_P(\mathbf{x})} (n(\mathbf{x}) + k)^{\frac{1}{2}} \mathbf{f} \right\|_{\mathcal{B}} \\ &\leq e^{\frac{1}{2} \|\phi\|^2} \|(N + n)^{\frac{n}{2}} \mathbf{f}\|_{\mathcal{B}}. \end{aligned}$$

Adding,

$$|\langle \phi | \mathbf{f} \rangle| \prod_{\mathbf{y} \in X} (1 + |\phi(\mathbf{y})|^4) \leq 2^{|X|} e^{\frac{1}{2} \|\phi\|^2} \|(N + 4|X|)^{2|X|} \mathbf{f}\|_{\mathcal{B}}. \quad \square$$

An approximate resolution of the identity

In the physics literature, the formal resolution of the identity

$$\mathbb{1} = \int \prod_{\mathbf{x} \in X} \left[\frac{d\phi^*(\mathbf{x})d\phi(\mathbf{x})}{2\pi i} \right] e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} |\phi\rangle \langle \phi|$$

is often used. See, for example [6, (1.123)]. However, for each $\phi \in L^2(T)$, the operator norm of $e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} |\phi\rangle \langle \phi|$ is exactly one. So the nature of the convergence of the integral in the above formal resolution of the identity is not clear. We now investigate the convergence more carefully.

Definition 2.25. For each $r > 0$, the measure $d\mu_r(\phi^*, \phi)$ on $L^2(X)$ is given by

$$d\mu_r(\phi^*, \phi) = \prod_{\mathbf{x} \in X} \left[\frac{d\phi^*(\mathbf{x})d\phi(\mathbf{x})}{2\pi i} \chi_r(|\phi(\mathbf{x})|) \right]$$

where χ_r is the characteristic function of the interval $[0, r]$. The measure $d\mu(\phi^*, \phi)$ on $L^2(X)$ is given by

$$d\mu(\phi^*, \phi) = \prod_{\mathbf{x} \in X} \left[\frac{d\phi^*(\mathbf{x})d\phi(\mathbf{x})}{2\pi i} \right].$$

Theorem 2.26. For each $r > 0$, let I_r be the operator that acts on \mathbf{f} in $\mathcal{B}(X)$ by

$$I_r \mathbf{f} = \int d\mu_r(\phi^*, \phi) e^{-\|\phi\|^2} |\phi\rangle \langle \phi | \mathbf{f} \rangle.$$

(a) For all $n \geq 0$ and all $Y \in X^n/S_n$,

$$I_r \delta_Y = \lambda_r(Y) \delta_Y$$

where

$$\lambda_r(Y) = \prod_{\mathbf{x} \in X} \frac{\Gamma_r(\mu_Y(\mathbf{x}))}{\mu_Y(\mathbf{x})!} \quad \text{with} \quad \Gamma_r(s) = \int_0^{r^2} dt e^{-t} t^s, \quad \text{for all } s > -1.$$

In particular, $0 \leq \lambda_r(Y) \leq 1$.

(b) I_r commutes with N .

(c) The operator norm of I_r is bounded by one for all r and I_r converges strongly to the identity operator as $r \rightarrow \infty$.

(d) For all n and r , the operator norm

$$\|(\mathbb{1} - I_r) P_n\| \leq |X| 2^{n+1} e^{-r^2/2}.$$

Here, P_n is the orthogonal projection from $\mathcal{B}(X)$ onto the direct sum

$$\bigoplus_{0 \leq m \leq n} \mathcal{B}_m(X).$$

(e) Suppose that $N^{2|X|}\mathbf{f}$ belongs to the Fock space \mathcal{B} . Then,

$$\mathbf{f} = \int d\mu(\phi^*, \phi) e^{-\|\phi\|^2} |\phi\rangle \langle \phi | \mathbf{f} \rangle$$

with the integral converging absolutely.

Proof. (a,b) To verify (a), let $Y \in X^n/S_n$ and observe that

$$\begin{aligned} I_r \delta_Y &= \int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} |\phi\rangle \langle \phi | \delta_Y \rangle \\ &= \sum_{n \geq 0} \sum_{Y' \in X^n/S_n} \delta_{Y', c_Y, c_Y} \int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \phi(Y') \phi^*(Y). \end{aligned}$$

We have

$$\begin{aligned} &\int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \phi^*(Y) \phi(Y') \\ &= \prod_{\mathbf{x} \in X} \left[\int \frac{d\phi^*(\mathbf{x}) d\phi(\mathbf{x})}{2\pi i} \chi_r(|\phi(\mathbf{x})|) e^{-|\phi(\mathbf{x})|^2} \phi^*(\mathbf{x})^{\mu_Y(\mathbf{x})} \phi(\mathbf{x})^{\mu_{Y'}(\mathbf{x})} \right] \\ &= \prod_{\mathbf{x} \in X} \left[\int_0^r d\rho \int_0^{2\pi} \frac{d\theta}{\pi} \rho e^{-\rho^2} \rho^{\mu_Y(\mathbf{x}) + \mu_{Y'}(\mathbf{x})} e^{i\theta(\mu_{Y'}(\mathbf{x}) - \mu_Y(\mathbf{x}))} \right] \\ &= \prod_{\mathbf{x} \in X} \left[\delta_{\mu_Y(\mathbf{x}), \mu_{Y'}(\mathbf{x})} \int_0^r d\rho 2\rho e^{-\rho^2} \rho^{2\mu_Y(\mathbf{x})} \right] \\ &= \delta_{Y, Y'} \prod_{\mathbf{x} \in X} \int_0^{r^2} dt e^{-t} t^{\mu_Y(\mathbf{x})} \\ &= \delta_{Y, Y'} \prod_{\mathbf{x} \in X} \Gamma_r(\mu_Y(\mathbf{x})). \end{aligned}$$

Consequently,

$$I_r \delta_Y = \delta_Y c_Y^2 \prod_{\mathbf{x} \in X} \Gamma_r(\mu_Y(\mathbf{x})) = \delta_Y \prod_{\mathbf{x} \in X} \frac{\Gamma_r(\mu_Y(\mathbf{x}))}{\mu_Y(\mathbf{x})!}.$$

Thus I_r is a diagonal operator in the orthonormal basis $\{\delta_Y\}$. Each diagonal entry

$$\lambda_r(Y) = \prod_{\mathbf{x} \in X} \left[\frac{1}{\mu_Y(\mathbf{x})!} \int_0^{r^2} dt e^{-t} t^{\mu_Y(\mathbf{x})} \right]$$

is between 0 and 1. Since $\{\delta_Y\}$ is a basis of eigenvectors for both I_r and N , they commute, which proves parts (a) and (b).

(c) Each

$$\lambda_r(Y) = \prod_{\mathbf{x} \in X} \left[\frac{1}{\mu_Y(\mathbf{x})!} \int_0^{r^2} dt e^{-t} t^{\mu_Y(\mathbf{x})} \right] \tag{2.2}$$

approaches 1 in the limit $r \rightarrow \infty$. By the Lebesgue dominated convergence theorem, if $f = \sum_n \sum_{Y \in X^n/S_n} f_Y \delta_Y$ is any vector in $\mathcal{B}(X)$, then

$$\lim_{r \rightarrow \infty} \|f - I_r f\|^2 = \lim_{r \rightarrow \infty} \sum_n \sum_{Y \in X^n/S_n} (1 - \lambda_r(Y))^2 |f_Y|^2 = 0.$$

The operator norm of I_r is bounded by 1 for all $r > 0$, because all of the eigenvalues of I_r are between 0 and 1. This completes the proof of part (c).

(d) We bound

$$\|(\mathbb{1} - I_r) P_n\| = \max_{|Y| \leq n} [1 - \lambda_r(Y)].$$

Fix any Y with $|Y| \leq n$ and set, for each $\mathbf{x} \in X$,

$$\beta_{\mathbf{x}} = \frac{1}{\mu_Y(\mathbf{x})!} \int_{r^2}^{\infty} dt e^{-t} t^{\mu_Y(\mathbf{x})}.$$

By (2.2)

$$1 - \lambda_r(Y) = 1 - \prod_{\mathbf{x} \in X} (1 - \beta_{\mathbf{x}}) \leq \sum_{\mathbf{x} \in X} \beta_{\mathbf{x}} \leq |X| \max_{\mathbf{x} \in X} \beta_{\mathbf{x}}.$$

The claim now follows from

$$\begin{aligned} \beta_{\mathbf{x}} &= \frac{2^{\mu_Y(\mathbf{x})}}{\mu_Y(\mathbf{x})!} \int_{r^2}^{\infty} dt e^{-t} \left(\frac{t}{2}\right)^{\mu_Y(\mathbf{x})} \leq 2^{\mu_Y(\mathbf{x})} \int_{r^2}^{\infty} dt e^{-t/2} = 2^{\mu_Y(\mathbf{x})+1} e^{-r^2/2} \\ &\leq 2^{n+1} e^{-r^2/2}. \end{aligned}$$

(e) By definition

$$I_r \mathbf{f} = \int d\mu(\phi^*, \phi) \prod_{\mathbf{x} \in X} \chi_r(|\phi(\mathbf{x})|) e^{-\|\phi\|^2} |\phi\rangle \langle \phi | \mathbf{f} \rangle.$$

By part (c) the left hand side converges to \mathbf{f} as $r \rightarrow \infty$. By Lemma 2.24, the norm of the integrand of the right hand side is bounded by

$$\left(\prod_{\mathbf{y} \in X} \frac{1}{1 + |\phi(\mathbf{y})|^4} \right) 2^{|X|} \|(N + 4|X|)^{2|X|} \mathbf{f}\|_{\mathcal{B}}$$

which is integrable with respect to $d\mu(\phi^*, \phi)$. Hence, as $r \rightarrow \infty$, the right hand side converges to $\int d\mu(\phi^*, \phi) e^{-\|\phi\|^2} |\phi\rangle \langle \phi | \mathbf{f} \rangle$. \square

The trace formula

Another commonly used formal property of coherent states is the trace formula (1.4). We now develop a rigorous, but limited, version of this formula that is adequate for our purposes.

Remark 2.27. We use the approximate identity I_r to prove a ‘‘cutoff’’ trace for any bounded operator that commutes with N . By (2.2), for each fixed $Y \in X^n/S_n$,

$\lim_{r \rightarrow \infty} \lambda_r(Y) = 1$. On the other hand, for each fixed r , there is a constant $C_{r,X} = r^2|X|$ such that

$$\max_{Y \in X^n/S_n} \lambda_r(Y) \leq \frac{1}{n!} C_{r,X}^n.$$

Thus the operator norm of I_r , restricted to $\mathcal{B}_n(X)$, is bounded by $\frac{1}{n!} C_{r,X}^n$. The dimension of $\mathcal{B}_n(X)$ is bounded by $|X|^n$. Therefore, for any bounded operator B that commutes with N , BI_r is trace class and

$$\text{Tr } BI_r = \lim_{n \rightarrow \infty} \text{Tr } BI_r P_n.$$

Proposition 2.28. (a) *Let B be a bounded operator on $\mathcal{B}(X)$ that commutes with N . Then, for all $r > 0$, BI_r is trace class and*

$$\text{Tr } BI_r = \int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \phi | B | \phi \rangle.$$

(b) *Let B be any operator on $\mathcal{B}(X)$ that commutes with N and obeys*

$$\|P^{(n)}B\|_{\mathcal{B}_n} \leq \text{const} (1+n)^{-2|X|}.$$

Then B is trace class and

$$\text{Tr } B = \int d\mu(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \phi | B | \phi \rangle.$$

Proof. (a) We have

$$\begin{aligned} \text{Tr } BI_r P_n &= \sum_{\text{deg } Y \leq n} \langle \delta_Y | BI_r | \delta_Y \rangle \\ &= \sum_{\text{deg } Y \leq n} \int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \delta_Y | B | \phi \rangle \langle \phi | \delta_Y \rangle \\ &= \int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \sum_{\text{deg } Y \leq n} \langle \phi | \delta_Y \rangle \langle \delta_Y | B | \phi \rangle \\ &= \int d\mu_r(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \phi | P_n B | \phi \rangle. \end{aligned}$$

Since B is a bounded operator, $e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} |\langle \phi | P_n B | \phi \rangle|$ is bounded uniformly in n and ϕ and the dominated convergence theorem provides the limit of the right hand side as $n \rightarrow \infty$. By Remark 2.27, the left hand side converges to $\text{Tr } BI_r$ as $n \rightarrow \infty$.

