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# Interface Instability under Forced Displacements

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**Abstract.** By applying linear response theory and the Onsager principle, the power (per unit area) needed to make a planar interface move with velocity $V$ is found to be equal to $V^2/\mu$, $\mu$ a mobility coefficient. To verify such a law, we study a one dimensional model where the interface is the stationary solution of a non local evolution equation, called an instanton. We then assign a penalty functional to orbits which deviate from solutions of the evolution equation and study the optimal way to displace the instanton. We find that the minimal penalty has the expression $V^2/\mu$ only when $V$ is small enough. Past a critical speed, there appear nucleations of the other phase ahead of the front, their number and location are identified in terms of the imposed speed.

## 1 Introduction

In a large variety of systems the power dissipated to force a motion with speed $V$ is given by the ratio $V^2/\mu$, $\mu$ a mobility coefficient, just think of Ohm's law in an electric circuit, or of a mechanical body moving in a viscous fluid or of the motion of a planar interface between two solid phases, the issue on which this paper is focused.

A general explanation of the law goes back to Onsager and linear response theory. Our purpose was to verify the validity or we should better say now, the limits of validity of the law in a model for interfaces. We restrict for technical reasons to one dimension (see Section 3 on this issue) and consider the non local evolution equation

$$u_t = f(u), \quad u(\cdot, 0) \text{ given}, \tag{1.1}$$

with $u_t$ the $t$-derivative of $u$ and the "force field" $f(u)$ given by

$$f(u) = J * u - A_\beta(u), \quad A_\beta(u) = \frac{1}{\beta} \operatorname{arctanh}(u), \quad J * u(x) = \int_{\mathbb{R}} J(x,y) u(y)\, dy.$$

We suppose $\beta > 1$ and that $J(x,y)$, $(x,y) \in \mathbb{R} \times \mathbb{R}$, is a smooth, symmetric, translational invariant probability kernel supported in $|y-x| \leq 1$. We also assume that $J(0,x)$ is a non increasing function whenever restricted to $x \geq 0$.

The two constant functions $m^{(\pm)}(x) \equiv \pm m_\beta$, with $m_\beta > 0$ solving the mean field equation $m_\beta = \tanh\{\beta m_\beta\}$ (recall $\beta > 1$) are stationary solutions of (1.1) and are interpreted as the two pure phases of the system, being the only "stable" stationary homogeneous solutions of (1.1) (the only other homogeneous, stationary solution $m(x) \equiv 0$ becomes unstable when $\beta$ increases past 1).

Interfaces, which are the objects of this paper, are defined as those stationary solutions of (1.1) which converge to $\pm m_\beta$ as $x \to \pm\infty$. Such solutions indeed exist for any $\beta > 1$, they are called instantons and denoted by $\bar{m}_\xi(x)$, $\xi$ a parameter called the center of the instanton $\bar{m}_\xi(x)$. They are obtained one from the other by a shift, so that calling $\bar{m} = \bar{m}_0$,

$$\bar{m}_\xi(x) = \bar{m}(x - \xi). \tag{1.2}$$

The instanton $\bar{m}$ satisfies

$$\bar{m}(x) = \tanh\left\{\beta J * \bar{m}(x)\right\}, \quad x \in \mathbb{R}. \tag{1.3}$$

It is an increasing, antisymmetric function which converges exponentially fast to $\pm m_\beta$ as $x \to \pm\infty$, see, e.g., [7], and there are $\alpha$ and $a$ positive so that

$$\lim_{x\to\infty} e^{\alpha x} \bar{m}'(x) = a, \tag{1.4}$$

see [6], Theorem 3.1. Moreover, any other solution of (1.3) which is definitively strictly positive [respectively negative] as $x \to \infty$ [respectively $x \to -\infty$], is a translate of $\bar{m}(x)$, see [8].

We next turn to the real issue of the paper. To impose a speed $v$ to the interface, we take $r$ and $t$ positive, $r/t = v$ (how to choose $r$ and $t$ will be discussed later) and consider the set

$$\mathcal{U}[r,t] = \Big\{u \in C^\infty\big(\mathbb{R} \times (0,t); (-1,1)\big) : \lim_{s\to 0^+} u(\cdot, s) = \bar{m},\ \lim_{s \to t^-} u(\cdot, s) = \bar{m}_r\Big\}. \tag{1.5}$$

Due to the stationarity of $\bar{m}$, no element in $\mathcal{U}[r,t]$ satisfies (1.1) and therefore there are other forces which must enter into play. Call $b = b(x,s)$, $x \in \mathbb{R}$, $0 \le s \le t$, an "external force", and consider the evolution equation

$$u_s = f(u) + b. \tag{1.6}$$

Existence and uniqueness for [the Cauchy problem for] (1.6) are proved in Appendix A. We are of course only interested in forces $b$ able to produce orbits in $\mathcal{U}[r,t]$. To select among them we introduce the action functional

$$I_t(u) = \frac{1}{4}\int_0^t \int_{\mathbb{R}} b(x,s)^2 \, dx \, ds, \tag{1.7}$$

where $b$, via (1.6), is a function of $u$ and of its time derivative. When writing (1.7), we have invoked the same general, linear response theory expression for dissipated power (with $\mu = 4$ for convenience) that we are putting under scrutiny. This should not be viewed however as a circular trap, because the principle is invoked at a "microscopic" (or better mesoscopic) level, while we want to investigate it at the macroscopic one. Moreover, in Section 4 we will discuss the question in a statistical mechanical context, where our model appears as a mesoscopic limit of Ising systems with Kac potentials and an expression structurally similar to (1.7) is rigorously proved by large deviation estimates. With such motivation we postulate that (1.7) is "the penalty functional". Then the cost of moving the instanton to $r$ in the time $t$ is defined as

$$\inf_{u\in\mathcal{U}[r,t]} I_t(u). \tag{1.8}$$

Let us turn now to the choice of $r$ and $t$, as the specification of $v$ only fixes their ratio. As we want to investigate macroscopic behaviors, we should consider a spatial scale where the instanton $\bar{m}$ looks like a sharp interface, namely like the step function $m_\beta\big(\mathbf{1}_{x\geq 0} - \mathbf{1}_{x<0}\big)$. Recalling that $\bar{m}(x)$ converges exponentially to $\pm m_\beta$ as $x \to \pm\infty$, we introduce a parameter $\epsilon > 0$ to scale distances $x \to \epsilon^{-1}x$ with the idea of eventually letting $\epsilon \to 0$. Time should then be taken equal to $\epsilon^{-1}r/v$, and if "the law $V^2/\mu$" is satisfied,

$$\text{energy dissipated} = \frac{v^2}{\mu}\frac{\epsilon^{-1}r}{v} = \frac{\epsilon^{-1}v}{\mu}. \tag{1.9}$$

To have a finite dissipation of energy we must then take $v$ of the order of $\epsilon$, which also agrees with the idea that the law $V^2/\mu$ should be investigated in the regime of small velocities. Another way to arrive at the same conclusion goes as follows: the expression $(V^2/\mu)T$ for the dissipated energy is invariant under parabolic scaling of space and time, it is therefore natural to use a parabolic scaling to derive it. With this in mind, we fix any pair $R$ and $T$ of positive numbers, and define the macroscopic work to displace the interface by $R$ in a time $T$ ($R$ the macroscopic space and $T$ the macroscopic time) as

$$W_-(R,T) = \liminf_{\epsilon\to 0} \inf_{u\in\mathcal{U}[\epsilon^{-1}R,\epsilon^{-2}T]} I_{\epsilon^{-2}T}(u), \tag{1.10}$$

$$W_+(R,T) = \limsup_{\epsilon\to 0} \inf_{u\in\mathcal{U}[\epsilon^{-1}R,\epsilon^{-2}T]} I_{\epsilon^{-2}T}(u). \tag{1.11}$$

We will prove that $W_-(R,T) = W_+(R,T)$ and compute their common value, the results are stated in the next section, together with an outline of their proofs. In Section 3 we discuss application of the theory to tunnelling, in Section 4 a formulation of the whole problem in a statistical mechanics setting. In the remaining sections we give the proofs.

## 2 Main results

Our first theorem is:

**Theorem 2.1** *There is a critical value $(V^2T)_c$ such that if $R^2/T \leq (V^2T)_c$, then*

$$W_-(R,T) = W_+(R,T) = \frac{R^2}{\mu T}, \quad \frac{1}{\mu} = \frac{\|\bar{m}'\|_2^2}{4} \tag{2.1}$$

*where $\bar{m}'$ is the derivative of $\bar{m}$ and $\|\cdot\|_2$ denotes the $L^2$ norm on $(\mathbb{R}, dx)$.*

An upper bound for $W_+(R,T)$ can be easily found by putting

$$u^\epsilon(x,t) = \bar m_{\epsilon V t}(x), \quad V = \frac{R}{T} \tag{2.2}$$

so that $u^\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$. Then $I_{\epsilon^{-2}T}(u^\epsilon)$ is independent of $\epsilon$ and equal to

$$\frac{1}{4}\|\bar m'\|_2^2 V^2 T \tag{2.3}$$

thus getting the same answer as in (2.1). We can easily rule out other ways to move continuously the instanton as more expensive. Indeed if, instead of (2.2), we choose

$$m^\epsilon(x,t) = \bar m_{\xi_\epsilon(t)}(x),$$

such that, for $v_\epsilon(t) := \dot\xi_\epsilon(t)$,

$$\int_0^{\epsilon^{-2}T} v_\epsilon(t)\,dt = \epsilon^{-1}R \tag{2.4}$$

then, with $\mu$ as in (2.1),

$$I_{\epsilon^{-2}T}(m^\epsilon) = \frac{1}{\mu}\int_0^{\epsilon^{-2}T} v_\epsilon^2(t)\,dt. \tag{2.5}$$

By computing the inf of $I_{\epsilon^{-2}T}$ in the class (2.4) we get that

$$\int_0^{\epsilon^{-2}T} v_\epsilon^2(t)\,dt \ge V^2 T, \quad \text{for all } v_\epsilon \text{ such that } \int_0^{\epsilon^{-2}T} v_\epsilon(t)\,dt = \epsilon^{-1}VT$$

which implies that (2.1) is optimal in the class (2.4). To prove the lower bound we thus need to examine more general orbits than mere shifts of the instanton.

Here comes another important issue, not touched so far in our discussion, namely "nucleations". By this we mean the appearance of droplets of the other phase inside one phase. We first define the free energy functional

$$\mathcal{F}(m) = \int_{\mathbb{R}} \phi_\beta(m)dx + \frac{1}{4}\int_{\mathbb{R}\times\mathbb{R}} J(x,y)[m(x)-m(y)]^2 dx\,dy, \tag{2.6}$$

where $\phi_\beta(m)$ is the "mean field excess free energy"

$$\phi_\beta(m) = \tilde\phi_\beta(m) - \min_{|s|\le 1}\tilde\phi_\beta(s), \quad \tilde\phi_\beta(m) = -\frac{m^2}{2} - \frac{1}{\beta}\mathcal{S}(m), \quad \beta > 1,$$

and $\mathcal{S}(m)$ the entropy:

$$\mathcal{S}(m) = -\frac{1-m}{2}\,\log\frac{1-m}{2} - \frac{1+m}{2}\,\log\frac{1+m}{2}.$$

By direct inspection $f(m) = -\dfrac{\delta \mathcal{F}(m)}{\delta m}$, the functional derivative of $\mathcal{F}$, so that (1.1) is the gradient flow associated to $\mathcal{F}(m)$. The gradient structure of the evolution has a very important role in the sequel, in particular the next theorem uses it in an essential way.

**Theorem 2.2** *For any $\vartheta > 0$ there is $\tau > 0$ and a function $\tilde{m}_{\epsilon,\tau}(x,s)$, $x \in \mathbb{R}$, $s \in [0, \tau\epsilon^{-3/2}]$, symmetric in $x$ for each $s$ and such that*

$$\tilde{m}_{\epsilon,\tau}(x,0) = m_\beta, \quad \tilde{m}_{\epsilon,\tau}(x,\tau\epsilon^{-3/2}) = \bar{m}_{\ell_\epsilon/2}(x),\ x \geq 0 \tag{2.7}$$

*where $e^{-\alpha\ell_\epsilon} = \epsilon^{3/2}$, $\alpha > 0$ as in (1.4), and*

$$I_{\tau\epsilon^{-3/2}}(\tilde{m}_{\epsilon,\tau}) \leq 2\mathcal{F}(\bar{m}) + \vartheta. \tag{2.8}$$

Theorem 2.2 follows from results proved in [2] and [3], as discussed briefly in Appendix B. It is now clear that (2.1) cannot keep its validity for all $V$. The key point is that the cost is quadratic in the velocity, so that, by creating more fronts, we can make them move with smaller velocity with the gain in cost covering the penalty for the nucleations, see Fig. 1.

To make this more precise, consider an orbit $m(x,t)$ with a nucleation at time 0 at position $\epsilon^{-1}(2/3)R$. We then divide the space in two parts, $x \leq \epsilon^{-1}(1/3)R$ and its complement. In the first one we set $m(x,t) = \bar{m}_{\epsilon(V/3)t}$ (the velocity being such that the front reaches $\epsilon^{-1}(1/3)R$ at the final time $\epsilon^{-2}\tau$). For $x > \epsilon^{-1}(1/3)R$, $m(x,t) = \tilde{m}_{\epsilon,\tau}(x - \epsilon^{-1}(2/3)R, t)$ for $t \leq \tau\epsilon^{-2/3}$ and for $t > \tau\epsilon^{-3/2}$, $m(x,t) = \bar{m}_{x(t)}(x)$, $x \geq \epsilon^{-1}(2/3)R$ and its symmetric image for $x < \epsilon^{-1}(2/3)R$; where $x(t) = \epsilon^{-1}(2/3)R + \ell_\epsilon + \epsilon(t - \epsilon^{-3/2}\tau)V/3$. Observe that for $t \in [\epsilon^{-3/2}\tau, \epsilon^{-2}\tau]$ and to leading order in $\epsilon$, $f(m)$ is given by $e^{-\alpha(\epsilon V t + \ell_\epsilon/2)}$ which implies that $\int_{\epsilon^{-3/2}\tau}^{\epsilon^{-2}\tau} f(m)^2$ vanishes in the limit $\epsilon \to 0$.

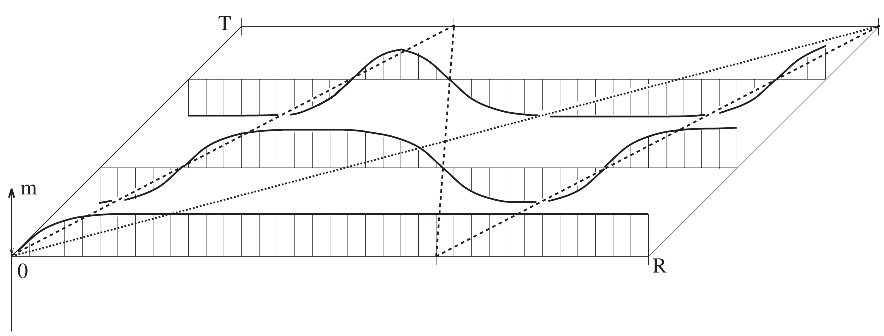


Figure 1. We depict for two possible trajectories the zero level sets in space time: Three fronts (dashed lines) and a single front (dotted line). Note that the single front has to move much faster. For the three front case we moreover show schematically the fronts initially, early after nucleation of a droplet, and shortly before they reach the final state.

