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A Nonlocal Diffusion Equation whose Solutions Develop a Free Boundary

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Abstract. Let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative, smooth compactly supported function such that $\int_{\mathbb{R}} J(r)dr = 1$. We consider the nonlocal diffusion problem

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy - u(x,t) \quad \text{in } \mathbb{R} \times [0,\infty)$$

with a nonnegative initial condition. Under suitable hypotheses we prove existence, uniqueness, as well as the validity of a comparison principle for solutions of this problem. Moreover we show that if $u(\cdot,0)$ is bounded and compactly supported, then $u(\cdot,t)$ is compactly supported for all positive times t. This implies the existence of a free boundary, analog to the corresponding one for the porous media equation, for this model.

1 Introduction

Let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative, smooth function with $\int_{\mathbb{R}} J(r)dr = 1$. Assume also that J is supported in $[-1,1]$, is strictly increasing in $[-1,0]$ and strictly decreasing in $[0,1]$.

Equations of the form

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}} J(x-y)u(y,t)dy - u(x,t), \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes, see [2], [4], [5], [6], [8]. As stated in [5] if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J * u)(x,t)$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}} J(y-x)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external sources, leads immediately to the fact that the density u satisfies equation (1.1).

Equation (1.1), so-called nonlocal diffusion equation, shares many properties with the classical heat equation

$$u_t = \Delta u$$

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such as: bounded stationary solutions are constant, a maximum principle holds for both of them and, even if J is compactly supported, perturbations propagate with infinite speed. By this we understand that if u is a nonnegative nontrivial solution, then $u(x,t) > 0$ for all $x \in \mathbb{R}$ and all $t > 0$ no matter whether the nontrivial initial condition $u(x,0)$ vanishes in some region.

Another classical equation that has been used to model diffusion is the well-known porous medium equation,

$$u_t = \Delta u^m$$

with $m > 1$. This equation also shares several properties with the heat equation but there is a fundamental difference, in this case if the initial data $u(\cdot,0)$ is compactly supported, then $u(\cdot,t)$ has compact support for all $t > 0$. In such a case, if the support of the initial condition is a finite interval, one can define the right and left free boundaries of the solution by

$$s_+(t) = \sup\{x \ / \ u(x,t) > 0\}$$

and

$$s_-(t) = \inf\{x \ / \ u(x,t) > 0\}$$

respectively. Properties and the behavior of the free boundary for the porous medium equation have been largely studied over the past years. See for example [1], [7] and the corresponding bibliography. It is worth mentioning that this phenomena also arises in the context of the Stefan problem, see [3] and the references therein.

The purpose of this note is to present a simple nonlocal model for diffusion whose solutions, with compactly supported bounded initial data, develop a free boundary. To do this we propose a model where the diffusion at a point depends on the density. The simplest situation we can think of is when the probability distribution of jumping from location y to location x is given by

$$J\left(\frac{x-y}{u(y,t)}\right)\frac{1}{u(y,t)}$$

when $u(y,t) > 0$ and 0 otherwise. In this case the rate at which individuals are arriving to position x from all other places is

$$\int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy$$

and the rate at which they are leaving location x to travel to all other sites is

$$-u(x,t) = -\int_{\mathbb{R}} J\left(\frac{y-x}{u(x,t)}\right) dy.$$

As before this consideration, in the absence of external sources, leads immediately to the fact that the density u has to satisfy

$$u_t(x,t) = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy - u(x,t).$$

As for the initial data, although we are mostly interested in functions $u(\cdot,0) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ it is more convenient, for technical reasons that will become clear later, to consider a slightly more general set of initial conditions. So in this paper we will deal with the problem

$$\begin{aligned} u_t(x,t) &= \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy - u(x,t) \ \text{ in } \ \mathbb{R}\times[0,\infty). \\ u(x,0) &= c + w_0(x) \ \text{ on } \ \mathbb{R}, \end{aligned} \tag{1.2}$$

where $c \geq 0$, $w_0 \in L^1(\mathbb{R})$ and $w_0 \geq 0$.