(b) As in part (a), but using part (e) of Theorem 2.26,

$$\text{Tr } BP_n = \int d\mu(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \phi | P_n B | \phi \rangle.$$

By Lemma 2.24,

$$\begin{aligned} |\langle \phi | P_n B | \phi \rangle| &\leq \left(\prod_{\mathbf{y} \in X} \frac{1}{1 + |\phi(\mathbf{y})|^4} \right) 2^{|X|} e^{\frac{1}{2} \|\phi\|^2} \left\| (N + 4|X|)^{2|X|} P_n B | \phi \right\|_B \\ &\leq e^{\|\phi\|^2} \left(\prod_{\mathbf{y} \in X} \frac{1}{1 + |\phi(\mathbf{y})|^4} \right) 2^{|X|} \left\| (N + 4|X|)^{2|X|} B \right\|. \end{aligned}$$

By the Lebesgue dominated convergence theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \int d\mu(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \phi | P_n B | \phi \rangle \\ = \int d\mu(\phi^*, \phi) e^{-\int d\mathbf{y} |\phi(\mathbf{y})|^2} \langle \phi | B | \phi \rangle. \end{aligned}$$

Since the dimension of $\mathcal{B}_n(X)$ is $\frac{(n+|X|-1)!}{n!(|X|-1)!} \leq \frac{(n+|X|-1)^{|X|-1}}{(|X|-1)!}$ and the operator norm of the restriction of B to $\mathcal{B}_n(X)$ is bounded by a constant times $(1+n)^{-2|X|}$, B is trace class and

$$\lim_{n \rightarrow \infty} \text{Tr} BP_n = \text{Tr} B. \quad \square$$

3. An integral representation of the partition function

Let h be a single particle operator on X and $v(\mathbf{x}_1, \mathbf{x}_2)$ a real, symmetric, pair potential which is repulsive in the sense of Definition 2.16. For the rest of this paper, except where otherwise stated, we write

$$K = K(h, v, X, \mu) = H_0(h, X) + V(v, X) - \mu N.$$

Recall that $H_0(h, X)$ and $V(v, X)$ were defined in Definition 2.2.

The first step in the formal derivation of the functional integral representation (1.1) is the application of the resolution of the identity (1.3) and the trace formula (1.4) to give the intermediate representation (1.5). Theorem 3.1, below, is a rigorous version of (1.5).

Theorem 3.1. *Suppose that the sequence $R(p)$ obeys*

$$\lim_{p \rightarrow \infty} p e^{-\frac{1}{2} R(p)^2} = 0.$$

Then,

$$\text{Tr} e^{-\beta K} = \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{T}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\int d\mathbf{y} |\phi_\tau(\mathbf{y})|^2} \right] \prod_{\tau \in \mathcal{T}_p} \langle \phi_{\tau-\varepsilon} | e^{-\varepsilon K} | \phi_\tau \rangle$$

where $\mathcal{T}_p = \{ \tau = q\varepsilon \mid q = 1, \dots, p \}$ and we use the conventions that $\varepsilon = \frac{\beta}{p}$ and $\phi_0 = \phi_{p\varepsilon} = \phi_\beta$.

In Proposition 3.2, below, we prove that the grand canonical partition function $\text{Tr } e^{-\beta K}$ is well-defined. Then we prove Lemma 3.4, which provides a rigorous multiple insertion of the approximate resolution of the identity in our context. The proof of Theorem 3.1 then follows Remark 3.5.

Proposition 3.2. *For any $\beta > 0$, the trace*

$$\text{Tr } e^{-\beta(H_0(h,X)+zV(v,X)-\mu N)}$$

on the Fock space $\mathcal{B}(X)$ is a holomorphic function of (z, μ) on $\{ z \in \mathbb{C} \mid \text{Re } z > 0 \} \times \mathbb{C}$.

Proof. Suppressing h, v and X ,

$$\text{Tr } e^{-\beta(H_0+zV-\mu N)} = \sum_{n \geq 0} \text{Tr}_n e^{-\beta(H_0+zV-\mu N)} = \sum_{n \geq 0} e^{\beta \mu n} \text{Tr}_n e^{-\beta(H_0+zV)} \quad (3.1)$$

where Tr_n denotes the trace on the n particle space \mathcal{B}_n . When restricted to \mathcal{B}_n , the Hamiltonian $H_0 + zV$ is an operator on a finite dimensional vector space. Therefore, each term $\text{Tr}_n e^{-\beta(H_0+zV)}$ is an entire function of (z, μ) on \mathbb{C}^2 . For each $n \geq 0$,

$$|e^{\beta \mu n} \text{Tr}_n e^{-\beta(H_0+zV)}| \leq e^{\beta |\text{Re } \mu| n} \dim \mathcal{B}_n(X) \|e^{-\beta(H_0+zV)}\|_{\mathcal{B}_n}$$

where $\| \cdot \|_{\mathcal{B}_n}$ is the operator norm on $\mathcal{B}_n(X)$. By the Trotter product formula,

$$\begin{aligned} \|e^{-\beta(H_0+zV)}\|_{\mathcal{B}_n} &= \lim_{m \rightarrow \infty} \left\| \left(e^{-\frac{\beta}{m}(H_0+\text{Re } zV)} e^{-i\frac{\beta}{m}\text{Im } zV} \right)^m \right\|_{\mathcal{B}_n} \\ &\leq \lim_{m \rightarrow \infty} \|e^{-\frac{\beta}{m}(H_0+\text{Re } zV)}\|_{\mathcal{B}_n}^m \|e^{-i\frac{\beta}{m}\text{Im } zV}\|_{\mathcal{B}_n}^m \\ &= \lim_{m \rightarrow \infty} \|e^{-\frac{\beta}{m}(H_0+\text{Re } zV)}\|_{\mathcal{B}_n}^m. \end{aligned}$$

By Proposition 2.17,

$$\|e^{-\frac{\beta}{m}(H_0+\text{Re } zV)}\|_{\mathcal{B}_n}^m \leq e^{-\beta[\lambda'_0 n + \frac{1}{2} \text{Re } z(\frac{\lambda_0}{|X|} n^2 - v_0 n)]}$$

with the $\lambda_0 = \lambda_0(v)$ of Definition 2.16. Since $\dim \mathcal{B}_n(X) \leq |X|^n$, the sum in (3.1) is absolutely convergent, uniformly for (z, μ) in compact subsets of $\{ z \in \mathbb{C} \mid \text{Re } z > 0 \} \times \mathbb{C}$. This gives the desired analyticity. \square

Remark 3.3. Observe that

$$\text{Tr } e^{-\beta(H_0(h,X)+zV(v,X)-\mu N)} = \infty$$

when $z = 0$ and μ is strictly bigger than the smallest eigenvalue of h . This indicates that the “free” limit $z \searrow 0$ is extremely singular.

Lemma 3.4. *Let $\beta > 0$ and let K be any self adjoint operator on $\mathcal{B}(X)$ that commutes with N and obeys*

$$K \geq a \left(\frac{N}{|X|} - \nu \right) N$$

for some constants $a > 0$ and ν . Also, let R be any map from \mathbb{N} to $(0, \infty)$ such that

$$\lim_{p \rightarrow \infty} p e^{-\frac{1}{2}R(p)^2} = 0.$$

Then,

$$\begin{aligned} \text{Tr } e^{-\beta K} &= \lim_{p \rightarrow \infty} \text{Tr} \left(e^{-\frac{\beta}{p}K} \mathbb{I}_{R(p)} \right)^{p-1} e^{-\frac{\beta}{p}K} \\ &= \lim_{p \rightarrow \infty} \text{Tr} \underbrace{e^{-\frac{\beta}{p}K} \mathbb{I}_{R(p)} e^{-\frac{\beta}{p}K} \mathbb{I}_{R(p)} \cdots \mathbb{I}_{R(p)} e^{-\frac{\beta}{p}K}}_{p \text{ exponentials } e^{-\frac{\beta}{p}K}} \end{aligned}$$

and

$$\langle \phi | e^{-\beta K} | \phi' \rangle = \lim_{p \rightarrow \infty} \left\langle \phi \left| \left(e^{-\frac{\beta}{p}K} \mathbb{I}_{R(p)} \right)^{p-1} e^{-\frac{\beta}{p}K} \right| \phi' \right\rangle$$

for all $\phi, \phi' \in L^2(X)$. Furthermore, for each $0 < \eta < 1$, there is a constant C_η , depending on η, β, a and ν , but independent of $\phi, \phi' \in L^2(X), p \in \mathbb{N}$ and X such that

$$\left| \left\langle \phi \left| \left(e^{-\frac{\beta}{p}K} \mathbb{I}_r \right)^{p-1} e^{-\frac{\beta}{p}K} \right| \phi' \right\rangle \right| \leq e^{\frac{\eta}{2}(\|\phi\|^2 + \|\phi'\|^2)} e^{C_\eta |X|}.$$

Proof. Introduce the local notation

$$A_i = \begin{cases} e^{-\frac{\beta}{p}K} & \text{if } i \text{ is odd} \\ \mathbb{I}_{R(p)} & \text{if } i \text{ is even} \end{cases} \quad B_i = \begin{cases} e^{-\frac{\beta}{p}K} & \text{if } i \text{ is odd} \\ \mathbb{1} & \text{if } i \text{ is even} \end{cases}$$

so that

$$\left(e^{-\frac{\beta}{p}K} \mathbb{I}_{R(p)} \right)^{p-1} e^{-\frac{\beta}{p}K} = \prod_{i=1}^{2p-1} A_i \quad \text{and} \quad e^{-\beta K} = \prod_{i=1}^{2p-1} B_i.$$

For any $n \in \mathbb{N}$,

$$\begin{aligned} &\left| \text{Tr} \left(\prod_{i=1}^{2p-1} A_i - e^{-\beta K} \right) \right| \\ &\leq \left| \text{Tr} \left(\prod_{i=1}^{2p-1} A_i - e^{-\beta K} \right) P_n \right| + \left| \text{Tr} \prod_{i=1}^{2p-1} A_i (\mathbb{1} - P_n) \right| + \left| \text{Tr } e^{-\beta K} (\mathbb{1} - P_n) \right| \\ &= \left| \text{Tr} \left(\prod_{i=1}^{2p-1} A_i - \prod_{i=1}^{2p-1} B_i \right) P_n \right| \tag{3.2} \end{aligned}$$

$$+ \left| \text{Tr} \prod_{i=1}^{2p-1} A_i (\mathbb{1} - P_n) \right| + \left| \text{Tr } e^{-\beta K} (\mathbb{1} - P_n) \right|. \tag{3.3}$$

Consider the first line, (3.2). Since $\|\mathbb{I}_{R(p)}\| \leq 1$, by part (c) of Theorem 2.26, and

$$K \geq \frac{a}{|X|} (N - \nu|X|)N \geq -\frac{a}{4}|X|\nu^2 \equiv -K_0$$

we have

$$\|A_i\|, \|B_i\| \leq \begin{cases} e^{\frac{\beta}{p}K_0} & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Since $A_q - B_q = I_{R(p)} - \mathbb{1}$ for $i = 2, 4, \dots, 2p - 2$ and is zero otherwise, we have, for all $n, p \in \mathbb{N}$,

$$\begin{aligned} \left\| \left(\prod_{i=1}^{2p-1} A_i - \prod_{i=1}^{2p-1} B_i \right) P_n \right\| &\leq \sum_{q=1}^{2p-1} \left\| \prod_{i=1}^{q-1} A_i (A_q - B_q) \prod_{i=q+1}^{2p-1} B_i P_n \right\| \\ &\leq (p-1) e^{K_0\beta} \| (I_{R(p)} - \mathbb{1}) P_n \| \\ &\leq p e^{K_0\beta} |X| 2^{n+1} e^{-R(p)^2/2}. \end{aligned}$$

Consequently,

$$\begin{aligned} \left| \text{Tr} \left(\left(e^{-\frac{\beta}{p}K} I_{R(p)} \right)^{p-1} e^{-\frac{\beta}{p}K} - e^{-\beta K} \right) P_n \right| &\leq p e^{K_0\beta} |X| 2^{n+1} e^{-R(p)^2/2} \text{Tr} P_n \\ &\leq p e^{K_0\beta} |X| 2^{n+1} e^{-R(p)^2/2} \frac{(n + |X|)!}{n!|X|!}. \end{aligned}$$

Hence, for any fixed $n \in \mathbb{N}$,

$$\lim_{p \rightarrow \infty} \left| \text{Tr} \left(\left(e^{-\frac{\beta}{p}K} I_{R(p)} \right)^{p-1} e^{-\frac{\beta}{p}K} - e^{-\beta K} \right) P_n \right| = 0.$$

Now we consider the second line, (3.3). For all $m \geq 1, K|_{\mathcal{B}_m} \geq a\left(\frac{m}{|X|} - \nu\right)m$ and

$$\left\| \prod_{i=1}^{2p-1} A_i \Big|_{\mathcal{B}_m} \right\|, \left\| e^{-\beta K} \Big|_{\mathcal{B}_m} \right\| \leq e^{-\beta a\left(\frac{m}{|X|} - \nu\right)m} \tag{3.4}$$

and it follows that

$$\left| \text{Tr} \prod_{i=1}^{2p-1} A_i \Big|_{\mathcal{B}_m} \right|, \text{Tr} e^{-\beta K} \Big|_{\mathcal{B}_m} \leq e^{-\beta a\left(\frac{m}{|X|} - \nu\right)m} \frac{(m + |X| - 1)!}{m!(|X| - 1)!}.$$

