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Spectral Analysis of Magnetic Laplacians on Conformally Cusp Manifolds

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Abstract. We consider an open manifold which is the interior of a compact manifold with boundary. Assuming gauge invariance, we classify magnetic fields with compact support into being trapping or non-trapping. We study spectral properties of the associated magnetic Laplacian for a class of Riemannian metrics which includes complete hyperbolic metrics of finite volume. When B is non-trapping, the magnetic Laplacian has nonempty essential spectrum. Using Mourre theory, we show the absence of singular continuous spectrum and the local finiteness of the point spectrum. When B is trapping, the spectrum is discrete and obeys the Weyl law. The existence of trapping magnetic fields with compact support depends on cohomological conditions, indicating a new and very strong long-range effect.

In the non-gauge invariant case, we exhibit a strong Aharonov–Bohm effect. On hyperbolic surfaces with at least two cusps, we show that the magnetic Laplacian associated to every magnetic field with compact support has purely discrete spectrum for some choices of the vector potential, while other choices lead to a situation of limiting absorption principle.

We also study perturbations of the metric. We show that in the Mourre theory it is not necessary to require a decay of the derivatives of the perturbation. This very singular perturbation is then brought closer to the perturbation of a potential.

1. Introduction

Let X be a smooth manifold of dimension n, diffeomorphic outside a compact set to a cylinder $(1,\infty)\times M$, where M is a possibly disconnected closed manifold. On X we consider asymptotically conformally cylindrical metrics, i.e., perturbations of the metric given near the border $\{\infty\}\times M$ by:

$$g_p = y^{-2p}(dy^2+h)\,, \quad y\to\infty \tag{1.1}$$

where h is a metric on M and $p > 0$. If $p = 1$ and h is flat, the ends are *cusps*, i.e., complete hyperbolic of finite volume. For $p > 1$ one gets the (incomplete) metric horns.

The refined properties of the essential spectrum of the Laplace–Beltrami operator $\Delta_p := d^*d$ have been studied by Froese and Hislop [10] in the complete case. For the unperturbed metric (1.1), they get

$$\sigma_{\mathrm{ess}}(\Delta_p) = \big[\kappa(p)\infty\big)\,, \quad \text{where} \quad \begin{cases} \kappa(p) = 0\,, & \text{for} \quad p < 1 \\ \kappa(1) = \left(\frac{n-1}{2}\right)^2 . \end{cases} \tag{1.2}$$

The singular continuous part of the spectrum is empty and the eigenvalues distinct from $\kappa(p)$ are of finite multiplicity and may accumulate only at $\kappa(p)$. Froese and Hislop actually show a limiting absorption principle, a stronger result, see also [8, 11, 12, 25] for the continuation of their ideas. Their approach relies on a positive commutator technique introduced by E. Mourre in [39], see also [2] and references therein. See for instance [20, 28] for different methods.

Consider more generally a conformal perturbation of the metric (1.1). Let $\rho \in \mathcal{C}^\infty(X, \mathbb{R})$ be such that $\inf_{y\in X}(\rho(y)) > -1$. Consider the same problem as above for the metric

$$\tilde{g}_p = (1 + \rho)g_p\,, \quad \text{for large } y\,. \tag{1.3}$$

To measure the size of the perturbation, we compare it to the lengths of geodesics. Let $L \in \mathcal{C}^\infty(X)$ be defined by

$$L \geq 1\,, \quad L(y) = \begin{cases} \frac{y^{1-p}}{1-p} & \text{for } p < 1 \\ \ln(y) & \text{for } p = 1 \end{cases}\,, \quad \text{for } y \text{ big enough}\,. \tag{1.4}$$

In [10], one essentially asks that

$$L^2\rho, L^2 d\rho \quad \text{and} \quad L^2\Delta_g\rho \quad \text{are in} \quad L^\infty(X)\,.$$

to obtain the absence of singular continuous spectrum and local finiteness of the point spectrum. On one hand, one knows from the perturbation of a Laplacian by a short-range potential V that only the speed of the decay of V is important to conserve these properties. On the other hand, in [15] and in a general setting, one shows that only the fact that ρ tends to 0 is enough to ensure the stability of the essential spectrum. Therefore, it is natural to ask whether the decay of the metric (without decay conditions on the derivatives) is enough to ensure the conservation of these properties. In this paper, we consider that $\rho = \rho_{\mathrm{sr}} + \rho_{\mathrm{lr}}$ decomposes in short-range and long-range components. We ask the long-range component to be radial. We also assume that there exists $\varepsilon > 0$ such that

$$\begin{gathered} L^{1+\varepsilon}\rho_{\mathrm{sr}} \quad \text{and} \quad d\rho_{\mathrm{sr}}, \Delta_g\rho_{\mathrm{sr}} \in L^\infty(X)\,, \\ L^\varepsilon\rho_{\mathrm{lr}},\, L^{1+\varepsilon}d\rho_{\mathrm{lr}} \quad \text{and} \quad \Delta_g\rho_{\mathrm{sr}} \in L^\infty(X)\,. \end{gathered} \tag{1.5}$$

Going from 2 to $1 + \varepsilon$ is not a significant improvement as it relies on the use of an optimal version of the Mourre theory instead of the original theory, see [2] and references therein. Nevertheless, the fact that the derivatives are asked only to be bounded and no longer to decay is a real improvement due to our method. We prove this result in Theorem 6.4. In the Mourre theory, one introduces a conjugate operator to study a given operator. The conjugate operator introduced in [10] is too rough to handle very singular perturbations. In our paper, we introduce a conjugate operator local in energy to avoid the problem. We believe that our approach could be implemented easily in the manifold settings from [5, 8, 10–12, 25, 27] to improve results on perturbations of the metric.

A well-known dynamical consequence of the absence of singular continuous spectrum and of the local finiteness of the point spectrum is that for an interval $\mathcal{J}$ that contains no eigenvalue of the Laplacian, for all $\chi \in \mathcal{C}_c^\infty(X)$ and $\phi \in L^2(X)$, the norm $\|\chi e^{it\Delta_p} E_{\mathcal{J}}(\Delta_p)\phi\|$ tends to 0 as t tends to $\pm\infty$. In other words, if you let evolve long enough a particle which is located at scattering energy, it eventually becomes located very far on the exits of the manifold. Add now a magnetic field B with compact support and look how strongly it can interact with the particle. Classically there is no interaction as B and the particle are located far from each other. One looks for a quantum effect.

The Euclidean intuition tells us that is no essential difference between the free Laplacian and the magnetic Laplacian Δ_A, where A is a magnetic potential arising from a magnetic field B with compact support. They still share the spectral properties of absence of singular continuous spectrum and local finiteness of the point spectrum, although a long-range effect does occur and destroys the asymptotic completeness of the couple (Δ, Δ_A); one needs to modify the wave operators to compare the two operators, see [31]. However, we point out in this paper that the situation is dramatically different in particular on hyperbolic manifolds of finite volume, even if the magnetic field is very small in size and with compact support. We now go into definitions and describe our results.

A magnetic field B is a smooth real exact 2-form on X. There exists a real 1-form A, called vector potential, satisfying $dA = B$. Set $d_A := d + iA\wedge : \mathcal{C}_c^\infty(X) \to \mathcal{C}_c^\infty(X, T^*X)$. The magnetic Laplacian on $\mathcal{C}_c^\infty(X)$ is given by $\Delta_A := d_A^* d_A$. When the manifold is complete, Δ_A is known to be essentially self-adjoint, see [46]. Given two vector potentials A and A' such that $A - A'$ is exact, the two magnetic Laplacians Δ_A and $\Delta_{A'}$ are unitarily equivalent, by gauge invariance. Hence when $H^1_{\mathrm{dR}}(X) = 0$, the spectral properties of the magnetic Laplacian do not depend on the choice of the vector potential, so we may write Δ_B instead of Δ_A.

The aim of this paper is the study of the spectrum of magnetic Laplacians on a manifold X with the metric (2.9), which includes the particular case (1.1). In this introduction we restrict the discussion to the complete case, i.e., $p \leq 1$. We focus first on the case of gauge invariance, i.e., $H^1_{\mathrm{dR}}(X) = 0$, and we simplify the presentation assuming that the boundary is connected. We classify magnetic fields.

Definition 1.1. Let X be the interior of a compact manifold with boundary $\overline{X}$. Suppose that $H^1_{\mathrm{dR}}(X) = 0$ and that $M = \partial\overline{X}$ is connected. Let B be a magnetic field on X which extends smoothly to a 2-form on $\overline{X}$. We say that B is *trapping* if

1) either B does not vanish identically on M, or

2) B vanishes on M but defines a non-integral cohomology class $[2\pi B]$ inside the relative cohomology group $H^2_{\mathrm{dR}}(X, M)$.

Otherwise, we say that B is *non-trapping*. This terminology is motivated by the spectral consequences a) and c) of Theorem 1.2. The definition can be generalized to the case where M is disconnected (Definition 7.1). The condition of B being trapping can be expressed in terms of any vector potential A (see Definition 3.2 and Lemma 7.2). When $H^1_{\mathrm{dR}}(X) \neq 0$, the trapping condition makes sense only for vector potentials, see Section 3.2 and Theorem 1.3.

Let us fix some notation. Given two Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, we denote by $\mathcal{B}(\mathscr{H}, \mathscr{K})$ and $\mathcal{K}(\mathscr{H}, \mathscr{K})$ the bounded and compact operators acting from $\mathscr{H}$ to $\mathscr{K}$, respectively. Given $s \geq 0$, let $\mathscr{L}_s$ be the domain of L^s equipped with the graph norm. We set $\mathscr{L}_{-s} := \mathscr{L}_s^*$ where the adjoint space is defined so that $\mathscr{L}_s \subset L^2(X, g_p) \subset \mathscr{L}_s^*$, using the Riesz lemma. Given a subset I of $\mathbb{R}$, let $I_\pm$ be the set of complex numbers $x \pm iy$, where $x \in I$ and $y > 0$. For simplicity, in this introduction we state our result only for the unperturbed metric (1.1).

Theorem 1.2. *Let $1 \geq p > 0$, g_p the metric given by* (1.1)*. Suppose that $H_1(X, \mathbb{Z}) = 0$ and that M is connected. Let B be a magnetic field which extends smoothly to $\overline{X}$. If B is trapping then:*

a) *The spectrum of Δ_B is purely discrete.*
b) *The asymptotic of its eigenvalues is given by*

$$N_{B,p}(\lambda) \approx \begin{cases} C_1 \lambda^{n/2} & \text{for } 1/n < p\,, \\ C_2 \lambda^{n/2} \log \lambda & \text{for } p = 1/n\,, \\ C_3 \lambda^{1/2p} & \text{for } 0 < p < 1/n \end{cases} \tag{1.6}$$

in the limit $\lambda \to \infty$, where C_3 is given in Theorem 4.2*, and*

$$C_1 = \frac{\mathrm{Vol}(X, g_p)\,\mathrm{Vol}(S^{n-1})}{n(2\pi)^n}\,, \quad C_2 = \frac{\mathrm{Vol}(M, h)\,\mathrm{Vol}(S^{n-1})}{2(2\pi)^n}\,. \tag{1.7}$$

If B is non-trapping with compact support in X then

c) *The essential spectrum of Δ_B is $[\kappa(p), \infty)$.*
d) *The singular continuous spectrum of Δ_B is empty.*
e) *The eigenvalues of Δ_B are of finite multiplicity and can accumulate only in $\{\kappa(p)\}$.*
f) *Let $\mathcal{J}$ a compact interval such that $\mathcal{J} \cap (\{\kappa(p)\} \cup \sigma_{\mathrm{pp}}(H)) = \emptyset$. Then, for all $s \in]1/2, 3/2[$ and all A such that $dA = B$, there is c such that*

$$\|(\Delta_A - z_1)^{-1} - (\Delta_A - z_2)^{-1}\|_{\mathcal{B}(\mathscr{L}_s, \mathscr{L}_{-s})} \leq c\|z_1 - z_2\|^{s-1/2}\,,$$

for all $z_1, z_2 \in \mathcal{J}_\pm$.

The statements a) and b) follow from general results from [38]. This part relies on the Melrose calculus of cusp pseudodifferential operators (see, e.g., [35]) and is proved in Theorem 4.2 for the perturbed metric (2.9). We start from the basic observation that for smooth vector potentials, the magnetic Laplacian belongs to the cusp calculus with positive weights. For this part, we can treat the metric (2.9) which is quasi-isometric (but not necessarily asymptotically equivalent) to (1.1). Moreover, the finite multiplicity of the point spectrum (which is possibly not locally finite) in e) follows from Appendix B for this class of metrics.

The statement *c*) follows directly from the analysis of the free case in Section 5. The perturbation of the metric is considered in Proposition 6.2 and relies on general results on stability of the essential spectrum shown in [15]. The points d), f) and e) rely on the use of an optimal version of Mourre theory, see [2]. They are developed in Theorem 6.4 for perturbations satisfying (1.5). Scattering theory under short-range perturbation of a potential and of a magnetic field is also considered.

The condition of being trapping (resp. non-trapping) is discussed in Section 7.1 and is equivalent to having empty (resp. non-empty) essential spectrum in the complete case. The terminology arises from the dynamical consequences of this theorem and should not be confused with the classical terminology. Indeed, when B is trapping, the spectrum of Δ_B is purely discrete and for all non-zero ϕ in $L^2(X)$, there is $\chi \in \mathcal{C}_c^\infty(X)$ such that $1/T \int_0^T \|\chi e^{it\Delta_B}\phi\|^2 dt$ tends to a non-zero constant as T tends to $\pm\infty$. On the other hand, taking $\mathcal{J}$ as in f), for all $\chi \in \mathcal{C}_c^\infty(X)$ one gets that $\chi e^{it\Delta_B} E_{\mathcal{J}}(\Delta_A)\phi$ tends to zero, when B is non-trapping and with compact support.

If $H^1_{\mathrm{dR}}(M) \neq 0$ (take $M = S^1$ for instance), there exist some trapping magnetic fields with *compact support*. We construct an explicit example in Proposition 7.3. We are able to construct some examples in dimension 2 and higher than 4 but there are topological obstructions in dimension 3, see Section 7.1. As pointed out above regarding the Euclidean case, the fact that a magnetic field with compact support can turn off the essential spectrum and even a situation of limiting absorption principle is somehow unexpected and should be understood as a strong long-range effect.

We discuss other interesting phenomena in Section 7.2. Consider $M = S^1$ and take a trapping magnetic field B with compact support and a coupling constant $g \in \mathbb{R}$. Now remark that Δ_{gB} is non-trapping if and only if g belongs to the discrete group $c_B\mathbb{Z}$, for a certain $c_B \neq 0$. When $g \notin c_B\mathbb{Z}$ and $p \geq 1/n$, the spectrum of Δ_{gB} is discrete and the eigenvalue asymptotics do not depend either on B or on g. It would be very interesting to know whether the asymptotics of embedded eigenvalues, or more likely of resonances, remain the same when $g \in c_B\mathbb{Z}$, (see [6] for the case $g = 0$). It would be also interesting to study the inverse spectral problem and ask if the magnetic field could be recovered from the knowledge of the whole spectrum, since the first term in the asymptotics of eigenvalues does not feel it.

Assume now that gauge invariance does not hold, i.e., $H^1_{\mathrm{dR}}(X) \neq 0$. In quantum mechanics, it is known that the choice of a vector potential has a physical meaning. This is known as the Aharonov–Bohm effect [1]. Two choices of magnetic potential may lead to in-equivalent magnetic Laplacians. In $\mathbb{R}^2$ with a bounded obstacle, this phenomenon can be seen through a difference of wave phase arising from two non-homotopic paths that circumvent the obstacle. Some long-range effect appears, for instance in the scattering matrix like in [42–44], in an inverse-scattering problem [40, 48] or in the semi-classical regime [3]. See also [22] for the influence of the obstacle on the bottom of the spectrum. In all of these cases, the essential spectrum remains the same.

In Section 7.3, we discuss the Aharonov–Bohm effect in our setting. In light of Theorem 1.2, one expects a drastic effect. We show that the choice of a vector potential can indeed have a significant spectral consequence. For one choice of vector potential, the essential spectrum could be empty and for another choice it could be a half-line. This phenomenon is generic for hyperbolic surfaces of finite volume, and also appears for hyperbolic 3-manifolds. We focus the presentation on magnetic fields B with compact support. We say that a smooth vector potential A (i.e., a smooth 1-form on $\overline{X}$) is trapping if Δ_A has compact resolvent, and non-trapping otherwise. By Theorems 4.2 and 6.4, for $p \leq 1$, this is equivalent to Definition 3.2. It also follows that when the metric is of type (1.1), A is trapping if and only if a)–b) of Theorem 1.2 hold for Δ_A, while A is non-trapping if and only if Δ_A satisfies c)–f) of Theorem 1.2.

Theorem 1.3. *Let X be a complete oriented hyperbolic surface of finite volume and B a smooth magnetic field on the compactification $\overline{X}$.*

- *If X has at least 2 cusps, then for all B there exists both trapping and non-trapping vector potentials A such that $B = dA$.*
- *If X has precisely 1 cusp, choose $B = dA = dA'$ where A, A' are smooth vector potentials for B on $\overline{X}$. Then*

$$A \text{ is trapping} \iff A' \text{ is trapping} \iff \int_X B \in 2\pi\mathbb{Z}\,.$$

This follows from Corollary 8.1. More general statements are valid also in dimension 3, see Section 8.

This implies on one hand that for a choice of A, as one has the points c), d) and e), a particle located at a scattering energy escapes from any compact set; on the other hand taking a trapping choice, the particle will behave like an eigenfunction and will remain bounded. It is interesting that the dimension 3 is exceptional in the Euclidean case [47] and that we are able to construct examples of such a behavior in any dimension.

In the first appendix, we discuss the key notion of C^1 regularity for the Mourre theory and make it suitable to the manifold context and for our choice of conjugate operator. As pointed in [13], this is a key hypothesis in the Mourre theory in order to apply the Virial theorem and deduce the local finiteness of the embedded eigenvalues. In the second appendix, we recall that (cusp) elliptic, not necessarily fully elliptic, cusp operators have L^2 eigenvalues of finite multiplicity. Finally, in the third appendix, we give a criteria of stability of the essential spectrum, by cutting a part of the space, encompassing incomplete manifolds. Some of the results concerning the essential spectrum and the asymptotic of eigenvalues were already present in the unpublished preprint [18].

2. Cusp geometry

2.1. Definitions

This section follows closely [38], see also [35]. Let $\overline{X}$ be a smooth n-dimensional compact manifold with closed boundary M, and $x : \overline{X} \to [0, \infty)$ a boundary-defining function. A *cusp metric* on $\overline{X}$ is a complete Riemannian metric g_0 on $X := \overline{X} \setminus M$ which in local coordinates near the boundary takes the form

$$g_0 = a_{00}(x,y)\frac{dx^2}{x^4} + \sum_{j=1}^{n-1} a_{0j}(x,y)\frac{dx}{x^2}dy_j + \sum_{i,j=1}^{n-1} a_{ij}(x,y)dy_i dy_j \tag{2.8}$$

such that the matrix $(a_{\alpha\beta})$ is smooth and non-degenerate down to $x = 0$. For example, if $a_{00} = 1$, $a_{0j} = 0$ and a_{ij} is independent of x, we get a product metric near M. If we set $y = 1/x$, a cusp metric is nothing but a quasi-isometric deformation of a cylindrical metric, with an asymptotic expansion for the coefficients in powers of y^{-1}. We will focus on the *conformally cusp* metric

$$g_p := x^{2p} g_0\,, \tag{2.9}$$

where $p > 0$. Note that (1.1) is a particular case of such metric.

Let $\mathcal{I} \subset \mathcal{C}^\infty(\overline{X})$ be the principal ideal generated by the function x. Recall [35] that a *cusp vector field* is a smooth vector field V on $\overline{X}$ such that $dx(V) \in \mathcal{I}^2$. The space of cusp vector fields forms a Lie subalgebra ${}^c\mathcal{V}$ of the Lie algebra $\mathcal{V}$ of smooth vector fields on $\overline{X}$. In fact, there exists a natural vector bundle ${}^cT\overline{X}$ over $\overline{X}$ whose space of smooth sections is ${}^c\mathcal{V}$, and a natural map ${}^cT\overline{X} \to TX$ which induces the inclusion ${}^c\mathcal{V} \hookrightarrow \mathcal{V}$. Let $E, F \to \overline{X}$ be smooth vector bundles. The space of cusp differential operators $\mathrm{Diff}_c(\overline{X}, E, F)$ is the space of those differential operators which in local trivializations can be written as composition of cusp vector fields and smooth bundle morphisms down to $x = 0$.

The *normal operator* of $P \in \mathrm{Diff}_c(\overline{X}, E, F)$ is the family of operators defined by

$$\mathbb{R} \ni \xi \mapsto \mathcal{N}(P)(\xi) := \left(e^{i\xi/x} P e^{-i\xi/x}\right)_{|x=0} \in \mathrm{Diff}\left(M, E_{|M}, F_{|M}\right).$$

Example 2.1. $\mathcal{N}(x^2\partial_x)(\xi) = i\xi$.

Note that $\ker \mathcal{N} = \mathcal{I} \cdot \mathrm{Diff}_c$, which we denote again by $\mathcal{I}$. The normal operator map is linear and multiplicative. It is also invariant under the conjugation by powers of x. Namely, if $P \in \mathrm{Diff}_c$ and $s \in \mathbb{C}$ then $x^s P x^{-s} \in \mathrm{Diff}_c$ and $\mathcal{N}(x^s P x^{-s}) = \mathcal{N}(P)$. Concerning taking the (formal) adjoint, one needs to specify the volume form on the boundary.

Lemma 2.2. *Let $P \in \mathrm{Diff}_c(\overline{X}, E, F)$ be a cusp operator and P^* its adjoint with respect to g_0. Then $\mathcal{N}(P^*)(\xi)$ is the adjoint of $\mathcal{N}(P)(\xi)$ with respect to the metric on $E_{|M}, F_{|M}$ induced by restriction, for the volume form ${a_0}^{1/2} \mathrm{vol}_{h_0}$, where the metric h_0 on M is defined from rewriting g_0 as in* (6.41).

The principal symbol of a cusp operator on X extends as a map on the cusp cotangent bundle down to $x = 0$. This implies that a cusp operator of positive order cannot be elliptic at $x = 0$ in the usual sense. A cusp operator is called cusp-elliptic if its principal symbol is invertible on ${}^cT^*\overline{X} \setminus \{0\}$ down to $x = 0$.

