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Nondeterministic Dynamics and Turbulent Transport*

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Abstract. Velocity fields occurring in fully developed turbulence are spatially rough and give rise to the phenomenon of nonuniqueness of trajectories in the motion of fluid particles even in the absence of noise. We review progress in understanding of this phenomenon in the context of synthetic gaussian velocity ensembles.

1 Richardson law

The paradigmatic case of deterministic dynamics is given by the flow induced by a smooth (possibly time dependent) vector field

$$\frac{d\mathbf{R}}{dt} \;=\; \mathbf{v}(t,\mathbf{R})\,. \tag{1}$$

Such flows often exhibit the phenomenon of *deterministic chaos* signaled by the exponential separation of nearby trajectories:

$$\rho(t) \sim e^{\lambda t}\rho(0)\,. \tag{2}$$

where $\rho = \mathbf{R}' - \mathbf{R}$ for two solutions $\mathbf{R}$ and $\mathbf{R}'$ of (1) and the equation holds for sufficiently small separations. The exponential separation leads to the lack of predictability in the long time behavior of the flow. However, such dynamics is still deterministic: in the limit of zero initial separation $\rho(0) \to 0$ the trajectories coincide: $\rho(t) \to 0$.

A very different phenomenon seems to occur in the dynamics of small test particles suspended in a turbulent fluid. Classical experiments on this phenomenon were performed by L.F. Richardson [1] in the 1920's. He was studying the motion of balloons released in the atmosphere. Such motion is described by the equation (1) where $\mathbf{v}(t,\mathbf{R})$ is the velocity field of the air.

The long time behavior of the trajectory of a single balloon once the mean drift is subtracted tends to be diffusive

$$\mathbf{R}(t)^2 \sim Dt\,. \tag{3}$$

Diffusive asymptotics is observed in a variety of dynamical systems and its occurrence here is not a surprise.

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The surprising observation that Richardson made concerned the relative motion of two balloons. Richardson found this separation to behave as

$$\rho(t)^2 \sim At^3 \tag{4}$$

for a wide range of separations from the smallest scale to the scale where the diffusive behavior (3) sets on. Thus the separation grows slower than exponentially but faster than ballistically. Moreover, the constant A in (4) seems to be *independent* on the initial separation $\rho(0)$, seemingly having a nonzero limit as $\rho(0) \to 0$! The Richardson law (4) seems quite well confirmed by later experiments as well as numerical simulations (for references see [2]) although the power 3 might only be approximately correct in three dimensions. We conclude that the motion of test particles in a turbulent fluid seems to exhibit properties that are very different from those observed in smooth dynamical systems. To understand the source of this discrepancy we need to digress on the properties of turbulent velocity fields.

2 Turbulent velocities

It is an experimental observation that the velocity field of a fluid in the regime of homogeneous isotropic turbulence exhibits approximate scale invariance on a wide range of length scales. Such fields are characterized by two length scales, the *dissipative scale* η and the *injection scale* L. For atmospheric flows such as considered by Richardson η can be of the order of a fraction of a millimeter whereas L can be of the order of a kilometer. For spatial scales between η and L the velocity field is approximately self similar: if we consider the difference $\Delta\mathbf{v}(t,\rho) = \mathbf{v}(t,\rho) - \mathbf{v}(t,0)$ then for $\eta << |\rho|, |\ell\rho| << L$

$$\Delta\mathbf{v}(t,\rho) \sim \ell^{-\alpha}\Delta\mathbf{v}(t,\ell\rho)\,. \tag{5}$$

where $\sim$ means statistically, i.e. as a spatial, temporal or ensemble average and $\alpha \sim \frac{1}{3}$. (5) holds only approximately for small moments

$$< (\Delta\mathbf{v}(t,\rho)\cdot\hat{\rho})^n > \sim C_n|\rho|^{n\alpha} \tag{6}$$

with significant corrections in the exponent for large n (so called intermittency). The ratio $\frac{L}{\eta}$ is proportional to $R^{\frac{3}{4}}$ where R is the Reynolds number of the flow (in atmospheric flows R can easily be of the order 10^8). Hence $\eta \to 0$ as $R \to \infty$ and in that limit the turbulent velocities loose their smoothness and become only Hölder continuous in their spatial dependence, that is, the difference of the velocity at two arbitrary nearby points scales as a sublinear power of the distance between the points. Although $R \to \infty$ is a mathematical limit it should be stressed that smoothness of the velocity field is not the right assumption for scales larger than a fraction of a millimeter in atmospheric flows. For such scales the right model is one of Hölder continuous velocities.