Most of the results contained in this note can be obtained in several dimensions without many changes in the elementary arguments but, we have chosen to treat the one-dimensional case for the sake of simplicity of the exposition.

We will address in this paper the questions of existence, uniqueness, comparison principles and some basic facts about the free boundary for solutions of problem (1.2). Several further questions, such as the decay rate of solutions, the speed at which the free boundary moves, the existence of the so-called waiting times for the free boundary and many others, are left open. Also one can consider equations involving a source term and to study, for example, the blow-up phenomena. We hope such questions can be answered by us or by someone else in the near future.

2 Existence and uniqueness

The existence and uniqueness result will be a consequence of Banach's fixed point theorem and it is convenient to give some preliminaries before giving its proof.

Fix $t_0 > 0$ and consider the Banach space $C([0,t_0];L^1)$ with the norm

$$|||w||| = \max_{0\leq t\leq t_0} \|w(\cdot,t)\|_{L^1}.$$

Let

$$X_{t_0} = \left\{ w \in C([0,t_0];L^1) \ / \ w \geq 0 \right\}$$

which is a closed subset of $C([0,t_0];L^1)$.

We will obtain the solution in the form $u(x,t) = w(x,t) + c$ where w is a fixed point of the operator $T_{w_0} : X_{t_0} \to X_{t_0}$ defined by

$$\begin{aligned} T_{w_0}(w)(x,t) = \ & \int_0^t e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{w(y,s)+c}\right) dy\, ds \\ & + e^{-t} w_0(x) - c(1-e^{-t}). \end{aligned}$$

The following lemma is the main ingredient of our proof.

Lemma 2.1 *Let* z_0, w_0 *be nonnegative functions such that* $w_0, z_0 \in L^1(\mathbb{R})$ *and* $w, z \in X_{t_0}$, *then*

$$|||T_{w_0}(w) - T_{z_0}(z)||| \leq (1 - e^{-t_0})|||w - z||| + ||w_0 - z_0||_{L^1(\mathbb{R})}.$$

Proof. We have

$$\begin{aligned}
&\int_{\mathbb{R}} |T_{w_0}(w)(x,t) - T_{z_0}(z)(x,t)|\, dx \\
&\leq \int_0^t e^{-(t-s)} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left(J\left(\frac{x-y}{w(y,s)+c}\right) - J\left(\frac{x-y}{z(y,s)+c}\right)\right) dy \right| dx\, ds \\
&\quad + e^{-t} \int_{\mathbb{R}} |w_0 - z_0|(y)\, dy.
\end{aligned}$$

Now set

$$A^+(s) = \{ y \,/\, w(y,s) \geq z(y,s) \}$$

and

$$A^-(s) = \{ y \,/\, w(y,s) < z(y,s) \}.$$

We have now

$$\begin{aligned}
&\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left(J\left(\frac{x-y}{w(y,s)+c}\right) - J\left(\frac{x-y}{z(y,s)+c}\right)\right) dy \right| dx \\
&\leq \int_{\mathbb{R}} \int_{A^+(s)} \left(J\left(\frac{x-y}{w(y,s)+c}\right) - J\left(\frac{x-y}{z(y,s)+c}\right)\right) dy\, dx \\
&+ \int_{\mathbb{R}} \int_{A^-(s)} \left(J\left(\frac{x-y}{z(y,s)+c}\right) - J\left(\frac{x-y}{w(y,s)+c}\right)\right) dy\, dx.
\end{aligned}$$

Since the integrands are nonnegative we can apply Fubini's theorem to get

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{A^+(s)} \left(J\left(\frac{x-y}{w(y,s)+c}\right) - J\left(\frac{x-y}{z(y,s)+c}\right)\right) dy\, dx \\
&= \int_{A^+(s)} (w(y,s) - z(y,s)) dy
\end{aligned}$$

and similarly for the integral over $A^-(s)$. Therefore we obtain

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left(J\left(\frac{x-y}{w(y,s)+c}\right) - J\left(\frac{x-y}{z(y,s)+c}\right)\right) dy \right| dx \leq \int_{\mathbb{R}} |w(y,s) - z(y,s)|\, dy.$$

Hence we get

$$|||T_{w_0}(w) - T_{z_0}(z)||| \le (1 - e^{-t_0})|||w - z||| + ||w_0 - z_0||_{L^1(\mathbb{R})}$$

as desired. □

We can state now the main result of this section.

