Annales Henri Poincaré

Poisson Statistics of Eigenvalues in the Hierarchical Anderson Model

Evgenij Kritchevski

Abstract. We study the eigenvalue statistics for the hieracharchial Anderson model of Molchanov [21–23,27,28]. We prove Poisson fluctuations at arbitrary disorder, when the the model has a spectral dimension d < 1. The proof is based on Minami's technique [25] and we give an elementary exposition of the probabilistic arguments.

1. Introduction

The models discussed in this paper fall into the following general framework. We are given a countable set $\mathbb{X}$, a bounded self-adjoint operator H_0 acting on the Hilbert space $l^2(\mathbb{X})$ and a random potential V_ω acting diagonally on $l^2(\mathbb{X})$:

$$(V_\omega \psi)(x) = \omega(x)\psi(x), \quad \psi \in l^2(\mathbb{X}), \quad x \in \mathbb{X}.$$

We assume that $\{\omega(x)\}_{x \in \mathbb{X}}$ are independent identically distributed (i.i.d.) random variables with a bounded density γ. Hence the random parameter ω is an element of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathbb{R}^{\mathbb{X}}$, $\mathcal{F}$ is the product Borel σ-algebra on Ω and $\mathbb{P}$ is the product probability measure $\mathbb{P} = \times_{x \in \mathbb{X}} \gamma(t)dt$. We consider the random discrete Schrödinger operator

$$H_\omega = H_0 + V_\omega.$$

The finite volume approximations to H_ω are given by an increasing sequence $(B_k)_{k \geq 1}$ of finite subsets of $\mathbb{X}, \bigcup_{k \geq 1} B_k = \mathbb{X}$, and a corresponding sequence of operators $(H_k^\omega)_{k \geq 1}$ approximating H_ω, such that the subspace $l^2(B_k)$ is invariant for H_k^ω. We are interested in the asymptotic behavior of the random eigenvalues

$$e_1^{\omega,k} \leq e_2^{\omega,k} \leq \cdots \leq e_{|B_k|}^{\omega,k},$$

of $H_k^\omega \upharpoonright l^2(B_k)$ as $k \to \infty$. Usually, the first step is to prove that there is a nonrandom probability measure μ^{av} on $\mathbb{R}$ such that, with probability one, the

random normalized eigenvalue counting measure

$$\mu_k^\omega = |B_k|^{-1} \sum_{j=1}^{|B_k|} \delta(e_j^{\omega,k}),\tag{1.1}$$

converges μ^{av} in the weak-* topology as $k \to \infty$. The measure μ^{av} is called the density of states for H_ω. For large k, the number of eigenvalues in a small interval $(e - \varepsilon, e + \varepsilon)$ around a point $e \in \mathbb{R}$ is then typically of the order of $|B_k|\,\mu^{av}((e + \varepsilon, e - \varepsilon))$. The fine eigenvalue statistics near e are then captured by the rescaled point measure

$$\xi_k^{\omega,e} = \sum_{i=1}^{|B_k|} \delta\big(|B_k|\,(e_i^{\omega,k} - e)\big).\tag{1.2}$$

Minami's technique [25] is a method allowing to prove that, in appropriate situations, $\xi_k^{\omega,e}$ is asymptotically a Poisson point process as $k \to \infty$. This means that for disjoint Borel sets $A_1, A_2, \cdots, A_m \subset \mathbb{R}$, the corresponding numbers of rescaled eigenvalues in each of the sets,

$$\xi_k^{\omega,e}(A_1), \xi_k^{\omega,e}(A_2), \dots, \xi_k^{\omega,e}(A_m),$$

are approximately independent Poisson random variables and hence the eigenvalues near e are uncorrelated.

Minami originally considered the Anderson tight-binding model on $\mathbb{Z}^{\mathrm{d}}$. In this case $\mathbb{X} = \mathbb{Z}^{\mathrm{d}}$ and H_0 is the discrete Laplacian:

$$(H_0\psi)(x) = \sum_{|y-x|=1} \psi(y), \quad \psi \in l^2(\mathbb{Z}^{\mathrm{d}}), \quad x \in \mathbb{Z}^{\mathrm{d}},\tag{1.3}$$

where $|x - y| = \sum_{j=1}^{\mathrm{d}} |x_j - y_j|$. He proved Poisson statistics of eigenvalues in the localized regime [18,25]. Minami's method has its origins in Molchanov's paper [26], where the first rigorous proof of the absence of energy level repulsion is given for a continuous one-dimensional model. After Minami's paper [25], the technique and its variations have been used to prove Poisson statistics of eigenvalues for different models [3,4,18,19,30]. In this paper, we combine existing and new results to prove Poisson statistics of eigenvalues for the hierarchical Anderson model (the precise definition of the model and the statement of our results are given in Section 3).

The probabilistic part of Minami's technique shared by most models is based on the theory of infinitely divisible point processes. As a result, one sometimes has to go though a substantial body of material also concerned with other questions e.g. [9,16] in order to extract the necessary results. One of our goals is to give a self-contained elementary exposition of the probabilistic part, only assuming standard material taught in a first graduate course on probability. The spectral part of the technique is based on decoupling, i.e. on approximating H_ω^k by a direct sum of a large number of statistically independent infinitesimal components. The analysis is specific to each model and the decoupling is possible only in an appropriate regime.

In Section 2, we discuss the necessary probabilistic preliminaries on Poisson point processes. In Section 3, we introduce the hierarchical Anderson model and we provide a complete proof of Poisson statistics of eigenvalues in the regime where the model has a spectral dimension d < 1. In the appendix, we outline, within our framework, Minami's original proof of Poisson statistics of eigenvalues for the Anderson model on $\mathbb{Z}^d$ in the localized regime.

2. Probabilistic preliminaries

2.1. Why the Poisson distribution

The Poisson distribution with parameter λ is the discrete probability measure $\mathbb{P}_\lambda$ on $\mathbb{N} = \{0, 1, 2, \ldots\}$ given by

$$\mathbb{P}_\lambda = e^{-\lambda} \sum_{r \in \mathbb{N}} \frac{\lambda^r}{r!} \delta(r) \,.$$

The simplest example where the Poisson distribution appears naturally in connection with the rescaled measure $\xi_k^{\omega, e}$ is the trivial case of a random discrete Schrödinger operator: $\mathbb{X} = \{1, 2, \ldots\}, H_0 = 0$ and the finite volume approximations are $B_k = \{1, \ldots, k\}, H_k^\omega = H_\omega \upharpoonright l^2(B_k)$. Then $H_k^\omega \upharpoonright l^2(B_k)$ has statistically independent eigenvalues $\{\omega(x)\}_{x \in B_k}$ and it follows from Kolmogorov's strong law of large numbers that for every Borel set $A \subset \mathbb{R}$,

$$\lim_{k \to \infty} \mu_k^\omega(A) = \mu^{av}(A) = \int_A \gamma(t) dt \,,$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$.

We denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$. Let us assume that γ is continuous at a point $e \in \mathbb{R}$ and that $\gamma(e) > 0$. If $A_1, A_2, \ldots, A_m \subset \mathbb{R}$ are disjoint bounded Borel sets, then the random vector

$$\left[\xi_k^{\omega, e}(A_1), \xi_k^{\omega, e}(A_2), \ldots, \xi_k^{\omega, e}(A_m) \right],$$

has a multinomial distribution

$$\mathbb{P}\left\{ \xi_k^{\omega, e}(A_1) = r_1, \xi_k^{\omega, e}(A_2) = r_2, \ldots, \xi_k^{\omega, e}(A_m) = r_m \right\}$$

$$= \frac{k!}{r_1! r_2! \ldots r_{m+1}!} q_{k,1}^{r_1} q_{k,2}^{r_2} \cdots q_{k,m+1}^{r_{m+1}}, \quad r_s = 0, \ldots, k, \quad \sum_{s=1}^{m+1} r_s = k \,,$$

where

$$q_{k,s} = \mathbb{P}\left\{ \left(k\omega(1) - e \right) \in A_s \right\} = \int_{e + k^{-1} A_s} \gamma(t) dt, \quad s = 1, \ldots, m+1 \,,$$

and $A_{m+1} = \mathbb{R} \backslash (\bigcup_{s=1}^m A_s)$. Continuity of γ at e yields that

$$\lim_{k \to \infty} k q_{k,s} = \gamma(e) \mathcal{L}(A_s) \,,$$

and hence

$$\lim_{k \to \infty} \mathbb{P}\big\{\xi_k^{\omega,e}(A_1) = r_1, \xi_k^{\omega,e}(A_2) = r_2, \ldots, \xi_k^{\omega,e}(A_m) = r_m\big\} = \prod_{s=1}^{m} \mathbb{P}_{\lambda_s}(\{r_s\}),$$

with $\lambda_s = \gamma(e)\mathcal{L}(A_s)$. Hence the random variables $\xi_k^{\omega,e}(A_s), s = 1, \ldots, m$, are asymptotically independent and have Poisson distributions $\mathbb{P}_{\lambda_s}$.

In nontrivial situations, the operator $H_0 \neq 0$ introduces statistical dependence to the eigenvalues of $H_k^\omega \upharpoonright l^2(B_k)$ and therefore the analysis of the rescaled measure $\xi_k^{\omega,e}$ is more involved. If this dependence is not too big in a suitable sense, then Minami's method allows to show that $\xi_k^{\omega,e}(A_s), s = 1, \ldots, m$ are still asymptotically independent Poisson random variables. In the next subsection, we discuss a general limit theorem needed for Minami's method.