Thus to leading order in $\epsilon$, $m$ has three fronts, the first from the left is the original one but moving with speed $V/3$ (which is one third of the original one), the second and third fronts are those produced by the nucleation. They move respectively to the left and to the right with same speed $V/3$. With such a choice the first two "collide with each other" at time $\epsilon^{-2}T$, while the third one reaches the final location $\epsilon^{-1}R$. In this way, at the final time we have just one front at $\epsilon^{-1}R$. A proof along these lines requires a "suitable slight modification" of the orbit described above (we need to adjust the velocities and to modify the orbit when the first two fronts become close to each other, in the sense of Theorem 2.2). With such a maquillage, it can be proved that the total cost in the limit $\epsilon \to 0$ converges to

$$2\mathcal{F}(\bar{m}) + 3\left\{\frac{1}{\mu}\left(\frac{V}{3}\right)^2 T\right\}. \tag{2.9}$$

By comparing (2.9) with the cost $V^2T/\mu$ of the motion without nucleations, we find (2.9) evidently winning for $V$ large. More precisely, we find equality if $V^2T = 3\mu\mathcal{F}(\bar{m})$ which is indeed the critical value $(V^2T)_c$ in Theorem 2.1. The above argument can be made rigorous (for brevity details are omitted) proving that besides $V^2T/\mu$ also (2.9) is an upper bound for $W_+(R,T)$. The argument can also be extended (again we omit the details) to prove upper bounds with any finite number $n$ of nucleations, the cost being

$$w_n(R,T) := n2\mathcal{F}(\bar{m}) + (2n+1)\left\{\frac{1}{\mu}\left(\frac{V}{2n+1}\right)^2 T\right\}. \tag{2.10}$$

We thus get the upper bound $W_+(R,T) \le \inf_{n\ge 0} w_n(R,T)$. The whole heart of the problem is to prove that this is also a lower bound, namely that there are no other strategies which give a smaller cost. The lower bound will be proved in the rest of the paper, here we just summarize the discussion by stating:

**Theorem 2.3** *For all $R$ and $T$, $W_+(R,T) = W_-(R,T) =: W(R,T)$ and, calling $V = R/T$,*

$$W(R,T) = w_n(R,T), \quad \text{if } \mathcal{F}(\bar{m})[(2n)^2-1] \le \frac{V^2T}{\mu} \le \mathcal{F}(\bar{m})\left(\left[2(n+1)\right]^2 - 1\right). \tag{2.11}$$

## 3 Tunnelling

The motivation for this research comes from tunnelling, in particular from questions raised by Stephan Luckhaus about multiple nucleations in stochastic evolutions where the order parameter is conserved. Shifting to one dimensions and to non conserved dynamics was (we hope) only a preliminary step. The next step will be to connect the present analysis to the tunnelling studied in [2] for the same one dimensional model we are considering here, but restricted to a finite interval $[-L, L]$ with Neumann boundary conditions. Tunnelling concerns orbits $u(x, t)$ which start from, say, the minus phase and end up at a final time $\tau$ in the plus phase, $u(x, 0) = m^{(-)}(x) = -m_\beta$, and $u(x, \tau) = m^{(+)}(x) = m_\beta$. The penalty in [2] is given by the same functional we are using here (but, of course, restricted to orbits in $[-L, L]$) and the cost of the tunnelling is defined as the inf over all $\tau$ of the inf over all orbits which tunnel in a time $\tau$.

The result found in [2] fits with many other results in the field, as the cost is equal to the finite volume free energy $\mathcal{F}_L(\hat{m}_L)$ of an "instanton-like saddle point" $\hat{m}_L$. Indeed $\hat{m}_L$ converges as $L \to \infty$ exponentially fast to $\bar{m}$. Moreover, the optimal strategy for minimizing the cost is to follow backwards in time the orbit which connects $\hat{m}_L$ to $m^{(-)}$ and, once past $\hat{m}_L$, to go along the orbit which connects $\hat{m}_L$ to $m^{(+)}$. Here comes the relation with the present paper, because for large $L$ these orbits are close to moving instantons, with the speed of their motion proportional, to leading orders, to $e^{-2\alpha L}$, $\alpha$ a positive parameter.

The familiar statement that the cost of tunnelling is equal to the energy of the saddle point depends critically on leaving unrestricted the time for tunnelling, but the result remains valid in the limit of large $L$ if we allow for exponentially growing times $\tau$. In experiments or simulations, infinite or exponentially growing times are clearly unrealistic and one forces in one way or the other the tunnelling to occur on faster times so that the event can be actually observed. But then the statistics over the systems where tunnelling has occurred will reflect the conditioning that they have occurred in the time span of the observation. The problem then involves the computation of the additional cost necessary for the interface to move fast enough. If our results can be extended, as we expect, to the model in [2], we would then have again a critical dependence on the time and only if it scales slowly on the scale $L^2$, the tunnelling will be described by a moving front, otherwise it will be characterized by many nucleations.

The applications of our results to realistic systems may only be valid of course when the front has really a planar structure. But on the other hand, tunnelling in a rectangular domain (say in $d = 2$ dimensions with Neumann boundary conditions) we believe occurs just as in one dimension. We expect in fact that the stationary solution which is spatially non homogeneous and has minimal energy is $f(x, y) = \hat{m}_L(x)$, supposing $x$ the direction of the longest side, $L$, of the rectangle. If this was actually true, then the arguments used in [2] would prove that the tunnelling event is just a planar front moving as in the $d = 1$ case.

The same questions can of course be framed in different contexts, maybe the most usual one is the Allen-Cahn equation and the Ginzburg-Landau functional. The cost of tunnelling under a time constraint has been recently investigated by Reznikoff, [10], for the functional

$$\int (u_t - \{\Delta u - V'(u)\})^2$$

where $V(u)$ is a double well potential and $u_t = \Delta u - V'(u)$ the Allen-Cahn equation. The analysis in [10] gives clear evidence that multiple nucleations are the most favorable strategy for tunnelling if times are sufficiently short, in total agreement with the picture we derive here.

## 4 A $d+1$ statistical mechanics setting

The model we are studying here has a clear statistical mechanics origin. Consider in fact the Ising model in $d = 1$ dimensions with a Kac potential, where the energy per spin is

$$-\frac{s(x)}{2}\sum_{y\neq x} J_\gamma(x,y)s(y) =: -\frac{s(x)}{2}V_x(s) \tag{4.1}$$

$s(z)$, $z \in \mathbb{Z}^d$, being $\pm1$ valued spins and $J_\gamma(x,y) = \gamma^d J(\gamma x, \gamma y)$, $J$ as in Section 1; $V_x(s)/2$ has then the interpretation of the "molecular magnetic field" at $x$ produced by the spins $s_y$, $y \neq x$. A first relation with our model comes from the fact that the free energy functional $\mathcal{F}(m)$ is the rate function for Gibbsian large deviations in the limit $\gamma \to 0$, see for instance [4].

Glauber dynamics is defined as the Markov process whose generator is determined by assigning flip rates $c_x(s)$ to the spins in such a way that the Gibbs measure is invariant (and a detailed balance condition, equivalent to self-adjointness of the generator, is satisfied). There is not a single choice for the rates, in the sequel it is convenient to assume

$$c_x(s) := \frac{e^{-s(x)V_x(s)}}{e^{-V_x(s)} + e^{+V_x(s)}}. \tag{4.2}$$

The $d+1$ setting in the title of this section refers to an interpretation of the Markov process in terms of a two dimensional Gibbs measure, one dimension referring to $\mathbb{Z}$, the space of sites of the spins, the other one, $\mathbb{R}$, to times. To implement it, consider for instance a "reference measure" which is the process where spins flip with rate $1/2$ independently of each other, which corresponds to (4.2) with $J = 0$. We can then use Girsanov formula (after restricting to "finite boxes") for the Radon-Nykodim derivative of the interacting process with respect to the free one, thus obtaining a $d = 2$ Hamiltonian.

Just like in equilibrium statistical mechanics, to have a Hamiltonian just defines the problem, the solution being still all the way ahead. A technique conceptually very powerful, but unfortunately only seldom really implementable, is renormalization group. The idea behind it, in the present context, is that, after coarse graining, the original system becomes a new system with low effective temperature. Its behavior is then ruled by the ground states of its effective Hamiltonian. The assumption that the interaction is a Kac potential is just what needed for implementing such a step of the renormalization group. Here it is convenient to coarse grain in space only, with blocks which scale to $\infty$, but having size smaller than $\gamma^{-1}$. At $\gamma > 0$ small enough, the effective Hamiltonian is then approximated by its limit value at $\gamma = 0$, which is the rate function for large deviations. This has been computed long ago by Comets, [5], the result is a quite complicated expression, that we have simplified here by assuming it given by the quadratic expression (1.7)–(1.6). We believe however that an analysis using the Comets functional could work as well and that it can be used to derive, by a perturbative analysis, also the behavior of the spins when $\gamma$ is small, but fixed. We hope to show all that in a forthcoming paper.

The $d+1$ Gibbsian interpretation of the problem stated in Section 1 has the following nice expression. We have a box $\mathbb{Z} \times [0, \epsilon^{-2}T]$ and we are giving boundary conditions on bottom and top. On the bottom we put in fact an interface at 0, on the top the interface is shifted by $\epsilon^{-1}R$. In elasticity this would be viewed as a shear problem. If $R$ and thus the shear is small, then the deformation is a well-defined straight line joining bottom and top, but if we increase $R$ then there are "fractures" which strongly resemble those appearing in totally different physical contexts.

## 5 Scheme of proofs

We have sketched in Section 2 the proof of the upper bound; as it is relatively easy to fill in the gaps, for brevity we omit the details, giving the upper bound for proved and thus, the proof of Theorem 2.3 will be completed once we prove:

**Proposition 5.1 (Lower bound)** *Let $P > \inf_{n\geq 0} w_n(R,T)$, and $\gamma > 0$. Then for any sequence $u_\epsilon \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ such that*

$$I_{\epsilon^{-2}T}(u_\epsilon) \leq P \tag{5.1}$$

*it holds that*

$$\liminf_{\epsilon\to 0} I_{\epsilon^{-2}T}(u_\epsilon) \geq \inf_{n\geq 0} w_n(R,T) - \gamma. \tag{5.2}$$

Of course $\gamma$ is redundant in (5.2) and (5.1) is not actually a restriction because we have already proved that there are sequences $u \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$ whose limsup is bounded by $\inf_{n\geq 0} w_n(R,T)$. Since

$$\liminf_{\epsilon\to 0} \inf_{u\in\mathcal{U}[\epsilon^{-1}R,\epsilon^{-2}T]} I_{\epsilon^{-2}T}(u) \geq \liminf_{\epsilon\to 0} \inf_{u\in\mathcal{U}^<[\epsilon^{-1}R,\epsilon^{-2}T]} I_{\epsilon^{-2}T}(u)$$

where

$$\mathcal{U}^<[\epsilon^{-1}R, \epsilon^{-2}T] = \Big\{u \in C^\infty(\mathbb{R}; (-1,1)) : u(\cdot,0) = \bar m, u(\cdot,\epsilon^{-2}T) \leq \bar m_{\epsilon^{-1}R}\Big\} \tag{5.3}$$

it will suffice to prove that for any $\gamma > 0$,

$$\liminf_{\epsilon\to 0} \inf_{u\in\mathcal{U}^<[\epsilon^{-1}R,\epsilon^{-2}T]} I_{\epsilon^{-2}T}(u) \geq \inf_{n\geq 0} w_n(R,T) - c\gamma \tag{5.4}$$

where $c$ is an absolute constant (determined only by the parameters entering in (1.1)).

Our strategy distinguishes two regimes: one is when the function $u(x,t)$ is everywhere "locally close" to an instanton (or to a reflected instanton); the other one, when instead $u(x,t)$ deviates from such a local equilibrium. In the first regime we study (1.6) regarding $b$ as a "small perturbation" and use spectral gap properties of the evolution linearized around an instanton. In such a linear approximation, we then obtain estimates for the penalty in agreement with the law "$V^2T/\mu$". The corrections to the linear approximation will also be proved to be under control. It thus remain to study the times when $u(\cdot,t)$ deviates from local equilibrium. Evidently these cannot be neglected because in such times there may occur the nucleations responsible for reaching the minimal cost. But, in any case, we need estimates which tell us that the intervals of time when the system is not in local equilibrium are bounded. We will start in the next sections from such an issue: we will first recall from the literature the Peierls estimates, which are a priori bounds on the spatial location of deviations from equilibrium in terms of the energy $\mathcal{F}$. By reversibility, we will bound $\mathcal{F}(u(\cdot,t)) \le P$, $P$ as in (5.1) and using the Peierls estimates, we will then bound the volume where the deviations from local equilibrium occur (contours) in terms of $\mathcal{F}(u(\cdot,t))$ and hence of $P$. We will then turn to another key point, namely upper bounds on the times of permanence outside local equilibrium. This is done in two steps. We first derive lower bounds on the energy gradients away from local equilibrium and in $\mathcal{F}(u(\cdot,t)) \le P$. These are lower bounds on the force which tries to restore local equilibrium, so that permanence of $u$ away from local equilibrium can only be achieved by applying a counter-force $b$. But since the total integral of $b^2$ is bounded by $P$, we then obtain an upper bound on the permanence outside local equilibrium.

We begin by defining local equilibrium, introducing the notion of contours and the Peierls estimates. We then define the "multi-instantons manifold", made by patching together several instantons. After that, we derive lower bounds on the energy gradients away from the multi-instantons manifold and finally estimates on permanence away from local equilibrium. At that point we will have all the ingredients necessary for proving Proposition 5.1.

## 6 Contours

In this section we recall from the literature notion and results which are extensively used in the sequel. Given $\ell > 0$, we denote by $\mathcal{D}^{(\ell)}$ the partition of $\mathbb{R}$ into the intervals $[n\ell,(n+1)\ell)$, $n \in \mathbb{Z}$, and by $Q_x^{(\ell)}$, $x \in \mathbb{R}$ the interval containing $x$. (Note that $x$ need not be the center of $Q_x^{(\ell)}$.) We say that $Q_x^{(\ell)}$, $Q_{x'}^{(\ell)}$ are connected, if the closures have nonempty intersection, i.e., $\overline{Q_x^{(\ell)}} \cap \overline{Q_{x'}^{(\ell)}} \ne \emptyset$. Now we define

$$m^{(\ell)}(x) := \fint_{Q_x^{(\ell)}} m(y)\,dy, \qquad \fint_{\Lambda} m(y)\,dy := \frac{1}{|\Lambda|}\int_{\Lambda} m(y)\,dy. \tag{6.1}$$

Given an "accuracy parameter" $\zeta > 0$, we then introduce

$$\eta^{(\zeta,\ell)}(m;x) = \begin{cases} \pm 1 & \text{if } |m^{(\ell)}(x) \mp m_\beta| \le \zeta, \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

For any $\Lambda \subseteq \mathbb{R}$ which is $D^{(\ell)}$-measurable we call

$$\begin{aligned} \mathcal{B}_0^{(\zeta,\ell,\Lambda)}(m) &:= \left\{ x \in \Lambda : \eta^{(\zeta,\ell)}(m;x) = 0 \right\}, \\ \mathcal{B}_\pm^{(\zeta,\ell,\Lambda)}(m) &:= \left\{ x \in \Lambda : |\eta^{(\zeta,\ell)}(m;x')| = \pm 1, \text{ there exists } x' \in \Lambda : \right. \\ &\qquad \left. \overline{Q_x^{(\ell)}} \cap \overline{Q_{x'}^{(\ell)}} \neq \emptyset \; \eta^{(\zeta,\ell)}(m;x') = -\eta^{(\zeta,\ell)}(m;x) \right\}, \\ \mathcal{B}^{(\zeta,\ell,\Lambda)}(m) &:= \mathcal{B}_+^{(\zeta,\ell,\Lambda)}(m) \cup \mathcal{B}_-^{(\zeta,\ell,\Lambda)}(m) \cup \mathcal{B}_0^{(\zeta,\ell,\Lambda)}(m). \end{aligned}$$

Calling $\ell_-$ and $\ell_+$ two values of the parameter $\ell$, with $\ell_+$ an integer multiple of $\ell_-$, we define a "phase indicator"

$$\Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = \begin{cases} \pm 1 & \text{if } \eta^{(\zeta,\ell_-)}(m;\cdot) = \pm 1 \text{ in } \left( Q_{x-\ell_+}^{(\ell_+)} \cup Q_x^{(\ell_+)} \cup Q_{x+\ell_+}^{(\ell_+)} \right), \\ 0 & \text{otherwise,} \end{cases}$$

and call contours of $m$ the connected components of the set $\{x : \Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = 0\}$. $\Gamma = [x_-, x_+)$ is a plus contour if $\eta^{(\zeta,\ell_-)}(m;x_\pm) = 1$, a minus contour if $\eta^{(\zeta,\ell_-)}(m;x_\pm) = -1$, otherwise it is called mixed.