Definition 2.3. A cusp operator is called *fully elliptic* if it is cusp-elliptic and if its normal operator is invertible for all values of $\xi \in \mathbb{R}$.

An operator $H \in x^{-l}\mathrm{Diff}_c^k(\overline{X}, E, F)$ is called a cusp differential operator of type (k, l).

Fix a product decomposition of X near M, compatible with the boundary-defining function x. This gives a splitting of the cusp cotangent bundle on $\overline{X}$ in a neighborhood of M:

$$ {}^cT^*\overline{X} \simeq T^*M \oplus \langle x^{-2}dx \rangle \,. \tag{2.10} $$

Lemma 2.4. *The de Rham differential $d : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X, T^*X)$ restricts to a cusp differential operator $d : \mathcal{C}^\infty(\overline{X}) \to \mathcal{C}^\infty(\overline{X}, {}^cT^*\overline{X})$. Its normal operator in the decomposition* (2.10) *is*

$$ \mathcal{N}(d)(\xi) = \begin{bmatrix} d^M \\ i\xi \end{bmatrix} $$

where d^M is the partial de Rham differential in the M factor of the product decomposition.

Proof. Let $\omega \in \mathcal{C}^\infty(\overline{X})$ and decompose $d\omega$ according to (2.10):

$$ d\omega = d^M\omega + \partial_x(\omega)dx = d^M\omega + x^2\partial_x(\omega)\frac{dx}{x^2} \,. $$

Since d^M commutes with x, it follows from the definition that $\mathcal{N}(d^M) = d^M$. The result follows using Example 2.1. □

2.2. Relative de Rham cohomology

Recall [4] that the cohomology of $\overline{X}$ and the relative cohomology groups of $(\overline{X}, M)$ (with real coefficients) can be computed using smooth differential forms as follows let $\Lambda^*(\overline{X})$ denote the space of forms smooth on $\overline{X}$ down to the boundary. Let $\Lambda^*(\overline{X}, M)$ denote the subspace of those forms whose pull-back to M vanishes. These spaces form complexes for the de Rham differential (because d commutes with pull-back to M) and their quotient is the de Rham complex of M:

$$ 0 \to \Lambda^*(\overline{X}, M) \hookrightarrow \Lambda^*(\overline{X}) \to \Lambda^*(M) \to 0 \,. $$

The induced long exact sequence in cohomology is just the long exact sequence of the pair $(\overline{X}, M)$.

2.3. Cusp de Rham cohomology

Notice that d preserves the space of cusp differential forms. Indeed, since d is a derivation and using Lemma 2.4, it suffices to check this property for a set of local generators of $\mathcal{C}^\infty(\overline{X}, {}^cT^*\overline{X})$. Choose local coordinates (y_j) on M and take as generators $x^{-2}dx$ and dy_j, which are closed.

Let ${}^cH^*(X)$ denote the cohomology of the complex of cusp differential forms $(\mathcal{C}^\infty(\overline{X}, \Lambda^*({}^cT\overline{X})), d)$ with respect to the de Rham differential.

Proposition 2.5. ${}^cH^k(X) = H^k(X) \oplus H^{k-1}(M)^2$.

Proof. The short exact sequence of de Rham complexes

$$0 \to \Lambda^*(T\overline{X}) \hookrightarrow \Lambda^*({}^cT\overline{X}) \to \Lambda^{*-1}(TM)^2 \to 0\,,$$

where the second map is given by

$$\Lambda^k({}^cT\overline{X}) \ni \omega \mapsto \Big((x^2\partial_x \lrcorner \omega)_{x=0}, \big(\partial_x(x^2\partial_x \lrcorner \omega)\big)_{x=0} \Big)\,, \tag{2.11}$$

gives rise to a long exact sequence in cohomology. Now the composition

$$\Lambda^*(T\overline{X}) \hookrightarrow \Lambda^*({}^cT\overline{X}) \hookrightarrow \Lambda^*(TX)$$

is a quasi-isomorphism, since de Rham cohomology can be computed either with smooth forms on $\overline{X}$, or with smooth forms on X. Thus in cohomology the map induced from $\Lambda^*(T\overline{X}) \hookrightarrow \Lambda^*({}^cT\overline{X})$ is injective. $\square$

3. The magnetic Laplacian

3.1. The magnetic Laplacian on a Riemannian manifold

A *magnetic field* B on the Riemannian manifold (X, g) is an exact real-valued 2-form. A *vector potential* A associated to B is a 1-form such that $dA = B$. We form the *magnetic Laplacian* acting on $\mathcal{C}^\infty(X)$:

$$\Delta_A := d_A^* d_A\,.$$

This formula makes sense for complex-valued 1-forms A. Note that when A is real, d_A is a metric connection on the trivial bundle $\underline{\mathbb{C}}$ with the canonical metric, and Δ_A is the connection Laplacian.

If we alter A by adding to it a real exact form, say $A' = A + df$, the resulting magnetic Laplacian satisfies

$$\Delta_{A'} = e^{-if}\Delta_A e^{if}$$

so it is unitarily equivalent to Δ_A in $L^2(X, g)$. Therefore if $H^1_{\mathrm{dR}}(X) = 0$ (for instance if $\pi_1(X)$ is finite; see [4]) then Δ_A depends, up to unitary equivalence, only on the magnetic field B. This property is called *gauge invariance*. For a more refined analysis of gauge invariance, see [19].

One usually encounters gauge invariance as a consequence of 1-connectedness (i.e., $\pi_1 = 0$). But in dimensions at least 4, every finitely presented group (in particular, every finite group) can be realized as π_1 of a compact manifold. Thus the hypothesis $\pi_1 = 0$ is unnecessarily restrictive, it is enough to assume that its abelianisation is finite.

While the properties of Δ_A in $\mathbb{R}^n$ with the flat metric are quite well understood, the (absence of) essential spectrum of magnetic Laplacians on other manifolds has not been much studied so far. One exception is the case of bounded geometry, studied in [26]. However our manifolds are not of bounded geometry because the injectivity radius tends to 0 at infinity.

3.2. Magnetic fields and cohomology

Recall that A is a (smooth) *cusp* 1*-form* (not to be confused with the notion of cusp form from automorphic form theory) on $\overline{X}$ if $A \in \mathcal{C}^\infty(X, T^*X)$ is a real-valued 1-form satisfying near ∂X

$$A = \varphi(x)\frac{dx}{x^2} + \theta(x) \tag{3.12}$$

where $\varphi \in \mathcal{C}^\infty(\overline{X})$ and $\theta \in \mathcal{C}^\infty([0,\varepsilon) \times M, \Lambda^1(M))$, or equivalently A is a smooth section in ${}^cT^*\overline{X}$ over $\overline{X}$.

Proposition 3.1. *Let B be a cusp* 2*-form. Suppose that B is exact on X, and its image by the map* (2.11) *is exact on M. Then there exists a smooth cusp* 1*-form A on $\overline{X}$ such that $dA = B$.*

Proof. Note that B is exact as a form on X, so $dB = 0$ on X. By continuity, $dB = 0$ on $\overline{X}$ (in the sense of cusp forms) so B defines a cusp cohomology 2-class. By hypothesis, this class maps to 0 by restriction to X. Now the pull-back of B to the level surfaces $\{x = \varepsilon\}$ is closed; by continuity, the image of B through the map (2.11) is closed on M. Assuming that this image is exact, it follows from Proposition 2.5 that B is exact as a cusp form. □

By Lemmata 2.2 and 2.4, Δ_A is a cusp differential operator of order $(2p, 2)$.

Definition 3.2. Let A be a (complex-valued) cusp vector potential. Given a connected component M_0 of M, we say A is a *trapping* vector potential on M_0 if

- either the restriction $\varphi_0 := \varphi(0)$ is not constant on M_0,
- or $\theta_0 := \theta(0)$ is not closed on M_0,
- or the cohomology class $[\theta_{0|M_0}] \in H^1_{\mathrm{dR}}(M_0)$ does not belong to the image of

$$2\pi H^1(M_0;\mathbb{Z}) \to H^1(M_0,\mathbb{C}) \simeq H^1_{\mathrm{dR}}(M_0) \otimes \mathbb{C}$$

and *non-trapping on* M_0 otherwise.

We say that A is *trapping* if it is trapping on each connected component of M. The vector potential is said to be *non-trapping* if it is non-trapping on at least one connected component of M. If A is non-trapping on all connected component of M, we say that it is *maximal non-trapping*.

Remark 3.3. The trapping notion can be expressed solely in terms of the magnetic field $B = dA$ when $H^1(X) = 0$, see Lemma 7.2.

We comment briefly the terminology. When A is real, constant in x near M, the multiplicity of the absolutely continuous part of the spectrum of Δ_A will be given by the number of connected component of M on which A is non-trapping. Hence, taking A maximal non-trapping maximizes the multiplicity of this part of the spectrum.

We refer to [4] for an exposition of cohomology with integer coefficients. The trapping property is determined only by the asymptotic behavior of A. More precisely, if A' is also of the form (3.12) with $\varphi(0) = 0$ and $\theta(0) = 0$ then A is a (non-)trapping vector potential if and only if $A + A'$ is.

The term "trapping" is motivated by dynamical consequences of Theorems 4.1 and 6.4 and has nothing to do with the classical trapping condition. This terminology is also supported by the examples given in Section 7.

For a trapping vector potential, $x^{2p}\Delta_A$ is a fully-elliptic cusp operator. In turn, this implies that Δ_A has empty essential spectrum so from a dynamical point of view, a particle can not diffuse, in other words it is trapped in the interior of X. Indeed, given a state $\phi \in L^2(X)$, there exists χ (the characteristic function of a compact subset of X) such that $1/T\int_0^T \|\chi e^{it\Delta_A}\phi\|^2 dt$ tends to a positive constant as T goes to infinity.

On the other hand, if A is a non-trapping vector potential, then Δ_A is not Fredholm between the appropriate cusp Sobolev spaces. If the metric is an exact cusp metric and complete, we show that Δ_A has nonempty essential spectrum also as an unbounded operator in L^2, given by $[\kappa(p), \infty)$ by Proposition 6.2. We go even further and under some condition of decay of φ and θ at infinity, we show that there is no singular continuous spectrum for the magnetic Laplacian and that the eigenvalues of $\mathbb{R} \setminus \{\kappa(p)\}$ are of finite multiplicity and can accumulate only in $\{\kappa(p)\}$. Therefore given a state ϕ which is not an eigenvalue of Δ_A, one obtains that for all χ, $\chi e^{it\Delta_A}\phi$ tends to 0 as $t \to \infty$.

When M is connected, the class of non-trapping vector potentials is a group under addition but that of trapping vector potential is not. When M is disconnected, none of these classes is closed under addition. Directly from the definition, we get however:

Remark 3.4. Let A be a maximal non-trapping vector potential and let A' be a 1-form smooth up to the boundary. Then A' is trapping if and only if $A + A'$ is.

Let B be a smooth magnetic field on $\overline{X}$ (i.e., a 2-form) whose pull-back to M vanishes. Since B is exact, it is also closed, thus it defines a relative de Rham class as in Subsection 2.2. If this class vanishes, we claim that there exists a vector potential A for B which is maximal non-trapping. Indeed, let $A \in \Lambda^1(\overline{X}, M)$ be any (relative) primitive of B. Then A is clearly a cusp form, the singular term $\phi(0)$ vanishes, and the pull-back of A to each boundary component vanishes by definition, in particular it defines the null 1-cohomology class. From Remark 3.4 we get

Corollary 3.5. *Let B be a cusp magnetic field. Let B' be a smooth magnetic field on $\overline{X}$ which vanishes on the boundary and which defines the zero relative cohomology class in $H^2(\overline{X}, M)$. Then B admits (non-)trapping vector potentials if and only if $B + B'$ does.*

Note that when $H^1(\overline{X}) \neq 0$, a given magnetic field may admit both trapping and non-trapping vector potentials. See Theorem 1.3 and Sections 7.3 and 8.

4. The trapping case

4.1. The absence of essential spectrum

In this section, given a smooth cusp 1-form, we discuss the link between its behavior at infinity and its trapping properties.

Theorem 4.1. *Let $p > 0$, g_p a metric on X given by (2.9) near ∂X and A a smooth cusp 1-form given by (3.12). Then Δ_A is a weighted cusp differential operator of order $(2p, 2)$. If A is trapping then Δ_A is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$, it has purely discrete spectrum and its domain is $x^{2p}H^2(X, g_p)$.*

If $p \leq 1$ then g_p is complete so Δ_A is essentially self-adjoint [46]. This fact remains true for a trapping A in the incomplete case, i.e., $p > 1$.

Proof. Using Lemma 2.4, we get

$$\mathcal{N}(d_A)(\xi) = \begin{bmatrix} d^M + i\theta_0 \\ i(\xi + \varphi_0) \end{bmatrix} .$$

Suppose that $x^{2p}\Delta_A$ is not fully elliptic, so there exists $\xi \in \mathbb{R}$ and $0 \neq u \in \ker(\mathcal{N}(x^{2p}\Delta_A)(\xi))$. By elliptic regularity, u is smooth. We replace M by one of its connected components on which u does not vanish identically, so we can suppose that M is connected. Using Lemma 2.2, by integration by parts with respect to the volume form ${a_0}^{1/2} dh_0$ on M and the metric h_0 on $\Lambda^1(M)$, we see that $u \in \ker(\mathcal{N}(\Delta_A)(\xi))$ implies $u \in \ker(\mathcal{N}(d_A)(\xi))$. Then

$$(\xi + \varphi_0)u = 0 \quad \text{and} \quad (d^M + i\theta_0)u = 0\,, \tag{4.13}$$

so u is a global parallel section in the trivial bundle $\underline{\mathbb{C}}$ over M, with respect to the connection $d^M + i\theta_0$. This implies

$$0 = (d^M)^2 u = d^M(-iu\theta_0) = -i(d^M u) \wedge \theta_0 - iud^M\theta_0 = -iud^M\theta_0\,.$$

By uniqueness of solutions of ordinary differential equations, u is never 0, so $d^M\theta_0 = 0$. Furthermore, from (4.13), we see that φ_0 equals the constant function $-\xi$. It remains to prove the assertion about the cohomology class $[\theta_0]$.

Let $\tilde{M}$ be the universal cover of M. Denote by $\tilde{u}$, $\tilde{\theta}_0$ the lifts of u, θ_0 to $\tilde{M}$. The equation $(d^M + i\theta_0)u = 0$ lifts to

$$(d^{\tilde{M}} + i\tilde{\theta}_0)\tilde{u} = 0\,. \tag{4.14}$$

The 1-form $\tilde{\theta}_0$ is closed on the simply connected manifold $\tilde{M}$, hence it is exact (by the universal coefficients formula, $H^1(\tilde{M}, \mathbb{C}) = H_1(\tilde{M}, \mathbb{C}) = H_1(\tilde{M}; \mathbb{Z}) \otimes \mathbb{C}$, and $H_1(\tilde{M}; \mathbb{Z})$ vanishes as it is the abelianisation of $\pi_1(\tilde{M})$). Let $v \in \mathcal{C}^\infty(\tilde{M})$ be a primitive of $\tilde{\theta}_0$, i.e., $d^{\tilde{M}} v = \tilde{\theta}_0$. Then, from (4.14), $\tilde{u} = Ce^{-iv}$ for some constant $C \neq 0$.

The fundamental group $\pi_1(M)$ acts to the right on $\tilde{M}$ via deck transformations. The condition that $\tilde{u}$ be the lift of u from M is the invariance under the action of $\pi_1(M)$, in other words

$$\tilde{u}(y) = \tilde{u}\big(y[\gamma]\big)$$

for all closed loops γ in M. This is obviously equivalent to

$$v\big(y[\gamma]\big) - v(y) \in 2\pi\mathbb{Z}\,, \quad \forall y \in \tilde{M}\,.$$

Let $\tilde{\gamma}$ be the lift of γ starting in y. Then

$$v\big(y[\gamma]\big) - v(y) = \int_{\tilde{\gamma}} d^{\tilde{M}} v = \int_{\tilde{\gamma}} \tilde{\theta}_0 = \int_\gamma \theta_0\,.$$

Thus the solution $\tilde{u}$ is $\pi_1(M)$-invariant if and only if the cocycle θ_0 evaluates to an integer multiple of 2π on each closed loop γ. These loops span $H_1(M; \mathbb{Z})$, so $[\theta_0]$ lives in the image of $H^1(M; \mathbb{Z})$ inside $H^1(M, \mathbb{C}) = \mathrm{Hom}(H_1(M; \mathbb{Z}), \mathbb{C})$. Therefore the solution u must be identically 0 unless φ_0 is constant, θ_0 is closed and $[\theta_0] \in 2\pi H^1(M; \mathbb{Z})$.

Conversely, if φ_0 is constant, θ_0 is closed and $[\theta_0] \in 2\pi H^1(M; \mathbb{Z})$ then $\tilde{u} = e^{-iv}$ as above is $\pi_1(M)$-invariant, so it is the lift of some $u \in \mathcal{C}^\infty(M)$ which belongs to $\ker(\mathcal{N}(x^{2p}\Delta_A)(\xi))$ for $-\xi$ equal to the constant value of φ_0.

The conclusion of the theorem is now a consequence of general properties of the cusp calculus [38, Theorem 17]. Namely, since Δ_A is fully elliptic, there exists an inverse in $x^{2p}\Psi_c^{-2}(X)$ (a micro-localized version of $\mathrm{Diff}_c(X)$) modulo compact operators. If $p > 0$, this pseudo-inverse is itself compact. The operators in the cusp calculus act by closure on a scale of Sobolev spaces. It follows easily that for $p \geq 0$, a symmetric fully elliptic cusp operator in $x^{2p}\Psi_c^2(X)$ is essentially self-adjoint, with domain $x^{2p}H_c^2(X)$. Thus Δ_A is self adjoint with compact inverse modulo compact operators, which shows that the spectrum is purely discrete. □

The cusp calculus [35] is a particular instance of Melrose's program of micro-localizing boundary fibration structures. It is a special case of the fibered-cusp calculus [33] and can be obtained using the groupoid techniques of [29].

4.2. Eigenvalue asymptotics for trapping magnetic Laplacians

If A is a trapping vector potential, the associated magnetic Laplacian has purely discrete spectrum. In this case we can give the first term in the eigenvalue growth law.

Theorem 4.2. *Let $p > 0$, g_p a metric on X given by (2.9) near ∂X and $A \in \mathcal{C}^\infty(X, T^*X)$ a complex-valued trapping vector potential in the sense of Definition 3.2. Then the eigenvalue counting function of Δ_A satisfies*

$$N_{A,p}(\lambda) \approx \begin{cases} C_1 \lambda^{n/2} & \text{for } 1/n < p < \infty\,, \\ C_2 \lambda^{n/2} \log \lambda & \text{for } p = 1/n\,, \\ C_3 \lambda^{1/2p} & \text{for } 0 < p < 1/n \end{cases} \tag{4.15}$$

in the limit $\lambda \to \infty$, where

$$\begin{aligned} C_1 &= \frac{\mathrm{Vol}(X, g_p)\,\mathrm{Vol}(S^{n-1})}{n(2\pi)^n}\,, \\ C_2 &= \frac{\mathrm{Vol}(M, {a_0}^{1/2} h_0)\,\mathrm{Vol}(S^{n-1})}{2(2\pi)^n}\,. \end{aligned} \tag{4.16}$$

If moreover we assume that g_0 is an exact cusp metric, then

$$C_3 = \frac{\Gamma\left(\frac{1-p}{2p}\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{1}{2p}\right)} \zeta\left(\Delta_A^{h_0}, \frac{1}{p} - 1\right)\,,$$

where $\Delta_A^{h_0}$ is the magnetic Laplacian on M with potential $A_{|M}$ with respect to the metric h_0 on M defined in Section 2.

We stress that the constants C_1 and C_2 do not depend on the choice of A or B, but only on the metric. This fact provides some very interesting coupling constant effect, see Section 7.2.

Note also that the hypotheses of the theorem are independent of the choice of the vector potential A inside the class of cusp 1-forms. Indeed, assume that $A' = A + dw$ for some $w \in \mathcal{C}^\infty(X)$ is again a cusp 1-form. Then dw must be itself a cusp form, so

$$dw = x^2 \partial_x w \frac{dx}{x^2} + d^M w \in \mathcal{C}^\infty(\overline{X}, {}^cT^*\overline{X})\,.$$

Write this as $dw = \varphi' \frac{dx}{x^2} + \theta'_x$. For each $x > 0$ the form θ'_x is exact. By the Hodge decomposition theorem, the space of exact forms on M is closed, so the limit θ'_0 is also exact. Now dw is an exact cusp form, in particular it is closed. This implies that $x^2 \partial_x \theta' = d^M \varphi'$. Setting $x = 0$ we deduce $d^M(\varphi')_{x=0} = 0$, or equivalently $\varphi'_{|x=0}$ is constant. Hence the conditions from Theorem 4.2 on the vector potential are satisfied simultaneously by A and A'.

Proof. From Theorem 4.1, the operator $x^{2p}\Delta_A$ is fully elliptic when A is trapping. The result follows directly from [38, Theorem 17]. Let us explain the idea: the complex powers Δ_A^{-s} belong to the cusp calculus, and are of trace class for sufficiently large real part of s. The trace of the complex powers is holomorphic for such s and extends meromorphically to $\mathbb{C}$ with two families of simple poles, coming from the principal symbol and from the boundary. The leading pole governs eigenvalue asymptotics, by the Delange theorem. In case the leading pole is double (by the superposition of the two families of poles), we get the logarithmic growth law.

The explicit computation of the constants (given by [38, Prop. 14 and Lemma 16]) is straightforward. □

5. Analysis of the free case for non-trapping potentials

We have shown in Theorems 4.1 and 4.2 that the essential spectrum of the magnetic Laplacian Δ_A is empty when A is trapping (see Definition 3.2), and we have computed the asymptotics of the eigenvalues. We now consider the case of a non-trapping vector potential A. One can guess that in this last case, the essential spectrum is not empty when the metric is complete. In this section we concentrate on the unperturbed metric (5.17) with the model non-trapping potential (5.18). We take advantage of the decomposition in low- and high-energy functions from Section 5.1. The computation of the essential spectrum is based on Proposition 5.1 and on the diagonalization of the magnetic Laplacian performed in Section 5.2. In Section 5.3, we construct a local conjugate operator and state the Mourre estimate (Theorem 5.6).