2.2. The Poisson point process and Grigelionis' limit theorem

Although $\xi_k^{\omega,e}$ as well as the other measures of interest to us are on $\mathbb{R}$, we discuss, for sake of clarity, the general situation of random point measures on a metric space S. We equip S with the Borel σ-algebra $\mathcal{B}_S$, i.e. the σ-algebra generated by open sets. We denote by $\mathcal{M}$ the set of all nonnegative Borel measures μ on $(S, \mathcal{B}_S)$ such that $\mu(A) < \infty$ for every bounded Borel set $A \subset S$. A measure $\mu \in \mathcal{M}$ is called a point measure if μ can be written in the form

$$\mu = \sum_{j \in J} \delta(x_j), \quad x_j \in S,$$

where J is a countable index set. We denote by $\mathcal{M}_p$ the set of all point measures on $(S, \mathcal{B}_S)$. A point process on S is map $\omega \to \mu^\omega$ from some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathcal{M}_p$ such that for every bounded Borel set $A \subset S$, the map $\omega \to \mu^\omega(A)$ is measurable. If μ^ω is a point process, then the map

$$\nu(B) = \mathbb{E}\mu^\omega(B), \quad B \in \mathcal{B}_S,$$

defines a measure on $(S, \mathcal{B}_S)$. The measure ν is called the intensity measure of the point process μ^ω.

Definition 2.1. Let $\nu \in \mathcal{M}$. A Poisson point process on S with intensity ν is a point process ξ^ω with the following properties:

1. for every bounded Borel set $A \subset S$, the random variable $\xi^\omega(A)$ has a Poisson distribution with parameter $\nu(A)$.
2. given disjoint bounded Borel sets $A_1, A_2, \ldots, A_m$ in S, the random variables $\xi^\omega(A_1), \xi^\omega(A_2), \ldots, \xi^\omega(A_m)$ are independent.

It can be shown [17] that given any $\nu \in \mathcal{M}$, there exists a Poisson process on S with intensity ν, constructed on a suitable probability space. The Poisson point process is an idealized model of noninteraction and the point process $\xi_k^{\omega,e}$ in the study of eigenvalue statistics never exactly verifies Conditions (1) and (2) of Definition 2.1.

Definition 2.2. A sequence ξ_k^ω of point processes on S, defined on the same probability space, is said to converge to a Poisson point process on S with intensity $\nu \in \mathcal{M}$ if for any given disjoint bounded Borel sets $A_1, A_2, \ldots, A_m$ in S, we have

$$\lim_{k\to\infty} \mathbb{P}\big\{\xi_k^\omega(A_1) = r_1, \xi_k^\omega(A_2) = r_2, \ldots, \xi_k^\omega(A_m) = r_m\big\} = \prod_{s=1}^m \mathbb{P}_{\nu(A_s)}(\{r_s\}), \quad (2.1)$$

for all $r_1, r_2, \ldots, r_m \in \mathbb{N}$.

Hence, in the previous subsection, the sequence of point processes $\xi_k^{\omega,e}$ on $\mathbb{R}$ converges to a Poisson process on $\mathbb{R}$ with intensity $\gamma(e)\mathcal{L}$. In general, it can be difficult to verify the Condition (2.1) directly and it is more convenient to verify an equivalent condition in terms of the characteristic functions, namely

$$\lim_{k\to\infty} \mathbb{E}e^{i\sum_{s=1}^m t_s\xi_k^\omega(A_s)} = \prod_{s=1}^m \exp\big(\nu(A_s)(e^{it_s} - 1)\big), \quad (2.2)$$

for all $t_1, t_2, \ldots, t_m \in \mathbb{R}$. Both (2.1) and (2.2) are equivalent to the usual definition of convergence in law for random vectors in $\mathbb{N}^m$.

The basic limit theorem guaranteeing the convergence of a sequence of point processes to a Poisson point processes is due to Griegelionis [13]. Originally formulated for step processes on $\mathbb{R}$, Grigelionis' theorem remains valid in more general settings and in our case it translates to:

Theorem 2.3 (Grigelionis, 1963). *Let $(n_k)_{k\geq 1}$ be a natural subsequence, let for each $k \geq 1$, $\xi_{k,1}^\omega, \xi_{k,2}^\omega, \ldots, \xi_{k,n_k}^\omega$ be independent point processes on S and let*

$$\xi_k^\omega = \sum_{j=1}^{n_k} \xi_{k,j}^\omega\,.$$

Let $\nu \in \mathcal{M}$ and assume that for every bounded Borel set $A \subset S$, we have

$$(1) \qquad\qquad \lim_{k\to\infty} \max_{1\leq j\leq n_k} \mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 1\right\} = 0\,,$$

$$(2) \qquad\qquad \lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 1\right\} = \nu(A)\,,$$

and

$$(3) \qquad\qquad \lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 2\right\} = 0\,.$$

Then ξ_k^ω converges to a Poisson point process on S with intensity ν.

Theorem 2.3 is well known and can be found in the literature e.g. [9, 16] as a corollary of more general results on point processes. For completeness, we include a self-contained proof here, following the original arguments of [13].

Proof. We use the standard notation $ab = \sum_{s=1}^{m} a_s b_s$, for $a, b \in \mathbb{R}^m$ and $|\alpha| = \sum_{s=1}^{m} \alpha_s$ for $\alpha \in \mathbb{N}^m$. We denote by $\{e_s\}_{s=1}^{m}$ the standard basis vectors of $\mathbb{R}^m$. Let $A_1, A_2, \ldots, A_m$ be given disjoint bounded Borel sets in S. Let X_k^ω be the random vector

$$X_k^\omega = \left[\xi_k^\omega(A_1), \xi_k^\omega(A_2), \ldots, \xi_k^\omega(A_m)\right],$$

and let $\phi_k : \mathbb{R}^m \to \mathbb{C}$ be the corresponding characteristic function

$$\phi_k(t) = \mathbb{E}e^{itX_k^\omega}, \quad t \in \mathbb{R}^m.$$

According to (2.2), we have to show that for all $t \in \mathbb{R}^m$,

$$\lim_{k \to \infty} \phi_k(t) = \prod_{s=1}^{m} \exp\left(\nu(A_s)(e^{it_s} - 1)\right). \tag{2.3}$$

We set

$$X_{k,j}^\omega = \left[\xi_{k,j}^\omega(A_1), \xi_{k,j}^\omega(A_2), \ldots, \xi_{k,j}^\omega(A_m)\right],$$
$$\phi_{k,j}(t) = \mathbb{E}e^{itX_{k,j}^\omega}, \quad t \in \mathbb{R}^m,$$

and

$$A = \bigcup_{s=1}^{m} A_s.$$

By assumption (1), there is a k_0 such that for $k \geq k_0$,

$$\max_{1 \leq j \leq n_k} \mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 1\right\} < 1/4.$$

Hence for $k \geq k_0$ and $1 \leq j \leq n_k$,

$$\left|\sum_{|\alpha| \geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1)\right| \leq 2\sum_{|\alpha| \geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\} = 2\mathbb{P}\left\{\xi_{k,j}^\omega(A) \geq 1\right\} < 1/2,$$

and we can write

$$\phi_{k,j}(t) = 1 + \sum_{|\alpha| \geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1)$$

$$= \exp\left(\sum_{|\alpha| \geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1) + E_{k,j}\right), \tag{2.4}$$

where

$$E_{k,j} = f\left(\sum_{|\alpha| \geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1)\right),$$

and $f(z) = \log(1 + z) - z$. The function f is analytic in the open disk $\{|z| < 1\}$ and

$$|f(z)| \leq C|z|^2 \quad \text{for} \quad |z| < 1/2, \tag{2.5}$$

where $0 < C < \infty$ is a numerical constant. Next, we write

$$\sum_{|\alpha| \geq 1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1) = \sum_{|\alpha|=1} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1) + F_{k,j}$$

$$= \sum_{s=1}^{m} \mathbb{P}\left\{X_{k,j}^\omega = e_s\right\}(e^{it_s} - 1) + F_{k,j} \qquad (2.6)$$

$$= \sum_{s=1}^{m} \mathbb{P}\left\{\xi_{k,j}^\omega(A_s) = 1\right\}(e^{it_s} - 1) + G_{k,j} + F_{k,j} \,,$$

where

$$F_{k,j} = \sum_{|\alpha| \geq 2} \mathbb{P}\left\{X_{k,j}^\omega = \alpha\right\}(e^{i\alpha t} - 1) \,,$$

and

$$G_{k,j} = \sum_{s=1}^{m} \left(\mathbb{P}\{X_{k,j}^\omega = e_s\} - \mathbb{P}\{\xi_{k,j}^\omega(A_s) = 1\}\right)(e^{it_s} - 1) \,.$$

Hence,

$$\phi_{k,j}(t) = \exp\left(\sum_{s=1}^{m} \mathbb{P}\left\{\xi_{k,j}^\omega(A_s) = 1\right\}(e^{it_s} - 1) + H_{k,j}\right),$$

where

$$H_{k,j} = E_{k,j} + F_{k,j} + G_{k,j} \,.$$

We then have, by independence, that

$$\phi_k(t) = \prod_{j=1}^{n_k} \phi_{k,j}(t)$$

$$= \exp\left(\sum_{s=1}^{m}\left(\sum_{j=1}^{n_k} \mathbb{P}\left\{\xi_{k,j}^\omega(A_s) = 1\right\}\right)(e^{it_s} - 1) + \sum_{j=1}^{n_k} H_{k,j}\right) \qquad (2.7)$$

The assumptions (2) and (3) imply that

$$\lim_{k \to \infty} \sum_{j=1}^{n_k} \mathbb{P}\{\xi_{k,j}^\omega(A_s) = 1\} = \nu(A_s) \,. \qquad (2.8)$$

