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Positive Energy-Momentum Theorem for AdS-Asymptotically Hyperbolic Manifolds

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Abstract. The aim of this paper is to prove a positive energy-momentum theorem under the (well known in general relativity) dominant energy condition, for AdS-asymptotically hyperbolic manifolds. These manifolds are by definition endowed with a Riemannian metric and a symmetric 2-tensor which respectively tend to the metric and second fundamental form of a standard hyperbolic slice in Anti-de Sitter space-time. There exists a positive mass theorem for asymptotically hyperbolic spin Riemannian manifolds (with zero extrinsic curvature), and we present an extension of this result for the non zero extrinsic curvature case.

1. The energy-momentum

1.1. Introduction

This paper proves a positive energy-momentum theorem under the (well known in general relativity) dominant energy condition, for AdS-asymptotically hyperbolic manifolds. An AdS-asymptotically hyperbolic manifold is by definition a manifold (M, g, k) such that at infinity, the Riemannian metric g and the symmetric 2-tensor k tend respectively to the metric and second fundamental form of a standard hyperbolic slice of Anti-de Sitter (AdS).

Chruściel and Nagy [19] recently defined the notion of energy-momentum of an asymptotically hyperbolic manifold, which generalizes the analogous notion in the asymptotically flat case. Besides Chruściel and Herzlich [15] recently proved a positive mass theorem for asymptotically hyperbolic spin Riemannian manifolds (with zero extrinsic curvature).

The aim of the present paper is to extend this result to the non-zero extrinsic curvature case.

1.2. Some definitions and notations

In the whole paper, we consider a spacelike hypersurface M in a (locally defined) Lorentzian manifold N. Using geodesic coordinates along M, we shall write a neighbourhood of M in N as a subset of $]-\epsilon, \epsilon[\times M$, endowed with the metric

$\gamma = -\mathrm{d}t^2 + g_t$. The Riemannian n-manifold M has induced metric $g_0 = g$ and second fundamental form $k := (-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}g_t)_{|t=0}$. We assume that (M, g, k) is AdS-asymptotically hyperbolic that is to say, the metric g and the second fundamental form k are asymptotic at infinity to the metric and the second fundamental form of a standard hyperbolic slice in AdS. More precisely we adopt the following

Definition 1.1. *(M, g, k) is said to be AdS-asymptotically hyperbolic if there exists some compact K, a positive number R and a homeomorphism called a chart at infinity $M \smallsetminus K \longrightarrow \mathbb{R}^n \smallsetminus B(0, R)$ such that in this chart we have*

$$\begin{cases} e := g - b = O(e^{-\tau r}), & \partial e = O(e^{-\tau r}), \quad \partial^2 e = O(e^{-\tau r}), \\ k = O(e^{-\tau r}), & \partial k = O(e^{-\tau r}), \end{cases}$$

for $\tau > n/2$ and where ∂ is taken with respect to the hyperbolic metric $b = \mathrm{d}r^2 + \sinh^2 r g_{\mathbb{S}^{n-1}}$ with $g_{\mathbb{S}^{n-1}}$ the standard metric of $\mathbb{S}^{n-1}$.

The model case is merely the standard embedding of the hyperbolic space $(\mathbb{H}^n, b, 0)$ in AdS space time, which will be denoted by $(\mathbb{R}^{n+1}, \beta = -\mathrm{d}t^2 + b_t)$.

The motivation for the definition of the energy-momentum comes from the study of the constraints map which by definition is

$$\begin{aligned} \Phi\colon\ \mathcal{M} \times \Gamma(S^2T^*M) &\longrightarrow \quad C^\infty(M) \times \Gamma(T^*M) \\ (h, p) \qquad &\longmapsto \begin{pmatrix} \mathrm{Scal}^h + (\mathrm{tr}_h\, p)^2 - |p|_h^2 \\ 2\,(\delta_h p + \mathrm{dtr}_h\, p) \end{pmatrix}, \end{aligned}$$

where $\mathcal{M}$ is the set of Riemannian metrics on the manifold M. Let us denote by $(\dot h, \dot p)$ an infinitesimal deformation of (h, p). Now if we take a couple $(f, \alpha) \in C^\infty(M) \times \Gamma(T^*M)$ then we compute

$$\begin{aligned} &\Big\langle (f, \alpha), (\Phi(h + \dot h, p + \dot p) - \Phi(h, p)) \Big\rangle \\ &\qquad = \delta(f(\delta \dot h + \mathrm{dtr}\, \dot h) + i_{\nabla f}\dot h - (\mathrm{tr}\, \dot h)\mathrm{d}f + 2i_\alpha \dot p - 2(\mathrm{tr}\, \dot p)\alpha) \\ &\qquad\quad + \delta(<p, \dot h>\alpha + <h, \dot h> i_\alpha p - 2i_{i_\alpha p}\dot h) \\ &\qquad\quad + \Big\langle \mathrm{d}\Phi^*_{(h,p)}(f, \alpha), (\dot h, \dot p) \Big\rangle + Q(f, \alpha, h, p, \dot h, \dot p), \end{aligned}$$

where $<,>$ is the metric extended to all tensors, δ is the h-divergence operator, $\mathrm{d}\Phi^*_{(h,p)}$ is the formal adjoint of the linearized constraints map at the point (h, p), traces are taken with respect to h and $Q(f, \alpha, h, p, \dot h, \dot p)$ is a remainder which is linear with respect to (f, α) and at least quadratic with respect to $(\dot h, \dot p)$. Now considering the constraints map along the hyperbolic space embedded in AdS, that is to say $(h, p) = (b, 0)$ and $(\dot h, \dot p) = (g - b = e, k)$ one finds

$$\begin{aligned} \langle (f, \alpha), (\Phi(g, k) - \Phi(b, 0)) \rangle &= \delta(f(\delta e + \mathrm{dtr}\, e) + i_{\nabla f} e - (\mathrm{tr}\, e)\mathrm{d}f + 2i_\alpha k - 2(\mathrm{tr}\, k)\alpha) \\ &\quad + \Big\langle \mathrm{d}\Phi^*_{(b,0)}(f, \alpha), (e, k) \Big\rangle + Q(f, \alpha, b, k, e). \end{aligned}$$

As a consequence if we assume that (M, g, k) is AdS-asymptotically hyperbolic and if the function $\langle (f, \alpha), (\Phi(g, k) - \Phi(b, 0)) \rangle$ is integrable on M with respect to the measure dVol_b, then the energy-momentum $\mathcal{H}$ can be defined as a linear form on $\operatorname{Ker} \mathrm{d}\Phi^*_{(b,0)}$

$$\mathcal{H}\colon (f,\alpha) \longmapsto \lim_{r\to\infty} \int_{S_r(b)} -f(\delta_b e + \mathrm{d}\operatorname{tr}_b e) - i_{\nabla^b f} e + (\operatorname{tr}_b e)\mathrm{d}f - 2 i_{\alpha^\sharp} k + 2(\operatorname{tr}_b k)\alpha.$$

The integrand in the formula of $\mathcal{H}$ is in index notation

$$f(e^i_{j,i} - e^i_{i,j}) - f^{,i} e_{ij} + (e^i_i) f_{,j} - 2\alpha^i k_{ij} + 2(k^i_i)\alpha_j,$$

where "," stands for the b-derivatives and where $h^i = b^{ij} h_j$ for any tensor h. This integrand is the same as the one of Chruściel and Nagy [19] since each Killing vector fields on AdS is decomposable into the sum of some normal and tangential components (with respect to a standard hyperbolic slice) which are in our case given by the couple (f,α) (see also [22]). More precisely, one can show, using Moncrief argument [26], that $\operatorname{Ker} \mathrm{d}\Phi^*_{(b,0)} \cong \mathfrak{Kill}(\mathrm{AdS})$ where $\mathfrak{Kill}(\mathrm{AdS})$ denotes the Lie algebra of Killing vector fields on AdS, since it satisfies the Einstein equations with a (negative) cosmological constant. The isometry group of AdS is $\mathrm{O}(n,2)$, and thereby $\mathfrak{Kill}(\mathrm{AdS}) \cong \mathfrak{so}(n,2) \cong N_b \oplus \mathfrak{so}(n,1) \cong N_b \oplus \mathfrak{Kill}(\mathbb{H}^n)$, where we have set $N_b = \{f \in C^\infty(M) \mid \operatorname{Hess} f = fb\}$. It is well known [19], [15] that the application

$$\begin{array}{lcl} \mathbb{R}^{n,1} & \longrightarrow & N_b \\ y_k & \longmapsto & x_k := y_{k|\mathbb{H}^n} \end{array},$$

(where $(y_k)_{k=0}^n$ are the standard coordinates) is an isometry, and the *mass* part of the energy-momentum is a linear form on N_b which is causal and positively oriented as soon as $\mathrm{Scal}^g \geq -n(n-1) = \mathrm{Scal}^b$. Remark that the sharpest integrability conditions in order to make $\mathcal{H}$ well defined and invariant under asymptotic isometries have been found by Chruściel and Nagy still in [19]. However for the sake of simplicity one can use instead of the integrability condition $\langle (f,\alpha), (\Phi(g,k) - \Phi(b,0))\rangle \in L^1(M, \mathrm{dVol}_b)$, the less general but more convenient condition $|\Phi(g,k) - \Phi(b,0)|\, e^r \in L^1(M, \mathrm{dVol}_b)$.

Remark 1.2. *In the asymptotically flat situation, the energy-momentum is also a linear form on* $\mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1)$ *where the first component corresponds to* translational *isometries* (*if a normal vector to the hypersurface is given, the normal component in the* $\mathbb{R}^{n,1}$ *part is nothing but the mass*) *and the second one to rotations. This interpretation gave rise to the respective terminology of linear and angular momentum. In the AdS-asymptotically hyperbolic situation, one cannot identify some linear momentum in the decomposition* $\mathfrak{so}(n,2) \cong \mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1)$*, since the first component* $\mathbb{R}^{n,1}$ *in* $\mathfrak{so}(n,2)$ *is not of translational nature. This whole first component of the energy is then called the* mass functional *and it only remains some* angular momentum. *Moreover physicists often call the limit of integrals* $\mathcal{H}(f,\alpha)$ global charges *and so the positive energy-momentum theorem could be consequently renamed global inequalities theorem. Some supplementary details on the physical interpretation of our result can be found in the forthcoming note by Chruściel and the author* [17].

1.3. Statement of the theorems and comments

As we have seen, given a chart at infinity, $\mathcal{H}$ can be considered as a vector of $\mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1)$ and will be denoted by $M \oplus \Xi$. The vector M is the mass part [15] of $\mathcal{H}$, and Ξ is the angular momentum. We will prove the existence of a Hermitian quadratic application

$$Q\colon\ \mathbb{C}^d \xrightarrow{\mathcal{K}} \mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1) \xrightarrow{\mathcal{H}} \mathbb{R}\ ,$$

which is nonetheless quite difficult to explicit in general.

We will also treat the case where the slice M has a compact inner boundary ∂M whose induced metric and second fundamental form are respectively denoted by $\breve{g}$ and $\breve{k}$. To this end, we have to define the vector field $\vec{k} := (-\operatorname{tr}\breve{k} + (n-1))e_0 + k(\nu)$ along the boundary ∂M. We can now state the first main result of this paper.

Theorem 1.3 (Positive energy-momentum). *Let (M^n, g, k) be an AdS-asymptotically hyperbolic spin Riemannian manifold satisfying the decay conditions stated in Section 1.2 and the following conditions*

(i) $\langle (f,\alpha), (\Phi(g,k) - \Phi(b,0))\rangle \in L^1(M, \mathrm{dVol}_b)$ *for every* $(f,\alpha) \in N_b \oplus \mathfrak{Kill}(M,b)$,

(ii) *the relative version of the dominant energy condition (cf. Section 2.2) holds, that is to say $(\Phi(g,k) - \Phi(b,0))$ is a positively oriented causal $(n+1)$-vector along M,*

(iii) *in the case where M has a compact boundary ∂M, we assume moreover that $\vec{k}$ is causal and positively oriented along ∂M.*

Then there exists a (hardly explicitable) map $\mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1) \longrightarrow \mathrm{Herm}(\mathbb{C}^d)$ which sends, under the assumptions (i)–(iii), the energy-momentum on a non-negative Hermitian quadratic form Q.

Moreover, in dimension $n = 3$, we can be more specific giving the explicit formula of Q in terms of the components $M \in \mathfrak{M} \subset M(2,\mathbb{C})$ (cf. Section 2.4 for the definition of $\mathfrak{M}$) and $\Xi \in \mathfrak{sl}(2,\mathbb{C})$ of the energy-momentum $\mathcal{H}$. More precisely,

$$Q = 2\begin{pmatrix} \widehat{M} & \Xi \\ \Xi^* & M \end{pmatrix},$$

where $\widehat{M}$ is the transposed comatrix of M.

In dimension $n = 3$, classical algebra results give the non-negativity of each principal minors of Q which provide a set of inequalities on the coefficients of $\mathcal{H}$ that are explicitly written in the appendix (cf. Section 5.1).

This result is new (even though many formal arguments where given by Gibbons, Hull and Warner in [20]) and based on the recent *global charge* definition of Chruściel and Nagy for AdS-asymptotically hyperbolic manifolds, which comes from the Hamiltonian description of General Relativity. Our approach is purely Riemannian, the Lorentzian connection and manifold introduced are auxiliary since everything is restricted to the Riemannian slice M (oppositely to [20]).

In the other hand, the positive mass theorem for Minkowski-asymptotically hyperbolic initial data sets of Chruściel, Jezierski and Łęski [16] is also different from ours, since their Riemannian hypersurface is supposed to be asymptotic at infinity to a standard hyperbolic slice of Minkowski space-time (in that case the extrinsic curvature does not tend to 0). Then considering the translational Killing vector fields of Minkowski, they defined a hyperbolic 4-momentum (usually called *Trautman-Bondi mass*) and proved that it is timelike and future directed under the dominant energy condition (and some other technical assumptions).

Remark finally that our result extends the positive mass theorem of Chruściel and Herzlich [15] in dimension n, since if one supposes that $k = 0$ then we recover their result: the mass functional M has to be time-like future directed.

As regards the rigidity part we have the

Theorem 1.4 (Rigidity). *Under the assumptions of the positive energy-momentum theorem, $Q = 0$ implies that (M, g, k) is isometrically embeddable in $\mathrm{AdS}^{n,1}$.*

This result is optimal in the sense that one could not reasonably hope better than being able to embbed isometrically our triple (M, g, k) in AdS.

Some additional but partial results will be proved (also for the Trautman-Bondi 4-momentum) in order to weaken the defining condition of rigidity (cf. Section 4).

1.4. Organisation of the paper

In Section 2, we give the necessary geometric background by recalling some basic facts on spinors and defining the Killing connection used in the remainder of the paper. We also prove the Bochner-Lichnerowicz-Weitzenbök-Witten formula with respect to our Killing connection and deduce an integration formula.

In Section 3, we prove the positive energy-momentum theorem: we remark that the boundary contribution of the integrated Bochner-Lichnerowicz-Weitzenbök-Witten formula can be identified to the *global charges* $\mathcal{H}(f, \alpha)$, for some choices of (f, α). This can be done using the same ideas as in [15] but in a Lorentzian situation, and extends the quite technical computations of [15] in a non-trivial way, since the algebraic structures are different (spinors, Hermitian scalar product, gauge etc...) and since new terms (involving the extrinsic curvature) appeared and had to be identified. We then make the analysis of the Dirac operator (we also treat the case where M has a compact boundary) which gives the non-negativity of the *global charges* $\mathcal{H}(f, \alpha)$ when the couple (f, α) comes from an imaginary Killing spinor of $\mathrm{AdS}^{n,1}$. Then we restrict to dimension $n = 3$, and completely study the imaginary Killing spinors of $\mathrm{AdS}^{3,1}$ in order to interpret the non-negativity of the *global charges* as the non-negativity of the Hermitian matrix Q on $\mathbb{C}^4$.

Section 4 is devoted to the proof of the *rigidity results*.

The last section is an appendix which gives the non-negativity of Q in dimension $n = 3$ seen through its coefficients, and proves some *rigidity results* for the Trautman-Bondi mass [16].

2. Geometric background

All the definitions and conventions of this section will be stated for any $n \geq 3$, where n is the dimension of the AdS-asymptotically hyperbolic slice (M, g, k), except if the dimension is explicitly mentioned to be 3. Recall that β denotes the AdS background metric.