Moreover we define for any measurable $\Lambda \subseteq \mathbb{R}$ and $m \in L^\infty(\mathbb{R};[-1,1])$ a local notion of energy by

$$\begin{aligned} \mathcal{F}(m_\Lambda | m_{\Lambda^c}) := & \int_\Lambda \phi_\beta(x) dx + \frac{1}{4} \int_{\Lambda \times \Lambda} J(x,y)[m(x) - m(y)]^2 \, dy \, dx \\ & + \frac{1}{2} \int_{\Lambda \times \Lambda^c} J(x,y)[m(x) - m(y)]^2 \, dy \, dx. \end{aligned}$$

The parameters $(\zeta, \ell_-, \ell_+)$ are called *compatible* with $(\zeta_0, c_1, \kappa) \in \mathbb{R}_+^3$ if $\zeta \in (0, \zeta_0)$, $\ell_- \le \kappa\zeta$, $\ell_+ \ge 1/\ell_-$, and if for any $D^{(\ell_-)}$-measurable set $\Lambda$ and any $m \in L^\infty(\mathbb{R};[-1,1])$

$$\mathcal{F}(m_\Lambda | m_{\Lambda^c}) \ge c_1 \zeta^2 |\mathcal{B}^{(\zeta,\ell_-,\Lambda)}(m)|.$$

**Theorem 6.1 ([2])** *There are positive constants $\zeta_0$, $c_1$, $\kappa$, $c_2$, and $\alpha$ so that if $(\zeta, \ell_-, \ell_+)$ is compatible with $(\zeta_0, c_1, \kappa)$, then for all $m \in L^\infty([-L, L];[-1,1])$,*

$$\mathcal{F}(m) \ge \sum_{\Gamma \text{ contour of } m} w_{\zeta,\ell_-,\ell_+}(\Gamma) \tag{6.3}$$

*where*

$$w_{\zeta,\ell_-,\ell_+}(\Gamma) = c_1\zeta^2\frac{\ell_-}{\ell_+}|\Gamma| \text{ if } \Gamma \text{ is a plus or a minus contour;}$$

$$w_{\zeta,\ell_-,\ell_+}(\Gamma) = \max\left\{c_1\zeta^2\tfrac{\ell_-}{\ell_+}|\Gamma|\ ;\ \mathcal{F}(\bar m) - c_2 e^{-\alpha\ell_+}\right\} \text{ if } \Gamma \text{ is a mixed contour.}$$

Let us conclude the section with some applications of Theorem 6.1. For any $u \in \mathcal{U}[\epsilon^{-1}R, \epsilon^{-2}T]$,

$$\sup_{t\le\epsilon^{-2}T}\Big(\mathcal{F}\big(u(\cdot,t)\big) - \mathcal{F}\big(u(\cdot,0)\big)\Big) \le I_{\epsilon^{-2}T}(u). \tag{6.4}$$

The proof follows directly from reversibility, see before Theorem 2.2, and it can be found in [1]. Combined with (5.1), (6.4) yields

$$\sup_{t\le\epsilon^{-2}T}\Big(\mathcal{F}\big(u(\cdot,t)\big) - \mathcal{F}\big(u(\cdot,0)\big)\Big) \le P. \tag{6.5}$$

Then, by Theorem 6.1, for $\zeta$ small enough,

$$\sum_{\Gamma_i \text{ contours of } u(\cdot,t)} |\Gamma_i| \le \frac{\ell_+}{c_1\ell_-}\zeta^{-2}(P + F(\bar m))\ , \tag{6.6}$$

$$\text{number of contours of } u(\cdot,t) \le \frac{1}{c_1\ell_-}\zeta^{-2}(P + F(\bar m)) =: N_{\max}\ , \tag{6.7}$$

$$\text{number of mixed of contours of } u(\cdot,t) \le \frac{P + F(\bar m)}{\mathcal{F}(\bar m) - c_2 e^{-\alpha\ell_+}} =: N_{\max}^{\text{mix}}\ . \tag{6.8}$$

## 7 Multi-instanton manifold

The instanton manifold is the set $\mathcal{M}^{(1)} = \{\bar m_\xi, \xi \in \mathbb{R}\}$. We extend the notion to the case of several coexisting instantons by defining the multi-instanton manifold $\mathcal{M}^{(k)}$, $k > 1$, as the set of all $\bar m_{\bar\xi}$, $\bar\xi = (\xi_1, \dots, \xi_k) \in \mathbb{R}^k$, $\xi_1 < \dots < \xi_k$, sufficiently apart from each other such that, setting $\xi_0 := -\infty$, $\xi_{k+1} := \infty$, the function

$$\bar m_{\bar\xi}(x) := \begin{cases} \bar m(x - \xi_j) & \text{if } x \in \left[\frac{\xi_{j-1}+\xi_j}{2}, \frac{\xi_{j+1}+\xi_j}{2}\right] \text{ and } j \text{ odd,} \\ \bar m(\xi_j - x) & \text{if } x \in \left[\frac{\xi_{j-1}+\xi_j}{2}, \frac{\xi_{j+1}+\xi_j}{2}\right] \text{ and } j \text{ even} \end{cases}$$

has exactly $k$ mixed contours.

We denote

$$\mathcal{M} = \bigcup_{k\ge1} \mathcal{M}^{(k)}. \tag{7.1}$$

To study "neighborhoods" of $\mathcal{M}$ we introduce the notion of "center of $m$" that we use here in a slightly different sense than usual:

**Definition.** *$\xi \in \mathbb{R}$ is a center of $m$ if $\xi \in \Gamma$, $\Gamma$ a mixed contour of $m$, and if*

$$\big(m - \bar{m}_\xi, \bar{m}'_\xi\big) = 0, \quad \textit{or, equivalently,} \quad \big(m, \bar{m}'_\xi\big) = 0 \tag{7.2}$$

*where $(\cdot,\cdot)$ denotes the scalar product in $L_2(\mathbb{R}, dx)$ and $\|\cdot\|_2$ the corresponding $L^2$ norm. $\xi$ is an odd, even, center if $\Gamma$ is a $(-,+)$, respectively $(+,-)$ mixed contour.*

**Remarks.** An odd center of $m$ specifies an element $\bar{m}_\xi \in \mathcal{M}^{(1)}$ such that the two directions, one pointing from $\bar{m}_\xi$ to $m$ and the other one along $\mathcal{M}^{(1)}$ are mutually $L^2$-orthogonal. If $\xi$ is even, same picture holds after a change of sign. Supposing $\Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = -1$ definitively as $x \to -\infty$, there is a first mixed contour coming from the left which is $(-,+)$, the next one is a $(+,-)$ and so on, this is the reason for naming the centers as odd and even.

The following theorem holds, see [7],

**Theorem 7.1** *If $\zeta$ (in the definition of contours) is small enough the following holds.*

- *Each mixed contour $\Gamma$ of $m$ contains a center of $m$.*
- *There is $\delta > 0$ so that if for some $\xi$ in a $(-,+)$ mixed contour $\Gamma$ of $m$ (analogous statement holding in the $(+,-)$ case), $\|\mathbf{1}_\Gamma(m - \bar{m}_\xi)\|_2 \le \delta$, then there is a unique center $\xi_m$ in $\Gamma$ and*

$$\int_{\mathbb{R}} \Big(\{m - \bar{m}_{\xi'}\}^2 - \{m - \bar{m}_{\xi_m}\}^2\Big) > 0, \quad \textit{for all } \xi' \in \Gamma,\ \xi' \neq \xi_m \tag{7.3}$$

  *and calling $v = m - \bar{m}_\xi$, $N_{v,\xi} = \dfrac{(v, \bar{m}'_\xi)}{(\bar{m}', \bar{m}')}$,*

$$\big|\xi_m - (\xi - N_{v,\xi})\big| \le c\|v\|_2^2, \quad |N_{v,\xi}| \le c\|v\|_2. \tag{7.4}$$

- *If also $\inf_\xi \|\mathbf{1}_\Gamma(n - \bar{m}_{\xi_n})\|_2 \le \delta$, then*

$$|\xi_m - \xi_n| \le c\|m - n\|_2. \tag{7.5}$$

By the first statement in Theorem 7.1 a function $m$ with $k$ mixed contours $\Gamma_1, \dots, \Gamma_k$ has (at least) one center in each one of the mixed contours; we denote by $\Xi$ the collection of all $\bar{\xi} = (\xi_1, \dots, \xi_k)$, $\xi_i < \xi_{i+1}$, $\xi_i$ a center of $m$ in $\Gamma_i$ and define

$$d_{\mathcal{M}}(m) = \inf_{\bar{\xi} \in \Xi} \|m - \bar{m}_{\bar{\xi}}\|_2. \tag{7.6}$$

If $m$ is close enough to $\mathcal{M}^{(k)}$, then the choice of $\bar{\xi}$ is unique. Note that this definition differs slightly from the usual definition of a distance of a point from a manifold, but the following lemma bounds this difference:

**Lemma 7.1** *For any $k$ there are $\delta > 0$ and $c$ so that if $m$ has $k$ mixed contours $\Gamma_1, \dots, \Gamma_k$ and $d_{\mathcal{M}}(m) \le \delta$, then*

$$d^2_{\mathcal{M}}(m) \ge \inf_{\bar\xi \in \Gamma_1 \times \cdots \times \Gamma_k} \|m - \bar m_{\bar\xi}\|_2^2 \ge d^2_{\mathcal{M}}(m) - c \sum_{i=1}^{k-1} e^{-\alpha\, \mathrm{dist}(\Gamma_{i+1}, \Gamma_i)/2} \tag{7.7}$$

*where $\alpha > 0$ is defined in* (1.4).

*Proof.* Call $\bar\xi^* = (\xi_1^*, \dots, \xi_k^*)$ the centers of $m$, which by Theorem 7.1 are uniquely defined (supposing $\delta > 0$ small enough). Let $A_i$, $i = 1, \dots, k$, be the decomposition of $\mathbb{R}$ defined by the midpoints of $\bar\xi^*$, then if $\bar\xi \in \Gamma_1 \times \cdots \times \Gamma_k$, and $\sigma_i = \pm 1$ if $i$ is odd, respectively even,

$$\|m - \bar m_{\bar\xi}\|_2^2 - d_{\mathcal{M}}(m)^2 = \sum_{i=1}^{k} \int_{A_i} \Big( \{m - \sigma_i \bar m_{\xi_i}\}^2 - \{m - \sigma_i \bar m_{\xi_i^*}\}^2 \Big).$$

By (7.3)

$$\int_{A_i} \Big( \{m - \sigma_i \bar m_{\xi_i}\}^2 - \{m - \sigma_i \bar m_{\xi_i^*}\}^2 \Big) \ge - \int_{A_i^c} \Big( \{m - \sigma_i \bar m_{\xi_i}\}^2 - \{m - \sigma_i \bar m_{\xi_i^*}\}^2 \Big)$$

hence (7.7) because of the exponential convergence of $\bar m(x)$ to $\pm m_\beta$ as $x \to \pm\infty$. □

## 8 Lower bounds on energy gradients

In this section we will investigate the structure of the energy levels of $\mathcal{F}(\cdot)$. In particular we will prove a lower bound on the energy gradient in terms of the distance from the manifolds $\mathcal{M}^{(k)}$:

**Theorem 8.1** *For any $\vartheta > 0$ there is $\rho > 0$ so that the following holds. Let $m \in L^\infty(\mathbb{R}; (-1,1))$ have an odd number $p$ of mixed contours, let $\mathcal{F}(m) \le P$ ($P$ as in Proposition* 5.1) *and let $d_{\mathcal{M}}(m)^2 \ge \vartheta$. Then*

$$\int_{\mathbb{R}} f(m)^2 \ge \rho. \tag{8.1}$$

The proof is given at the end of the section, after several preliminary estimates, but before we state a corollary of Theorem 8.1 on the "permanence away from equilibrium" which will be essential in the sequel.

**Theorem 8.2** *Let $u$ satisfy* (5.1), *then for any $\vartheta > 0$ there is $\rho > 0$ so that, if $d_{\mathcal{M}}(u(\cdot, t)) \ge \vartheta$ when $t \in [t_0, t_1]$, $0 \le t_0 < t_1 \le \epsilon^{-2} T$, then necessarily $t_1 - t_0 \le \dfrac{8}{3} \dfrac{P}{\rho}$.*

*Proof.* Let $\rho$ be the parameter associated to $\vartheta$ by Theorem 8.1. Then

$$\int_{t_0}^{t_1} \int_{\mathbb{R}} |f(u)|^2 \geq [t_1 - t_0]\rho.$$

We estimate

$$\begin{aligned}
\mathcal{F}(u(t_1)) - \mathcal{F}(u(t_0)) &= \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(u(s))\,\mathrm{d}s \\
&= \int_{t_0}^{t_1} \int_{\mathbb{R}} (f(u(s)) + b(s)) \frac{\delta \mathcal{F}}{\delta u}(u(s)) \\
&\leq - \int_{t_0}^{t_1} \|f(u)\|_2^2 + \int_{t_0}^{t_1} \|f(u)\|_2 \|b(s)\|_2 \\
&\leq \int_{t_0}^{t_1} \left( -\frac{3}{4}\|f(u)\|_2^2 + \|b(s)\|_2^2 \right) \mathrm{d}s \leq -\frac{3}{4}[t_1 - t_0]\rho + P.
\end{aligned}$$

With the help of (6.5) we can estimate

$$\mathcal{F}(u(t_1)) - \mathcal{F}(u(t_0)) \geq - \left( \sup_{s \in [0, \epsilon^{-2}T]} [\mathcal{F}(u(s))] - \mathcal{F}(\bar{m}) \right) \geq -P,$$

and we conclude the proof of Theorem 8.2. □

We start the proof of Theorem 8.1 by a general outline of its strategy. By analogy with the Allen-Cahn equation, it can be conjectured that the stationary, spatially non homogeneous solutions of (1.1) are either the instanton $\bar{m}$ (and its translates) or periodic functions, which then have infinite energy. The assumption in the theorem excludes both possibilities, thus leading to the conclusion that the functions $m$ to consider are such that $f(m)$ is not identically 0. As we will see it is possible to reach the same conclusion avoiding the above characterization of the stationary solutions of (1.1). It still remains, however, to quantify the condition $f(m) \not\equiv 0$ in the sense of the inequality (8.1). This will be done using continuity and compactness, the argument being that once we know that $\int f(m)^2 > 0$ for each $m$ in the set defined in Theorem 8.1, then also the inf (in the same set) is non zero. Continuity and compactness require to work in weak $L^2$ spaces, which, on the other hand, do not fit well in our context, as for instance the function $m \to f(m)^2$ is not weakly continuous.

Besides such "technical problems", anyway the proof of (8.1) cannot go too smoothly. Suppose $m$ has $2k+1$, $k \geq 1$, mixed contours. Then it is known that the orbit starting from $m$ converges to an instanton, as a consequence $f(m) \not\equiv 0$ and $\int f(m)^2 > 0$. However the integral may be arbitrarily small if the mixed contours in $m$ are very far apart from each other and in each of them $m$ looks like an instanton or its reverse. Such a possibility however will be excluded by the condition $d_{\mathcal{M}}(m)^2 \geq \vartheta$, showing that such an assumption must complement the information that $f(m) \not\equiv 0$. The analysis of the condition $d_{\mathcal{M}}(m)^2 \geq \vartheta$ will distinguish whether the deviations of $m$ from $\bar{m}_{\bar{\xi}}$ are localized in a neighborhood of the contours of $m$ or in the complement, and we will start by examining the former case.

We will denote space intervals and contours by the letter Q, in order to distinguish them from time intervals, which will be denoted by the letter I.

Let $Q$, $Q_j$ and $B^{\pm}_{k,j}$ be intervals of the form $Q = [a,b)$, $Q_j = [a-j, b+j)$, $B^{-}_{k,j} = [a-j-k, a-j)$, $B^{+}_{k,j} = [b+j, b+j+k)$ with $a, b, j, k$ all in $\ell_+\mathbb{N}$. Then, given $\vartheta > 0$, we set

$$U_{Q,j,\vartheta} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : Q \text{ is a mixed } -,+ \text{ contour for } m \text{ and } \inf_{\xi \in Q} \int_{Q_j} |m - \bar{m}_\xi|^2 \geq \vartheta \Big\}, \tag{8.2}$$

$$V_{k,j} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : \eta^{(\zeta,\ell_-)}(m;x) = \pm 1 \text{ for all } x \in B^{\pm}_{k,j} \Big\}. \tag{8.3}$$

**Lemma 8.1** *For any $\vartheta > 0$, $Q$ and $Q_j$ as above, there is $k$ so that*

$$\int_{Q_{k+j}} |f(m)| > 0 \quad \textit{for any } m \in U_{Q,j,\vartheta} \cap V_{k,j}. \tag{8.4}$$

*Proof.* Define

$$K_h = U_{Q,j,\vartheta} \cap V_{h,j} \cap \{m : \int_{Q_{h+j}} |f(m)| = 0\}. \tag{8.5}$$

The proof of (8.4) is then equivalent to showing that for some $h$, $K_h = \emptyset$. We rewrite $\int_{Q_{h+j}} |f(m)| = 0$ as $\int_{Q_{h+j}} |m - \tanh\{\beta J * m\}| = 0$ and, since $m = \tanh\{\beta J * m\}$ in $Q_{h+j}$,

$$K_h = U^*_{Q,j,\vartheta} \cap V_{h,j} \cap \{m : \int_{Q_{h+j}} |f(m)| = 0\}$$

where

$$U^*_{Q,j,\vartheta} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : Q \text{ is a mixed } -,+ \text{ contour for } m \text{ and } \inf_{\xi \in Q} \int_{Q_{j+h}} |\tanh\{\beta J * m\} - \bar{m}_\xi|^2 \geq \vartheta \Big\}.$$

The advantage of having $U^*_{Q,j,\vartheta}$ is that this set is closed (in the weak $L^2$ topology) and, more importantly, the same $K_h$ is weakly closed in $L^2_{\text{loc}}$. Since $K_h$ is contained in the unit ball of $L^\infty$, $K_h$ is also weakly $L^2_{\text{loc}}$ compact. By compactness of such a space,

$$\{\bigcap_h K_h = \emptyset\} \quad \Leftrightarrow \quad \{K_h = \emptyset \text{ for some } h\}.$$

We have thus reduced the proof of the lemma to showing that $\bigcap_h K_h = \emptyset$.