If we impose the stronger condition $m \geq n$ with $\frac{n}{|X|} \geq 2\nu$, the last inequality becomes

$$\left| \text{Tr} \prod_{i=1}^{2p-1} A_i \Big|_{\mathcal{B}_m} \right|, \text{Tr} e^{-\beta K} \Big|_{\mathcal{B}_m} \leq e^{-\frac{\alpha}{2|X|} m^2 \beta} \frac{(m + |X| - 1)!}{m!(|X| - 1)!}$$

and

$$\frac{(m + |X| - 1)!}{m!(|X| - 1)!} \leq \frac{(m + |X| - 1)^{|X|-1}}{(|X| - 1)!} \leq \frac{(cm)^{|X|-1}}{(|X| - 1)!} \tag{3.5}$$

where, $c = \frac{1}{2\nu} + 1$. Now, we have, if $n \geq 2|X|\nu$,

$$\begin{aligned} & \left| \text{Tr} \prod_{i=1}^{2p-1} A_i(\mathbb{1} - P_n) \right|, \quad \text{Tr} e^{-\beta K}(\mathbb{1} - P_n) \\ & \leq \sum_{m>n} e^{-\frac{a}{2|X|} m^2 \beta} \frac{(cm)^{|X|-1}}{(|X|-1)!} \\ & \leq \int_n^\infty dt e^{-\frac{a}{2|X|} t^2 \beta} \frac{(ct)^{|X|-1}}{(|X|-1)!} \tag{3.6} \\ & \leq e^{-\frac{a}{4|X|} n^2 \beta} \int_0^\infty dt e^{-\frac{a}{4|X|} t^2 \beta} \frac{(ct)^{|X|-1}}{(|X|-1)!} \\ & \leq C e^{-\frac{a}{4|X|} n^2 \beta} \end{aligned}$$

for some positive constant C , depending only on β, a, ν and $|X|$. The first claim now follows from the observation that this converges to zero as $n \rightarrow \infty$, uniformly in $R(p)$ and p . In fact this proves convergence in trace norm and hence convergence in operator norm and also weak convergence, so that this also proves the second claim.

Finally, we prove the bound. By (3.4) and Proposition 2.22,

$$\begin{aligned} \left| \left\langle \phi \left| \left(e^{-\frac{\beta}{p} K} \mathbf{I}_R \right)^{p-1} e^{-\frac{\beta}{p} K} \right| \phi' \right\rangle \right| & \leq \sum_{m=0}^\infty \frac{\|\phi\|^m}{\sqrt{m!}} e^{-\beta a(\frac{m}{|X|} - \nu)m} \frac{\|\phi'\|^m}{\sqrt{m!}} \\ & \leq \sum_{m=0}^\infty \frac{t^m}{m!} e^{-\beta a(\frac{m}{|X|} - \nu)m} \end{aligned}$$

where $t = \frac{1}{2}(\|\phi\|^2 + \|\phi'\|^2)$. Observe that, for any $\gamma > 0$,

$$\begin{aligned} a\beta \left(\frac{m^2}{|X|} - \nu m - \gamma m \right) & = \frac{a\beta}{|X|} \left(m - \frac{\nu + \gamma}{2}|X| \right)^2 - \frac{a}{4}\beta(\nu + \gamma)^2|X| \\ & \geq -\frac{a\beta}{4}(\nu + \gamma)^2|X|. \end{aligned}$$

Thus

$$\begin{aligned} \left| \left\langle \phi \left| \left(e^{-\frac{\beta}{p} K} \mathbf{I}_R \right)^{p-1} e^{-\frac{\beta}{p} K} \right| \phi' \right\rangle \right| & \leq \sum_{m=0}^\infty \frac{t^m}{m!} e^{-a\beta\gamma m} e^{-a\beta(\frac{m^2}{|X|} - \nu m - \gamma m)} \\ & \leq \sum_{m=0}^\infty \frac{t^m}{m!} e^{-a\beta\gamma m} e^{\frac{a\beta}{4}(\nu + \gamma)^2|X|} \\ & = e^{-a\beta\gamma t} e^{\frac{a\beta}{4}(\nu + \gamma)^2|X|}. \end{aligned}$$

It now suffices to choose γ so that $\eta = e^{-a\beta\gamma}$ and then set $C_\eta = \frac{a\beta}{4}(\nu + \gamma)^2$. \square

Remark 3.5. If $R(p) \geq c|\ln p|^{\frac{1}{2} + \epsilon}$, then $\lim_{p \rightarrow \infty} p e^{-R(p)^2} = 0$. Also if $R(p) \geq c|\ln p|^{\frac{1}{2}}$ with $c > 1$, then $\lim_{p \rightarrow \infty} p e^{-R(p)^2} = 0$.

Proof of Theorem 3.1. By (3.6), the strong convergence of I_r to $\mathbb{1}$, and Proposition 2.28.a

$$\begin{aligned} \text{Tr } e^{-\beta K} &= \lim_{p \rightarrow \infty} \text{Tr } e^{-\beta K} I_{R(p)} \\ &= \lim_{p \rightarrow \infty} \int d\mu_{R(p)}(\phi_0^*, \phi_0) e^{-\int d\mathbf{y} |\phi_0(\mathbf{y})|^2} \langle \phi_0 | e^{-\beta K} | \phi_0 \rangle . \end{aligned}$$

It follows from Lemma 3.4 and the dominated convergence theorem that

$$\text{Tr } e^{-\beta K} = \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{T}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\int d\mathbf{y} |\phi_\tau(\mathbf{y})|^2} \right] \prod_{\tau \in \mathcal{T}_p} \langle \phi_{\tau-\varepsilon} | e^{-\varepsilon K} | \phi_\tau \rangle$$

as desired. □

The logarithm of $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$

Theorem 3.1 is a rigorous version of the intermediate representation (1.5). As discussed in the introduction, it now remains to show that one may replace $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$ by $e^{\int d\mathbf{y} \alpha(\mathbf{y})^* \phi(\mathbf{y}) - \varepsilon K(\alpha^*, \phi)}$ in the formula for $\text{Tr } e^{-\beta K}$ of Theorem 3.1. In Theorem 3.13, below, we show that this is indeed the case, provided $R(p)$ is chosen appropriately. To prepare for that, we explicitly find the logarithm $F(\varepsilon, \alpha^*, \phi) = \ln \langle \alpha | e^{-\varepsilon K} | \phi \rangle$ at least for α and ϕ not too large, and show that

$$F(\varepsilon, \alpha^*, \phi) = \int_X d\mathbf{x} \alpha(\mathbf{x})^* \phi(\mathbf{x}) - \varepsilon K(\alpha^*, \phi) + O(\varepsilon^2).$$

This expression is the same, to order ε , as

$$\begin{aligned} F(\varepsilon, \alpha^*, \phi) &= \iint_{X^2} d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* j(\varepsilon; \mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \\ &\quad - \frac{1}{2} \varepsilon \iint_{X^2} d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) + O(\varepsilon^2) \end{aligned}$$

provided $j(\varepsilon) = \mathbb{1} - \varepsilon(h - \mu) + O(\varepsilon^2)$. For application of renormalization group methods, the latter form is more convenient. So we show it too. We typically use the supremum norm

$$|\phi|_X = \max_{\mathbf{x} \in X} |\phi(\mathbf{x})|$$

to measure the size of the field ϕ and the norm

$$\|h\|_{1,\infty} = \max_{\mathbf{x} \in X} \int_X d\mathbf{y} |h(\mathbf{x}, \mathbf{y})|$$

to measure the size of (symmetric) integral operators on $L^2(X)$.

Proposition 3.6. *For each $\varepsilon > 0$, there is an analytic function $F(\varepsilon, \alpha^*, \phi)$ such that*

$$\langle \alpha | e^{-\varepsilon K} | \phi \rangle = e^{F(\varepsilon, \alpha^*, \phi)}$$

on the domain $|\alpha|_X, |\phi|_X < C_\varepsilon$ where $C_\varepsilon = [8e^{\varepsilon(\|h\|_{1,\infty} + \mu + v_0)} \varepsilon \|v\|_{1,\infty}]^{-1/2}$ with $v_0 = \max_{\mathbf{x} \in X} v(\mathbf{x}, \mathbf{x})$.

Let $c_j > 0$ and $j(\varepsilon; \mathbf{x}, \mathbf{y})$ be the kernel of an operator obeying

$$\|j(\varepsilon) - e^{-\varepsilon(h-\mu)}\|_{1,\infty} \leq c_j \varepsilon^2.$$

Define the function $\mathcal{F}_1(\varepsilon, \alpha^*, \phi)$ by

$$F(\varepsilon, \alpha^*, \phi) = \iint_{X^2} d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* j(\varepsilon; \mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) - \frac{\varepsilon}{2} \iint_{X^2} d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) + \mathcal{F}_1(\varepsilon, \alpha^*, \phi).$$

For every $\varepsilon_0 > 0$ there is a constant const (depending only on $\varepsilon_0, \|h\|_{1,\infty}, v, c_j$ and μ) such that for all $0 < \varepsilon < \varepsilon_0$

$$|\mathcal{F}_1(\varepsilon, \alpha^*, \phi)| \leq \text{const} \varepsilon^2 (R^2 + \|v\|_{1,\infty}^2 R^6) |X|$$

for all $|\alpha|_X, |\phi|_X \leq R \leq \frac{1}{2} C_\varepsilon$.

An immediate consequence is

Corollary 3.7. We use the notation of Proposition 3.6. Define the function $\mathcal{F}_0(\varepsilon, \alpha^*, \phi)$ by

$$F(\varepsilon, \alpha^*, \phi) = \int_X d\mathbf{x} \alpha(\mathbf{x})^* \phi(\mathbf{x}) - \varepsilon K(\alpha^*, \phi) + \mathcal{F}_0(\varepsilon, \alpha^*, \phi)$$

where

$$K(\alpha^*, \phi) = \iint_{X^2} d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) - \mu \int_X d\mathbf{x} \alpha(\mathbf{x})^* \phi(\mathbf{x}) + \frac{1}{2} \iint_{X^2} d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}).$$

For every $\varepsilon_0 > 0$ there is a constant const (depending only on $\varepsilon_0, \|h\|_{1,\infty}, v$ and μ) such that for all $0 < \varepsilon < \varepsilon_0$

$$|\mathcal{F}_0(\varepsilon, \alpha^*, \phi)| \leq \text{const} \varepsilon^2 (R^2 + \|v\|_{1,\infty}^2 R^6) |X|$$

for all $|\alpha|_X, |\phi|_X \leq R \leq \frac{1}{2} C_\varepsilon$.

We now prove a number of lemmas leading up to the proof of Proposition 3.6, following Lemma 3.11.

Lemma 3.8. Let $\varepsilon > 0$. There exists a function $F(\varepsilon, \alpha^*, \phi)$, analytic in α^* and ϕ in a neighbourhood of the origin, such that

$$\langle \alpha | e^{-\varepsilon K} | \phi \rangle = e^{F(\varepsilon, \alpha^*, \phi)}.$$

F satisfies the differential equation

$$\frac{\partial}{\partial \varepsilon} F = -\mathcal{K} \left(\alpha^*, \frac{\partial}{\partial \alpha^*} \right) F - \frac{1}{2} \iint_X d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \frac{\partial F}{\partial \alpha(\mathbf{x})^*} \frac{\partial F}{\partial \alpha(\mathbf{y})^*}$$

with the initial condition

$$F(0, \alpha^*, \phi) = \int_X d\mathbf{x} \alpha(\mathbf{x})^* \phi(\mathbf{x}).$$

Here,

$$\begin{aligned} \mathcal{K} \left(\alpha^*, \frac{\partial}{\partial \alpha^*} \right) &= \iint_X dx dy \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{y})^*} - \mu \int_X dx \alpha(\mathbf{x})^* \frac{\partial}{\partial \alpha(\mathbf{x})^*} \\ &+ \frac{1}{2} \iint_X dx dy \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{x})^*} \frac{\partial}{\partial \alpha(\mathbf{y})^*}. \end{aligned}$$