We claim that

$$\lim_{k \to \infty} \sum_{j=1}^{n_k} H_{k,j} = 0 \,. \qquad (2.9)$$

If (2.9) holds, then (2.8), (2.9) and (2.7) together yield the desired conclusion (2.3) and we are done. We now prove (2.9). We have

$$|F_{k,j}| \leq 2\mathbb{P}\{\xi_{k,j}^\omega(A) \geq 2\} \,, \qquad (2.10)$$

and the bound (2.5) yields

$$|E_{k,j}| \le C \left(2 \sum_{|\alpha| \ge 1} \mathbb{P}\left\{ X_{k,j}^{\omega} = \alpha \right\} \right)^2 = 4C \left(\mathbb{P}\left\{ \xi_{k,j}^{\omega}(A) \ge 1 \right\} \right)^2. \qquad (2.11)$$

To estimate $|G_{k,j}|$, note that

$$\left\{ X_{k,j}^{\omega} = e_s \right\} \subset \left\{ \xi_{k,j}^{\omega}(A_s) = 1 \right\},$$

and

$$\left(\left\{ \xi_{k,j}^{\omega}(A_s) = 1 \right\} \setminus \left\{ X_{k,j}^{\omega} = e_s \right\} \right) \subset \left\{ \xi_{k,j}^{\omega}(A) \ge 2 \right\}.$$

Hence

$$|G_{k,j}| \le 2m \mathbb{P}\left\{ \xi_{k,j}^{\omega}(A) \ge 2 \right\}. \qquad (2.12)$$

We now combine the bounds (2.11), (2.10) and (2.12) to get

$$\left| \sum_{j=1}^{n_k} H_{k,j} \right| \le (2m+2) \sum_{j=1}^{n_k} \mathbb{P}\left\{ \xi_{k,j}^{\omega}(A) \ge 2 \right\}$$

$$+ 4C \left(\max_{1 \le j \le n_k} \mathbb{P}\left\{ \xi_{k,j}^{\omega}(A) \ge 1 \right\} \right) \sum_{j=1}^{n_k} \mathbb{P}\left\{ \xi_{k,j}^{\omega}(A) \ge 1 \right\}.$$

The assumptions (1), (2) and (3) imply that the right hand side of last inequality converges to zero as $k \to \infty$, completing the proof.   □

2.3. Corollaries of Grigelionis' limit theorem

For the point processes ξ^{ω} on $S = \mathbb{R}$ arising in the study of eigenvalue statistics, it is sometimes more natural to obtain information about the Poisson integrals $\int_{\mathbb{R}} \mathrm{Im}(t - z)^{-1} d\xi^{\omega}(t)$, $\mathrm{Im}\, z > 0$, rather than about the events $\{ \xi^{\omega}(A) \ge 1 \}$ and $\{ \xi^{\omega}(A) \ge 2 \}$. In this subsection, we replace the Conditions (2) and (3) of Theorem 2.3 by sufficient conditions in terms of the Poisson integrals. We refer the reader to [15] for the general theory of Poisson integrals and their applications to spectral theory.

For a positive Borel measure μ on S and a Borel function $f : S \to [0, \infty)$, we set

$$\mathcal{I}(\mu, f) = \int_{t \ne t'} f(t) f(t') d\mu(t) d\mu(t').$$

If $\mu = \sum_j \delta(t_j)$ is a point measure on S and $f(t) = 1_A(t)$ is the indicator function of a bounded Borel set $A \subset S$, then we have

$$\mathcal{I}(\mu, 1_A) = \sum_{i \ne j} 1_A(t_i) 1_A(t_j) = \mu(A)\left(\mu(A) - 1 \right),$$

and therefore $\mathcal{I}(\mu, 1_A) \neq 0 \Leftrightarrow \mu(A) \geq 2$. If ξ^ω is a point process on S, then

$$\sum_{l \geq 2} \mathbb{P}\{\xi^\omega(A) \geq l\} = \sum_{l \geq 2} (l-1)\mathbb{P}\{\xi^\omega(A) = l\}$$

$$\leq \sum_{l \geq 2} l(l-1)\mathbb{P}\{\xi^\omega(A) = l\}$$

$$= \mathbb{E}\mathcal{I}(\xi^\omega, 1_A).$$

Since

$$\mathbb{P}\{\xi^\omega(A) \geq 1\} = \mathbb{E}\xi^\omega(A) - \sum_{l \geq 2} \mathbb{P}\{\xi^\omega(A) \geq l\},$$

we conclude that the conditions

$$(2') \quad \lim_{k \to \infty} \sum_{j=1}^{n_k} \mathbb{E}\xi^\omega_{k,j}(A) = \nu(A),$$

and

$$(3') \quad \lim_{k \to \infty} \sum_{j=1}^{n_k} \mathbb{E}\mathcal{I}(\xi^\omega_{k,j}, 1_A) = 0,$$

together imply Conditions (2) and (3) of Theorem 2.3. The next step is to replace, in (2') and (3'), the quantity $\mathbb{E}\xi^\omega_{k,j}(A)$ by $\mathbb{E}\int f d\xi^\omega_{k,j}$ for f in a sufficiently rich family F of functions.

Theorem 2.4. *For each $k \geq 1$, let $\xi^\omega_{k,1}, \xi^\omega_{k,2}, \ldots, \xi^\omega_{k,n_k}$ be point processes on S and let $\xi^{av}_k = \sum_{j=1}^{n_k} \mathbb{E}\xi^\omega_{k,j}$. Let $\nu \in \mathcal{M}$. Suppose that there is a measure $\mu \in \mathcal{M}$ s.t. that ν and $(\xi^{av}_k)_{k \geq 1}$ are absolutely continuous with respect to μ, with uniformly bounded densities, i.e. there is a constant $0 < C < \infty$ such that for all bounded Borel sets $A \subset S$,*

$$\nu(A) \leq C\mu(A),$$

and

$$\xi^{av}_k(A) \leq C\mu(A), \quad k \geq 1.$$

Suppose that $F \subset L_1(S, \mu)$ is a family of functions such that finite linear combinations of functions in F are dense in $L_1(S, \mu)$ and such that for every bounded Borel set $A \subset S$, there exists $f \in F$ with $f \geq 1_A$. Suppose that for all $f \in F$, we have

$$(2'') \quad \lim_{k \to \infty} \int f d\xi^{av}_k = \int f d\nu,$$

and

$$(3'') \quad \lim_{k \to \infty} \sum_{j=1}^{n_k} \mathbb{E}\mathcal{I}(\xi^\omega_{k,j}, f) = 0.$$

Then (2') and (3') hold for all bounded Borel sets $A \subset S$.

Proof. Let A be a bounded Borel set. Let $\varepsilon > 0$. There is a finite linear combination $g = \sum_i c_i f_i, f_i \in F$, with $\int |g - 1_A| \, d\mu < \varepsilon$. Then $\left| \int g d\nu - \nu(A) \right| < C\varepsilon$ and $\left| \int g d\xi_k^{av} - \xi_k^{av}(A) \right| < C\varepsilon$. Since $\lim_{k\to\infty} \int g d\xi_k^{av} = \int g d\nu$, we have

$$\nu(A) - 2C\varepsilon \leq \liminf_{k\to\infty} \xi_k^{av}(A) \leq \limsup_{k\to\infty} \xi_k^{av}(A) \leq \nu(A) + 2C\varepsilon \,,$$

and (2') is obtained after letting $\varepsilon \downarrow 0$. Now let $f \in F$ be such that $f \geq 1_A$. Since, $\mathcal{I}(\xi_{k,j}^\omega, 1_A) \leq I(\xi_{k,j}^\omega, f)$, (3') follows from (3"). $\qquad\square$

The special case when $S = \mathbb{R}$, $\mu = \mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$, $\nu = \lambda\mathcal{L}$ for a $\lambda > 0$ and F is the family of functions $\left\{ \mathrm{Im}(t - z)^{-1} \right\}_{\mathrm{Im}z > 0}$ yields

Theorem 2.5. *Let $(n_k)_{k\geq 1}$ be a natural subsequence, let for each $k \geq 1$, $\xi_{k,1}^\omega, \xi_{k,2}^\omega$, $\ldots, \xi_{k,n_k}^\omega$ be independent point processes on $\mathbb{R}$ and let*

$$\xi_k^\omega = \sum_{j=1}^{n_k} \xi_{k,j}^\omega \,.$$

We make the following four hypotheses:

(H0): *there is a constant $0 < C < \infty$ such that for all $k \geq 1$ and every bounded Borel set $A \subset \mathbb{R}$,*

$$\sum_{j=1}^{n_k} \mathbb{E}\xi_{k,j}^\omega(A) \leq C\mathcal{L}(A) \,.$$

(H1): *for every bounded Borel set $A \subset \mathbb{R}$,*

$$\lim_{k\to\infty} \max_{1\leq j\leq n_k} \mathbb{P}\left\{ \xi_{k,j}^\omega(A) \geq 1 \right\} = 0 \,.$$

(H2): *there is a constant $0 < \lambda < \infty$ such that for $\mathrm{Im}z > 0$,*

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{E} \int_{\mathbb{R}} \mathrm{Im}(t - z)^{-1} d\xi_{k,j}^\omega(t) = \pi\lambda \,.$$

(H3): *for $\mathrm{Im}z > 0$,*

$$\lim_{k\to\infty} \sum_{j=1}^{n_k} \mathbb{E} \int_{t\neq t'} \mathrm{Im}(t - z)^{-1}\mathrm{Im}(t' - z)^{-1} d\xi_{k,j}^\omega(t) d\xi_{k,j}^\omega(t') = 0 \,.$$

Then ξ_k^ω converges to a Poisson point process on $\mathbb{R}$ with intensity $\lambda\mathcal{L}$.