Suppose that $m \in \bigcap_h K_h$. Then $m = \tanh\{\beta J * m\}$ almost everywhere, while, simultaneously, $\eta^{(\zeta,\ell_-)}(m;x) = \pm 1$, eventually as $x \to \pm\infty$. Then $m = \overline{m}_\xi$ for some $\xi \in \mathbb{R}$ and since $Q$ is a mixed contour for $m$, $\xi \in Q$, which contradicts $m \in U^*_{Q,j,\vartheta}$, hence $\bigcap_h K_h = \emptyset$. □

**Proposition 8.1** *For any $\vartheta > 0$, $Q$ and $Q_j$ let $k$ be as in Lemma* 8.1*. Then there is $\rho > 0$ so that*

$$\inf_{m \in U_{Q,j,\vartheta} \cap V_{k,j}} \int_{Q_{k+j}} f(m)^2 \geq \rho. \tag{8.6}$$

*Proof.* Suppose by contradiction that the inf is 0. Then there is a sequence $m_n \in U_{Q,j,\vartheta} \cap V_{k,j}$ such that

$$\lim_{n\to\infty} \int_{Q_{k+j}} f(m_n)^2 = 0 \tag{8.7}$$

and which converges weakly in $L^2_{\text{loc}}$, say $m_n \rightharpoonup \hat{m}$. As $J(0,\cdot)$ is smooth and has support in the unit ball, this implies that $J * m_n \to J * \hat{m}$ strongly in $L^2_{\text{loc}}$ and pointwise. From (8.7) we derive

$$A_\beta(m_n) \to J * \hat{m} \text{ in } L^2(Q_{k+j}).$$

Since the function tanh is uniformly Lipschitz continuous, we get $m_n \to \tanh(\beta J * \hat{m})$ in $L^2(Q_{k+j})$. Therefore

$$\lim_{n\to\infty} \hat{m} = \tanh(\beta J * \hat{m}) \text{ in } Q_{k+j}, \tag{8.8}$$

and $f(\hat{m})(x) = 0$ for all $x \in Q_{k+j}$. By (8.8), $\hat{m} \in U_{Q,j,\vartheta}$; moreover $\hat{m} \in V_{k,j}$ because the latter is weak $L^2$ closed, hence $\hat{m} \in U_{Q,j,\vartheta} \cap V_{k,j}$. We have already seen that $f(\hat{m})(x) = 0$ for all $x \in Q_{k+j}$ and this, by Lemma 8.1, leads to a contradiction. Thus $\rho > 0$. □

The analogues of $U_{Q,j,\vartheta}$ and $V_{k,j}$ when the external conditions are in the plus or in the minus phase are

$$U^{\pm}_{Q,j,\vartheta} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : Q \text{ is a } \pm \text{ contour for } m \text{ and } \int_{Q_j} |m \mp m_\beta|^2 \geq \vartheta \Big\} \tag{8.9}$$

$$V^{\pm}_{k,j} = \Big\{ m \in L^\infty(\mathbb{R}, (-1,1)) : \eta^{(\zeta,\ell_-)}(m;x) = \pm 1 \text{ for all } x \in B^-_{k,j} \cup B^+_{k,j} \Big\}. \tag{8.10}$$

The previous arguments can be adapted to prove (details are omitted):

**Proposition 8.2** *For any $\vartheta > 0$, $Q$ and $Q_j$ there are $k$ and $\rho > 0$ so that*

$$\inf_{m \in U^{\pm}_{Q,j,\vartheta} \cap V^{\pm}_{k,j}} \int_{Q_{k+j}} f(m)^2 \geq \rho. \tag{8.11}$$

Given an interval $Q$ and a function $\hat{m} \in L^\infty(\mathbb{R}, (-1,1))$, we denote by $\hat{L}$ the operator on $L^2(Q; dx)$ defined by

$$\hat{L}\psi(x) = \int_Q J(x,y)\psi(y) + \frac{1}{\beta(1-\hat{m}(x)^2)}\psi(x), \quad x \in Q. \tag{8.12}$$

$\hat{L}$ is obtained by linearizing around $\hat{m}$ the "partial dynamics" $u_t = f(u)$ in $Q$, $u = \hat{m}$ outside $Q$. In [2] it is proved that:

**Proposition 8.3** *There are $c$, $\alpha$ and $\omega$ all positive so that the following holds. Let $Q = [a,b)$, $a, b \in \ell_+\mathbb{N}$, $b$ possibly equal to $+\infty$, and $m \in L^\infty(\mathbb{R}, (-1,1)$, $\Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = 1$ for all $x \in Q$. Then:*

- *There is a unique solution $\hat{m}$ of*

$$\begin{aligned} &\hat{m}(x) = \tanh\{\beta J * \hat{m}(x)\}, \quad \Theta^{(\zeta,\ell_-,\ell_+)}(\hat{m};x) = 1, && \text{for all } x \in Q \\ &\hat{m}(x) = m(x), && \text{for all } x \notin Q \end{aligned} \tag{8.13}$$

- *$\hat{m}$ is a smooth function on $Q$ and*

$$|\hat{m}(x) - m_\beta| \leq c e^{-\alpha \operatorname{dist}(x, Q^c)} \tag{8.14}$$

- *$\hat{L}$ is self-adjoint in $L^2(Q)$ and its spectrum lies in $(-\infty, -\omega]$, $\omega > 0$.*

**Lemma 8.2** *There is $c^* > 0$ so that for any $Q$, $m$ and $\hat{m}$ as in Proposition 8.3,*

$$\int_Q f(m)^2 \geq c^* \int_Q |m - \hat{m}|^2 \tag{8.15}$$

*Proof.* Let $a \in (m_\beta, 1)$, $A_\beta(a) \geq 4$,

$$c_a := \inf_{x \neq y \in [0,a]} \frac{|A_\beta(x) - A_\beta(y)|}{|x-y|}, \quad c_a < 1$$

the last inequality because $A'_\beta(0) < 1$ for $\beta > 1$. Suppose also $\zeta$ so small that $\hat{m}(x) < a$ for all $x \in Q$ and

$$16\zeta < c_a; \quad \frac{\omega^2}{2} > \frac{16\zeta}{c_a}; \quad \frac{16\zeta}{c_a} < \kappa$$

where $\kappa \in (0, [1+a]/2)$ is such that

$$c^2\kappa^2 \le \frac{\omega^2}{2}, \quad c := A'_\beta([1+a]/2). \tag{8.16}$$

We then call

$$Q_\kappa := \{x \in Q : |m(x) - \hat{m}(x)| > \kappa\}. \tag{8.17}$$

Since $\Theta^{(\zeta,\ell_-,\ell_+)}(m;x) = 1$ for all $x \in Q$, if $\zeta$ is small enough, $|J*(m-\hat{m})| \le 4\zeta$ on $Q$, i.e., including $Q_\kappa$ as well.

We are going to prove that

$$f(m)^2 \ge |J*(m-\hat{m})|^2 + \frac{c_a^2}{4}|m-\hat{m}|^2 \quad \text{on } Q_\kappa. \tag{8.18}$$

We distinguish two cases. Case 1: $x \in Q_\kappa$ and $|m(x)| \le a$. Then, since $f(\hat{m}) = 0$ on $Q_\kappa$,

$$\begin{aligned}|f(m)| &= |\{A_\beta(m) - A_\beta(\hat{m})\} - J*(m-\hat{m})| \ge |A_\beta(m) - A_\beta(\hat{m})| - |J*(m-\hat{m})| \\ &\ge c_a|m-\hat{m}| + |J*(m-\hat{m})| - 8\zeta \ge \frac{c_a}{2}|m-\hat{m}| + |J*(m-\hat{m})|.\end{aligned}$$

Case 2: $x \in Q_\kappa$ and $|m(x)| > a$. Then, recalling that $A_\beta(a) \ge 4$, $A_\beta(a) \ge |m-\hat{m}| + 8\zeta$ and

$$|f(m)| \ge A_\beta(a) + |J*(m-\hat{m})| - 8\zeta \ge |J*(m-\hat{m})| + |m-\hat{m}|$$

which concludes the proof of (8.18) because $c_a \le 1$.

We write

$$\begin{aligned}\int_Q f(m)^2 &\ge \int_{Q_\kappa} f(m)^2 + \epsilon \int_{Q\setminus Q_\kappa} f(m)^2 \\ &\ge \int_{Q_\kappa} \frac{c_a^2}{4}(m-\hat{m})^2 + [J*(m-\hat{m})]^2 + \epsilon\int_{Q\setminus Q_\kappa} f(m)^2\end{aligned} \tag{8.19}$$

with $\epsilon > 0$ to be specified later. In $Q \setminus Q_\kappa$ we linearize around $\hat{m}$ and recalling that $\hat{m} \le a$, $\max A'_\beta(\hat{m}) \le A'_\beta(a) \le c$, $c$ as in (8.16), we obtain

$$\begin{aligned}\int_{Q\setminus Q_\kappa} f(m)^2 &= \int_{Q\setminus Q_\kappa} |f(m) - f(\hat{m})|^2 \\ &\ge \int_{Q\setminus Q_\kappa} \Big(\hat{L}(m-\hat{m})\Big)^2 - c^2|m-\hat{m}|^4 \\ &\ge \int_Q \Big(\hat{L}(m-\hat{m})\Big)^2 - \int_{Q_\kappa}\Big(\hat{L}(m-\hat{m})\Big)^2 - c^2\kappa^2\int_{Q\setminus Q_\kappa}|m-\hat{m}|^2.\end{aligned}$$

Using again that $\hat{m} \le a$, $\max A'_\beta(\hat{m}) \le A'_\beta(a) \le c$,

$$\int_{Q_\kappa}\Big(\hat{L}(m-\hat{m})\Big)^2 \le 2\int_{Q_\kappa}|J*(m-\hat{m})|^2 + c^2\,(m-\hat{m})^2\,.$$

We now choose $\epsilon > 0$ so that $2\epsilon < 1$ and $2c^2\epsilon \le c_a^2/4$, then getting from (8.19) and (8.16)

$$\int_Q f(m)^2 \ge \epsilon \int_Q \Big(\hat{L}(m-\hat{m})\Big)^2 - \epsilon\frac{\omega^2}{2}\int_{Q\setminus Q_\kappa} |m-\hat{m}|^2. \tag{8.20}$$

By Proposition 8.3

$$\int_Q [\hat{L}\psi(x)]^2 \ge \omega^2 \int_Q \psi^2, \quad \text{for any } \psi \in L^2(Q) \tag{8.21}$$

hence (8.15) because $m - \hat{m} = 0$ on $\mathbb{R}\setminus Q$. □

*Proof of Theorem* 8.1. Without loss of generality, we may suppose $\vartheta > 0$ as small as required by the arguments below. By Theorem 6.1, $m$ has at most $N_{\max}$ contours, with at most $N_{\max}^{\text{mix}}$ among them which are mixed contours.

We start from the case when there is a $(+,+)$ contour $Q$. Calling $Q_- := \{x \in Q : \eta^{(\zeta,\ell_-)}(m;x) < 1\}$, since $Q_- \neq \emptyset$ because $\eta^{(\zeta,\ell_-)}(m;x) < 1$ somewhere in $Q$, by definition of contours,

$$|⨍_{Q_-} [m(x) - m_\beta]| \ge \zeta$$

and, by Cauchy-Schwartz,

$$\int_Q |m(x) - \bar{m}_\beta|^2 \ge \ell_-(\frac{\zeta}{2})^2 \ge \vartheta \tag{8.22}$$

for $\vartheta$ small enough.

We take $j = 0$, $Q_j = Q$ and call $k_1$ the smallest value of $k$ for which Proposition 8.2 applies with $Q$, $\vartheta$ and $j = 0$. If in $Q_{k_1}$ there are no contours Proposition 8.2 yields (8.1). If on the contrary, there are contours, according to cases, we will apply either Proposition 8.2 or Proposition 8.1, as it will be explained after (8.23) below. To this end, we call $\bar{m}_{\bar{\xi}}$ the element of $\mathcal{M}$ with $\bar{\xi}$ centers of $m$. Observe that $|\bar{m}_{\bar{\xi}}(x) - m_\beta| \le ce^{-\alpha\ell_+}$ for $x \in Q$, by definition of contours and because of the decay properties of $\bar{m}$. Then

$$|⨍_{Q_-} [m(x) - \bar{m}_{\bar{\xi}}]| \ge |⨍_{Q_-} [m(x) - m_\beta]| - |⨍_{Q_-} [\bar{m}_{\bar{\xi}} - m_\beta]| \ge \zeta - ce^{-\alpha\ell_+} \ge \frac{\zeta}{2}$$

and, analogously to (8.22),

$$\int_Q |m(x) - \bar{m}_{\bar{\xi}}|^2 \ge \ell_-(\frac{\zeta}{2})^2 \ge \vartheta. \tag{8.23}$$

We now continue the previous argument. If in $Q_{k_1}$ there are contours besides $Q$, we take $j = k_1$ and call $k_2$ the smallest $k$ for which either Proposition 8.2 or

Proposition 8.1 can be applied with $Q$, $\vartheta$ and $j = k_1$. Again, if in $B^{\pm}_{k_2}$ there are contours, we call $j = k_2$ and repeat the procedure. As there are at most $N_{\max}$ contours, the iteration is finite and the final $j$ and $k$ are bounded in terms of $P$ and $\vartheta$ only. Let $\rho$ be the value corresponding to such parameters, hence (8.1) holds for such $m$ with the above value of $\rho$.

Same argument applies when there is a $--$ contour, and we are left with the case with only mixed contours, say there are $p \le N^{\min}_{\max}$ mixed contours. Fix $j^*$ so that

$$ce^{-\alpha\ell+j^*} \le \vartheta^2. \tag{8.24}$$

We distinguish two cases. Case 1 is when there is a mixed contour $Q$ such that

$$\int_{Q_{j^*}} |m - \bar{m}_{\bar{\xi}}|^2 \ge \frac{\vartheta}{2p}. \tag{8.25}$$

In this case using Proposition 8.1 we can proceed as before, getting again (8.1) with the new value of $\rho$.

We are then reduced to case 2, where calling $\Lambda$ the complement of the union of $Q^{(i)}_{j^*}$, $Q^{(i)}$ the $i$-th contour,

$$\int_{\Lambda} |m - \bar{m}_{\bar{\xi}}|^2 \ge \frac{\vartheta}{2}. \tag{8.26}$$

$\Lambda$ is a union of intervals, let $Q_0$ be one such that

$$\int_{Q_0} |m - \bar{m}_{\bar{\xi}}|^2 \ge \frac{\vartheta}{2(p+1)}. \tag{8.27}$$

Call $Q$ the interval containing $Q_0$ between two consecutive contours. By applying Lemma 8.2, we get, using (8.24),

$$\int_{Q} |f(m)|^2 \ge c\int_{Q_0} |m - \hat{m}|^2 \ge c\int_{Q_0} |m - \bar{m}_{\bar{\xi}}|^2 - c'\vartheta^2 \ge c''\vartheta. \tag{8.28}$$

Theorem 8.1 is proved. □

## 9 Good and bad time intervals

In this section we introduce an analogue for times of the notion of contours. To this end we partition the time axis $\mathbb{R}_+$ into intervals $\{S[j, j+1),\ j \in \mathbb{N}\}$ of length $S > 0$. The analogue of the function $\eta^{(\zeta,\ell_-)}(m;x)$, here denoted by $\phi^{(\delta,S)}(u;t)$, $\delta > 0$, is defined as

$$\phi^{(\delta,S)}(u;t) = \begin{cases} 1 & \text{if } \displaystyle\int_{jS}^{(j+1)S} \|b(s)\|_2^2 < \delta \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } t \in S[j, j+1). \tag{9.1}$$

The role of $\Theta^{(\zeta,\ell_-,\ell_+)}(m;x)$ is played by $\Phi^{(\delta,S)}(u;t)$, defined equal to 1 if $\phi^{(\delta,S)}(u;s) = 1$ for all $s \in S[j-1,j+1)$ and $= 0$ otherwise. We define $G_{\rm tot} = \{t \le \epsilon^{-2}T : \Phi^{(\delta,S)}(u;t) = 1\}$ and call $t$ a "good time" and $S[j,j+1)$ a good interval if they are contained in $G_{\rm tot}$. Bad times and bad intervals are defined complementary.