Proof. Set $\lambda_0 = \lambda_0(v)$ as in Definition 2.16. Since, by Propositions 2.22 and 2.17,

$$\begin{aligned} |\langle \alpha | e^{-\varepsilon K} | \phi \rangle| &\leq \sum_{m=0}^{\infty} \frac{\|\alpha\|^m}{\sqrt{m!}} \| e^{-\varepsilon K} |_{\mathcal{B}_m} \| \frac{\|\phi\|^m}{\sqrt{m!}} \\ &\leq \sum_{m=0}^{\infty} \frac{\|\alpha\|^m}{\sqrt{m!}} e^{-\varepsilon(\frac{\lambda_0}{2} \frac{m}{|\mathbf{x}|} + \lambda'_0 - \frac{v_0}{2} - \mu)m} \frac{\|\phi\|^m}{\sqrt{m!}} \\ &\leq \sum_{m=0}^{\infty} \frac{1}{2^m m!} (\|\alpha\|^2 + \|\phi\|^2)^m e^{-\varepsilon(\frac{\lambda_0}{2} \frac{m}{|\mathbf{x}|} + \lambda'_0 - \frac{v_0}{2} - \mu)m} \end{aligned}$$

the Taylor series expansion of $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$ converges for all $\alpha, \phi \in L^2(X)$ so that $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$ is an entire function of α^* and ϕ . Since $\langle \alpha | \phi \rangle = e^{\int \alpha^*(\mathbf{x})\phi(\mathbf{x}) dx} \neq 0$, the matrix element has the representation

$$\langle \alpha | e^{-\varepsilon K} | \phi \rangle = e^{F(\varepsilon, \alpha^*, \phi)}$$

in a neighbourhood of 0, with $F(\varepsilon, \alpha^*, \phi)$ is analytic in α^*, ϕ . We have

$$\begin{aligned} e^{F(\varepsilon, \alpha^*, \phi)} \frac{\partial}{\partial \varepsilon} F &= \frac{\partial}{\partial \varepsilon} e^{F(\varepsilon, \alpha^*, \phi)} = \frac{\partial}{\partial \varepsilon} \langle \alpha | e^{-\varepsilon K} | \phi \rangle = -\langle \alpha | K e^{-\varepsilon K} | \phi \rangle \\ &= -\left\langle \alpha \left| \left[\iint_{X^2} dx dy \psi^\dagger(\mathbf{x}) h(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) - \mu \int_X dx \psi^\dagger(\mathbf{x}) \psi(\mathbf{x}) \right. \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \iint_{X^2} dx dy \psi^\dagger(\mathbf{x}) \psi^\dagger(\mathbf{y}) v(\mathbf{x}, \mathbf{y}) \psi(\mathbf{x}) \psi(\mathbf{y}) \right] e^{-\varepsilon K} \right| \phi \rangle. \end{aligned}$$

By Proposition 2.20, the first term

$$\begin{aligned} &\left\langle \alpha \left| \left[\iint_{X^2} dx dy \psi^\dagger(\mathbf{x}) h(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \right] e^{-\varepsilon K} \right| \phi \right\rangle \\ &= \iint_{X^2} dx dy h(\mathbf{x}, \mathbf{y}) \langle \alpha | \psi^\dagger(\mathbf{x}) \psi(\mathbf{y}) e^{-\varepsilon K} | \phi \rangle \\ &= \iint_{X^2} dx dy \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \langle \alpha | \psi(\mathbf{y}) e^{-\varepsilon K} | \phi \rangle \\ &= \iint_{X^2} dx dy \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{y})^*} \langle \alpha | e^{-\varepsilon K} | \phi \rangle \\ &= \iint_{X^2} dx dy \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{y})^*} e^{F(\varepsilon, \alpha^*, \phi)}. \end{aligned}$$

Treating the other two terms similarly,

$$e^{F(\varepsilon, \alpha^*, \phi)} \frac{\partial}{\partial \varepsilon} F = - \left[\iint_{X^2} d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{y})^*} - \mu \int_X d\mathbf{x} \alpha(\mathbf{x})^* \frac{\partial}{\partial \alpha(\mathbf{x})^*} + \frac{1}{2} \iint_{X^2} d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{x})^*} \frac{\partial}{\partial \alpha(\mathbf{y})^*} \right] e^{F(\varepsilon, \alpha^*, \phi)}.$$

The differential equation for F follows. □

Lemma 3.9. *The function $F(\varepsilon, \alpha^*, \phi)$ of Lemma 3.8 has an expansion*

$$F(\varepsilon, \alpha^*, \phi) = \sum_{n=1}^{\infty} \int_{X^{2n}} d^n \tilde{\mathbf{x}} d^n \tilde{\mathbf{y}} e^{n\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^n \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i)$$

where $\tilde{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$
 $\tilde{\mathbf{y}} = (\mathbf{y}_1, \dots, \mathbf{y}_n)$

in powers of the fields α^* and ϕ . Furthermore

$$F_1(\varepsilon, \mathbf{x}_1, \mathbf{y}_1) = e^{-\varepsilon h(\mathbf{x}_1, \mathbf{y}_1)}$$

$$F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = -\frac{1}{2} \int_0^\varepsilon d\tau \int_{X^n} d^n \tilde{\mathbf{x}}' (e^{-(\varepsilon-\tau)H_n})(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') \sum_{m=1}^{n-1} S_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}} \\ \times \sum_{j=1}^m \sum_{k=m+1}^n v(\mathbf{x}'_j, \mathbf{x}'_k) F_m(\tau, \tilde{\mathbf{x}}'_{[\leq m]}, \tilde{\mathbf{y}}_{[\leq m]}) F_{n-m}(\tau, \tilde{\mathbf{x}}'_{[> m]}, \tilde{\mathbf{y}}_{[> m]})$$

where $S_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}}$ denotes independent symmetrization in the \mathbf{x} variables and the \mathbf{y} variables,

$$H_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = \sum_{k=1}^n h(\mathbf{x}_k, \mathbf{x}'_k) \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq k}} \delta_{\mathbf{x}_\ell}(\mathbf{x}'_\ell) + \frac{1}{2} \sum_{\substack{j, k=1 \\ j \neq k}}^n v(\mathbf{x}_j, \mathbf{x}_k) \prod_{1 \leq \ell \leq n} \delta_{\mathbf{x}_\ell}(\mathbf{x}'_\ell)$$

and

$$\tilde{\mathbf{x}}' = (\mathbf{x}'_1, \dots, \mathbf{x}'_n) \quad \tilde{\mathbf{x}}_{[\leq m]} = (\mathbf{x}_1, \dots, \mathbf{x}_m) \quad \text{and} \quad \tilde{\mathbf{x}}_{[> m]} = (\mathbf{x}_{m+1}, \dots, \mathbf{x}_n).$$

Proof. Expand F in the power series

$$F(\varepsilon, \alpha^*, \phi) = \sum_{n=1}^{\infty} \int d^n \tilde{\mathbf{x}} e^{n\varepsilon\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \phi) \prod_{i=1}^n \alpha(\mathbf{x}_i)^*$$

in α^* with coefficients $F_n(\varepsilon, \tilde{\mathbf{x}}, \phi)$ that are symmetric under permutation of the \mathbf{x}_k 's. The constant term is zero because $\langle \alpha | e^{-\varepsilon K} | \phi \rangle |_{\alpha=\phi=0} = 1$. Each $F_n(\varepsilon, \tilde{\mathbf{x}}, \phi)$ has degree n in ϕ because the fact that $e^{-\varepsilon K}$ preserves particle number implies that $F(\varepsilon, e^{-i\theta} \alpha^*, e^{i\theta} \phi) = F(\varepsilon, \alpha^*, \phi)$ for all real θ . Observe that

$$e^{n\varepsilon\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \phi) = \frac{1}{n!} \frac{\partial}{\partial \alpha(\mathbf{x}_1)^*} \cdots \frac{\partial}{\partial \alpha(\mathbf{x}_n)^*} F(\varepsilon, \alpha^*, \phi) \Big|_{\alpha=0}$$

and

$$\begin{aligned} & \frac{1}{n!} \frac{\partial}{\partial \alpha(\mathbf{x}_1)^*} \cdots \frac{\partial}{\partial \alpha(\mathbf{x}_n)^*} \iint_{X^2} d\mathbf{x} d\mathbf{y} \alpha(\mathbf{x})^* h(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{y})^*} F(\varepsilon, \alpha^*, \phi) \Big|_{\alpha=0} \\ &= \sum_{k=1}^n \frac{1}{n!} \int_X d\mathbf{y} h(\mathbf{x}_k, \mathbf{y}) \frac{\partial}{\partial \alpha(\mathbf{y})^*} \prod_{\ell \neq k} \frac{\partial}{\partial \alpha(\mathbf{x}_\ell)^*} F(\varepsilon, \alpha^*, \phi) \Big|_{\alpha=0} \\ &= \sum_{k=1}^n \frac{1}{n!} \int_X d\mathbf{x}'_k h(\mathbf{x}_k, \mathbf{x}'_k) e^{n\varepsilon\mu} F_n(\varepsilon, \mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{x}'_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_n, \phi) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{n!} \frac{\partial}{\partial \alpha(\mathbf{x}_1)^*} \cdots \frac{\partial}{\partial \alpha(\mathbf{x}_n)^*} G(\alpha^*) H(\alpha^*) \\ &= \sum_{m=0}^n S \left[\frac{1}{m!} \prod_{\ell=1}^m \frac{\partial}{\partial \alpha(\mathbf{x}_\ell)^*} G(\alpha^*) \right] \left[\frac{1}{(n-m)!} \prod_{\ell=m+1}^n \frac{\partial}{\partial \alpha(\mathbf{x}_\ell)^*} H(\alpha^*) \right] \end{aligned}$$

where S denotes the symmetrization operator in the variables \mathbf{x}_k . Thus Lemma 3.8 gives the system of equations

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} F_1(\varepsilon, \mathbf{x}_1, \phi) &= - \int d\mathbf{x}'_1 h(\mathbf{x}_1, \mathbf{x}'_1) F_1(\varepsilon, \mathbf{x}'_1, \phi) \quad \text{and for } n > 1 \\ \frac{\partial}{\partial \varepsilon} F_n(\varepsilon, \tilde{\mathbf{x}}, \phi) &= - \int d^n \tilde{\mathbf{x}}' H_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') F_n(\varepsilon, \tilde{\mathbf{x}}', \phi) - \frac{1}{2} \sum_{m=1}^{n-1} S \sum_{j=1}^m \sum_{k=m+1}^n v(\mathbf{x}_j, \mathbf{x}_k) \\ &\quad \times F_m(\varepsilon, \mathbf{x}_1, \dots, \mathbf{x}_m, \phi) F_{n-m}(\varepsilon, \mathbf{x}_{m+1}, \dots, \mathbf{x}_n, \phi) \end{aligned} \tag{3.7}$$

with the initial condition

$$F_1(0, \mathbf{x}_1, \phi) = \phi(\mathbf{x}_1), \quad F_n(0, \tilde{\mathbf{x}}, \phi) = 0 \quad \text{for } n > 1.$$

The “integral” form of these equations is

$$\begin{aligned} F_1(\varepsilon, \mathbf{x}_1, \phi) &= (e^{-\varepsilon h} \phi)(\mathbf{x}_1) \\ F_n(\varepsilon, \tilde{\mathbf{x}}, \phi) &= -\frac{1}{2} \int_0^\varepsilon d\tau \int d\tilde{\mathbf{x}}' (e^{-(\varepsilon-\tau)H_n})(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') \sum_{m=1}^{n-1} S \sum_{j=1}^m \sum_{k=m+1}^n v(\mathbf{x}'_j, \mathbf{x}'_k) \\ &\quad F_m(\tau, \mathbf{x}'_1, \dots, \mathbf{x}'_m, \phi) F_{n-m}(\tau, \mathbf{x}'_{m+1}, \dots, \mathbf{x}'_n, \phi). \end{aligned}$$

We remark that the fact that $F_n(\varepsilon, \tilde{\mathbf{x}}, \phi)$ is of degree n in ϕ also follows by induction on n from these equations. Writing

$$F_n(\varepsilon, \tilde{\mathbf{x}}, \phi) = \int d\tilde{\mathbf{y}} F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^n \phi(\mathbf{y}_i)$$

for each $n \geq 1$, with the $F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$'s symmetric under permutations of the \mathbf{y}_k 's too, defines the functions of the lemma. \square

We now estimate the norm of the operator $e^{-\varepsilon H_n}$ of Lemma 3.9, acting on functions $F : X^n \times X^n \rightarrow \mathbb{C}$. The space of functions is equipped with the norm

$$\|F\|_{1,\infty} = \max_{1 \leq j \leq 2n} \max_{\mathbf{x}_j \in X} \int_{X^{2n-1}} |F(\mathbf{x}_1, \dots, \mathbf{x}_{2n})| \prod_{\substack{1 \leq i \leq 2n \\ i \neq j}} dx_i .$$

Lemma 3.10. *Write*

$$H_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = h_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') + V_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}')$$

where

$$h_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = \sum_{k=1}^n h(\mathbf{x}_k, \mathbf{x}'_k) \prod_{\substack{1 \leq \ell \leq n \\ \ell \neq k}} \delta_{\mathbf{x}_\ell}(\mathbf{x}'_\ell)$$

$$V_n(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = \frac{1}{2} \sum_{\substack{j,k=1 \\ j \neq k}}^n v(\mathbf{x}_j, \mathbf{x}_k) \prod_{1 \leq \ell \leq n} \delta_{\mathbf{x}_\ell}(\mathbf{x}'_\ell) .$$

For all $F : X^n \times X^n \rightarrow \mathbb{C}$, we have

- (a) $\|e^{-\varepsilon h_n} F\|_{1,\infty} \leq e^{n\varepsilon \|h\|_{1,\infty}} \|F\|_{1,\infty}$
- (b) $\|e^{-\varepsilon V_n} F\|_{1,\infty} \leq e^{\frac{1}{2}n\varepsilon v_0} \|F\|_{1,\infty}$ where $v_0 = \max_{\mathbf{x} \in X} v(\mathbf{x}, \mathbf{x})$
- (c) $\|e^{-\varepsilon H_n} F\|_{1,\infty} \leq e^{n\varepsilon (\|h\|_{1,\infty} + \frac{1}{2}v_0)} \|F\|_{1,\infty}$
- (d) $\|H_n F\|_{1,\infty} \leq n(\|h\|_{1,\infty} + \frac{1}{2}(n-1)v|_X) \|F\|_{1,\infty}$.