2.1. Connections and curvatures

Let $\nabla, \overline{\nabla}$ denote respectively the Levi-Civita connections of γ and g. Let us take a spinor field $\psi \in \Gamma(\Sigma)$ (where $\Sigma := \Sigma N_{|M}$) and a vector field $X \in \Gamma(TM)$, then

$$\begin{cases} \nabla_X \psi &= \overline{\nabla}_X \psi - \frac{1}{2} k(X) \cdot e_0 \cdot \psi \\ \langle k(X), Y \rangle_\gamma &= \langle \nabla_X Y, e_0 \rangle_\gamma \end{cases} .$$

In these formulae $\cdot$ denotes the Clifford action with respect to the metric γ, and $e_0 = \partial_t$. We will use different notations when we have to make the difference between the Clifford action with respect to the metric γ or β.

Definition 2.1. *The Killing equation on a spinor field $\tau \in \Gamma(\Sigma)$ is*

$$\widehat{D}_X \tau := D_X \tau + \frac{i}{2} X \cdot_\beta \tau = 0 \quad \forall X \in \Gamma(TM),$$

where D denotes the Levi-Civita connection of AdS *along M. Such a $\widehat{D}$-parallel spinor field is called a β-imaginary Killing spinor and we denote $\tau \in IKS(\Sigma)$. In the same way, a $\widehat{\nabla}$-parallel spinor field (where $\widehat{\nabla}_X := \nabla_X + \frac{\mathrm{i}}{2} X \cdot_\gamma$) is called a γ-imaginary Killing spinor.*

Notice that the equation $\widehat{D}\tau = 0$ is neither the Killing equation in AdS nor in $\mathbb{H}^n$, but the Killing equation in AdS along $\mathbb{H}^n$ (in particular the imaginary Killing spinors considered here are not the one of [3], [15]).

Now if $R, \widehat{R}$ are the respective curvatures of ∇ and $\widehat{\nabla}$, we have the relation

$$\widehat{R}_{X,Y} = R_{X,Y} - \frac{1}{4}(X \cdot Y - Y \cdot X) \cdot \quad ,$$

where we use the convention of [24] for the curvature.

2.2. Bochner-Lichnerowicz-Weitzenbök-Witten formula and the dominant energy condition

From now on $(e_k)_{k=0}^n$ is an orthonormal basis at the point with respect to the metric γ. We define the Dirac-Witten operators

$$\mathfrak{D}\psi = \sum_{k=1}^n e_k \cdot \nabla_{e_k} \psi, \quad \widehat{\mathfrak{D}}\psi = \sum_{k=1}^n e_k \cdot \widehat{\nabla}_{e_k} \psi,$$

where n is the dimension of the spacelike slice.

Lemma 2.2 (Bochner-Lichnerowicz-Weitzenbök-Witten formula).

$$\widehat{\mathfrak{D}}^*\widehat{\mathfrak{D}} = \widehat{\nabla}^*\widehat{\nabla} + \widehat{\mathfrak{R}},$$

where $\widehat{\mathfrak{R}} := \frac{1}{4}\left(\operatorname{Scal}^\gamma + n(n-1) + 4\operatorname{Ric}^\gamma(e_0,e_0) + 2e_0\cdot\operatorname{Ric}^\gamma(e_0)\right)$.

Proof. The Dirac-Witten operator $\mathfrak{D}$ is clearly formally self adjoint, and we have the classical Bochner-Lichnerowicz-Weitzenbök formula (cf. [23], [28] for instance) $\mathfrak{D}^*\mathfrak{D} = \mathfrak{D}^2 = \nabla^*\nabla + \mathfrak{R}$, where $\mathfrak{R} := \frac{1}{4}\left(\operatorname{Scal}^\gamma + 4\operatorname{Ric}^\gamma(e_0,e_0) + 2e_0\cdot\operatorname{Ric}^\gamma(e_0)\right)$. We also know that $\widehat{\mathfrak{D}} = \mathfrak{D} - \mathbf{i}\frac{n}{2}$ and so we get

$$\widehat{\mathfrak{D}}^*\widehat{\mathfrak{D}} = \nabla^*\nabla + \mathfrak{R} + \frac{n^2}{4},$$

but finally remarking that $\widehat{\nabla}^*\widehat{\nabla} = \nabla^*\nabla + \frac{n}{4}$, we obtain our formula. □

We derive an integration formula from the Bochner-Lichnerowicz-Weitzenbök-Witten identity considering the 1-form θ on M defined by $\theta(X) = \left\langle \widehat{\nabla}_X\psi + X\cdot\widehat{\mathfrak{D}}\psi, \psi\right\rangle_\gamma$, where ψ is a spinor field. Straightforward computations lead to the following g-divergence formula

$$\operatorname{div}\theta = \left\langle \widehat{\mathfrak{D}}\psi, \widehat{\mathfrak{D}}\psi\right\rangle_\gamma - \left\langle \widehat{\mathfrak{R}}\psi, \psi\right\rangle_\gamma - \left\langle \widehat{\nabla}\psi, \widehat{\nabla}\psi\right\rangle_\gamma.$$

Let S_r the g-geodesic sphere of radius r and centered in a point of M. The radius r is supposed to be as large as necessary. We denote by M_r the interior domain of S_r and ν_r the (pointing outside) unit normal. Integrating our divergence formula over M_r and using Stokes theorem, we get

$$\int_{M_r}\left|\widehat{\mathfrak{D}}\psi\right|_\gamma^2 = \int_{M_r}\left(\left|\widehat{\nabla}\psi\right|_\gamma^2 + \left\langle\widehat{\mathfrak{R}}\psi,\psi\right\rangle_\gamma\right) - \int_{S_r}\left\langle \widehat{\nabla}_{\nu_r}\psi + \nu_r\cdot\widehat{\mathfrak{D}}\psi, \psi\right\rangle_\gamma \mathrm{dVol}_{S_r} \quad .$$

Let us now consider the Einstein tensor $G = \operatorname{Ric}^\gamma - \frac{1}{2}\operatorname{Scal}^\gamma\gamma$ with respect to the metric γ. The dominant energy condition [33] says that the speed of energy flow of matter is always less than the speed of light. More precisely, for every positively oriented time-like vector field v, the energy-momentum current of density of matter $-G(v,.)^\sharp$ must be time-like or null, with the same orientation as v. The assumption we make in order to prove the positive energy-momentum theorem is a relative version of the dominant energy condition: $-\left(G - \frac{n(n-1)}{2}\gamma\right)(e_0)$ *is a positively oriented time-like or null vector along* M. Some easy computations give

$$\operatorname{Scal}^\gamma = 2(G(e_0,e_0) - \operatorname{Ric}^\gamma(e_0,e_0))$$
$$e_0\cdot\operatorname{Ric}^\gamma(e_0) = e_0\cdot G_{|TM}(e_0) - \operatorname{Ric}^\gamma(e_0,e_0),$$

where $G_{|TM}(e_0) = \sum_{k=1}^{3} G(e_0, e_k)e_k$. Thereby

$$\begin{aligned}
\widehat{\mathfrak{R}} &= \frac{1}{4}\left(2G(e_0,e_0) + (n(n-1)) + 2e_0 \cdot G_{|TM}(e_0)\right) \cdot \\
&= \frac{1}{2}\left(\left(G(e_0,e_0) + \frac{n(n-1)}{2}\right) e_0 - G_{|TM}(e_0)\right) \cdot e_0 \cdot \\
&= \frac{1}{2}\left(\left(G(e_0,e_0) - \frac{n(n-1)}{2}\gamma(e_0,e_0)\right) e_0 - G_{|TM}(e_0)\right) \cdot e_0 \cdot \\
&= -\frac{1}{2}\left(G - \frac{n(n-1)}{2}\gamma\right)(e_0) \cdot e_0 \cdot \quad .
\end{aligned}$$

Our assumption gives the non negativity of the spinorial endomorphism $\widehat{\mathfrak{R}}$ that is to say $\left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle \geq 0$ for every spinor field ψ.

Remark 2.3. *We can express the dominant energy condition in terms of the constraints as in Section* 1.3 *since*

$$-\left(G - \frac{n(n-1)}{2}\gamma\right)(e_0) = \frac{1}{2}\left(\Phi(g,k) - \Phi(b,0)\right).$$

2.3. Spinorial gauge

In the same way as Andersson and Dahl [3], but in a Lorentzian situation, we compare spinors in Σ (along M) with respect to the two different metrics β and γ. This can be done according to [13] as soon as the tubular neighbourhood of M in N is small enough. Consequently we suppose that both metrics are written in Gaussian coordinates $\beta = -\mathrm{d}t^2 + g_t$, $\gamma = -\mathrm{d}t^2 + b_t$ on $]-\epsilon, +\epsilon[\, \times M$ for ϵ small enough. We define the spinorial gauge $\mathcal{A} \in \Gamma(\mathrm{End}(\mathbb{T}))$ with the relations

$$\begin{cases} \gamma(\mathcal{A}X, \mathcal{A}Y) = \beta(X,Y) \\ \gamma(\mathcal{A}X, Y) \quad = \gamma(X, \mathcal{A}Y) \end{cases},$$

where $\mathbb{T}$ is TN restricted to M. The first relation says that $\mathcal{A}$ sends β-orthonormal frames on γ-orthonormal frames whereas the second one means that the endomorphism $\mathcal{A}$ is symmetric. We notice that these relations are only satisfied along $M = \{t = 0\}$ and can also be written in the following way

$$\begin{cases} \mathcal{A}e_0 = e_0 \\ g(\mathcal{A}X, \mathcal{A}Y) = b(X,Y) \\ g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y) \end{cases}.$$

Consequently $\mathcal{A}$ is an application $\mathrm{P}_{\mathrm{SO}_0(n,1)}(\beta)_{|M} \to \mathrm{P}_{\mathrm{SO}_0(n,1)}(\gamma)_{|M}$, which can be covered by an application still denoted by $\mathcal{A}\colon \mathrm{P}_{\mathrm{Spin}_0(n,1)}(\beta)_{|M} \to \mathrm{P}_{\mathrm{Spin}_0(n,1)}(\gamma)_{|M}$. This application carries β-spinors on γ-spinors so that we have the compatibility relation about the Clifford actions of β and γ

$$\mathcal{A}(X \cdot_\beta \sigma) = (\mathcal{A}X) \cdot_\gamma (\mathcal{A}\sigma),$$

for every $X \in \Gamma(\mathbb{T}), \sigma \in \Gamma(\Sigma)$ and where $\cdot_\beta, \cdot_\gamma$ denotes the Clifford actions respectively of β and γ. Remark that our gauge is more sophisticated that the one of [3] since it deals with the trace of Lorentzian structures (metrics, spinors, Hermitian scalar product etc...) along the spacelike slice M.

We define a new connection $\widetilde{\nabla} X = \mathcal{A}(\overline{D}\mathcal{A}^{-1}X)$ along M. It is easy to check that $\widetilde{\nabla}$ is g-metric and has torsion $\widetilde{T}(X,Y) = -((\overline{D}_X\mathcal{A})\mathcal{A}^{-1}Y - (\overline{D}_Y\mathcal{A})\mathcal{A}^{-1}X)$. We extract some formulae for later use

$$2g\left(\widetilde{\nabla}_X Y - \overline{\nabla}_X Y, Z\right) = g\left(\widetilde{T}(X,Y),Z\right) - g\left(\widetilde{T}(X,Z),Y\right) - g\left(\widetilde{T}(Y,Z),X\right).$$

Now we intend to compare the connections $\overline{\nabla}$ and $\widetilde{\nabla}$ on Σ. $(\sigma_s)_s$ denotes the spinorial frame corresponding to the orthonormal frame $(e_k)_{k=0}^n$, and $\overline{\omega}, \widetilde{\omega}$ are the connection 1-forms respectively of $\overline{\nabla}$ and $\widetilde{\nabla}$

$$\begin{aligned} \overline{\omega}_{ij} &= g(\overline{\nabla} e_i, e_j) \\ \widetilde{\omega}_{ij} &= g(\widetilde{\nabla} e_i, e_j) \ , \end{aligned}$$

and if we take a general spinor $\varphi = \varphi^s \sigma_s$, their derivatives are given by

$$\begin{aligned} \overline{\nabla}\varphi &= \mathrm{d}\varphi^s \otimes \sigma_s + \frac{1}{2}\sum_{i<j} \overline{\omega}_{ij} \otimes e_i \cdot_\gamma e_j \cdot_\gamma \varphi \\ \widetilde{\nabla}\varphi &= \mathrm{d}\varphi^s \otimes \sigma_s + \frac{1}{2}\sum_{i<j} \widetilde{\omega}_{ij} \otimes e_i \cdot_\gamma e_j \cdot_\gamma \varphi \ , \end{aligned}$$

and as a consequence

$$(\overline{\nabla} - \widetilde{\nabla})\varphi = \frac{1}{4}\sum_{i,j=0}^{n} (\overline{\omega}_{ij} - \widetilde{\omega}_{ij}) \otimes e_i \cdot_\gamma e_j \cdot_\gamma \varphi \quad .$$

2.4. Tangent and spinor bundles

In this paper, the model spaces $\mathrm{AdS}^{n,1}$ and $\mathbb{H}^n$ are considered as symmetric spaces:

$$\mathbb{H}^n = \mathrm{Spin}_0(n,1)/\,\mathrm{Spin}(n) \hookrightarrow \mathrm{AdS}^{n,1} = \mathrm{Spin}_0(n,2)/\,\mathrm{Spin}_0(n,1) \ ,$$

so that every section of the spinor bundle Σ of AdS restricted to $\mathbb{H}^n$, can be seen as a function $\mathrm{Spin}_0(n,1) \longrightarrow \mathbb{C}^d$ which is $\mathrm{Spin}(n)$-equivariant (with d depending upon n). We can be more explicit when we take $n = 3$ (this fact is due to the exceptional isomorphisms of Lie groups below)

$$\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2) \hookrightarrow \mathrm{AdS}^{3,1} = \mathrm{Spin}_0(3,2)/\,\mathrm{Spin}_0(3,1) \ ,$$

with $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ and $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}_0(3,1)$.

The spinor bundle of AdS is $\Sigma_{\mathrm{AdS}} = \mathrm{Spin}_0(3,2) \times_{\tilde{\rho}} \mathbb{C}^4$, where $\mathrm{Spin}_0(3,2)$ is the bundle of the $\mathrm{Spin}_0(3,1)$-frames in AdS, and $\tilde{\rho}$ is the standard representation of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \overline{\mathbb{C}^2}'$. In other words

$$\begin{aligned} \tilde{\rho}\colon\ \mathrm{SL}(2,\mathbb{C}) &\longrightarrow \mathrm{M}_4(\mathbb{C}) \\ \tilde{g} \quad &\longmapsto \begin{pmatrix} \tilde{g} & 0 \\ 0 & (\tilde{g}^*)^{-1} \end{pmatrix} \ , \end{aligned}$$

where $A^* = {}^t\overline{A}$, $A \in \mathrm{M}_2(\mathbb{C})$. When we restrict this bundle to the hypersurface $\mathbb{H}^3$ we have $\Sigma = \mathrm{SL}(2,\mathbb{C}) \times_{\tilde{\rho}_{|\,\mathrm{SU}(2)}} \mathbb{C}^4$.

Proposition 2.4. *Σ and $\mathbb{H}^3 \times \mathbb{C}^4$ are isomorphic thanks to the following trivialisation:*

$$\begin{aligned} T\colon \quad \Sigma \quad &\longrightarrow \mathbb{H}^3 \times \mathbb{C}^4 \\ \{\tilde{e}, w\} &\longmapsto ([\tilde{e}], \tilde{\rho}(\tilde{e})w) \quad , \end{aligned}$$

where $\{\tilde{e}, w\}$ denotes the class of $(\tilde{e}, w) \in \mathrm{SL}(2,\mathbb{C}) \times \mathbb{C}^4$ in Σ, and $[\tilde{e}]$ denotes the class of $\tilde{e} \in \mathrm{SL}(2,\mathbb{C})$ in $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$.

The construction of $\mathbb{T}_{\mathrm{AdS}}$, the tangent bundle of AdS, is quite similar to the construction of the spinor bundle. Still noticing that the principal bundle of $\mathrm{SO}_0(3,1)$-frames in AdS is isomorphic to $\mathrm{SO}_0(3,2)$, we write $\mathbb{T}_{\mathrm{AdS}} = \mathrm{SO}_0(3,2) \times_\rho \mathbb{R}^4$, where ρ is the standard representation of $\mathrm{SO}_0(3,1)$ on $\mathbb{R}^4$. By restriction to the hypersurface $\mathbb{H}^3$, we obtain $\mathbb{T} = \mathrm{SO}_0(3,1) \times_{\rho_{|\,\mathrm{SO}(3)}} \mathbb{R}^4$, where $\mathrm{SO}(3)$ is by definition the isotropy group of f_0 if $(f_k)_{k=0}^3$ denotes the canonical basis of $\mathbb{R}^4$.