Theorem 2.5 is implicitly derived in [25] and is suitable for applications to eigenvalue statistics of general random discrete Schrödinger operators.

3. Poisson statistics of eigenvalues in the hierarchical Anderson model

3.1. Definition of the model and its basic properties

In this subsection, we review the definition and the basic properties of the hierarchical Anderson model. For additional information, we refer the reader to [21–23, 27, 28]. Theorems 3.1 and 3.2 collect, for reference purposes, the main known results on the hierarchical Anderson model and are stated without proof.

We consider the set $\mathbb{X} = \{0, 1, 2, \dots\}$. Given an integer $n \geq 2$, $\mathbb{X}$ has a metric space structure with the distance $d : \mathbb{X} \times \mathbb{X} \to [0, \infty)$

$$d(x, y) = \min \left\{ r : q(x, n^r) = q(y, n^r) \right\},$$

where $q(x, n^r)$ denotes the quotient of the division of x by n^r. The (closed) ball with center x and radius r is denoted by

$$B(x, r) = \left\{ y \in \mathbb{X} : d(x, y) \leq r \right\}.$$

The main property of d is that two balls of the same radius are either disjoint or identical, and that each $B(x, r + 1)$ is a disjoint union of n balls of radius r.

For $x \in \mathbb{X}$, the unit vector $\delta_x \in l^2(\mathbb{X})$ denotes the Kronecker delta function at x: $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \neq x$. For each integer $r \geq 1$, we set $E_r : l^2(\mathbb{X}) \to l^2(\mathbb{X})$,

$$(E_r \psi)(x) = n^{-r} \sum_{d(y,x) \leq r} \psi(y).$$

Thus E_r is the orthogonal projection onto the subspace of $l^2(\mathbb{X})$ consisting of functions that are constant on every ball of radius r. The hierarchical Laplacian is then defined by the formula

$$\Delta = \sum_{r=1}^{\infty} p_r E_r,$$

where $(p_r)_{r \geq 1}$ is a given sequence such that $p_r > 0$ and $\sum_{r=1}^{\infty} p_r = 1$. We assume that

$$\frac{C_1}{\rho^r} \leq p_r \leq \frac{C_2}{\rho^r},$$

for some fixed constants $\rho > 1, C_1 > 0, C_2 > 0$. The number

$$\mathrm{d} = \mathrm{d}(n, \rho) = 2 \frac{\log n}{\log \rho}, \tag{3.1}$$

is called the spectral dimension of Δ. The following theorem [21, 28] summarizes some of the spectral features of Δ.

Theorem 3.1. Δ *is a bounded self-adjoint operator on* $l^2(\mathbb{X})$ *and its spectrum consists of infinitely degenerate isolated eigenvalues*

$$\lambda_0 = 0, \quad \lambda_1 = p_1, \quad \lambda_2 = p_1 + p_2, \quad \lambda_3 = p_1 + p_2 + p_3, \dots$$

and of their accumulation point $\lambda_\infty = 1$, which is not an eigenvalue. For each $x \in \mathbb{X}$,

$$\sum_{y \in \mathbb{X}} \langle \delta_x | \Delta \delta_y \rangle = 1 \,,$$

and hence Δ generates a random walk on $\mathbb{X}$. The random walk is recurrent when $\mathrm{d} \leq 2$ and transient when $\mathrm{d} > 2$.

The hierarchical Anderson model is the random discrete Schrödinger operator

$$H_\omega = \Delta + V_\omega \,,$$

as in the framework of the introduction, with $H_0 = \Delta$. If the set $\{\omega(x) : x \in \mathbb{X}\}$ is unbounded, then V_ω and H_ω are unbounded self-adjoint operators with the domain

$$\mathcal{D}_\omega = \left\{ \psi : \sum_{x \in \mathbb{X}} |\psi(x)|^2 \left(1 + |\omega(x)|^2\right) < \infty \right\}.$$

Theorem 3.2. *H_ω has the following generic spectral properties.*

(1) [22, 23] *If the support of γ is connected, $\mathrm{supp}(\gamma) = [\mathrm{a}, \mathrm{b}]$, then for $\mathbb{P}$-a.e. $\omega \in \Omega$, the spectrum of H_ω is given by*

$$\Sigma = \bigcup_{r=0}^{\infty} [\lambda_r + a, \lambda_r + b] \,.$$

(2) [22] *If the model has a spectral dimension $\mathrm{d} < 4$ then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the spectrum of H_ω is dense pure-point in Σ.*

(3) [28] *For any spectral dimension $\mathrm{d} < \infty$, the same conclusion as in (2) holds provided the random variables $\omega(x)$ have a Cauchy distribution, i.e. the density $\gamma(t)$ is of the special form:*

$$\gamma(t) = \frac{1}{\pi} \frac{v}{(u - t)^2 + v^2} \,, \tag{3.2}$$

for some $u \in \mathbb{R}, v > 0$.

3.2. The density of states

We denote by $C_0(\mathbb{R})$ the space of continuous functions $f : \mathbb{R} \to \mathbb{C}$ vanishing at infinity, i.e. $\lim_{|t| \to \infty} |f(t)| = 0$. If $(\nu_k)_{k \geq 1}$ and ν are Borel probability measures on $\mathbb{R}$, we say that ν_k converges to ν in the weak-* topology if for every $f \in C_0(\mathbb{R})$,

$$\lim_{k \to \infty} \int f(t) d\nu_k(t) = \int f(t) d\nu(t) \,.$$

The finite volume approximations to H_ω are defined as follows. We fix $x_0 \in \mathbb{X}$ and we consider the increasing sequence of balls

$$B_k = B(x_0, k) \quad k \geq 0 \,.$$

Each B_k has then size $|B_k| = n^k$. We define H_k^ω to be the truncated operator

$$H_k^\omega = \sum_{s=1}^{k} p_s E_s + V_\omega \,.$$

Note that the subspace

$$l^2(B_k) = \left\{ \psi \in l^2(\mathbb{X}) : \psi(x) = 0 \text{ for } x \notin B_k \right\},$$

is invariant for H_k^ω. The normalized eigenvalue counting measure μ_k^ω is then given by (1.1). The averaged spectral measure for H_ω is the unique Borel probability measure μ^{av} on $\mathbb{R}$ defined by

$$\int f(t) d\mu^{av}(t) = \mathbb{E}\langle \delta_{x_0} | f(H_\omega) \delta_{x_0} \rangle, \quad f \in C_0(\mathbb{R}). \tag{3.3}$$

By symmetry, $\int f(t) d\mu^{av}(t) = \mathbb{E}\langle \delta_x | f(H_\omega) \delta_x \rangle$ for all $x \in \mathbb{X}$. The content of the following theorem is that the averaged spectral measure μ^{av} is naturally interpreted as the density of states for H_ω.

Theorem 3.3. *For $\mathbb{P}$-a.e. $\omega \in \Omega, \mu_k^\omega \to \mu^{av}$ in the weak-* topology as $k \to \infty$, i.e. there is a set $\widetilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$ such that for all $\omega \in \widetilde{\Omega}$ and $f \in C_0(\mathbb{R})$ we have*

$$\lim_{k \to \infty} \int f(t) d\mu_k^\omega(t) = \int f(t) d\mu^{av}(t).$$

We start the proof of Theorem 3.3 with resolvent bounds. Since $H_r^\omega = H_{r-1}^\omega + p_r E_r$, the resolvent identity yields

$$(H_{r-1}^\omega - z)^{-1} - (H_r^\omega - z)^{-1} = p_r (H_{r-1}^\omega - z)^{-1} E_r (H_r^\omega - z)^{-1},$$

for $z \in \mathbb{C} \backslash \mathbb{R}$. Therefore:

$$\left\| (H_{r-1}^\omega - z)^{-1} - (H_r^\omega - z)^{-1} \right\| \leq |\mathrm{Im} z|^{-2} p_r, \quad z \in \mathbb{C} \backslash \mathbb{R}. \tag{3.4}$$

Iterating (3.4) yields for $r < k$,

$$\left\| (H_r^\omega - z)^{-1} - (H_k^\omega - z)^{-1} \right\| \leq |\mathrm{Im} z|^{-2} \sum_{s=r+1}^k p_s, \quad z \in \mathbb{C} \backslash \mathbb{R}. \tag{3.5}$$

and letting $k \to \infty$,

$$\left\| (H_r^\omega - z)^{-1} - (H_\omega - z)^{-1} \right\| \leq |\mathrm{Im} z|^{-2} \sum_{s=r+1}^\infty p_s, \quad z \in \mathbb{C} \backslash \mathbb{R}. \tag{3.6}$$

Proposition 3.4. *For every $z \in \mathbb{C} \backslash \mathbb{R}$ there is a set $\Omega_z \in \mathcal{F}$, with $\mathbb{P}(\Omega_z) = 1$ and such that for all $\omega \in \Omega_z$, the difference*

$$D_{k,\omega} = \int (t - z)^{-1} d\mu_k^\omega(t) - \int (t - z)^{-1} d\mu^{av}(t),$$

converges to 0 as $k \to \infty$.