In Section 5.4 we introduce the classes of perturbation under which we later give a limiting absorption principle. The perturbation theory is developed in Section 6. We refer to Proposition 6.2 for the question of the essential spectrum and to Theorem 6.4 for its refined analysis under short/long range perturbations.

We localize the computation on the end $X' := (0, \varepsilon) \times M \subset X$. We assume that on $X' := (0, \varepsilon) \times M \subset X$ we have

$$g_p = x^{2p}\left(\frac{dx^2}{x^4} + h_0\right), \tag{5.17}$$

$$A_{\mathrm{f}} = C dx/x^2 + \theta_0\,, \tag{5.18}$$

where C is a constant, θ_0 is closed and independent of x, and the cohomology class $[\theta_0] \in H^1(M)$ is an integer multiple of 2π. By a change of gauge, one may assume that $C = 0$. Indeed, it is enough to subtract from A_{f} the exact 1-form $d(-C/x)$.

5.1. The high and low energy functions decomposition

Set $d_{\tilde\theta} := d^M + i\theta_0\wedge$. We now decompose the L^2 space as follows:

$$L^2(X') = \mathscr{H}_{\mathrm{l}} \oplus \mathscr{H}_{\mathrm{h}}\,, \tag{5.19}$$

where $\mathscr{H}_{\mathrm{l}} := \mathscr{K} \otimes \ker(d_{\tilde\theta})$ with $\mathscr{K} := L^2((0,\varepsilon), x^{np-2}dx)$, and where $\mathscr{H}_{\mathrm{h}} = \mathscr{K} \hat\otimes \ker(d_{\tilde\theta})^{\perp}$. We do not emphasize the dependence on ε for these spaces as the properties we are studying are independent of ε. The subscripts l, h stand for low and high energy, respectively. The importance of the *low energy functions* space $\mathscr{H}_{\mathrm{l}}$ is underlined by the following:

Proposition 5.1. *The magnetic Laplacian* $\Delta_{A_{\mathrm{f}}}$ *stabilizes the decomposition* (5.19) *of* $L^2(X')$. *Let* $\Delta^l_{A_{\mathrm{f}}}$ *and* $\Delta^h_{A_{\mathrm{f}}}$ *be the Friedrichs extensions of the restrictions of* $\Delta_{A_{\mathrm{f}}}$

to smooth compactly supported functions in $\mathscr{H}_{\mathrm{l}}$*, respectively in* $\mathscr{H}_{\mathrm{h}}$*. Then* $\Delta^h_{A_{\mathrm{f}}}$ *has compact resolvent, while*

$$\Delta^l_{A_{\mathrm{f}}} = \left(D^*D + c_0^2 x^{2-2p}\right) \otimes 1\,,$$

where $c_0 := ((2-n)p-1)/2$*, and* $D := x^{2-p}\partial_x - c_0 x^{1-p}$ *acts in* $\mathscr{K}_{k,\varepsilon}$*.*

By Proposition C.3, the essential spectrum is prescribed by the space of low energy functions. This kind of decomposition can be found in various places in the literature, see for instance [32] for applications to finite-volume negatively-curved manifolds.

Remark 5.2. The dimension of the kernel of $d_{\tilde\theta}$ equals the number of connected components of M on which A_f non-trapping. Indeed, take such a connected component M_1 and let v be a primitive of θ_0 on its universal cover. On different sheets of the cover, v changes by $2\pi\mathbb{Z}$ so e^{iv} is a well-defined function on M_1 which spans $\ker(d_{\tilde\theta})$. This decomposition is also valid in the trapping case, only that the low energy functions space is then 0. Using Proposition C.3, we obtain the emptiness of the essential spectrum for the unperturbed metric and along the way a special case of Theorem 4.1 (we do not recover the full result in this way, since the metric (2.9) can *not* be reached perturbatively from the metric (5.17)).

Proof. We decompose the space of 1-forms as the direct sum (2.10). Recall that δ_M is the adjoint of d^M with respect to h_0. We compute

$$\begin{aligned} d_{A_{\mathrm{f}}} &= \begin{bmatrix} d^M + i\theta_0\wedge \\ x^2\partial_x \end{bmatrix} \\ d^*_{A_{\mathrm{f}}} &= x^{-np}\begin{bmatrix}\delta_M - i\theta_0\lrcorner & -x^2\partial_x\end{bmatrix} x^{(n-2)p} \\ \Delta_{A_{\mathrm{f}}} &= x^{-2p}\left(d_{\tilde\theta}{}^* d_{\tilde\theta} - (x^2\partial_x)^2 - (n-2)px(x^2\partial_x)0\right). \end{aligned} \tag{5.20}$$

On the Riemannian manifold (M,h), $d_{\tilde\theta}{}^* d_{\tilde\theta}$ is non-negative with discrete spectrum. Since $-(x^2\partial_x)^2 - (n-2)px(x^2\partial_x) = (x^2\partial_x)^*(x^2\partial_x)$ is non-negative, one has that $\Delta^h_{A_{\mathrm{f}}} \geq \varepsilon^{-2p}\lambda_1$, where λ_1 is the first non-zero eigenvalue of $d_{\tilde\theta}{}^* d_{\tilde\theta}$. By Proposition C.3, the essential spectrum is independent of ε. By letting $\varepsilon \to 0$ we see that it is empty; thus $\Delta^h_{A_{\mathrm{f}}}$ has compact resolvent. The assertion on $\Delta^l_{A_{\mathrm{f}}}$ is a straightforward computation. □

5.2. Diagonalization of the free magnetic Laplacian

In order to analyze the spectral properties of $\Delta_{A_{\mathrm{f}}}$ on (X, g_p), where A_{f} is given by (5.18) with $C = 0$ and g_p by (5.17), we go into some "Euclidean variables". We concentrate on the complete case, i.e., $p \leq 1$. We start with (5.19) and work on $\mathscr{K}$. The first unitary transformation is

$$L^2(x^{np-2}dx) \to L^2(x^{p-2}dx) \qquad \phi \mapsto x^{(n-1)p/2}\phi\,.$$

Then we proceed with the change of variables $z := L(x)$, where L is given by (6.44). Therefore, $\mathscr{K}$ is unitarily sent into $L^2\left((c,\infty), dz\right)$ for a certain c. We indicate operators and spaces obtained in the new variable with a subscript 0.

Thanks to this transformation, we are pursuing our analysis on the manifold $X_0 = X$ endowed with the Riemannian metric

$$dr^2 + h\,, \quad r \to \infty\,, \tag{5.21}$$

on the end $X_0' = [1/2, \infty) \times M$. The subscript r stands for radial. The magnetic Laplacian is unitarily sent into an elliptic operator of order 2 denoted by Δ_0. On $\mathcal{C}_c^\infty(X_0')$, it acts by

$$\Delta_0 := Q_p \otimes d_{\tilde{\theta}}{}^* d_{\tilde{\theta}} + (-\partial_r^2 + V_p) \otimes 1\,, \tag{5.22}$$

on the completed tensor product $L^2([1/2,\infty), dr)\hat{\otimes}L^2(M,h)$, where

$$V_p(r) = \begin{cases} \big((n-1)/2\big)^2 \\ c_0 r^{-2} \end{cases} \quad \text{and} \quad Q_p(r) = \begin{cases} e^{2r} \\ \big((1-p)r\big)^{2p/(1-p)} \end{cases} \quad \text{for} \quad \begin{cases} p = 1 \\ p < 1\,. \end{cases}$$

Recall that c_0 is defined in Proposition 5.1 and $d_{\tilde{\theta}} = d_M + i\theta_0\wedge$.

We denote also by L_0 the operator of multiplication corresponding to L, given by (6.44), in the new variable r. It is bounded from below by a positive constant, equals 1 on the compact part and on the trapping ends, and equals $r \mapsto r$ on the non-trapping ends.

Let $\mathscr{H}_0^s := \mathcal{D}((1+\Delta_0)^{s/2})$ for $s > 0$. By identifying $\mathscr{H}_0$ with $\mathscr{H}_0^*$ by the Riesz isomorphism, by duality, we define $\mathscr{H}_0^s$ for $s < 0$ with $\mathscr{H}_0^{-s}$. We need the next well-known fact.

Lemma 5.3. *For every $\gamma \in \mathcal{C}_c^\infty(X)$, we have that $\gamma : \mathscr{H}^s \subset \mathscr{H}^s$ for all $s \in \mathbb{R}$.*

Proof. A computation gives that there is c such that $\|(\Delta_0+i)\gamma\varphi\| \le c\|(\Delta_0+i)\varphi\|$, for all $\varphi \in \mathcal{C}_c^\infty(X_0)$. Since $\mathcal{C}_c^\infty(X_0)$ is a core for Δ_0, we get the result for $s = 2$. By induction we get it for $s \in 2\mathbb{N}$. Duality and interpolation give it for $s \in \mathbb{R}$. □

5.3. The local conjugate operator

In this section, we construct some conjugate operators in order to establish a Mourre estimate. By mimicking the case of the Laplacian, see [10], one may use the following localization of the generator of dilations. Let $\xi \in \mathcal{C}^\infty([1/2,\infty))$ such that the support of ξ is contained in $[2,\infty)$ and that $\xi(r) = r$ for $r \ge 3$ and let $\tilde{\chi} \in \mathcal{C}^\infty([1/2,\infty))$ with support in $[1,\infty)$, which equals 1 on $[2,\infty)$. By abuse of notation, we denote $\tilde{\chi} \otimes 1 \in \mathcal{C}^\infty(X_0)$ with the same symbol. Let $\chi := 1 - \tilde{\chi}$. On $\mathcal{C}_c^\infty(X_0)$ we set:

$$S_\infty := \big(-i(\xi\partial_r + \partial_r\xi) \otimes P_0\big)\tilde{\chi}\,, \tag{5.23}$$

where P_0 is the orthogonal projection onto $\ker(d_{\tilde{\theta}})$. The presence of P_0 comes from the decomposition in low and high energies.

Remark 5.4. By considering a C_0-group associated to a vector field on $\mathbb{R}$ like in [2, Section 4.2], one shows that $-i(\xi\partial_r + \partial_r\xi)$ is essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R})$. This C_0-group acts trivially away from the support of $\tilde{\chi}$ and then it is easy to construct another C_0-group G_0 which acts like the first on $L^2(\operatorname{supp}(\tilde{\chi})) \otimes \ker(d_{\tilde{\theta}})$, and trivially on the rest of $L^2(X_0)$. Let $\overline{S_\infty}$ be the generator of G_0. Since $G_0(t)$

leaves invariant $\mathcal{C}_c^\infty(X_0)$ the Nelson lemma implies that $\overline{S_\infty}$ is the closure of S_∞, in other words $\overline{S_\infty}$ is essentially self-adjoint on $\mathcal{C}_c^\infty(X_0)$. This kind of approach has been used in [5] for instance.

Therefore, we denote below also by S_∞ the self-adjoint closure of the operator defined by (5.23) on $\mathcal{C}_c^\infty(X_0)$.

However this operator is *not* suitable for very singular perturbations like that of the metric considered in this paper. To solve this problem, one should consider a conjugate operator more "local in energy". Concerning the Mourre estimate, as it is local in energy for the Laplacian, one needs only a conjugate operator which fits well on this level of energy. Considering singular perturbation theory, the presence of differentials in (5.23) is a serious obstruction; the idea is to replace the conjugate operator with a multiplication operator in the analysis of perturbations, therefore reducing the rôle of derivatives within it. The approach has been used for Dirac operators for instance in [16] to treat very singular perturbations. The case of Schrödinger operators is summarized in [2, Theorem 7.6.8]. We set:

$$S_R := \tilde{\chi}\Big(\big(\Phi_R(-i\partial_r)\xi + \xi\Phi_R(-i\partial_r)\big) \otimes P_0\Big)\tilde{\chi}\,, \tag{5.24}$$

where $\Phi_R(x) := \Phi(x/R)$ for some $\Phi \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying $\Phi(x) = x$ for all $x \in [-1,1]$. The operator $\Phi_R(-i\partial_r)$ is defined on $\mathbb{R}$ by $\mathscr{F}^{-1}\Phi_R\mathscr{F}$, where $\mathscr{F}$ is the unitary Fourier transform. Let us also denote by S_R the closure of this operator.

Unlike (5.23), S_R does not stabilize $\mathcal{C}_c^\infty(X_0)$ because $\Phi_R(-i\partial_r)$ acts like a convolution with a function with non-compact support. This subspace is sent into $\tilde{\chi}\mathscr{S}(\mathbb{R})$, where $\mathscr{S}(\mathbb{R})$ denotes the Schwartz space. To motivate the subscript R, note that S_R tends strongly in the resolvent sense to S, as R goes to infinity. We give some properties of S_R. The point (2) is essential to be able to replace S_R by L_0 in the theory of perturbations. The point (3) is convenient to be able to express a limiting absorption principle in terms of L_0, which is very explicit. Of course these two points are false for S_∞ and this explains why we can go further in the perturbation theory compared to the standard approach.

Lemma 5.5. *Let S_R denote the closure of the unbounded operator* (5.24).

1. *For all $R \in [1,\infty]$, the operator S_R is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$.*
2. *For R finite, $L_0^{-2}S_R^2 : \mathcal{C}_c^\infty(X_0) \to \mathcal{D}(\Delta_0)$ extends to a bounded operator in $\mathcal{D}(\Delta_0)$.*
3. *For R finite, $\mathcal{D}(L_0^s) \subset \mathcal{D}(|S_R|^s)$ for all $s \in [0,2]$.*

Proof. The case $R = \infty$ is discussed in Remark 5.4, so assume that R is finite. We compare S_R with L_0, defined in Section 5.2, which is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$. Noting that it stabilizes the decomposition (5.19), we write also by $L_{0,1}$ its restriction to $\mathscr{H}_1$, which is simply the multiplication by r. On $\mathcal{C}_c^\infty(X_0)$,

$$S_R = \tilde{\chi}\Big(2\Phi_R(-i\partial_r)\xi L_{0,1}^{-1} + \big[\xi, \Phi_R(-i\partial_r)\big]L_{0,1}^{-1}\Big) \otimes P_0\tilde{\chi}L_0.$$

Noting that $\xi L_{0,1}^{-1}$ is bounded and that $\xi' \in L^\infty$ and using Lemma 5.11, we get $\|S_R\varphi\| \leq a\|L_0\varphi\|$, for all $\varphi \in \mathcal{C}_c^\infty(X_0)$.

On the other hand, $[S_R, L_0]$ is equal to the bounded operator

$$[S_R, L_0] = \tilde{\chi}\Big(\big[\Phi_R(-i\partial_r), L_{0,\mathrm{l}}\big]\xi L_{0,\mathrm{l}}^{-1}\Big) \otimes P_0\tilde{\chi}L_0 + L_0\tilde{\chi}\Big(L_{0,\mathrm{l}}^{-1}\xi\big[\Phi_R(-i\partial_r), L_{0,\mathrm{l}}\big]\Big) \otimes P_0\tilde{\chi}\,.$$

This gives $|\langle S_R\varphi, L_0\varphi\rangle - \langle L_0\varphi, S_R\varphi\rangle| \leq b\|L_0^{1/2}\varphi\|^2$, for all $\varphi \in \mathcal{C}_c^\infty(X_0)$. Finally, one uses [41, Theorem X.37] to conclude that S_R is essentially self-adjoint.

On $\mathcal{C}_c^\infty(X_0)$, we have

$$L_0^{-2}S_R^2 = \Big(2\chi L_{0,\mathrm{l}}^{-1}\xi\Phi_R(-i\partial_r)\chi \otimes P_0 + \chi L_{0,\mathrm{l}}^{-1}\big[\Phi_R(-i\partial_r), \xi\big]\chi \otimes P_0\Big)^2\,. \tag{5.25}$$

All these terms are bounded in $L^2(X_0)$ by Lemma 5.11 and by density. We now compute $\Delta_0 L_0^{-2}S_R^2$. By Lemma 5.3, it is enough to show that $\Phi_R(-i\partial_r)$ and $[\Phi_R(-i\partial_r), \xi]$ stabilize the domain Δ in $L^2(\mathbb{R})$. The first one commutes with Δ. For the second one, we compute on $\mathcal{C}_c^\infty(\mathbb{R})$. Since $[\Phi_R(-i\partial_r), \xi]$ is bounded in $L^2(\mathbb{R})$, it is enough to show that the commutator $[\Delta, [\Phi_R(-i\partial_r), \xi]]$ is also bounded in $L^2(\mathbb{R})$. By Jacobi's identity, it is equal to $[\Phi_R(-i\partial_r), [\Delta, \xi]] = [\Phi_R(-i\partial_r), 2\xi'\partial_r + \xi''] = 2\Phi_R(-i\partial_r)\partial_r\xi' - 2\Phi_R(-i\partial_r)\xi'' + 2\xi'\Phi_R(-i\partial_r) + \xi''$. This is a bounded operator in $L^2(\mathbb{R})$ and we get point (2).

We now note that (5.25) is bounded in $L^2(X_0)$. Then, since $S_R^2 L_0^{-2}$ is also bounded, we get $\|S_R^2\varphi\|^2 \leq c\|L_0^2\varphi\|$ for all $\varphi \in \mathcal{C}^\infty(X_0)$. Taking a Cauchy sequence, we deduce $\mathcal{D}(L_0^2) \subset \mathcal{D}(S_R^2)$. An argument of interpolation gives point (3). $\square$

The aim of this section is the following Mourre estimate.

Theorem 5.6. *Let $R \in [1, \infty]$. Then $e^{itS_R}\mathscr{H}_0^2 \subset \mathscr{H}_0^2$ and $\Delta_0 \in \mathcal{C}^2(S_R, \mathscr{H}_0^2, \mathscr{H}_0)$. Given an interval $\mathcal{J}$ inside $\sigma_{\mathrm{ess}}(\Delta_0)$, there exist $\varepsilon_R > 0$ and a compact operator K_R such that*

$$E_{\mathcal{J}}(\Delta_0)[\Delta_0, iS_R]E_{\mathcal{J}}(\Delta_0) \geq \big(4\inf(\mathcal{J}) - \varepsilon_R\big)E_{\mathcal{J}}(\Delta_0) + K_R$$

holds in the sense of forms, and such that ε_R tends to 0 as R goes to infinity.

Proof. The regularity assumptions follow from Lemmata 5.9 and 5.10. The left hand side of (5.30) is the commutator $[\Delta_0, iS_R]$ in the sense of forms. It extends to a bounded operator in $\mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0^{-2})$ since $\mathcal{C}_c^\infty(X_0)$ is a core for Δ_0. We can then apply the spectral measure and obtain the inequality using Lemma 5.8. $\square$

Compared to the method from [10, Lemma 2.3] (for the case $R = \infty$), we have a relatively more direct proof based on Lemma 5.8. However this has no real impact on applications of the theory.

We now go in a series of lemmata to prove this theorem. Given a commutator $[A, B]$, we denote its closure by $[A, B]_0$.

Lemma 5.7. *For $R = \infty$, the commutators $[\Delta_0, iS_\infty]_0$ and $[[\Delta_0, iS_\infty], iS_\infty]_0$ belong to $\mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0)$. For R finite, $[\Delta_0, iS_R]_0$ and $[[\Delta_0, iS_\infty], iS_R]_0$ belong to $\mathcal{B}(\mathscr{H}_0)$. Moreover, if $p = 1$, all higher commutators extend to bounded operators in $\mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0)$ for $R = \infty$ and in $\mathcal{B}(\mathscr{H}_0)$ for $R < \infty$.*

Proof. Let $\varphi \in \mathcal{C}_c^\infty(X_0')$ such that $\varphi = \varphi_{\mathbb{R}} \otimes \varphi_M$ where $\varphi_M \in \mathcal{C}^\infty(M)$ and $\varphi_{\mathbb{R}} \in \mathcal{C}_c^\infty([1/2,\infty))$. Note that $P_0\varphi_M \in \mathcal{C}^\infty(M)$ by the Hodge decomposition, since $d_A^2 = 0$. Applying the brackets to φ, by a straightforward computation, we get

$$[\Delta_0, iS_\infty] = -\left(4\xi'\partial_r^2 + 4\xi''\partial_r + \xi''' - 2V_p'\right)\tilde{\chi} \otimes P_0\,. \tag{5.26}$$

By linearity and density, we get $\|[\Delta_0, iS_\infty]\varphi\| \le C\|(\Delta_0+i)\varphi\|$ for all $\varphi \in \mathcal{C}_c^\infty(X_0')$. Take now $\varphi \in \mathcal{C}_c^\infty(X_0)$, considering the support of the commutator, we get

$$\begin{aligned}\|[\Delta_0, iS_\infty]\varphi\| &= \|[\Delta_0, iS_\infty]\tilde{\tilde{\Xi}}\varphi\| \\ &\le C\|(\Delta_0+i)\varphi\| + \|[\Delta_0, \tilde{\tilde{\Xi}}]\varphi\| \le C'\|(\Delta_0+i)\varphi\|\end{aligned}$$

where $\tilde{\tilde{\Xi}} \in \mathcal{C}^\infty(X)$ with support in X_0' such that $\tilde{\tilde{\Xi}}|_{[1,\infty)\times M} = 1$. Therefore, since $\mathcal{C}_c^\infty(X_0)$ is a core for Δ_0, we conclude $[\Delta_0, iS_\infty]_0 \in \mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0)$.

In the same way we compute

$$\begin{aligned}\big[[\Delta_0, iS_\infty], iS_\infty\big] = -\Big(&\big(16(\xi')^2 - 8\xi\xi''\big)\partial_r^2 + (24\xi'\xi'' - 8\xi\xi''')\partial_r \\ &+ 4(\xi'')^2 + 4\xi''' - 2\xi'''' - 4V_p''\Big)\tilde{\chi} \otimes P_0\end{aligned} \tag{5.27}$$

and get $[[\Delta_0, iS_\infty], iS_\infty]_0 \in \mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0)$. For $p = 1$, the boundedness of higher commutators follows easily by induction ($V_p = 0$ in this case).

We compute next the commutators of Δ_0 with S_R. As above we compute for $\varphi = \varphi_r \otimes \varphi_M$. For brevity, we write Φ_R instead of $\Phi_R(-i\partial_r)$.