Proposition 2.5. *$\mathbb{T}$ and $\mathbb{H}^3 \times \mathbb{R}^4$ are isomorphic thanks to the following trivialisation:*

$$\begin{aligned} T\colon \quad \mathbb{T} \quad &\longrightarrow \mathbb{H}^3 \times \mathbb{R}^4 \\ \{e, u\} &\longmapsto ([e], \rho(e)u) \quad , \end{aligned}$$

where $\{e, u\}$ denotes the class of $(e, u) \in \mathrm{SO}_0(3,1) \times \mathbb{R}^4$ in $\mathbb{T}$, and $[e]$ denotes the class of $e \in \mathrm{SO}_0(3,1)$ in $\mathbb{H}^3 = \mathrm{SO}_0(3,1)/\,\mathrm{SO}(3)$.

We are going to define the Clifford action on Σ, in the same way as in [28]. To this end, we denote by $(\mathbb{R}^4, q)$ the Minkowski space-time of signature (3,1), where $q = -\mathrm{d}y_0^2 + \mathrm{d}y_1^2 + \mathrm{d}y_2^2 + \mathrm{d}y_3^2$. This space is isometric to a subspace of $\mathrm{M}_2(\mathbb{C})$ via

$$\begin{aligned} \Lambda\colon \quad (\mathbb{R}^4, q) \quad &\longrightarrow \mathfrak{M} := (\{A \in \mathrm{M}_2(\mathbb{C}) \mid A^* = A\}, -\det) \\ y = (y_i)_{i=0}^3 &\longmapsto \qquad \begin{pmatrix} y_0 + y_1 & y_2 + iy_3 \\ y_2 - iy_3 & y_0 - y_1 \end{pmatrix} \quad . \end{aligned}$$

We have thus the following real vector space isomorphisms:

$$\begin{aligned} \mathrm{M}_2(\mathbb{C}) &\cong \mathfrak{u}(2) \oplus \mathfrak{M} \\ \mathfrak{sl}_2(\mathbb{C}) &\cong \mathfrak{su}(2) \oplus (\mathfrak{M} \cap \mathfrak{sl}_2(\mathbb{C})) \\ &\cong \mathfrak{su}(2) \oplus \mathfrak{G} \ , \end{aligned}$$

and $\mathfrak{G} \cong \mathbb{R}^3$. In order to make the value of the sectional curvature of $\mathbb{H}^3$ equal to -1, when we consider $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$ as a symmetric space, we have to consider $\mathbb{R}^4$ endowed with $4q$ and not q, and consequently the embedding of the Clifford algebra $\mathrm{C}\ell_{3,1}$ in $\mathrm{M}_4(\mathbb{C})$ becomes

$$\Theta\colon X \in \mathfrak{M} \longmapsto \begin{pmatrix} 0 & 2X \\ 2\widehat{X} & 0 \end{pmatrix},$$

where $\widehat{X}$ means the transposed comatrix of X.

It will be convenient to see $\mathbb{T}$ as $\mathrm{SL}(2,\mathbb{C}) \times_\mu \mathfrak{M}$, where μ is the universal covering of $\mathrm{SO}_0(3,1)$ by $\mathrm{SL}(2,\mathbb{C})$, which is given by:

$$\begin{array}{llll} \mu\colon & \mathrm{SL}(2,\mathbb{C}) \longrightarrow & \mathrm{SO}_0(3,1) & \\ & \tilde{g} \qquad \longmapsto & (\tilde{g}\colon X \in \mathfrak{M} \mapsto \tilde{g} X \tilde{g}^*) & . \end{array}$$

We can now define the Clifford action. Let us take $e \in \mathrm{SO}_0(3,1)$ and $\tilde{e} \in \mathrm{SL}(2,\mathbb{C})$ such that $e = \mu(\tilde{e})$. A vector $X = X[e]$ tangent at the point $[e] = [\tilde{e}] \in \mathbb{H}^3$, is a class $\{e,u\} \in \mathbb{T}$. A spinor $\sigma = \sigma[\tilde{e}]$ at the same point is likewise a class $\{\tilde{e},w\} \in \Sigma$. The result of the Clifford action of X on σ is the spinor $(X\cdot\sigma)[\tilde{e}] = \{e,u\}\cdot\{\tilde{e},w\} = \{\tilde{e},\Theta(u)w\}$. We define a sesquilinear inner product (not definite positive) $(\cdot,\cdot)$ on $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \overline{\mathbb{C}^2}'$ as in [28] $(\xi,\eta) := \langle \xi_1,\eta_2\rangle_{\mathbb{C}^2} + \langle \xi_2,\eta_1\rangle_{\mathbb{C}^2}$, where $\xi = \binom{\xi_1}{\xi_2}, \eta = \binom{\eta_1}{\eta_2} \in \mathbb{C}^4$ and where $\langle\cdot,\cdot\rangle_{\mathbb{C}^2}$ is the standard Hermitian product on $\mathbb{C}^2$. This induces a sesquilinear product on Σ by $(\{\tilde{e},\xi\},\{\tilde{e},\eta\}) := (\xi,\eta)$. In the same way we define a scalar product on Σ setting

$$\begin{aligned} \langle \{\tilde{e},\xi\},\{\tilde{e},\eta\}\rangle &:= \left(\tfrac{1}{2} f_0 \cdot \{\tilde{e},\xi\},\{\tilde{e},\eta\}\right) \\ &= (\{\tilde{e},\tfrac{1}{2}\Theta(f_0)\xi\},\{\tilde{e},\eta\}) \\ &= \langle \xi,\eta\rangle_{\mathbb{C}^4}\,, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^4}$ denotes the standard Hermitian product on $\mathbb{C}^4$.

Since $\mathrm{SL}(2,\mathbb{C})$ is the 2-sheeted covering of $\mathrm{SO}_0(3,1)$, there exists a natural (left) action of $\mathrm{SL}(2,\mathbb{C})$ on Σ which is derived from the natural (left) action of $\mathrm{SO}_0(3,1)$ on $\mathbb{T}$: the action of the group of the isometries of AdS preserving the slice $\mathbb{H}^3$ that is $\tilde{g}*\{\tilde{e},w\} = \{\tilde{g}\tilde{e},w\}$, with $\tilde{g} \in \mathrm{SL}(2,\mathbb{C})$ and $\sigma[\tilde{e}] = \{\tilde{e},w\}$ a spinor at $[\tilde{e}]$. To have the action on a section $\sigma \in \Gamma(\Sigma)$ we set as usual $(\tilde{g}*\sigma)[\tilde{e}] = \tilde{g}*\sigma(\tilde{g}^{-1}\tilde{e})$.

3. Positive energy-momentum theorem

In this section, the dimension will be $n \geq 3$ expect if n is explicitly mentioned to be 3.

Moreover f will denote a smooth cutoff function which is 0 on M except on a small neighbourhood of the infinity boundary of M where $f \equiv 1$, and $H(a)$, $H_-(a)$ are Hilbert spaces of spinor fields defined in the subsections below. We will prove the

Proposition 3.1. *For every β-imaginary Killing spinor $\sigma \in IKS(\Sigma)$ there exists a unique γ-spinor field $\xi_0 \in H(a)$ (resp. $\in H_-(a)$ if M has a boundary) such that*

$$\xi = f\mathcal{A}\sigma + \xi_0 \in \operatorname{Ker}\widehat{\mathfrak{D}} \ (\textit{resp.} \ \in \operatorname{Ker}\widehat{\mathfrak{D}} \cap H_-(a)) \quad \textit{and} \quad \mathcal{H}(V_\sigma,\alpha_\sigma) \geq 0,$$

where $V_\sigma = \langle\sigma,\sigma\rangle$ and $\alpha_\sigma(X) = \langle X\cdot e_0\cdot\sigma,\sigma\rangle$.

In Section 3.3, it will proved that the couple (V_σ,α_σ) belongs to $N_b \oplus \mathfrak{Kill}(\mathbb{H}^n)$ so that $\mathcal{H}(V_\sigma,\alpha_\sigma)$ is actually well defined.

The computations we will make in Section 3.1 prove that if we integrate the Bochner-Lichnerowicz-Weitzenbök-Witten formula with an asymptotically imaginary Killing spinor $f\mathcal{A}\sigma$, then the boundary integrals tend to some *global charge*

$\mathcal{H}(V_\sigma, \alpha_\sigma)$, when r goes to infinity. In fact it is still true if we *perturb* $f\mathcal{A}\sigma$ with a smooth compactly supported spinor field ξ_0 (that is to say if we consider $f\mathcal{A}\sigma + \xi_0$ instead of $f\mathcal{A}\sigma$). Actually we will show in Section 3.2 that we can find a *perturbation* ξ_0 in a relevant Hilbert space such that ξ_0 has no contribution at infinity, and $f\mathcal{A}\sigma + \xi_0$ belongs to the kernel of $\widehat{\mathfrak{D}}$.

This will naturally imply the non-negativity of $\mathcal{H}(V_\sigma, \alpha_\sigma)$ when σ is a *β-imaginary Killing spinor*. This is the reason why we focus on the study of the Killing equation in Section 3.3 so as to interpret the non-negativity of the $\mathcal{H}(V_\sigma, \alpha_\sigma)$.

3.1. Energy-momentum and imaginary Killing spinors

The aim of this section is to show the

Proposition 3.2. *Let $\xi = f\mathcal{A}\sigma + \xi_0$, where $\sigma \in IKS(\Sigma)$ and ξ_0 is a compactly supported spinor field. Then we have*

$$\begin{aligned}\mathcal{H}(V_\sigma, \alpha_\sigma) &= 4 \lim_{r \to +\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r}\xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma \\ &= 4 \int_M \left(\left|\widehat{\nabla}\xi\right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle_\gamma \right) - 4 \int_M \left|\widehat{\mathfrak{D}}\xi\right|_\gamma^2 .\end{aligned}$$

Remark that the only important data is the exact *β-imaginary Killing spinor* σ involved in the definition of the couple $(V_\sigma, \alpha_\sigma)$.

Proof. Remember that

$$\int_{M_r} \left|\widehat{\mathfrak{D}}\psi\right|_\gamma^2 = \int_{M_r} \left(\left|\widehat{\nabla}\psi\right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle_\gamma \right) - \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r}\psi + \mathcal{A}\nu_r \cdot_\gamma \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma \mathrm{dVol}_{S_r} ,$$

where ν_r denotes the b-normal of S_r, $e_0 = \partial_t$ and we set $e_1 = \mathcal{A}\nu_r$ for the remainder of the proof.

We have to work on the expression $\left\langle \widehat{\nabla}_{\mathcal{A}\nu_r}\psi + \mathcal{A}\nu_r \cdot_\gamma \widehat{\mathfrak{D}}\psi, \psi \right\rangle_\gamma$ in order to identify the integrand used to compute the energy-momentum for some couple (f, α). We start with noticing that $e_1 \cdot_\gamma e_1 \cdot_\gamma \widehat{\nabla}_{e_1} = -\widehat{\nabla}_{e_1} = -\widehat{\nabla}_{\mathcal{A}\nu_r}$ so that

$$\widehat{\nabla}_{\mathcal{A}\nu_r}\psi + \mathcal{A}\nu_r \cdot_\gamma \widehat{\mathfrak{D}}\psi = \mathcal{A}\nu_r \cdot_\gamma \left(\sum_{j=2}^n e_j \cdot_\gamma \widehat{\nabla}_{e_j} \right) \psi.$$

From now on we work on

$$\left\langle \mathcal{A}\nu_r \cdot_\gamma \left(\sum_{j=2}^n e_j \cdot_\gamma \widehat{\nabla}_{e_j} \right) \cdot_\gamma \mathcal{A}\sigma, \mathcal{A}\sigma \right\rangle_\gamma .$$

Let us take σ a *β-imaginary Killing spinor*, that is to say a spinor field solution, by definition, of $\widehat{D}_X\sigma = D_X\sigma + \frac{\mathrm{i}}{2} X \cdot_\beta \sigma = 0$, for every vector field $X \in \Gamma(TM)$.

Consider f a smooth cutoff function which is 0 on M except on a compact neighbourhood of the infinity boundary of M where $f \equiv 1$. Then we have

$$\begin{aligned}\widehat{\nabla}_X(f\mathcal{A}\sigma) &= \mathrm{d}f(X)\mathcal{A}\sigma + f\widehat{\nabla}_X(\mathcal{A}\sigma)\\ &= \mathrm{d}f(X)\mathcal{A}\sigma + f(\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma)\\ &\quad + f\left(\widetilde{\nabla}_X + \tfrac{\mathbf{i}}{2}\cdot_\gamma - \tfrac{1}{2}k(X)\cdot_\gamma e_0\cdot_\gamma\right)(\mathcal{A}\sigma),\end{aligned}$$

but since $\widetilde{\nabla}_X(\mathcal{A}\sigma) = \mathcal{A}\overline{D}_X\sigma = -\frac{\mathbf{i}}{2}\mathcal{A}(X\cdot_\beta\sigma) = -\frac{i}{2}(\mathcal{A}X)\cdot_\gamma(\mathcal{A}\sigma)$, we obtain

$$\widehat{\nabla}_X(f\mathcal{A}\sigma) = \mathrm{d}f(X)\mathcal{A}\sigma + f(\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma) - \frac{1}{2}f\left(k(X)\cdot_\gamma e_0 + \mathbf{i}(\mathcal{A}-Id)X\right)\cdot_\gamma(\mathcal{A}\sigma),$$

that we restrict to the neighbourhood where $f \equiv 1$

$$\widehat{\nabla}_X(\mathcal{A}\sigma) = (\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma) - \frac{1}{2}(k(X)\cdot_\gamma e_0 + \mathbf{i}(\mathcal{A}-Id)X)\cdot_\gamma(\mathcal{A}\sigma).$$

As a consequence our boundary term becomes for r great enough

$$\sum_{j=2}^n \left\langle \mathcal{A}\nu_r\cdot_\gamma e_j\cdot_\gamma\left((\overline{\nabla}_{e_j} - \widetilde{\nabla}_{e_j}) - \frac{1}{2}(k(e_j)\cdot_\gamma e_0 + \mathbf{i}(\mathcal{A}-Id)e_j)\cdot_\gamma\right)(\mathcal{A}\sigma), \mathcal{A}\sigma\right\rangle_\gamma .$$

We will estimate this boundary term in several steps. From the decay assumptions stated in Section 1.2, the gauge is supposed to be of the form $\mathcal{A} = Id + B + O(|B|^2)$, where B has the same decay to 0 as $e = g - b$. In the following $(\epsilon_j = \mathcal{A}^{-1}e_j)_{j=0}^n$ is a β-orthonormal frame.

We begin with the easiest term

$$\begin{aligned}&\sum_{j=2}^n \langle \mathcal{A}\nu_r\cdot_\gamma e_j\cdot_\gamma k(e_j)\cdot_\gamma e_0\cdot_\gamma(\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma\\ &\qquad= \sum_{j=2}^n \langle \mathcal{A}\nu_r\cdot_\gamma \mathcal{A}\epsilon_j\cdot_\gamma k(\mathcal{A}\epsilon_j)\cdot_\gamma \mathcal{A}\epsilon_0\cdot_\gamma(\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma\\ &\qquad= \sum_{j=2}^n \left\langle \nu_r\cdot_\beta \epsilon_j\cdot_\beta \mathcal{A}^{-1}\circ k\circ\mathcal{A}(\epsilon_j)\cdot_\beta\epsilon_0\cdot_\beta\sigma, \sigma\right\rangle_\beta .\end{aligned}$$

But we note that $\mathcal{A}^{-1}\circ k\circ\mathcal{A} = k - B\circ k + k\circ B + O(|B|^2)$. Now B has the same decay as k so $B\circ k + k\circ B = O(|B|^2)$, terms that we can neglect since the energy-momentum is computed by a limit procedure of integrals over large spheres. We conclude that $\mathcal{A}^{-1}\circ k\circ\mathcal{A} \approx k$, where for convenience the relation $\diamond \approx \star$ means that $|\diamond - \star|$ is at least a $O(e^{-2\tau r})$ when r goes to infinity. Moreover

$$\begin{aligned}\nu_r\cdot_\beta\sum_{j=2}^n \epsilon_j\cdot_\beta k(\epsilon_j) &= \epsilon_1\cdot_\beta\left(\sum_{j=1}^n \epsilon_j\cdot_\beta k(\epsilon_j) - \epsilon_1\cdot_\beta k(\epsilon_1)\right)\\ &= k(\nu_r) - (\operatorname{tr}_b k)\nu_r,\end{aligned}$$

which implies

$$\sum_{j=2}^{n} \langle \mathcal{A}\nu_r \cdot_\gamma e_j \cdot_\gamma k(e_j) \cdot_\gamma e_0 \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \approx \langle k(\nu_r) - (\mathrm{tr}_b\, k)\nu_r \cdot_\beta \epsilon_0 \cdot_\beta \sigma, \sigma\rangle_\beta$$
$$= (i_{\alpha_\sigma} k - (\mathrm{tr}_b\, k)\alpha_\sigma)\,(\nu_r),$$

where $\alpha_\sigma(X) = \langle X \cdot_\beta \epsilon_0 \cdot_\beta \sigma, \sigma\rangle_\beta$.