Theorem 2.1 *For every nonnegative $w_0 \in L^1$ and every constant $c \ge 0$, there exists a unique solution u, such that $(u - c) \in C([0,\infty); L^1)$, of 1.2. Moreover, the solution verifies $u(x,t) \ge c$ and preserves the total mass above c, that is*

$$\int_{\mathbb{R}} (u(y,t) - c)\, dy = \int_{\mathbb{R}} w_0(y)\, dy \textit{ for all } t \ge 0. \tag{2.1}$$

Proof. We check first that T_{w_0} maps X_{t_0} into X_{t_0}. Since $w \ge 0$ we have

$$J\left(\frac{x-y}{w(y,s)+c}\right) \ge J\left(\frac{x-y}{c}\right)$$

and hence

$$\begin{aligned} T_{w_0}(w)(x,t) \ge \quad & \int_0^t e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{c}\right) dy\, ds \\ & + e^{-t} w_0(x) - c(1 - e^{-t}) = e^{-t} w_0(x) \ge 0. \end{aligned} \tag{2.2}$$

Taking $z_0 \equiv 0$, $z \equiv 0$ in Lemma 2.1 we get that $T_{w_0}(w) \in C([0,t_0]; L^1)$.

Now taking $z_0 \equiv w_0$ in Lemma 2.1 we get that T_{w_0} is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem.

We finally prove that if $u = w + c$ is the solution, then the integral in x of w is preserved. Since

$$0 = \int_0^t e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{c}\right) dy\, ds - c(1 - e^{-t}),$$

we can write

$$w(x,t) = \int_0^t e^{-(t-s)} \int_{\mathbb{R}} (J\left(\frac{x-y}{w(y,s)+c}\right) - J\left(\frac{x-y}{c}\right))\, dy\, ds + e^{-t} w_0(x).$$

The integrand in the above formula is nonnegative so we can integrate in x and apply Fubini's theorem to obtain

$$\int_{\mathbb{R}} w(x,t) dx = \int_0^t e^{-(t-s)} \int_{\mathbb{R}} w(y,s)\, dy\, ds + e^{-t} \int_{\mathbb{R}} w_0(x) dx \tag{2.3}$$

from where it follows that
$$\frac{d}{dt}\int_{\mathbb{R}} w(x,t)dx = 0$$
and the theorem is proved. □

We will need in what follows the following lemma which is a direct corollary of the proof of Theorem 2.1 and is a first version of the comparison principle of Section 3 below.

Lemma 2.2 *With the above notation if* $0 \le w(x,0) \le M$ *for all* $x \in \mathbb{R}$*, then* $w(x,t) \le M$ *for all* $(x,t) \in \mathbb{R} \times [0,\infty)$*.*

Proof. Under the given hypotheses one has that if $w(x,t) \le M$, then

$$T_{w_0}(w)(x,t) =$$

$$\int_0^t e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{w(y,s)+c}\right) dy\, ds + e^{-t}w_0(x) - c(1-e^{-t})$$

$$\le \int_0^t e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{M+c}\right) dy\, ds + e^{-t}M - c(1-e^{-t}) = M.$$

The lemma follows by the uniqueness of the fixed point for T_{w_0}. □

Lemma 2.1, Theorem 2.1, Lemma 2.2 and their proofs have several immediate consequences that we state as a series of remarks for the sake of future references.