As Φ_R is not a local operator, we first note that the commutator $[\Delta_0, S_R]$ could be taken in the operator sense. Indeed, $\tilde{\chi}$ sends φ_r to $\mathcal{C}_c^\infty(\mathbb{R})$ (note that $[1/2,\infty)$ is injected in a canonical way into $\mathbb{R}$), then $\Phi_R\xi + \xi\Phi_R$ sends to the Schwartz space $\mathscr{S}(\mathbb{R})$ and finally $\tilde{\chi}$ sends to $\tilde{\chi}\mathscr{S}(\mathbb{R})$ which belongs to $\mathcal{D}(\Delta_0)$.

We compute $[\partial_r^2, \tilde{\chi}(\Phi_R\xi + \xi\Phi_R)\tilde{\chi}] \otimes P_0$. Against $\varphi_r \otimes \varphi_M$, we have:

$$\begin{aligned}[\partial_r^2, \tilde{\chi}\Phi_R\xi + \xi\Phi_R\tilde{\chi}] &= \Xi[\partial_r^2, \tilde{\chi}\Phi_R\xi + \xi\Phi_R\tilde{\chi}]\Xi = \Xi[\partial_r^2, \Phi_R r + r\Phi_R]\Xi + \Psi_{\mathrm{comp}} \\ &= 4\Xi\partial_r\Phi_R\Xi + \Psi_{\mathrm{comp}} = 4\tilde{\chi}\partial_r\Phi_R\tilde{\chi} + \Psi_{\mathrm{comp}}\,,\end{aligned} \tag{5.28}$$

where Ψ_{comp} denotes a pseudo-differential operator with compact support such that its support in position is in the interior of X_0'. For $p < 1$, the potential part V_p arises. We treat its first commutator:

$$\begin{aligned}[V_p, \tilde{\chi}\Phi_R\xi + \xi\Phi_R\tilde{\chi}] &= \tilde{\Xi}[V_p, 2\Phi_R r - i\Phi_R']\tilde{\Xi} + \Psi_{\mathrm{comp}} \\ &= \tilde{\chi}\big(2[V_p, \Phi_R]\xi - i[V_p, \Phi_R']\big)\tilde{\chi} + \Psi_{\mathrm{comp}}\,,\end{aligned} \tag{5.29}$$

where $\Phi_R' = \Phi_R'(-i\partial_r)$. Applying Lemma 5.11, we get that $[V_p, \Phi_R]\xi$ and $[V_p, \Phi_R']$ are bounded in $L^2(\mathbb{R})$ also. Therefore, using like above $\tilde{\Xi}$, we get $\|[\Delta_0, iS_R]\varphi\| \le C\|\varphi\|$ for all $\varphi \in \mathcal{C}_c^\infty(X_0)$. This implies that $[\Delta_0, iS_R]_0 \in \mathcal{B}(\mathscr{H}_0)$.

For higher commutators, the n-th commutator with $\tilde{\chi}\Phi_R\xi + \xi\Phi_R\tilde{\chi}$ is given by $2^n\tilde{\chi}\partial_r^n\Phi_R\tilde{\chi} + \Psi_{\mathrm{comp}}$. Note that $\partial_r^n\Phi_R$ is a compactly supported function of ∂_r, so the contribution of this term is always bounded.

Consider now the second commutator of V_p. As above, since we work up to $\tilde{\Xi}$, $\tilde{\chi}$ and Ψ_{comp}, it is enough to show that the next commutator defined on $\mathscr{S}(\mathbb{R})$

extend to bounded operators in $L^2(\mathbb{R})$. We treat only the most singular part of the second commutator:

$$\begin{aligned}\big[[V_p,\Phi_R]r,\Phi_R' r\big] &= \big[[V_p,\Phi_R],\Phi_R r\big]r + [V_p,\Phi_R][r,\Phi_R]r\\ &= \big[[V_p,\Phi_R],\Phi_R\big]r^2 + \Phi_R\big[[V_p,\Phi_R],r\big]r - i[V_p,\Phi_R]\Phi_R' r\\ &= \big[[V_p,\Phi_R],\Phi_R\big]r^2 - \Phi_R\big[[r,\Phi_R],V_p\big]r - i[V_p,\Phi_R][\Phi_R',r]\\ &\quad - i[V_p,\Phi_R]r\Phi_R'\\ &= \big[[V_p,\Phi_R],\Phi_R\big]r^2 + i\Phi_R[\Phi_R',V_p]r + [V_p,\Phi_R]\Phi_R'' - i[V_p,\Phi_R]r\Phi_R'.\end{aligned}$$

These terms extend to bounded operators by Lemma 5.11. □

The following lemma is the key-stone for the Mourre estimate.

Lemma 5.8. *For all $R \in [1,\infty]$, there exists $K \in \mathcal{K}(\mathscr{H}_0^2,\mathscr{H}_0^{-2})$ and $N_R \in \mathcal{B}(\mathscr{H}_0^2,\mathscr{H}_0^{-2})$ such that*

$$\begin{aligned}\langle \Delta_0\varphi, iS_R\varphi\rangle + \langle iS_R\varphi,\Delta_0\varphi\rangle = 4\Big\langle\varphi,\big(\Delta_0 - \inf\big(\sigma_{\mathrm{ess}}(\Delta_0)\big)\varphi\Big\rangle&\\ + \langle\varphi,N_R\varphi\rangle + \langle\varphi,K\varphi\rangle,&\end{aligned}\tag{5.30}$$

for all $\varphi\in\mathcal{C}_c^\infty(X_0)$ and such that $\|N_R\|_{\mathcal{B}(\mathscr{H}_0^2,\mathscr{H}_0^{-2})}$ tends to 0 as R goes to infinity.

Proof. First note that the essential spectrum of Δ_0 is $[V_p(\infty),\infty)$, by Proposition 6.2. We now act in three steps. Let Ξ be like in the proof of Lemma 5.7 and let $\varphi\in\mathcal{C}_c^\infty(X_0)$. Since $\tilde\chi$ and Ξ have disjoint supports, one has for all R that $\langle\varphi,[\Delta_0,iS_R]\varphi\rangle = \langle\widetilde\Xi\varphi,[\Delta_0,iS_R]\widetilde\Xi\varphi\rangle$.

For the first step, we start with $R=\infty$. By (5.26) and since $\xi'=1$ on $[2,\infty)$, the Rellich–Kondrakov lemma gives

$$\big\langle\widetilde\Xi\varphi,[\Delta_0,iS_\infty]\widetilde\Xi\varphi\big\rangle = \Big\langle\widetilde\Xi\varphi,4\big(\Delta_0 - V_p(\infty)\big)(1\otimes P_0)\widetilde\Xi\varphi\Big\rangle + \langle\varphi,K_1\varphi\rangle \tag{5.31}$$

for a certain $K_1\in\mathcal{K}(\mathscr{H}_0^2,\mathscr{H}_0^{-2})$. Indeed, $1-\xi', V_p - V_p(\infty), \xi'', \xi''', V_p'$ belong to $\mathcal{K}(\mathscr{H}_0^1,\mathscr{H}_0)$ since they tend to 0 at infinity.

We consider now R finite. We add (5.28) and (5.29). We have

$$\big\langle\widetilde\Xi\varphi,[\Delta_0,iS_R]\widetilde\Xi\varphi\big\rangle = \Big\langle\widetilde\Xi\varphi,4\big(\Delta_0 - V_p(\infty) - T_R\big)(1\otimes P_0)\widetilde\Xi\varphi\Big\rangle + \langle\varphi,K_2\varphi\rangle \tag{5.32}$$

for a certain $K_2 = K_2(R)\in\mathcal{K}(\mathscr{H}_0^2,\mathscr{H}_0^{-2})$ and with $T_R = \partial_r(\partial_r - \Phi_R(\partial_r))$. The compactness of K_2 follows by noticing that $L_0^{-1}\in\mathcal{K}(\mathscr{H}_0^2,\mathscr{H}_0)$ and that $L_0[V_p,iS_R]_0\in\mathcal{B}(\mathscr{H}_0,\mathscr{H}_0)$, by Lemma 5.11. We control the size of T_R by showing that $\|\widetilde\Xi T_R(1\otimes P_0)\widetilde\Xi\|_{\mathcal{B}(\mathscr{H}_0^2,\mathscr{H}_0^{-2})}$ tends to 0 as R goes to infinity. By Lemma 5.3, one has that $\widetilde\Xi$ stabilizes $\mathscr{H}_0^{\pm2}$, therefore $-\partial_r^2\widetilde\Xi$ belongs to $\mathcal{B}(\mathscr{H}_0^2,L^2(\mathbb{R}))$. It remains to note that $(-\partial_r^2+i)^{-2}T_R$ tends to 0 in norm by functional calculus, as R goes to infinity.

The second step is to control the high energy functions part. Consider the Friedrichs extension of Δ_0 on $\mathscr{H}_{0,h} := L^2([1/2,\infty))\otimes P_0^\perp L^2(M)$. We have:

$$\big\langle\widetilde\Xi\varphi,\Delta_0P_0^\perp\widetilde\Xi\varphi\big\rangle = \big\langle(\Delta_0P_0^\perp+i)^{-1}(\Delta_0P_0^\perp+i)\widetilde\Xi\varphi,\Delta_0P_0^\perp\widetilde\Xi\varphi\big\rangle = \langle\varphi,K_1\varphi\rangle \tag{5.33}$$

where $K_1 \in \mathcal{K}(\mathscr{H}_0^2, \mathscr{H}_0^{-2})$. Indeed, note first that $\Delta_0 P_0^\perp \tilde{\chi} \in \mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_{0,h})$ and that $(\Delta_0 P_0^\perp + i)^{-1} \in \mathcal{K}(\mathscr{H}_{0,h})$, since Q_p in (5.22) goes to infinity. Therefore the left hand side belongs $\mathcal{K}(\mathscr{H}_0^2, \mathscr{H}_0)$ and the right hand side belongs to $\mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0)$.

The third step is to come back on the whole manifold. It is enough to note that $[\Delta_0, \tilde{\chi}] \in \mathcal{K}(\mathscr{H}_0^1, \mathscr{H}_0)$ and to add (5.31), (5.32) with (5.33). $\square$

We now turn to the regularity assumptions. Lemma A.2 plays a central rôle.

Lemma 5.9. *For $R \in [1, \infty]$, one has $\Delta_0 \in \mathcal{C}^1(S_R)$ and $e^{itS_R}\mathcal{D}(\Delta_0) \subset \mathcal{D}(\Delta_0)$.*

Proof. We start by showing that $\Delta_0 \in \mathcal{C}^1(S_R)$. We check the hypothesis of Lemma A.2. Let $\chi_n(r) := \chi(r/n)$ and $\mathscr{D} = \mathcal{C}_c^\infty(X_0)$. Remark that $\operatorname{supp}(\chi_n') \subset [n, 2n]$ and that $\xi\chi_n^{(k)}$ tends strongly to 0 on $L^2(\mathbb{R}^+)$, for any $k \geq 1$. By the uniform boundedness principle, this implies that $\sup_n \|\chi_n\|_{\mathcal{D}(H)}$ is finite.

Remark 5.4 and Lemma 5.5 give that $\mathscr{D}$ is a core for S_R. Assumption (1) is obvious, assumption (2) holds since $(1-\chi_n)$ has support in $[2n, \infty)$ and assumption (3) follows from the fact that H is elliptic, so the resolvent of Δ_0 sends $\mathscr{D}$ into $\mathcal{C}^\infty(X_0)$. The point (A.6) follows from Lemma 5.8. We now show that (A.5) is true. Let $\phi \in \mathcal{C}^\infty(X_0) \cap \mathcal{D}(\Delta_0)$. We have $[\Delta_0, \chi_n]\phi = [\Delta_0, \chi_n]\tilde{\chi}\phi = 2\chi_n' \partial_r \tilde{\chi}\phi + \chi_n'' \tilde{\chi}\phi$. We have $iS_\infty[\Delta_0, \chi_n]\phi = 2\xi\chi_n'\partial_r^2 P_0 \tilde{\chi}\phi + (4\chi\xi_n'' + \xi'\chi_n')\partial_r P_0\tilde{\chi}\phi + (2\xi\chi_n''' + \xi'\chi_n'')P_0\tilde{\chi}\phi$ and, for a finite R, we get $iS_R[\Delta_0, \chi_n]\phi = \tilde{\chi}(2\Phi_R(\partial_r)\xi + [\xi, \Phi_R(\partial_r)])(2\chi_n'\partial_r P_0\tilde{\chi}\phi + \chi_n'' P_0 \tilde{\chi}\phi)$. Both terms are tending to 0 because of the previous remark, Lemma 5.3 and the fact that $[\xi, \Phi_R(\partial_r)]$ is bounded by Lemma 5.11. From that, we can apply the lemma and obtain $H \in \mathcal{C}^1(S_R)$.

By Lemma 5.7, we have that $[\Delta_0, iS_R]_0 \in \mathcal{B}(\mathscr{H}_0^2, \mathscr{H}_0)$ and [13, Lemma 2] gives that $e^{itA}\mathscr{H}_0^2 \subset \mathscr{H}_0^2$. $\square$

The invariance of the domain under the group e^{itS_R} implies that $e^{itS_R}\mathscr{H}_0^s \subset \mathscr{H}_0^s$ for $s \in [-2, 2]$ by duality and interpolation. This allows one to define the class $\mathcal{C}^k(S_R, \mathscr{H}_0^s, \mathscr{H}_0^{-s})$ for $s \in [-2, 2]$, for instance; we recall that a self-adjoint operator H is in this class if $t \mapsto e^{itS_R} H e^{-itS_R}$ is strongly C^k from $\mathscr{H}_0^s$ to $\mathscr{H}_0^{-s}$.

Lemma 5.10. *Let $R \in [1, \infty]$. Then Δ_0 belongs to $\mathcal{C}^2(S_R, \mathscr{H}_0^2, \mathscr{H}_0)$ for $p \leq 1$.*

Proof. From Lemma 5.7, the commutators with S_R extend to bounded operators from $\mathscr{H}_0^2$ to $\mathscr{H}_0$. $\square$

We finally give an estimation of commutator that we have used above.

Lemma 5.11. *Let $f \in \mathcal{C}^0(\mathbb{R})$ with polynomial growth, $\Phi_j \in \mathcal{C}_c^\infty(\mathbb{R})$ and $g \in \mathcal{C}^k(\mathbb{R})$ with bounded derivatives. Let $k \geq 1$. Assume that $\sup_{t\in\mathbb{R}, |s-t|\leq 1} |f(t) g^{(l)}(s)| < \infty$, for all $1 \leq l \leq k$. Then the operator $f[\Phi_1(-i\partial_r), [\Phi_2(-i\partial_r) \ldots [\Phi_k(-i\partial_r), g] \ldots]$, defined on $\mathcal{C}_c^\infty(\mathbb{R})$, extends also to a bounded operator.*

Proof. Take $k = 1$. We denote with a hat the unital Fourier transform. We get

$$\Big(f\big[\Phi(-i\partial_r), g\big]\varphi\Big)(t) = \frac{1}{2\pi}\int \widehat{\Phi}(s-t) f(t)\big(g(s) - g(t)\big)\varphi(s)ds\,,$$

for $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$. In order to show that the L^2 norm of the left hand side is uniformly bounded by $\|f\|_2$, we separate the integral in $|s-t|$ lower and bigger than one. We start with the first part. Recalling $t^k\widehat{\Phi}(t) = (-1)^k\widehat{\Phi^{(k)}}(t)$, Φ is replaced by Φ' when we divide by $|s-t|$ the term with g. Now since $\widehat{\Phi'} \in L^1$, $\sup_{t\in\mathbb{R},|s-t|\le 1} |f(t)(g(s)-g(t))/(s-t)|$ is finite, and the convolution by a L^1 function is bounded in L^2, we control this part of the integral. We turn to the part $|s-t| \geq 1$. Let $R \in \mathbb{N}$ such that $|f(t)| \le C(1+|t|^R)$. We let appear the Fourier transform of $\Phi^{(R+1)}$. Now to conclude, note that $\sup_{s,t\in\mathbb{R}} |(g(s)-g(t))/(s-t)|$ is finite and that $t \mapsto |f(t)/(s-t)^R|$ is in L^∞ uniformly in s. For higher k, one repeats the same decomposition and let appear the terms in the l-th derivative of g by regrouping terms. □

5.4. A short-range and long-range class of perturbations

In the early versions of Mourre theory, one asked $[[H, S_R], S_R]$ to be $H-$bounded to obtain refined results of the resolvent like the limiting absorption principle and the Hölder regularity of the resolvent. In this section, we check the optimal class of regularity $\mathcal{C}^{1,1}(S_R)$, for R finite. This is a weak version of the two-commutators hypothesis. We refer to [2] for definition and properties. This is the optimal class of operators which give a limiting absorption principle for H in some optimal Besov spaces associated to the conjugate operator S_R.

The operator Δ_0 belongs to $\mathcal{C}^2(S_R)$ by Lemma 5.10 and therefore also to $\mathcal{C}^{1,1}(S_R)$. We now consider perturbations of Δ_0 which are also in $\mathcal{C}^{1,1}(S_R)$. We define two classes.

Consider a symmetric differential operator $T : \mathcal{D}(\Delta_0) \to \mathcal{D}(\Delta_0)^*$. Take $\theta_{\mathrm{sr}} \in \mathcal{C}_c^\infty((0,\infty))$ not identically 0; V is said to be *short-range* if

$$\int_1^\infty \left\| \theta_{\mathrm{sr}}\left(\frac{L_0}{r}\right) T \right\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)} dr < \infty \tag{5.34}$$

and to be *long-range* if

$$\begin{aligned}\int_1^\infty \left\| [T, L_0]\theta_{\mathrm{lr}}\left(\frac{L_0}{r}\right)\right\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)} + \left\|\tilde{\Xi}[T, P_0]L_0\theta_{\mathrm{lr}}\left(\frac{L_0}{r}\right)\tilde{\Xi}\right\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)}\\ + \left\|\tilde{\Xi}[T, \partial_r]P_0L_0\theta_{\mathrm{lr}}\left(\frac{L_0}{r}\right)\tilde{\Xi}\right\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)} \frac{dr}{r} < \infty \end{aligned}\tag{5.35}$$

where θ_{lr} is the characteristic function of $[1,\infty)$ in $\mathbb{R}$ and where $\widetilde{\Xi} \in \mathcal{C}^\infty(X)$ with support in X_0' such that $\widetilde{\Xi}|_{[1,\infty)\times M} = 1$.

The first condition is evidently satisfied if there is $\varepsilon > 0$ such that

$$\|L_0^{1+\varepsilon}T\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)} < \infty \tag{5.36}$$

and the second one if

$$\begin{aligned}\|L_0^{\varepsilon}[T, L_0]\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)} + \|L_0^{1+\varepsilon}\widetilde{\Xi}[T, P_0]\widetilde{\Xi}\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)}\\ + \|L_0^{1+\varepsilon}\widetilde{\Xi}[T, \partial_r]P_0\widetilde{\Xi}\|_{\mathcal{B}(\mathcal{D}(\Delta_0),\mathcal{D}(\Delta_0)^*)} < \infty\end{aligned} \tag{5.37}$$

The condition with P_0 essentially tells that the non-radial part of T is a short-range perturbation. This is why we will ask the long-range perturbation to be radial. To show that the first class is in $\mathcal{C}^{1,1}(S_R)$ for R finite, one use [2, Theorem 7.5.8]. The hypotheses are satisfied thanks to Lemmata 5.5 and 5.14. Concerning the second class, one shows that $[T, S_R] \in \mathcal{C}^{0,1}(S_R)$ by using [2, Proposition 7.5.7] (see the proof of [2, Proposition 7.6.8] for instance).

We go back to the x coordinate. For $\mathscr{G} = \mathcal{D}(\Delta^{1/2})$, the short and long-range perturbation of the electric and magnetic perturbations are given by:

Lemma 5.12. *Let $V \in L^\infty(X)$ and $\tilde{A} \in L^\infty(X, T^*X)$. If $\|L^{1+\varepsilon}\tilde{A}\|_\infty < \infty$ (respectively $\|L^{1+\varepsilon}V\|_\infty < \infty$) then the perturbation $(d^*_{A_{\mathrm{f}}}(i\tilde{A}\wedge) + (i\tilde{A}\wedge)^* d_{A_{\mathrm{f}}})$ (resp. V) is short-range in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$. If $\tilde{A}$ is radial, $\|L^\varepsilon \tilde{A}\| < \infty$ and $\|L^{1+\varepsilon}x^{2-p}\partial_x \tilde{A}\| < \infty$, where these norms are in $\mathcal{B}(L^2(X, g), L^2(X, \Lambda^1, g))$ (respectively V radial, and $\|L^{1+\varepsilon}x^{2-p}\partial_x V\|_\infty < \infty$) then the same perturbation is long-range in $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$.*

Proof. We deal with the magnetic perturbation. Start with the short-range. We have $\langle L^{1+\varepsilon}(i\tilde{A}\wedge)^* d_{A_{\mathrm{f}}})f\rangle = \langle d_A f, L^{1+\varepsilon}(i\tilde{A}\wedge)f\rangle$ and on the other hand, we have $\langle L^{1+\varepsilon}d^*_{A_{\mathrm{f}}}(i\tilde{A}\wedge)f, f\rangle = \langle [L^{1+\varepsilon}, d^*_{A_{\mathrm{f}}}]L^{-\varepsilon}L^\varepsilon(i\tilde{A}\wedge)f, f\rangle + \langle L^{1+\varepsilon}(i\tilde{A}\wedge)f, d_A f\rangle$. This is bounded by $\|f\| + \|d_A f\|^2$ uniformly in $f \in \mathcal{C}_c^\infty(X)$.