The second term we study is

$$\begin{aligned}
&\mathbf{i}\sum_{j=2}^{n} \langle \mathcal{A}\nu_r \cdot_\gamma e_j \cdot_\gamma (\mathcal{A} - Id)(e_j) \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \\
&\qquad = \mathbf{i}\sum_{j=2}^{n} \langle \mathcal{A}\nu_r \cdot_\gamma \mathcal{A}\epsilon_j \cdot_\gamma (\mathcal{A} - Id)(\mathcal{A}\epsilon_j) \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \\
&\qquad = \mathbf{i}\sum_{j=2}^{n} \left\langle \nu_r \cdot_\beta \epsilon_j \cdot_\beta \mathcal{A}^{-1} \circ (\mathcal{A} - Id) \circ \mathcal{A}(\epsilon_j) \cdot_\beta \sigma, \sigma\right\rangle_\beta \\
&\qquad \approx \mathbf{i}\sum_{j=2}^{n} \langle \nu_r \cdot_\beta \epsilon_j \cdot_\beta B(\epsilon_j) \cdot_\beta \sigma, \sigma\rangle_\beta\,,
\end{aligned}$$

but thanks to the same property as above

$$\nu_r \cdot_\beta \sum_{j=2}^{n} \epsilon_j \cdot_\beta B(\epsilon_j) = B(\nu_r) - (\mathrm{tr}_b\, B)\nu_r,$$

which induces

$$\mathbf{i}\sum_{j=2}^{n} \langle \mathcal{A}\nu_r \cdot_\gamma e_j \cdot_\gamma (\mathcal{A} - Id)(e_j) \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \approx \mathbf{i}\,\langle B(\nu_r) - (\mathrm{tr}_b\, B)\nu_r \cdot_\beta \sigma, \sigma\rangle_\beta$$
$$= (i_{\nabla V_\sigma} B - (\mathrm{tr}_b\, B)\mathrm{d}V_\sigma)\,(\nu_r),$$

where $\mathrm{d}V_\sigma(X) = \mathbf{i}\,\langle X \cdot_\beta \sigma, \sigma\rangle_\beta$.

The last term we have to study is certainly the most difficult (summation convention $k \in \{2, 3, \ldots, n\}$, $l \in \{1, 2, \ldots, n\}$, $m \in \{1, 2, \ldots, n\}$)

$$\begin{aligned}
&\left\langle \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma (\overline{\nabla}_{e_k} - \widetilde{\nabla}_{e_k})(\mathcal{A}\sigma), \mathcal{A}\sigma\right\rangle_\gamma \\
&\qquad = \frac{1}{4}\,\langle (\overline{\omega}_{lm} - \widetilde{\omega}_{lm})(e_k)\mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma e_l \cdot_\gamma e_m \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \\
&\qquad = \frac{1}{4}\,\langle (\overline{\omega}_{lm} - \widetilde{\omega}_{lm}) \circ \mathcal{A}(\epsilon_k)\nu_r \cdot_\beta \epsilon_k \cdot_\beta \epsilon_l \cdot_\beta \epsilon_m \cdot_\beta \sigma, \sigma\rangle_\beta \\
&\qquad = \frac{1}{4} S
\end{aligned}$$

$$
\begin{aligned}
S = & \sum_{k,l,m=2}^{n} \langle (\overline{\omega}_{lm} - \widetilde{\omega}_{lm})(e_k) \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma e_l \cdot_\gamma e_m \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma \\
& + 2 \sum_{k,l=2}^{n} \langle (\overline{\omega}_{1l} - \widetilde{\omega}_{1l})(e_k) \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma e_1 \cdot_\gamma e_l \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma \\
& = S_1 + 2S_2.
\end{aligned}
$$

We will give estimates of each S_k, keeping in mind that they are real and that every term that is at least $O(|B|^2)$ can be neglected when $r \to +\infty$ for the computations of the *global charge integrals*.

Estimate of S_1

$$
S_1 = \sum_{k,l,m=2}^{n} \langle (\overline{\omega}_{lm} - \widetilde{\omega}_{lm})(e_k) \mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma e_l \cdot_\gamma e_m \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma .
$$

We can keep only the subscripts $l \neq m$ because of the skew-symmetry of $(\omega - \widetilde{\omega})$. Besides if we suppose that $k = l$, we have terms like $\langle \mathcal{A}\nu_r \cdot_\gamma e_m \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma \rangle_\gamma$ which belong to $\mathbf{i}\mathbb{R}$. So we can sum over k, l, m distinct subscripts without any loss of generality. On the other hand

$$
(\overline{\omega}_{lm} - \widetilde{\omega}_{lm})(e_k) = \frac{1}{2} \Big(-g(\widetilde{T}(e_k, e_l), e_m) + g(\widetilde{T}(e_k, e_m), e_l) + g(\widetilde{T}(e_l, e_m), e_k) \Big)
$$

where the two last terms of the right-hand side member are symmetric with respect to (l, k), so they vanish when we sum over k and l distinct. Consequently

$$
\begin{aligned}
& (\overline{\omega}_{lm} - \widetilde{\omega}_{lm})(e_k) \epsilon_k \cdot_\beta \epsilon_l \cdot_\beta \epsilon_m \\
& \qquad = \frac{1}{2} b \left(\mathcal{A}^{-1}(\overline{D}_{e_k}\mathcal{A})\epsilon_l - \mathcal{A}^{-1}(\overline{D}_{e_l}\mathcal{A})\epsilon_k, \epsilon_m \right) \epsilon_k \cdot_\beta \epsilon_l \cdot_\beta \epsilon_m \\
& \qquad = b \left(\mathcal{A}^{-1}(\overline{D}_{e_k}\mathcal{A})\epsilon_l, \epsilon_m \right) \epsilon_k \cdot_\beta \epsilon_l \cdot_\beta \epsilon_m ,
\end{aligned}
$$

but

$$
\begin{aligned}
b(\mathcal{A}^{-1}(\overline{D}_{e_k}\mathcal{A})\epsilon_l, \epsilon_m) &= b(\mathcal{A}^{-1}(\overline{D}_{e_k}(\mathcal{A}\epsilon_l) - \mathcal{A}\overline{D}_{e_k}\epsilon_l), \epsilon_m) \\
&\approx b(\overline{D}_{\epsilon_k}(B\epsilon_l) - B(\overline{D}_{\epsilon_k}\epsilon_l), \epsilon_m) \\
&= b((\overline{D}_{\epsilon_k}B)\epsilon_l, \epsilon_m),
\end{aligned}
$$

expression which is symmetric with respect to (l, m), since $\overline{D}B$ is a symmetric endomorphism. Consequently

$$
\sum_{k,l,m \text{ distinct}} (\overline{\omega}_{lm} - \widetilde{\omega}_{lm})(e_k) \epsilon_k \cdot_\beta \epsilon_l \cdot_\beta \epsilon_m \approx 0,
$$

when $r \to +\infty$.

Estimate of S_2

$$
\begin{aligned}
S_2 &= \sum_{k,l=2}^{n} \langle (\overline{\omega}_{1l} - \widetilde{\omega}_{1l})(e_k)\mathcal{A}\nu_r \cdot_\gamma e_k \cdot_\gamma e_1 \cdot_\gamma e_l \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \\
&= -\sum_{k=2}^{n} \langle (\overline{\omega}_{1k} - \widetilde{\omega}_{1k})(e_k)\sigma, \sigma\rangle_\beta \\
&\quad + \sum_{k\neq l} \langle (\overline{\omega}_{1l} - \widetilde{\omega}_{1l})(e_k) e_k \cdot_\gamma e_l \cdot_\gamma (\mathcal{A}\sigma), \mathcal{A}\sigma\rangle_\gamma \,,
\end{aligned}
$$

but the second sum is in $\mathbf{i}\mathbb{R}$, so it remains

$$
\Re e(S_2) = -\left(\sum_{k=1}^{n} (\overline{\omega}_{1k} - \widetilde{\omega}_{1k})(e_k)\right) V_\sigma .
$$

We only have to compute

$$
\begin{aligned}
-\sum_{k=1}^{n} (\overline{\omega}_{1k} - \widetilde{\omega}_{1k})(e_k) &= \sum_{k=1}^{n} g((\overline{D}_{e_1}\mathcal{A})\mathcal{A}^{-1}e_k, e_k - (\overline{D}_{e_k}\mathcal{A})\mathcal{A}^{-1}e_1, e_k) \\
&= S_2' - S_2'' .
\end{aligned}
$$

We focus on

$$
\begin{aligned}
S_2'' &= \sum_{k=1}^{n} g((\overline{D}_{e_k}\mathcal{A})\mathcal{A}^{-1}e_1, e_k) \\
&= \sum_{k=1}^{n} b(\mathcal{A}^{-1}(\overline{D}_{e_k}\mathcal{A})\epsilon_1, \epsilon_k) \\
&= \sum_{k=1}^{n} b(\mathcal{A}^{-1}\overline{D}_{e_k}(\mathcal{A}\epsilon_1) - \overline{D}_{e_k}\epsilon_1, \epsilon_k) \\
&\approx \sum_{k=1}^{n} b((\overline{D}_{e_k}B)\epsilon_1, \epsilon_k) \\
&\approx \sum_{k=1}^{n} b(\epsilon_1, (\overline{D}_{\epsilon_k}B)\epsilon_k) \\
&= -\operatorname{div}_b B(\nu_r).
\end{aligned}
$$

As regards the first term S_2', we decompose the gauge endomorphism $\mathcal{A}$ as follows: $\mathcal{A}\epsilon_i = \sum_{k=0}^{n} \mathcal{A}_i^k \epsilon_k$. We remind that $\mathcal{A}\epsilon_0 = \epsilon_0$, $\mathcal{A}(TM) \subset TM$ and so we have $\mathcal{A}_0^k = \mathcal{A}_k^0 = 0$, $k \geq 1$.

$$\begin{aligned}
S_2' &= \sum_{k=1}^{n} g((\overline{D}_{e_1}\mathcal{A})\mathcal{A}^{-1}e_k, e_k) \\
&= \sum_{k=1}^{n} b((\mathcal{A}^{-1}\overline{D}_{e_1}\mathcal{A})\epsilon_k, \epsilon_k) \\
&= \sum_{k=1}^{n} b(\mathcal{A}^{-1}\overline{D}_{e_1}(\mathcal{A}\epsilon_k) - \overline{D}_{e_1}\epsilon_k, \epsilon_k) \\
&\approx \sum_{k,l=1}^{n} (e_1 \cdot \mathcal{A}_k^l)\,\{b(\epsilon_l, \epsilon_k) - b(B\epsilon_l, \epsilon_k)\} - \sum_{k=1}^{n} b(\overline{D}_{e_1}\epsilon_k, \epsilon_k) \\
&\quad + \sum_{k,l=1}^{n} \mathcal{A}_k^l \left\{b(\overline{D}_{e_1}\epsilon_l, \epsilon_k) - b(B\overline{D}_{e_1}\epsilon_l, \epsilon_k)\right\} \\
&\approx (\epsilon_1 \cdot \operatorname{tr}_b B) - \sum_{k=1}^{n} b(\overline{D}_{e_1}\epsilon_k, \epsilon_k) \\
&\quad + \sum_{k,l=1}^{n} (\delta_l^k + B_k^l)\left\{b(\overline{D}_{e_1}\epsilon_l, \epsilon_k) - b(B\overline{D}_{e_1}\epsilon_l, \epsilon_k)\right\} \\
&\approx (\epsilon_1 \cdot \operatorname{tr}_b B) - \sum_{k=1}^{n} \left\{b(B\overline{D}_{e_1}\epsilon_k, \epsilon_k) - b(\overline{D}_{e_1}\epsilon_k, B\epsilon_k)\right\} \\
&= \mathrm{d}(\operatorname{tr}_b B)(\nu_r),
\end{aligned}$$

that entails

$$\Re e(S_2) \approx V_\sigma(\mathrm{d}(\operatorname{tr}_b B) + \operatorname{div}_b B)(\nu_r).$$

We can conclude, taking $B = -\frac{1}{2}e$, that the real part of our boundary integrand is nothing but

$$\frac{1}{4}\left(-V_\sigma(\delta_b e + \mathrm{d}\operatorname{tr}_b e) - i_{\nabla^b V_\sigma} e + (\operatorname{tr}_b e)\mathrm{d}V_\sigma - 2i_{\alpha_\sigma^\sharp} k + 2(\operatorname{tr}_b k)\alpha_\sigma\right)(\nu_r),$$

what achieves the proof. □

3.2. Analysis of $\widehat{\mathfrak{D}}$

This section is devoted to the study of the analytical properties of $\widehat{\mathfrak{D}}$. The first paragraph deals with the case where M has no boundary whereas the second one deals with the case where M has a compact and connected boundary denoted as usual by ∂M.

3.2.1. The boundaryless case.

Proposition 3.3. *For every $\sigma \in IKS(\Sigma)$, there exists a unique $\xi_0 \in H(a)$ such that $\xi = f\mathcal{A}\sigma + \xi_0 \in \operatorname{Ker}\widehat{\mathfrak{D}}$ and*

$$\mathcal{H}(V_\sigma, \alpha_\sigma) = \lim_{r\to+\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r}\xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma = 4\int_M \left(\left|\widehat{\nabla}\xi\right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle_\gamma \right) \geq 0.$$

Proof. We study in a usual way the analytical properties of $\widehat{\mathfrak{D}}$. Let us consider $C_0^\infty(\Sigma) = C_0^\infty$ the space of smooth and compactly supported spinors. We define a sesquilinear form on C_0^∞ by

$$a(\varphi,\psi) = \int_M \left\langle \widehat{\mathfrak{D}}\varphi, \widehat{\mathfrak{D}}\psi \right\rangle_\gamma \mathrm{d}\mu_g,$$

where $\mathrm{d}\mu_g$ denotes the standard volume form of the metric g. The form a is clearly bounded and non-negative on C_0^∞. We define the usual Sobolev space

$$H^1(\Sigma) = \left\{ \psi \in \Sigma \;\middle|\; \int_M |\psi|_\gamma^2 + |\nabla\psi|_\gamma^2 < \infty \right\}.$$

Definition 3.4. *We set* $H(a) := \overline{C_0^\infty}^a$.