Proof. (a) The kernel of $e^{-\varepsilon h_n}$ is

$$e^{-\varepsilon h_n}(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') = \prod_{k=1}^n e^{-\varepsilon h(\mathbf{x}_k, \mathbf{x}'_k)} .$$

So we may view $e^{-\varepsilon h_n} F$ as

$$\overbrace{e^{-\varepsilon h} \otimes \dots \otimes e^{-\varepsilon h}}^{n \text{ factors}} \otimes \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{n \text{ factors}} = \prod_{k=1}^n \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{k-1 \text{ factors}} \otimes e^{-\varepsilon h} \otimes \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{n-k \text{ factors}}$$

acting of F , viewed as an element of $L^2(X^{2n})$. The bounds

$$\begin{aligned} & \sup_{\mathbf{x}} \int d\mathbf{x}' \int d\mathbf{z}_1 \dots d\mathbf{z}_m |L(\mathbf{x}, \mathbf{x}')| |G(\mathbf{x}', \mathbf{z}_1, \dots, \mathbf{z}_m)| \\ &= \sup_{\mathbf{x}} \int d\mathbf{x}' |L(\mathbf{x}, \mathbf{x}')| \int d\mathbf{z}_1 \dots d\mathbf{z}_m |G(\mathbf{x}', \mathbf{z}_1, \dots, \mathbf{z}_m)| \\ &\leq \|L\|_{1,\infty} \|G\|_{1,\infty} \end{aligned}$$

and

$$\begin{aligned} & \sup_{\mathbf{z}_m} \int d\mathbf{x} d\mathbf{x}' \int d\mathbf{z}_1 \dots d\mathbf{z}_{m-1} |L(\mathbf{x}, \mathbf{x}')| |G(\mathbf{x}', \mathbf{z}_1, \dots, \mathbf{z}_m)| \\ &= \sup_{\mathbf{z}_m} \int d\mathbf{x}' d\mathbf{z}_1 \dots d\mathbf{z}_{m-1} \left[\int d\mathbf{x} |L(\mathbf{x}, \mathbf{x}')| \right] |G(\mathbf{x}', \mathbf{z}_1, \dots, \mathbf{z}_m)| \\ &\leq \|L\|_{1,\infty} \|G\|_{1,\infty} \end{aligned}$$

imply that

$$\left\| \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{k-1 \text{ factors}} \otimes L \otimes \overbrace{\mathbb{1} \otimes \dots \otimes \mathbb{1}}^{n-k \text{ factors}} \otimes F \right\|_{1,\infty} \leq \|L\|_{1,\infty} \|F\|_{1,\infty}. \tag{3.8}$$

Part (a) now follows by repeated application of (3.8) in conjunction with

$$\begin{aligned} \int d\mathbf{x}' |e^{-\varepsilon h}(\mathbf{x}', \mathbf{x})| &= \int d\mathbf{x}' |e^{-\varepsilon h}(\mathbf{x}, \mathbf{x}')| \leq \sum_{n=0}^{\infty} \int d\mathbf{x}' \frac{1}{n!} \varepsilon^n |h^n(\mathbf{x}, \mathbf{x}')| \\ &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \varepsilon^n \|h\|_{1,\infty}^n = e^{\varepsilon \|h\|_{1,\infty}}. \end{aligned} \tag{3.9}$$

(b) Since $v(\mathbf{x}, \mathbf{y})$ is the kernel of a positive definite operator

$$\sum_{\substack{j,k=1 \\ j \neq k}}^n v(\mathbf{x}_j, \mathbf{x}_k) = \sum_{j,k=1}^n v(\mathbf{x}_j, \mathbf{x}_k) - \sum_{j=1}^n v(\mathbf{x}_j, \mathbf{x}_j) \geq -v_0 n \tag{3.10}$$

so that

$$|e^{-\varepsilon V_n} F(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})| \leq e^{\frac{1}{2} n \varepsilon v_0} |F(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})|.$$

(c) follows from the Trotter product formula

$$e^{-\varepsilon H_n} F = \lim_{p \rightarrow \infty} (e^{-\frac{\varepsilon}{p} h_n} e^{-\frac{\varepsilon}{p} V_n})^p F$$

and repeated application of parts (a) and (b).

(d) By (3.8),

$$\|h_n F\|_{1,\infty} \leq n \|h\|_{1,\infty} \|F\|_{1,\infty}.$$

Since

$$\left| \sum_{\substack{j,k=1 \\ j \neq k}}^n v(\mathbf{x}_j, \mathbf{x}_k) \right| \leq n(n-1) |v|_X$$

we also have

$$\|V_n F\|_{1,\infty} \leq \frac{1}{2} n(n-1) |v|_X \|F\|_{1,\infty}$$

and the claim follows. □

Lemma 3.11. *The functions $F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ of Lemma 3.9 obey*

$$\|F_n(\varepsilon, \cdot, \cdot)\|_{1,\infty} \leq (8\varepsilon\|v\|_{1,\infty})^{n-1} e^{n\varepsilon K_1} \frac{1}{n^3}$$

where $K_1 = \|\mathbf{h}\|_{1,\infty} + \frac{v_0}{2}$. Furthermore there is a constant const , depending only on $\|\mathbf{h}\|_{1,\infty}$ and $\|v\|_{1,\infty}$, such that

$$F_2(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = -\frac{1}{4}\varepsilon v(\mathbf{x}_1, \mathbf{x}_2) [\delta_{\mathbf{x}_1}(\mathbf{y}_1)\delta_{\mathbf{x}_2}(\mathbf{y}_2) + \delta_{\mathbf{x}_1}(\mathbf{y}_2)\delta_{\mathbf{x}_2}(\mathbf{y}_1)] + F_{2+}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2)$$

with

$$\|F_{2+}(\varepsilon, \cdot, \cdot)\|_{1,\infty} \leq \text{const} \varepsilon^2 \|v\|_{1,\infty} e^{2\varepsilon K_1}.$$

Proof. We first prove the bound on $\|F_n(\varepsilon, \cdot, \cdot)\|_{1,\infty}$ by induction on n . The case $n = 1$ follows immediately from (3.9). So assume that the bound has been proven for all $m < n$. In general, if

$$F(\mathbf{x}_1, \dots, \mathbf{x}_{n+m}) = \int d\mathbf{y}_1 d\mathbf{y}_2 G(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_1) \omega(\mathbf{y}_1, \mathbf{y}_2) H(\mathbf{y}_2, \mathbf{x}_{n+1}, \dots, \mathbf{x}_{n+m})$$

then

$$\|F\|_{1,\infty} \leq \|G\|_{1,\infty} \|\omega\|_{1,\infty} \|H\|_{1,\infty}.$$

This is proven by repeated application of

$$\begin{aligned} \left| \int d\mathbf{x}_2 d\mathbf{y} f(\mathbf{x}_1, \mathbf{y}) g(\mathbf{y}, \mathbf{x}_2) \right| &\leq \int d\mathbf{y} |f(\mathbf{x}_1, \mathbf{y})| \int d\mathbf{x}_2 |g(\mathbf{y}, \mathbf{x}_2)| \\ &\leq \int d\mathbf{y} |f(\mathbf{x}_1, \mathbf{y})| \|g\|_{1,\infty} \\ &\leq \|f\|_{1,\infty} \|g\|_{1,\infty}. \end{aligned}$$

Hence, by the inductive hypothesis, Lemma 3.9 and part (c) of Lemma 3.10,

$$\begin{aligned} \|F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})\|_{1,\infty} &\leq \frac{1}{2} \left\| \int_0^\varepsilon d\tau \int d\tilde{\mathbf{x}}' (e^{-(\varepsilon-\tau)H_n})(\tilde{\mathbf{x}}, \tilde{\mathbf{x}}') \sum_{m=1}^{n-1} S_{\tilde{\mathbf{x}}, \tilde{\mathbf{y}}} \sum_{j=1}^m \sum_{k=m+1}^n \right. \\ &\quad \left. v(\mathbf{x}'_j, \mathbf{x}'_k) F_m(\tau, \tilde{\mathbf{x}}'_{[\leq m]}, \tilde{\mathbf{y}}'_{[\leq m]}) F_{n-m}(\tau, \tilde{\mathbf{x}}'_{[> m]}, \tilde{\mathbf{y}}'_{[> m]}) \right\|_{1,\infty} \\ &\leq \frac{1}{2} \int_0^\varepsilon d\tau e^{(\varepsilon-\tau)K_1 n} \sum_{m=1}^{n-1} \sum_{j=1}^m \sum_{k=m+1}^n \|v\|_{1,\infty} \\ &\quad \times \|F_m(\tau, \cdot)\|_{1,\infty} \|F_{n-m}(\tau, \cdot)\|_{1,\infty} \\ &\leq \frac{1}{2} e^{n\varepsilon K_1} (8\|v\|_{1,\infty})^{n-2} \int_0^\varepsilon d\tau \tau^{n-2} \sum_{m=1}^{n-1} \sum_{j=1}^m \\ &\quad \sum_{k=m+1}^n \|v\|_{1,\infty} \frac{1}{m^3} \frac{1}{(n-m)^3} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{16} e^{n\varepsilon K_1} (8\varepsilon \|v\|_{1,\infty})^{n-1} \frac{1}{n-1} \sum_{m=1}^{n-1} \frac{1}{m^2} \frac{1}{(n-m)^2} \\ &\leq \frac{1}{16} e^{n\varepsilon K_1} (8\varepsilon \|v\|_{1,\infty})^{n-1} \frac{1}{n-1} \frac{8}{n^2} \\ &\leq (8\varepsilon \|v\|_{1,\infty})^{n-1} e^{n\varepsilon K_1} \frac{1}{n^3}. \end{aligned}$$

The equation for $n = 2$ in Lemma 3.9 yields

$$\begin{aligned} F_2(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) &= -\frac{1}{2} \int_0^\varepsilon d\tau \int d\mathbf{x}'_1 d\mathbf{x}'_2 (e^{-(\varepsilon-\tau)H_2})(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) v(\mathbf{x}'_1, \mathbf{x}'_2) \\ &\quad \frac{1}{2} [e^{-\tau h}(\mathbf{x}'_1, \mathbf{y}_1) e^{-\tau h}(\mathbf{x}'_2, \mathbf{y}_2) + e^{-\tau h}(\mathbf{x}'_1, \mathbf{y}_2) e^{-\tau h}(\mathbf{x}'_2, \mathbf{y}_1)] \\ &= -\frac{1}{4} \varepsilon v(\mathbf{x}_1, \mathbf{x}_2) [\delta_{\mathbf{x}_1}(\mathbf{y}_1) \delta_{\mathbf{x}_2}(\mathbf{y}_2) + \delta_{\mathbf{x}_1}(\mathbf{y}_2) \delta_{\mathbf{x}_2}(\mathbf{y}_1)] \\ &\quad + F_{2+}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) \end{aligned}$$

where the second order Taylor remainder

$$F_{2+}(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) = \int_0^\varepsilon d\tau' (\varepsilon - \tau') \frac{\partial^2}{\partial \varepsilon^2} F_2(\tau', \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2). \tag{3.11}$$

By (3.7)

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} F_2(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) &= - \int d\mathbf{x}'_1 d\mathbf{x}'_2 H_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) F_2(\varepsilon, \mathbf{x}'_1, \mathbf{x}'_2, \mathbf{y}_1, \mathbf{y}_2) \\ &\quad - \frac{1}{2} S_{\mathbf{x},\mathbf{y}} v(\mathbf{x}_1, \mathbf{x}_2) F_1(\varepsilon, \mathbf{x}_1, \mathbf{y}_1) F_1(\varepsilon, \mathbf{x}_2, \mathbf{y}_2) \end{aligned}$$

and hence

$$\begin{aligned} \frac{\partial^2}{\partial \varepsilon^2} F_2(\varepsilon, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2) &= - \int d\mathbf{x}'_1 d\mathbf{x}'_2 H_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}'_1, \mathbf{x}'_2) \frac{\partial}{\partial \varepsilon} F_2(\varepsilon, \mathbf{x}'_1, \mathbf{x}'_2, \mathbf{y}_1, \mathbf{y}_2) \\ &\quad - S_{\mathbf{x},\mathbf{y}} v(\mathbf{x}_1, \mathbf{x}_2) F_1(\varepsilon, \mathbf{x}_1, \mathbf{y}_1) \frac{\partial}{\partial \varepsilon} F_1(\varepsilon, \mathbf{x}_2, \mathbf{y}_2). \end{aligned}$$