We deal now with the long-range perturbation by checking (5.37). The condition with L^ε is treated as above. In the variable of the free metric g_p (5.17), ∂_r is given by $\partial_L := x^{2-p}\partial_x - (n-1)p/2\, x^{1-p}$. We extend ∂_L on 1-forms by setting $\partial_L = x^{2-p}\partial_x - (n+1)p/2\, x^{1-p}$. Note it is symmetric on 1-forms with compact support on the cusp. First, we have on smooth functions with compact support on the cusp that:

$$[d_{A_{\mathrm{f}}}, \partial_L]P_0 = \big(2(1-p)x^{1-p}d_{A_{\mathrm{f}}} + c(1-p)x^{2(1-p)}x^{p-2}dx \wedge\big)P_0 \,. \tag{5.38}$$

Here, we used $d_{A_{\mathrm{f}}}P_0 = (dx\wedge\partial_x\cdot)P_0$. Note that $x^{p-2}dx\wedge$ is a bounded operator from function to 1-forms. In the following, we drop P_0 and Ξ to lighten the notation. For $\varphi \in \mathcal{C}_c^\infty(X)$, we have

$$\begin{aligned}\Big\langle L^{1+\varepsilon}\varphi, \big[d^*_{A_{\mathrm{f}}}(i\tilde{A}\wedge) + (i\tilde{A}\wedge)^* d_{A_{\mathrm{f}}}\big]\varphi\Big\rangle &= \big\langle [\partial_L, d_{A_{\mathrm{f}}}]L^{1+\varepsilon}\varphi, \tilde{A}\wedge\varphi\big\rangle \\ &\quad + \big\langle \tilde{A}\wedge L^{1+\varepsilon}\varphi, [d_{A_{\mathrm{f}}}, \partial_L]\varphi\big\rangle \\ &\quad + 1\big\langle d_{A_{\mathrm{f}}}L^{1+\varepsilon}\varphi, [\tilde{A}\wedge, \partial_L]\big\rangle \\ &\quad + \big\langle [\partial_L, \tilde{A}\wedge]L^{1+\varepsilon}\varphi, d_{A_{\mathrm{f}}}\varphi\big\rangle \,.\end{aligned}$$

Once $d_{A_{\mathrm{f}}}$ commuted with $L^{1+\varepsilon}$, the two last terms are controlled by the assumption on $\|[\partial_L, \tilde{A}\wedge]L^{1+\varepsilon}\|$. The two first ones are 0 for $p = 1$ using (5.38). When $p < 1$, note that $x^{1-p}L^{1+\varepsilon} = cL^\varepsilon$ and control the term using $L^\varepsilon\tilde{A}$ bounded. □

We now describe the perturbation of the metric following the two classes. We keep the notation from Theorem 6.4. We introduce the canonical unitary transformation due to the change of measure. Set ρ to be ρ_{sr}, ρ_{lr} or ρ_{t}. Let U be the operator of multiplication by $(1+\rho)^{-n/4}$ in $L^2(X, g)$ and V the operator of multiplication by $(1+\rho)^{(2-n)/4}$ in $L^2(X, T^*X, g)$. The operator U is a unitary operator from $L^2(X,g)$ onto $L^2(X,\tilde{g})$ and V a unitary operator from $L^2(X,T^*X,g)$ onto $L^2(X,T^*X,\tilde{g})$.

Lemma 5.13. *Let $\widetilde{\Delta}_{A_{\mathrm{f}}}$ be the magnetic Laplacian of vector potential A_{f} acting in $L^2(X,\tilde{g})$. Let $W_0 := U^{-1}\widetilde{\Delta}_{A_{\mathrm{f}}}U$. Then,*

1. *On $\mathcal{C}_c^\infty(X)$, W_0 acts by $Ud^*_{A_{\mathrm{f}}}V^2 d_{A_{\mathrm{f}}}U$. In $L^2(X,g)$, it is essentially self-adjoint and its domain is $\mathcal{D}(\Delta_{A_{\mathrm{f}}})$.*
2. *One has $(W_0-\Delta_{A_{\mathrm{f}}}+i)^{-1}-(\Delta_{A_{\mathrm{f}}}+i)^{-1}$ is compact.*
3. *For ρ_{sr} and ρ_{t}, W_0 is a short-range perturbation of the magnetic Laplacian $\Delta_{A_{\mathrm{f}}}$ in the space $\mathcal{B}(\mathcal{D}(\Delta_{A_{\mathrm{f}}}),\mathcal{D}(\Delta_{A_{\mathrm{f}}})^*)$.*
4. *For ρ_{lr}, W_0 is a long-range perturbation of $\Delta_{A_{\mathrm{f}}}$ in $\mathcal{B}(\mathcal{D}(\Delta_{A_{\mathrm{f}}}),\mathcal{D}(\Delta_{A_{\mathrm{f}}})^*)$.*

Proof. We write $\widetilde{\Delta}_{A_{\mathrm{f}}}$ with the help of the operator $d_{A_{\mathrm{f}}}$. Since the manifold is complete, $\widetilde{\Delta}_{A_{\mathrm{f}}}$ is essentially self-adjoint on $\mathcal{C}_c^\infty(X)$. In particular it corresponds to $\tilde{d}^*_{A_{\mathrm{f}}}\tilde{d}_{A_{\mathrm{f}}}$, the Friedrichs extension. This is equal to $U^2d^*_{A_{\mathrm{f}}}V^2d_{A_{\mathrm{f}}}$ in $L^2(X,\tilde{g})$. Now remark that $(1+\rho)^\alpha$ stabilizes $\mathcal{D}(\Delta_{A_{\mathrm{f}}})$, $\mathcal{D}(d_{A_{\mathrm{f}}})$ and $\mathcal{D}(d^*_{A_{\mathrm{f}}})$, for all $\alpha\in\mathbb{R}$ to obtain the first point.

We now compare the two operators in $L^2(X,g)$. We compute on $\mathcal{C}_c^\infty(X)$.

$$\begin{aligned} D := W_0-\Delta_{A_{\mathrm{f}}} &= U^{-1}U^2d^*_{A_{\mathrm{f}}}V^2d_{A_{\mathrm{f}}}U - d^*_{A_{\mathrm{f}}}d_{A_{\mathrm{f}}}\\ &= U^{-1}\widetilde{\Delta}_{A_{\mathrm{f}}}(U-1)+Ud^*_{A_{\mathrm{f}}}(V^2-1)d_{A_{\mathrm{f}}}+(U-1)\Delta_{A_{\mathrm{f}}}\,. \end{aligned}\tag{5.39}$$

We focus on point (3). The two first terms need a justification. We start with the first term.

$$L^{1+\varepsilon}U^{-1}\widetilde{\Delta}_{A_{\mathrm{f}}}(U-1) = U^{-1}\big(\widetilde{\Delta}_{A_{\mathrm{f}}}+[L^{1+\varepsilon},\widetilde{\Delta}_{A_{\mathrm{f}}}]L^{-1-\varepsilon}\big)L^{1+\varepsilon}(U-1)\,.$$

Using Lemma 5.14 and the invariance of the domain under $(1+\rho)^\alpha$, we obtain that $(\widetilde{\Delta}_{A_{\mathrm{f}}}+[L^{1+\varepsilon},\widetilde{\Delta}_{A_{\mathrm{f}}}]L^{-\varepsilon})^*$ is bounded from $\mathcal{D}(\Delta_{A_{\mathrm{f}}})$ to $L^2(X,g)$. Again using properties of $(1+\rho)^\alpha$, for all $\varphi\in\mathcal{C}_c^\infty(X)$ we get

$$\begin{aligned} &\big|\big\langle\varphi, L^{1+\varepsilon}U^{-1}\widetilde{\Delta}_{A_{\mathrm{f}}}(U-1)\varphi\big\rangle\big|\\ &\quad = \Big|\Big\langle\big(\widetilde{\Delta}_{A_{\mathrm{f}}}+[L^{1+\varepsilon},\widetilde{\Delta}_{A_{\mathrm{f}}}]L^{-1-\varepsilon}\big)^*U^{-1}\varphi, L^{1+\varepsilon}(U-1)\varphi\Big\rangle\Big| \le c\|(\Delta_{A_{\mathrm{f}}}+i)\varphi\|^2\,. \end{aligned}$$

For the second term, we have $L^{1+\varepsilon}(Ud^*_{A_{\mathrm{f}}}(V^2-1)d_{A_{\mathrm{f}}}) = U(d^*_{A_{\mathrm{f}}}+[L^{1+\varepsilon},d^*_{A_{\mathrm{f}}}]L^{-1-\varepsilon})$ $L^{1+\varepsilon}(V^2-1)d_{A_{\mathrm{f}}}$. By Lemma 5.14 and the invariance of $\mathcal{D}(d_{A_{\mathrm{f}}})$ by ρ^α, we obtain

$$\begin{aligned} &\Big|\Big\langle\varphi, L^{1+\varepsilon}\big(Ud^*_{A_{\mathrm{f}}}(V^2-1)d_{A_{\mathrm{f}}}\big)\varphi\Big\rangle\Big|\\ &\qquad = \Big|\Big\langle\big(d_{A_{\mathrm{f}}}+L^{-1-\varepsilon}[d_{A_{\mathrm{f}}},L^{1+\varepsilon}]\big)U\varphi, L^{1+\varepsilon}(V^2-1)d_{A_{\mathrm{f}}}\varphi\Big\rangle\Big|\\ &\qquad \le c\|(\Delta_{A_{\mathrm{f}}}+i)\varphi\|^2\,, \end{aligned}$$

for all $\varphi\in\mathcal{C}_c^\infty(X)$. To finish, use the fact that $\mathcal{C}_c^\infty(X)$ is a core for $\Delta_{A_{\mathrm{f}}}$.

We now deal with (4) by checking (5.37). The real point to check is that $\|L^{1+\varepsilon}\tilde{\Xi}[D,\partial_L]P_0\tilde{\Xi}\|_{\mathcal{B}(\mathcal{D}(\Delta),\mathcal{D}(\Delta)^*)}$ is finite. We take ∂_L like in the proof of Lemma 5.12. First,

$$[d^*d,\partial_L]P_0 = c(1-p)x^{3(1-p)}P_0\,.\tag{5.40}$$

We start with the easy part of D. We drop $\tilde{\Xi}$ and P_0 for clarity. We have:

$$\begin{aligned}\Big\langle L^{1+\varepsilon}\varphi, \big[(U-1)d^*d, \partial_L\big]\varphi\Big\rangle \\ = \big\langle L^{1+\varepsilon}\varphi, [U,\partial_L]d^*d\varphi\big\rangle + \big\langle L^{1+\varepsilon}(U-1)\varphi, [d^*d,\partial_L]\varphi\big\rangle\,.\end{aligned}$$

This is bounded by $\|(\Delta+i)\varphi\|^2$. Indeed, the first term follows since $L^{1+\varepsilon}[U,\partial_L]$ is bounded in $L^2(X,g)$. The second one is 0 for $p=1$ and equals otherwise to $c\langle\varphi, L^{\varepsilon-2}(U-1)\varphi\rangle$ by (5.40). Turn now to:

$$\begin{aligned}\Big\langle L^{1+\varepsilon}\varphi, \big[Ud^*(V^2-1)d, \partial_L\big]\varphi\Big\rangle &= \big\langle dUL^{1+\varepsilon}\varphi, [V^2,\partial_L]d\varphi\big\rangle \\ &\quad + \big\langle [U,\partial_L]L^{1+\varepsilon}\varphi, d^*(V^2-1)d\varphi\big\rangle \\ &\quad + \big\langle (V^2-1)dUL^{1+\varepsilon}\varphi, [d,\partial_L]\varphi\big\rangle \\ &\quad + \big\langle [d,\partial_L]UL^{1+\varepsilon}\varphi, (V^2-1)d\varphi\big\rangle\,.\end{aligned}$$

The first is controlled by commuting $L^{1+\varepsilon}$ with d like above and by using that $L^{1+\varepsilon}[V^2,\partial_L]$ is bounded in $L^2(X,\Lambda^1,g)$. For the second one, $[U,\partial_L]L^{1+\varepsilon}$ is bounded in $L^2(X,\Lambda^1,g)$. Turn the two last ones and use (5.40), for $p=1$ this is 0. Focus on the very last one for example. Now commute $L^{1+\varepsilon}$ with d like above. The most singular term being $\langle dU\varphi, x^{1-p}L^{1+\varepsilon}(V^2-1)d\varphi\rangle$. Now remember that $x^{1-p}L^{1+\varepsilon} = cL^{\varepsilon}$ and use the fact that $L^{\varepsilon}(V^2-1)$ is bounded to control it. To conclude, repeat the same arguments for $[U^{-1}\tilde{\Delta}(U-1),\partial_L]$.

We turn to point (2), W_0 and Δ_{A_f} have the same domain. We take the proof of Lemma 6.3 replacing $\mathcal{G}$ with this domain. We then obtain a rigorous version of (6.42). Therefore, it remains to check that $W_0 - \Delta_{A_\mathrm{f}} \in \mathcal{K}(\mathcal{D}(\Delta_{A_\mathrm{f}}), \mathcal{D}(\Delta_{A_\mathrm{f}})^*)$. This comes directly using Rellich–Kondrakov lemma and (5.39). □

Finally, we gather various technicalities concerning the operator L.

Lemma 5.14. *We have that dL is with support in $(0,\varepsilon)\times M_{\mathrm{nt}}$ and $dL = f(x)dx$ where $f:(0,\varepsilon)\to\mathbb{R}$ such that f is 0 in a neighborhood of ε and such that $f(x) = -x^{p-2}$ for x small enough. Moreover:*

1. *The operator $L^{-\varepsilon}d(L^{1+\varepsilon})\wedge$ belongs to $\mathcal{B}(L^2(X,g), L^2(X,T^*X,g))$ and the commutator $L^{-\varepsilon}[\Delta_{A_\mathrm{f}}, L^{1+\varepsilon}]$ with initial domain $\mathcal{C}_c^\infty(X)$ extends to a bounded operator in $\mathcal{B}(\mathcal{D}(\Delta_{A_\mathrm{f}}), L^2(X,g))$.*
2. *$e^{itL}\mathcal{D}(\Delta_{A_\mathrm{f}}) \subset \mathcal{D}(\Delta_{A_\mathrm{f}})$ and $\|e^{itL}\|_{\mathcal{B}(\mathcal{D}(\Delta_{A_\mathrm{f}}))} \le c(1+t^2)$.*
3. *$L^{-1-\varepsilon}\mathcal{D}(\Delta_{A_\mathrm{f}}) \subset \mathcal{D}(\Delta_{A_\mathrm{f}})$.*

Proof. With the diagonalization of Section 5.2, the operator Δ_{A_f} is given by (5.22). The operator L corresponds to the operator L_0 of multiplication by $r\otimes 1_{M_{\mathrm{nt}}}$ on (c,∞) in this variable and by 1 on the rest of the manifold. Hence, points (1) and (3) are easily obtained. Moreover $e^{itL_0}/(1+t^2)$ and its first and second derivative belong to $L^2(X_0)$, uniformly in t, from which (2) follows. □

6. The non-trapping case for perturbed metrics

6.1. The essential spectrum

In this section, we compute the essential spectrum of a magnetic Laplacian given by a non-trapping vector potential. Unlike the trapping case, it is non-empty in the complete case. To show this, we apply perturbation techniques to the results of the previous section.

We restrict ourselves to conformal perturbations of *exact* cusp metrics. In a fixed product decomposition of X near M we rewrite (2.8) as

$$g_0 = a\left(\frac{dx}{x^2} + \alpha(x)\right)^2 + h(x) \tag{6.41}$$

where $a \in \mathcal{C}^\infty(\overline{X})$, $\alpha \in \mathcal{C}^\infty([0,\varepsilon) \times M, \Lambda^1(M))$ and $h \in \mathcal{C}^\infty([0,\varepsilon) \times M, S^2TM)$. By [38, Lemma 6], the function $a_0 := a(0)$, the metric $h_0 := h(0)$ and the class (modulo exact forms) of the 1-form $\alpha_0 := \alpha(0)$, defined on M, are independent of the chosen product decomposition and of the boundary-defining function x inside the fixed cusp structure.

Definition 6.1. The metric g_0 is called *exact* if $a_0 = 1$ and α_0 is an exact 1-form.

If $\alpha_0 = df$ is exact, then by replacing x with the boundary-defining function $x' = x/(1+xf)$ inside the same cusp structure, we can as well assume that $\alpha_0 = 0$ (see [38]). It follows that g_0 is quasi-isometric to a cylindrical metric near infinity.

Proposition 6.2. *Let $(X, \tilde{g}_p)$ be a Riemannian manifold with a conformal* exact *cusp metric $\tilde{g}_p := (1+\rho)g_p$, where g_0 is exact, $g_p = x^{2p}g_0$ and*

$$\rho \in L^\infty(X;\mathbb{R}), \quad \inf_{x\in X} \rho(x) > -1, \quad \rho(x) \to 0, \quad as \ \ x \to 0\,.$$

Let A be a non-trapping vector potential given by (3.12)*. Then*

- *For $0 < p \leq 1$, the Friedrichs extension of Δ_A has essential spectrum $\sigma_{\mathrm{ess}}(\Delta_A) = [\kappa(p), \infty)$, where $\kappa(p) = 0$ for $p < 1$ and $\kappa(1) = (n-1)/2$. Moreover, if ρ is smooth, the Δ_A is essential self-adjoint on $\mathcal{C}_c^\infty(X)$.*
- *If $p > 1$ and $\tilde{g}_p := g_p$ is the unperturbed metric given in* (1.1)*, then every self-adjoint extension of Δ_A has empty essential spectrum.*

Note that for $p > 1$, the unperturbed metric g_p given in (1.1) is essentially of *metric horn* type [30].

Proof. Using the Weyl theorem, Lemma 6.3 and by changing the gauge, we can suppose without loss of generality that A is of the form (5.18) with $C = 0$.

We start with the complete case. The essential self-adjointness follows from [46]. In the exact case, g_p is quasi-isometric to the metric (5.17). Using [15, Theorem 9.4] (see Theorem 9.5 for the case of the Laplacian), to compute the essential spectrum we may replace $h(x)$ in (6.41) by the metric $h_0 := h(0)$ on M, extended to a symmetric 2-tensor constant in x near M, and we may set $\rho = 0$. By Proposition C.3, computing $\sigma_{\mathrm{ess}}(\Delta_A)$ is the same as computing $\sigma_{\mathrm{ess}}(\Delta_A^l)$ on X'

of Proposition 5.1. By the results of Section 5.2, this is given on $L^2(\mathbb{R}^+)$ by $\sigma_{\mathrm{ess}}(-\Delta + V_p) = [V_p(\infty), \infty)$.

Let now $p > 1$. The metric is no longer complete and (X, g_p) is not proper; one can not apply [15, Theorem 9.4]. By Lemma B.1 and by the Krein formula, all self-adjoint extensions have the same essential spectrum. So it is enough to consider the Friedrichs extension of Δ_A. We now use Propositions 5.1 and C.3. The operator D^*D is non-negative, so the spectrum of Δ_A^0 is contained in $[\varepsilon^{2-2p}c_0, \infty)$. By Proposition C.3, the essential spectrum does not depend on the choice of ε. Now we remark that $p > 1$ implies $c_0 \neq 0$. Indeed, the equality would imply that $1/p \in \mathbb{Z}$, which is impossible. Thus by letting $\varepsilon \to 0$ we conclude that the essential spectrum is empty. □

We have used above a general lemma about compact perturbations of magnetic Laplacians:

Lemma 6.3. *Let A and A' in $L^\infty(X, T^*X)$ be two magnetic fields on a smooth Riemannian manifold (X, g) (possibly incomplete) with a measurable metric g. Suppose that $A - A'$ belongs to $L_0^\infty(X, T^*X)$. Let $\Delta_A = d_A{}^* d_A$ and $\Delta_{A'} = d_{A'}{}^* d_{A'}$. Then $(\Delta_A + i)^{-1} - (\Delta_{A'} + i)^{-1}$ is compact.*

Here, $L_0^\infty(X, T^*X)$ denotes the space of those forms of $L^\infty(X, T^*X)$ which are norm limit of compactly supported forms. Note that this lemma holds without any modification for a C^1 manifold equipped with a (RM) structure, see [15, Section 9.3]. Unlike the result on the stability of the essential spectrum of the (magnetic) Laplacian from [15, Theorem 9.5], we do not ask for the completeness of the manifold. A magnetic perturbation is much less singular than a perturbation of the metric.

Proof. Note that the form domain of Δ_A and of Δ_A is given by $\mathscr{G} := \mathcal{D}(d)$, because A and A' are in $L^\infty(X, T^*X)$. We write with a tilde the extension of the magnetic Laplacians to $\mathcal{B}(\mathscr{G}, \mathscr{G}^*)$. We aim to give a rigorous meaning to

$$(\Delta_A + i)^{-1} - (\Delta_{A'} + i)^{-1} = (\Delta_A + i)^{-1}(\Delta_{A'} - \Delta_A)(\Delta_{A'} + i)^{-1} . \tag{6.42}$$

We have $(\Delta_A + i)^{-1*}\mathscr{H} \subset \mathscr{G}$. This allows one to deduce that $(\Delta_A + i)^{-1}$ extends to a unique continuous operator $\mathscr{G}^* \to \mathscr{H}$. We denote it for the moment by R. From $R(\Delta_A + i)u = u$ for $u \in \mathcal{D}(\Delta_A)$ we get, by density of $\mathcal{D}(\Delta_A)$ in $\mathscr{G}$ and continuity, $R(\widetilde{\Delta_A + i})u = u$ for $u \in \mathscr{G}$, in particular

$$(\Delta_{A'} + i)^{-1} = R(\widetilde{\Delta_A} + i)(\Delta_{A'} + i)^{-1} .$$

Clearly,

$$(\Delta_A + i)^{-1} = (\Delta_A + i)^{-1}(\Delta_{A'} + i)(\Delta_{A'} + i)^{-1} = R(\widetilde{\Delta_{A'}} + i)(\Delta_{A'} + i)^{-1} .$$

We subtract the last two relations to get

$$(\Delta_A + i)^{-1} - (\Delta_{A'} + i)^{-1} = R(\widetilde{\Delta_{A'}} - \widetilde{\Delta_A})(\Delta_{A'} + i)^{-1}$$

Since R is uniquely determined as the extension of $(\Delta_A + i)^{-1}$ to a continuous map $\mathscr{G}^* \to \mathscr{H}$, one may keep the notation $(\Delta_A + i)^{-1}$ for it. With this convention, the rigorous version of (6.42) that we shall use is:

$$(\Delta_A + i)^{-1} - (\Delta_{A'} + i)^{-1} = (\Delta_A + i)^{-1}(\widetilde{\Delta}_{A'} - \widetilde{\Delta}_A)(\Delta_{A'} + i)^{-1} . \qquad (6.43)$$

Since $A'' = i(A - A') \in L^\infty_0(X, T^*X)$, the Rellich–Kondrakov lemma gives that $A''\wedge$ belongs to $\mathcal{K}(\mathscr{G}, L^2(X, T^*X))$. Therefore, $\widetilde{\Delta}_{A'} - \widetilde{\Delta}_A = (A''\wedge)^* d_A - d_A^* A''\wedge \in \mathcal{K}(\mathscr{G}, \mathscr{G}^*)$. This gives the announced compactness. □

6.2. The spectral and scattering theory

In this section, we refine the study of the essential spectrum given in Proposition 6.2 for non-trapping vector potential. As the essential spectrum arises only in the complete case, we will suppose that $p \leq 1$.