It is well known that the differential equation (1) need not have a unique solution if $\mathbf{v}$ is only Hölder continuous. Indeed, the standard textbook example $v = x^\alpha$, $\alpha \in (0,1)$, in one dimension has two solutions $x = 0$ and $x = ((1-\alpha)t)^{\frac{1}{1-\alpha}}$ starting from the origin. It is therefore natural to expect non-uniqueness of trajectories in the turbulent fluid and seek an understanding of Richardson's observations from this phenomenon.

There is also a natural timescale related to the spatial scale r fluctuations ("eddies") in the velocity field. This is the so-called eddy turnover time $\tau(r) = \frac{r}{|\Delta\mathbf{v}(r)|} \sim r^{1-2\alpha} \equiv r^{2\beta}$. For turbulent velocities $\beta \sim \frac{1}{3}$. Thus it would be natural to expect decorrelation of velocity differences in such time scales:

$$< \Delta\mathbf{v}(t,\rho)\Delta\mathbf{v}(0,\rho) >= |\rho|^{2\alpha} f(\frac{t}{\tau(|\rho|)}) \tag{7}$$

with f falling off at infinity. There seems to be, however, scant experimental data on (7), in particular on whether it might be true only in the Lagrangian frame.

3 Synthetic velocity ensembles

To gain further understanding how Richardson's observations might be explained in terms of the roughness of the turbulent velocities we will consider the equation (1) with a velocity field with explicitly given gaussian statistics mimicking the properties of real turbulent velocities described in the preceding section. It should be noted however that high Reynolds number solutions to the Navier-Stokes equations are far from gaussian so gaussianity is a simplifying assumption.

Thus we assume $\mathbf{v}(x,t)$ is gaussian with zero mean and covariance satisfying (7). Concretely, we take

$$\left\langle v^i(t,\mathbf{r})\, v^j(t',\mathbf{r}')\right\rangle = A \int_{L^{-1}<|k|<\eta^{-1}} \mathrm{e}^{-|t-t'|B\, k^{2\beta}}\ \frac{\mathrm{e}^{i\,\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^{d+2\alpha}}\, P^{ij}(\mathbf{k},\wp)\, \frac{d\mathbf{k}}{(2\pi)^d}\,. \tag{8}$$

Let us discuss the various parameters in (8).

1. L is the largest length scale in the problem, modeling the injection scale of turbulent velocities. It dominates the mean square velocity field:

$$\left\langle \mathbf{v}(t,\mathbf{r})^2 \right\rangle = \text{const} \int_{L^{-1}<|\mathbf{k}|<\eta^{-1}} |\mathbf{k}|^{-2\alpha-d} \frac{d\mathbf{k}}{(2\pi)^d} \sim L^{2\alpha}$$

Therefore we expect strong L dependence in the one particle motion.

2. η is the smallest length scale in the problem, modeling the dissipative scale of turbulent velocities. For $\eta \neq 0$ the velocities are smooth, but as $\eta \to 0$ they loose their smoothness and become only Hölder:

$$\left\langle \Delta\mathbf{v}(t,\mathbf{r})^2 \right\rangle = \text{const} \int_{L^{-1}<|\mathbf{k}|<\eta^{-1}} (1-\cos(\mathbf{k}\cdot\mathbf{r}))|\mathbf{k}|^{-2\alpha-d} \frac{d\mathbf{k}}{(2\pi)^d}$$

which, as $\eta \to 0$ is proportional to $|\mathbf{r}|^{2\alpha}$ for $|\mathbf{r}| << L$. Note that this velocity difference has both the $\eta \to 0$ and the Ł $\to \infty$ limits. In those limits our velocity ensemble becomes scale invariant:

$$\Delta\mathbf{v}(t,\mathbf{r}) \sim \lambda^{-\alpha}\Delta\mathbf{v}(\lambda^{2\beta}t,\lambda\mathbf{r}) \tag{9}$$

where $\sim$ means in law. It is believed that real turbulent velocities, i.e. solutions of the Navier-Stokes equations, have the $\eta \to 0$ limit but not the Ł $\to \infty$ limit. Indeed, it is believed that

$$\left\langle (\hat{\mathbf{r}}\cdot\Delta\mathbf{v}(t,\mathbf{r}))^n \right\rangle = A_n(\frac{L}{|\mathbf{r}|})^{\zeta_n}|\mathbf{r}|^{\frac{n}{3}}(1+o(\frac{|\mathbf{r}|}{L}))$$

with $\zeta_n > 0$ for $n > 3$. This aspect (*intermittency*) of turbulent velocities is thus not modeled by our gaussian ensemble. Since the two particle motion (15) involves velocity differences we expect it to be less sensitive to the large scales than the one particle motion.