Using part (d) of Lemma 3.10 and the bounds on F_1 and F_2 already proven, we have

$$\begin{aligned} &\left\| \frac{\partial}{\partial \varepsilon} F_2(\varepsilon, \cdot, \cdot, \cdot) \right\|_{1,\infty} \\ &\leq \|H_2 F_2(\varepsilon, \cdot, \cdot, \cdot)\|_{1,\infty} + \frac{1}{2} \|v\|_{1,\infty} \|F_1(\varepsilon, \cdot, \cdot, \cdot)\|_{1,\infty}^2 \\ &\leq 2 \left(\|h\|_{1,\infty} + \frac{1}{2} |v|_X \right) \|F_2(\varepsilon, \cdot, \cdot, \cdot)\|_{1,\infty} + \frac{1}{2} \|v\|_{1,\infty} \|F_1(\varepsilon, \cdot, \cdot, \cdot)\|_{1,\infty}^2 \\ &\leq \frac{1}{2} \|v\|_{1,\infty} \left[1 + 4\varepsilon \left(\|h\|_{1,\infty} + \frac{1}{2} |v|_X \right) \right] e^{2\varepsilon K_1}. \end{aligned}$$

Using the bound

$$\left\| \frac{\partial}{\partial \varepsilon} F_1(\varepsilon, \cdot, \cdot) \right\|_{1,\infty} = \|h e^{-\varepsilon h}\|_{1,\infty} \leq \|h\|_{1,\infty} e^{\varepsilon K_1}$$

we similarly get

$$\begin{aligned} & \left\| \frac{\partial^2}{\partial \varepsilon^2} F_2(\varepsilon, \cdot, \cdot) \right\|_{1,\infty} \\ & \leq \left\| H_2 \frac{\partial}{\partial \varepsilon} F_2(\varepsilon, \cdot, \cdot) \right\|_{1,\infty} + \|v\|_{1,\infty} \|F_1(\varepsilon, \cdot, \cdot)\|_{1,\infty} \left\| \frac{\partial}{\partial \varepsilon} F_1(\varepsilon, \cdot, \cdot) \right\|_{1,\infty} \\ & \leq \left(\|h\|_{1,\infty} + \frac{1}{2} |v|_X \right) \|v\|_{1,\infty} \left[1 + 4\varepsilon \left(\|h\|_{1,\infty} + \frac{1}{2} |v|_X \right) \right] e^{2\varepsilon K_1} \\ & \quad + \|v\|_{1,\infty} \|h\|_{1,\infty} e^{2\varepsilon K_1} \\ & \leq \left[\|v\|_{1,\infty} \left(2\|h\|_{1,\infty} + \frac{1}{2} |v|_X \right) + 4\varepsilon \|v\|_{1,\infty} \left(\|h\|_{1,\infty} + \frac{1}{2} |v|_X \right)^2 \right] e^{2\varepsilon K_1} \\ & \leq \text{const } \|v\|_{1,\infty} e^{2\varepsilon K_1}. \end{aligned}$$

Hence, by (3.11)

$$\|F_{2+}(\varepsilon, \cdot, \cdot)\|_{1,\infty} \leq \text{const } \varepsilon^2 \|v\|_{1,\infty} e^{2\varepsilon K_1}$$

as desired. □

Proof of Proposition 3.6. We routinely write $\tilde{\mathbf{x}}$ for $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ and $\tilde{\mathbf{y}}$ for $(\mathbf{y}_1, \dots, \mathbf{y}_n)$. The value of n should be clear from the context. By Lemma 3.11,

$$\begin{aligned} & \sum_{n=3}^{\infty} \int d\tilde{\mathbf{x}} d\tilde{\mathbf{y}} \left| e^{n\varepsilon\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^n \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \right| \\ & \leq \sum_{n=3}^{\infty} |X| e^{n\varepsilon\mu} \|F_n(\varepsilon, \cdot, \cdot)\|_{1,\infty} |\alpha|_X^n |\phi|_X^n \\ & \leq |X| \sum_{n=3}^{\infty} (8\varepsilon \|v\|_{1,\infty})^{n-1} e^{n\varepsilon(K_1+\mu)} |\alpha|_X^n |\phi|_X^n. \end{aligned} \tag{3.12}$$

This gives us the desired domain of analyticity. Since $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$ and $e^{F(\varepsilon, \alpha^*, \phi)}$ are both analytic on this domain and since they agree for all sufficiently small α, ϕ , they coincide on the full domain.

Set

$$\begin{aligned} \mathcal{F}_1(\varepsilon, \alpha^*, \phi) &= \int d\mathbf{x}d\mathbf{y} [e^{\varepsilon\mu} F_1(\varepsilon, \mathbf{x}, \mathbf{y}) - j(\varepsilon; \mathbf{x}, \mathbf{y})] \alpha(\mathbf{x})^* \phi(\mathbf{y}) \\ &\quad + \int d^2\tilde{\mathbf{x}}d^2\tilde{\mathbf{y}} [e^{2\varepsilon\mu} - 1] F_2(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^2 \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \\ &\quad + \int d^2\tilde{\mathbf{x}}d^2\tilde{\mathbf{y}} F_{2+}(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^2 \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \\ &\quad + \sum_{n=3}^{\infty} \int d^n\tilde{\mathbf{x}}d^n\tilde{\mathbf{y}} e^{n\varepsilon\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^n \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \end{aligned}$$

so that

$$\begin{aligned} F(\varepsilon, \alpha^*, \phi) &= \iint_{X^2} d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* j(\varepsilon; \mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \\ &\quad - \frac{\varepsilon}{2} \iint_{X^2} d\mathbf{x}d\mathbf{y} \alpha(\mathbf{x})^* \alpha(\mathbf{y})^* v(\mathbf{x}, \mathbf{y}) \phi(\mathbf{x}) \phi(\mathbf{y}) + \mathcal{F}_1(\varepsilon, \alpha^*, \phi). \end{aligned}$$

If $|\alpha|_X, |\phi|_X \leq R$, then, by (3.12),

$$\begin{aligned} &\left| \sum_{n=3}^{\infty} \int d^n\tilde{\mathbf{x}}d^n\tilde{\mathbf{y}} e^{n\varepsilon\mu} F_n(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^n \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \right| \\ &\leq |X| \sum_{n=3}^{\infty} (8\varepsilon\|v\|_{1,\infty})^{n-1} e^{n\varepsilon(K_1+\mu)} R^{2n} \\ &\leq \frac{64 |X| \varepsilon^2 \|v\|_{1,\infty}^2 e^{3\varepsilon(K_1+\mu)} R^6}{1 - 8\varepsilon\|v\|_{1,\infty} e^{\varepsilon(K_1+\mu)} R^2} \\ &\leq 128 \varepsilon^2 R^6 |X| \|v\|_{1,\infty}^2 e^{3\varepsilon(K_1+\mu)}. \end{aligned}$$

Similarly, by Lemma 3.11,

$$\begin{aligned} \left| \int d^2\tilde{\mathbf{x}}d^2\tilde{\mathbf{y}} F_{2+}(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^2 \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \right| &\leq |X| \|F_{2+}(\varepsilon, \cdot, \cdot, \cdot)\|_{1,\infty} R^4 \\ &\leq \text{const } \varepsilon^2 R^4 |X| \|v\|_{1,\infty} e^{2\varepsilon(K_1+|\mu|)} \end{aligned}$$

and

$$\begin{aligned} &\left| \int d^2\tilde{\mathbf{x}}d^2\tilde{\mathbf{y}} [e^{2\varepsilon\mu} - 1] F_2(\varepsilon, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \prod_{i=1}^2 \alpha(\mathbf{x}_i)^* \phi(\mathbf{y}_i) \right| \\ &\leq 2\varepsilon|\mu| |X| \|F_2(\varepsilon, \cdot, \cdot, \cdot)\|_{1,\infty} R^4 e^{2\varepsilon|\mu|} \\ &\leq 2|\mu|\varepsilon^2 R^4 |X| \|v\|_{1,\infty} e^{2\varepsilon(K_1+|\mu|)} \end{aligned}$$

and

$$\begin{aligned} & \left| \int dx dy [e^{\varepsilon\mu} F_1(\varepsilon, \mathbf{x}, \mathbf{y}) - j(\varepsilon; \mathbf{x}, \mathbf{y})] \alpha(\mathbf{x})^* \phi(\mathbf{y}) \right| \\ & \leq |X| \|e^{-\varepsilon(h-\mu)} - j(\varepsilon)\|_{1,\infty} R^2 \\ & \leq c_j \varepsilon^2 R^2 |X|. \end{aligned}$$

Since

$$R^4 \|v\|_{1,\infty} = R (R^3 \|v\|_{1,\infty}) \leq \frac{1}{2} (R^2 + \|v\|_{1,\infty}^2 R^6)$$

the desired bound on \mathcal{F}_1 follows. □

Example 3.12. Here is a simple example that shows that $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$ can have zeroes so that the logarithm of $\langle \alpha | e^{-\varepsilon K} | \phi \rangle$ need not be defined for all $\alpha, \phi \in L^2(X)$. If the finite set X , which plays the role of space here, consists of just a single point, then each n -particle space \mathcal{B}_n with $n \geq 1$ has dimension exactly one. So any operator that commutes with the number operator must be a function of the number operator. In particular, $K = H_0 + V - \mu N$, which is of degree two in the annihilation operators and of degree two in the creation operators, is a polynomial in N of degree two. As a simple example, we take $K = N^2 - N$. Then

$$\langle \alpha | e^{-\varepsilon(N^2-N)} | \phi \rangle = \sum_{n=0}^{\infty} \frac{(\alpha^* \phi)^n}{n!} e^{-\varepsilon n(n-1)}.$$

Set

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-\varepsilon n(n-1)}.$$

Observe that f fulfills the functional equation

$$f'(z) = f(e^{-2\varepsilon} z)$$

since

$$\begin{aligned} f(e^{-2\varepsilon} z) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-2\varepsilon n} e^{-\varepsilon n(n-1)} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-\varepsilon n(n+1)} = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^{n+1}}{(n+1)!} e^{-\varepsilon n(n+1)} \\ &= \frac{d}{dz} \sum_{n=1}^{\infty} \frac{z^n}{n!} e^{-\varepsilon n(n-1)} = \frac{d}{dz} f(z). \end{aligned}$$

We claim that f necessarily has a zero on the negative real axis, somewhere between 0 and $-\frac{1}{1-e^{-2\varepsilon}} = -\frac{1}{2\varepsilon+O(\varepsilon^2)}$.

Proof. Set $\kappa = e^{-2\varepsilon} < 1$, so that $f'(z) = f(\kappa z)$. By inspection, $f(x) > 0$ for all $x \geq 0$. Now assume that f has no zero on the negative real axis. Then

$$h(x) = -\log f(-x)$$

is well defined for all real x and fulfills the equation

$$h'(x) = \frac{f'(-x)}{f(-x)} = \frac{f'(-\kappa x)}{f(-x)} = e^{h(x)-h(\kappa x)}.$$

In particular $h'(x) > 0$ for all $x \in \mathbb{R}$ and h is monotonically increasing. The second derivative

$$h''(x) = (h'(x) - \kappa h'(\kappa x)) e^{h(x)-h(\kappa x)} = (e^{h(x)-h(\kappa x)} - \kappa e^{h(\kappa x)-h(\kappa^2 x)}) e^{h(x)-h(\kappa x)}.$$

As $\kappa < 1$ and $h(0) = 0$, we have $h''(0) = 1 - \kappa > 0$. Next we show that $h''(x) > 0$ for all $x \geq 0$. If this were not the case we would have a smallest positive zero x_0 of h'' . Then h' would be monotonically increasing in $[0, x_0]$. By the formula for h'' above

$$e^{h(x_0)-h(\kappa x_0)} - \kappa e^{h(\kappa x_0)-h(\kappa^2 x_0)} = 0$$

so that

$$h(x_0) - h(\kappa x_0) < h(\kappa x_0) - h(\kappa^2 x_0).$$

By the mean value theorem there exist $\xi_1 \in [\kappa x_0, x_0]$ and $\xi_2 \in [\kappa^2 x_0, \kappa x_0]$ such that

$$h(x_0) - h(\kappa x_0) = h'(\xi_1) (1 - \kappa) x_0$$

$$h(\kappa x_0) - h(\kappa^2 x_0) = h'(\xi_2) (1 - \kappa) \kappa x_0.$$

Then

$$h'(\xi_1) (1 - \kappa) x_0 < h'(\xi_2) (1 - \kappa) \kappa x_0$$

and

$$h'(\xi_1) < h'(\xi_2) \kappa.$$

As $\xi_2 \leq \xi_1 \leq x_0$ and $\kappa < 1$ this contradicts the monotonicity of h' on $[0, x_0]$.