Remark 3.5 (Weighted Poincaré inequality).

$$\exists \omega \in L^1_{\mathrm{loc}}(M, \mathrm{dVol}_g) \;\; \mathrm{ess}_M \inf \omega > 0 \;\; \forall u \in C_0^1 \;\; \int_M \omega |u|^2 \,\mathrm{dVol}_g \le \int_M |\widehat{\nabla} u|^2 \,\mathrm{dVol}_g\,.$$

The symmetric part of the connection $\widehat{\nabla}$ *is given by* $\Gamma_X = \frac{1}{2}\{k(X)\cdot e_0 - \mathbf{i}X\}\,\cdot$, *and so satisfies the conditions* (*cf.* [7]) *in order to have the existence of a weighted Poincaré inequality that is to say* $\Gamma \in L^n_{\mathrm{loc}}(M)$ *and* $\lim\sup_{x\to 0} |x\Gamma_x| < \frac{n-1}{2}$. *Such a weighted Poincaré inequality insures the continuity of the embedding of* $H(a) \hookrightarrow H^1_{loc}$. *This claim is true even when* M *has a compact boundary so that we will make its proof when there is a boundary. Indeed consider* $(\psi_k)_{k\in\mathbb{N}} \in (C_0^\infty)^{\mathbb{N}}$ *a Cauchy sequence with respect to the form* a *whose elements satisfy the boundary condition* $F(\psi_k) = -\psi_k$ (*cf. next section for the definition of the boundary endomorphism* F). *Then we have* (*the vector field* $\vec{k}$ *will also be defined in the next section*)

$$\int_M \left|\widehat{\mathfrak{D}}\psi_k\right|_\gamma^2 = \int \left(\left|\widehat{\nabla}\psi_k\right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\psi_k, \psi_k \right\rangle_\gamma \right) + \frac{1}{2}\int_{\partial M} \left\langle e_0 \cdot \vec{k}\cdot \psi_k, \psi_k \right\rangle ,$$

and thus thanks to the weighted Poincaré inequality

$$\forall \Omega \subset M \quad |\Omega| < \infty \quad \psi_k \xrightarrow{L^2(\Omega)} \psi \quad \text{and} \quad \widehat{\nabla}\psi_k \xrightarrow{L^2(M)} \rho\,.$$

Now let us take a $\varphi \in C_0^1$ *such that* $\mathrm{Supp}\,\varphi \subset K \subset (M \setminus \partial M)$ (K *compact without boundary*) *and then*

$$\begin{array}{ccc}
\int_K \left\langle \widehat{\nabla}^*\varphi, \psi_k \right\rangle & = & \int_K \left\langle \varphi, \widehat{\nabla}\psi_k \right\rangle , \\
\Big\downarrow{\scriptstyle k\to\infty} & & \Big\downarrow{\scriptstyle k\to\infty} \\
\int_K \left\langle \widehat{\nabla}^*\varphi, \psi \right\rangle & = & \int_K \langle \varphi, \rho \rangle
\end{array}$$

and therefore $\rho = \widehat{\nabla}\psi$ *in the distributional sense.*

We notice that for r great enough and for $\sigma \in IKS(\Sigma)$

$$\begin{aligned}\widehat{\nabla}_X(\mathcal{A}\sigma) &= \nabla_X(\mathcal{A}\sigma) + \frac{\mathbf{i}}{2} X \cdot_\gamma (\mathcal{A}\sigma)\\ &= (\overline{\nabla}_X - \widetilde{\nabla}_X)(\mathcal{A}\sigma) - \frac{1}{2}(k(X)\cdot_\gamma e_0 + \mathbf{i}(\mathcal{A} - Id)X)\cdot_\gamma (\mathcal{A}\sigma).\end{aligned}$$

But the relations

$$\begin{cases} \widetilde{T}(X,Y) = -((\overline{D}_X\mathcal{A})\mathcal{A}^{-1}Y - (\overline{D}_Y\mathcal{A})\mathcal{A}^{-1}X) \\ 2g(\widetilde{\nabla}_X Y - \overline{\nabla}_X Y, Z) = g(\widetilde{T}(X,Y),Z) - g(\widetilde{T}(X,Z),Y) - g(\widetilde{T}(Y,Z),X) \end{cases}$$

tell us that $|(\overline{\omega}_{ij} - \widetilde{\omega_{ij}})(e_k)| \leq C|\mathcal{A}^{-1}||\overline{D}\mathcal{A}|$. We get an estimate

$$|\widehat{\mathfrak{D}}(\mathcal{A}\sigma)| \leq C|\mathcal{A}|(|\overline{D}\mathcal{A}| + |\mathcal{A} - Id| + |k|)|\sigma| \in L^2(M, \mathrm{d}\mu_g),$$

which infers that $\widehat{\mathfrak{D}}(f\mathcal{A}\sigma) \in L^2(M, \mathrm{d}\mu_g)$. We now consider the linear form l on $H(a)$ defined by

$$l(\psi) = \int_M \left\langle \widehat{\mathfrak{D}}(f\mathcal{A}\sigma), \widehat{\mathfrak{D}}\psi \right\rangle_\gamma \mathrm{d}\mu_g.$$

Thanks to our estimate above we get $|l(\psi)|^2 \leq \left\| \widehat{\mathfrak{D}}(f\mathcal{A}\sigma)\right\|^2_{L^2} a(\psi,\psi)$, that gives the continuity of l in $H(a)$. We can claim, thanks to Lax-Milgram theorem, that there exists a unique $\xi_0 \in H(a)$ such that $l = a(-\xi_0, \cdot)$. In other words

$$\int_M \left\langle (\widehat{\mathfrak{D}})^* \widehat{\mathfrak{D}}(f\mathcal{A}\sigma + \xi_0), \psi \right\rangle_\gamma = 0.$$

Since $\widehat{\mathfrak{D}}^* = \widehat{\mathfrak{D}} + \mathbf{i}n$, we have in the distributional sense $(\widehat{\mathfrak{D}} + \mathbf{i}n)\widehat{\mathfrak{D}}\xi = 0$, where we have set $\xi = f\mathcal{A}\sigma + \xi_0$. By an elliptic regularity argument, $\widehat{\mathfrak{D}}\xi$ is in fact smooth and $(\widehat{\mathfrak{D}})^k\xi$ are L^2, for every $k \in \mathbb{N}$. It follows

$$\begin{aligned}\int_M \left\langle (\widehat{\mathfrak{D}})^2\xi, (\widehat{\mathfrak{D}})^2\xi \right\rangle_\gamma &= \int_M \left\langle (\widehat{\mathfrak{D}} + \mathbf{i}n)(\widehat{\mathfrak{D}})^2\xi, \widehat{\mathfrak{D}}\xi \right\rangle_\gamma \\ &= \int_M \left\langle \widehat{\mathfrak{D}}(\widehat{\mathfrak{D}} + \mathbf{i}n)\widehat{\mathfrak{D}}\xi, \widehat{\mathfrak{D}}\xi \right\rangle_\gamma \\ &= 0,\end{aligned}$$

that implies $(\widehat{\mathfrak{D}})^2\xi = 0$, but we already know that $(\widehat{\mathfrak{D}} + \mathbf{i}n)\widehat{\mathfrak{D}}\xi = 0$, and thereby $\widehat{\mathfrak{D}}\xi = 0$. We now apply our integration formula to ξ

$$\begin{aligned}\mathcal{H}(V_\sigma, \alpha_\sigma) &= \lim_{r\to+\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r}\xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma \\ &= 4\int_M \left(\left|\widehat{\nabla}\xi\right|^2_\gamma + \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle_\gamma \right) - 4\int_M \left|\widehat{\mathfrak{D}}\xi\right|^2_\gamma \\ &= 4\int_M \left(\left|\widehat{\nabla}\xi\right|^2_\gamma + \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle_\gamma \right) \geq 0,\end{aligned}$$

and the proof is complete. $\square$

3.2.2. The non-empty boundary case. We will consider, in this section, a Riemannian slice M that has a non-empty inner compact boundary ∂M. $\breve{g}, \breve{\nabla}, \breve{k}$ will denote respectively the induced metric, the connection and the second fundamental form which is defined by

$$\overline{\nabla}_X Y = \breve{\nabla}_X\, Y - \breve{k}(X,Y)\nu$$
$$\overline{\nabla}_X \psi = \breve{\nabla}_X\, \psi - \frac{1}{2}\breve{k}(X)\cdot\nu\cdot\psi,$$

where ν is the normal to ∂M pointing toward infinity (that is to say pointing inside), and $\cdot$ still denotes the Clifford action with respect to the metric γ. Consequently our integration formula has another boundary term

$$\int_{M_r}\left|\widehat{\mathfrak{D}}\psi\right|^2_\gamma = \int_{M_r}\left(\left|\widehat{\nabla}\psi\right|^2_\gamma + \left\langle\widehat{\mathfrak{R}}\psi,\psi\right\rangle_\gamma\right) - \int_{S_r}\left\langle\widehat{\nabla}_{\mathcal{A}\nu_r}\psi + \mathcal{A}\nu_r\cdot\widehat{\mathfrak{D}}\psi,\psi\right\rangle_\gamma + \int_{\partial M}\left\langle\widehat{\nabla}_\nu\psi + \nu\cdot\widehat{\mathfrak{D}}\psi,\psi\right\rangle_\gamma .$$

But if ψ is a compactly supported smooth spinor field then, making $r\to\infty$ one finds

$$\int_M\left|\widehat{\mathfrak{D}}\psi\right|^2_\gamma = \int_M\left(\left|\widehat{\nabla}\psi\right|^2_\gamma + \left\langle\widehat{\mathfrak{R}}\psi,\psi\right\rangle_\gamma\right) + \int_{\partial M}\left\langle\widehat{\nabla}_\nu\psi + \nu\cdot\widehat{\mathfrak{D}}\psi,\psi\right\rangle_\gamma .$$

We then have to estimate the boundary integrand $\left\langle\widehat{\nabla}_\nu\psi + \nu\cdot\widehat{\mathfrak{D}}\psi,\psi\right\rangle$.

Lemma 3.6. *If $(\nu = e_1, e_2, \ldots, e_n)$ is a local orthonormal frame of $TM_{|\partial M}$ then*

$$\widehat{\nabla}_\nu\psi + \nu\cdot\widehat{\mathfrak{D}}\psi = \nu\cdot\sum_{k=2}^n \widehat{\nabla}_{e_k}\psi \quad .$$

Proof. Just remark that $\widehat{\nabla}_\nu\psi = -e_1\cdot e_1\cdot\widehat{\nabla}_{e_1}\psi$. □

Lemma 3.7. *Keeping our orthonormal frame $(\nu = e_1, e_2, \ldots, e_n)$, we have*

$$\widehat{\nabla}_\nu\psi + \nu\cdot\widehat{\mathfrak{D}}\psi = \sum_{k=2}^n \nu\cdot e_k\cdot\breve{\nabla}_{e_k}\,\psi + \frac{1}{2}\left\{-\operatorname{tr}\breve{k} - (n-1)\mathbf{i}\nu + (\operatorname{tr}k)\nu\cdot e_0 - k(\nu)\cdot e_0\right\}\cdot\psi \quad .$$

Proof. Using the formula above, we then express $\widehat{\nabla}$ in term of the $(n-1)$-dimensional connection and second form, and the n-dimensional second form. □

Let us define $F\in\operatorname{End}(\Sigma_{|\partial M})$ by $F(\psi) = \mathbf{i}\nu\cdot\psi$. We sum up some basic properties of F in the following

Proposition 3.8. *The endomorphism F is symmetric, isometric with respect to $\langle\cdot,\cdot\rangle$, commutes to the action of $\nu\cdot$ and anticommutes to each $e_k\cdot$, $(k\neq 1)$.*

Lemma 3.9. *If $F(\psi) = -\psi$ then*

$$\left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_{|\partial M} = \frac{1}{2}\left\langle e_0 \cdot \Big((-\operatorname{tr}\breve{k} + (n-1))e_0 + k(\nu) \Big) \cdot \psi, \psi \right\rangle .$$

Proof. Using the proposition above we know that $\nu \cdot e_k \, (k \neq 1)$ anticommutes with F and the formula follows since F respects $\langle \cdot, \cdot \rangle$. □

Assumption. *Let us suppose that the 4-vector $\vec{k} := (-\operatorname{tr}\breve{k} + (n-1))e_0 + k(\nu)$ is causal and positively oriented, that is to say $\gamma(\vec{k}, \vec{k}) \leq 0$ and $\operatorname{tr}\breve{k} \leq (n-1)$.*

This assumption (which is exactly the same as for $\widehat{\mathfrak{R}}$) guarantees the non-negativity of the boundary integrand term $\left\langle \widehat{\nabla}_\nu \psi + \nu \cdot \widehat{\mathfrak{D}}\psi, \psi \right\rangle_{|\partial M} = \frac{1}{2}\left\langle e_0 \cdot \vec{k} \cdot \psi, \psi \right\rangle$, whenever the boundary condition $F(\psi) = -\psi$ is satisfied. Although this assumption is vectorial, it clearly extends the one given in [15].

Let us define $H_-(a) = \{\psi \in H(a) \mid F(\psi) = -\psi\}$ where $H(a)$ has been defined in Section 3.6.1. Still taking ψ a compactly supported smooth spinor field in $H_-(a)$, we have

$$a(\psi, \psi) = \int_M \left(\left| \widehat{\nabla}\psi \right|_\gamma^2 + \left\langle \widehat{\mathfrak{R}}\psi, \psi \right\rangle_\gamma \right) + \frac{1}{2}\int_{\partial M} \left\langle e_0 \cdot \vec{k} \cdot \psi, \psi \right\rangle ,$$

whose each single term is non-negative tanks to our assumption.

We consider the linear form l on $H_-(a)$ defined by

$$l(\psi) = \int_M \left\langle \widehat{\mathfrak{D}}(f\mathcal{A}\sigma), \widehat{\mathfrak{D}}\psi \right\rangle_\gamma \mathrm{d}\mu_g .$$

It still is a continuous linear form on the Hilbert space $H_-(a)$ (it is complete since the condition $F(\psi) = -\psi$ is closed) and applying again Lax-Milgram theorem we get the existence of a unique $\xi_0 \in H(a)$ such that $l = a(-\xi_0, \cdot)$. In other words

$$\forall \psi \in H_-(a) \quad \int_M \left\langle \chi, \widehat{\mathfrak{D}}\psi \right\rangle = 0,$$

where we have set $\xi = f\mathcal{A}\sigma + \xi_0$ and $\chi = \widehat{\mathfrak{D}}\xi$.

For any $\psi \in C_0^1$ we have

$$\int_M \left\langle \chi, \widehat{\mathfrak{D}}\psi \right\rangle = 0 = \int_M \left\langle \widehat{\mathfrak{D}}^* \chi, \psi \right\rangle + \int_{\partial M} \langle \nu \cdot \chi, \psi \rangle .$$

But remembering that $C_0^\infty(M \setminus \partial M)$ the space of smooth spinor fields compactly supported in $M \setminus \partial M$ is dense in $L^2(M)$ then we obtain that $\widehat{\mathfrak{D}}^* \chi = 0$ and $\chi \in H_+(a) = \{\psi \in H(a) \mid F(\psi) = +\psi\}$. By ellipticity χ is smooth and $\widehat{\mathfrak{D}}^k \chi \in L^2(M)$

for every $k \in \mathbb{N}$. Finally we notice that

$$\begin{aligned}\int_M \left|\widehat{\mathfrak{D}}\chi\right|^2 &= \int_M \left\langle \widehat{\mathfrak{D}}^*\widehat{\mathfrak{D}}\chi, \chi \right\rangle + \int_{\partial M} \left\langle \nu \cdot \widehat{\mathfrak{D}}\chi, \chi \right\rangle \\ &= 0 + \int_{\partial M} \langle -\mathbf{i}n\nu \cdot \chi, \chi \rangle \\ &= -n \int_{\partial M} |\chi|^2 ,\end{aligned}$$

and therefore $\widehat{\mathfrak{D}}\chi = 0$ which implies that $\chi = 0$. We can conclude with the

Proposition 3.10. *For every $\sigma \in IKS(\Sigma)$ there exists a unique $\xi_0 \in H_-(a)$ such that $\xi = f\mathcal{A}\sigma + \xi_0 \in \operatorname{Ker}\widehat{\mathfrak{D}} \cap H_-(a)$ and*

$$\begin{aligned}\mathcal{H}(V_\sigma, \alpha_\sigma) &= \lim_{r\to+\infty} \int_{S_r} \left\langle \widehat{\nabla}_{\mathcal{A}\nu_r}\xi + \mathcal{A}\nu_r \cdot \widehat{\mathfrak{D}}\xi, \xi \right\rangle_\gamma \\ &= 4 \int_M \left(\left|\widehat{\nabla}\xi\right|^2_\gamma + \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle_\gamma \right) + 2 \int_{\partial M} \left\langle e_0 \cdot \vec{k} \cdot \psi, \psi \right\rangle \geq 0.\end{aligned}$$

3.3. Imaginary Killing spinors

In general (that is to say whatever the dimension), the spinor bundle under consideration is $\Sigma = \mathrm{Spin}_0(n,1)/\,\mathrm{Spin}(n) \times_\rho \mathbb{C}^d$ for a certain spinorial representation ρ and some integer d depending upon $n+1$. It is known that Σ is trivialized by the space of imaginary Killing spinors of $\mathrm{AdS}^{n,1}$ along $\mathbb{H}^n$ (cf. [14] for instance). Now consider some $\sigma \in IKS(\Sigma)$, and define the function $V_\sigma = \langle \sigma, \sigma \rangle$ and the real 1-form $\alpha_\sigma(X) = \langle X \cdot e_0 \cdot \sigma, \sigma \rangle$. We can easily compute the first derivative of α_σ

$$D_X \alpha_\sigma(Y) = \frac{\mathbf{i}}{2} \langle (X \cdot Y - Y \cdot X) \cdot e_0 \cdot \sigma, \sigma \rangle ,$$

which is a real skew symmetric 2-form and hence α_σ is a Killing form on $\mathbb{H}^n$. Furthermore, it is clear that $V_\sigma \in N_b \cong \mathbb{R}^{n,1}$ [15]. By this way, we merely define a quadratic application

$$IKS(\Sigma) \longrightarrow \mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1) \ .$$

Now the complex isomorphism $\mathbb{C}^d \xrightarrow{\sim} IKS(\Sigma)$ is quite difficult to explicit (it may be possible by the means of harmonic analysis, and it will be the aim of a future paper) because of the non-explicit character of the Clifford action when the dimension is arbitrary. However we have a formal Hermitian quadratic application

$$\mathcal{K}\colon\ \mathbb{C}^d \longrightarrow IKS(\Sigma) \longrightarrow \mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1) \ ,$$

that we will explicit when the dimension of the slice is $n = 3$ (because of exceptional isomorphisms of Lie groups).