3. The factor

$$P^{ij}(\mathbf{k},\wp) = \frac{1-\wp}{d-1}\delta^{ij} + \frac{\wp d-1}{d-1}\frac{k^ik^j}{k^2}$$

involves the compressibility degree $0 \le \wp \le 1$ which can be used to interpolate between incompressible velocities corresponding to $\wp = 0$ and ones corresponding to $\wp = 1$ which are gradients of a potential (in one dimension necessarily $\wp = 1$).

4. The two parameters A and B can be used to discuss limiting cases of the ensemble. For $B = 0$ we have the *frozen ensemble* of time independent velocities

$$\left\langle v^i(t,\mathbf{r})\, v^j(t',\mathbf{r}')\right\rangle \;=\; A\int \frac{\mathrm{e}^{\,i\,\mathbf{k}\cdot(\mathbf{r}-\mathbf{x}')}}{k^{d+2\alpha}}\; P^{ij}(\mathbf{k},\wp)\;\frac{d\mathbf{k}}{(2\pi)^d}\,, \tag{10}$$

whereas in the opposite limit $A \to \infty$, $B \to \infty$ with $\frac{A}{B}$ held constant one obtains the *Kraichnan ensemble* of delta correlated velocities

$$\left\langle v^i(t,\mathbf{r})\, v^j(t',\mathbf{r}')\right\rangle \;=\; C\;\delta(t-t')\int \frac{\mathrm{e}^{\,i\,\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^{d+\xi}}\; P^{ij}(\mathbf{k},\wp)\;\frac{d\mathbf{k}}{(2\pi)^d} \equiv \delta(t-t')D^{ij}(\mathbf{r}-\mathbf{r}')\,, \tag{11}$$

which are Hölder continuous with any exponent smaller than $\frac{1}{2}\xi$ with

$$\xi = 2(\alpha+\beta).$$

4 Kraichnan ensemble

The one and two particle motion can be analyzed completely explicitly in the Kraichnan model. The first question we must face is the meaning of the equation (1) when the velocity field is not smooth enough (Lipschitz) for the uniqueness results of differential equations to be applicable. That is, we need to regularize (1). There are two natural ways to do this. The first is to reintroduce the viscous scale η and study the $\eta \to 0$ limit. The second is to introduce molecular noise, i.e. to replace (1) by the stochastic equation

$$\frac{d\mathbf{R}}{dt} \;=\; \mathbf{v}(t,\mathbf{R}) + \sqrt{2\kappa}\dot{\beta}\,. \tag{12}$$

where β is Brownian motion. Both regularizations and their mixture are physically relevant, the relative size of the viscosity and diffusivity is described by the Peclet number. There are also some subtleties in the order of the limits, see [3]. Here we content ourselves to study the equation (12).

The solution of (12) with a fixed realization of $\mathbf{v}$ defines a Markov process $\mathbf{x}(t|\mathbf{v})$. (This was done rigorously in [4].) Let us denote by $P_t(\mathbf{x},\mathbf{y}|\mathbf{v})$ its transition probability density. It satisfies the equation

$$\partial_t P_t = (\kappa\Delta - \mathbf{v}\cdot\nabla)P_t. \tag{13}$$

Due to the white noise nature of $\mathbf{v}$ one has to be a bit careful with the product in (13): the right definition is in the Stratonowich sense. This means in particular that the average of P_t over the velocity realizations satisfies the equation

$$\partial_t\Big\langle P_t\Big\rangle = (\kappa\Delta + \tfrac{1}{2}D^{ij}(0)\partial_i\partial_j)\Big\langle P_t\Big\rangle. \tag{14}$$

Note how the drift term in (13) contributes a second order term involving the velocity two point function. This leads to a renormalization of the molecular diffusion coefficient κ by the "eddy diffusion constant"

$$\kappa_{\rm ren} = \kappa + \kappa_{\rm eddy}$$

where as in (9) we get that $\kappa_{\rm eddy}$ is proportional to L^ξ. Thus the single particle motion is diffusive, given by eq. (3), and the diffusion persists even in the limit of vanishing molecular diffusivity $\kappa \to 0$ and is dominated by the large scale fluctuations of the velocity field.