**Choice of parameters.** Given $R$ and $T$, i.e., the macroscopic displacement of the interface and the time interval when it occurs, we call

$$n^* = 1 + \frac{2P}{\mathcal{F}(\bar m)}. \tag{9.2}$$

$n^*$ is an upper bound for the total number of fronts, considering that each nucleation produces two fronts, it costs more than $> \mathcal{F}(\bar m)$, as we will see and $P$ is an upper bound for the cost of the orbit, see Proposition 5.1. By the same Proposition 5.1, the proof of Theorem 2.3 follows from showing that (5.4) holds, we thus fix arbitrarily $\gamma > 0$ and then determine $\ell^* > 0$ so that

$$\left|\mathcal{F}\big(\bar m_{(-\ell^*,\ell^*)}\big) - 2\mathcal{F}(\bar m)\right| \le \frac{\gamma}{10^3 (n^*)^3}, \quad \bar m_{(-\ell^*,\ell^*)} = \mathbf{1}_{x\ge 0}\bar m_{\ell^*} - \mathbf{1}_{x<0}\bar m_{-\ell^*}. \tag{9.3}$$

By the $L^2$-continuity of $\mathcal{F}(\cdot)$, there is $\vartheta$ so that for all $m$ such that $d_{\mathcal{M}}(m) \le \vartheta$ and with centers $(\xi_1,\dots,\xi_n)$, $n \le n^*$, $\xi_{i+1} - \xi_i \ge 2\ell^*$,

$$\left|\mathcal{F}(m) - k\mathcal{F}(\bar m)\right| \le \frac{\gamma}{10^3 (n^*)^2}. \tag{9.4}$$

It remains to fix $\delta$ and $S$ in (9.1): $\delta$ will be "small" and $S$ "large" but their exact choice is rather intricate. The problem here comes from "incomplete nucleations", we need in fact to "discard" those where the distance $\ell$ between the centers of the nucleating instantons is smaller than $2\ell^*$.

**Proposition 9.1** *There is $\tau > 0$ so that for any positive $\ell \le \ell^*$, the solution $v(x,s)$ of* (1.1) *starting from $\bar m_{(-\ell,\ell)}$ verifies*

$$\sup_{x\in\mathbb{R}} |v(x,\tau) - m_\beta| \le \vartheta.$$

The proof is "essentially contained" in [3], for brevity we omit the details. By a barrier lemma and the comparison theorem, see Appendix A in [1], we also have (again details are omitted):

**Proposition 9.2** *There is $L > 0$ for which the following holds. Let $\ell$ and $\tau$ as in Proposition* 9.1 *and $\bar\xi = (\xi_1,\dots,\xi_n)$, $n \le n^*$. Call $\mathcal{I}$ the set of all even $i$ such that $\xi_{i+1} - \xi_i \le \ell$. Suppose $\mathcal{I}$ non void and that for $j \notin \mathcal{I}$, $\xi_{j+1} - \xi_j \ge L$. Then the solution $w(x,t)$ of* (1.1) *which starts from $\bar m_{\bar\xi}$ is such that*

$$\sup_{x\in\mathbb{R}} |w(x,\tau) - \bar m_{\bar\xi^*}(x)| \le \vartheta \tag{9.5}$$

*where $\bar\xi^*$ is obtained from $\bar\xi$ by dropping all pairs $\xi_i, \xi_{i+1}$, $i \in \mathcal{I}$.*

By a continuity argument, see Theorem C.1, (again details are omitted):

**Proposition 9.3** *Let $\ell$, $\tau$, $L$, $\bar\xi$ and $\bar\xi^*$ as in Proposition 9.2. Then there is $\alpha > 0$ such that if*

$$\|m - \bar m_{\bar\xi}\|_2 \le \vartheta, \qquad \int_0^\tau \|b\|_2^2 \le \alpha \tag{9.6}$$

*then the solution $w^{(b,m)}(x,t)$ of (1.6) with force $b$ and which starts from $m$ is such that*

$$\|w^{(b,m)}(x,\tau) - \bar m_{\bar\xi^*}(x)\|_2 \le 4\vartheta. \tag{9.7}$$

**Choice of $S$ and $\delta$.** Let $\rho$ be the parameter associated to $\vartheta$ by Theorem 8.2, then

$$S > 10^3 \max\big\{\tau, \frac{8}{3}P\rho^{-1}, s', s'', \frac{4}{\omega}\big\} \tag{9.8}$$

with $s'$ and $s''$ as in Appendix C, $\omega$ as in Appendix D. We finally choose $\delta$ so that

$$\delta = 10^{-3} \min\big\{\alpha, \frac{\vartheta}{c_{11.1}S}\big\}, \quad \alpha \text{ and } c_{11.1} \text{ as in Proposition 9.3 and Proposition 11.1.} \tag{9.9}$$

**Theorem 9.1** *Let $u$ satisfy (5.1) and let $\delta$ and $S$ as above. Then:*

$$\textit{number of bad time intervals} \le \frac{P}{2\delta}. \tag{9.10}$$

*If $S[j, j+1)$ is a good time interval, there is $t_1 \in S[j - \frac{1}{2}, j - \frac{1}{4})$ such that $d_{\mathcal{M}}(u(\cdot, t_1)) \le \vartheta$.*

*Proof.* Suppose $I$ is a bad interval, call $I^-$ the previous time interval. By definition, the inequality (9.1) cannot hold for both $I$ and $I^-$, otherwise $I$ would be good, hence (9.10), recalling that $P \ge I_{\epsilon^{-2}T}(u)$. The second statement follows from Theorem 8.2 and (9.8). □

## 10 Subsolutions

Having fixed an orbit $u$ as in Proposition 5.1, we define once for all $b := u_t - f(u)$ and consider an orbit $m(x,t)$, such that $m(\cdot, 0) = \bar m(\cdot)$, obtained by patching together solutions of (1.6) with forcing term $b$ as above. We decompose the time axis into intervals $I_i$ and define $m(x,t)$, $t \in I_i$, as the solution of (1.6) starting from $m(\cdot, s^+)$, $s$ the left end point in $I_i$. $m(\cdot, s^+)$ may be either equal to $m(\cdot, s^-)$ or $m(\cdot, s) < m(\cdot, s^-)$, according to cases. As proved in Appendix A there is an existence and uniqueness theorem for these Cauchy problems so that the definition is well posed and, by the validity of a comparison theorem,

$$m(x,t) \le u(x,t), \quad x \in \mathbb{R}, t \in [0, \epsilon^{-2}T]. \tag{10.1}$$

The new orbit $m$ is not necessarily in $\mathcal{U}[\epsilon^{-2}T, \epsilon^{-1}R]$, but it is in $\mathcal{U}^{<}[\epsilon^{-2}T, \epsilon^{-1}R]$, see (5.4)–(5.3).

Inequalities play an important role and we will often use the following notion. We define a partial order by setting

$$(\xi_1, \dots, \xi_k) \geq (\xi'_1, \dots, \xi'_{k'}) \quad \Leftrightarrow \quad \bar{m}_{(\xi_1,\dots,\xi_k)} \geq \bar{m}_{(\xi'_1,\dots,\xi'_{k'})} \tag{10.2}$$

In particular, if $k = k'$,

$$(\xi_1, \dots, \xi_k) \geq (\xi'_1, \dots, \xi'_k) \quad \Leftrightarrow \quad \xi_i \leq \xi'_i,\ i \text{ odd}, \quad \xi_i \geq \xi'_i,\ i \text{ even}. \tag{10.3}$$

We will use different strategies in the bad and the good time intervals. We start from the latter, calling $jS$ the left end point of a maximal connected component $G$ of $G_{\text{tot}}$. We will choose a time $t_{\text{in}} \in [(j-1/2)S, jS]$ which depends on the orbit $m$ and it is such that $m(\cdot, t_{\text{in}})$ is "very nice" and we will then study $m(\cdot, t)$, $t \geq t_{\text{in}}$, via the evolution equation (1.6) which it satisfies, taking advantage of the fact that when times are good, the "external force" $b$ is small. The choice of $m(x, t_{\text{in}})$ is aimed at a perturbative analysis, based on the linearization of (1.6) around the manifold $\mathcal{M}$ and the choice of $t_{\text{in}}$ is critical. Let $t_1$ be the smallest time $\geq (j-1/2)S$ when $d_{\mathcal{M}}(m(\cdot, t)) \leq \vartheta$. Then $t_1 \leq (j-1/4)S$ by Theorem 9.1. For $\vartheta$ small enough, $m$ has only mixed contours, their number, denoted by $k$, being odd. Call $\bar{\xi} = (\xi_1, \dots, \xi_k)$ its centers, ordered increasingly. We distinguish three cases, with Case 1) when $\xi_{j+1} - \xi_j > 2|\log \epsilon^{-1}|^2$ for all $j$: we do not need in Case 1) to modify $m$, so that $t_{\text{in}} = t_1$ and $m(\cdot, t^+_{\text{in}}) = m(\cdot, t^-_{\text{in}})$, in the remaining cases, instead, (10.1) will hold as a strict inequality.

In Cases 2) and 3) we erase from $\bar{\xi}$ all pairs $\xi_i, \xi_{i+1}$ with $i$ odd, such that $\xi_{i+1} - \xi_i \leq 2|\log \epsilon|^2$, calling $\bar{\xi}_1$ the new configuration. Since we are erasing pairs $\xi_i, \xi_{i+1}$ with $i$ odd, then $\bar{m}_{\bar{\xi}_1} \leq \bar{m}_{\bar{\xi}}$. With $\ell^*$ as in the paragraph "Choice of parameters" in Section 9, we then look at all even $j$ in $\bar{\xi}_1$ such that $2\ell^* \leq \xi_{j+1} - \xi_j \leq 2|\log \epsilon|^2$ and move each $\xi_j, \xi_{j+1}$ to $\xi'_j, \xi'_{j+1}$ where

$$\xi'_j + \xi'_{j+1} = \xi_j + \xi_{j+1}, \quad \xi'_{j+1} - \xi'_j = 2|\log \epsilon|^2.$$

We call $\bar{\xi}_2$ the configuration obtained in this way and $\bar{\xi}_3$ the one obtained from $\bar{\xi}_2$ by the same procedure used to define $\bar{\xi}_1$ starting from $\bar{\xi}$. In $\bar{\xi}_3$ the pairs $\xi_i, \xi_{i+1}$ with $i$ even either verify $\xi_{i+1} - \xi_i \geq 2|\log \epsilon|^2$ or $\xi_{i+1} - \xi_i \leq 2\ell^*$. Case 2) is when $\xi_{i+1} - \xi_i \geq 2|\log \epsilon|^2$ for all $i$, while Case 3) covers the remaining possibilities. We define

$$\tilde{m}(x, t_1) = \min\big\{m(x, t_1), \bar{m}_{\bar{\xi}_3}(x)\big\}.$$

In Case 2) $t_{\text{in}} = t_1$ and $m(\cdot, t^+_{\text{in}}) = \tilde{m}(\cdot, t_1)$, while in Case 3) $t_{\text{in}} = t_1 + \tau$, $\tau$ as in Proposition 9.1, and $m(\cdot, t^+_{\text{in}})$ is the solution at time $t_1 + \tau$ of (1.6) starting from $\tilde{m}(\cdot, t_1)$ at time $t_1$.

**Proposition 10.1** *For all $\epsilon > 0$ small enough, the centers of $m(\cdot, t^+_{\text{in}})$ have mutual distance $\geq |\log \epsilon|^2$ and*

$$d_{\mathcal{M}}\big(m(\cdot, t^+_{\text{in}})\big) \leq 6\vartheta.$$

*Proof.* By definition of $t_1$, $d_{\mathcal{M}}(m(\cdot,t_1)) \le \vartheta$. In Case 1) the centers of $m(\cdot,t_{\rm in})$ have mutual distance $\ge 2|\log\epsilon^{-1}|^2$, hence the statements in the proposition. In Case 2), by construction the elements of $\bar\xi_3$ have distance $\ge 2|\log\epsilon^{-1}|^2$ and $\bar m_{\bar\xi_3} \le \bar m_{\bar\xi}$. We have

$$\|\tilde m - \bar m_{\bar\xi_3}\|_2 \le \|m - \bar m_{\bar\xi}\|_2. \tag{10.4}$$

In fact, $\tilde m(x) = \bar m_{\bar\xi_3}(x)$ unless $\tilde m(x) = m(x) < \bar m_{\bar\xi_3}(x)$, and (10.4) follows recalling that $\bar m_{\bar\xi_3} \le \bar m_{\bar\xi}$. Recalling that $\|m(\cdot,t_1) - \bar m_{\bar\xi}\|_2 \le \vartheta$, by definition of $t_1$, $\|\tilde m - \bar m_{\bar\xi_3}\|_2 \le \theta$ and, denoting by $h$ the number of elements in $\bar\xi_3$ and by $\Gamma_i$ the mixed contours of $m$, by (7.7),

$$\begin{aligned} d_{\mathcal{M}}\big(m(\cdot,t_{\rm in})\big) &\le \|\tilde m - \bar m_{\bar\xi_3}\|_2 + c\sum_{i=1}^{h-1} e^{-\alpha\,{\rm dist}(\Gamma_{i+1},\Gamma_i)/2} \\ &\le \vartheta + cn^* e^{-\alpha|\log\epsilon^{-1}|^2/2} \le 2\vartheta \end{aligned}$$

for $\epsilon$ small enough. In Case 3), by (10.4) and Proposition 9.3,

$$d_{\mathcal{M}}\big(m(\cdot,t^+_{\rm in})\big) \le 4\vartheta + cn^* e^{-\alpha|\log\epsilon^{-1}|^2/2} \le 6\vartheta.$$

Moreover the centers of $m(\cdot,t^+_{\rm in})$ differ from the corresponding ones in $m(\cdot,t_1)$ at most by $2\vartheta$, as it follows from Proposition 9.2 and Theorem 7.1. Proposition 10.1 is proved. □

## 11 Estimates by linearizion

In this section we will study the solutions of (1.6) in a maximal connected component $G$ of the good times set, $G_{\rm tot}$,

$$G = [j, j^*]S \subset G_{\rm tot}, \tag{11.1}$$

see Section 9 and Section 10 for the relevant definitions. We will start from the first good time interval $[j, j+1]S$ contained in $G$ and then iterate the argument to the successive ones.

**Setup.** As explained at the beginning of Section 10, we actually study an orbit $m(x,t)$ solution of (1.6) for $t \ge t^+_{\rm in}$, $t^+_{\rm in} \in [j-1/2, j-1/4]S$, which starts from $m(\cdot,t^+_{\rm in})$. After a careful choice of $t_{\rm in}$ and after using inequalities, we have seen that we may suppose $m(\cdot,t_{\rm in})$ as having an odd number $k$ of mixed contours at mutual distance $\ge |\log\epsilon^{-1}|^2$; moreover $d_{\mathcal{M}}(m(\cdot,t^+_{\rm in})) \le 6\vartheta$. Finally, by definition of good intervals, the force $b(x,t)$ is such that

$$\int_{hS}^{(h+1)S} \|b(\cdot,s)\|_2^2 \le \delta, \quad h \in \{j-1, j\}. \tag{11.2}$$

**Choice of parameters.** In the sequel $\omega > 0$ is the "spectral gap parameter" defined in Appendix D; $s'$, $\alpha'$ and $M$ are as in Theorem C.2 of Appendix C;

$C(M) = \sup_{m\in[0,M]} A''_\beta(m)$; $\alpha''$ and $s''$ of Theorem C.3 of Appendix D are such that $\epsilon = \epsilon_1$ with $\epsilon_1 < \frac{\omega}{8C(M)c_1}$; $\alpha^* := \min\{\alpha', \alpha''\}$. Recall also that $S \geq \max\{s', s''\}$, see (9.8).