We give below our main result in the study of the nature of the essential spectrum and in scattering theory under short-range perturbation. It is a consequence of the Mourre theory [39] with an improvement for the regularity of the boundary value of the resolvent, see [14] and references therein.

We treat some conformal perturbation of the metric (5.17). To our knowledge, this is the weakest hypothesis of perturbation of a metric obtained so far using Mourre theory. Compared to previous approaches, we use a conjugate operator which is local in energy and therefore can be compared directly to a multiplication operator. We believe that this procedure can be implemented to all known Mourre estimates on manifolds to improve the results obtained by perturbation of the metric.

We fix A_f a non-trapping vector potential of the form (5.18). By a change of gauge, one can suppose that $A_\mathrm{f} = \theta_0$ which is constant in a neighborhood of M. Let M_t (resp. M_nt) be the union of the connected components of M on which A_f is trapping (resp. non-trapping). Let L be the operator of multiplication by a smooth function $L \geq 1$ which measures the length of a geodesic going to infinity in the directions where θ_0 is non-trapping:

$$L(x) = \begin{cases} -\ln(x) & \text{for } p = 1\,, \\ -\frac{x^{p-1}}{p-1} & \text{for } p < 1 \end{cases} \qquad (6.44)$$

on $(0, \varepsilon/4) \times M_\mathrm{nt}$ for small x, and $L = 1$ on the trapping part $\big((0, \varepsilon/2) \times M_\mathrm{nt}\big)^c$. Given $s \geq 0$, let $\mathscr{L}_s$ be the domain of L^s equipped with the graph norm. We set $\mathscr{L}_{-s} := \mathscr{L}_s^*$. Using the Riesz theorem, we obtain the scale of spaces $\mathscr{L}_s \subset L^2(X, \tilde{g}) \subset \mathscr{L}_s^*$, with dense embeddings and where $\tilde{g}$ is defined in the theorem below. Given a subset I of $\mathbb{R}$, let $I_\pm$ be the set of complex number $x \pm iy$, where $x \in I$ and $y > 0$. The thresholds $\{\kappa(p)\}$ are given in Proposition 6.2.

For shorthand, perturbations of *short-range* type (resp. *trapping* type) are denoted with the subscript sr (resp. t); they are supported in $(0, \varepsilon) \times M_\mathrm{nt}$ (resp. in $((0, \varepsilon/2) \times M_\mathrm{nt})^c$). We stress that the class of "trapping type" perturbations is also of short-range nature, in the sense described in Section 5.4, even if no decay is required. This is a rather amusing phenomenon, linked to the fact that no essential spectrum arises from the trapping cusps. The subscript lr denotes *long-range* type perturbations, also with support in $(0,\varepsilon)\times M_{\mathrm{nt}}$. We ask such perturbations to be *radial*, i.e., independent of the variables in M. In other words, a perturbation W_{lr} satisfies $W_{\mathrm{lr}}(x,m)=W_{\mathrm{lr}}(x,m')$ for all $m,m'\in M$.

Theorem 6.4. *Fix $\varepsilon>0$. Let X be endowed with the metric $\tilde g=(1+\rho_{\mathrm{sr}}+\rho_{\mathrm{lr}}+\rho_{\mathrm{t}})g$, where $g=g_p$ is given in (5.17) for some $0<p\le 1$; ρ_{sr}, ρ_{lr} and ρ_{t} belong to $\mathcal{C}^\infty(X)$, such that*

$$\inf_{x\in X}\big(\rho_{\mathrm{sr}}(x)+\rho_{\mathrm{lr}}(x)+\rho_{\mathrm{t}}(x)\big)>-1\,,\quad \rho_{\mathrm{t}}(x)=o(1)\quad \textit{as}\ \ x\to 0$$

and such that

$$L^{1+\varepsilon}\rho_{\mathrm{sr}}, d_{A_{\mathrm{f}}}\rho_{\mathrm{sr}}, \Delta_{A_{\mathrm{f}}}\rho_{\mathrm{sr}}, L^{\varepsilon}\rho_{\mathrm{lr}}, L^{1+\varepsilon}d_{A_{\mathrm{f}}}\rho_{\mathrm{lr}}, \Delta_{A_{\mathrm{f}}}\rho_{\mathrm{lr}}, d_{A_{\mathrm{f}}}\rho_{\mathrm{t}}, \Delta_{A_{\mathrm{f}}}\rho_{\mathrm{t}}\quad \textit{belong to}\ \ L^\infty.$$

In $\mathscr{H}=L^2(X,\tilde g)$, let $\widetilde\Delta_A$ be the magnetic Laplacian with a non trapping potential $A=A_{\mathrm{f}}+A_{\mathrm{lr}}+A_{\mathrm{sr}}+A_{\mathrm{t}}$, where A_{f} is as in (5.18), A_{sr}, A_{lr} and A_{t} are in $\mathcal{C}^\infty(X,T^*X)$ such that:

$$\|L^{1+\varepsilon}A_{\mathrm{sr}}\|_\infty,\ \|L^{\varepsilon}A_{\mathrm{lr}}\|_\infty,\ \|L^{1+\varepsilon}\mathcal{L}_{x^{(2-p)}\partial_x}A_{\mathrm{lr}}\|_\infty<\infty\quad \textit{and}\quad A_{\mathrm{t}}=o(1)\,,$$

where $\mathcal{L}$ denotes the Lie derivative. Let $V=V_{\mathrm{loc}}+V_{\mathrm{sr}}+V_{\mathrm{lr}}+V_{\mathrm{t}}$ and V_{lr} be some potentials, where V_{loc} is measurable with compact support and $\widetilde\Delta_{A_{\mathrm{f}}}$-compact and V_{sr}, V_{lr} and V_{t} are in $L^\infty(X)$ such that:

$$\|L^{1+\varepsilon}V_{\mathrm{sr}}\|_\infty,\ \|L^{1+\varepsilon}d_{A_{\mathrm{f}}}V_{\mathrm{lr}}\|_\infty<\infty\quad \textit{and}\quad V_{\mathrm{t}}=o(1)\,.$$

Consider the magnetic Schrödinger operators $H_0=\widetilde\Delta_{A_{\mathrm{f}}+A_{\mathrm{lr}}}+V_{\mathrm{lr}}$ and $H=\widetilde\Delta_A+V$. Then

1. *H has no singular continuous spectrum.*
2. *The eigenvalues of $\mathbb{R}\setminus\{\kappa(p)\}$ have finite multiplicity and no accumulation points outside $\{\kappa(p)\}$.*
3. *Let $\mathcal{J}$ a compact interval such that $\mathcal{J}\cap(\{\kappa(p)\}\cup\sigma_{\mathrm{pp}}(H))=\emptyset$. Then, for all $s\in(1/2,3/2)$, there exists c such that*
$$\|(H-z_1)^{-1}-(H-z_2)^{-1}\|_{\mathcal{B}(\mathscr{L}_s,\mathscr{L}_{-s})}\le c\|z_1-z_2\|^{s-1/2}\,,$$
for all $z_1,z_2\in\mathcal{J}_\pm$.
4. *Let $\mathcal{J}=\mathbb{R}\setminus\{\kappa(p)\}$ and let E_0 and E be the continuous spectral component of H_0 and H, respectively. Then, the wave operators defined as the strong limit*
$$\Omega_\pm=\mathrm{s-}\lim_{t\to\pm\infty}e^{itH}e^{-itH_0}E_0(\mathcal{J})$$
exist and are complete, i.e., $\Omega_\pm\mathscr{H}=E(\mathcal{J})\mathscr{H}$.

Remark 6.5. Any smooth 1-form A on $\overline{X}$ is a short-range perturbation of a free vector potential A_{f} as in (5.18).

Remark 6.6. If one is interested only in the free metric g_p, the conclusions of the theorem hold for V_{loc} in the wider class of $\widetilde\Delta_{A_{\mathrm{f}}}$-form compact perturbations, using [2, Theorem 7.5.4]. Similar results should hold for a smooth metric on $\overline{X}$ and Dirichlet boundary conditions, as in [5].

Proof. We start with $\Delta_{A_{\mathrm{f}}}$ in $L^2(X,g)$. In Section 5.3, we transform it unitarily into Δ_0 given by (5.22). For R finite, we construct a conjugate operator S_R to Δ_0 given by (5.24). Theorem 5.6 gives a Mourre estimate for Δ_0 and the regularity of Δ_0 compared to the self-adjoint operator S_R. We go back by unitary transform into $L^2(X,g)$. Since the dependence on R is no longer important, we denote simply by S the image of the conjugate operator S_R. Therefore, we have $\Delta_{A_{\mathrm{f}}} \in \mathcal{C}^2(S, \mathcal{D}(\Delta_{A_{\mathrm{f}}}), L^2(X,g))$ and given $\mathcal{J}$ an open interval included in $\sigma_{\mathrm{ess}}(H_0)$, there is $c>0$ and a compact operator K such that the inequality

$$E_{\mathcal{J}}(T)[T,iS]E_{\mathcal{J}}(T) \geq cE_{\mathcal{J}}(T) + K \tag{6.45}$$

holds in the sense of forms in $L^2(X,g)$, for $T = \Delta_{A_{\mathrm{f}}}$.

Let W_0 be the unitary conjugate of $\widetilde{\Delta}_{A_{\mathrm{f}}}$ acting in $L^2(X,g)$. By Lemma 5.13, $W_0 \in \mathcal{C}^{1,1}(S, \mathcal{D}(\Delta_{A_{\mathrm{f}}}), \mathcal{D}(\Delta_{A_{\mathrm{f}}})^*)$ is a sum of short and long-range perturbation as described in Section 5.4. In particular, we get $W_0 \in \mathcal{C}^1_{\mathrm{u}}(S, \mathcal{D}(\Delta_{A_{\mathrm{f}}}), \mathcal{D}(\Delta_{A_{\mathrm{f}}})^*)$. By the point (2) of Lemma 5.13 and [2, Theorem 7.2.9] the inequality (6.45) holds for T (up to changing c and K).

We now go into $L^2(X,\tilde{g})$ using U defined before Lemma 5.13. We write the conjugate operator obtained in this way by $\tilde{S}$. Therefore, $\tilde{\Delta}_{A_{\mathrm{f}}}$ belongs to $\mathcal{C}^{1,1}(\tilde{S}, \mathcal{D}(\Delta_{\tilde{A}_{\mathrm{f}}}), \mathcal{D}(\Delta_{\tilde{A}_{\mathrm{f}}})^*)$ and given $\mathcal{J}$ an open interval included in $\sigma_{\mathrm{ess}}(H_0)$, there is $c>0$ and a compact operator K such that

$$E_{\mathcal{J}}(\tilde{T})[\tilde{T},i\tilde{S}]E_{\mathcal{J}}(\tilde{T}) \geq cE_{\mathcal{J}}(\tilde{T}) + K \tag{6.46}$$

holds in the sense of forms in $L^2(X,\tilde{g})$ for $\tilde{T} = \tilde{\Delta}_{A_{\mathrm{f}}}$. We now add the perturbation given by $A_{\mathrm{sr}}, A_{\mathrm{lr}}, A_{\mathrm{t}}, V_{\mathrm{sr}}, V_{\mathrm{lr}}$, and V_{t}. Note that H has the same domain as H_0 and that $(H+i)^{-1} - (H_0+i)^{-1}$ is compact by Rellich–Kondrakov lemma and Lemma 6.3. By Lemma 5.12, we obtain $H \in \mathcal{C}^{1,1}(S, \mathcal{D}(H), \mathcal{D}(H)^*)$. As above, the inequality 6.46 is true for $\tilde{T} = H$.

We now deduce the different claims of the theorem. The first comes from [2, Theorem 7.5.2]. The second ones is a consequence of the Virial theorem. For the third point first note that $\mathscr{L}_s \subset \mathcal{D}(|A|^s)$ for $s \in [0,2]$ by Lemma 5.5 and use [14] for instance (see references therein). Finally, the last point follows from [2, Theorem 7.6.11]. □

7. The non-stability of the essential spectrum and of the situation of limiting absorption principle

In $\mathbb{R}^n$, with the flat metric, it is well-known that only the behavior of the magnetic field at infinity plays a rôle in the computation of the essential spectrum. Moreover, [26, Theorem 4.1] states that the non-emptiness of the essential spectrum is preserved by the addition of a bounded magnetic field, even if it can become purely punctual. Concerning a compactly support magnetic field, the essential spectrum remains the same, see [36]. However, it is well-known that one obtains a long-range effect from it, in other words it acts on particles which have support away from it.

In the case of $\mathbb{R}^n$ with a hole of some kind, this phenomena are of special physical interests and are related to the Aharonov–Bohm effect, see Section 7.3 and references therein.

In contrast with the Euclidean setting, Theorem 4.2 indicates that in general the essential spectrum may vanish under *compactly supported* perturbations of the magnetic field. In the next sections, we discuss this effect both with and without the hypothesis of gauge invariance, and we investigate the coupling constant effect.

7.1. The case $H^1(X) = 0$

In this section, we assume gauge invariance. We first characterize trapping condition in terms of the magnetic field, see Definition 3.2 for the case of a magnetic potential. We recall that if $H^1(X) = 0$, given a magnetic field B the spectral properties of the magnetic Laplacian Δ_A will not depend on the choice of vector potential A such that $dA = B$. Indeed, given A, A' such that $dA = dA'$, the operators Δ_A and $\Delta_{A'}$ are unitarily equivalent by a gauge transformation. Therefore, we denote the magnetic Laplacian by Δ_B and express the condition of being (non-)trapping in function of B.

Let $p > 0$ and let X be the interior a compact manifold $\overline{X}$ endowed with the metric g_p given by (2.9). For simplicity, assume that B is a smooth 2-form on $\overline{X}$ such that its restriction to X is exact. Then there exists $A \in \mathcal{C}^\infty(\overline{X}, T^*\overline{X})$ such that $B = dA$ (since the cohomology of the de Rham complex on $\overline{X}$ equals the singular cohomology of $\overline{X}$, hence that of X). Let

$$M = \sqcup_{\alpha\in\mathcal{A}} M_\alpha$$

be the decomposition of the boundary M into its connected components. Set

$$\mathcal{A}_0 := \left\{\alpha \in \mathcal{A}; H^1(M_\alpha;\mathbb{R}) = 0\right\}.$$

For some $\mathcal{B} \subset \mathcal{A}$ set $M_\mathcal{B} = \sqcup_{\beta\in\mathcal{B}} M_\beta$ and consider the long exact cohomology sequence of the pair $(\overline{X}, M_\mathcal{B})$ with real coefficients:

$$H^1\left(\overline{X};\mathbb{R}\right) \longrightarrow H^1\left(M_\mathcal{B};\mathbb{R}\right) \xrightarrow{\partial} H^2\left(\overline{X}, M_\mathcal{B};\mathbb{R}\right) \xrightarrow{i} H^2\left(\overline{X};\mathbb{R}\right)$$

Since we assume that $H^1(X;\mathbb{R}) = 0$ it follows that the connecting map ∂ is injective. If B vanishes under pull-back to $M_\mathcal{B}$ then (since it is exact on X) it defines a class in $H^2(\overline{X}, M_\mathcal{B};\mathbb{R})$ which vanishes under the map i, so it belongs to the image of the injection ∂. We denote by $[B]_\beta$ the component of $[B]$ inside $\partial H^1(M_\beta) \subset H^2(\overline{X}, M_\mathcal{B};\mathbb{R})$.

Definition 7.1. Assume $H^1(X) = 0$. Let B be a smooth exact 2-form on $\overline{X}$. Denote by $\mathcal{B}$ the set of those $\alpha \in \mathcal{A}$ such that B vanishes identically on M_α. The field B is called *trapping* if for each $\beta \in \mathcal{B}$, the component $[B]_\beta \in \partial H^1(M_\beta) \subset H^2(\overline{X}, M_\mathcal{B};\mathbb{R})$ is not integral, i.e., it does not live in the image of the map of multiplication by 2π

$$H^2\left(\overline{X}, M_\mathcal{B};\mathbb{Z}\right) \xrightarrow{2\pi\cdot} H^2\left(\overline{X}, M_\mathcal{B};\mathbb{R}\right),$$

and *non-trapping* otherwise.

This definition is consistent with Definition 1.1 when M is connected. Note that if B is trapping then $\mathcal{B}$ must contain the index set $\mathcal{A}_0$ defined above.

In order to apply Theorem 6.4 and Theorem 4.2 we use the following lemma:

Lemma 7.2. 1. *Let A be a smooth vector potential on $\overline{X}$ such that $dA = 0$ in a neighbourhood of $M = \partial X$. Then there exists a smooth vector potential A',* constant in x *in a neighborhood of M, such that $A = A'$ on M and $d(A - A') = 0$.*

2. *Assume that $H^1(\overline{X}, \mathbb{R})$ vanishes. Let B be a trapping magnetic field on $\overline{X}$. Then every vector potential for B will be trapping.*
3. *Assume moreover that $H_1(X; \mathbb{Z}) = 0$. Let B be non-trapping such that $\imath_M^* B = 0$. Then every vector potential for B will be non-trapping.*

Recall that $\pi_1(X) = 0 \Longrightarrow H_1(X; \mathbb{Z}) = 0 \Longrightarrow H^1_{\mathrm{dR}}(X) = 0$.

Proof. 1) Let us show that one can choose A to be *constant in x* near the boundary, in the sense that near M it is the pull-back of a form from M under the projection $\pi : [0, \varepsilon) \times M \to M$ for ε small enough. Indeed, $A - \pi^* \imath_M^* A$ is closed on the cylinder $[0, \varepsilon) \times M$ and vanishes when pulled-back to M. Now M is a deformation-retract of the above cylinder, so the map of restriction to M induces an isomorphism in cohomology and thus the cohomology class $[A - \pi^* \imath_M^* A] \in H^1([0, \varepsilon) \times M)$ must be zero. Let $f \in \mathcal{C}^\infty(\overline{X}, \mathbb{R})$ be a primitive of this form for $x \leq \varepsilon/2$, then $A - df$ is the desired constant representative.

2) Consider the commutative diagram

$$\begin{array}{ccccc} H^1\big(M_{\mathcal{B}};\big) & \xrightarrow{\ \partial\ } & H^2\big(\overline{X}, M_{\mathcal{B}}; \mathbb{R}\big) & \longrightarrow & H^2\big(\overline{X}, \mathbb{R}\big) \\ \uparrow{\scriptstyle 2\pi\cdot} & & \uparrow{\scriptstyle 2\pi\cdot} & & \uparrow{\scriptstyle 2\pi\cdot} \\ H^1\big(M_{\mathcal{B}}; \mathbb{Z}\big) & \xrightarrow{\ \partial\ } & H^2\big(\overline{X}, M_{\mathcal{B}}; \mathbb{Z}\big) & \longrightarrow & H^2\big(\overline{X}, \mathbb{Z}\big) \end{array} \tag{7.47}$$

where the horizontal maps come form the long exact sequence of the pair $(\overline{X}, M)$ and the vertical maps are multiplication by 2π. Let B be trapping and choose a vector potential A for B, smooth on $\overline{X}$. We claim that A is trapping. Indeed, A is not closed on the components M_α, $\alpha \in \mathcal{A} \setminus \mathcal{B}$, while it is closed on $M_{\mathcal{B}}$. We note that $\partial[A_{|M_{\mathcal{B}}}] = [dA] = [B]$, so $\partial[A_{|M_\beta}] = [B]_\beta$. Assume that for some $\beta \in \mathcal{B}$, the class $[A_{|M_\beta}]$ were integral. Then using the first square from diagram (7.47), it would follow that $[B]_\beta$ was also integral, contradiction.

3) If B is non-trapping, there exists $\beta \in \mathcal{B}$ and $b \in H^2(\overline{X}, M_{\mathcal{B}}, \mathbb{Z})$ with $[B]_\beta = 2\pi b$. The image of $[B]_\beta \in \partial H^1(M_\beta, \mathbb{R})$ in $H^2(\overline{X}, \mathbb{R})$ is zero, thus b maps to a torsion element in $H^2(\overline{X}, \mathbb{Z})$. From $H_1(X, \mathbb{Z}) = 0$ we see using the universal coefficients theorem

$$0 \to \operatorname{Ext}\big(H_1(\overline{X}, \mathbb{Z}), \mathbb{Z}\big) \to H^2(\overline{X}; \mathbb{Z}) \to \operatorname{Hom}\big(H_2(\overline{X}, \mathbb{Z}), \mathbb{Z}\big) \to 0$$

that $H^2(\overline{X}; \mathbb{Z})$ is torsion-free. Thus b comes from some $a \in H^1(M_{\mathcal{B}}, \mathbb{Z})$. By commutativity we have $\partial(2\pi a) = [B]_\beta$. Since ∂ is an injection, it follows that every vector potential smooth on $\overline{X}$ for B will define an integral cohomology class on M_β, thus will be non-trapping. □

A spectacular example is a compactly supported magnetic field which induces very strong long-range effects. If $H^1(\overline{X};\mathbb{Z}) = 0$ and B be is an exact 2-form with compact support in X, then B is maximal non-trapping (see Definition 3.2) if and only if its class in $H^2(\overline{X}, M;\mathbb{R})$ is integral. We will stress more on this aspect in the following section.

We can also construct compactly supported magnetic fields for which the consequences of Theorem 4.2 hold true. We summarize this fact in the next proposition and give an explicit construction in the proof.

Proposition 7.3. *Let X be the interior of a compact manifold $\overline{X}$ with boundary $M = \partial\overline{X}$, endowed with a conformally cusp metric g_p. Assume that $H^1(X) = 0$ and $H^1(M_j) \neq 0$ for every connected component M_j of the boundary. Then there exists a non-zero smooth magnetic field B with* compact support *such that the essential spectrum of Δ_B is empty and such that for $p \geq 1/n$ the growth law of the eigenvalues does not depend on B and is given by* (4.15)*. Such fields B are generic inside compactly supported magnetic fields.*

Proof. We construct A like in (3.12) satisfying the hypotheses of Theorem 4.2. We take φ_0 to be constant. Let $\psi \in \mathcal{C}^\infty([0,\varepsilon))$ be a cut-off function such that $\psi(x) = 0$ for $x \in [3\varepsilon/4, \varepsilon)$ and $\psi(x) = 1$ for $x \in [0, \varepsilon/2)$. Since $H^1(M_j) \neq 0$, there exists a closed 1-form β_j on M which is not exact. Up to multiplying β_j by a real constant, we can assume that the cohomology class $[\beta_j] \in H^1_{\mathrm{dR}}(M_j)$ does not belong to the image of $2\pi H^1(M_j;\mathbb{Z}) \to H^1(M_j;\mathbb{R}) \simeq H^1_{\mathrm{dR}}(M_j)$. Let β denote the form on M which equals β_j on M_j. Choose A to be $\psi(x)\beta$ for $\varepsilon > x > 0$ and extend it by 0 to X. The magnetic field $B = dA = \psi'(x)dx \wedge \beta$ has compact support in X.