The 3-dimensional case

The aim of this section is to solve explicitly the Killing equation of Section 2.2. As a matter of fact, representation theory provides us good candidates for the imaginary Killing spinors. Thanks to Schur's lemma, we have an isomorphism

$$\begin{array}{lcl} \mathbb{C}^2 & \longrightarrow & \mathrm{Hom}^{\mathrm{SU}(2)}(\mathbb{C}^2, \mathbb{C}^2 \oplus \mathbb{C}^2) \\ \binom{z_1}{z_2} & \longmapsto & \binom{z_1 \mathrm{I}_2}{z_2 \mathrm{I}_2} \end{array} .$$

We are now considering two families of spinors which are derived from representation theory. To this end, we will denote $w \otimes z \in \mathbb{C}^2 \otimes \mathrm{Hom}^{\mathrm{SU}(2)}(\mathbb{C}^2, \mathbb{C}^2 \oplus \mathbb{C}^2)$ thanks to the isomorphism above.

Definition 3.11. *Let $w \otimes z \in \mathbb{C}^2 \otimes \mathbb{C}^2$ and set $\sigma^{-1}_{w\otimes z}[\tilde{g}] = \left\{\tilde{g}, z(\tilde{g}^{-1} w)\right\}$, $\sigma^{*}_{w\otimes z}[\tilde{g}] = \{\tilde{g}, z(\tilde{g}^{*} w)\}$.*

Let us consider a spinor field $\tau \in \Gamma(\Sigma)$ and a vector field $X \in \Gamma(\mathbb{T})$ tangent to $\mathbb{H}^3$. We can write $\tau[\tilde{g}] = \{\tilde{g}, v(\tilde{g})\}$ and $X[g] = \{g, \zeta(g)\}$, where $v\colon \mathbb{H}^3 \longrightarrow \mathbb{C}^4$ and $\zeta\colon \mathbb{H}^3 \longrightarrow \mathfrak{G}$ are respectively SO(3) and SU(2)-equivariant functions. We can now differentiate τ in the direction of X and write down

$$(D_X \tau)[\tilde{g}] = \{\tilde{g}, v_*(X)_{[\tilde{g}]} + \tilde{\rho}_* \circ s^* \theta(\zeta)_{[\tilde{g}]} v[\tilde{g}]\},$$

where θ is the connection 1-form of the bundle of $\mathrm{SL}(2,\mathbb{C})$-frames, restricted to $\mathbb{H}^3$. If one remembers that θ is only the projection on the first factor in the decomposition $\mathfrak{sl}_2(\mathbb{C}) \cong \mathfrak{su}(2) \oplus \mathfrak{G}$, we can conclude that $\tilde{\rho}_* \circ s^* \theta(\zeta)_{[\tilde{g}]} v[\tilde{g}]$ vanishes. Besides we will apply this formula to spinors in $\left\{\sigma^{-1}_{w\otimes z}, \sigma^{*}_{u\otimes z}, w, u \in \mathbb{C}^2\right\}$ so that we can only derive at the point $\tilde{g} = 1$ unity in $\mathrm{SL}(2,\mathbb{C})$ since we have the

Proposition 3.12. *The set $\left\{\sigma^{-1}_{w\otimes z}, \sigma^{*}_{u\otimes z}, w, u \in \mathbb{C}^2\right\}$ is stable under the $\mathrm{SL}(2,\mathbb{C})$ action. More precisely for every $\tilde{e} \in \mathrm{SL}(2,\mathbb{C})$ we have $\tilde{e} * \sigma^{-1}_{w\otimes z} = \sigma^{-1}_{\tilde{e}w\otimes z}$ and $\tilde{e} * \sigma^{*}_{u\otimes z} = \sigma^{*}_{(\tilde{e}^*)^{-1}u\otimes z}$.*

We obtain

$$\begin{cases} (D_X \sigma^{-1}_{w\otimes z})[1] = \{1, -z(\zeta w)\} \\ (D_X \sigma^{*}_{u\otimes z})[1] = \{1, z(\zeta u)\} \end{cases} ,$$

where $\zeta = \zeta(1)$. We also compute the Clifford action of X on $\sigma^{-1}_{w\otimes z}, \sigma^{*}_{u\otimes z}$ at the point 1:

$$\begin{cases} X \cdot \sigma^{-1}_{w\otimes z}[1] = \{1, \Theta(\zeta) z(w)\} \\ X \cdot \sigma^{*}_{u\otimes z}[1] = \{1, \Theta(\zeta) z(u)\} \end{cases} .$$

We must precise $\Theta_{|\mathfrak{G}} : \zeta \longmapsto \begin{pmatrix} 0 & 2\zeta \\ -2\zeta & 0 \end{pmatrix}$, and if we introduce the sections $\sigma^{-1}_{w\otimes\binom{1}{-i}}$ and $\sigma^{*}_{w\otimes\binom{1}{i}}$, for any $w \in \mathbb{C}^2$, we have on one hand

$$\begin{cases} -\frac{i}{2} X \cdot \sigma^{-1}_{w\otimes\binom{1}{-i}}[1] = -i\,\{1, -i\zeta w \oplus -\zeta w\} = \{1, -\zeta w \oplus i\zeta w\} \\ -\frac{i}{2} X \cdot \sigma^{*}_{u\otimes\binom{1}{i}}[1] \;\; = -i\,\{1, i\zeta u \oplus -\zeta u\} \quad\; = \{1, \zeta u \oplus i\zeta u\} \end{cases} ,$$

and on the other hand

$$\begin{cases} \left(D_X \sigma^{-1}_{w\otimes\binom{1}{-i}}\right)[1] = \{1, -\zeta w \oplus i\zeta w\} \\ \left(D_X \sigma^{*}_{u\otimes\binom{1}{i}}\right)[1] \ = \{1, \zeta u \oplus i\zeta u\} \quad . \end{cases}$$

Since $\left\{\sigma^{-1}_{w\otimes\binom{1}{-i}} + \sigma^{*}_{u\otimes\binom{1}{i}} \;\middle|\; w, u \in \mathbb{C}^2\right\}$ is a 4-dimensional complex vector space, we obviously obtain the

Proposition 3.13. *The space of imaginary Killing spinors denoted by $IKS(\Sigma)$ is generated by*

$$\left\{\sigma^{-1}_{w\otimes\binom{1}{-i}}, \sigma^{*}_{u\otimes\binom{1}{i}}, w, u \in \mathbb{C}^2\right\}.$$

Let σ an imaginary Killing spinor and set $V_\sigma := <\sigma, \sigma>$ which is a function on $\mathbb{H}^3$, and if e_0 denotes a unit normal of $\mathbb{H}^3$ in AdS, we set $\alpha_\sigma(Y) := \langle Y \cdot e_0 \cdot \sigma, \sigma\rangle$ which is a real 1-form on $\mathbb{H}^3$. The goal of the two next paragraphs is to define some $\mathrm{SL}(2,\mathbb{C})$-equivariant application

$$\begin{aligned} \mathcal{K}\colon\ IKS(\Sigma) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow (\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^{*\mathbb{R}} \\ w \oplus u &\longmapsto \mathcal{K}_{w\oplus u} := (V_{w\oplus u} \oplus \alpha_{w\oplus u}) \quad . \end{aligned}$$

The functions V_σ when $n = 3$

We compute the functions V_σ which are by definition

$$\begin{aligned} V_\sigma[\tilde{g}] &= \left|\sigma^{-1}_{w\otimes\binom{1}{-i}}[\tilde{g}]\right|^2_{\mathbb{C}^4} + \left|\sigma^{*}_{u\otimes\binom{1}{i}}[\tilde{g}]\right|^2_{\mathbb{C}^4} + 2\Re e\left(\left\langle \sigma^{-1}_{w\otimes\binom{1}{-i}}[\tilde{g}], \sigma^{*}_{u\otimes\binom{1}{i}}[\tilde{g}]\right\rangle_{\mathbb{C}^4}\right) \\ &= 2\left|\tilde{g}^{-1}w\right|^2_{\mathbb{C}^2} + 2\left|\tilde{g}^* u\right|^2_{\mathbb{C}^2} . \end{aligned}$$

Remark 3.14. $\sigma^{-1}_{w\otimes\binom{1}{-i}}$ *and* $\sigma^{*}_{u\otimes\binom{1}{i}}$ *are orthogonal spinors for every $u, w \in \mathbb{C}^2$.*

If $\tilde{g} \in \mathrm{SL}(2,\mathbb{C})$, the corresponding base point is $\tilde{g}\tilde{g}^* \in \mathbb{H}^3 \subset \mathfrak{M} \cong \mathbb{R}^{3,1}$ whose coordinates are given by $(x_k)_{k=0}^3 = \Lambda^{-1}(\tilde{g}\tilde{g}^*)$.

Proposition 3.15. *V_σ is a causal element of N_b.*

Proof. Let $U = \binom{u_1}{-\overline{w_2}} \in \mathbb{C}^2, V = \binom{u_2}{\overline{w_1}} \in \mathbb{C}^2$. We notice that

$$V_\sigma[\tilde{g}] = x_0(|U|^2 + |V|^2) + x_1(|U|^2 - |V|^2) + 2x_2\Re e(<U, V>) - 2x_3\Im m(<U, V>),$$

so that the norm of V_σ is $|V_\sigma[\tilde{g}]|^2 = 4\left(|<U, V>|^2 - |U|^2|V|^2\right) \le 0$, thanks to the Cauchy-Schwarz inequality for the standard Hermitian form on $\mathbb{C}^2$. $\square$

More conceptually we see that $V_\sigma[\tilde{g}] = 2(w^*\widehat{W}w + u^*Wu)$, where we have set $W := \tilde{g}\tilde{g}^* \in \mathbb{H}^3 \subset \mathfrak{M}$. Thereby we can define by extension an application

$$\begin{aligned} \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow \mathfrak{M}^* \\ w \oplus u \quad &\longmapsto \left(V_{w\oplus u} : W \mapsto 2(w^*\widehat{W}w + u^*Wu)\right) \quad . \end{aligned}$$

The 1-forms α_σ when $n = 3$

The positively oriented unit normal of $\mathbb{H}^3$ in AdS is given by $e_0[\tilde{g}] = \left\{\tilde{g}, \frac{1}{2}\mu(\tilde{g})I_2\right\}$ and for any $\xi \in \mathfrak{G}$ satisfying $-\det\xi = 1$ we set $X^\xi[\tilde{g}] = \left\{\tilde{g}, \frac{1}{2}\mu(\tilde{g})\xi\right\}$. Just remember that $\alpha_\sigma(X^\xi)_{[\tilde{g}]} := \left\langle X^\xi \cdot e_0 \cdot \sigma, \sigma\right\rangle_{[\tilde{g}]}$. As we suppose that $\sigma \in IKS(\Sigma)$, we can easily compute the first derivative of α_σ

$$D_{X^\eta}\alpha_\sigma(X^\xi)_{[\tilde{g}]} = \frac{\mathbf{i}}{2}\left\langle (X^\eta \cdot X^\xi - X^\xi \cdot X^\eta) \cdot e_0 \cdot \sigma, \sigma\right\rangle_{[\tilde{g}]},$$

which is a real skew symmetric 2-form and hence α_σ is a Killing form on $\mathbb{H}^3$. From now on we set $\alpha_\sigma = (\alpha_\sigma)_1$ and $D\alpha_\sigma = (D\alpha_\sigma)_1$, that we will write as function of $w \oplus u$. After some computations we find

$$\begin{cases} \alpha_\sigma(\xi) & = 2(w^*\xi u + u^*\xi w) \\ D\alpha_\sigma(\eta,\xi) & = (w^*(\xi\eta - \eta\xi)u - u^*(\xi\eta - \eta\xi)w) \end{cases}.$$

We have to notice that $\xi\eta - \eta\xi \in \mathbf{i}\mathfrak{G}$ so that $D\alpha_\sigma$ is naturally a linear form on $\mathbf{i}\mathfrak{G}$. As a consequence we define, thanks to the Killing 1-form α_σ, the following application

$$\begin{aligned} \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow \mathfrak{sl}_2(\mathbb{C})^{*\mathbb{R}} \\ w \oplus u &\longmapsto (\alpha_{w\oplus u} : \xi \mapsto 2(w^*\xi u + u^*\xi^* w)) \end{aligned},$$

where ${}^{*\mathbb{R}}$ stands for the duality with respect to the reals. We then define

$$\mathcal{K}_{w\oplus u} = V_{w\oplus u} \oplus \alpha_{w\oplus u}$$

and conclude with the

Proposition 3.16. *The application $\mathcal{K}$ is $\mathrm{SL}(2,\mathbb{C})$-equivariant. More precisely, for every $\tilde{e} \in \mathrm{SL}(2,\mathbb{C})$*

$$\mathcal{K}_{\tilde{e}*(w\oplus u)} = \left(V_{w\oplus u} \circ \mu(\tilde{e}^{-1})\right) \oplus \left(\alpha_{w\oplus u} \circ \mathrm{Ad}(\tilde{e}^*)\right).$$

Proof. We must compute for every $W \in \mathfrak{M}$ and $\xi \in \mathfrak{sl}_2(\mathbb{C})$

$$\begin{aligned} \mathcal{K}_{\tilde{e}*(w\oplus u)}(W,\xi) &= \mathcal{K}_{\tilde{e}w\oplus(\tilde{e}^*)^{-1}u}(W,\xi) \\ &= 2(w^*\tilde{e}^*\widehat{W}\tilde{e}w + u^*\tilde{e}^{-1}W(\tilde{e}^*)^{-1}u + w^*\tilde{e}^*\xi(\tilde{e}^*)^{-1}u + u^*\tilde{e}^{-1}\xi^*\tilde{e}w) \\ &= V_{w\oplus u} \circ \mu(\tilde{e}^{-1})(W) \oplus \alpha_{w\oplus u} \circ \mathrm{Ad}(\tilde{e}^*)(\xi). \end{aligned}$$ □

Remark 3.17 (The norm of imaginary Killing spinors). *Classical considerations on Lie algebras show that $\mathfrak{so}(3,2)$ endowed with its Killing form, is isometric to $(\mathfrak{M}, -\det) \oplus (\mathfrak{sl}_2(\mathbb{C}), -\Re e(\det))$ which is a 10-dimensional real vector space of signature (6,4). The norm of $\mathcal{K}(w \oplus u)$ with respect to the Killing form is, up to a multiplicative and positive constant $|\mathcal{K}(w \oplus u)|^2 = |{<}U,V{>}|^2 - |U|^2|V|^2 + \Re e(\chi^2)$, where we have set $\chi = \overline{u_1}w_1 + \overline{u_2}w_2$. Besides, if $V_{w\oplus u}$ is isotropic in $\mathfrak{M}$ then $\alpha_{w\oplus u}$ and $\mathcal{K}(w\oplus u)$ are also isotropic respectively in $\mathfrak{sl}_2(\mathbb{C})^*$ and $(\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^*$. Indeed the equality case in the Cauchy-Schwarz inequality occurs if and only if U and V satisfy $\det_{\mathbb{C}^2}(U,V) = \bar{\chi} = 0$.*

3.4. End of the proof

Whatever the dimension is, we obtain a Hermitian quadratic application

$$Q\colon\ \mathbb{C}^d \xrightarrow{\mathcal{K}} \mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1) \xrightarrow{\mathcal{H}} \mathbb{R}\ ,$$

which has to be non-negative in vertue of the non-negativity results of Sections 3.2.1 and 3.2.2. This completes the proof of the first part of the (positivity) Theorem 1.3. It is also important to notice that in the zero extrinsic curvature case, that is $k \equiv 0$, then $\mathcal{H}(f,\alpha) = \mathcal{H}(f,0)$ which is merely the hyperbolic mass functional, and thereby we recover the result of [15].