**Notation.** We denote by $\chi$ the characteristic function of $A_{\alpha^*}$, where $\alpha^*$ is defined above and

$$A_{\alpha^*} := \Big\{ x \in \mathbb{R} : \int_{(j-1)S}^{(j+1)S} b^2(x,s)\mathrm{d}s \leq \alpha^* \Big\} \tag{11.3}$$

noting that

$$|A^c_{\alpha^*}| \leq \frac{1}{\alpha^*} \int_{(j-1)S}^{(j+1)S} \|b(s)\|_2^2 \,. \tag{11.4}$$

Calling $\bar\xi(t) = (\xi_1(t), \dots, \xi_k(t))$ the centers of $m(\cdot,t)$, $t \geq t_{\rm in}$, we define the approximate centers $\tilde\xi(t) = (\tilde\xi_1(t), \dots, \tilde\xi_k(t))$ and the deviation $u(\cdot,t)$, in the usual way except for inserting the characteristic function $\chi$:

$$\big(\chi \bar m'_{\tilde\xi_i(t)}, [m(\cdot,t) - \sigma_i \bar m_{\tilde\xi_i(t)}]\big) = 0, \quad u(\cdot,t) = m(\cdot,t) - \bar m_{\tilde\xi(t)} \tag{11.5}$$

with $\sigma_i = 1$ [$\sigma_i = -1$] if $i$ is odd, [even], and $\tilde\xi_i(t)$ in the $i$-th mixed contour of $m(\cdot,t)$ (as we will see $m(\cdot,t)$ has only mixed contours).

Finally we call $\Lambda_i(t)$, $i = 1, \dots, k$, the open intervals $\frac{1}{2}\big(\tilde\xi_{i-1}(t)+\tilde\xi_i(t), \tilde\xi_{i+1}(t)+\tilde\xi_i(t)\big)$, $\tilde\xi_0(t) = -\infty$ and $\tilde\xi_{k+1}(t) = +\infty$.

**Remarks.** We have

$$|\tilde\xi_i(t) - \xi_i(t)| + \|u(\cdot,t) - \{m(\cdot,t) - \bar m_{\bar\xi(t)}\}\|_2 \leq \frac{c}{\alpha^*} \int_{(j-1)S}^{(j+1)S} \|b(s)\|_2^2. \tag{11.6}$$

We sketch the proof for the case of one contour only. The extension to the general case is straightforward, as the centers have distance $\geq |\log\epsilon|^2$.

Denote by $\xi(\chi m)$ and $\xi(m)$ the centers of $\chi m$ and respectively $m$. We estimate by (7.5) $|\xi(\chi m) - \xi(m)| \leq c|A^c_{\alpha^*}|$. According to its definition, $\tilde\xi(t)$ may be different from $\xi(\chi m)$, but for $d^2_{\mathcal M}(\chi m)$ small enough the function

$$\xi \to (\bar m'_\xi, \chi m - \bar m_\xi)$$

has nonzero derivative at its unique zero $\xi(\chi m)$. As $(\bar m'_{\tilde\xi}, \chi m - \bar m_{\tilde\xi}) \leq c'|A^c_{\alpha^*}|$, we get with a possibly different constant (11.6).

The variational inequality (5.4) requires lower bounds on $\tilde\xi(t)$ in the sense of (10.2). We will thus prove in the sequel upper bounds for displacements of centers with $i$ odd and lower bounds for those with $i$ even.

**Proposition 11.1** *There is a constant $c_{11.1} > 0$, so that for $\vartheta$ and $\delta$ small enough and for all $t \in [t_{\rm in}, (j+1)S]$,*

$$\|u(\cdot,t)\|_2^2 \le e^{-(t-t_{\rm in})\omega/2}\|u(\cdot,t_{\rm in})\|_2^2 + c_{11.1} S U_j^2 \tag{11.7}$$

$$\sigma_i[\xi_i(t)-\xi_i(t_{\rm in})] \le -\frac{1}{\|\bar m'\|_2^2}\int_{t_{\rm in}}^t (b, \bar m'_{\xi_i(t)}) + c_{11.1}\big[\|u(\cdot,t_{\rm in})\|_2^2 + SU_j^2\big] \tag{11.8}$$

*where $i = 1,\dots,k$ and*

$$U_j^2 = \int_{(j-1)S}^{(j+1)S} \|b(\cdot,s)\|_2^2 + R_{\max}, \quad R_{\max} = c_{11.1} e^{-\alpha|\log\epsilon|^2/2}. \tag{11.9}$$

Note that $R_{\max} \to 0$ as $\epsilon \to 0$.

*Proof.* Let

$$L\colon L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (Lu)(x) := (J*u)(x) - A'_\beta\big(\bar m_{\tilde\xi(t)}(x)\big)u(x).$$

Note that the coefficient of the local part depends on $t$. For $x \in \Lambda_i$, (see the paragraph "Notation" above)

$$\frac{du(x,t)}{dt} = \sigma_i \dot{\tilde\xi}_i(t)\bar m'_{\tilde\xi_i(t)} + Lu(x,t) + u^2(x,t)\int_0^1 A''_\beta\left(\bar m_{\tilde\xi(t)} + \lambda u(x,t)\right)d\lambda + b(x,t) \tag{11.10}$$

Multiply (11.10) by $u(\cdot,t)\chi$ and integrate over space. Note that $\chi$ depends on the time interval we are considering, but since such interval is here fixed, $\chi$ does no longer depend on time. Since $\chi^2 = \chi$, we obtain

$$\frac{\mathrm d}{\mathrm dt}\left(\frac12\|u\chi\|_2^2\right) = (u\chi, Lu) + \left(\chi u\,,\, u^2\int_0^1 A''_\beta\big(\bar m_{\tilde\xi(t)} + \lambda u(\cdot,t)\big)d\lambda\right) + (\chi u, b) + R$$

$$R = R(t) = \sum_{i=1}^k \sigma_i\dot{\tilde\xi}_i(t)\big(\mathbf{1}_{\Lambda_i(t)}\bar m'_{\tilde\xi_i(t)}, \chi u\big) \tag{11.11}$$

By (11.4),

$$|(u\chi, Lu) - (u\chi, L(u\chi))| \le \int_{A^c_{\alpha^*}\times\mathbb{R}} J(x-y)|u(x)||u(y)| \le 4|A^c_{\alpha^*}| \le \frac{4}{\alpha^*}\int_{(j-1)S}^{(j+1)S}\|b(s)\|_2^2.$$

On the other hand $(u\chi, L(u\chi)) \le -\omega\|u\chi\|_2^2$ by the spectral gap property of the operator $L$ proved in Appendix D. We use Theorem C.3 to bound the cubic term in

(11.11) and recalling the "Choice of parameters" in the beginning of this section, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\chi\|_2^2\right)$$
$$\leq -\omega\|u\chi\|_2^2 + C(M)(\epsilon_1 + c_1\|u\|_2^{2/3})\|\chi u\|_2^2 + (\chi u, b) + c\int_{(j-1)S}^{(j+1)S}\|b(s)\|_2^2 + R.$$

Let

$$\tau := \inf\left\{t : \|u(\cdot,t)\|_2^{2/3} > \frac{\omega}{8C(M)c_1}\right\}. \tag{11.12}$$

Bounding $|(\chi u, b)| \leq \dfrac{2\|b\|_2^2}{\omega} + \dfrac{\omega\|\chi u\|_2^2}{4}$, for all times $t \in [t_{\rm in}, (j+1)S]$ such that $t < \tau$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u\chi\|_2^2\right) \leq -\frac{\omega}{2}\|u\chi\|_2^2 + \frac{2}{\omega}\|b\|^2 + c\int_{(j-1)S}^{(j+1)S}\|b(s)\|_2^2 + R,$$

i.e., for $t^* = \min\{\tau, (j+1)S\}$ we obtain

$$\|\chi u(\cdot,t^*)\|_2^2 \leq e^{-(t^*-t_{\rm in})\omega/2}\|u(\cdot,t_{\rm in})\|_2^2 + c'S\Big(\int_{(j-1)S}^{(j+1)S}\|b(s)\|_2^2 + R_{\max}\Big)$$

with $R_{\max}$ defined in (11.9). The bound comes from (11.5) after estimating $\tilde\xi_{i+1}(t) - \tilde\xi_i(t) > |\log\epsilon|^2/2$. The latter holds for $\epsilon > 0$ small enough using Theorem C.1 and because the time interval we are considering and the force are uniformly bounded in $\epsilon$, (a posteriori, see (11.8), we will see that displacements are of the order of $\vartheta$).

Since $|u^2| \leq 4$,

$$\|u\|_2^2 \leq \|\chi u\|_2^2 + 4|A^c_{\alpha^*}| \leq \|\chi u\|_2^2 + \frac{4}{\alpha^*}\int_{(j-1)S}^{(j+1)S}\|b(s)\|_2^2$$
$$\|u(\cdot,t^*)\|_2^2 \leq e^{-(t^*-t_{\rm in})\omega/2}\|u(\cdot,t_{\rm in})\|_2^2 + c''SU_j^2.$$

The last term is bounded by $c''S(2\delta + R_{\max})$, $S\delta \leq 10^{-3}\vartheta$ by (9.9), so that for $\delta$, $\vartheta$ and $\epsilon$ small enough, the r.h.s. in the last equation is $< (\dfrac{\omega}{8C(M)c_1})^3$, thus $\tau \geq t^*$, $\tau$ as in (11.12) and $t^* = (j+1)S$. The proof of (11.7) is complete.

To prove (11.8), we multiply (11.10) by $\chi\bar m'_{\tilde\xi_i(t)}$ and estimate $(u_t, \chi\bar m'_{\tilde\xi_i(t)})$ by first writing (11.5) as

$$\big(\chi\bar m'_{\tilde\xi_i(t)}, u\big) = \big(\chi\bar m'_{\tilde\xi_i(t)}, [\sigma_i\bar m_{\tilde\xi_i(t)} - \bar m_{\tilde\xi(t)}]\big) \tag{11.13}$$

and then differentiating it on time. We then get

$$\begin{aligned}\big(\chi\bar m'_{\tilde\xi_i(t)},u_t\big) = \dot{\tilde\xi}_i(t)\Big\{\big(\chi\bar m''_{\tilde\xi_i(t)},u\big)+\big(\chi\bar m''_{\tilde\xi_i(t)},[\bar m_{\tilde\xi(t)}-\sigma_i\bar m_{\tilde\xi_i(t)}]\big)\Big\}\\ -\sum_{j\neq i}\big(\chi\mathbf{1}_{\Lambda_j(t)}\bar m'_{\tilde\xi_i(t)},[\sigma_i\dot{\tilde\xi}_i(t)\bar m'_{\tilde\xi_i(t)}-\sigma_j\dot{\tilde\xi}_j(t)\bar m'_{\tilde\xi_j(t)}]\big)\end{aligned} \tag{11.14}$$

The second term on the r.h.s. of (11.10) gives

$$(Lu,\chi\bar m'_{\tilde\xi_i(t)}) = (u,L\bar m'_{\tilde\xi_i(t)})-(Lu,(1-\chi)\bar m'_{\tilde\xi_i(t)}). \tag{11.15}$$

Note that the kernel of the operator obtained by linearizing around $\bar m$ is spanned by $\bar m'$. As the centers have distance $\geq|\log\epsilon|^2$, the exponential convergence of both $\bar m$ and $\bar m'$ implies that $c_{11.1}>0$ can be chosen in such a way that $|L\bar m'_{\tilde\xi_i(t)}|\leq R_{\max}$. As the second term in (11.15) is bounded by $c|A^c_{\alpha^*}|$, $c$ a positive constant, we then obtain from (11.10),

$$\begin{aligned}\sigma_i\dot{\tilde\xi}_i\Big\{\|\bar m'_{\tilde\xi_i(t)}\chi\|_2^2-\sigma_i(u,\chi\bar m''_{\tilde\xi})-\sigma_i\big(\chi\bar m''_{\tilde\xi_i(t)},[\bar m_{\tilde\xi(t)}-\sigma_i\bar m_{\tilde\xi_i(t)}]\big)\Big\}\\ +\sum_{j\neq i}\big(\chi\mathbf{1}_{\Lambda_j(t)}\bar m'_{\tilde\xi_i(t)},[\sigma_i\dot{\tilde\xi}_i(t)\bar m'_{\tilde\xi_i(t)}-\sigma_j\dot{\tilde\xi}_j(t)\bar m'_{\tilde\xi_j(t)}]\big)\\ \leq -(b,\chi\bar m'_{\tilde\xi_i(t)})+c|A^c_\alpha|+c'\cdot C(M)\|u\chi\|_2^2+R_{\max}\end{aligned} \tag{11.16}$$

which has the form

$$\sigma_i\|\bar m'\|_2^2\,\dot{\tilde\xi}_i\leq\beta_i+\sum_{j=1}^k a_{i,j}|\dot{\tilde\xi}_j| \tag{11.17}$$

with $\beta_i$ and $a_{i,j}>0$ identified by (11.16). We will prove that

$$a:=\frac{1}{\|\bar m'\|_2^2}\max_{1\leq i\leq k}\sum_{h=1}^k a_{i,h}<1 \tag{11.18}$$

so that

$$\sigma_i\|\bar m'\|_2^2\,\dot{\tilde\xi}_i\leq\beta_i+\frac{a}{(1-a)}\max_{i=1,\dots,k}|\beta_i|. \tag{11.19}$$

Using (11.7) we have

$$|\beta_i+\big(b,\bar m'_{\tilde\xi_i(t)}\big)|\leq c\Big\{SU_j^2+\|1-\chi\|_2\|b\|_2+e^{-(t-t_{\rm in})\omega/2}\|u(\cdot,t_{\rm in})\|_2^2\Big\}. \tag{11.20}$$

To bound $a$, we bound $|(u,\chi\bar m''_{\tilde\xi})|\leq c\|u\|_2$ and, after some computations which are omitted, $a\leq\Big(e^{-(t-t_{\rm in})\omega/2}\|u(\cdot,t_{\rm in})\|_2^2+SU_j^2\Big)^{1/2}$ and

$$\sigma_i\|\bar m'\|_2^2\,\dot{\tilde\xi}_i\leq\beta_i+c\|b\|_2\Big(e^{-(t-t_{\rm in})\omega/2}\|u(\cdot,t_{\rm in})\|_2^2+SU_j^2\Big)^{1/2}. \tag{11.21}$$

We bound the last term as 1/2 the sum of the squares, then integrating over time we finally get (11.8), after using (11.6) to relate $\tilde\xi$ to $\bar\xi$. □

By (9.9), $c_{11.1} S U_j^2 \leq \vartheta$; by (9.8), $e^{-\omega S/2} \leq 1/2$, then by (11.7) we get, supposing $\epsilon$ small enough,

$$\|u(\cdot,(j+1)S)\|_2^2 \leq e^{-\omega S/2}\, \|u(t_{\rm in})\|_2^2 + c_{11.1} S U_j^2 \leq 4\vartheta. \tag{11.22}$$

Since $\tilde\xi_{i+1}((j+1)S) - \tilde\xi_i((j+1)S) \geq |\log\epsilon|^2/2$, as we have seen in the course of the proof of Proposition 11.1, it then follows from (7.7) that for $\epsilon$ small enough,

$$d_{\mathcal M}(m(\cdot,(j+1)S)) \leq 5\vartheta. \tag{11.23}$$

We set

$$v_i^0(t) = \sigma_i \frac{1}{\|\bar m'\|_2^2}\, \Big| \int_{t_{\rm in}}^t (b, \bar m'_{\xi_i(t)}) \Big| \tag{11.24}$$

$$v_i(t) = v_i^0(t) + \sigma_i c_{11.1} \Big( U_j^2 + \|u(\cdot,t_{\rm in})\|_2^2 \Big) \tag{11.25}$$

$$r_i(t) = \xi_i(t_{\rm in}) + \int_{t_{\rm in}}^t v_i(s), \quad \bar r(t) = \big(r_1(t),\dots,r_k(t)\big) \tag{11.26}$$

observing that $\bar\xi(t) \geq \bar r(t)$, for $t \in [t_{\rm in},(j+1)S]$. We then define $\bar r\big([(j+1)S]^+\big)$ by erasing in $\bar r\big((j+1)S\big)$ all pairs $r_{i+1}\big((j+1)S\big) - r_i\big((j+1)S\big) \leq |\log\epsilon|^2$. We will recall this by saying that particles $i$ and $i+1$ have collided and, due to the collision, they have disappeared, (in the next section we will then write $r_i(t) = r_{i+1}(t) = \emptyset$ for $t > (j+1)S$, but here we will not use such notation).