Since $h(0) = 0$ and $h''(x) > 0$ for all $x \geq 0$ we have $h(\kappa x) \leq \kappa h(x)$ for all $x \geq 0$ and therefore $h(x) - h(\kappa x) \geq (1 - \kappa) h(x)$. In particular

$$h'(x) = e^{h(x)-h(\kappa x)} \geq e^{(1-\kappa)h(x)} \Rightarrow e^{-(1-\kappa)h(x)} h'(x) \geq 1$$

$$\Rightarrow -\frac{1}{1-\kappa} \frac{d}{dx} e^{-(1-\kappa)h(x)} \geq 1.$$

Integrating both sides and using the initial condition $h(0) = 0$ gives

$$-\frac{1}{1-\kappa} [e^{-(1-\kappa)h(x)} - 1] \geq x \Rightarrow e^{-(1-\kappa)h(x)} \leq 1 - (1-\kappa)x$$

for all $x \geq 0$. Thus $h(x)$ must have a pole at some $0 < x < \frac{1}{1-\kappa}$ and $f(x)$ must have a zero at some $-\frac{1}{1-\kappa} < x < 0$. □

A functional integral representation of $\text{Tr } e^{-\beta K}$

Let h be a single particle operator on X and $v(\mathbf{x}_1, \mathbf{x}_2)$ a real, symmetric, pair potential which is repulsive in the sense of Definition 2.16. One of our precise formulations of the standard physics representation (1.1) for the partition function $\text{Tr } e^{-\beta K}$, where $K = H_0(h, X) + V(v, X) - \mu N$ is the second quantized Hamiltonian of a boson gas, is

Theorem 3.13. *Suppose that the sequence $R(p)$ obeys*

$$\lim_{p \rightarrow \infty} p e^{-\frac{1}{2}R(p)^2} = 0 \quad \text{and} \quad R(p) < p^{\frac{1}{24|X|}}.$$

Then

$$\text{Tr } e^{-\beta K} = \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{T}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\int d\mathbf{y} [\phi_\tau^*(\mathbf{y}) - \phi_{\tau-\varepsilon}^*(\mathbf{y})] \phi_\tau(\mathbf{y})} e^{-\varepsilon K(\phi_{\tau-\varepsilon}^*, \phi_\tau)} \right]$$

with the conventions that $\varepsilon = \frac{\beta}{p}$ and $\phi_0 = \phi_{p\varepsilon}$. The function $K(\alpha^*, \phi)$ was defined in Corollary 3.7.

This theorem will be proven after Example 3.17. During the course of the proof, we will modify the factors $\langle \phi_{\tau-\varepsilon} | e^{-\varepsilon K} | \phi_\tau \rangle$ of the integrand in Theorem 3.1, using the representation of these factors in Corollary 3.7. In Proposition 3.16, below, we develop some machinery to assist in proving that such modifications do not change the limit.

Definition 3.14. Let $r > 0$. Define, for $\mathcal{I} : \mathbb{C}^{2|X|} \rightarrow \mathbb{C}$, the seminorm

$$\|\mathcal{I}\|_r = \sup_{\substack{\alpha, \phi \in \mathbb{C}^X \\ |\alpha|_X, |\phi|_X \leq r}} |\mathcal{I}(\alpha, \phi)|.$$

The “ r -product” of $\mathcal{I}, \mathcal{J} : \mathbb{C}^{2|X|} \rightarrow \mathbb{C}$, with $\|\mathcal{I}\|_r, \|\mathcal{J}\|_r < \infty$ is defined to be

$$(\mathcal{I} *_r \mathcal{J})(\alpha, \gamma) = \int \mathcal{I}(\alpha, \phi) \mathcal{J}(\phi, \gamma) d\mu_r(\phi^*, \phi)$$

which is just the convolution with respect to the measure $d\mu_r$. The q^{th} power with respect to this product is denoted

$$\mathcal{I}^{*_r q} = \overbrace{\mathcal{I} *_r \mathcal{I} *_r \cdots *_r \mathcal{I}}^{q \text{ factors}}.$$

Example 3.15. For each $\varepsilon > 0$, set

$$\mathcal{I}_\varepsilon(\alpha, \phi) = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} e^{F(\varepsilon, \alpha^*, \phi)} = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \langle \alpha | e^{-\varepsilon K} | \phi \rangle.$$

Theorem 3.1 states that, for $R(p)$ obeying $\lim_{p \rightarrow \infty} p e^{-\frac{1}{2}R(p)^2} = 0$,

$$\text{Tr } e^{-\beta K} = \lim_{p \rightarrow \infty} \int d\mu_r(\phi^*, \phi) \mathcal{I}_\varepsilon^{*_r p}(\phi, \phi) \Big|_{\substack{r=R(p) \\ \varepsilon=\beta/p}}.$$

The operator K is bounded below. Suppose that $K \geq -K_0 \mathbb{1}$. Then

$$|\langle \alpha | e^{-\varepsilon K} | \phi \rangle| \leq \|\alpha\| e^{\varepsilon K_0} \|\phi\| = e^{\frac{1}{2}\|\alpha\|^2 + \frac{1}{2}\|\phi\|^2} e^{\varepsilon K_0}$$

implies that

$$\|\mathcal{I}_\varepsilon\|_r \leq e^{\varepsilon K_0}$$

for all $r > 0$. Furthermore

$$\mathcal{I}_\varepsilon^{*r,q}(\alpha, \phi) = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \left\langle \alpha \left| (e^{-\varepsilon K} \mathbf{I}_r)^{q-1} e^{-\varepsilon K} \right| \phi \right\rangle.$$

Thus, by part (c) of Theorem 2.26,

$$\|\mathcal{I}_\varepsilon^{*r,q}\|_r \leq e^{q\varepsilon K_0}$$

for all $r > 0$.

Proposition 3.16. *Let $K_0, \varepsilon, \zeta > 0$ and $0 < \kappa < 1$ and $r, C_\beta \geq 1$ obey*

$$C_\beta (\pi r^2)^{3|X|} \zeta^{1-\kappa} \leq \varepsilon.$$

Let $\mathcal{I}, \tilde{\mathcal{I}} : \mathbb{C}^{2|X|} \rightarrow \mathbb{C}$ obey

$$\|\mathcal{I} - \tilde{\mathcal{I}}\|_r \leq \zeta \quad \|\mathcal{I}^{*r,q}\|_r \leq e^{q\varepsilon K_0} \quad \text{for all } q \in \mathbb{N}.$$

Then, for all $q \in \mathbb{N}$ with $q \leq \frac{C_\beta}{\varepsilon}$,

$$\begin{aligned} \|\tilde{\mathcal{I}}^{*r,q}\|_r &\leq e^{q\varepsilon(K_0 + \zeta^\kappa)} \\ \|\tilde{\mathcal{I}}^{*r,q} - \mathcal{I}^{*r,q}\|_r &\leq \zeta^\kappa e^{q\varepsilon(K_0 + \zeta^\kappa)} \\ \int d\mu_r(\phi^*, \phi) \left| \tilde{\mathcal{I}}^{*r,q}(\phi, \phi) - \mathcal{I}^{*r,q}(\phi, \phi) \right| &\leq \zeta^\kappa e^{q\varepsilon(K_0 + \zeta^\kappa)}. \end{aligned}$$

Proof. For notational convenience, we suppress the subscript r on $*_r$. We first prove, by induction on q , that

$$\|\tilde{\mathcal{I}}^{*q}\|_r \leq (A + B)^q \quad \|\tilde{\mathcal{I}}^{*q} - \mathcal{I}^{*q}\|_r \leq qB(A + B)^{q-1} \tag{3.13}$$

where $A = e^{\varepsilon K_0}$ and $B = (\pi r^2)^{2|X|} \zeta$. The case $q = 1$ is obvious. So assume that these bounds hold when q is replaced by $\ell < q$. Then

$$\tilde{\mathcal{I}}^{*q} = \sum_{\ell=0}^{q-1} \mathcal{I}^{*\ell} * (\tilde{\mathcal{I}} - \mathcal{I}) * \tilde{\mathcal{I}}^{*q-\ell-1} + \mathcal{I}^{*q}.$$

Since

$$\int d\mu_r(\phi^*, \phi) \leq (\pi r^2)^{|X|}$$

we have

$$\begin{aligned} \|\tilde{\mathcal{I}}^{*q}\|_r &\leq \sum_{\ell=0}^{q-1} \|\mathcal{I}^{*\ell}\|_r (\pi r^2)^{|X|} \|\tilde{\mathcal{I}} - \mathcal{I}\|_r (\pi r^2)^{|X|} \|\tilde{\mathcal{I}}^{*q-\ell-1}\|_r + \|\mathcal{I}^{*q}\|_r \\ &\leq \sum_{\ell=0}^{q-1} A^\ell B (A + B)^{q-\ell-1} + A^q \end{aligned}$$

and

$$\|\tilde{\mathcal{I}}^{*q} - \mathcal{I}^{*q}\|_r \leq \sum_{\ell=0}^{q-1} A^\ell B (A+B)^{q-\ell-1}.$$

Then (3.13) follows from

$$\begin{aligned} (A+B)^q &= \sum_{\ell=0}^{q-1} A^\ell (A+B)^{q-\ell} - \sum_{\ell=0}^{q-1} A^{\ell+1} (A+B)^{q-\ell-1} + A^q \\ &= \sum_{\ell=0}^{q-1} A^\ell B (A+B)^{q-\ell-1} + A^q \end{aligned}$$

and

$$(A+B)^q - A^q = \int_0^1 dt \frac{d}{dt} (A+tB)^q \leq qB(A+B)^{q-1}.$$

To complete the proof, we observe that

$$\begin{aligned} A+B &= e^{\varepsilon K_0} + (\pi r^2)^{2|X|} \zeta \leq e^{\varepsilon K_0} \left(1 + (\pi r^2)^{2|X|} \zeta\right) \leq e^{\varepsilon K_0} (1 + \varepsilon \zeta^\kappa) \\ &\leq e^{\varepsilon(K_0 + \zeta^\kappa)} \end{aligned}$$

and

$$qB, qB(\pi r^2)^{|X|} \leq \frac{C_\beta}{\varepsilon} (\pi r^2)^{3|X|} \zeta \leq \zeta^\kappa. \quad \square$$

Example 3.17. Let

$$\mathcal{I}_\varepsilon(\alpha, \phi) = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} e^{F(\varepsilon, \alpha^*, \phi)} = e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \langle \alpha \mid e^{-\varepsilon K} \mid \phi \rangle$$

be as in Example 3.15 and set

$$\begin{aligned} \tilde{\mathcal{I}}_\varepsilon(\alpha, \phi) &= e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} e^{F(\varepsilon, \alpha^*, \phi) - \mathcal{F}_0(\varepsilon, \alpha^*, \phi)} \\ &= e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \langle \alpha \mid e^{-\varepsilon K} \mid \phi \rangle e^{-\mathcal{F}_0(\varepsilon, \alpha^*, \phi)} \end{aligned}$$

where \mathcal{F}_0 was defined in Corollary 3.7. Observe that

$$\tilde{\mathcal{I}}_\varepsilon(\alpha, \phi) = \exp \left\{ -\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2 + \int d\mathbf{x} \alpha^*(\mathbf{x})\phi(\mathbf{x}) - \varepsilon K(\alpha^*, \phi) \right\}.$$

Let r satisfy $r \leq \frac{1}{2} [8e^{\varepsilon(\|h\|_{1,\infty} + \mu + v_0)} \varepsilon \|v\|_{1,\infty}]^{-1/2}$. Then, by Corollary 3.7,

$$|\mathcal{F}_0(\varepsilon, \alpha^*, \phi)| \leq \text{const } \varepsilon^2 (r^2 + \|v\|_{1,\infty}^2 r^6) |X| \quad \text{for all } |\alpha|_X, |\phi|_X \leq r.$$

Consequently (assuming that $r > 1$ and allowing the constant to depend on $\|v\|_{1,\infty}$ too)

$$\begin{aligned} \|\mathcal{I}_\varepsilon - \tilde{\mathcal{I}}_\varepsilon\|_r &= \sup_{|\alpha|_X, |\phi|_X \leq r} \left| e^{-\frac{1}{2}\|\alpha\|^2 - \frac{1}{2}\|\phi\|^2} \langle \alpha \mid e^{-\varepsilon K} \mid \phi \rangle [1 - e^{-\mathcal{F}_0(\varepsilon, \alpha^*, \phi)}] \right| \\ &\leq e^{\varepsilon K_0} \text{const } \varepsilon^2 r^6 |X| e^{\text{const } \varepsilon^2 r^6 |X|}. \end{aligned}$$