Let us next consider the motion of two particles in the velocity field picked randomly from the Kraichnan ensemble. Denoting the position of the first particle by $\mathbf{R}$ and the second by $\mathbf{R}+\rho$ where ρ is the separation vector of the particles the latter satisfies the equation

$$\frac{d\rho}{dt} = \mathbf{v}(t,\rho+\mathbf{R}) - \mathbf{v}(t,\mathbf{R})\,, \tag{15}$$

We may write this in terms of the velocity field in the frame of the first particle, the so called *quasi-Lagrangian velocity*

$$\mathbf{v}^{\mathrm{qL}}(t,\mathbf{r}) = \mathbf{v}(t,\mathbf{r}+\mathbf{R}(t))\,, \tag{16}$$

so that

$$\frac{d\rho}{dt} = \Delta\mathbf{v}^{\mathrm{qL}}(t,\rho)\,, \tag{17}$$

In general, the probability law of the quasi-Lagrangian velocity field is a complicated function of the law of $\mathbf{v}$, in particular not gaussian. However, in the Kraichnan model due to the delta correlation in time, it is not hard to check that the quasi-Lagrangian velocity field has the same law as the velocity itself. Therefore ρ satisfies the single particle equation with the velocity $\mathbf{v}$ replaced by the velocity difference $\Delta\mathbf{v}$. Since the spatial part of the covariance of the latter is

$$2(D^{ij}(0) - D^{ij}(\rho)) \equiv 2d^{ij}(\rho)$$

we conclude that in the limit of vanishing molecular diffusivity the transition probability density for ρ averaged over the $\mathbf{v}$, $\mathcal{P}_t(\rho,\rho')$ satisfies

$$\partial_t\mathcal{P}_t = d^{ij}(\rho)\partial_i\partial_j\mathcal{P}_t \tag{18}$$

i.e. a diffusion equation where the diffusion constant depends on the separation of the particles. In the limit $\eta \to 0$ and $L \to \infty$ the function $d^{ij}(\rho)$ is homogeneous of degree ξ i.e.

$$d^{ij}(\rho) = |\rho|^{\xi} d^{ij}(\hat{\rho}).$$

This leads to the scaling

$$\mathcal{P}_t(\rho_0,\rho) = \lambda^d \mathcal{P}_{\lambda^{2-\xi}t}(\lambda\rho_0,\lambda\rho) \tag{19}$$

which implies

$$\left\langle \rho(t,\rho_0)^2 \right\rangle = t^{\frac{2}{2-\xi}} f\Big(\frac{|\rho_0|}{t^{\frac{1}{2-\xi}}}\Big). \tag{20}$$

The function f turns out to be sensitive to the compressibility degree of the velocity field [5].

For incompressible velocities $f(0) \neq 0$ and therefore

$$< \rho(t,\rho_0)^2 >= Bt^{\frac{2}{2-\xi}}$$

where

$$\lim_{\rho_0\to 0} B > 0.$$

This fits to Richardson's observations and yields the Richardson law provided $\xi = \frac{4}{3}$ which in turn follows from $\alpha = \frac{1}{3} = \beta$, the Kolmogorov values.

Since (20) represents the average over velocity realizations of the trajectory separation we may conclude that with nonzero probability the trajectories in a *fixed* incompressible velocity field are non unique. Thus for such velocities, the limit of the Markov process $\mathbf{x}(t|\mathbf{v})$ as the molecular diffusivity $\kappa \to 0$ remains a stochastic process. The solution to the deterministic looking differential equation (1) is not a single trajectory, but a probability measure on a set of trajectories!

If we reintroduce the viscous scale η below which the velocity field is smooth, the equation (20) remains true for separations larger than η. For smaller separations the standard exponential separation (2) holds, with an explicitly calculable Lyapunov exponent λ [10].

The Kraichnan model has served as a testing ground for many other ideas in turbulence. Just to mention one, the study of the multi-particle motion leads to understanding the problem of intermittency in the statistics of a tracer concentration carried by the velocity field [6], [7], [8], [9]. However, the delta correlation in time of the velocity field is certainly an unphysical idealization and therefore it would be nice to be able to relax this assumption and test the robustness of the mechanisms uncovered in the Kraichnan model. Thus we will turn to the study of the time correlated self similar ensemble (8).