By (11.8) the centers $\bar\xi$ of $m(\cdot,(j+1)S)$ are $\geq \bar r\big([(j+1)S]^+\big)$, in the sense of (10.2) and we set

$$m\big(x,[(j+1)S]^+\big) = \min\big\{ m(x,(j+1)S), \bar m_{[\bar r((j+1)S)]^+}(x) \big\}. \tag{11.27}$$

For $\epsilon$ small enough,

$$d_{\mathcal M}\Big( m\big(\cdot,[(j+1)S]^+\big)\Big) 6\vartheta. \tag{11.28}$$

Moreover the centers of $m\big(x,[(j+1)S]^+\big)$ have mutual distance $\geq |\log\epsilon|^2$. We are thus in the same setup as in Proposition 11.1, which can then be iterated to all the intervals of $G$. Hence for $h \in \mathbb N$ such that $(j+1) < h \leq j^*$, see (11.1),

$$\begin{aligned} \|u(\cdot,hS)\|_2^2 &\leq e^{-\omega S/2} \|u(\cdot,(h-1)S)\|_2^2 + c_{11.1} S \Big( \int_{(h-1)S}^{hS} \|b\|^2 + R_{\max} \Big) \\ &\leq c_{11.1} S e^{\omega S/2} \Big( \int_{(j-1)S}^{hS} e^{-\omega(hS-s)/2} \|b(s)\|^2 + R_{\max} \Big) \\ &\quad + e^{-\omega(h-(j-1))S/2} \|u(\cdot,t_{\rm in})\|_2^2 \end{aligned} \tag{11.29}$$

and for $t \in [h,h+1)S$, (11.25) yields

$$\begin{aligned} v_i(t) &= v_i^0(t) + \sigma_i c_{11.1} e^{-\omega(h-(j-1))S/2} \|u(\cdot,t_{\rm in})\|_2^2 \\ &\quad + \sigma_i c_{11.1} S e^{\omega S/2} \Big( \int_{(j-1)S}^{t} e^{-\omega(t-s)/2} \{ \|b\|^2 + R_{\max} \} \Big), \end{aligned} \tag{11.30}$$

hence

$$\Big|r_i(t) - \{\xi_i(t_{\rm in}) + \int_0^t v_i^0(s)\}\Big| \le c\|u(\cdot,t_{\rm in})\|_2^2 \\ + cSe^{\omega S/2}\Big(\int_{(j-1)S}^t \{\|b\|^2 + R_{\max}\}\Big). \qquad (11.31)$$

We summarize what proved so far, by saying that we have introduced auxiliary particles orbits $\bar r(t) = \big(r_1(t),\dots,r_k(t)\big)$ which starts from $\bar\xi(t_{\rm in})$. The particles move with velocity $v_i(t)$ and collide disappearing once they are at mutual distance $\le |\log\epsilon|^2$, after such time we write $r_i(t)=\emptyset$ for the disappeared particle. We recall the relation between $\bar r(t)$ and the function $m(x,t)$ in the following proposition:

**Proposition 11.2** *The centers $\bar\xi$ of $m(x,[hS]^+)$, $h\in\{j+1,\dots,j^*\}$, satisfy $\bar\xi \ge \bar r([hS]^+)$* (see (10.2))*, and $d_{\mathcal M}(m(\cdot,[hS]^+)) \le 6\vartheta$.*

## 12 Displacements in the bad intervals and total cost

We have defined the auxiliary process $\bar r(t)$ for $t\in G$, with $G$ as in (11.1), and we want to extend the definition to all times $t\in[0,\epsilon^{-2}T]$.

We use the following notation: $t_0 = j^*S$ is the right end point in $G$; $j'S$ the left end point of the next good time period $G'$; $t_1\in[j'-1/2, j'-1/4]S$ is the time associated to $G'$ as defined in Section 10; we write $\bar r(t_0) = \big(r_1(t_0),\dots,r_k(t_0)\big)$ and $\bar\xi = (\xi_1,\dots,\xi_h)$ the centers of $m(\cdot,t_1)$. We recall that $r_{i+1}(t_0) - r_i(t_0) \ge |\log\epsilon|^2$, $i=1,\dots,k$ and that $d_{\mathcal M}(m(\cdot,t_0)) \le 6\vartheta$ and $d_{\mathcal M}(m(\cdot,t_1) \le \vartheta$.

We then define $\bar r(t) = \bar r(t_0)$ for $t\in[t_0,t_1)$ and will use Theorem 12.1 below to extend the definition to $t\ge t_1$.

Both the maximal length of the bad interval and the field $b$ applied there are bounded by the total cost. Therefore the displacement of the already existing contours during the bad interval is bounded, and the newly nucleated fronts are close to each other. This is formalized in the next theorem.

**Theorem 12.1** *The number $h$ of centers of $m(\cdot,t_1)$ is odd and $h\ge k$. There is $K$ and an increasing sequence $i_1,\dots,i_k$ in $\{1,\dots,h\}$ so that $|\xi_{i_j} - r_j(t_0)| \le K$. Let $p = h-k$ and $\{\ell_1,\dots,\ell_p\} = \{1,\dots,h\}\setminus\{i_1,\dots,i_k\}$, then $\xi_{\ell_{i+1}} - \xi_{\ell_i} \le K$ for all $i$ odd in $\{1,\dots,p\}$.*

*Proof.* Call $m^0(x,t)$, $t\ge t_0$, the solution of (1.1) which starts from $m_0 = m(\cdot,t_0^+)$ at time $t_0$. By regarding (1.1) as (1.6) with $b=0$, we can apply the analysis of Section 11 so that, for $\epsilon$ small enough, $m^0(x,t_1)$ has $k$ centers $\bar\xi^0 = (\xi_1^0,\dots,\xi_k^0)$, $|\xi_i^0 - r_i(t_0)| \le 1$ and $d_{\mathcal M}(m^0(\cdot,t_1)) \le 6\vartheta$.

By Theorem C.4 in Appendix C,

$$\|m(\cdot,t_1) - m^0(\cdot,t_1)\|_2^2 \le e^{3|t_1-t_0|}\int_{t_1}^{t_0} \|b(t)\|_2^2\ , \qquad (12.1)$$

so that

$$\|\bar m_{\bar\xi} - \bar m_{\bar\xi^0}\|_2 \le P\, e^{3|t_1-t_0|} + 12\vartheta\ . \tag{12.2}$$

There is $a > 0$ so that $\bar m_{\bar\xi^0}(x) < -\frac{m_\beta}{2}$ for $x < \xi_1^0 - a$; $\bar m_{\bar\xi^0}(x) > \frac{m_\beta}{2}$ for $x \in (\xi_1^0 + a, \xi_2^0 - a)$ and so on. The same property evidently holds for $\bar m_{\bar\xi}$ so that the upper bound (12.2) induces an upper bound on the volume where $\bar m_{\bar\xi}$ and $\bar m_{\bar\xi^0}$ have a mismatch in the above sense, hence the statements in the theorem observing that by Theorem 9.1, $t_1 - t_0 \le \frac{P}{2\delta}S$. In particular, the sequence $i_1, \dots, i_k$ can be defined as follows. For $j$ odd call $i_j$ the odd label such that

$$\min_{i \text{ odd}} |\xi_i - \xi_j^0| = |\xi_{i_j} - \xi_j^0|$$

$i_j$, $j$ even, being defined analogously. The elements $i_1, \dots, i_k$ are mutually distinct for $\epsilon$ small enough because $\xi_{i+1}^0 - \xi_i^0 \ge |\log\epsilon|^2 - 2$. □

We identify the labels $1, \dots, k$ of the particles in $\bar r(t_0)$ with the sequence $i_1, \dots, i_k$ defined in Theorem 12.1. We now refer to Cases 1) to 3) listed in Section 10. In Case 1), where $t_{\rm in} = t_1$, we define $r_j(t_{\rm in}) = \xi_{i_j}$ and add particles at positions $\xi_{\ell_i}$, $i = 1, \dots, p$ according to Theorem 12.1. In this way $\bar r(t)$ has a discontinuity at time $t_1$, as the positions of the old particles may have been displaced by $\le K$ and moreover because new particles may have been added. In Case 2) $t_{\rm in} = t_1$ and a new configuration $\xi'$ has been defined in terms of $\xi$ by first shifting apart till distance $|\log\epsilon^{-1}|^2$ all pairs in $\xi$ at distance $\in [\ell^*, |\log\epsilon^{-1}|^2]$ and then by erasing all colliding particles. We define $\bar r(t_{\rm in})$ by setting $r_j(t_{\rm in}) = \xi'_{i_j}$ if the particle $i_j$ has not collided, and otherwise $r_j(t_{\rm in}) = \emptyset$. We complete the definition by adding particles at positions $\xi'_{\ell_i}$, $i = 1, \dots, p$, provided they have not collided. In Case 3) we let first run (1.6) for a time $\tau$ and then repeat the above procedure, we refer to Section 10 for details.

It is convenient to say that at all times there are $n^*$ particles present so that $\bar r(t) = \big(r_1(t), \dots, r_{n^*}(t)\big)$ but the existing ones are only those such that $r_i(t) \ne \emptyset$. We use a labeling of the particles so that whenever existent, $r_i(t) < r_j(t)$ if $i < j$.

By iteration the above rules define $\bar r(t)$ at all times $t \in [0, \epsilon^{-2}T]$. $r_i(t) \ne \emptyset$ has velocity $v_i(t) = 0$ in the intervals $(t_0, t_1)$ and otherwise $v_i(t)$ is given by (11.25).

$r_i(t)$ may have discontinuities at the beginning of the new good periods, the jumps being bounded by a constant $K$. When a pair of particles is created the two have distance $|\log\epsilon|^2$. Two particles collide, disappearing, when they are at mutual distance $|\log\epsilon|^2$. These types of discontinuous motion are not counted by $v_i(t)$, which can be interpreted as absolutely continuous part of the velocity.

The constraint (5.3) implies that the total displacement of the centers is at least $\epsilon^{-1}R$. In order to derive from this information a constraint for the $v_i^0(t)$ defined in (11.24), we have to take into account the error made when replacing $v_i$ by $v_0^i$ (see (11.30)), the displacement during bad intervals, and finally the displacement due to nucleation and collision of droplets. Therefore we obtain

$$\sum_{i=1}^{n^*} \int_{\{t:r_i(t)\neq\emptyset\}} |v_i^0(t)| \geq \epsilon^{-1}R - \left( cn^* \int_0^{\epsilon^{-2}T} \{\|b(s)\|_2^2 + R_{\max}\} + \frac{P}{2\delta}K + n^* 4|\log\epsilon|^2 \right). \tag{12.3}$$

We next compute the total cost. We have

$$\|b(t)\|_2^2 \geq \sum_{i:r_i(t)\neq\emptyset} \{\frac{1}{\|\bar{m}'\|_2^2}(b, \bar{m}'_{r_i(t)})^2 - ce^{-\alpha|\log\epsilon|^2/2}\} \tag{12.4}$$

so that

$$\|b(t)\|_2^2 \geq \sum_{i:r_i(t)\neq\emptyset} \|\bar{m}'\|_2^2 \, v_i^0(t)^2 - ce^{-\alpha|\log\epsilon|^2/2} \tag{12.5}$$

and, recalling that the mobility $\mu = 4\|\bar{m}'\|_2^{-2}$,

$$\frac{1}{4}\int_{G_{\mathrm{tot}}} \|b(t)\|_2^2 \geq \int_{G_{\mathrm{tot}}} \sum_{i:r_i(t)\neq\emptyset} \frac{v_i^0(t)^2}{\mu} - ce^{-\alpha|\log\epsilon|^2/2}\epsilon^{-2}T. \tag{12.6}$$

The cost of the bad times between two successive good periods is completely neglected if no nucleation occurs otherwise, with the same notation as in Theorem 12.1, we estimate by reversibility

$$\frac{1}{4}\int_{t_0}^{t_1} \|b(t)\|_2^2 \geq \mathcal{F}(m(\cdot,t_1)) - \mathcal{F}(m(\cdot,t_0)) \tag{12.7}$$

and by (9.4),

$$\mathcal{F}(m(\cdot,t_1)) - \mathcal{F}(m(\cdot,t_0)) \geq (h-k)\mathcal{F}(\bar{m}) - \frac{2\gamma}{10^3(n^*)^2}. \tag{12.8}$$

Thus

$$\frac{1}{4}\int_0^{\epsilon^{-2}T} \|b(t)\|_2^2 \geq \int_{G_{\mathrm{tot}}} \sum_{i:r_i(t)\neq\emptyset} \frac{v_i^0(t)^2}{\mu} + n\mathcal{F}(\bar{m}) - ce^{-\alpha|\log\epsilon|^2/2}\epsilon^{-2}T - \frac{2\gamma}{10^3 n^*} \tag{12.9}$$

where $n/2$ is the total number of nucleations and because $h-k \leq n^*$ and there are at most $n^*$ of such times. We now observe that the inf over $\{v_i^0(\cdot)\}$ of the right-hand side of (12.9) under the constraint (12.3), converges in the limit $\epsilon \to 0$ to $\inf_h w_h(R,T) - \frac{2\gamma}{10^3 n^*}$ which proves (5.4) thus concluding the proof of the lower bound.

We now consider the auxiliary variational problem of finding the inf over $\{v_i^0(\cdot)\}$ of the right-hand side of (12.9) under the constraint (12.3). Keep the number of particles $n$ fixed, $n \le n^*$, and let $t_i$ be the lifetime of the $i$-th particle, i.e., $t_i \le T\epsilon^{-2}$. If we keep $(t_i)_{i=1,\dots,n}$ fixed, then we see immediately that the velocity of each particle must be constant throughout its lifetime. Let $v_i$ be this constant velocity. For the auxiliary problem we get that a minimizer must fulfill the constraint as equality, hence we have to minimize $\sum_{i=1}^{n} v_i^2 t_i$ under $\sum_{i=1}^{n} v_i t_i = R\epsilon^{-1}$, which leads to $v_i = \lambda$, $i = 1,\dots,n$. As $\lambda$ satisfies $\lambda = \epsilon^{-1} R(\sum_{i=1}^{n} t_i)^{-1}$, we get $t_i = T\epsilon^{-2}$ for a minimizer, so that the minimum of the auxiliary problem for $n$ fixed converges in the limit $\epsilon \to 0$ to $w_n(R,T)$. Optimizing over the number of particles proves (5.4), thus concluding the proof of the lower bound.

## A Existence and uniqueness theorems

We will study here the Cauchy problem

$$\frac{du}{dt} = J * u - A_\beta(u) + b, \quad u(x,0) = u_0(x). \tag{A.1}$$

In Theorem A.1 below we will prove existence and uniqueness in $C(\mathbb{R};(-1,1))$ for $b \in C(\mathbb{R} \times \mathbb{R}_+;\mathbb{R})$. Observe that since $J(x,y) \ge 0$, a comparison theorem holds for (A.1).

**Theorem A.1** *There exists one and only one solution $u \in C(\mathbb{R} \times \mathbb{R}_+;(-1,1))$ of* (A.1).

*Proof.* Let $u_n^\pm(x,t)$, $n > 1$, be the functions equal respectively to $\pm 1$ on $[-n,n]^c$ which solve

$$\frac{du_n^\pm}{dt} = J * u_n^\pm - A_\beta(u_n^\pm) + b, \quad u_n^\pm(x,0) = u_0(x), \;\; x \in [-n,n]. \tag{A.2}$$

Existence and uniqueness of $u_n^\pm(x,t)$ follow from standard methods, moreover using the comparison theorem,

$$-1 < u_n^-(x,t) \le u_n^+(x,t) < 1, \quad x \in [-n,n]. \tag{A.3}$$

Call

$$\psi_n(x,t) = u_n^+(x,t) - u_n^-(x,t) \tag{A.4}$$

then, in $[-n,n]$

$$\frac{1}{2}\frac{d\psi_n^2}{dt} = \psi_n J * \psi_n - \psi_n\{A_\beta(u_n^+) - A_\beta(u_n^-)\} \le \frac{1}{2}\{\psi_n^2 + J * \psi_n^2\}.$$

Let $L_n f(x) = \mathbf{1}_{|x|\le n}[J * f - f]$, then

$$\frac{\psi_n^2(x,t)}{2} \le \int_{[-n,n]^c} e^{(3/2\,+L_n)t}(x,y)\psi_n^2(y,0) + 4\int_0^t \int_{[-n,n]^c} e^{(3/2\,+L_n)(t-s)}(x,y) \quad \text{(A.5)}$$

which shows that for any $x$ and $t$, $\psi_n(x,t) \to 0$ as $n \to \infty$. Since, by the comparison theorem, any $u \in C(\mathbb{R} \times \mathbb{R}_+; (-1,1))$ solution of (A.1) is such that $u_n^-(x,t) \le u(x,t) \le u_n^+(x,t)$, the theorem is proved. □

## B Nucleation and collapse of droplets

In this appendix we sketch the proof of Theorem 2.2, which, as mentioned in the text, uses heavily reversibility. We shorthand by $\bar m_\xi^{\mathcal{R}}$ the function equal to $\bar m_\xi(x)$ for $x \ge 0$ and to $\bar m_\xi(-x)$ for $x \le 0$ and call $T_t(m)$ the solution of (1.1) which at time 0 is equal to $m$. Let $\alpha$ be as in (1.4).