Proof of Theorem 3.13. We apply Proposition 3.16 with $\mathcal{I} = \mathcal{I}_\varepsilon$, $\tilde{\mathcal{I}} = \tilde{\mathcal{I}}_\varepsilon$, as in Examples 3.15 and 3.17, $\zeta = \varepsilon^{3/2}$, $r = R(p)$, $p = \frac{\beta}{\varepsilon}$, $\kappa = \frac{1}{12}$ and $C_\beta = \beta$. If ε is sufficiently small, the three hypotheses of the Proposition are satisfied because then

$$C_\beta (\pi r^2)^{3|X|} \zeta^{1-\kappa} = \beta \pi^{3|X|} R(p)^{6|X|} \varepsilon^{\frac{3}{2}(1-\frac{1}{12})} \leq \beta \pi^{3|X|} \left(\frac{\beta}{\varepsilon}\right)^{\frac{6}{24}} \varepsilon^{\frac{33}{24}} \leq \varepsilon$$

and, by Example 3.17,

$$\|\mathcal{I} - \tilde{\mathcal{I}}\|_r \leq \text{const } \varepsilon^2 \left(\frac{\beta}{\varepsilon}\right)^{\frac{1}{4|X|}} |X| e^{\varepsilon K_0} e^{\text{const } |X| \varepsilon^2 (\beta/\varepsilon)^{\frac{1}{4|X|}}} \leq \varepsilon^{3/2} = \zeta$$

and, by Example 3.15,

$$\|\mathcal{I}^{*,q}\|_r \leq e^{q\varepsilon K_0}.$$

By the last conclusion of Proposition 3.16,

$$\lim_{p \rightarrow \infty} \int d\mu_r(\alpha^*, \alpha) \left| \tilde{\mathcal{I}}_\varepsilon^{*,p}(\alpha, \alpha) - \mathcal{I}_\varepsilon^{*,p}(\alpha, \alpha) \right|_{\substack{r=R(p) \\ \varepsilon=\beta/p}} = 0.$$

Recall, from Example 3.15, that

$$\text{Tr } e^{-\beta K} = \lim_{p \rightarrow \infty} \int d\mu_r(\phi^*, \phi) \mathcal{I}_\varepsilon^{*,p}(\phi, \phi) \Big|_{\substack{r=R(p) \\ \varepsilon=\beta/p}}$$

so that

$$\begin{aligned} \text{Tr } e^{-\beta K} &= \lim_{p \rightarrow \infty} \int d\mu_r(\phi^*, \phi) \tilde{\mathcal{I}}_\varepsilon^{*,p}(\phi, \phi) \Big|_{\substack{r=R(p) \\ \varepsilon=\beta/p}} \\ &= \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{I}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\frac{1}{2}\|\phi_{\tau-\varepsilon}\|^2 - \frac{1}{2}\|\phi_\tau\|^2 + \int d\mathbf{y} \phi_{\tau-\varepsilon}^*(\mathbf{y}) \phi_\tau(\mathbf{y})} \right. \\ &\quad \left. \times e^{-\varepsilon K(\phi_{\tau-\varepsilon}^*, \phi_\tau)} \right] \\ &= \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{I}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\|\phi_\tau\|^2 + \int d\mathbf{y} \phi_{\tau-\varepsilon}^*(\mathbf{y}) \phi_\tau(\mathbf{y})} \right. \\ &\quad \left. \times e^{-\varepsilon K(\phi_{\tau-\varepsilon}^*, \phi_\tau)} \right] \\ &= \lim_{p \rightarrow \infty} \int \prod_{\tau \in \mathcal{I}_p} \left[d\mu_{R(p)}(\phi_\tau^*, \phi_\tau) e^{-\int d\mathbf{y} [\phi_\tau^*(\mathbf{y}) - \phi_{\tau-\varepsilon}^*(\mathbf{y})] \phi_\tau(\mathbf{y})} \right. \\ &\quad \left. \times e^{-\varepsilon K(\phi_{\tau-\varepsilon}^*, \phi_\tau)} \right]. \end{aligned}$$

□

Appendix A. A Cameron style model computation

We consider the formal infinite dimensional complex Gaussian measure

$$\frac{1}{N} \prod_{0 \leq \tau < 1} \frac{d\phi_\tau^* d\phi_\tau}{2\pi i} e^{\int_0^1 d\tau \phi_\tau^*(\partial + \mu)\phi_\tau}$$

where $\mu < 0$,

$$\partial\phi_\tau = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} [\phi_{\tau+\varepsilon} - \phi_\tau]$$

and the normalization constant N is chosen so that the integral of the function 1 with respect to this measure is one. This is the formal Gaussian measure of (1.1) when $\beta = 1$, \mathbb{R}^3 is replaced by a single point and $h = v = 0$. Suppose one attempts to construct this measure as the limit

$$\lim_{p \rightarrow \infty} \frac{1}{N_p} \prod_{\tau \in \mathbb{T}_p} \frac{d\phi_\tau^* d\phi_\tau}{2\pi i} \exp \left\{ \frac{1}{p} \sum_{\tau \in \mathbb{T}_p} [\phi_\tau^* p(\phi_{\tau+1/p} - \phi_\tau) + \mu \phi_\tau^* \phi_\tau] \right\} \quad (\text{A.1})$$

of finite dimensional complex Gaussian measures. Here, $\mathbb{T}_p = \{ \frac{q}{p} \mid q = 0, \dots, p-1 \}$ and we use the convention that $\phi_1 = \phi_0$. The normalization constant

$$N_p = \int d\nu_p(\phi^*, \phi)$$

where

$$d\nu_p(\phi^*, \phi) = \prod_{\tau \in \mathbb{T}_p} \frac{d\phi_\tau^* d\phi_\tau}{2\pi i} \exp \left\{ \frac{1}{p} \sum_{\tau \in \mathbb{T}_p} [\phi_\tau^* p(\phi_{\tau+1/p} - \phi_\tau) + \mu \phi_\tau^* \phi_\tau] \right\}.$$

The following proposition shows that (A.1) cannot be a well-defined complex measure.

Proposition A.1. *Let $\mu < 0$. Then*

$$\lim_{p \rightarrow \infty} N_p = \frac{1}{e^{-\mu} - 1}.$$

However, if p is a multiple of 8 and is large enough, then

$$\int |d\nu_p(\phi^*, \phi)| > 10^{p/8}.$$

Proof. Think of \mathbb{T}_p as the discrete torus $\frac{1}{p}\mathbb{Z}/\mathbb{Z}$. Denote by $L^2(\mathbb{T}_p)$ the p complex dimensional Hilbert space of functions on \mathbb{T}_p with the usual inner product $(\psi, \phi) = \sum_{\tau \in \mathbb{T}_p} \psi_\tau^* \phi_\tau$. Define the operator ∂_p on $L^2(\mathbb{T}_p)$ by

$$(\partial_p \phi)_\tau = p(\phi_{\tau+1/p} - \phi_\tau).$$

Set, for each $0 \leq n \leq p-1$

$$e_n(\tau) = e^{2n\pi i \tau}$$

Then $\{e_n(\tau)\}_{n=0, \dots, p-1}$ is an (orthogonal) basis for $L^2(\mathbb{T}_p)$ and each $e_n(\tau)$ is an eigenvector for any translation invariant operator on $L^2(\mathbb{T}_p)$. Since

$$\partial_p e_n(\tau) = p(e^{2\pi \frac{n}{p} i} - 1)e_n(\tau)$$

the eigenvalues of ∂_p are

$$\lambda_{n,p} = p(e^{2\pi \frac{n}{p} i} - 1), \quad n = 0, 1, \dots, p-1.$$

The $2p$ real dimensional Gaussian integral

$$N_p = \int \prod_{\tau \in T_p} \frac{d\phi_\tau^* d\phi_\tau}{2\pi i} e^{\frac{1}{p}(\phi, (\partial_p + \mu)\phi)} = \prod_{n=0}^{p-1} \left[-\frac{1}{p}(\lambda_{n,p} + \mu) \right]^{-1}.$$

We have

$$\begin{aligned} \lim_{p \rightarrow \infty} \prod_{n=0}^{p-1} \left[-\frac{1}{p}(\lambda_{n,p} + \mu) \right] &= \lim_{p \rightarrow \infty} \prod_{n=0}^{p-1} \left[1 - \frac{\mu}{p} - e^{2\pi \frac{n}{p} i} \right] \\ &= \lim_{p \rightarrow \infty} \left(1 - \frac{\mu}{p} \right)^p - 1 = e^{-\mu} - 1 \end{aligned}$$

since

$$\prod_{n=0}^{p-1} (z - e^{i \frac{2\pi n}{p}}) = z^p - 1.$$

On the other hand, assuming that p is divisible by 8,

$$\int |d\nu_p(\phi^*, \phi)| = \int \prod_{\tau \in T_p} \frac{d\phi_\tau^* d\phi_\tau}{2\pi i} |e^{\frac{1}{p}(\phi, (\partial_p + \mu)\phi)}| = \prod_{n=0}^{p-1} \left[-\frac{1}{p} \operatorname{Re}(\lambda_{n,p} + \mu) \right]^{-1}$$

and

$$\prod_{n=0}^{p-1} \left[-\frac{1}{p} \operatorname{Re}(\lambda_{n,p} + \mu) \right] = \prod_{n=0}^{p-1} \left[C_p - \cos\left(2\pi \frac{n}{p}\right) \right]$$

where $C_p = 1 - \frac{\mu}{p}$

$$\begin{aligned} &= (C_p - 1)(C_p + 1) \prod_{n=1}^{\frac{p}{2}-1} \left[C_p - \cos\left(2\pi \frac{n}{p}\right) \right]^2 \\ &\qquad\qquad\qquad \text{since } \cos\left(2\pi \frac{p-n}{p}\right) = \cos\left(2\pi \frac{n}{p}\right) \\ &= (C_p - 1)C_p^2(C_p + 1) \prod_{n=1}^{\frac{p}{4}-1} \left[C_p^2 - \cos^2\left(2\pi \frac{n}{p}\right) \right]^2 \\ &\qquad\qquad\qquad \text{since } \cos\left(2\pi \frac{p/2-n}{p}\right) = -\cos\left(2\pi \frac{n}{p}\right) \\ &= (C_p - 1)C_p^2(C_p + 1) \left(C_p^2 - \frac{1}{2}\right)^2 \prod_{n=1}^{\frac{p}{8}-1} \left[C_p^2 - \cos^2\left(2\pi \frac{n}{p}\right) \right]^2 \left[C_p^2 - \sin^2\left(2\pi \frac{n}{p}\right) \right]^2 \\ &\qquad\qquad\qquad \text{since } \cos\left(2\pi \frac{p/4-n}{p}\right) = \sin\left(2\pi \frac{n}{p}\right). \end{aligned}$$

Now

$$\begin{aligned} \left[C_p^2 - \cos^2 \left(2\pi \frac{n}{p} \right) \right]^2 \left[C_p^2 - \sin^2 \left(2\pi \frac{n}{p} \right) \right]^2 &= \left[C_p^4 - C_p^2 + \sin^2 \left(2\pi \frac{n}{p} \right) \cos^2 \left(2\pi \frac{n}{p} \right) \right]^2 \\ &= \left[C_p^2 \left(-2\frac{\mu}{p} + \frac{\mu^2}{p^2} \right) + \frac{1}{4} \sin^2 \left(4\pi \frac{n}{p} \right) \right]^2 \\ &\leq \frac{1}{10} \end{aligned}$$

if p is large enough, since $\lim_{p \rightarrow \infty} C_p = 1$ and $\lim_{p \rightarrow \infty} \left(-2\frac{\mu}{p} + \frac{\mu^2}{p^2} \right) = 0$. If p is large enough, we also have

$$(C_p - 1)C_p^2(C_p + 1) \left(C_p^2 - \frac{1}{2} \right)^2 = -\frac{\mu}{p} C_p^2(C_p + 1) \left(C_p^2 - \frac{1}{2} \right)^2 < \frac{1}{10}. \quad \square$$

References

- [1] T. Balaban, *A low temperature expansion and “spin wave picture” for classical N -vector models*, in *Constructive Physics*, V. Rivasseau, editor, Springer, 1995.
- [2] T. Balaban, J. Feldman, H. Knörrer and E. Trubowitz, *A functional integral representation for many Boson systems. II: Correlation functions*, *Annales Henri Poincaré* **9** (2008), 1275–1307.
- [3] R. Cameron, *The Ilstow and Feynman integrals*, *J. Analyse Math.* **10** (1962/63), 287–361.
- [4] J. Fröhlich, B. Simon and T. Spencer, *Infrared bounds, phase transitions and continuous symmetry breaking*, *Commun. Math. Phys.* **50** (1976), 79–95.
- [5] I. M. Gel’fand and N. Ya. Vilenkin, *Generalized Functions. Applications of Harmonic Analysis*, Vol. 4, Acad. Press (1968).
- [6] J. W. Negele and H. Orland, *Quantum Many-Particle Systems*, Addison-Wesley (1988).

Tadeusz Balaban
 Department of Mathematics
 Rutgers
 The State University of New Jersey
 110 Frelinghuysen Rd
 Piscataway, NJ 08854-8019
 USA
 e-mail: tbalaban@math.rutgers.edu

Joel Feldman
 Department of Mathematics
 University of British Columbia
 Vancouver, B.C. V6T 1Z2
 Canada
 e-mail: feldman@math.ubc.ca

Horst Knörrer and Eugene Trubowitz
Mathematik
ETH-Zentrum
CH-8092 Zürich
Switzerland
e-mail: knoerr@math.ethz.ch
trub@math.ethz.ch

Communicated by Vincent Rivasseau.

Submitted: August 20, 2007.

Accepted: July 1, 2008.