Remark 2.1 *Solutions of* 1.2 *depend continuously on the initial condition in the following sense. If* u *and* v *are solutions of* 1.2*, then*

$$\max_{0\le t\le t_0} \|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R})} \le e^{t_0}||u(\cdot,0) - v(\cdot,0)||_{L^1(\mathbb{R})}$$

for all $t_0 \ge 0$*.*

Remark 2.2 *The function* u *is a solution of* 1.2 *if and only if*

$$u(x,t) = \int_0^t e^{-(t-s)} \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,s)}\right) dy\, ds + e^{-t}u(x,0).$$

Remark 2.3 *From the previous remark and Lemma* 2.2 *we get that if* $c > 0$ *and* $u(\cdot,0) \in C^k(\mathbb{R})$ *with* $0 \le k \le \infty$*, then* $u(\cdot,t) \in C^k(\mathbb{R})$ *for all* $t \ge 0$*. Moreover if* $u(\cdot,0)$ *is a compactly supported* C^1 *function, then there exists a constant* K *depending on* c*,* J *and* w_0 *such that*

$$|u_t(x,t)|\,,\ \left|\frac{\partial u}{\partial x}(x,t)\right| \le K.$$

Remark 2.4 *A consequence of Remark* 2.3 *and of* (2.1) *is that if* $c > 0$ *and* w_0 *is a compactly supported* C^1 *function, then*

$$\lim_{|x|\to\infty} u(x,t) = c \text{ uniformly on compact intervals } [0,T].$$

Remark 2.5 *It follows from inequality* (2.2) *that*

$$w(x,t) \geq e^{-t} w(x,0).$$

In particular, in the case that $u(\cdot,0) \in L^1(\mathbb{R})$*, the support of* $u(\cdot,t)$ *does not shrink as time increases. By this we understand that if* $u(x_0,t_0) > 0$*, then* $u(x_0,t) > 0$ *for all* $t \geq t_0$*.*

3 Comparison Principle

Comparison principles like the one below have proven to be a very useful tool in studying diffusion problems.

Theorem 3.1 *Let* u *and* v *be continuous solutions of* 1.2*. If*

$$u(x,0) \leq v(x,0) \text{ for all } x \in \mathbb{R},$$

then

$$u(x,t) \leq v(x,t) \text{ for all } (x,t) \in \mathbb{R}\times[0,\infty). \tag{3.1}$$

Proof. We assume first that

$$u(x,0) = c + w(x,0) \quad \text{and} \quad v(x,0) = d + z(x,0)$$

with $0 < c < d$ and $u(x,0) < v(x,0)$. Moreover we assume for a moment that $w(x,0)$ and $z(x,0)$ are compactly supported C^1 functions. In this case there exists $\delta > 0$ such that $u(x,0) + \delta < v(x,0)$. Assume, for a contradiction that the conclusion does not hold. In view of Remark 2.4 we have that there exists a time $t_0 > 0$ and a point $x_0 \in \mathbb{R}$ such that $u(x_0,t_0) = v(x_0,t_0)$ and $u(x,t) \leq v(x,t)$ for all $(x,t) \in \mathbb{R}\times[0,t_0]$.

Let us consider the set $B = \{x \in \mathbb{R} \,/\, u(x,t_0) = v(x,t_0)\}$. Clearly B is nonempty and closed.

Let $x_1 \in B$. We have then

$$0 \leq (u-v)_t(x_1,t_0) = \int_{\mathbb{R}} \left(J\left(\frac{x_1-y}{u(y,t_0)}\right) - J\left(\frac{x_1-y}{v(y,t_0)}\right)\right) dy \leq 0$$

which implies

$$u(y,t_0) = v(y,t_0) \text{ for all } y \in (x_1 - c, x_1 + c).$$

Hence B is open. It follows that $B = \mathbb{R}$ which is the desired contradiction since $(u(\cdot, t_0) - c) \in L^1(\mathbb{R})$.

We now get rid of the extra hypothesis that $w(x, 0)$ and $z(x, 0)$ are compactly supported C^1 functions. In order to do this let $w_n(x, 0)$ and $z_n(x, 0)$ be sequences of compactly supported C^1 functions such that $w_n(x, 0) \to w(x, 0)$ and $z_n(x, 0) \to z(x, 0)$ in $L^1(\mathbb{R})$ as $n \to \infty$ and, moreover, $u_n(x, 0) = c + w_n(x, 0) < v_n(x, 0) = d + z_n(x, 0)$. Let u_n and v_n be the solutions with initial data $u_n(x, 0)$ and $v_n(x, 0)$ respectively. By the previous argument one has $u_n \leq v_n$ an the result follows by letting $n \to \infty$ in view of Remark 2.1.