5 Beyond the Kraichnan ensemble

Let us now consider the equation for the trajectory separation (16) for more general velocity fields [11]. If we wish to model this problem with velocities taken from the self similar ensemble (8) we need to address the question of the difference of the statistics of $\mathbf{v}$ and $\mathbf{v}^{\mathrm{qL}}$. In general, these agree only in the delta correlated case. We have two choices:

(a) Model the Eulerian velocity $\mathbf{v}$ by (8).

(b) Model the quasi-Lagrangian velocity $\mathbf{v}^{\mathrm{qL}}$ by (8).

We consider first the case (b). This means we replace (16) by the equation

$$\frac{d\rho}{dt} = \Delta\mathbf{v}(t,\rho)\,, \tag{21}$$

where $\mathbf{v}$ has the statistics (8) which implies

$$\left\langle \Delta v^i(t,\mathbf{r})\,\Delta v^j(t',\mathbf{r}')\right\rangle \;= \\ 2A\int_{|k|>L^{-1}} \mathrm{e}^{-|t-t'|B\,k^{2\beta}}\;\frac{1-\cos(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}'))}{k^{d+2\alpha}}\;P^{ij}(\mathbf{k},\wp)\;\frac{d\mathbf{k}}{(2\pi)^d} \tag{22}$$

where we took $\eta = 0$.

In the Kraichnan model the separation process ρ is self similar as seen in (19). Let us therefore define the rescaled separation

$$\rho_\lambda(t) = \lambda^{-\sigma}\rho(\lambda t) \tag{23}$$

and study its dynamics. We obtain

$$\frac{d\rho_\lambda}{dt} = \Delta\mathbf{v}_\lambda(t, \rho_\lambda)\,, \tag{24}$$

where

$$\Delta\mathbf{v}_\lambda(t, \rho) = \lambda^{1-\sigma}\Delta\mathbf{v}(\lambda t, \lambda^\sigma \rho). \tag{25}$$

Now $\Delta\mathbf{v}_\lambda$ is distributed again as in (22) with the parameters

$$\begin{aligned} A(\lambda) &= \lambda^{2(1-(1-\alpha)\sigma)}A \\ B(\lambda) &= \lambda^{1-2\beta\sigma}B \\ L(\lambda) &= \lambda^{-\sigma}L. \end{aligned}$$

We can now ask when is this rescaled velocity field $\Delta\mathbf{v}_\lambda$ identical in law to $\Delta\mathbf{v}$. This happens when

$$L = \infty$$

and either

(1) $\alpha + 2\beta = 1$, $\sigma = \frac{1}{1-\alpha}$

or

(2) $B = 0$, $\sigma = \frac{1}{1-\alpha}$

or

(3) $A \to \infty$ $B \to \infty$ with $\frac{A}{B}$ fixed, $\sigma = \frac{1}{2(1-\alpha-\beta)}$

In all these cases we conclude that $\rho(t, \rho_0)$ and $\rho_\lambda(t, \lambda^{-\sigma}\rho_0)$ should be identically distributed and therefore

$$\langle \rho(t, \rho_0)^2 \rangle = t^{2\sigma} g(\frac{\rho_0}{t^\sigma}) \tag{26}$$

where the scaling function g would again presumably depend on the compressibility and possibly also the exponents α and β.

The case (1) consists of a line of fixed points under the scaling (25), whereas the case(2) is the frozen ensemble and case (3) the Kraichnan ensemble. Note that the Kolmogorov value $(\frac{1}{3}, \frac{1}{3})$ lies on the line (1) and gives rise to the Richardson value $\sigma = \frac{3}{2}$ in case $g(0) > 0$.

Let us finally ask the question of what can be said of the behavior of the pair separation when the parameters don't correspond to the fixed point values above i.e. let $\alpha+2\beta \neq 1$ and A, B finite, nonzero. As usual in such contexts we expect the short and long time behavior to be controlled by appropriate fixed points. Indeed, since $\lambda \to 0$ behavior of ρ_λ gives the short time asymptotics and $\lambda \to \infty$ behavior the long time asymptotics we need to study what happens to the velocity $\mathbf{\Delta v}_\lambda$ of (25) in these limits. It will be convenient to distinguish various regions in the α, β plane as depicted in Figure 1.