By Theorem 4.2, Δ_A has purely discrete spectrum with the Weyl asymptotic law eigenvalues independent of B.

The relative cohomology class $[B]$ lives in the direct sum $\oplus_j \partial H^1(M_j,\mathbb{R}) \subset H^2(\overline{X}, M, \mathbb{R})$. The field B is non-trapping if at least one of its components in this decomposition lives in the image of $H^1(M_j,\mathbb{Z})$. Since we assume all $H^1(M_j,\mathbb{R})$ to be nonzero, the space of non-trapping magnetic fields is a finite union of subspaces of codimension at least 1. □

We now show that the cohomological hypothesis about X and M can be satisfied in all dimensions greater than or equal to 2, and different from 3.

In dimension 2, take $X = \mathbb{R}^2$ endowed with the metric (2.9). Consider, for instance, the metric $r^{-2p}(dr^2 + d\sigma^2)$ given in polar coordinates. Here M is the circle at infinity and $x = 1/r$ for large r. Thus $b_1(X) = 0$ while $b_1(M) \neq 0$. The product of this manifold with a closed, connected, simply connected manifold Y of dimension k yields an example in dimension $2 + k$ with the same properties. Indeed, by the Künneth formula, the first cohomology group of $\mathbb{R}^2 \times Y$ vanishes, while $H^1(S^1 \times Y) \simeq H^1(S^1) = \mathbb{Z}$. Clearly k cannot be 1 since the only closed manifold in dimension 1 is the circle. Thus the dimension 3 is actually exceptional.

For orientable X of dimension 3, the assumptions $H^1(X) = 0$ and $H^1(M) \neq 0$ cannot be simultaneously fulfilled. Indeed, we have the following long exact sequence (valid actually regardless of the dimension of X)

$$H^1(\overline{X}) \xrightarrow{i_M} H^1(M) \xrightarrow{\delta} H^2(\overline{X}, M)\,. \tag{7.48}$$

If $\dim(X) = 3$, the spaces $H^1(\overline{X})$ and $H^2(\overline{X}, M)$ are isomorphic by Poincaré duality, hence $H^1(\overline{X}) = 0$ implies $H^2(\overline{X}, M) = 0$ and so (by exactness) $H^1(M) = 0$. It should be possible to build a non-orientable example in dimension 3 such that one could apply Proposition 7.3 but we were not able to construct one.

We finally give an example not covered by Lemma 7.2 but such that the conclusion of Theorem 4.2 holds. We considered so far magnetic fields on $\overline{X}$ with vector potential smooth on $\overline{X}$. One may consider also more singular magnetic fields arising from Proposition 3.1.

7.1.1. Example. Let X be any conformally cusp manifold, without any cohomological assumptions. Suppose that $B = df \wedge dx/x^2$ where f is a function on X smooth down to the boundary M of X. Assume that f is not constant on any connected component of M. Then the essential spectrum of the magnetic operator (which is well-defined by B if $H^1(X) = 0$) is empty. This follows from the fact that $A := f dx/x^2$ is trapping.

Note that the pull-back to the border of the above magnetic field is zero.

7.2. The coupling constant effect

In flat Euclidean space it is shown in [24], under some technical hypotheses, that the spectrum has a limit as the coupling constant tends to infinity. In contrast, in the next example, we exhibit the creation of essential spectrum for periodic values of the coupling constant. We will focus on the properties of Δ_{gB} for some coupling constant $g \in \mathbb{R}$. In order to be able to exploit the two sides of this work we concentrate here on the metric (5.17). We assume that M is *connected*, $H^1(X, \mathbb{Z}) = 0$ and $H^1(M) \neq 0$.

Given B a magnetic potential with compact support, gB is non-trapping if and only its class in $H^2(\overline{X}, M; \mathbb{R})$ is integral. Let G_B be the discrete subgroup of those $g \in \mathbb{R}$ such that gB is non-trapping. As this subgroup is possibly $\{0\}$, we start with some exact form B which represents a nonzero cohomology class in $H^2(\overline{X}, M; \mathbb{Z})$; then by exactness of the relative cohomology long sequence, $[B]$ lives in the image of the injection $H^1(M; R) \xrightarrow{\partial} H^2(\overline{X}, M; \mathbb{R})$. With these restrictions, G_B is a non-zero discrete subgroup of $\mathbb{Q}$. Now we apply Theorem 4.2 for the trapping case and Theorem 6.4 for the non-trapping case. We obtain that

1. For $g \in G_B$, the essential spectrum of Δ_{gB} is given by $[\kappa(p), \infty)$, where $1\kappa(p)$ is defined in Proposition 6.2. The spectrum of Δ_{gB} has no singular continuous part and the eigenvalues of $\mathbb{R} \setminus \{\kappa(p)\}$ are of finite multiplicity and can accumulate only in $\kappa(p)$.
2. For $g \notin G_B$, the spectrum Δ_{gB} is discrete and if $p \geq 1/n$, the asymptotic of the eigenvalues depend neither on g nor on B.

We now describe the long-range effect regarding the coupling constant. Take a state $\phi \in L^2(X)$ such that ϕ is not an eigenvalue of the free Laplacian Δ_0 and is located in a energy higher than $\kappa(p)$. Since the Fourier transform of an absolutely continuous measure (comparing to the Lebesgue measure) tends to 0 at infinity, we obtain, for each $g \in G_{gB}$ and for each χ operator of multiplication by the characteristic function of compact support that $\chi e^{it\Delta_{gB}}\phi \to 0$ as $t \to \infty$. If one considers χ being 1 above the support of the magnetic field, then after some time the norm of ϕ above this zone is arbitrary small. Classically, the particle stops interacting with the magnetic field. Let us denote by ϕ' the particle at this moment. Now, if we switch on the interaction with intensity as small as one desires one gets $g \notin G_{gB}$ and then the spectrum of Δ_{gB} is discrete. Therefore there exists χ' operator of multiplication by the characteristic function of compact support such that $1/T \int_0^T \|\chi' e^{it\Delta_{gB}}\phi'\|^2 dt$ tends to a positive constant, as $T \to \infty$. The particle is caught by the magnetic field even thought they are far from being able to interact classically.

In other words, switching on the interaction of the magnetic field with compact support has destroyed the situation of limiting absorption principle. This is a strong long-range effect.

For the sake of utmost concreteness, take $X = \mathbb{R}^2$ endowed with the metric $r^{-2p}(dr^2 + d\theta^2)$ in polar coordinates, for r big enough and $1 \geq p > 0$. The border M is S^1 and $H^2(\overline{X}, M; \mathbb{R}) \simeq \mathbb{Z}$. Then for every closed 2-form B with compact support and non-zero integral, the group G_B defined above is non-zero.

7.3. The case $H^1(X) \neq 0$, the Aharonov–Bohm effect

Gauge invariance does not hold in this case, so one expects some sort of Aharonov–Bohm effect [1]. Indeed, given two vectors potential arising from the same magnetic field, the associated magnetic Laplacians Δ_A and $\Delta_{A'}$ might be unitarily in-equivalent. The vector potential acquires therefore a certain physical meaning in this case.

In flat $\mathbb{R}^n$ with holes, some long-range effect appears, for instance in the scattering matrix like in [42–44, 47], in an inverse-scattering problem [40, 48] or in the semi-classical regime [3]. See also [22] for the influence of the obstacle on the bottom of the spectrum.

In all the above cases the essential spectrum remains the same. In light of Proposition 7.3, one can expect a much stronger effect in our context. We now give some examples of magnetic fields with compact support such that there exists a non-trapping vector potential A, constant in x in a neighborhood of M, and a trapping vector potential A', such that $dA = dA' = B$. To ease the presentation, we stick to the metric (5.17). For A, one applies Theorem 6.4 and obtain that the essential spectrum of Δ_A is given by $[\kappa(p), \infty)$, that Δ_A has no singular continuous part and the eigenvalues of $\mathbb{R} \setminus \{\kappa(p)\}$ have finite multiplicity and can accumulate only to $\{\kappa(p)\}$. For A', one applies Theorem 4.2 to get the discreteness of the spectrum of $\Delta_{A'}$ and to obtain that the asymptotic of eigenvalues depends neither on A nor on B for $n \geq 1/p$. In the next section, we describe how generic this situation is for hyperbolic manifolds of dimension 2 and 3.

The easy step is to construct the non-trapping vector potential A constant in a neighborhood of M, more precisely we construct A to be *maximal* non-trapping, see Definition 3.2. Indeed, prescribe a closed 1-form θ on M defining integral cohomology 1-classes on each component of the boundary, and extend it smoothly to X, constant in x in a neighborhood on M, like in the proof of Proposition 7.3. The magnetic field $B := dA$ has then compact support and one may apply Theorem 6.4 for Δ_A. We now construct A' by adding to A a closed form α, smooth on $\overline{X}$. Since α is closed, A' and A define the same magnetic field. In light of Remark 3.4, A' is trapping if and only if α is. We can then apply Theorem 4.2 to $\Delta_{A'}$. It remains to show that closed, trapping α do exist. We start with a concrete example.

Example 7.4. Consider the manifold $\overline{X} = (S^1)^{n-1} \times [0,1]$ with a metric g_p as in (5.17) near the two boundary components. Let $\theta_i \in \mathbb{R}$ be variables on the torus $(S^1)^{n-1}$, so $e^{i\theta_i} \in S^1$. Take the vector potential A to be 0, it is (maximal) non-trapping. Choose now $\alpha = A'$ to be the closed form $\mu d\theta_1$ for some $\mu \in \mathbb{R}$. It is constant in a neighborhood of $(S^1)^{n-1}$. The class $[i_M^*(A')]$ is an integer multiple of 2π if and only if $\mu \in \mathbb{Z}$. In other words, A' is non-trapping if and only if $\mu \in \mathbb{R} \setminus \mathbb{Z}$. Note that here the magnetic field B vanishes.

In order to show the existence of such α in a more general setting, we assume that the first Betti number of each connected component of the boundary is non-zero. It is enough to find some closed α, smooth on $\overline{X}$, which on each boundary component represents a non-zero cohomology class. Then, up to a multiplication by a constant, α will be trapping. When X is orientable and $\dim(X)$ is 2 or 3, one proceeds as follows.

Proposition 7.5. *Let $\overline{X}$ be a compact manifold with non-empty boundary M. Assume that one of the following hypotheses holds:*

1. $\dim(X) = 2$ *and M is disconnected;*
2. $\dim(X) = 2$ *and X is non-orientable;*
3. $\dim(X) = 3$*, X is orientable and none of the connected components of M are spheres.*

Then there exists a closed smooth form $\alpha \in \Lambda^1(\overline{X})$, constant in x near the boundary, such that for all connected components M_j of M, the class $[\alpha_{|M_j}] \in H^1(M_j;\mathbb{R})$ is non-zero.

Proof. Any cohomology class on $\overline{X}$ admits a smooth representative α up to the boundary. Moreover since α is closed, one can choose α to be constant in x near the boundary using Lemma 7.2. Thus, in cohomological terms, the proposition is equivalent to finding a class $[\alpha] \in H^1(\overline{X})$ whose pull-back to each connected component of M is non-zero, i.e., $H^1(M_j) \ni i_{M_j}[\alpha] \neq 0$.

Consider first the case $\dim(X) = 2$. If X is non-orientable, $H^2(\overline{X}, M) = 0$ so δ is the zero map. If X is oriented, the boundary components are all oriented circles, so $H^1(M_j) \simeq \mathbb{R}$; the compactly supported cohomology $H^2(\overline{X}, M)$ is isomorphic to $\mathbb{R}$ via the integration map, and the boundary map $\delta : H^1(M) \to H^2(\overline{X}, M)$ restricted to $H^1(M_j)$ is just the identity map of $\mathbb{R}$ under these identifications. Thus the kernel of δ is made of ν-tuples (where ν is the number of boundary components) $(a_1, \ldots, a_\nu)$ of real numbers, with the constraint $\sum a_j = 0$ in the orientable case. By exactness, this space is also the image of the restriction map $H^1(\overline{X}) \to H^1(M)$. Clearly there exist such tuples with non-zero entries, provided $\nu \geq 2$ in the orientable case. Thus the conclusion follows for $\dim(X) = 2$.

Assume now that $\dim(X) = 3$. Then the maps i_M and δ from the relative long exact sequence (7.48) are dual to each other under the intersection pairing on M, respectively on $H^1(\overline{X}) \times H^2(\overline{X}, M)$:

$$\int_M i_M(\alpha) \wedge \beta = \int_X \alpha \wedge \delta\beta\,.$$

These bilinear pairings are non-degenerate by Poincaré duality; in particular since the pairing on $H^1(M)$ is skew-symmetric, it defines a symplectic form. It follows easily that the subspace

$$L := i_M\big(H^1(X)\big) \subset H^1(M)$$

is a Lagrangian subspace (i.e., it is a maximal isotropic subspace for the symplectic form). Now the symplectic vector space $H^1(M)$ splits into the direct sum of symplectic vector spaces $H^1(M_j)$, By hypothesis, the genus g_j of the oriented surface M_j is at least 1 so $H^1(M_j)$ is non-zero for all j. It is clear that the projection of L on each $H^1(M_j)$ must be non-zero, otherwise L would not be maximal. Hence, there exists an element of $L = i_M(H^1(X))$ which restrict to non-zero classes in each $H^1(M_j)$, as desired. □

Remark 7.6. If one is interested in some coupling constant effect, it is interesting to choose the closed form α so that every $[\alpha_{|M_j}]$ are non-zero integral classes. In dimension 2, this amounts to choosing non-zero integers with zero sum. In dimension 3, as $L = i_M(H^1(X)) = i_M(H^1(X, \mathbb{Z})) \otimes \mathbb{R}$ is spanned by integer classes, we can find an integer class in L with non-zero projection on all $H^1(M_j)$. For real g, the vector potential $g\alpha$ is therefore non-trapping precisely for g in a discrete subgroup $g_0\mathbb{Z}$ for some $g_0 \in \mathbb{Q}$.

8. Application to hyperbolic manifolds

We now examine in more detail how this Aharonov–Bohm effect arises in the context of hyperbolic manifolds of finite volume in dimension 2 and 3.

These are conformally cusp manifold with $p = 1$, with unperturbed metric of the form (1.1) and such that every component M_j of the boundary is a circle when $\dim(X) = 2$, respectively a flat torus when $\dim(X) = 3$: indeed, outside a compact set, the metric takes the form $g = dt^2 + e^{-2t}h$, where $t \in [0, \infty)$, and this is of the form (1.1) after the change of variables $x := e^{-t}$. We denote by $\overline{X}$ the compactification of X by requiring that $x := e^{-t}$ be a boundary-defining function for "infinity". These manifolds and their boundary components always have non-zero first Betti number.

For complete hyperbolic *surfaces* with cusps, every smooth 2-form B on $\overline{X}$ must be exact because $H^2(\overline{X};\mathbb{R})$ is always zero for a non-closed surface. Call A a smooth primitive of B. If B vanishes at M then A is necessarily closed over M. In terms of cohomology classes, we have $[B] = \delta_{\mathbb{R}}[A_{|M}]$ where $\delta_{\mathbb{R}}$ is the connecting morphism from the sequence (7.48) with real coefficients. Notice that A can be chosen to define an integer class on M if and only if $[B]$ is integer. Indeed, $H^2(\overline{X};\mathbb{Z})$ is also 0 for a compact surface with non-empty boundary, so if $[B]$ is integer, it must lie in the image of $\delta_{\mathbb{Z}}$ for the sequence (7.48) with integer coefficients. Conversely, if $[A]$ is integer, i.e., $[A] = 2\pi[A_{\mathbb{Z}}]$ (see diagram (7.47) with M in the place of $M_{\mathcal{B}}$, where the horizontal maps are now *surjective*) then $[B] = 2\pi\delta_{\mathbb{Z}}[A_{\mathbb{Z}}]$ is also integer. We summarize these remarks in the following

Corollary 8.1. *Let B be a smooth 2-form on the compactification of a complete hyperbolic surface X with cusps, and denote by $[B] \in H^2(\overline{X}, M, \mathbb{R})$ its relative cohomology class.*

- *If either X is non-orientable, or X is orientable with at least two cusps, then B admits both trapping and non-trapping vector potentials.*
- *If X is orientable with precisely one cusp, then B admits only non-trapping vector potentials if $[B]$ is integral, while if $[B]$ is not integral then B admits only trapping vector potentials.*

Proof. First note that B is closed since it is of maximal degree; it is exact since $H^2(\overline{X}) = 0$ for every surface with boundary; moreover its pull-back to the 1-dimensional boundary also vanishes, so B defines a relative de Rham class. By Corollary 3.5, the existence of trapping and non-trapping vector potentials depends only on this class.

If X is orientable and has precisely 1 cusp, then the map $\delta : H^1(M) \to H^2(\overline{X}, M)$ is an isomorphism both for real and for integer coefficients. Thus $[B]$ is integer if and only if $[A_{|M}]$ is integer. Since the boundary is connected, A is trapping if and only if the cohomology class of its restriction to the boundary is non-integer.

If X is oriented and has at least two cusps, identify $H^2(\overline{X}, M)$ and each $H^1(M_j)$ with $\mathbb{Z}$, so that the boundary map restricted to $H^1(M_j)$ is the identity. We can write $[B]$ first as a sum $\sum \alpha_j$ of non-integer numbers, then also as a sum where at least one term is integer. Let A be a 1-form on $\overline{X}$ which restricts to closed forms of cohomology class α_j on $M_j = S^1$. Then $B - dA$ represents the 0 class in $H^2(\overline{X}, M)$, so after adding to A a form vanishing at the boundary, we can assume that $B = dA$. Now when all α_j are non-integers, A is trapping, while in the other case it is non-trapping as claimed.

If X is non-orientable, the class $[B]$ vanishes. It is enough to find trapping and non-trapping vector potentials for the zero magnetic field, which is done as in the orientable case. □

When $\dim(X) = 3$, we have:

Corollary 8.2. *Let X be an orientable complete hyperbolic 3-manifold of finite volume. Then every magnetic field B smooth on the compactification $\overline{X}$ admits trapping vector potentials.*

Assume that the pull-back of B to the boundary M vanishes. If X has precisely one cusp, then there exists a rational (i.e., containing integer classes) infinite cyclic subgroup $G \subset H^2(\overline{X}, M, \mathbb{R})$ so that B admits a non-trapping vector potential if and only if $[B] \in G$. In general, one of the following alternative statements holds:

1. *Either every magnetic field smooth on $\overline{X}$ and vanishing at M admits a non-trapping vector potential, or*
2. *Generically, magnetic fields smooth on $\overline{X}$ and vanishing at M do not admit non-trapping vector potentials.*

There exists moreover $q \in \mathbb{Z}^*$ such that if $[B]$ is integer, then qB admits non-trapping vector potentials.

Proof. For the existence of trapping vector potentials we use the closed form α from Proposition 7.5. Let A be any vector potential for B. It suffices to note that for $u \in \mathbb{R}$, the form $A + u\alpha$ is another vector potential for B, which is trapping on each connected component of M except possibly for some discrete values of u.

Let h denote the number of cusps of X. Both the Lagrangian subspace $L \subset H^1(M)$ and the image space $\partial H^1(M) \subset H^2(\overline{X}, M)$ have dimension h. By hypothesis, the cohomology class of B on $\overline{X}$ is 0 so by exactness of (7.47), the relative cohomology class $[B]$ lives in $\partial H^1(M)$.

Assume first that X has precisely one cusp. Let A be a vector potential for B (smooth on $\overline{X}$). We can change A by adding to it any class in the line L without changing $[B]$. Notice that $H^1(M) = \mathbb{Z}^2$. The line L has an integer generator (given by the image of $H^1(\overline{X}, \mathbb{Z}) \to H^1(M, \mathbb{Z})$). Without loss of generality, we can assume that L is not the horizontal axis in $\mathbb{Z}^2$. It follows that the translates of all integer points in $\mathbb{Z}^2$ in directions parallel to L form a discrete subgroup of $\mathbb{Q}$. Thus B admits non-trapping vector potentials if and only if the cohomology class $[B]$ inside the 1-dimensional image $\partial H^1(M)$ lives inside a certain infinite cyclic discrete subgroup. In particular, if B is irrational (i.e., no positive integer multiple of B is an integral class) then B does not admit non-trapping vector potentials.

In the general case, assume first that there exists a boundary component M_j so that L projects surjectively onto $H^1(M_j, \mathbb{R})$. Let A be a vector potential for an arbitrary magnetic field B which vanishes at M. Let $[A'_j] \in H^1(M_j, \mathbb{R})$ be such that $[A]_{H^1(M_j)} + [A'_j]$ is integer. Let $[A'] \in L$ be an element whose component in $H^1(M_j)$ is $[A'_j]$. Choose a representative A' and extend it to a smooth 1-form on $\overline{X}$, constant in x near the boundary. Then $A + A'$ is a non-trapping vector potential. From the definition of L, the form dA' defines the zero class in relative cohomology, so from Corollary 3.5 we get the assertion on B.

If the assumption on L is not fulfilled, we claim that

$$\hat{L}_j := L \cap \oplus_{i \neq j} H^1(M_i)$$

has dimension $h-1$ for all j. Indeed, this dimension cannot be h (since the projection of L on $H^1(M_j)$ is not zero) and it cannot be $h-2$ (since the projection is not surjective). It follows easily that $\hat{L}_j$ is a Lagrangian subspace of $\oplus_{i\neq j} H^1(M_i)$. Let v be a vector in $L \setminus \hat{L}_j$. The component $\hat{v}_j$ of v in $\oplus_{i\neq j} H^1(M_i)$ is clearly orthogonal (with respect to the symplectic form) to $\hat{L}_j$, so by maximality it must belong to $\hat{L}_j$. Thus we may subtract this component to obtain, for each j, a non-zero element of $L \cap H^1(M_j)$. These elements may be taken integral since L has integer generators. Since $L = \ker \partial$, it follows that the image of ∂ is the direct sum of the images $\partial(H^1(M_j))$. As in the case of only one cusp, we see that B has a non-trapping potential if and only if at least one of the components of $[B]$ in this decomposition belong to a certain cyclic subgroup containing integer classes.