In order to prove the theorem in the general case pick strictly decreasing sequences a_n and b_n such that $0 < a_n < b_n$ and $b_n \to 0$ as $n \to \infty$. Let u_n and v_n be the solutions with initial conditions $u_n(x, 0) = u(x, 0) + a_n$ and $v_n(x, 0) = v(x, 0) + b_n$ respectively. According to the previous argument one has $u_n \leq v_n$. Moreover $u_{n+1} \leq u_n$ and $v_{n+1} \leq v_n$. By Remark 2.2, after an application of the monotone convergence theorem, it follows that $u_n(x, t) \to u(x, t)$ and $v_n(x, t) \to v(x, t)$ as $n \to \infty$ and the theorem is proved. □

An immediate consequence of the comparison principle and Remark 2.4 is the following corollary that extends Remark 2.4 to the case $c = 0$.

Corollary 3.1 *If $c = 0$ and w_0 is a compactly supported C^1 function, then*

$$\lim_{|x| \to \infty} u(x, t) = 0 \text{ uniformly on compact intervals } [0, T].$$

4 The free boundary

In this section we will prove that solutions of (1.2), with compactly supported continuous initial data, do have a free boundary in the sense that

$$s_+(t) = \sup\{x \;/\; u(x, t) > 0\} < +\infty$$

and

$$s_-(t) = \inf\{x \;/\; u(x, t) > 0\} > -\infty$$

for all $t \geq 0$. It follows from Remark 2.5 that s_+ and s_- are nondecreasing and nonincreasing functions respectively. Moreover we will also prove in this section that the supports of $u(\cdot, t)$ eventually fill at least half a ray of the space, in particular either $\lim_{t \to \infty} s_+(t) = \infty$ or $\lim_{t \to \infty} s_-(t) = -\infty$. In the case that J is even, that is the case of an isotropic media, the supports eventually cover the whole of $\mathbb{R}$.

The following theorem implies the existence of free boundaries.

Theorem 4.1 *If $u(\cdot, 0)$ is compactly supported and bounded then $u(\cdot, t)$ is also compactly supported for all $t \geq 0$.*

Proof. Due to the scaling invariance of the equation, namely if $u(x,t)$ is a solution then for any $\lambda > 0$ the function $v_\lambda(x,t) = \lambda u(\frac{x}{\lambda}, t)$ is also a solution, we can restrict ourselves to initial data supported in $[-1,1]$ and such that $\sup_{x\in\mathbb{R}} u(x,0) \le 1$.

We note first that

$$u_t(x,t) \le \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy. \tag{4.1}$$

Therefore, since $0 \le u \le 1$, we get by (4.1) that

$$u(x,t) \le \frac{1}{2} \text{ for all } t \le \frac{1}{2} \text{ and all } x \text{ such that } |x| \ge 1.$$

Now if $|x| \ge 2$ and $t \le \frac{1}{2}$ we have that $|x-y| \le u(y,t)$ implies that $|y| \ge 1$ and hence $u(y,t) \le \frac{1}{2}$. Therefore, again by (4.1), we have

$$u(x,t) \le \frac{1}{4} \text{ for all } t \le \frac{1}{2} \text{ and all } x \text{ such that } |x| \ge 2.$$

We look now at the case $|x| \ge 2 + \frac{1}{2}$ and $t \le \frac{1}{2}$. In this case $|x-y| \le u(y,t)$ implies that $|y| \ge 2$ and hence $u(y,t) \le \frac{1}{4}$. Again by (4.1), we have

$$u(x,t) \le \frac{1}{8} \text{ for all } t \le \frac{1}{2} \text{ and all } x \text{ such that } |x| \ge 2 + \frac{1}{2}.$$

Repeating this procedure we obtain by induction that for any integer $n \ge 1$ one has

$$u(x,t) \le \frac{1}{2^{n+2}} \text{ for all } t \le \frac{1}{2} \text{ and all } x \text{ such that } |x| \ge 2 + \sum_{k=1}^{n} \frac{1}{2^k}.$$

It follows that the support of $u(\cdot,t)$ is contained in the interval $[-3,3]$ for all $t \le \frac{1}{2}$ as we wanted to prove. □

In order to prove our next result we need a preliminary lemma.