In dimension $n = 3$, we can be more specific giving the explicit formula of Q in terms of the components of the energy-momentum $\mathcal{H}$. More precisely, on one hand we have found a quadratic application

$$\begin{aligned}\mathcal{K}\colon\ IKS(\Sigma) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 &\longrightarrow (\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^{*\mathbb{R}} \cong \operatorname{Ker} \mathrm{d}\Phi^*_{(b,0)}\\ w \oplus u &\longmapsto (V_{w\oplus u} \oplus \alpha_{w\oplus u})\ \ ,\end{aligned}$$

which is $\mathrm{SL}(2,\mathbb{C})$-equivariant. On the other hand we know that the energy-momentum functional $\mathcal{H}$ can be seen as a real linear form on $(\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C}))^{*\mathbb{R}}$ that is to say, as a vector $\mathcal{H} = M \oplus \Xi \in \mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C})$. In the following, we will adopt the notations $\Xi = N \oplus \mathbf{i}R \in \mathfrak{G} \oplus \mathbf{i}\mathfrak{G}$, and $M = \Lambda(m_0, m)$, $N = \Lambda(0,n)$, $R = \Lambda(0,r)$, where Λ is the isomorphism defined in Section 2.4. Now applying the non-negativity results of Section 3.2.1 or 3.2.2, we know that (even if our AdS-asymptotically hyperbolic manifold has a compact boundary such that $\vec{k}$ is causal and positively oriented)

$$\forall \sigma \in IKS(\Sigma) \quad \mathcal{H}(V_\sigma, \alpha_\sigma) \geq 0.$$

In other words, for each $w \oplus u \in \mathbb{C}^4$, we have $\mathcal{H}(\mathcal{K}_{w\oplus u}) \geq 0$. But the complete study of $IKS(\Sigma)$ of Section 3.3 implies that actually

$$\begin{aligned}\mathcal{H}(\mathcal{K}_{w\oplus u}) &= V_{w\oplus u}(M) + \alpha_{w\oplus u}(\Xi)\\ &= 2(w^*\widehat{M}w + u^*Mu) + 2(w^*\Xi u + u^*\Xi^* w),\end{aligned}$$

and consequently the application $w \oplus u \longmapsto \mathcal{H}(\mathcal{K}_{w\oplus u})$ is a Hermitian form on $\mathbb{C}^2 \oplus \mathbb{C}^2$ whose matrix is

$$Q = 2\begin{pmatrix}\widehat{M} & \Xi\\ \Xi^* & M\end{pmatrix} = 2\begin{pmatrix}\Lambda(m_0,-m) & \Lambda(0,n) + \mathbf{i}\Lambda(0,r)\\ \Lambda(0,n) - \mathbf{i}\Lambda(0,r) & \Lambda(m_0,m)\end{pmatrix}.$$

It is easy to conclude since we have the identity

$$\forall w \oplus u \in \mathbb{C}^4 \quad \mathcal{H}(V_{w\oplus u} \oplus \alpha_{w\oplus u}) = Q(w\oplus u, w\oplus u) \geq 0,$$

which ends the proof of

Theorem 1.3 *Let (M^n, g, k) be an AdS-asymptotically hyperbolic spin Riemannian manifold satisfying the decay conditions stated in Section* 1.2 *and the following conditions*

(i) $\langle (f,\alpha), (\Phi(g,k) - \Phi(b,0))\rangle \in L^1(M, \mathrm{dVol}_b)$ *for every* $(f,\alpha) \in N_b \oplus \mathfrak{Kill}(M,b)$,

(ii) *the relative version of the dominant energy condition* (*cf. Section* 2.2) *holds, that is to say* $(\Phi(g,k) - \Phi(b,0))$ *is a positively oriented causal* $(n+1)$*-vector along* M,

(iii) *in the case where* M *has a compact boundary* ∂M, *we assume moreover that* $\vec{k}$ *is causal and positively oriented along* ∂M.

Then there exists a (*hardly explicitable*) *map* $\mathbb{R}^{n,1} \oplus \mathfrak{so}(n,1) \longrightarrow \mathrm{Herm}(C^d)$ *which sends, under the assumptions* (i)–(iii), *the energy-momentum on a non-negative Hermitian form* Q.

Moreover, when $n = 3$, *we can explicit* Q *in terms of the components of the energy-momentum as described above.*

The end of this section is devoted to the 3-dimensional case.

As the invariance of the energy-momentum under asymptotic hyperbolic isometries was proved in [19], one can be interested in the description of the orbit of Q under the action of $\mathrm{SL}(2,\mathbb{C})$.

Proposition 3.18. *If* M *is timelike, there exists a* (*non-unique*) *representative element of the orbit of* $\mathcal{H} = M \oplus \Xi$ *under the natural action* (*cf. Section* 3.3) *of* $\mathrm{SL}(2,\mathbb{C})$ *on* $\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C})$ *which can be written*

$$m_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus n_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{i} \begin{pmatrix} r_1 & r_2 \\ r_2 & -r_1 \end{pmatrix}, \quad m_0, n_1, r_1, r_2 \in \mathbb{R}.$$

The positive energy-momentum theorem then reduces to $m_0 \geq \sqrt{(|n_1| + |r_2|)^2 + r_1^2}$.

Proof. Let us suppose that $M \in \mathfrak{M}$ is timelike. Thus considering the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathfrak{M} \oplus \mathfrak{sl}_2(\mathbb{C})$ (cf. Section 3.3), then there exists an element in the orbit of $\mathcal{H}$ that can be written $m_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \Xi'$. Since the isotropy group of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is $\mathrm{SU}(2)$ whose action on $\mathfrak{G}$ is transitive, then there exists an element in the orbit of $\mathcal{H}$ that can be written $m_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus n_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{i}R'$. But the isotropy group of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the one parameter group $\left\{ \begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{-\mathbf{i}\theta} \end{pmatrix}, \theta \in \mathbb{R} \right\}$. Finally there exists an element (not unique since the isotropy group of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is isomorphic to $\mathbb{Z}_2$) in the orbit of $\mathcal{H}$ that can be written as announced in the proposition. The corresponding Hermitian matrix is

$$Q = 2 \begin{pmatrix} m_0 & 0 & n_1 + \mathbf{i}r_1 & \mathbf{i}r_2 \\ 0 & m_0 & \mathbf{i}r_2 & -n_1 - \mathbf{i}r_1 \\ n_1 - \mathbf{i}r_1 & -\mathbf{i}r_2 & m_0 & 0 \\ -\mathbf{i}r_2 & -n_1 + \mathbf{i}r_1 & 0 & m_0 \end{pmatrix}.$$

Since Q is non negative we have

$$\begin{aligned} m_0 &\geq 0 \\ m_0(m_0^2 - (n_1^2 + r_1^2 + r_2^2)) &\geq 0 \\ (m_0^2 - (n_1^2 + r_1^2 + r_2^2))^2 &\geq 4(n_1 r_2)^2, \end{aligned}$$

which can be summarized with $m_0 \geq \sqrt{(|n_1| + |r_2|)^2 + r_1^2}$. □

Remark 3.19. *The $\{t = t_0\}$ slices of the Kerr-AdS metrics are AdS-asymptotically hyperbolic and parametrized by 2 real parameters: the mass and the angular momentum. The proposition above then shows that there exists some energy-momenta that could not be obtained by the action of $\mathrm{SL}(2, \mathbb{C})$ on a Kerr-AdS solution. As a consequence, an interesting question would be to find some (new) AdS-asymptotically hyperbolic metrics which have an energy-momentum of the form given in the proposition above with non-zero coefficients m_0, n_1, r_1, r_2, and which satisfy the dominant energy condition or the (stronger) cosmological vacuum constraints.*

4. Rigidity theorems

Theorem 1.4 *Under the assumptions of the positive energy-momentum theorem, $Q = 0$ implies that (M, g, k) is isometrically embeddable in $\mathrm{AdS}^{n,1}$.*

Proof. The vanishing of Q implies that our spinor bundle Σ is trivialized by a basis of γ-*imaginary Killing spinors*. We denote by ξ any γ-*imaginary Killing spinors* of this basis. We will need the following spinorial Gauss-Codazzi equation.

Proposition 4.1. *For every $X, Y \in \Gamma(TM)$ we have*

$$R^{\gamma}_{X,Y} = R^{g}_{X,Y} - \frac{1}{2}\left(\mathrm{d}^{\overline{\nabla}} k(X,Y) \cdot e_0 + \frac{1}{2}\big(k(X) \cdot k(Y) - k(Y) \cdot k(X)\big)\right) \cdot \quad ,$$

where $\cdot$ denotes the Clifford action with respect to the metric γ.

Proof of the proposition. It is a straightforward computation where we use vector fields X, Y satisfying at the point $\overline{\nabla}_X Y = \overline{\nabla}_Y X = 0$.

$$\begin{aligned} \nabla_X \nabla_Y &= \nabla_X \left(\overline{\nabla}_Y - \frac{1}{2} k(Y) \cdot e_0 \cdot\right) \\ &= \overline{\nabla}_X \overline{\nabla}_Y - \frac{1}{2} k(X) \cdot e_0 \cdot \overline{\nabla}_Y \\ &\quad - \frac{1}{2}\left(\nabla_X k(Y) \cdot e_0 \cdot + k(Y) \cdot (\nabla_X e_0) \cdot + k(Y) \cdot e_0 \cdot \nabla_X\right) \\ &= \overline{\nabla}_X \overline{\nabla}_Y - \frac{1}{2}\left(k(X) \cdot e_0 \cdot \overline{\nabla}_Y + k(Y) \cdot e_0 \cdot \overline{\nabla}_X - k \circ k(X,Y) e_0 \cdot\right) \\ &\quad - \left(\frac{1}{2} \overline{\nabla}_X k(Y) \cdot e_0 - \frac{1}{4} k(Y) \cdot k(X)\right) \cdot \quad , \end{aligned}$$

and the curvature formula above follows. □

Using the fact that ξ is a γ-imaginary Killing spinor one gets

$$\begin{aligned}\Big\langle R^g_{X,Y}\xi - \frac{1}{4}\Big(X\cdot Y - Y\cdot X + k(X)\cdot k(Y) - k(Y)\cdot k(X)\Big)\cdot\xi,\xi\Big\rangle \\ = \frac{1}{2}\left\langle \mathrm{d}^{\nabla}k(X,Y)\cdot e_0\cdot\xi,\xi\right\rangle,\end{aligned}$$

where $\left\langle R^g_{X,Y}\xi,\xi\right\rangle$ and $\Big\langle\Big(X\cdot Y - Y\cdot X + k(X)\cdot k(Y) - k(Y)\cdot k(X)\Big)\cdot\xi,\xi\Big\rangle$ are purely imaginary terms whereas $\left\langle \mathrm{d}^{\nabla}k(X,Y)\cdot e_0\cdot\xi,\xi\right\rangle$ is real. As a consequence $\left\langle \mathrm{d}^{\nabla}k(X,Y)\cdot e_0\cdot\xi,\xi\right\rangle = 0$ for any ξ of our γ-imaginary Killing spinor basis and so $\mathrm{d}^{\nabla}k = 0$. This implies

$$R^g_{X,Y} = \frac{1}{4}\big(X\cdot Y - Y\cdot X + k(X)\cdot k(Y) - k(Y)\cdot k(X)\big)\cdot \quad ,$$

and using the natural isomorphism between $\mathrm{C}\ell_0(\mathbb{R}^{3,1})$ and $\Lambda^2(\mathbb{R}^{3,1})$ (cf. [24] proposition 6.2) we get that

$$\begin{aligned}R^g &= \frac{1}{2}\big(g \owedge g + k \owedge k\big)\\ \mathrm{d}^{\nabla}k &= 0.\end{aligned}$$

Let us denote by V the function $<\xi,\xi>$, α the real 1-form defined by $\alpha(Y) = \langle Y\cdot e_0\cdot\xi,\xi\rangle$. Then the couple $(V,W) := (V,-\alpha^\sharp)$ is a Killing Initial Data (KID) [10]. If we consider $(\widetilde{M},\tilde{g},\tilde{k})$ the universal Riemannian covering of (M,g,k), then we can make the Killing development of $(\widetilde{M},\tilde{g},\tilde{k})$ with respect to the KID $(\widetilde{V},\widetilde{W})$ which by definition is $\mathbb{R}\times\widetilde{M}$ endowed with the Lorentzian metric $\tilde{\gamma} = \Big(-\widetilde{V}^2 + |\widetilde{W}|^2\Big)\,\mathrm{d}u^2 + 2\widetilde{W}^\flat\odot\mathrm{d}u + \tilde{g}$. By construction, $\widetilde{M}$ is embedded in $(\mathbb{R}\times\widetilde{M},\tilde{\gamma})$ with induced metric $\tilde{g}$ and second fundamental form $\tilde{k}$. Besides $\mathbb{R}\times\widetilde{M}$ is the universal covering of N, and $\tilde{\gamma}$ which has sectional curvature -1, is a stationary solution of the vacuum Einstein equations with cosmological constant that is to say $G^{\tilde{\gamma}} = \frac{n(n-1)}{2}\tilde{\gamma}$. But $(\widetilde{M},\tilde{g})$ is complete since (M,g) is complete and therefore [1] $(\mathbb{R}\times\widetilde{M},\tilde{\gamma})$ is geodesically complete. It follows that $(\mathbb{R}\times\widetilde{M},\tilde{\gamma})$ is $\mathrm{AdS}^{n,1}$ (in vertue of Proposition 23 [p. 227] of [27]). It only remains to show that M is simply connected. We know that $\mathbb{R}\times\widetilde{M}\cong\mathbb{R}^{n+1}$ and thereby using the following compactly supported de Rham cohomology isomorphisms $\{0\} = H^2_{dR,c}(\mathbb{R}\times\widetilde{M}) = H^2_{dR,c}(\mathbb{R}^{n+1}) = H^1_{dR,c}(\widetilde{M})$ (cf. Proposition 4.7 and Corollary 4.7.1 [p. 39] of [12] for instance), we obtain that $\widetilde{M}$ has only one asymptotic end. This last fact compels the universal covering map $\widetilde{M}\to M$ to be trivial and as a consequence $(M,g,k)\equiv(\widetilde{M},\tilde{g},\tilde{k})$ is isometrically embedded in $\mathrm{AdS}^{n,1}\equiv(\widetilde{N},\tilde{\gamma})$. This completes the proof of the theorem. $\square$

A natural and less restriction on Q is to assume that it is degenerate, and one can wonder whether this implies rigidity. The end of this section is devoted to a study of the rigidity case in dimension $n = 3$. Namely we prove the following

Theorem 4.2. *Let us suppose that* (M^3, g, k) *satisfies the assumptions of the positive energy-momentum theorem and that the matrix* Q *is degenerate. Then there exists some* $\widehat{\nabla}$*-parallel spinor field* ξ *such that* $\left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle = 0$ *and consequently* (M, g, k) *is isometrically embeddable in a stationary pp-wave space-time.*

If furthermore the constant function (ξ, ξ) *is non-zero then* (M, g, k) *admits a vacuum Killing development which is a solution of the Einstein equations (with the cosmological constant* -3*).*

Remark 4.3. *A pp-wave space-time is a Lorentzian manifolds such that its stress-energy tensor satisfies* $T_{\mu\nu} = \lambda Z_\mu \otimes Z_\nu$ *where* Z^μ *is an isotropic Killing vector field and* λ *a function on the manifold. Some results were also proved by Siklos in* [32] *and by Leitner in* [25] *for Lorentzian manifolds admitting a Killing spinor.*

Proof. The degenerate character of Q implies the existence of a non-zero $w \oplus u \in \mathbb{C}^4$ and a unique ξ_0 such that $\xi = f\mathcal{A}\sigma_{w\oplus u} + \xi_0$ satisfies the conditions

$$\begin{aligned} \widehat{\nabla}\xi &= 0 \\ \left\langle \widehat{\mathfrak{R}}\xi, \xi \right\rangle &= 0. \end{aligned}$$

By the same argument as above we get that $\left\langle \mathrm{d}^{\overline{\nabla}} k(X, Y) \cdot e_0 \cdot \xi, \xi \right\rangle = 0$ (which can also be thought as $\overline{\nabla}_X k(Y, \alpha) = \overline{\nabla}_Y k(X, \alpha)$). Now since ξ is $\widehat{\nabla}$-parallel we get

$$\begin{aligned} \Re e \left\langle \sum_{k=1}^{3} e_k \cdot R^\gamma_{X,e_k} \xi, Y \cdot \xi \right\rangle &= \frac{1}{4} \Re e \left\langle \sum_{k=1}^{3} e_k \cdot (X \cdot e_k - e_k \cdot X) \cdot \xi, Y \cdot \xi \right\rangle \\ &= \langle X, Y \rangle \qquad \forall X, Y \in \Gamma(TM). \end{aligned}$$