Proof. Let $\varepsilon > 0$ be given. We take $r = r(\varepsilon, z)$ big enough so that

$$|\mathrm{Im} z|^{-2} \sum_{s=r+1}^\infty p_s < \varepsilon/2. \tag{3.7}$$

Then for $r < k$,

$$
\begin{aligned}
D_{k,\omega} &= |B_k|^{-1} \sum_{x \in B_k} \left\langle \delta_x | (H_k^\omega - z)^{-1} \delta_x \right\rangle - \mathbb{E}\left\langle \delta_{x_0} | (H_\omega - z)^{-1} \delta_{x_0} \right\rangle \\
&= \left\{ |B_k|^{-1} \sum_{x \in B_k} \left\langle \delta_x | \left((H_k^\omega - z)^{-1} - (H_r^\omega - z)^{-1} \right) \delta_x \right\rangle \right\} \\
&\quad + \left\{ |B_k|^{-1} \sum_{x \in B_k} \left\langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \right\rangle - \mathbb{E}\left\langle \delta_{x_0} | (H_\omega - z)^{-1} \delta_{x_0} \right\rangle \right\} \\
&= I_{k,\omega} + II_{k,\omega} \,.
\end{aligned}
\tag{3.8}
$$

The bounds (3.5) and (3.7) yield $|I_{k,\omega}| < \varepsilon/2$. We proceed with estimating $|II_{k,\omega}|$. Note that B_k is a disjoint union of n^{k-r} balls of radius r,

$$
B_k = \bigcup_{j=1}^{n^{k-r}} B_{k,j} \,,
$$

and therefore

$$
l^2(B_k) = \bigoplus_{j=1}^{n^{k-r}} l^2(B_{k,j}) \,.
$$

Since each subspace $l^2(B_{k,j})$ is invariant for H_r^ω, we can write

$$
|B_k|^{-1} \sum_{x \in B_k} \left\langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \right\rangle = \frac{1}{n^{k-r}} \sum_{j=1}^{n^{k-r}} n^{-r} \sum_{x \in B_{k,j}} \left\langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \right\rangle,
$$

and recognize that the right hand side is an average of n^{k-r} i.i.d. bounded random variables. Hence, Kolmogorov's strong law of large numbers yields that there is a set $\Omega_{z,\varepsilon} \in \mathcal{F}$ with $\mathbb{P}(\Omega_{z,\varepsilon}) = 1$ and such that for all $\omega \in \Omega_{z,\varepsilon}$,

$$
\lim_{k \to \infty} |B_k|^{-1} \sum_{x \in B_k} \left\langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \right\rangle = n^{-r} \sum_{x \in \widetilde{B}} \mathbb{E}\left\langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \right\rangle, \tag{3.9}
$$

where $\widetilde{B}$ is some fixed ball of radius r. The bounds (3.5) and (3.7) yield

$$
\left| \left\langle \delta_x | (H_r^\omega - z)^{-1} \delta_x \right\rangle - \left\langle \delta_x | (H_\omega - z)^{-1} \delta_x \right\rangle \right| < \varepsilon/2 \,,
$$

which combined with (3.9) yields

$$
\limsup_{k \to \infty} |II_{k,\omega}| < \varepsilon/2 \,.
$$

Hence for $\omega \in \Omega_{z,\varepsilon}$, $\limsup_{k \to \infty} |D_{k,\omega}| < \varepsilon$, and the statement follows after taking $\Omega_z = \bigcap_{m=1}^{\infty} \Omega_{z,1/m}$. $\qquad\square$

Theorem 3.3 is a consequence of Proposition 3.4 and a density argument. Let G be a countable dense set in $\mathbb{C}\backslash\mathbb{R}$. Since any function $f \in C_0(\mathbb{R})$ can be uniformly approximated by finite linear combinations of the functions $t \to (t - z)^{-1}$, with z ranging through G, Theorem 3.3 follows after taking $\widetilde{\Omega} = \bigcap_{z \in G} \Omega_z$.

Remarks on Theorem 3.3. There is no restriction on the spectral dimension d. Also, the theorem and the above proof remain valid without the assumption that the random variables $\omega(x)$ have a density γ.

3.3. Fine eigenvalue statistics

For our study of fine eigenvalue statistics, we need the following two well-known general estimates for random discrete Schrödinger operators. For both estimates, the density γ plays a fundamental role.

Lemma 3.5 (Wegner estimate [31]). *Let M_0 be any self-adjoint operator on $l^2(\mathbb{X})$ and let*

$$M_\omega = M_0 + V_\omega \,.$$

Then for every bounded Borel measurable function $h : \mathbb{R} \to [0, \infty)$ and $x \in \mathbb{X}$,

$$\mathbb{E}\langle \delta_x | h(M_\omega)\delta_x \rangle \leq \|\gamma\|_\infty \int h(t)dt \,. \tag{3.10}$$

Hence, if ν_ω is the spectral measure for δ_x and M_ω and $\nu^{av} = \mathbb{E}\nu_\omega$ is the corresponding averaged measure, then ν^{av} is absolutely continuous with respect to Lebesgue measure,

$$d\nu^{av}(t) = \upsilon(t)dt \,,$$

and

$$\|\upsilon\|_\infty \leq \|\gamma\|_\infty \,.$$

Lemma 3.6 (Minami's estimate [4, 12, 25]). *Let M_0 be any self-adjoint operator on $l^2(\mathbb{X})$ and let*

$$M_\omega = M_0 + V_\omega \,.$$

Then for every $x, y \in \mathbb{X}$ and $\mathrm{Im}z > 0$

$$\mathbb{E} \det \begin{pmatrix} \langle \delta_x | \mathrm{Im}(M_\omega - z)^{-1}\delta_x \rangle & \langle \delta_x | \mathrm{Im}(M_\omega - z)^{-1}\delta_y \rangle \\ \langle \delta_y | \mathrm{Im}(M_\omega - z)^{-1}\delta_x \rangle & \langle \delta_y | \mathrm{Im}(M_\omega - z)^{-1}\delta_y \rangle \end{pmatrix} \leq \pi^2 \|\gamma\|_\infty^2 \,. \tag{3.11}$$

Wegner estimate yields that μ^{av} is absolutely continuous with respect to Lebesgue measure,

$$d\mu^{av}(t) = \eta(t)dt \,,$$

and

$$\|\eta\|_\infty \leq \|\gamma\|_\infty \,.$$

If $e \in \sum$ and $\varepsilon > 0$ are given, then in view of Theorem 3.3 we expect the number of eigenvalues of $H_k^\omega \upharpoonright l^2(B_k)$ in the interval $(e - \varepsilon, e + \varepsilon)$,

$$\# \left\{ i : e_i^{\omega,k} \in (e - \varepsilon, e + \varepsilon) \right\} \,,$$

to have typical size of order $|B_k| \, \mu^{av}(e - \varepsilon, e + \varepsilon)$ for large k. The precise statistical behavior of the eigenvalues $e_j^{\omega,k}$ near e is captured by the rescaled measure $\xi_k^{\omega,e}$ given by (1.2). We make the following regularity assumption on e: for $\mathrm{Im}z > 0$,

$$\lim_{\varepsilon \downarrow 0} \int \mathrm{Im}(t - e - \varepsilon z)^{-1}\eta(t)dt = \pi\eta(e) \,. \tag{3.12}$$

For example, if η is continuous at e, then (3.12) holds. However, it is in general a difficult problem to establish the continuity of η for random discrete Schrödinger operators. In the case of the Cauchy random potential (3.2), η is known to be analytic [24]. If the Fourier transform of $\gamma(t)$ decays exponentially, then it is possible [7] to prove analyticity of η after increasing the disorder, i.e. replacing V_ω with σV_ω for a sufficiently large σ. When continuity of η is not available, one appeals to a classical theorem in harmonic analysis (see for example [20]), due to Fatou, guaranteeing that (3.12) holds for Lebesgue almost all $e \in \mathbb{R}$. We now state our main result.

Theorem 3.7. *Assume that the hierarchical model has a spectral dimension* $\mathrm{d} < 1$. *Assume that* $\eta(e) > 0$ *and that* e *verifies the regularity Condition* (3.12). *Then* $\xi_k^{\omega,e}$ *converges to a Poisson point process on* $\mathbb{R}$ *with intensity* $\eta(e)\mathcal{L}$.

Remarks on Theorem 3.7. Our theorem is the analogue of Minami's result for the localized Anderson model on $\mathbb{Z}^{\mathrm{d}}$ (see the appendix). The proof of Poisson statistics for the hierarchical model is technically simpler than the corresponding proofs for the Anderson model on $\mathbb{Z}^{\mathrm{d}}$, because of the low spectral dimension assumption and because of the high degree of self-similarity of the hierarchical model.

The rest of the section is devoted to the proof of Theorem 3.7. The main idea is to approximate H_k^ω with H_r^ω for $r < k$, as in the proof of Theorem 3.3. This time we choose r to depend on k, $r = r_k$, such that

$$\lim_{k\to\infty} \frac{r_k}{k} = c \,, \tag{3.13}$$

where

$$\mathrm{d} < c < 1 \,. \tag{3.14}$$

Let

$$\widetilde{e}_1^{\,\omega,k} \leq \widetilde{e}_2^{\,\omega,k} \leq \cdots \leq \widetilde{e}_{|B_k|}^{\,\omega,k} \,,$$

denote the eigenvalues of $H_{r_k}^\omega \restriction l^2(B_k)$ and let

$$\widetilde{\xi}_k^{\,\omega,e} = \sum_{i=1}^{|B_k|} \delta\big(\, |B_k|\,(\widetilde{e}_i^{\,\omega,k} - e)\big) \,,$$

be the corresponding rescaled measure near e. Since B_k is a disjoint union of n^{k-r_k} balls of radius r_k,

$$B_k = \bigcup_{j=1}^{n^{k-r_k}} B_{k,j}$$

we have the corresponding direct sum decomposition

$$H_{r_k}^\omega \restriction l^2(B_k) = \bigoplus_{j=1}^{n^{k-r_k}} H_{r_k}^\omega \restriction l^2(B_{k,j}) \,.$$

Therefore the point process $\widetilde{\xi}_k^{\omega,e}$ is the sum of n^{k-r_k} independent point processes,

$$\widetilde{\xi}_k^{\omega,e} = \sum_{j=1}^{n^{k-r_k}} \widetilde{\xi}_{k,j}^{\omega,e},$$

where

$$\widetilde{\xi}_{k,j}^{\omega,e} = \sum_{l=1}^{n^{r_k}} \delta\big(|B_k|\,(\widetilde{e}_l^{\,\omega,k,j} - e)\big),$$

and $\widetilde{e}_l^{\,\omega,k,j}$, $l = 1, \ldots, n^{r_k}$ are the eigenvalues of $H_{r_k}^\omega \restriction l^2(B_{k,j})$.