Set q to be the least common denominator of the generators of these subgroups for all j. If $[B]$ is integer, it follows that every vector potential for qB must be maximal non-trapping. □

Appendix A. The C^1 condition in the Mourre theory

In this appendix, we give a general criterion of its own interest to check the, somehow abstract, hypothesis of regularity $\mathcal{C}^1$ which is a key notion in the Virial theorem within Mourre's theory, see [2] and [13]. Let A and H be two self-adjoint operators in a Hilbert space $\mathscr{H}$. The commutator $[H, iA]$ is defined in the sense of forms on $\mathcal{D}(A)\cap\mathcal{D}(H)$. Suppose that the commutator $[H, iA]$ extends to $\mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ and denote by $[H, iA]_0$ the extension. Suppose also that the following Mourre estimate holds true on an open interval $\mathcal{I}$, i.e., there is a constant $c > 0$ and a compact operator K such that

$$E_{\mathcal{I}}(H)[H, iA]_0 E_{\mathcal{I}}(H) \geq cE_{\mathcal{I}}(H) + K\,, \tag{A.1}$$

where $E_{\mathcal{I}}(H)$ denotes the spectral measure of H above $\mathcal{I}$. Take now $\lambda \in \mathcal{I}$ which is not an eigenvalue of H. Set $\mathcal{I}_n := (\lambda - 1/n, \lambda + 1/n)$. Then $E_{\mathcal{I}_n}(H)$ tends strongly to 0 as $n \to \infty$, so $E_{\mathcal{I}_n}(H) K E_{\mathcal{I}_n}(H)$ tends in norm to 0. Hence for n big enough and for some $0 < c' \leq c$, one gets the strict Mourre estimate

$$E_{\mathcal{I}_n}(H)[H, iA]_0 E_{\mathcal{I}_n}(H) \geq c' E_{\mathcal{I}_n}(H)\,. \tag{A.2}$$

By supposing that $H \in \mathcal{C}^1(A)$ (see below) or that $e^{itA}\mathcal{D}(H) \subset \mathcal{D}(H)$, the Virial theorem holds true, i.e., $\langle f, [H, iA]_0 f\rangle = 0$ for every eigenvector f of H. Note that f has no reason to lie in $\mathcal{D}(A)$ and that the expansion of the commutator $[H - \lambda, iA]$ over f is formal.

The Virial theorem is crucial to the study of embedded eigenvalues of H. Assuming (A.1), it implies the local finiteness of the point spectrum of H over $\mathcal{I}$, i.e., that the sum of the multiplicities of the eigenvalues of H inside $\mathcal{I}$ is finite.

To see this, apply (A.1) to a infinite sequence $(f_n)_{n\in\mathbb{N}}$ of orthonormal eigenvectors of H. Then, since $\langle f_n, Kf_n\rangle$ tends to 0 as n goes to infinity, one obtains a contradiction with the positivity of c. Assuming (A.2), the Virial theorem implies directly that H has no eigenvalue in $\mathcal{I}_n$.

We stress that the hypothesis $[H, iA]_0 \in \mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ does not imply the Virial theorem. A counterexample is given in [13]. If one adds some conditions on the second order commutator of H and A, e.g., like in [7], one deduces from (A.2) a limiting absorption principle and therefore the absence of eigenvalues in $\mathcal{I}_n$. In turn, assuming (A.1), we deduce that the set of eigenvalues of H in $\mathcal{I}$ is closed. It is not known whether the multiplicity of the point spectrum must be locally finite when the Virial theorem does not hold.

Checking the Virial theorem, or a sufficient condition for it like the C^1 condition, is sometimes omitted in the Mourre analysis in a manifold context. To our knowledge, no result exists actually to show directly the C^1 regularity in a manifold context. On a class of exponentially growing manifolds, Bouclet [5] circumvents the problem by showing a stronger fact, i.e., the invariance of the domain. This method does not seem to work for our local conjugate operator, see Section 5.3. Besides giving an abstract criterion for the $\mathcal{C}^1$ condition, we will explain under which additional condition we can recover the invariance of the domain from it.

Given $z \in \rho(H)$, we denote by $R(z) = (H - z)^{-1}$. For $k \in \mathbb{N}$, we recall that $H \in \mathcal{C}^k(A)$ if for one $z \notin \sigma(H)$ (then for all $z \notin \sigma(H)$) the map $t \mapsto e^{-itA}R(z)e^{itA}$ is C^k in the strong topology. We recall a result following from Lemma 6.2.9 and Theorem 6.2.10 of [2].

Theorem A.1. *Let A and H be two self-adjoint operators in the Hilbert space $\mathscr{H}$. The following points are equivalent:*

1. *$H \in \mathcal{C}^1(A)$.*
2. *For one (then for all) $z \notin \sigma(H)$, there is a finite c such that*
$$\left|\left\langle Af, R(z)f\right\rangle - \left\langle R(\overline{z})f, Af\right\rangle\right| \le c\|f\|^2, \quad \text{for all} \quad f \in \mathcal{D}(A)\,. \tag{A.3}$$
3. a. *There is a finite c such that for all $f \in \mathcal{D}(A) \cap \mathcal{D}(H)$:*
$$|\langle Af, Hf\rangle - \langle Hf, Af\rangle| \le c\big(\|Hf\|^2 + \|f\|^2\big)\,. \tag{A.4}$$
   b. *For some (then for all) $z \notin \sigma(H)$, the set $\{f \in \mathcal{D}(A) \mid R(z)f \in \mathcal{D}(A)$ and $R(\overline{z})f \in \mathcal{D}(A)\}$ is a core for A.*

Note that in practice, condition (3.a) is usually easy to check and follows from the construction of the conjugate operator. The condition (3.b) could be more delicate. This is addressed in the next lemma, inspired by [5].

Lemma A.2. *Let $\mathscr{D}$ be a subspace of $\mathscr{H}$ such that $\mathscr{D} \subset \mathcal{D}(H) \cap \mathcal{D}(A)$, $\mathscr{D}$ is a core for A and $H\mathscr{D} \subset \mathscr{D}$. Let $(\chi_n)_{n\in\mathbb{N}}$ be a family of bounded operators such that*

1. *$\chi_n\mathscr{D} \subset \mathscr{D}$, χ_n tends strongly to 1 as $n \to \infty$, and $\sup_n \|\chi_n\|_{\mathcal{D}(H)} < \infty$.*
2. *$A\chi_n f \to Af$, for all $f \in \mathscr{D}$, as $n \to \infty$.*
3. *There is $z \notin \sigma(H)$, such that $\chi_n R(z)\mathscr{D} \subset \mathscr{D}$ and $\chi_n R(\overline{z})\mathscr{D} \subset \mathscr{D}$.*

Suppose also that for all $f \in \mathscr{D}$

$$\lim_{n\to\infty} A[H,\chi_n]R(z)f = 0 \quad \text{and} \quad \lim_{n\to\infty} A[H,\chi_n]R(\overline{z})f = 0\,. \tag{A.5}$$

Finally, suppose that there is a finite c such that

$$|\langle Af, Hf\rangle - \langle Hf, Af\rangle| \le c\big(\|Hf\|^2 + \|f\|^2\big)\,, \quad \forall f \in \mathscr{D}\,. \tag{A.6}$$

Then one has $H \in \mathcal{C}^1(A)$.

Note that (A.5) is well defined by expanding the commutator $[H, \chi_n]$ and by using (3) and $H\mathscr{D} \subset \mathscr{D}$.

Proof. By polarization and by applying (A.6) to $\chi_n R(\overline{z})f$ and to $\chi_n R(z)f$, with $f \in \mathscr{D}$, we see that there exists $c < \infty$ such that

$$\begin{aligned}\big|\big\langle A\chi_n R(\overline{z})f, H\chi_n R(z)f\big\rangle - &\big\langle H\chi_n R(\overline{z})f, A\chi_n R(z)f\big\rangle\big| \\ &\le c\|(H+i)\chi_n R(\overline{z})f\| \cdot \|(H+i)\chi_n R(z)f\|\,, \end{aligned}\tag{A.7}$$

for all $f \in \mathscr{D}$. By condition (1), the right-hand side is bounded by $C\|f\|^2$ for some C. We expand the left hand side of (A.7) by commuting H with χ_n:

$$\begin{aligned}\big|\big\langle \chi_n R(\overline{z})f, A\chi_n f\big\rangle - &\big\langle A\chi_n f, \chi_n R(z)f\big\rangle \\ &+ \big\langle \chi_n R(\overline{z})f, A[H,\chi_n]R(z)f\big\rangle - \big\langle A[H,\chi_n]R(\overline{z})f, \chi_n R(z)f\big\rangle\big|\,.\end{aligned}$$

Using (A.5), the second line vanishes as n goes to infinity. Taking in account the assumptions (1) and (2), we deduce:

$$\big|\big\langle R(\overline{z})f, Af\big\rangle - \big\langle Af, R(z)f\big\rangle\big| \le C\|f\|^2\,, \quad \forall f \in \mathscr{D}\,.$$

Finally, since $\mathscr{D}$ is a core for A, we obtain (A.3). We conclude that $H \in \mathcal{C}^1(A)$. □

The hypotheses of the lemma are easily satisfied in a manifold context with H being the Laplacian and A its conjugate operator, constructed as a localization on the ends of the generator of dilatation like for instance in [10]. Let $\mathscr{D} = \mathcal{C}_c^\infty(X)$ and χ_n a family of operators of multiplication by smooth cut-off functions with compact support. The fact that A is self-adjoint comes usually by some consideration of C_0-group associated to some vector fields and using the Nelson lemma and the invariance of $\mathscr{D}$ under the C_0-group give that $\mathscr{D}$ is a core for A, see Remark 5.4. The hypothesis (3) follows then by elliptic regularity. The only point to really check is (A.5). At this point one needs to choose more carefully the family χ_n. In Lemma 5.9, we show that the hypotheses of Lemma A.2 hold for the standard conjugate operator and for our local conjugate operator.

The invariance of the domain is desirable in order to deal, in a more convenient way, directly with operators and no longer with resolvents. On a manifold the C_0-group e^{itA} is not explicit and it could be delicate to deal with the domain of H directly. However, one may obtain this invariance of the domain using [13] and a $C^1(A)$ condition. We recall:

Lemma A.3. *If $H \in \mathcal{C}^1(A)$ and $[H, iA] : \mathcal{D}(H) \to \mathscr{H}$ then $e^{itA}\mathcal{D}(H) \subset \mathcal{D}(H)$, for all $t \in \mathbb{R}$.*

In light of this lemma, one understands better the importance of having some $C^1(A)$ criteria. On one hand, one can easily check the invariance of the domain. On the other hand, if the commutator belongs only to $\mathcal{B}(\mathcal{D}(H), \mathcal{D}(H)^*)$ and not $\mathcal{B}(\mathcal{D}(H), \mathscr{H})$, one may turn to another version of the Mourre theory like in [2] when H has a spectral gap or like in [17, 45] in the other case.

Appendix B. Finite multiplicity of L^2 eigenvalues

In order to classify all maximal symmetric extension of a given cusp-elliptic operator H (see Section 2.1 for definitions), one computes the defect indices, i.e., $\dim\ker(H^* \pm i)$. If they are equal and finite, one concludes that all maximal symmetric extensions of H are self-adjoint. By the Krein formulae, one hence obtains that the difference of the resolvent of two maximal extensions is finite rank. This implies by Weyl's theorem that the essential spectrum is the same for all self-adjoint extensions. Moreover, by Birman's theorem, the wave operators associated to a pair of such extensions exist and are complete. On the other hand, note that if the defect indices are not finite, one may have maximal symmetric extensions which are not self-adjoint.

It is also interesting to control the multiplicity of eigenvalues embedded in the essential spectrum.

In the next lemma we assume that X is a conformally cusp manifold with respect to the metric (2.9). We fix a vector bundle E over $\overline{X}$ (for instance the bundle of cups differential forms, although in this paper we only use the case where E is the trivial bundle $\mathbb{C}$) endowed with a smooth metric up to $\partial\overline{X} = M$.

Lemma B.1. *Let Δ, acting on $\mathcal{C}_c^\infty(X, E)$, be a cusp-elliptic differential operator in $x^{-2p}\mathrm{Diff}^k(X, E)$ for some $p, k > 0$. Then the dimension of any L^2-eigenspace of Δ^* is finite.*

Remember that if the operator Δ is bounded from below, then the defect indices are the same. This lemma guarantees that they are also finite. This point is not obvious when the manifold is not complete even if Δ is a Laplacian. Of course, this result is based on ideas that can be traced back to [34] and which are today quite standard. This lemma generalizes a result of [18].

Proof. We start by noticing that Δ can be regarded as an unbounded operator in a larger L^2 space. Namely, let L^2_ε be the completion of $\mathcal{C}_c^\infty(X, E)$ with respect to the volume form $e^{-\frac{2\varepsilon}{x}} dg_p$ for some $\varepsilon > 0$. Clearly then L^2_ε contains L^2. A distributional solution of $\Delta - \lambda$ in L^2 is evidently also a distributional solution of $\Delta - \lambda$ in L^2_ε. Thus the conclusion will follow by showing that Δ has in L^2_ε a unique closed extension with purely discrete spectrum. The strategy for this is by now clear. First we conjugate Δ through the isometry

$$L^2_\varepsilon \to L^2\,, \quad \phi \mapsto e^{-\frac{\varepsilon}{x}}\phi\,.$$

We get an unbounded operator $e^{-\frac{\varepsilon}{x}}\Delta e^{\frac{\varepsilon}{x}}$ in L^2, which is unitarily equivalent to Δ (acting in L^2_ε). Essentially from the definition, see Section 2.1,

$$\mathcal{N}\big(x^{2p}e^{-\frac{\varepsilon}{x}}\Delta e^{\frac{\varepsilon}{x}}\big)(\xi) = \mathcal{N}\big(x^{2p}\Delta\big)(\xi + i\varepsilon)\,.$$

The normal operator is a polynomial in ξ, in particular it is entire. Then by analytic Fredholm theory [34, Prop. 5.3], the set of complex values of ξ for which $\mathcal{N}(x^{2p}\Delta)(\xi)$ is not invertible, is discrete. Thus there exists $\varepsilon > 0$ such that $\mathcal{N}(x^{2p}\Delta)(\xi + i\varepsilon)$ is invertible for all $\xi \in \mathbb{R}$. For such ε the operator Δ in L^2_ε is unitarily equivalent to a fully elliptic cusp operator of order $(k, 2p)$ in L^2. It is then a general fact about the cusp algebra [38, Theorem 17] that such an operator has a unique closed extension and admits a compact inverse modulo compact operators in L^2_ε. In particular, its eigenvalues have finite multiplicity. As noted above, the eigenspaces of Δ in L^2 are contained in the eigenspaces of Δ in L^2_ε for the same eigenvalue. □

As a corollary, the magnetic Laplacians for the metric (2.9) and for vector potentials (3.12) which are smooth cusp 1-forms, have finite multiplicity eigenvalues.

Appendix C. Stability of the essential spectrum

It is well-known that the essential spectrum of an elliptic differential operator on a complete manifold can be computed by cutting out a compact part and studying the Dirichlet extension of the remaining operator on the non-compact part (see, e.g., [9]). This result is obvious using Zhislin sequences, but the approach from loc. cit. fails in the non-complete case. For completeness, we give below a proof which has the advantage to hold in a wider context and for a wider class of operator, pseudodifferential operators for instance.

We start with a general lemma. We recall that a *Weyl sequence* for a couple (H, λ) with H a self-adjoint operator and $\lambda \in \mathbb{R}$, is a sequence $\varphi_n \in \mathcal{D}(H)$ such that $\|\varphi_n\| = 1$, $\varphi_n \rightharpoonup 0$ (weakly) and such that $(H - \lambda)\varphi_n \to 0$, as n goes to infinity. It is well-known that $\lambda \in \sigma_{\mathrm{ess}}(H)$ if and only if there is Weyl sequence for (H, λ).

Lemma C.1. *Let H be a self-adjoint operator in a Hilbert space $\mathscr{H}$. Let φ_n be a Weyl sequence for the couple (H, λ). Suppose that there is a closed operator Φ in $\mathscr{H}$ such that:*

1. *$\Phi\mathcal{D}(H) \subset \mathcal{D}(H)$,*
2. *$\Phi(H + i)^{-1}$ is compact,*
3. *$[H, \Phi]$ is a compact operator from $\mathcal{D}(H)$ to $\mathscr{H}$.*

Then there is $\widetilde{\varphi}_n \in \mathcal{D}(H)$ such that $(1 - \Phi)\widetilde{\varphi}_n$ is a Weyl sequence for (H, λ).

Proof. First we note (2) implies that $\Phi\varphi_n$ goes to 0. Indeed, we have $\Phi\varphi_n = \Phi(H + i)^{-1}((H - \lambda)\varphi_n + (i + \lambda)\varphi_n 1)$ and the bracket goes weakly to 0. Similarly, using (3) we get that $[H, \Phi]\varphi_n \to 0$. Therefore we obtain $\|(1 - \Phi)\varphi_n\| \geq 1/2$ for n

large enough. We set $\widetilde{\varphi}_n := \varphi_n/\|(1-\Phi)\varphi_n\|$. Note that $(1-\Phi)\widetilde{\varphi}_n \rightharpoonup 0$. Finally, $(H-\lambda)(1-\Phi)\widetilde{\varphi}_n \to 0$ since $[H,\Phi]\varphi_n \to 0$. $\square$

This shows that the essential spectrum is given by a "non-compact" part of the space. We now focus on Friedrichs extension. Given a dense subspace $\mathscr{D}$ of a Hilbert space $\mathscr{H}$ and a positive symmetric operator on $\mathscr{D}$. Let $\mathscr{H}_1$ be the completion of $\mathscr{D}$ under the norm given by $\mathscr{Q}(\varphi)^2 = \langle H\varphi, \varphi\rangle + \|\varphi\|^2$. The domain of the Friedrichs extension of H, is given by $\mathcal{D}(H_{\mathscr{F}}) = \{f \in \mathscr{H}_1 \mid \mathscr{D} \ni g \mapsto \langle Hg, f\rangle + \langle g, f\rangle$ extends to a norm continuous function on $\mathscr{H}\}$. For each $f \in \mathcal{D}(H_{\mathscr{F}})$, there is a unique u_f such that $\langle Hg, f\rangle + \langle g, f\rangle = \langle g, u_f\rangle$, by Riesz theorem. The Friedrichs extension of H is defined by setting $H_{\mathscr{F}} f := u_f - f$. It is a self-adjoint extension of H, see [41].

Let (X, g) be a smooth Riemannian with distance d. We fix K a smooth compact sub-manifold of X of same dimension. We endow it with the induced Riemannian metric. We set $X' = X \setminus K$. In the following, we embed $L^2(X')$ in $L^2(X)$. We will need the next definition within the proof.

Definition C.2. We say that Φ is a (smooth) *cut-off function* for K if $\Phi \in \mathcal{C}_c^\infty(X)$ and $\Phi|_K = 1$. We say that it is an ε-cut-off is $\operatorname{supp}(\Phi) \subset B(K, \varepsilon)$.

We are now able to give a result of stability of the essential spectrum.

Proposition C.3. *Let d be a differential form of order 1 on $C_c^\infty(X) \to C^\infty(X, \Lambda^1)$ with injective symbol away from the 0 section of the cotangent bundle. We denote by d_X and $d_{X'}$ the closure of d in $L^2(X)$ and $L^2(X')$, respectively. Consider $\Delta_X := d_X^* d_X$ and $\Delta_{X'} = d_{X'}^* d_{X'}$, the Friedrichs extensions of the operator d^*d, acting on $\mathcal{C}_c^\infty(X)$ and $\mathcal{C}_c^\infty(X')$, respectively. One has $\sigma_{\mathrm{ess}}(\Delta_X) = \sigma_{\mathrm{ess}}(\Delta_{X'})$.*

Proof. Let $f \in L^2(B(K,\varepsilon)^c)$ and let Φ be a ε-cut-off for K. We first show that $f \in \mathcal{D}(d_X)$ if and only if $f \in \mathcal{D}(d_{X'})$. Suppose that $f \in \mathcal{D}(d_X)$, then for all $\eta > 0$, there is $\varphi \in \mathcal{C}_c^\infty(X)$ such that $\|f - \varphi_n\| + \|df - d\varphi_n\| < \eta$. Because of the support of f, one obtain that $\|\Phi\varphi\| < \eta\|\Phi\|_\infty$ and that $\|[d,\Phi]\varphi\| < \eta\|[d,\Phi]\|_\infty$. Therefore $(1-\Phi)\varphi_n \in \mathcal{C}_c^\infty(X')$ and is Cauchy in $\mathcal{D}(d_{X'})$, endow with the graph norm. By uniqueness of the limit, one obtains that $f \in \mathcal{D}(d_{X'})$ and that $d_X f = d_{X'} f$. The opposite implication is obvious. Using again the ε-cut-off, one shows that $g \in \mathcal{D}(d_X^*)$ if and only if $g \in \mathcal{D}(d_{X'}^*)$ and that $d_X^* g = d_{X'}^* g$ for $g \in L^2(\Lambda^1(B(K,\varepsilon)^c)$. Finally, we obtain that $f \in \mathcal{D}(\Delta_X)$ if and only if $f \in \mathcal{D}(\Delta_{X'})$ and that $\Delta_X f = \Delta_{X'} f$, for $f \in L^2(B(K,\varepsilon)^c)$.

From the definition of the Friedrichs extension and the injectivity of the symbol of d, the domain of Δ_X, $\Delta_{X'}$ is contained in $H_0^1(X) \cap H^2_{\mathrm{loc}}(X)$, respectively in $H_0^1(X') \cap H^2_{\mathrm{loc}}(X')$. By taking the same Φ as above and using the Rellich–Kondrakov lemma, the hypotheses of Lemma C.1 are satisfied. Finally, we apply it to Δ_X and $\Delta_{X'}$ and since the Weyl sequence is with support away from K, the first part of the proof gives us the double inclusion of the essential spectra. $\square$

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