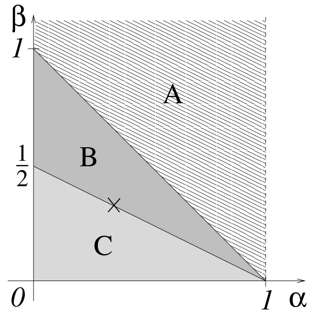

Fig. 1

Let us consider first $\lambda \to \infty$ at $L = \infty$. Taking $\sigma = \frac{1}{2(1-\alpha-\beta)}$ fixes the ratio $\frac{A(\lambda)}{B(\lambda)}$ with $A(\lambda),\ B(\lambda) \to \infty$ if $\alpha + \beta > 1$ (domain A in Fig. 1) or if $\alpha + 2\beta < 1$ (domain C in Fig. 1). The latter case leads to a non-singular Kraichnan ensemble of velocity differences with $\xi = 2(\alpha + \beta)$ whereas the former one does not (it would correspond to $\xi > 2,\ L = \infty$). We may then expect that

$$\lim_{\lambda\to\infty} \lambda^{d\sigma}\, \mathcal{P}(\lambda^\sigma \rho_0, \lambda^\sigma \rho; \lambda t) = \mathcal{P}^{\rm Kr}(\rho_0, \rho; t) \qquad \text{for } \sigma = \tfrac{1}{2(1-\alpha-\beta)} \text{ and } \alpha + 2\beta < 1 \tag{27}$$

where $\mathcal{P}^{\rm Kr}$ pertains to the Kraichnan model with $\xi = 2(\alpha + \beta)$. This is indeed consistent with the scaling properties of the Kraichnan model dispersion.

Taking $\sigma = \frac{1}{1-\alpha}$ keeps $A(\lambda)$ constant while $B(\lambda) \to 0$ if $\alpha+2\beta > 1$ (domains A and B in Fig.1). We then expect that

$$\lim_{\lambda\to\infty} \lambda^\sigma \mathcal{P}(\lambda^\sigma \rho_0, \lambda^\sigma \rho; \lambda t) = \mathcal{P}^{\rm fr}(\rho_0, \rho; t)\ , \text{ for } \sigma = \tfrac{1}{1-\alpha} \text{ and } \alpha + 2\beta > 1, \tag{28}$$

where $\mathcal{P}^{\rm fr}$ stands for the PDF of the frozen velocity model with Hölder exponent α.

Inquiring about the short-time asymptotics of the trajectory dispersion reverses the asymptotics. We should then have

$$\lim_{\lambda\to 0} \lambda^\sigma\, \mathcal{P}(\lambda^\sigma \rho_0, \lambda^\sigma \rho; \lambda t) \quad = \mathcal{P}^{\rm fr}(\rho_0, \rho; t) \text{ for } \sigma = \tfrac{1}{1-\alpha} \text{ and } \alpha + 2\beta < 1 \tag{29}$$

(i.e. in domain C in Fig. 1) with the same value of the Hölder exponent α, and

$$\lim_{\lambda\to 0} \lambda^\sigma \mathcal{P}(\lambda^\sigma\rho_0, \lambda^\sigma\rho; \lambda t) = \mathcal{P}^{\rm Kr}(\rho_0,\rho;t)$$
$$\text{for } \sigma = \frac{1}{2(1-\alpha-\beta)} \text{ and } \begin{cases}\alpha+\beta<1,\\ \alpha+2\beta>1\end{cases} \quad (30)$$

(i.e. in domain B in Fig. 1) with $\xi = 2(\alpha+\beta)$ for the Kraichnan model. Again, this is consistent with the scaling of the limiting PDF's.

To summarize, the scale invariance of the statistics of the pair dispersion although broken away from the $\alpha + 2\beta = 1$ line, should be restored at long and short times. In the regions B and C the long (short) time asymptotics of the separation is controlled by the frozen (Kraichnan) and Kraichnan (frozen) fixed points respectively. In the region A, although the velocities there possess equal time statistics that is only Hölder continuous, the trajectories seem to be unique due to the rapid decrease of time correlations in short scales.

It is worth stressing, that the relations (27) to (30) are conjectural. The PDF $\mathcal{P}$ is a complicated nonlinear functional of the velocity statistics and the conjectured relations assume their continuity in an appropriate topology, which is not obvious. In particular, since the convergence of the rescaled velocity covariances to the one of the Kraichnan model is very slow at long distances, there is a potential threat for the corresponding convergence of the rescaled PDF $\mathcal{P}(\rho_0,\rho;t)$ coming from the contribution of trajectories that venture far apart, if such contributions are important. Similarly, the slow convergence to the frozen model at long distances could create problems for the corresponding convergence of the rescaled pair dispersion PDF. Whether such effects invalidate some of the conclusions (27) to (30) could be, in principle, studied in perturbation theory around the Kraichnan or frozen model.