The proof of Theorem 3.7 is organized as follows. We first establish that the point processes $\xi_k^{\omega,e}$ and $\widetilde{\xi}_k^{\omega,e}$ are asymptotically close in the following sense:

Proposition 3.8. *For every $f \in L_1(\mathbb{R}, dt)$,*

$$\lim_{k\to\infty} \mathbb{E}\left|\int f d\widetilde{\xi}_k^{\omega,e} - \int f d\xi_k^{\omega,e}\right| = 0. \tag{3.15}$$

Corollary 3.9. *Let $A_1, A_2, \ldots, A_m$ be given disjoint bounded Borel sets in $\mathbb{R}$. Let X_k^ω and $\widetilde{X}_k^\omega$ be the random vectors*

$$X_k^\omega = \left[\xi_k^{\omega,e}(A_1), \xi_k^{\omega,e}(A_2), \ldots, \xi_k^{\omega,e}(A_m)\right],$$
$$\widetilde{X}_k^\omega = \left[\widetilde{\xi}_k^{\omega,e}(A_1), \widetilde{\xi}_k^{\omega,e}(A_2), \ldots, \widetilde{\xi}_k^{\omega,e}(A_m)\right].$$

and let $\phi_k, \widetilde{\phi}_k : \mathbb{R}^m \to \mathbb{C}$ be the corresponding characteristic functions

$$\phi_k(t) = \mathbb{E}e^{itX_k^\omega}, \quad \widetilde{\phi}_k(t) = \mathbb{E}e^{it\widetilde{X}_k^\omega}, \quad t \in \mathbb{R}^m.$$

Then for all $t \in \mathbb{R}^m$,

$$\lim_{k\to\infty} \left|\phi_k(t) - \widetilde{\phi}_k(t)\right| = 0.$$

Then we establish

Proposition 3.10. *The point process $\widetilde{\xi}_k^{\omega,e}$ converges to a Poisson point process on $\mathbb{R}$ with intensity $\eta(e)\mathcal{L}$.*

Proposition 3.10 and Corollary 3.9 together imply Theorem 3.7. The Wegner estimate plays a crucial role in the proof of Propositions 3.8 and 3.10. For every Borel set $A \subset \mathbb{R}$, we have $\xi_k^{\omega,e}(A) = \sum_{x \in B_k}\langle\delta_x|f(H_k^\omega)\delta_x\rangle$, where $f(t) = 1_A(|B_k|\,(t-e))$. Wegner estimate (3.10) yields that for all $x \in B_k$,

$$\mathbb{E}\langle\delta_x|f(H_k^\omega)\delta_x\rangle \leq \|\gamma\|_\infty \int f(t)dt = \|\gamma\|_\infty |B_k|^{-1}\mathcal{L}(A). \tag{3.16}$$

Summing (3.16) over all $x \in B_k$ yields

$$\mathbb{E}\xi_k^{\omega,e}(A) \leq \|\gamma\|_\infty \mathcal{L}(A). \tag{3.17}$$

Similarly

$$\mathbb{E}\widetilde{\xi}_k^{\omega,e}(A) \leq \|\gamma\|_\infty \mathcal{L}(A). \tag{3.18}$$

Proof of Proposition 3.8. Step 1: We first prove (3.15) for the family of functions

$$g_z(t) = \text{Im}(t-z)^{-1}, \quad \text{Im}z > 0.$$

Setting

$$z_k = e + |B_k|^{-1} z, \tag{3.19}$$

we have

$$\int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e} = |B_k|^{-1} \text{Im} \sum_{x \in B_k} \left\langle \delta_x \big| \big((H_{r_k}^\omega - z_k)^{-1} - (H_k^\omega - z_k)^{-1}\big)\delta_x \right\rangle.$$

Hence

$$\left| \int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e} \right| \leq |\text{Im}z_k|^{-2} \sum_{s=r_k+1}^{\infty} p_s = const\,|\text{Im}z|^{-2} \left(\frac{n^{2k}}{\rho^{r_k}} \right)$$

The formulas (3.13) and (3.14) imply that for large enough k, $\rho^{r_k} \geq \rho^{c'k}$ where $d < c' < c < 1$. Therefore

$$\frac{n^{2k}}{\rho^{r_k}} \leq \left(\frac{n^2}{\rho^{c'}} \right)^k,$$

and $\frac{n^2}{\rho^{c'}} < 1$ because of the formula (3.1). This proves (3.15).

Step 2: To prove (3.15) for general $f \in L_1(\mathbb{R}, dt)$, note that $span\,\{g_z, \text{Im}z > 0\}$ is dense in $L^1(\mathbb{R}, dt)$. Hence given $\varepsilon > 0$, there is a finite linear combination

$$g(t) = \sum_{j=1}^{p} a_j \text{Im}(t-z^{(j)})^{-1}, \quad \text{Im}z^{(j)} > 0,$$

with

$$\int_{\mathbb{R}} |f(t) - g(t)|\, dt \leq \varepsilon.$$

The triangle inequality

$$\mathbb{E} \left| \int f d\xi_k^{\omega,e} - \int f d\widetilde{\xi}_k^{\omega,e}(t) \right| \leq \mathbb{E} \int |f - g|\, d\xi_k^{\omega,e}$$
$$+ \mathbb{E} \left| \int g d\xi_k - \int g d\widetilde{\xi}_k^{\omega,e} \right| + \mathbb{E} \int |g - f|\, d\widetilde{\xi}_k^{\omega,e},$$

together with Step 1 and the bounds (3.17) and (3.18) imply

$$\limsup_{k \to \infty} \mathbb{E} \left| \int f d\xi_k^{\omega,e} - \int f d\widetilde{\xi}_k^{\omega,e}(t) \right| \leq 2\,\|\gamma\|_\infty \,\varepsilon,$$

and (3.15) follows after letting $\varepsilon \downarrow 0$. $\qquad\qquad\square$

Proof of Proposition 3.10. I suffices to show that $\widetilde{\xi}_k^{\omega,e}$ and the $\widetilde{\xi}_{k,j}^{\omega,e}$ verify the four hypotheses of Theorem 2.5.

(H0): holds because of the bound (3.18).

(H1): we need to to establish that for every bounded Borel set $A \subset \mathbb{R}$,

$$\lim_{k \to \infty} \max_{1 \leq j \leq n^{k-r_k}} \mathbb{P}\big(\widetilde{\xi}_{k,j}^{\omega,e}(A) \geq 1\big) = 0. \tag{3.20}$$

Proof. Chebyshev's inequality and the bound (3.16) yield

$$\mathbb{P}\big(\widetilde{\xi}_{k,j}^{\omega,e}(A) \geq 1\big) \leq \mathbb{E}\widetilde{\xi}_{k,j}^{\omega,e}(A)$$
$$\leq \frac{|B_{k,j}|}{|B_k|} \, \|\gamma\|_\infty \, \mathcal{L}(A)$$
$$= n^{r_k-k} \, \|\gamma\|_\infty \, \mathcal{L}(A) \,,$$

and (3.20) follows. $\square$

(H2): We need to establish that for all $\text{Im} z > 0$,

$$\lim_{k \to \infty} \mathbb{E} \int \text{Im}(t-z)^{-1} d\widetilde{\xi}_k^{\omega,e}(t) = \pi \eta(e) \,.$$

Proof. We have

$$\mathbb{E} \int \text{Im}(t-z)^{-1} d\widetilde{\xi}_k^{\omega,e}(t) = |B_k|^{-1} \mathbb{E} \text{Im} \sum_{x \in B_k} \big\langle \delta_x | (H_{r_k}^\omega - z_k)^{-1} \delta_x \big\rangle$$
$$= |B_k|^{-1} \mathbb{E} \text{Im} \sum_{x \in B_k} \big\langle \delta_x \big| \big((H_{r_k}^\omega - z_k)^{-1} - (H^\omega - z_k)^{-1}\big) \delta_x \big\rangle$$
$$+ \mathbb{E} \text{Im} \big\langle \delta_{x_0} | (H^\omega - z_k)^{-1} \delta_{x_0} \big\rangle$$
$$= I_{k,\omega} + II_{k,\omega} \,.$$

Now $II_{k,\omega} \to \pi \eta(e)$ by 3.12 and $I_{k,\omega} \to 0$, as in the proof of Proposition 3.8. $\square$

(H3): We need to establish that for every function $g_z(t) = \text{Im}(t-z)^{-1}, \text{Im} z > 0$,

$$\lim_{k \to \infty} \sum_{j=1}^{n^{k-r_k}} \mathbb{E}\mathcal{I}(\widetilde{\xi}_{k,j}^{\omega,e}, g_z) = 0 \,. \tag{3.21}$$