**Theorem B.1** *There is $V > 0$ and, for any $\zeta > 0$ there is $L_\zeta > 0$ so that for all $\ell \ge L_\zeta$,*

$$\|T_t(\bar m_\ell^{\mathcal{R}}) - m_\beta\|_2 \le \zeta, \quad \textit{for all } t \ge t^* := 10\,\frac{e^{2\alpha\ell}}{2\alpha V}. \quad \text{(B.1)}$$

We will discuss briefly the proof of Theorem B.1, after showing how it can be used to prove Theorem 2.2.

The force field $f(m)$ defined by the r.h.s. of (1.1) is continuous in $L^2 \cap \{\|m\|_\infty \le m_\beta\}$ (by the comparison theorem $\|T_t(\bar m_\ell^{\mathcal{R}})\|_\infty \le m_\beta$ for all $t \ge 0$), so that

$$\|f\big(T_{t^*}(\bar m_\ell^{\mathcal{R}})\big)\|_2 + \|f\big(T_{t^*}(\bar m_\ell^{\mathcal{R}})\big)\|_\infty \le o_1(\zeta) \quad \text{(B.2)}$$

where $o_1(\zeta)$ vanishes when $\zeta \to 0$. The linear interpolation

$$u(x,t) = t\,T_{t^*}(\bar m_\ell^{\mathcal{R}})(x) + (1-t)\,m_\beta, \quad 0 \le t \le 1 \quad \text{(B.3)}$$

has then a cost

$$\frac{1}{4}\int_0^1 \Big(u_t - f(u)\Big)^2 \le o_2(\zeta) \quad \text{(B.4)}$$

and defining $u(\cdot, t+1) = T_{t^*-t}(\bar m^{\mathcal{R}})$, for $t \in [0, t^*]$, we then get by reversibility and because $\mathcal{F}(T_{t^*}(\bar m_\ell^{\mathcal{R}})) \ge 0$,

$$\frac{1}{4}\int_0^{t^*+1} \Big(u_t - f(u)\Big)^2 \le o_2(\zeta) + \mathcal{F}\big(\bar m_\ell^{\mathcal{R}}\big) \quad \text{(B.5)}$$

and, by (1.4),

$$\mathcal{F}\big(\bar m_\ell^{\mathcal{R}}\big) \le 2\mathcal{F}\big(\bar m\big) + ce^{-\alpha\ell/2}. \tag{B.6}$$

After a $C^\infty$ regularization of the orbit $u$ (which can be such that the additional cost is bounded by $\zeta$), we then obtain an upper bound for the cost of connecting $m_\beta$ to $\bar m_\ell^{\mathcal{R}}$ in a time $t^*+1$, which is given by

$$2\mathcal{F}\big(\bar m\big) + ce^{-\alpha\ell/2} + o_2(\zeta) + \zeta. \tag{B.7}$$

To prove Theorem 2.2 we fix $\zeta$ so that $o_2(\zeta)+\zeta \le \vartheta/2$. With $\ell_\epsilon$ as in the statement of Theorem 2.2, for $\epsilon$ small enough, $ce^{-\alpha\ell_\epsilon/4} < \vartheta/2$ and $\ell_\epsilon/2 \ge L_\zeta$. By taking $\ell = \ell_\epsilon/2$ we thus complete the proof of Theorem 2.2, pending the validity of Theorem B.1.

In [2] Theorem B.1 is proved for the semigroup $S_t(m)$ which solves the analogue of (1.1)

$$u_t = -u + \tanh\{J * u\} =: g(u) \tag{B.8}$$

restricted to a finite interval with Neumann boundary conditions.

The two evolutions, (B.8) and (1.1), share many properties, in particular they have same stationary solutions, time monotonicity of $\mathcal{F}$ and the comparison theorem. Nonetheless the proof of Theorem B.1 does not follow from its analogue for (B.8) and requires a proof, which however is nothing but a lengthy yet uneventful extension of the one in [2]. For brevity we omit it here, also because it will be contained in a paper in preparation by Bellettini, De Masi and Presutti where the minimizing sequences of the tunnelling event are characterized. By reversibility this problem is related to an accurate description of the orbits where two instantons collapse converging to the plus state, an analysis which includes a proof of Theorem B.1.

## C A priori estimates

We write

$$A_{\alpha,t_1,t_2} = \Big\{x \in \mathbb{R} : \int_{t_1}^{t_2} b^2(x,t) \le \alpha\Big\}, \quad \alpha > 0, \ \ 0 \le t_1 < t_2 \tag{C.1}$$

and denote by $m(x,t) \in L^\infty(\mathbb{R}\times\mathbb{R}^+;(-1,1))$ a solution of (1.6).

**Theorem C.1** *For any $\alpha > 0$ and any $t > t_1 \ge 0$,*

$$\sup_{x\in A_{\alpha,t_1,t}} |m(x,t) - m^0(x,t)| \le ce^{\|J\|_\infty (t-t_1)}\Big(\sqrt{\alpha(t-t_1)} + (t-t_1)|A^c|\Big)^{1/2} \tag{C.2}$$

*where $c = 8\|J\|_\infty + 1$ and $m^0(x,s)$, $s \ge t_1$, is the solution of (1.1) such that $m^0(x,t_1) = m(x,t_1)$.*

*Proof.* The proof is a simple adaptation of a proof in [3] for finite volumes. Shorthand $A = A_{\alpha'',t_1,t}$ and call $\phi(x,s) = m(x,s) - m^0(x,s)$, $w(s) := \sup_{x\in A} |\phi(x,s)|$. Then,

$$\frac{1}{2}\frac{d}{ds}\phi(x,s)^2 \le |\phi(x,s)|\,\|J\|_\infty\Big(w(s) + 2|A^c|\Big) + 2|b(x,s)| \tag{C.3}$$

having used that $|\phi| \le 2$ and that $\phi(x,t)[A_\beta(m(x,t)) - A_\beta(m^0(x,t))] \ge 0$.

For any $x \in A$, we integrate (C.3) over time, getting

$$w(s)^2 \le 2\|J\|_\infty \int_{t_1}^{s} w(s')^2 + 8(s-t_1)\|J\|_\infty|A^c| + 4[(s-t_1)\alpha]^{1/2} \tag{C.4}$$

hence (C.2). □

**Theorem C.2** *There are $M \in (0,1)$, $\alpha' > 0$ and $s' > 0$ so that for any $t_1 \ge 0$ and any $t_2 > t_1 + s'$,*

$$|m(x,t)| \le M, \quad \text{for all } x \in A_{\alpha',t_1,t_2} \text{ and } t \in [t_1+s',t_2]. \tag{C.5}$$

*Proof.* We will first prove that $m(x,t) \le M$ and since the proof that $m(x,t) \ge -M$ is completely analogous, we will then have proved (C.5). Call $b^+(x,t) = \max\{b(x,t),0\}$ and $v(x,t)$, $t \ge t_1$, the solution of

$$v_t = 1 - A_\beta(v) + b^+, \quad v(x,t_1) = 1. \tag{C.6}$$

Then, $m(x,t) \le v(x,t)$. Let now $v^0(t)$, $t \ge t_1$ solve

$$v^0_t = 1 - A_\beta(v^0), \quad v^0(t_1) = 1 \tag{C.7}$$

and let $w(x,t) := v^0(t) + \int_{t_1}^{t} b^+(x,s)$. Then $v(x,t) \le w(x,t)$. Indeed, since $w \ge v^0$,

$$\frac{dw}{dt} = 1 - A_\beta(v^0) + b^+ \ge 1 - A_\beta(w) + b^+.$$

We have thus proved that $w$ is a super-solution of (C.6) and hence $m(x,t) \le w(x,t)$.

Since $\lim_{t\to\infty} v^0(t) < 1$, there are $s'$ and $M_0 < 1$ so that $v^0(t_1+s') = M_0$. We choose $\alpha'$ so that $\sqrt{\alpha' s'} + M_0 = M_1 < 1$, and Theorem C.2 is proved. □

**Theorem C.3** *There are $c_1$ and $c_2$ positive so that the following holds. For any $\epsilon > 0$, there are $\alpha'' > 0$ and $s'' > 0$ so that for any $t_1 \ge 0$, $t_2 > t_1 + s''$ and $\xi(t)$,*

$$|m(x,t) - \bar m_{\xi(t)}(x)| \le \epsilon + c_1\|m(\cdot,t) - \bar m_{\xi(t)}\|_2^{2/3} + c_2|A^c|^{1/2} \tag{C.8}$$

*for all $x \in A_{\alpha'',t_1,t_2}$ and $t \in [t_1+s'',t_2]$.*

*Proof.* The function

$$u(x,t) = [m(x,t) - \bar{m}_{\xi(t)}(x)]\mathbf{1}_{x\in A} + [m^0(x,t) - \bar{m}_{\xi(t)}(x)]\mathbf{1}_{x\in A^c}, \quad t \in [t_1 + s'', t_2] \tag{C.9}$$

verifies the condition

$$|u(x,t) - u(y,t)| \le \rho + c_3|x-y| \tag{C.10}$$

where, calling $C = 2ce^{\|J\|_\infty s''}[\left(\alpha'' s''\right)^{1/4} + |A^c|^{1/2}]$ an upper bound of the r.h.s. of (C.2),

$$\rho = 2C + 2e^{-s''/\beta}; \quad c_3 = \|\bar{m}'\|_\infty + \beta\|J'\|_\infty \tag{C.11}$$

since

$$|m^0(x,T) - m^0(y,T)| \le 2e^{-s''/\beta} + \beta\|J'\|_\infty\,|x-y| \tag{C.12}$$

as proved in [3]. In [3] it is also proved that:

**Lemma C.1** *Let $f \in L^2(\mathbb{R},[-1,1])$ be such that there are $\rho \ge 0$ and $c_3 > 0$ so that*

$$|f(x) - f(y)| \le \rho + c_3\,|x-y| \tag{C.13}$$

*then*

$$\|f\|_\infty \le \rho + \frac{3c_3^{1/3}}{\sqrt{8}}\|f\|_2^{2/3}. \tag{C.14}$$

Given $\epsilon > 0$ we choose $s''$ so that $2e^{-s''/\beta} \le \frac{\epsilon}{2}$ and $\alpha''$ so that

$$4ce^{\|J\|_\infty s''}\left(\alpha'' s''\right)^{1/4} \le \frac{\epsilon}{2}. \tag{C.15}$$

(C.14) yields (C.8) with $c_2 = 4ce^{\|J\|_\infty s''} + \frac{6c_3^{1/3}}{\sqrt{8}}$ and $c_1 = \frac{3c_3^{1/3}}{\sqrt{8}}$ and $c_3$ as in (C.11). □

**Theorem C.4** *Let $m$ solve (1.6) with forcing $b \in L^2$ and let $m_0$ solve (1.1); suppose $m(\cdot,t_0) = m_0(\cdot,t_0)$ and that $m_0(x,t_0)$ converges exponentially fast to $\pm m_\beta$ as $x \to \pm\infty$. Then for any $t_1 > t_0$*

$$\|m(\cdot,t_1) - m_0(\cdot,t_1)\|_2^2 \le e^{3|t_1-t_0|}\int_{t_1}^{t_0}\|b(t)\|_2^2. \tag{C.16}$$

*Proof.* Let $u := m - m_0$. We multiply the difference of (1.1) and (1.6) by $u$ and obtain

$$\begin{aligned}\frac{d}{dt}\|u(\cdot,t)\|_2^2 = 2\|u^2(\cdot,t)\|_2^2 - \int\int J(x-y)(u(x,t)-u(y,t))^2\mathrm{d}x\mathrm{d}y \\ + 2(b,u) - 2([A_\beta(m) - A_\beta(m_0)],[m-m_0]).\end{aligned}$$

Noting that $A_\beta$ is monotone and that $|2(b,u)| \le \|b\|_2^2 + \|u\|_2^2$, we get

$$\frac{d}{dt}t\|m(\cdot,t) - m_0(\cdot,t)\|_2^2 \le 3\|m(\cdot,t) - m_0(\cdot,t)\|_2^2 + \|b(t)\|_2^2.$$

which, by the Gronwall's inequality, yields (C.16). □

## D Spectral gap estimates

Given $\bar\xi = (\xi_1, \dots \xi_n)$, $\xi_i < \xi_{i+1}$, $i = 1, \dots, n-1$, call $\Lambda_j = \frac{1}{2}[\xi_j + \xi_{j-1}, \xi_{j+1} + \xi_j]$, with $\xi_0 = -\infty$ and $\xi_{n+1} = \infty$ and denote by $1_j$ the characteristic function of $\Lambda_j$. We then define

$$V_{\bar\xi}^{\perp} := \Big\{ u \in L^2(\mathbb{R}) : \, (u1_j, \bar m'_{\xi_j}) = 0, \;\; j = 1, \dots, n \Big\}. \tag{D.1}$$

Let $L \equiv L_{\bar\xi}$ be the linear operator on $L^2(\mathbb{R})$ defined as

$$L\psi(x) = J * \psi(x) - A'_\beta\big(m_{\bar\xi}(x)\big)\psi(x). \tag{D.2}$$

Due to the symmetry of $J(x,y)$, $L$ is self-adjoint. We set

$$-\omega_{\bar\xi} := \sup_{u \in V_{\bar\xi}^{\perp} : \|u\|_2 = 1} (u, Lu). \tag{D.3}$$

When $n = 1$, $\bar\xi = \xi$, $\omega_\xi = \omega_1 > 0$ is independent of $\xi$ and equal to the spectral gap of $L$, hence the title of this appendix (but notice that $\omega_{\bar\xi}$ is not the spectral gap when $n > 1$, the spectral gap vanishing as the mutual distance of the element of $\xi$ diverges). Call finally

$$D_{\bar\xi} := \min_{j=1,\dots,n-1} \big\{ \xi_{j+1} - \xi_j \big\}. \tag{D.4}$$

**Theorem D.1** *There are $\omega$ and $c$ positive so that*

$$\omega_{\bar\xi} \ge \omega - \frac{cn}{\sqrt{D_{\bar\xi}}}. \tag{D.5}$$

*Proof.* Let $k_j$, $j = 1, \dots, n-1$ be integers such that

$$|k_j - \frac{\xi_j + \xi_{j+1}}{2}| \le \sqrt{D_{\bar\xi}}, \quad \|u\mathbf{1}_{[k_j-1,k_j+1]}\|_2 \le \frac{10}{\sqrt{D_{\bar\xi}}} \tag{D.6}$$

whose existence follows from the condition $\|u\|_2 = 1$. Calling $k_0 = -\infty$, $k_n = +\infty$ and $u_j = \mathbf{1}_{[k_{j-1},k_j]}u$, we have

$$(u, Lu) = \sum_{j=1}^{n-1} (u_j, Lu_j) + \sum_{j=1}^{n-1} (u_j, J * u_{j+1}). \tag{D.7}$$

Since the $L^2$ norm of the operator $J$ is $\le 1$, by (D.6) the last term is bounded by $100nD_{\bar\xi}^{-1}$.

For $j$ odd,

$$(u_j, Lu_j) \le -\omega_1 \| u_j - \frac{(\bar m'_{\xi_j}, u_j)}{(\bar m'_{\xi_j}, \bar m'_{\xi_j})} \bar m'_{\xi_j} \|_2^2 \le -\omega_1 \|u_j\|_2^2 + \omega_1 \| \frac{(\bar m'_{\xi_j}, [u - u_j])}{(\bar m'_{\xi_j}, \bar m'_{\xi_j})} \bar m'_{\xi_j} \|_2^2$$
$$\le -\omega_1 \|u_j\|_2^2 + c e^{-\alpha \sqrt{D_{\bar\xi}}} \tag{D.8}$$

because $(\bar m'_{\xi_j}, u) = 0$ and $\bar m'(x) \le c' e^{-\alpha|x|}$. An analogous argument holds for $j$ even and the theorem is proved. □

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