Let us finally note that the regions B and C have a simple interpretation in terms of the so called eddy turnover time. This is the typical timescale of length scale r fluctuations in the velocity field $\tau_{eddy}(r) = \frac{r}{|\Delta \mathbf{v}(r)|} \sim r^{1-2\alpha}$. The correlation time for such fluctuations in our ensemble is $\tau_c(r) = r^{2\beta}$. Thus on the line $\alpha + 2\beta = 1$ these times coincide, whereas e.g. in the region B $\tau_{eddy}(r) > \tau_c(r)$ for small separations i.e. such fluctuations are decorrelated, in accordance with the picture that this region in small times should be governed by the Kraichnan fixed point.

Let us now turn to the case (a) above i.e. modeling the Eulerian velocity field by the gaussian self similar ensemble. The quasi-Lagrangian velocity field governing the evolution of the pair separation is defined as the Eulerian $\mathbf{v}$ relative to the frame of reference of one of the particles. The motion of this particle, as we saw before, is dominated by the large scale fluctuations in $\mathbf{v}$ that have variance of the order $L^{2\alpha}$. Indeed, for short times this motion is ballistic as can be seen by rewriting the trajectory equation (1) as

$$\mathbf{R}(t) \;=\; \int_0^t \mathbf{v}(s,0)\,ds \;+\; \int_0^t [\mathbf{v}(s,\mathbf{R}(s)) - \mathbf{v}(s,0)]\,ds\,. \tag{31}$$

Since the r.m.s. value of velocity $v(s,0)$ in the ensemble (8) is proportional to L^α, we may expect that, for fixed t, the first integral is of the order L^α . On the other hand, the r.m.s. equal-time velocity differences on scales much smaller than L are of the order *distance*$^\alpha$. In particular, on the scales $\sim L^\alpha$ they are of the order L^{α^2} and the second integral is of the order $L^{\alpha^2} \ll L^\alpha$. More precisely, let us observe that the Gaussian process with the components $L^{-\alpha}\mathbf{v}(t,0)$ and $L^{-\alpha^2}[\mathbf{v}(t,L^\alpha\mathbf{r}) - \mathbf{v}(t,0)]$ converges in law when $L \to \infty$ to the t-independent Gaussian process $(\mathbf{v}_0,\, \mathbf{w}(\mathbf{r}))$ with the 2-point functions

$$\begin{aligned}
\langle\, \mathbf{v}_0\, \mathbf{v}_0 \,\rangle \quad &= \quad A \int \frac{1}{k_1^{d+2\alpha}}\, \frac{d\mathbf{k}}{(2\pi)^d}\,, \\
\langle\, \mathbf{w}(\mathbf{r})\, \mathbf{w}(\mathbf{r}') \,\rangle \quad &= \quad A \int \frac{(1-\mathrm{e}^{i\mathbf{k}\cdot\mathbf{r}})(1-\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{r}'})}{k^{d+2\alpha}}\, \frac{d\mathbf{k}}{(2\pi)^d}\,, \\
\langle\, \mathbf{v}_0\, \mathbf{w}(\mathbf{r}) \,\rangle \quad &= \quad 0\,.
\end{aligned} \tag{32}$$

Note the independence of $\mathbf{v}_0$ and $\mathbf{w}(\mathbf{r})$. It is then natural to conjecture that the following convergences in law take place, describing the leading terms in the single trajectory statistics for large L:

$$L^{-\alpha}\mathbf{r}(t) \quad \underset{L\to\infty}{\longrightarrow} \quad \mathbf{v}_0 t\,, \tag{33}$$

$$L^{-\alpha^2}[\mathbf{R}(t) - L^\alpha \mathbf{v}_0 t] \quad \underset{L\to\infty}{\longrightarrow} \quad \int_0^t \mathbf{w}(\mathbf{v}_0 s)\,ds\,. \tag{34}$$

Turning next to the pair separation, we expect it to show L dependence in its statistics due to the strong L dependence of the trajectory of the reference particle which we dent expect to decouple as in the Kraichnan case. It appears quite difficult to study this L dependence in the general gaussian ensemble above and particularly for long times. Here we will consider just a simple case of (8), namely the frozen model in one dimension.