Proof. We have,

$$|B_k|^2 \, I(\widetilde{\xi}_{k,j}^{\omega,e}, g_z) = \left(\sum_{x \in B_{k,j}} \big\langle \delta_x | \text{Im}(H_{r_k}^\omega - z_k)^{-1} \delta_x \big\rangle \right)^2$$
$$- \sum_{x \in B_{k,j}} \big\langle \delta_x \big| \big(\text{Im}(H_{r_k}^\omega - z_k)^{-1}\big)^2 \delta_x \big\rangle$$
$$= \sum_{x,y \in B_{k,j}} \det \begin{pmatrix} \big\langle \delta_x | \text{Im}(H_{r_k}^\omega - z_k)^{-1} \delta_x \big\rangle & \big\langle \delta_x | \text{Im}(H_{r_k}^\omega - z_k)^{-1} \delta_y \big\rangle \\ \big\langle \delta_y | \text{Im}(H_{r_k}^\omega - z_k)^{-1} \delta_x \big\rangle & \big\langle \delta_y | \text{Im}(H_{r_k}^\omega - z_k)^{-1} \delta_y \big\rangle \end{pmatrix} \,.$$

Using Minami's estimate (3.11) we get the bounds

$$|B_k|^2 \, \mathbb{E} I(\widetilde{\xi}_{k,j}^{\omega,e}, g_z) \leq \pi^2 \, \|\gamma\|_\infty^2 \, |B_{k,j}|^2 \ ,$$

and hence

$$\sum_{j=1}^{n^{k-r_k}} \mathbb{E} I(\widetilde{\xi}_{k,j}^{\omega,e}, g_z) \leq \pi^2 \, \|\gamma\|_\infty^2 \, n^{-r_k} \ ,$$

which yields (3.21). $\qquad\qquad\square$

$\square$

Appendix A. Minami's proof of Poisson statistics for the localized Anderson model on $\mathbb{Z}^{\mathrm{d}}$

For a rectangle $B \subset \mathbb{Z}^{\mathrm{d}}$, we denote by H_B^ω the restriction of H_ω to $l^2(B)$ with Dirichlet boundary conditions: i.e. $\langle \delta_x | H_B^\omega \delta_y \rangle = \langle \delta_x | H_\omega \delta_y \rangle$ if both $x, y \in B$, and $\langle \delta_x | H_B^\omega \delta_y \rangle = 0$ otherwise. For $k \geq 1$, let B_k be the rectangle $\{ x \in \mathbb{Z}^{\mathrm{d}} : \max_{i=1,\ldots,\mathrm{d}} |x_i| \leq k \}$, and let $H_k^\omega = H_{B_k}^\omega$. As before, $e_1^{\omega,k} \leq e_2^{\omega,k} \leq \cdots \leq e_{|B_k|}^{\omega,k}$, are the eigenvalues of $H_k^\omega \upharpoonright l^2(B_k)$, μ_k^ω is the corresponding normalized counting measure given by (1.1) and $\xi_k^{\omega,e}$ is the rescaled measure near e given by (1.2). We refer the reader to [18] for a discussion of the regime where both space and energy are rescaled. The averaged spectral measure for H_ω is given by (3.3) and the Wegner estimate yields that μ^{av} has a bounded density $\eta(t)$ with respect to $\mathcal{L}$. A basic result for the Anderson model is that for $\mathbb{P}$-a.e. $\omega \in \Omega$, the spectrum of H_ω is equal to $[-2\mathrm{d}, 2\mathrm{d}] + \mathrm{supp}(\gamma) = \mathrm{supp}(\mu^{av})$ and μ_k^ω converges to μ^{av} in the weak-* topology as $k \to \infty$ [6, 8, 29].

Theorem A.1 (Minami, 1996). *Assume that there are constants $0 < C < \infty, 0 < D < \infty$ and $0 < s < 1$ such that*

$$\mathbb{E} \left| \langle \delta_x | (H_B^\omega - z)^{-1} \delta_y \rangle \right|^s \leq C e^{-D|x-y|} \,, \quad x, y, \in \mathbb{Z}^{\mathrm{d}} \,, \tag{A.1}$$

for all z with $e_1 < \mathrm{Re} z < e_2, \mathrm{Im} z \neq 0$ and for all rectangles $B \subset \mathbb{Z}^{\mathrm{d}}$. Assume that $e \in (e_1, e_2)$ verifies the regularity Condition (3.12) and that $\eta(e) > 0$. Then $\xi_k^{\omega,e}$ converges to a Poisson point process on $\mathbb{R}$ with intensity $\eta(e)\mathcal{L}$.

We refer the reader to [14] for a discussion of the set of e for which $\eta(e) > 0$. Condition (A.1) is called fractional-moments localization. It implies that within (e_1, e_2), for $\mathbb{P}$-a.e. $\omega \in \Omega$ the spectrum of H_ω, if any, is pure-point with exponentially decaying eigenfunctions [1, 2]. For $\mathrm{d} = 1$, Condition (A.1) holds for all energy intervals (e_1, e_2) [25]. In dimensions $\mathrm{d} \geq 2$, condition (A.1) is obtained by either moving the energy interval (e_1, e_2) to $\pm\infty$ or by increasing the disorder. The two main techniques for proving that are the multiscale analysis [10, 11] and the Aizenman–Molchanov theory [1].

Proof of Theorem A.1. We fix $\alpha \in (0,1)$ and for each k, we make a partition

$$B_k = \bigcup_{j=1}^{n_k} B_{k,j} \,,$$

where $B_{k,j}$ are disjoint rectangles with side $\sim (2k)^\alpha$. Hence $n_k \sim k^{d(1-\alpha)}$. Let $\widetilde{e}_l^{\,\omega,k,j}, l = 1, \ldots, |B_{k,j}|$ denote the eigenvalues of $H_{B_{k,j}}^\omega \restriction l^2(B_{k,j})$ and let

$$\widetilde{\xi}_{k,j}^{\omega,e} = \sum_{l=1}^{|B_{k,j}|} \delta\big(|B_k| \, (\widetilde{e}_l^{\,\omega,k,j} - e) \big) \,,$$

$$\widetilde{\xi}_k^{\omega,e} = \sum_{j=1}^{n_k} \widetilde{\xi}_{k,j}^{\omega,e} \,.$$

Hence the point process $\widetilde{\xi}_k^{\omega,e}$ is the sum of n_k independent point processes $\widetilde{\xi}_{k,j}^{\omega,e}$. As in Section 3, Theorem A.1 follows from the following two propositions.

Proposition A.2. *For every* $f \in L_1(\mathbb{R}, dt)$,

$$\lim_{k\to\infty} \mathbb{E} \left| \int f d\widetilde{\xi}_k^{\omega,e} - \int f d\xi_k^{\omega,e} \right| = 0 \,. \tag{A.2}$$

Proposition A.3. *The point process* $\widetilde{\xi}_k^{\omega,e}$ *converges to a Poisson point process on* $\mathbb{R}$ *with intensity* $\eta(e)\mathcal{L}$.

Proof of Theorem A.2. As in the proof of Proposition 3.8, it is enough to prove (A.2) for the family of functions

$$g_z(t) = \mathrm{Im}(t - z)^{-1}, \quad \mathrm{Im}\, z > 0 \,.$$

We set

$$z_k = e + |B_k|^{-1} z \,. \tag{A.3}$$

Then

$$\int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e}$$

$$= |B_k|^{-1} \mathrm{Im} \sum_{j=1}^{n_k} \sum_{x \in B_{k,j}} \left\langle \delta_x \left| \left((H_{B_{k,j}}^\omega - z_k)^{-1} - (H_k^\omega - z_k)^{-1} \right) \delta_x \right. \right\rangle \,.$$

Let $v_k = \beta \ln k$, where $\beta > 0$ is a fixed big enough constant to be specified later. We set

$$\mathrm{int}(B_{k,j}) = \big\{ x \in B_{k,j} : dist(x, \partial B_{k,j}) \geq v_k \big\} \,,$$

and

$$\mathrm{wall}(B_{k,j}) = \big\{ x \in B_{k,j} : dist(x, \partial B_{k,j}) < v_k \big\} \,.$$

Then

$$\mathbb{E} \left| \int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\xi_k^{\omega,e} \right| \leq \mathbb{E} \left| I_{k,\omega} \right| + \mathbb{E} \left| II_{k,\omega} \right| \,,$$

where

$$I_{k,\omega} = |B_k|^{-1} \operatorname{Im} \sum_{j=1}^{n_k} \sum_{x \in \mathrm{wall}(B_{k,j})} \left\langle \delta_x \middle| \left((H_{B_{k,j}}^\omega - z_k)^{-1} - (H_k^\omega - z_k)^{-1} \right) \delta_x \right\rangle,$$

$$II_{k,\omega} = |B_k|^{-1} \operatorname{Im} \sum_{j=1}^{n_k} \sum_{x \in \mathrm{int}(B_{k,j})} \left\langle \delta_x \middle| \left((H_{B_{k,j}}^\omega - z_k)^{-1} - (H_k^\omega - z_k)^{-1} \right) \delta_x \right\rangle.$$

The Wegner estimate (3.18) yields that

$$\mathbb{E} |I_{k,\omega}| \le 2\pi \|\gamma\|_\infty |B_k|^{-1} \sum_{j=1}^{n_k} |\mathrm{wall}(B_{k,j})|,$$

and the right hand side converges to zero as $k \to \infty$.