Lemma 4.1 *If $u(x,0)$ is continuous and not constant, then the function*

$$M(t) = \max_{x\in\mathbb{R}} u(x,t)$$

is strictly decreasing.

Proof. It is clear, by comparison with a constant, that $M(t)$ decreases as t increases. Moreover by Remark 2.5 one has $M(t) > c$ for all $t \ge 0$. Fix $t_0 \ge 0$ and let $t_1 > t_0$. Let us consider the set

$$C = \{x \,/\, u(x,t_1) = M(t_0)\}.$$

The set C is clearly closed. Since $u(x,t) \le M(t_0)$ for all $t \ge t_0$ we have that at any point $x_0 \in C$ one must have

$$0 \le u_t(x_0, t_1) = \int_{\mathbb{R}} J\left(\frac{x_0 - y}{u(y,t_1)}\right) dy - u(x_0, t_1) \le 0.$$

This implies that $u(x, t_1) = M(t_0)$ for all x in a neighborhood of x_0 and hence C is open. Consequently either $C = \mathbb{R}$ or C is empty. It is clear that $C \neq \mathbb{R}$, so $C = \emptyset$ and the lemma is proved. □

We are now in a position to prove that at least one of the free boundaries go to infinity.

Theorem 4.2 *Let u be the solution of problem* 1.2 *with $c = 0$ and $w_0 \neq 0$. Then either*

$$\lim_{t\to\infty} s_+(t) = \infty \qquad \text{or} \qquad \lim_{t\to\infty} s_-(t) = -\infty$$

and the supports of $u(\cdot, t)$ eventually cover an infinite half-ray of $\mathbb{R}$. If J is an even function the supports eventually cover the whole of $\mathbb{R}$.

Proof. By comparison, and the invariance under translations of the equation, it is enough to prove the theorem under the assumptions that $w_0 \in C^1$, its support is the interval $[-A, A]$ and it is symmetric with respect to the origin.

We claim first that the support of $u(\cdot, t)$ is not uniformly bounded. Assume for a contradiction that there exists $L > 0$ such that $u(x,t) = 0$ for all x such that $|x| \ge L$ and all $t \ge 0$. Since $\int_{\mathbb{R}} u(x,t)dx = \int_{\mathbb{R}} u(x,0)dx > 0$ there exists $C > 0$ such that

$$\lim_{t\to\infty} M(t) = C.$$

Let $v(x,0)$ be a smooth function supported in $[-L-1, L+1]$ such that $0 \le v(x,0) \le C$ and $v(x,0) \equiv C$ if $x \in [-L, L]$. Let us denote by $v(x,t)$ the solution of (1.2) with this initial condition. By Lemma 4.1 we have that

$$\max_{x\in\mathbb{R}} v(x,1) < C.$$

Now for any integer $n > 0$ let $v_n(x,0)$ be a smooth compactly function supported in $[-L-2, L+2]$ such that $0 \le v(x,0) \le C + \frac{1}{n}$ and $v_n(x,0) \equiv C + \frac{1}{n}$ if $x \in [-L, L]$. Assume further that

$$v_{n+1}(x,0) \le v_n(x,0)$$

and denote by $v_n(x,t)$ the solution of (1.2) with initial condition $v_n(x,0)$. By comparison it follows that

$$v_{n+1}(x,t) \le v_n(x,t).$$

Using Remark 2.2 and the monotone convergence theorem one has

$$v_n(x,1) \to v(x,1) \text{ in } [-L-2, L+2] \text{ as } t \to \infty.$$

Moreover, being the limit continuous the convergence is uniform by Dini's theorem. Consequently there exists n_0 such that

$$\max_{x \in \mathbb{R}} v_{n_0}(x,1) < C.$$

On the other hand there exists t_0 such that

$$u(x,t_0) \le v_{n_0}(x,0).$$

This implies, by comparison, that

$$\max_{x \in \mathbb{R}} u(x,t_0+1) < C$$

a contradiction that proves the claim.