On the other hand a direct computation leads to

$$\begin{aligned} &\sum_{k=1}^{3} e_k \cdot R^\gamma_{X,e_k} \\ &= \frac{1}{2} \sum_{l,m=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) e_l \cdot e_0 \cdot e_m \cdot \ + \frac{1}{2} \sum_{l,m=1}^{3} R^\gamma(X, e_l, e_l, e_m) e_l \cdot e_l \cdot e_m \cdot \\ &= \frac{1}{2} \sum_{l,m=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) e_l \cdot e_0 \cdot e_m \cdot \\ &\quad - \frac{1}{2} \sum_{l,m=1}^{3} \{ R^g(X, e_l, e_l, e_m) - k(X, e_l) k(e_l, e_m) + k(X, e_m) k(e_l, e_l) \} \cdot e_m \cdot \end{aligned}$$

$$= \frac{1}{2} \sum_{l,m=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) e_l \cdot e_0 \cdot e_m \cdot \; - \frac{1}{2}(E(X) - 2X) \cdot \quad ,$$

where we have set $E = \mathrm{Ric}^g + 2g + (\mathrm{tr}\, k)k - k \circ k$. It is then clear that

$$\Re e \left\langle \sum_{k,l=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) e_l \cdot e_0 \cdot e_m \cdot \psi, Y \cdot \psi \right\rangle = VE(X, Y).$$

In the following computation we will set $Y = e_s$. We recall that

$$\begin{aligned}
&\Re e \left\langle \sum_{l,m=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) e_l \cdot e_0 \cdot e_m \cdot \psi, e_s \cdot \psi \right\rangle \\
&\qquad = \sum_{l,m=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) \Re e \left\langle e_l \cdot e_0 \cdot e_m \cdot \psi, e_s \cdot \psi \right\rangle \\
&\qquad = \sum_{l=1}^{3} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_l) \Re e \left\langle e_l \cdot e_0 \cdot e_l \cdot \psi, e_s \cdot \psi \right\rangle \\
&\qquad\quad + \sum_{l \neq m} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) \Re e \left\langle e_l \cdot e_0 \cdot e_m \cdot \psi, e_s \cdot \psi \right\rangle \\
&\qquad = (I + II)(X, e_s),
\end{aligned}$$

and we will treat I and II separately for convenience. The easiest one is

$$\begin{aligned}
I(X, e_s) &= - \left\langle e_s \cdot e_0 \cdot \psi, \cdot \psi \right\rangle \sum_{l=1}^{3} \left(\overline{\nabla}_X k(e_l, e_l) - \overline{\nabla}_{e_l} k(X, e_l) \right) \\
&= - \left((\delta_g k + \mathrm{d}\mathrm{tr}_g\, k) \otimes \alpha \right)(X, e_s).
\end{aligned}$$

Thereby we can conclude that $I = -(\delta_g k + \mathrm{d}\mathrm{tr}_g\, k) \otimes \alpha$. We compute now $II(X, e_s)$.

$$\begin{aligned}
II(X, e_s) = &\sum_{l \neq s} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_s) \Re e \left\langle e_l \cdot e_0 \cdot e_s \cdot \psi, e_s \cdot \psi \right\rangle \\
&+ \sum_{m \neq s} \mathrm{d}^{\overline{\nabla}} k(X, e_s, e_m) \Re e \left\langle e_s \cdot e_0 \cdot e_m \cdot \psi, e_s \cdot \psi \right\rangle \\
&+ \sum_{l \neq m,\, l \neq s,\, m \neq s} \mathrm{d}^{\overline{\nabla}} k(X, e_l, e_m) \Re e \left\langle e_l \cdot e_0 \cdot e_m \cdot \psi, e_s \cdot \psi \right\rangle,
\end{aligned}$$

but the last sum is zero since $\langle e_k \cdot e_0 \cdot e_m \cdot \psi, e_s \cdot \psi \rangle$ is purely imaginary whenever k, m, s are distinct indices. Thereby it comes out that $II(X, e_s) = \big(\overline{\nabla}_{e_s} k(X, \alpha) - \overline{\nabla}_\alpha k(X, e_s)\big)$, so that we can conclude

$$II(X, Y) = \overline{\nabla}_Y k(X, \alpha) - \overline{\nabla}_\alpha k(X, Y),$$

and consequently

$$\begin{aligned} V\Big(\mathrm{Ric}^g + 2g + (\mathrm{tr}_g\, k)k - k\circ k\Big)(X,Y) &= -\Big((\delta_g k + \mathrm{dtr}_g\, k)\otimes\alpha\Big)(X,Y) \\ &\quad + \Big(\overline{\nabla}_Y k(X,\alpha) - \overline{\nabla}_\alpha k(X,Y)\Big). \end{aligned}$$

Moreover the couple (V,α) satisfies the following differential equations

$$\begin{aligned} \overline{\nabla}_X \alpha(Y) &= Vk(X,Y) + \frac{\mathbf{i}}{2}\big((X\cdot Y - Y\cdot X)\cdot\xi,\xi\big) \\ \delta_g^*\alpha &= Vk \\ \mathrm{d}V(X) &= k(X,\alpha) + \mathbf{i}\,\langle X\cdot\psi,\psi\rangle \end{aligned}$$

and

$$\begin{aligned} &(\mathrm{Hess}^g V)(X,Y) \\ &\quad= \overline{\nabla}_Y k(X,\alpha) - V(k\circ k)(X,Y) + Vg(X,Y) + \overline{\nabla}_X\alpha(k(Y)) + \overline{\nabla}_Y\alpha(k(X)) \\ &\quad= \overline{\nabla}_Y k(X,\alpha) - \overline{\nabla}_\alpha k(X,Y) - V(k\circ k)(X,Y) + Vg(X,Y) + \mathcal{L}_\alpha k(Y,X) \\ &\quad= V\Big(\mathrm{Ric}^g + 3g + (\mathrm{tr}_g\, k)k - 2(k\circ k)\Big)(X,Y) \\ &\qquad + \Big((\delta_g k + \mathrm{dtr}_g\, k)\otimes\alpha\Big)(X,Y) + \mathcal{L}_\alpha k(Y,X)\ . \end{aligned}$$

It is clear that the couple $(V,W) := (V,-\alpha^\sharp)$ satisfies the first KID equation [10], and using the second KID equation for defining the symmetric tensor τ of [10], namely

$$\begin{aligned} V\left(\tau - \frac{1}{2}(\mathrm{tr}_g\,\tau - \rho)g\right) &= V\Big(\mathrm{Ric}^g + (\mathrm{tr}_g\, k)k - 2(k\circ k)\Big) - \mathcal{L}_W k - (\mathrm{Hess}^g\, V) \\ &= \Big((\delta_g k + \mathrm{dtr}_g\, k)\otimes W^\flat\Big) - 3Vg, \end{aligned}$$

where $2\rho := \mathrm{Scal}^g + (\mathrm{tr}\, k)^2 - |k|^2$. Taking the trace of last equation one gets $\mathrm{tr}_g\,\tau - \rho = 12$ and consequently

$$V\tau = \Big((\delta_g k + \mathrm{dtr}_g\, k)\otimes W^\flat\Big) + 3Vg.$$

Now the equation $\langle\widehat{\mathfrak{R}}\xi,\xi\rangle = 0$ implies $\Big(\mathrm{Scal}^g + 6 + (\mathrm{tr}_g\, k)^2 - |k|_g^2\Big) = 2\,|\delta_g k + \mathrm{dtr}_g\, k|$.

We also know that $V\Big(\mathrm{Scal}^g + 6 + (\mathrm{tr}_g\, k)^2 - |k|_g^2\Big) = 2\,\big\langle(\delta_g k + \mathrm{dtr}_g\, k), W^\flat\big\rangle$, and thereby it is clear that there exists some function on M denoted by ϑ such that $W = \vartheta(\delta_g k + \mathrm{dtr}_g\, k)$ and so

$$V\Big(\mathrm{Scal}^g + 6 + (\mathrm{tr}_g\, k)^2 - |k|_g^2\Big) = 2\,|\vartheta|\,|\delta_g k + \mathrm{dtr}_g\, k|^2$$

and therefore

$$2V\,|\delta_g k + \mathrm{dtr}_g\, k| = \Big(2\rho + 6\Big)\,|W|\,,$$

which shows that in the Killing development the Killing vector field (V, W) will be colinear to the cosmological constraints 4-vector $(2\rho+6, 2(\delta_g k + \mathrm{dtr}_g\, k))$ which is isotropic. It follows that the Killing vector field (N, W) is also isotropic in the Killing development. We finally obtain the relation $V^2(\tau - 3g) = \frac{1}{2}(2\rho+6)W^\flat \otimes W^\flat$ which means that the Killing development is a stationary pp-wave space-time.

Supposing furthermore that the constant function (ξ, ξ) is non-zero, we get $(\delta_g k + \mathrm{dtr}_g\, k)(\xi, \xi) = 0$ and so $(\delta_g k + \mathrm{dtr}_g\, k) = 0$ by tracing the equation $\Big\langle \mathrm{d}^{\overline{\nabla}} k(X, Y) \cdot e_0 \cdot \xi, \xi \Big\rangle = 0$. It follows by the dominant energy condition that $\mathrm{Scal}^g + (\mathrm{tr}\, k)^2 - |k|^2 = -6$, and we obtain finally that $\tau = 3g$. Thereby the couple (V, W) is a cosmological vacuum KID. It is known [10] that in that case (M, g, k) has a cosmological vacuum Killing development denoted by $(\overline{N}, \overline{\gamma})$ which is a stationary 4-dimensional Lorentzian manifold satisfying $G^{\overline{\gamma}} = 3\overline{\gamma}$ and carrying a Killing vector field which is the natural extension of the KID (V, W). □

Remark 4.4. *It is clear that expecting $m_0 = 0$ so as to define the rigidity situation is much stronger than expecting the degenerate character of Q. A good issue would certainly be to use the geometry at infinity in the same way as in* [18] *but in the AdS-asymptotically hyperbolic context, in order to prove under the degenerate character of Q the existence of an isometric embedding of (M, g, k) in* AdS.

5. Appendix

5.1. Non-negativity of Q seen through its coefficients when $n = 3$

Classical linear algebra results state that every principal minor of Q must be non-negative which give rise to a set of inequalities on the coefficients of $\mathcal{H}$.

$$\begin{aligned} m_0 + m_1 &\geq 0 \\ m_0 - m_1 &\geq 0 \end{aligned}$$

$$\begin{aligned} m_0^2 - |m|^2 &\geq 0 \\ (m_0 + m_1)^2 - (n_2 + r_3)^2 - (r_2 - n_3)^2 &\geq 0 \\ (m_0 - m_1)^2 - (n_2 - r_3)^2 - (r_2 + n_3)^2 &\geq 0 \\ m_0^2 - m_1^2 - n_1^2 - r_1^2 &\geq 0 \end{aligned}$$

$$\begin{aligned} &(m_0 + m_1)(m_0^2 - (|m|^2 + n_1^2 + r_1^2)) - (m_0 - m_1)((n_2 + r_3)^2 + (n_3 - r_2)^2) \\ &\qquad - 2((n_2 + r_3)(m_2 n_1 + m_3 r_1) + (-n_3 + r_2)(m_2 r_1 - m_3 n_1) \geq 0 \\ &(m_0 - m_1)(m_0^2 - (|m|^2 + n_1^2 + r_1^2)) - (m_0 + m_1)((n_2 - r_3)^2 + (n_3 + r_2)^2) \\ &\qquad + 2((n_2 - r_3)(m_2 n_1 - m_3 r_1) + (n_3 + r_2)(m_2 r_1 + m_3 n_1) \geq 0 \end{aligned}$$

$$\begin{aligned} &(m_0^2 - (|m|^2 + |n|^2 + |r|^2))^2 - 4(|m|^2|n|^2 + |m|^2|r|^2 + |n|^2|r|^2) \\ &+ 4(<m,n>^2 + <m,r>^2 + <n,r>^2) + 8m_0 \det_{\mathbb{R}^3}(m,n,r) \qquad \geq 0. \end{aligned}$$

5.2. Rigidity results for the Trautman-Bondi mass

Oppositely to the rest of the paper, we consider here the situation of the *Trautman-Bondi mass* [16], namely (M, g, k) is assumed to be Minkowski-asymptotically hyperbolic which means that the triple (M, g, k) is asymptotic at infinity to a standard hyperbolic slice of Minkowski space-time. It has been proved (cf. Theorem 5.4 of [16]) that the *Trautman-Bondi four-momentum* p_μ is timelike and future directed under the dominant energy condition (and some other technical assumptions). The aim of this section is to prove some rigidity results for the *Trautman-Bondi four-momentum* which are analogous to the statements of Section 4. More precisely

Theorem 5.1. *Under the assumptions of Theorem* 5.4 *of* [16]*, and if the component* p_0 *of the Trautman-Bondi four-momentum* p_μ *vanishes, then* (M, g, k) *can be isometrically embedded in Minkowski space-time.*

Proof. This can be done in the same way as our rigidity theorem: since p_μ is timelike, the condition $p_0 = 0$ implies that p_μ actually vanishes. Consequently there exists a basis of ∇-parallel spinor fields on M, where ∇ is the connection on some cylinder $]-\epsilon, +\epsilon[\, \times M$ endowed with some Lorentzian metric $\gamma = -\mathrm{d}t^2 + g_t$ (such that M has induced metric g and extrinsic curvature k satisfying the conditions of [16]). Now if $\overline{\nabla}$ denotes the Levi-Civita connection of $g = g_0$, we still have the relation $\nabla_X \xi = \overline{\nabla}_X \xi - \frac{1}{2} k(X) \cdot e_0 \cdot \xi$ where $\cdot$ is the Clifford action with respect to γ. Our spinorial Gauss-Codazzi formula is still valid, that is

$$R^\gamma_{X,Y} = R^g_{X,Y} - \frac{1}{2}\left(\mathrm{d}^{\overline{\nabla}} k(X,Y) \cdot e_0 + \frac{1}{2}\big(k(X) \cdot k(Y) - k(Y) \cdot k(X)\big)\right) \cdot = 0 \quad ,$$

and so

$$R^g = \frac{1}{2} k \owedge k$$
$$\mathrm{d}^{\overline{\nabla}} k = 0.$$

Furthermore, the couple $(V, W) := (V, -\alpha^\sharp)$ is a vacuum KID if one defines $V = <\xi, \xi>$ and the real 1-form α by $\alpha(Y) = \langle Y \cdot e_0 \cdot \xi, \xi \rangle$. We consider again the Killing development of $(\widetilde{M}, \widetilde{g}, \widetilde{k})$ with respect to $(\widetilde{V}, \widetilde{W})$, and observe that it must be a geodesically complete stationary solution of vacuum Einstein equations of zero sectional curvature and thereby must be Minkowski space-time (cf. Proposition 23 [p. 227] of [27]). Now the same cohomological arguments give $(\widetilde{M}, \widetilde{g}, \widetilde{k}) = (M, g, k)$ which is by construction embedded in its Killing development that is Minkowski. $\square$

Theorem 5.2. *Let us suppose that* (M, g, k) *satisfies the assumptions of Theorem* 5.4 *of* [16] *and that* p_μ *is null. Then there exists some* ∇*-parallel spinor field* ξ *such that* $\langle \Re \xi, \xi \rangle = 0$ *and consequently* (M, g, k) *is isometrically embeddable in a stationary pp-wave space-time.*

If furthermore the constant function (ξ, ξ) is non-zero then (M, g, k) admits a vacuum Killing development which is a stationary solution of the Einstein equations.

Proof. p_μ is null implies the existence of a spinor field ξ satisfying the conditions

$$\nabla\xi = 0$$
$$\langle \Re\xi, \xi\rangle = 0.$$

Then in the same way as in the last Theorem of Section 4, but defining here the 2-tensor $E =: \mathrm{Ric}^g + (\operatorname{tr} k)k - k \circ k$ we obtain that the couple (V, W) is a vacuum KID and the corresponding Killing development satisfies $V^2\tau = \rho(W^\flat \otimes W^\flat)$ which means that it is a stationary pp-wave space-time.

Still using the same computations as in the last Theorem of Section 4 and assuming that the constant function (ξ, ξ) is non-zero we find that the constraints equations are satisfied (because of the dominant energy condition) and that $\tau = 0$. Thereby (V, W) is a vacuum KID and it is known that in this case (M, g, k) has a stationary vacuum Killing development. □

Remark 5.3. *It is clear that expecting $p_0 = 0$ so as to define the rigidity situation is much stronger than expecting the null character of p_μ. As in our situation (cf. the remark at the end of Section 4), a good issue would certainly be to use the* geometry at infinity *in the same way as in* [18] *but in the Minkowski-asymptotically hyperbolic context, in order to prove under the equality case of Theorem 5.4 of* [16], *the existence of an isometric embedding of (M, g, k) in Minkowski.*

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