To estimate $\mathbb{E} |II_{k,\omega}|$, we use the resolvent identity

$$\left\langle \delta_x \middle| \left((H_{B_{k,j}}^\omega - z_k)^{-1} - (H_k^\omega - z_k)^{-1} \right) \delta_x \right\rangle$$
$$= \sum_{(y,y')} \langle \delta_x | (H_{B_{k,j}}^\omega - z_k)^{-1} \delta_y \rangle \langle \delta_{y'} | (H_k^\omega - z_k)^{-1} \delta_x \rangle,$$

where the sum is over all pairs (y, y'), with $y \in \partial B_{k,j}, y' \notin B_{k,j}$ and $|y - y'| = 1$. Hence,

$$\mathbb{E} |II_{k,\omega}|$$
$$\le |B_k|^{-1} \sum_{j=1}^{n_k} \sum_{x \in \mathrm{int}(B_{k,j})} \sum_{(y,y')} \mathbb{E} \left| \langle \delta_x | (H_{B_{k,j}}^\omega - z_k)^{-1} \delta_y \rangle \langle \delta_{y'} | (H_k^\omega - z_k)^{-1} \delta_x \rangle \right|.$$
(A.4)

For k large enough so that $e_1 < \operatorname{Re} z_k < e_2$, we use the main assumption (A.1) together with the bound

$$\left| \langle \delta_x | (H_{B_{k,j}}^\omega - z_k)^{-1} \delta_y \rangle \langle \delta_{y'} | (H_k^\omega - z_k)^{-1} \delta_x \rangle \right| \le (\operatorname{Im} z_k)^{-2} = (|B_k| / \operatorname{Im} z)^2,$$

to obtain

$$\mathbb{E} \left| \langle \delta_x | (H_{B_{k,j}}^\omega - z_k)^{-1} \delta_y \rangle \langle \delta_{y'} | (H_k^\omega - z_k)^{-1} \delta_x \rangle \right|$$
$$\le (|B_k| / \operatorname{Im} z)^{2(1-s/2)} \mathbb{E} \left| \langle \delta_x | (H_{B_{k,j}}^\omega - z_k)^{-1} \delta_y \rangle \langle \delta_{y'} | (H_k^\omega - z_k)^{-1} \delta_x \rangle \right|^{s/2}$$
$$\le (|B_k| / \operatorname{Im} z)^{2(1-s/2)} \left(\mathbb{E} \left| \langle \delta_x | (H_{B_{k,j}}^\omega - z_k)^{-1} \delta_y \rangle \right|^s \right)^{1/2}$$
$$\left(\mathbb{E} \left| \langle \delta_{y'} | (H_k^\omega - z_k)^{-1} \delta_x \rangle \right|^s \right)^{1/2}$$
$$\le (|B_k| / \operatorname{Im} z)^{2(1-s/2)} C e^{-D v_k}.$$
(A.5)

Since, in (A.4), there are $O(k^{\alpha(d-1)})$ pairs (y, y') for each $B_{k,j}$, the bounds (A.4) and (A.5) yield

$$\mathbb{E}\, |II_{k,\omega}| \le O(k^{\alpha(d-1)}\, |B_k|^{2(1-s/2)}\, e^{-Dv_k})$$
$$= O(k^{\alpha(d-1)+2d(1-s/2)}e^{-D\beta \ln k})$$

Hence, if we choose $\beta > D^{-1}\left(\alpha(d-1) + 2d(1-s/2)\right)$, then $\mathbb{E}\, |II_{k,\omega}| \to 0$ as $k \to \infty$. $\qquad\square$

Proof of Proposition A.3. As in the proof of Propositon 3.10, it suffices to show that $\widetilde{\xi}_k^{\omega,e}$ and the $\widetilde{\xi}_{k,j}^{\omega,e}$ verify the four hypotheses of Theorem 2.5. The proof of (H0), (H1) and (H3) is the same as in Propositon 3.10. It remains to show that (H2) holds, i.e. for $\mathrm{Im}\, z > 0$,

$$\lim_{k\to\infty} \mathbb{E} \int g_z d\widetilde{\xi}_k^{\omega,e} = \pi\eta(e)\,. \qquad (A.6)$$

The argument of the proof of Proposition A.2, with H_k^ω replaced by H_ω, yields that

$$\lim_{k\to\infty} \mathbb{E} \left(\int g_z d\widetilde{\xi}_k^{\omega,e} - \int g_z d\mu^{av} \right) = 0\,, \qquad (A.7)$$

and then (A.6) follows from (A.7) and (3.12). $\qquad\square$

$\hfill\square$

Acknowledgements

References

[1] M. Aizenman and S. Molchanov, *Localization at large disorder and at extreme energies: An elementary derivation*, Comm. Math. Phys. **157** (1993), 245–278.

[2] M. Aizenman, J. Schenker, R. Friedrich, and D. Hundertmark, *Finite-volume fractional moment criteria for Anderson localization*, Comm. Math. Phys. **224** (2001), 219–253.

[3] M. Aizenman and S. Warzel, *The canopy graph and level statistics for random operators on trees*, Mathematical Physics, Analysis and Geometry **9**, no. 4, (2006).

[4] J. V. Bellissard, P. D. Hislop, and G. Stolz, *Correlations estimates in the lattice Anderson model*, to appear in J. Statist. Phys.

[5] A. Bovier, *The density of states in the Anderson model at weak disorder: A renormalization group analysis of the hierarchical model*, J. Statist. Phys. **59** (1990), no. 3–4, 745–779.

[6] R. Carmona and J. Lacroix, *Spectral Theory of Random Schrödinger Operators*, Birkhäuser Boston, Inc., Boston, MA, 1990.

[7] F. Constantinescu, J. Fröhlich and T. Spencer, *Analyticity of the density of states and replica method for random Schrödinger operators on a lattice*, J. Statist. Phys. **34** (1984), no. 3–4, 571–596.

[8] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Texts and Monographs in Physics. Springer Study Edition. Springer-Verlag, Berlin, 1987.

[9] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*, Springer Series in Statistics. Springer-Verlag, New York, 1988.

[10] H. von Dreifus and A. Klein, *A new proof of localization in the Anderson tight binding model*, Comm. Math. Phys. **124** (1989), 285–299.

[11] J. Fröhlich and T. Spencer, *Absence of diffusion in the Anderson tight binding model for large disorder or low energy*, Comm. Math. Phys. **88** (1983), 151–184.

[12] G. M. Graf and A. Vaghi, *A Remark on the estimate of a determinant by Minami*, Lett. Math. Phys. **79** (2007), no. 1, 17–22.

[13] B. Grigelionis, *On the convergence of sums of random step processes to a Poisson process*, Theory of Probability and its Applications **8** (1963), 177–182.

[14] P. D. Hislop and P. Müller, *A lower bound for the density of states of the lattice Anderson model*, http://front.math.ucdavis.edu/0705.1707.

[15] V. Jaksic, *Topics in spectral theory. Open Quantum Systems I. The Hamiltonian Approach*, Lecture Notes in Mathematics 1880 (2006), 235–312, Springer.

[16] O. Kallenberg, *Random Measures*, Academic Press. 4th. ed. 1986.

[17] J. F. C. Kingman, *Poisson Processes*, Oxford Studies in Probability, 3. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[18] R. Killip and F. Nagano, *Eigenfunction statistics in the localized Anderson model*, Ann. Henri Poincaré **8** (2007), 27–36.

[19] R. Killip and M. Stoiciu, *Eigenvalue statistics for CMV matrices: From Poisson to clock via circular beta ensembles*, arXiv preprint - math-ph/0608002.

[20] P. Koosis, *Introduction to H_p Spaces*, Second edition. Cambridge Tracts in Mathematics, 115. Cambridge University Press, Cambridge, 1998.

[21] E. Kritchevski, *Spectral localization in the hierarchical Anderson model*, Proc. Amer. Math. Soc. **135** (2007), 1431–1440.

[22] E. Kritchevski, *Hierarchical Anderson model*, CRM Proceedings and Lecture Notes **42** (2007), 309–322.

[23] E. Kritchevski, PhD Thesis, McGill University. In preparation.

[24] P. Lloyd, *Exactly solvable model of electronic states in a three-dimensional disordered Hamiltonian: Non-existence of localized states*, J. Phys. C: Solid State Phys. **2** no. 10 (Oct. 1969), 1717–1725.

[25] N. Minami, *Local fluctuation of the spectrum of a multidimensional Anderson tight-binding model*, Comm. Math. Phys. **177** (1996), 709–725.

[26] S. Molchanov, *The local structure of the spectrum of the one-dimensional Schrödinger operator*, Comm. Math. Phys. **78** (1981), 429–446.

[27] S. Molchanov, Lectures on random media. Lectures on probability theory (Saint-Flour, 1992), 242–411, Lecture Notes in Math., 1581, Springer, Berlin, 1994.

[28] S. Molchanov, *Hierarchical random matrices and operators. Application to Anderson model. Multidimensional statistical analysis and theory of random matrices*, (Bowling Green, OH, 1996), 179–194, VSP, Utrecht, 1996.

[29] L. Pastur and A. Figotin, *Spectra of random and almost-periodic operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 297. Springer-Verlag, Berlin, 1992.

[30] M. Stoiciu, *Poisson statistics for eigenvalues: From random Schrödinger operators to random CMV matrices*, CRM Proceedings and Lecture Notes **42** (2007), 465–475.

[31] F. Wegner, *Bounds on the density of states in disordered systems*, Z. Phys. B. **44** (1981), 9–15.

Evgenij Kritchevski
Department of Mathematics and Statistics
McGill University
805 Sherbrooke Street West Montreal
QC, H3A 2K6
Canada
e-mail: `ekritc@math.mcgill.ca`