We are ready now to prove the statement of the theorem.

We claim that if there exists $x_0 \ge A$ such that $u(x_0,t) = 0$ for all $t \ge 0$, then

$$u(x,t) = 0 \text{ for all } (x,t) \in [x_0,\infty) \times [0,\infty).$$

Indeed, let $d > 0$ and we will prove that

$$u(x,t) \le d \text{ for all } x \ge x_0 \text{ and all } t \ge 0. \tag{4.2}$$

Since $u(x_0,t) \equiv 0$ one has

$$u(x,t) \le |x - x_0| \text{ for all } x \in \mathbb{R} \text{ and all } t \ge 0.$$

Moreover $u(x,0) = 0$ for all $x \ge x_0$. So if (4.2) does not hold, using Corollary 3.1, there exists a point $x_1 \in \mathbb{R}$ with $x_1 \ge x_0 + d$ and a time $t_1 > 0$ such that $u(x_1,t_1) = d$ and $u(x,t) \le d$ for all $(x,t) \in \mathbb{R} \times [0,t_1]$. As in the proof of Theorem 3.1 we consider the set

$$B = \{x \ge x_0 + d \;/\; u(x,t_1) = d\}$$

which is clearly closed. Also at a point $x_2 \in B$ one has

$$0 \le (d-u)_t(x_2,t_1) = \int_{\mathbb{R}} \left(J\left(\frac{x_2 - y}{d}\right) - J\left(\frac{x_2 - y}{u(y,t_0)}\right)\right) dy \le 0$$

which implies

$$u(y,t_0) = d \text{ for all } y \in (x_2 - d, x_2 + d).$$

It follows that B is open and hence $B = [x_0, \infty)$ which is a contradiction that proves (4.2). Since $d > 0$ was chosen arbitrarily the claim follows. An analog of the above claim holds for points $-x_1 < -A$ such that $u(-x_1, t) = 0$ for all $t \geq 0$. Such a points x_0 and x_1 can not exist simultaneously because this contradicts the fact that the supports of $u(\cdot, t)$ are not uniformly bounded. This, plus the fact that if J and $u(\cdot, 0)$ are even functions then $u(\cdot, t)$ is even for all $t \geq 0$, proves the theorem. □

Finally we give an example of a nonsymmetric function J such that the supports of solutions $u(\cdot, t)$, with compactly supported bounded initial data, do not eventually cover the whole of $\mathbb{R}$.

We will show that for a special choice of J the function

$$u(x) = x_+ = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ x & \text{if} \quad x \geq 0 \end{cases}$$

satisfies

$$0 = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y)}\right) dy - u(x). \tag{4.3}$$

It is immediate that if $x \leq 0$, then

$$\int_{\mathbb{R}} J\left(\frac{x-y}{u(y)}\right) dy = 0$$

and hence (4.3) is satisfied.

As for the case $x > 0$ we have that $\frac{|x-y|}{y_+} \leq 1$ implies $0 < x \leq 2y$ and hence

$$\begin{aligned} &\int_{\mathbb{R}} J\left(\frac{x-y}{u(y)}\right) dy - u(x) \\ &= \int_{\frac{x}{2}}^{\infty} J\left(\frac{x}{y} - 1\right) dy - x \\ &= x\left(\int_{-1}^{1} J(r) \frac{dr}{(1+r)^2} - 1\right). \end{aligned}$$

Now we choose J such that, in addition to the hypotheses already made, satisfies

$$\int_{-1}^{1} J(r) \frac{dr}{(1+r)^2} = 1$$

and (4.3) also holds.

The desired example follows now by a comparison argument, like the one of the proof of Theorem 3.1, using the function x_+, or a translation of it, as a barrier.

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