We shall consider two particle trajectories $x(t)$ and $x(t)+\rho_0$ staring at time zero at origin and $\rho_0 > 0$, respectively, and we shall try to estimate the behavior of their separation $\rho(t)$. First notice that $\rho(t) \geq 0$, i.e. the order of the particles on the line will never change. For large L, the dominant events are when the velocities of the particles and at the intermediate points are all of the order L^α and of the same sign during the time interval $(0,t)$. Let us suppose that they are positive (the case of negative velocities can be treated in a symmetric way).

The crucial fact resulting from the one-dimensional geometry is the identity

$$\int_0^{\rho_0} \frac{d\rho}{v(\rho)} \;=\; \int_{x(t)}^{x(t)+\rho(t)} \frac{d\rho}{v(\rho)}\,. \tag{35}$$

The left hand side is the time Δt that the first particle takes to reach the initial position of the second one. The best way to understand the above identity is by releasing the second particle after the delay Δt so that both particles move subsequently together. The delay changes nothing in the movement of the second particle since the velocity field is frozen. The delayed particle will then arrive at position $x(t)$ at time t (together with the first particle) and at position $x(t)+\rho(t)$ at time $t+\Delta t$. But the right hand side of Eq. (35) is the time that the second particle takes to move from $x(t)$ to $x(t)+\rho(t)$. Hence the identity which may be also proven more formally by noticing that the time derivative of its right hand side vanishes.

Writing for large L

$$x(t) \;=\; L^{\alpha} v_0 t \,+\, \mathcal{O}(L^{\alpha^2})\,, \qquad v(x(t)) \;=\; L^{\alpha} v_0 \,+\, L^{\alpha^2} w(v_0 t) \,+\, \mathcal{O}(L^{\alpha^3})\,, \tag{36}$$

see relations (33) and (34), and anticipating that $\rho(t)=\mathcal{O}(1)$, Eq. (35) may be approximated as

$$\frac{\rho_0}{L^{\alpha} v_0} \;=\; \frac{\rho(t)}{L^{\alpha} v_0 + L^{\alpha^2} w(v_0 t)} \,+\, \mathcal{O}(L^{\alpha^3-2\alpha}) \tag{37}$$

from which we infer that

$$\rho(t)-\rho_0 \;=\; L^{\alpha^2-\alpha} \rho_0\, v_0^{-1} w(v_0 t) \;+\; \mathcal{O}(L^{\alpha^3-\alpha})\,. \tag{38}$$

The process $w(x)$ is the two-sided **fractional Brownian motion**, i.e. the Gaussian process with mean zero and 2-point function

$$\langle w(x)\, w(y)\rangle \;=\; \tfrac{1}{2}(x^{2\alpha}+y^{2\alpha}-|x-y|^{2\alpha}) \;\equiv\; G(x,y) \tag{39}$$

for $x,\, y \geq 0$. Note the scale invariance under $w(x) \mapsto \mu^{-\alpha} w(\mu x)$. The precise conjecture would then assert the convergence in law

$$L^{\alpha(1-\alpha)}\,[\rho(t)-\rho_0] \quad \underset{L\to\infty}{\longrightarrow} \quad \rho_0\, v_0^{\alpha-1}\, w(t)\,. \tag{40}$$

Note that the above calculations indicate that not only a single particle motion, but also the separation of trajectories in the Eulerian frozen one-dimensional velocity ensemble are dominated by the scale L velocities, i.e. by the large eddy sweeping, but the effect on the separation is inverse to that on the single particle motion. Whereas the latter one becomes very fast for large L, the trajectory separation becomes essentially frozen to the initial value in a localization-type effect. It would be interesting to know if the localizing tendency persists in the more general Eulerian Gaussian ensembles (8).

6 Outlook

The Richardson law for pair dispersion seems to be intimately connected to the irregularity of turbulent velocity fields. The framework of classical chaos is not applicable in such situations, being restricted to the viscosity dominated scales where the velocity field presumably is smooth.

The simple gaussian ensemble of velocities that are delta correlated in time (the Kraichnan model) reproduces this law and so does the model where quasi-Lagrangian velocities are modeled by a more general scale invariant gaussian ensemble with time correlations. Such gaussian models with time correlations however lead to strong sweeping effects by the large scale velocity fields and might not be good models for turbulent velocities where the sweeping effects should be less pronounced. It is still a major open problem how to quantitatively understand the Richardson law in real turbulent velocity fields.

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